EXISTENTIAL CLOSURE OF GRAPHS

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by
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A thesis submitted to the
School of Graduate Studies
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, Newfoundland

Submitted: Summer 2011

Dedication

To the memory of my "father" who was a "father" by all the meaning and to the honour of my beautiful "mother".

Abstract

We study the n-existential closure property of graphs which was first considered by Erdős and Rényi in 1963. A graph G is said to be n-existentially closed, abbreviated as n-e.c., if for each pair (A, B) of disjoint subsets of $V(G)$ with $|A| + |B| \leq n$ there exists a vertex in $V(G) \setminus (A \cup B)$ which is adjacent to each vertex in A and to no vertex in B. Accordingly, we call the largest integer n (if it exists) for which a given graph G is n-e.c. the existential closure number of G, and we denote it by $\Xi(G)$. Despite the fact that there are many graphs that are n-e.c., only a handful of explicit families of graphs with the property have been found.

Recently the property has become a subject of renewed interest and several techniques have appeared in the literature to construct n-e.c. graphs. These techniques benefit from design theory, finite geometry, probability theory, matrix theory, as well as computer search over classes of graphs that are likely to contain n-e.c. graphs. In this thesis we focus on two subjects: obtaining 3-existentially closed graphs using graph operations and investigating the n-e.c. property of the block intersection graphs of infinite designs.

In 2001 Bonato and Cameron examined several graph operations to see which operations could be used to construct n-e.c. graphs from given n-e.c. graphs, and showed that the symmetric difference of two 3-e.c. graphs is a 3-e.c. graph. In 2008

another 3-e.c. preserving graph operation was introduced by Baker et al. We have taken a different approach to the construction of Baker et al. that enables us to relax the requirement that the two graphs considered be both 3-e.c. We formulate the construction as the modular graph product denoted by $\diamond$ and we determine necessary and sufficient conditions for the graph $G\diamond H$ to be 3-e.c. given that H itself is a 3-e.c. graph. We then use this operation to construct new classes of 3-e.c. graphs of the form $G\diamond H$ where G is not necessarily 3-e.c. The classes that we consider are those for which G is either a complete multipartite graph or a strongly regular graph. The graphs G for which we show that $G\diamond H$ is 3-e.c. can have as few as four vertices, which represents an improvement in comparison to when G is required to be 3-e.c.

As part of an effort to find n-e.c. graphs, Forbes et al. first considered the block intersection graphs of Steiner triple systems, and later McKay and Pike studied the n-e.c. property of graphs arising from BIBDs. We extend the study of the n-existential closure property of block intersection graphs of designs to infinite designs. An infinite t-(v, k, λ) design $\mathcal{D}$ is a design with an infinitely many points while k, t and λ can be either finite or infinite. The block intersection graph of a design $\mathcal{D}$ denoted by $G_{\mathcal{D}}$ is a graph with the block set of $\mathcal{D}$ as the vertex set and two vertices of $G_{\mathcal{D}}$ are adjacent if their corresponding blocks share a point. These graphs have infinite vertex sets and have motivated us to investigate whether we can use the construction to find another construction of the Rado graph (the countably infinite random graph that is known to be n-e.c. for all n).

We suppose that t and λ are finite and solve the problem in two cases: when k is finite and when k is infinite. If k is finite, then for such an infinite design $\mathcal{D}$ we show that $\Xi(G_{\mathcal{D}}) = \min\{t, \lfloor\frac{k-1}{t-1}\rfloor+1\}$ if $\lambda = 1$ and $2 \leq t \leq k$, and $2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil\frac{k}{t}\rceil\}$ if $\lambda \geq 2$ and $2 \leq t \leq k-1$. Our results show that block intersection graphs of such infinite designs are different from countably infinite random graphs as n is bounded for the n-existential closure property of the block intersection graphs of such infinite designs.

If k is infinite and $(t, \lambda) \neq (1, 1)$, then for each non-negative integer n, we show that there exists a t-(v, v, λ) design $\mathcal{D}$ such that $\Xi(G_{\mathcal{D}}) = n$. We also show that there exists a t-(v, v, λ) design $\mathcal{D}'$ such that $G_{\mathcal{D}'}$ is n-e.c. for each non-negative integer n. This implies the existence of t-$(\aleph_0, \aleph_0, \lambda)$ designs whose block intersection graphs are isomorphic to the Rado graph. However, if $k < v$, then $\Xi(G_{\mathcal{D}}) \leq \min\{\ell, t\}$ where ℓ is the smallest cardinal such that there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$.

Acknowledgements

It is an honour to extend my gratitude to all the people who made this thesis possible with my humble acknowledgments. Foremost, I would like to extend my thanks to my supervisor Dr. David Pike for his advice, encouragement, and guidance from the starting day of this research until today. I particularly appreciate him for his understanding and for providing me with the time and space whenever I needed.

Many thanks go to Dr. Danny Dyer for his inspiration and for coordinating weekly combinatorial seminars and to Dr. Nabil Shalaby for his support and for teaching me the necessary knowledge of Design Theory. I gratefully acknowledge Dr. Daniel Horsley for his collaboration and crucial contribution to this thesis. I also wish to thank Dr. Peter Dukes and Dr. Catharine Baker for examining this thesis and for their helpful comments and suggestions.

I am grateful to Dr. Chris Radford, the head of the Department of Mathematics and Statistics, Dr. Jie Xia and Dr. J. C. Loredo-Osti, the former and current graduate officers, for their concerns regarding graduate students. My words of thanks go to the staff of the department and particularly to Ms. Rosalind English.

I also would like to thank my former supervisor Dr. Mehdi Hashemi who showed me the beauty of Graph Theory and for his encouragements. Likewise, I am thankful to Dr. Mehdi Dehghan for all the help and support that he has provided to me.

Special thanks go to Dr. Öznur Yaşar for helping me while she was studying here, and my deepest thanks go to Ms. Daniela Silvesan for helping me whenever I needed.

My deepest gratitude goes to my family for their love and support throughout my life: my dearest father for the unflagging care and love he gave me and weekly follow ups on my progress; my beautiful mother for her love and endless sacrifices to keep the family warm and running; my kind brother for all his support and encouragement; my beautiful, strong, and capable sisters for their love and encouragements.

I also would like to thank all my friends around the world for their care, help, and all the happy moments that we have shared.

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Chapter 1

Introduction

In this chapter, we present the required definitions and terminology used throughout the thesis as well as a review of previous results that are related to the thesis. These mainly include results on the existential closure property of graphs.

1.1 Existential Closure: Definition and Origin

The n-existential closure property was originally studied in 1963 by Erdős and Rényi [22] where they showed that almost all graphs have the property. The n-existential closure property is defined as follows:

Definition 1.1 *A graph G is said to be n-existentially closed, or n-e.c., if for each*

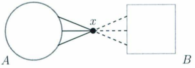

Figure 1.1: The n-existential closure property.

pair (A, B) of disjoint subsets of $V(G)$ with $|A| + |B| \leq n$ there exists a vertex $x \in V(G) \setminus (A \cup B)$ which is adjacent to each vertex in A and to no vertex in B.

Equivalently, a graph G with vertex set $V(G)$ is said to be n-existentially closed, or n-e.c., if for each proper subset S of $V(G)$ with cardinality $|S| = n$ and each subset T of S, there exists some vertex x not in S that is adjacent to each vertex of T but to none of the vertices of $S \setminus T$.

The n-existential closure property of a graph is illustrated in Figure 1.1. The solid edges between x and the set A show adjacency between x and all the vertices in A, and the dashed edges between x and the set B indicate non-adjacency between x and all the vertices in B.

It is then clear from the definition that a graph G is 1-e.c. if and only if for any vertex u of G, there exists a vertex adjacent to u and there is a vertex non-adjacent to u. Equivalently, a graph G is 1-e.c. if and only if it has no isolated vertex and has no universal vertex. Similarly, a graph G is 2-e.c. if and only if for any set S of two

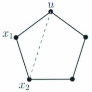

Figure 1.2: Example of a 1-e.c. graph.

vertices, there are four other vertices that are joined to the vertices of S in all four possible ways.

Definition 1.2 *[29] If it exists, we call the largest integer n for which a given graph G is n-e.c. the existential closure number of G, and we denote it by $\Xi(G)$.*

For example, a graph G that is 4-e.c. but is not 5-e.c. has existential closure number 4; $\Xi(G) = 4$. Also, for a cycle C_m with length $m \geq 4$, $\Xi(C_m) = 1$ since C_m is 1-e.c. but not 2-e.c. This is because for any vertex u of C_m with $m \geq 4$, there is a vertex x_1 adjacent to u and there is a vertex x_2 non-adjacent to u (see Figure 1.2); however, for any pair of adjacent vertices of C_m with $m \geq 4$, there is no vertex adjacent to both of them.

Although we will not focus on directed graphs in this thesis, we remark that the n-existential closure property has been defined for directed graphs as well and mostly has been considered for complete directed graphs, which are referred to as tournaments. For the definition, results, and open problems the reader is referred to [11, 13, 28].

Some of the early research regarding existential closure of graphs was asymptotic and probabilistic in nature. The probability space $G\left(m, \frac{1}{2}\right)$ consists of graphs with vertices $\{0, 1, \ldots, m-1\}$ so that two distinct vertices are joined independently and with probability $\frac{1}{2}$. The following theorem was first proved in [22] and states that almost all finite graphs have the n-e.c. property; for a proof see [9].

Theorem 1.3 *For a fixed integer $n > 1$*

1. *with probability 1 as $m \to \infty$, a graph $G \in G(m, \frac{1}{2})$ satisfies the n-e.c. property,*

2. *if m is chosen so that $\binom{m}{n}\frac{1}{2^n}(1 - \frac{1}{2^n})^{m-n} < 1$, then there is an n-e.c. graph of order m.*

Although Theorem 1.3 implies that for a fixed integer n, there are many examples of n-e.c. graphs, to date, only a handful families of graphs have been found to have the property.

Theorem 1.4 below is useful in the study of n-e.c. graphs; the proof is trivial.

Theorem 1.4 *For a fixed integer n, if the graph G is n-e.c., then*

1. *the graph G is m-e.c. for all $1 \leq m \leq n-1$,*

2. *the graph G has order at least $n + 2^n$, and at least $n2^{n-1}$ edges,*

3. *the graph $\overline{G}$ is n-e.c. where $\overline{G}$ is the complement of G.*

Most of the n-e.c. graphs known to date are strongly regular or vertex-transitive; these properties are defined as follows.

Definition 1.5 *A k-regular graph G in which each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours is called a strongly regular graph; we say that G is a $SRG(v, k, \lambda, \mu)$ with $v = |V(G)|$.*

Definition 1.6 *A graph G is vertex-transitive if for every pair $x, y \in V(G)$ there is an automorphism of G that maps x to y.*

Similarly, a graph G is edge-transitive if for all $e_1, e_2 \in E(G)$ there is an automorphism of G that maps the endpoints of e_1 to the endpoints of e_2.

1.2 Minimum Orders

A challenge in the search for n-e.c. graphs is to find such graphs on small orders. With $m_{ec}(n)$ we denote the minimum order of an n-e.c. graph. By the second item in Theorem 1.3, for each positive integer n, n-e.c. graphs exist and hence $m_{ec}(n)$

is well-defined. One can easily find the smallest non-isomorphic graphs which have existential closure number 1 (see Figure 1.3).

Figure 1.3: P_4, $\overline{C_4}$ and C_4 are the smallest 1-e.c. graphs.

As we have shown in Figure 1.3 and is proved in [10], $m_{ec}(1) = 4$. It is also known that $m_{ec}(2) = 9$ since the graph $K_3 \square K_3$ is the unique smallest graph with existential closure number 2 [10]; see Figure 1.4. For years, the Paley graph of order 29, $P(29)$, was the smallest known 3-e.c. graph (Paley graphs will be discussed in Section 1.3). Later in 2001, Bonato and Cameron showed that $m_{ec}(3) \geq 20$, and they also found two non-isomorphic 3-e.c. graphs of order 28 by a computer search through the vertex-transitive graphs of order 20 and up [10]. Very recently, Gordinowicz and Prałat have improved the lower bound for $m_{ec}(3)$ by eliminating the values 20, 21, 22 and 23 [27].

Theorem 1.7 *[10, 27]* $24 \leq m_{ec}(3) \leq 28$.

Similarly, Bonato and Costea have conducted a computer search among the class of strongly regular graphs and vertex-transitive graphs of orders between 24 and 30.

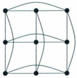

Figure 1.4: $K_3 \square K_3$ is the unique smallest 2-e.c. graph.

They have found two 3-e.c. graphs of order 30 which are the complements of each other. They also have found several non-isomorphic 3-e.c. graphs of order 28 by deleting vertices of $P(29)$ and adding edges to the resulting graphs [12].

1.3 Existential Closure and Paley Graphs

Because Paley graphs were the very first families of graphs that were discovered to contain n-e.c. members for all integers n, we will review some of their history. Paley graphs are self-complementary, strongly regular, and vertex and edge-transitive. Blass et al. first proved that Paley graphs contain members satisfying Axiom n [7]. A graph G is said to satisfy Axiom n if for any two n-sets A and B of vertices, there is a vertex in $V(G) \setminus (A \cup B)$ that is adjacent to each vertex in A and to no vertex in B.

Finite fields of order q (i.e., fields that contain a finite number of elements and which are denoted by $\mathbb{F}_q$) are important in various branches of mathematics including combinatorics. There is exactly one finite field up to isomorphism of size p^k for each prime p and positive integer k. Paley graphs of order q, q a prime power, have as their vertex sets the elements of $\mathbb{F}_q$.

Definition 1.8 *The Paley graph of order q where q is a prime power with $q \equiv 1 \pmod 4$ is a graph denoted $P(q)$ whose vertices are the elements of the finite field $\mathbb{F}_q$ in which two distinct vertices x and y are joined if and only if $x - y = z^2$ for some $z \in \mathbb{F}_q$.*

Theorem 1.9 *[7, 8] If $q > n^2 2^{2n-2}$, then $P(q)$ is n-e.c.*

Given that Paley graphs of order q are n-e.c. for sufficiently large q, by using higher order residues on finite fields other classes of graphs which are called cubic and quadruple Paley graphs have been generated that are n-e.c. for sufficiently large vertex set.

Definition 1.10 *For $q \equiv 1 \pmod 3$, a prime power, the cubic Paley graph, $P^3(q)$ is defined as follows: the vertices of $P^3(q)$ are the elements of the finite field $\mathbb{F}_q$, and two vertices x and y are adjacent if and only if $x - y = z^3$ for some $z \in \mathbb{F}_q$. Also, for $q \equiv 1 \pmod 8$ a prime power, the quadruple Paley graph $P^4(q)$ is defined as follows: the vertices of $P^4(q)$ are the elements of the finite field $\mathbb{F}_q$, and two vertices x and y are adjacent if and only if $x - y = z^4$ for some $z \in \mathbb{F}_q$.*

Cubic and quadruple Paley graphs were first introduced as generalised Paley graphs in [1] where they were shown to be n-e.c. for sufficiently large q. Later on, it was proved that cubic Paley graphs are n-e.c. whenever $q \geq n^2 2^{4n-2}$ and quadruple Paley graphs are n-e.c. whenever $q \geq 9n^2 6^{2n-2}$ [2].

Theorem 1.11 *[2] Let $q \equiv 1 \pmod 3$ be a prime power. If $q \geq n^2 2^{4n-2}$, then $P^3(q)$ has the n-e.c. property.*

Theorem 1.12 *[2] Let $q \equiv 1 \pmod 8$ be a prime power. If $q \geq 9n^2 6^{2n-2}$, then $P^4(q)$ has the n-e.c. property.*

Also, let $q = p^r$ be a prime power such that $p \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$, let $\mathbb{F}_q$ denote the finite field with q elements, and let z be a generator of the multiplicative group of the field. The graph $P^*(q)$ has vertex set $\mathbb{F}_q$, and two vertices x and y are adjacent if $x - y = z^j$ where $j \equiv 0$ or $1 \pmod 4$. Kisielewicz and Peisert have shown that for sufficiently large q, $P^*(q)$ is n-e.c.

Theorem 1.13 *[32] If $q \geq 8n^2 2^{8n}$, then $P^*(q)$ is n-e.c.*

1.4 Existential Closure and Graph Products

Binary graph operations such as Cartesian product or the join of two graphs produce a new graph when given two graphs G and H. It is natural to ask if it is possible to produce an n-e.c. graph by some graph operations that are applied to two n-e.c. graphs. It has been shown that such operations are rare and moreover, there is no such operation known for $n \geq 4$.

Bonato and Cameron in 2003 examined several common binary graph operations to see which ones preserve the n-e.c. adjacency property. Of the operations that they examined, although some of them are shown to preserve the 1-e.c. or 2-e.c. property, only the symmetric difference operation was shown to preserve the 3-e.c. adjacency property.

Definition 1.14 *The symmetric difference of two graphs G and H, $G \triangle H$, is a graph with vertex set $V(G) \times V(H)$ and for two vertices $(a, b), (c, d) \in V(G \triangle H)$, $(a, b)(c, d) \in E(G \triangle H)$ if and only if exactly one of $ac \in E(G)$ or $bd \in E(H)$.*

Theorem 1.15 *[10] If G and H are 3-e.c. graphs, then $G \triangle H$ is 3-e.c.*

Also in 2003, Baker et al. presented another binary graph operation that preserves the 3-e.c. adjacency property [3]. We will discuss this operation in detail later in Chapter 2 where we show that in fact this operation is even stronger and can produce 3-e.c. graphs from two graphs, only one of which needs to be 3-e.c.

1.5 Existential Closure and the Rado Graph

As we have already mentioned, the n-e.c. adjacency property was originally studied in 1963 by Erdős and Rényi [22], at which time they observed the uniqueness of the countably infinite random graph.

Definition 1.16 *A graph G is said to be existentially closed, or e.c., if it is n-e.c. for all positive integers n.*

The countably infinite random graph is known to be e.c. as first shown in [22]. A random graph on a given set X of vertices can be chosen by deciding, independently with probability $\frac{1}{2}$ whether each unordered pair of vertices should be joined by an edge or not. Any two countably infinite random graphs having the e.c. property are isomorphic, and the Rado graph, also known as the random graph or the Erdős-Renyi graph, is the unique (up to isomorphism) countably infinite random graph R.

In an attempt to describe the meaning of existential closure of finite graphs, Bonato [9] has stated that "with the example of R in mind, if a finite graph G is n-e.c., then G may be viewed as a finitary version of R".

The Rado graph was initially constructed in 1964 by Richard Rado [36]. Since then, additional explicit representations have appeared in the literature; see [18]. For more information on the random graph one can refer to [16, 17]. In Chapter 4, we present a new construction of the Rado graph by considering the block intersection graphs of infinite combinatorial designs with certain parameters.

1.6 Existential Closure and Combinatorial Designs

In searching for n-e.c. graphs, researchers have tried to construct such graphs using combinatorial and geometrical structures. Combinatorial designs such as Steiner systems, balanced incomplete block designs, affine designs, and Hadamard designs are mostly considered.

Definition 1.17 *A t-(v,k,λ) design is a v-set of points V with a collection $\mathcal{B}$ of k-subsets called blocks with the property that every t-subset of the point set is contained in precisely λ blocks. A Steiner system is a t-$(v,k,1)$ design and is denoted by $S(t,k,v)$.*

The parameter λ is referred to as the index of the design, v is referred to as the order of the design, and the number of blocks of a design is denoted by b; $b = |\mathcal{B}|$. In studying designs, there are two families of designs, Steiner triple systems and balanced incomplete block designs, that are of most interest.

Definition 1.18 *A 2-(v,k,λ) design is called a balanced incomplete block design and is denoted by BIBD(v,k,λ).*

Definition 1.19 *A 2-$(v, 3, 1)$ design is called a Steiner triple system of order v and is denoted by $STS(v)$.*

Definition 1.20 *An affine plane of order q is a 2-$(q^2, q, 1)$ design.*

The blocks of an affine plane are referred to as the lines of the plane. Affine planes are frequently used in the study of designs. Hadamard designs are another kind of design that we briefly present here. Both of these designs can be used to construct n-e.c. graphs.

Definition 1.21 *A Hadamard matrix of order n is an $n \times n$ matrix H with entries from $\{\pm 1\}$ such that $HH^T = nI_n$.*

Fisher proved that in any BIBD(v, k, λ), $v \leq b$ (Fisher's inequality) [24]. The extreme case of the inequality gives symmetric designs.

Definition 1.22 *In a BIBD(v, k, λ), if $b = v$ then the design is said to be symmetric.*

As the following theorem states, Hadamard matrices and symmetric designs are interrelated.

Theorem 1.23 *[6] There exists a Hadamard matrix of order $4m$ if and only if there exists a symmetric BIBD$(4m - 1, 2m - 1, m - 1)$.*

Definition 1.24 *A symmetric BIBD*$(4m-1, 2m-1, m-1)$ *is called a Hadamard design.*

Definition 1.25 *If* $q \equiv 3 \pmod 4$*, the vertices of the Paley tournament* $\overrightarrow{P(q)}$ *are the elements of the finite field* $\mathbb{F}_q$ *and there is a directed edge from a vertex* x *to another vertex* y *if and only if* $y - x = z^2$ *for some* $z \in \mathbb{F}_q$*.*

In 2002, Fon-Der-Flaass presented a prolific construction of strongly regular graphs using affine planes [25]. Later on Cameron and Stark presented a prolific construction of strongly regular graphs with the n-e.c. property by considering Hadamard designs obtained from Paley tournaments [19] rather than affine planes as presented in [25]. In fact by probabilistic methods they have shown that:

Theorem 1.26 *Suppose that* q *is a prime power such that* $q \equiv 3 \pmod 4$*. There are non-isomorphic SRG*$((q+1)^2, q(q+1)/2, (q^2-1)/4, (q^2-1)/4)$ *which are* n*-e.c. whenever* $q \geq 16n^2 2^{2n}$*.*

Given a combinatorial design, there are several ways to obtain graphs from it, one of which is constructing its block intersection graph.

Definition 1.27 *The block intersection graph of a design* $\mathcal{D}$ *is the graph denoted by* $G_{\mathcal{D}}$*, having vertex set the set of blocks* $\mathcal{B}$*, and two vertices are adjacent if and only if their corresponding blocks share at least one point of* V*.*

The fact that the block intersection graphs of Steiner triple systems are strongly regular has motivated research on these classes of graphs in order to obtain new n-e.c. graphs. However, the results show that n-e.c. graphs arising from these designs are rare. Here are two main results by Forbes et al.:

Theorem 1.28 *[26] The block intersection graph of a Steiner triple system of order v is 2-existentially closed if and only if $v \geq 13$.*

Theorem 1.29 *[26] The block intersection graph of a Steiner triple system of order v can be 3-e.c. only if $v = 19$ or 21.*

In 2004, it was shown that there are precisely $11,084,874,829$ non-isomorphic Steiner triple systems of order 19 [31]. Using a computer search among those that have non-trivial automorphism group, Forbes et al. found two non-isomorphic STS(19) whose block intersection graphs are 3-existentially closed [26]. Very recently, in was confirmed that there are exactly two STS(19) with 3-e.c. block intersection graphs [21]. The case $v = 21$ is still open for no STS(21) with 3-e.c. block intersection graph has been found, and there is no enumeration of non-isomorphic Steiner triple systems of oder 21 in order to conduct a computer search.

Later, McKay and Pike considered the existential closure property of the block intersection graphs of balanced incomplete block designs in general, and they presented bounds on the parameters of such designs whose block intersections are n-existentially closed [33].

Definition 1.30 *A design is said to be a simple design if it does not contain repeated blocks.*

By Lemma 1.31 below, if we want to consider the n-existential closure property of the block intersection graphs of combinatorial designs, we assume that our designs are simple.

Lemma 1.31 *[33] If $n \geq 2$ and $\mathcal{D}$ is a BIBD(v, k, λ) such that $G_{\mathcal{D}}$ is n-e.c., then $\mathcal{D}$ is simple.*

McKay and Pike have found bounds on n and v in order for the block intersection graph of the BIBD(v, k, λ) to be n-e.c.

Theorem 1.32 *[33] The block intersection graph of a BIBD(v, k, λ) with $k \geq 3$ is 2-e.c. if and only if $v \geq k^2 + k - 1$.*

While Theorem 1.32 establishes a lower bound on v for a BIBD(v, k, λ) with $k \geq 3$ to have a 2-e.c. block intersection graph, the following theorem establishes upper bounds on v for the case $n \geq 3$ by considering two possibilities for λ: $\lambda = 1$ and $\lambda \geq 2$.

Theorem 1.33 *[33] Let $n \geq 3$ and $\mathcal{D}$ be a BIBD(v, k, λ) for which the block intersection graph is n-e.c. If $\lambda = 1$ then $v \leq k^4 - nk^3 + (2n-2)k^2 - nk + k + 1$, and if $\lambda \geq 2$ then $v \leq \lambda k^4 - \lambda n k^3 + (\lambda + 1)(n-1)k^2 - nk + k + 1$.*

It was also determined that if the block intersection of a BIBD(v, k, λ) is n-e.c., then n cannot exceed k.

Theorem 1.34 *[33] If $\mathcal{D}$ is a BIBD(v, k, λ) such that $G_{\mathcal{D}}$ is n-e.c., then $n \leq k$ for $\lambda = 1$ and $n \leq \lfloor \frac{k+1}{2} \rfloor$ for $\lambda \geq 2$.*

The n-e.c. graphs arising as incidence graphs of partial planes resulting from affine planes are another example of graphs with the property being constructed from combinatorial designs.

Definition 1.35 *A partial plane results from an affine plane by deleting some set of the lines of the affine plane.*

Definition 1.36 *If $\mathcal{P}$ is a partial plane resulting from an affine plane, then the collinearity (or point) graph of $\mathcal{P}$ is the graph with vertices equal to the points of the affine plane, with two points joined if they are joined by a line of $\mathcal{P}$.*

In 2003, using geometric methods, new explicit examples of 3-e.c. graphs were presented which are the collinearity graphs of partial planes derived from affine planes [3].

In 2008 using probabilistic and geometric techniques, new examples of graphs with the n-e.c. adjacency property which are the collinearity graphs of certain partial planes derived from affine planes of even order were given by Baker et. al. [4].

In 2009, using random constructions new infinite classes of regular n-e.c. graphs arising from resolvable BIBD$(v, k, 1)$ were presented [5]. A resolvable design is defined as follows.

Definition 1.37 *A BIBD(v, k, λ) is resolvable if the blocks can be arranged into r sets so that $(b/r) = (v/k)$ blocks of each set are disjoint and contain in their union each element of the point set exactly once. The sets are called resolution classes or parallel classes.*

1.7 Infinite Designs

In Chapters 3 and 4, we will extend the study of the n-existential closure property of block intersection graphs of designs to infinite designs. The vertex sets of such graphs are infinite, which motivates research in order to construct infinite graphs with certain properties.

In this section, we introduce infinite designs as described in [20] by Cameron and Webb. With an infinite design we mean a design whose point set is infinite and the other parameters can either be finite or infinite. However, throughout this thesis, we assume that t and λ are finite integers.

Definition 1.38 *[20] A t-(v, k, Λ) design is a v-set of points with a collection $\mathcal{B}$ of k-subsets called blocks, with the properties that:*

1. *no block is a strict subset of any other block;*
2. *the cardinality of the set of points missed by a block is non-zero, and is independent of the block;*
3. *if i, j are non-negative integers with $i + j \leq t$, then the cardinality, denoted by $\lambda_{i,j}$, of the set of blocks containing all of i given points $x_1, ..., x_i$ and none of j given points $y_1, ..., y_j$ (where the xs and ys are all distinct) depends only on i and j, and not on the chosen points.*

We have $\Lambda = \left(\lambda_{i,j}\right)_{t+1 \times t+1}$, where $\lambda_{i,j}$ is defined as above for $i+j \leq t$, and undefined for $i + j \geq t + 1$. In particular, $\lambda_{t,0} = \lambda$, $\lambda_{1,0} = r$ where r is the number of block containing a point of the design, and $\lambda_{0,0} = b$.

Note that in the definition above, the matrix Λ is for general infinite designs, but as we are dealing with finite values for t and λ in this thesis, it suffices for us to refer to infinite t-(v, k, λ) designs. This is because when t and λ are finite, if a structure satisfies the first condition in the definition, and every set of t points is a subset of precisely λ blocks, then the third condition also holds for the structure by Theorem 3.1 and Proposition 4.1 stated in [20]. So Definition 1.38 turns out to be as follows when t and λ are finite:

Definition 1.39 *[20] A t-(v, k, λ) design is a v-set of points with a collection $\mathcal{B}$ of k-subsets called blocks, with the properties that:*

1. *no block is a strict subset subset of any other block;*
2. *the cardinality of the set of points missed by a block is non-zero, and is independent of the block;*
3. *each set of t points is a subset of exactly λ blocks.*

To see the full description of conditions in order for an infinite structure to be an infinite design the reader is referred to [20].

1.8 Outline of Thesis

At this point we have briefly reviewed the background, terminology, and motivations and particularly on n-existential closure property. In the following chapters we present our results and advances in this topic as is outlined here.

We mainly focus on two problems: constructing new families of 3-e.c. graphs using modular graph product and finding the existential closure number of block intersection graphs of infinite designs.

In Chapter 2 we will consider the modular graph product and will show that it can be applied on two graphs one of which is not necessarily 3-e.c. to obtain a new 3-e.c. graph. The operation has been first considered by Baker et al. where they showed that it preserves the 3-e.c. property [3].

In Chapters 3 and 4 we study the n-existential closure property of block intersection graphs of infinite t-(v, k, λ) designs with finite t and λ. We will show that if the block size is finite, then n is bounded above for the block intersection graph of infinite designs to be n-e.c. In contrast, we will establish that there are infinite designs with infinite block size whose block intersection graphs are e.c.

In Chapter 5 we present some open problems and some potential research areas on the n-existential closure property and infinite designs.

Chapter 2

Modular Product and Existential Closure

2.1 Introduction

The scarcity of other readily recognised families of n-e.c. graphs for arbitrary n has motivated research into classes of graphs that are n-e.c. for small values of n; however, it is not easy to find explicit example of such graphs even for $n = 3$. A graph G is 3-existentially closed if for each 3-set S of vertices, there are eight additional vertices that are joined to the vertices of S in all possible ways. Although the property is straightforward to define and almost all graphs are 3-e.c., it is not easy to find explicit example of such graphs.

It has been shown that every 3-e.c. graph has at least 24 vertices and examples of 3-e.c. graphs of order 28 have been found [10, 27]. In 2001, Hadamard matrices of order $4m$ with odd $m > 1$ were used to obtain 3-e.c. SRG$(16m^2, 8m^2 - 2m, 4m^2 - 2m, 4m^2 - 2m)$ [14]. Also in 2001, Baker et al. presented new 3-e.c. graphs arising from collinearity graphs of partial planes resulting from affine planes [3]. In 2002, Cameron and Stark presented a family of 3-e.c. graphs, however, the smallest such graphs produced have at least 84,953,089 vertices [19]. Recently, another construction of 3-e.c. graphs of order at least p^d for prime $p \geq 7$ and $d \geq 5$ was presented using quadrances (a quadrance between points $X = (x_1, \ldots, x_d)$ and $Y = (y_1, \ldots, y_d)$ in $\mathbb{Z}_p^d$ is the number $Q(X,Y) = (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2$) [38]. Also it was confirmed that there are only two STS(19) with 3-e.c. block intersection graphs [21, 26].

As part of an effort to find new explicit examples of finite n-e.c. graphs, Bonato and Cameron examined several common binary graph operations to see which operations preserve the n-e.c. property for $n \geq 1$ [10]. They showed that the symmetric difference of two 3-e.c. graphs is a 3-e.c. graph. Baker et al. subsequently introduced another graph construction which is 3-e.c. preserving [3].

In this chapter, we take a different approach to the construction in [3] that enables us to relax the requirement that the two graphs considered be both 3-e.c. We formulate the construction as a binary non-commutative graph operation denoted by the symbol $\diamond$ and we determine necessary and sufficient conditions for the graph $G \diamond H$ to be 3-e.c., given that H itself is a 3-e.c. graph. We then use this operation to construct new classes of 3-e.c. graphs of the form $G \diamond H$ when G is not necessarily a 3-e.c. graph. In particular, the classes that we consider are those for which G is either a complete multipartite graph or a strongly regular graph. The graph G for which we show that $G \diamond H$ is 3-e.c. can have as few as four vertices, which represents an improvement in comparison to when G is required to be 3-e.c.

The results of this chapter are accepted for publication in *The Australasian Journal of Combinatorics* [35].

2.2 The Modular Product and a Characterisation Theorem

If G and H are two graphs, then we let $G \diamond H$ represent the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, u) and (y, v) are adjacent if

(a) $xy \in E(G)$ and $uv \in E(H)$, or

(b) $xy \notin E(G)$ and $uv \notin E(H)$.

It so happens that $G \Diamond H$ is the complement of a construction that was introduced by Vizing in 1974 [39]. In keeping with [37,40], we shall refer to $G \Diamond H$ as the modular product of G and H.

Unless stated otherwise, we shall generally assume that the graph G has a loop at each vertex and also that H is 3-e.c. When describing the graph $G \Diamond H$, for each vertex $x \in V(G)$ let H_x be the subgraph of $G \Diamond H$ that is isomorphic to H and consists of all vertices of the form (x, u) where $u \in V(H)$. Since the vertices of H_x can be considered to be indexed by $V(G)$, we will often use the notation u_x to denote the vertex (x, u). Two vertices $u_x = (x, u) \in H_x$ and $v_y = (y, v) \in H_y$ will be said to be congruent if $u = v$; otherwise they are incongruent. An example of $G \Diamond H$ is illustrated in Figure 1, for $G = K_{1,2}$ and $H = \overline{K_{1,2}}$.

It can be easily deduced that $\overline{G \Diamond H} = G \Diamond \overline{H}$ where $\overline{G}$ is the simple complement of G (the complement of a loop is a non-loop and the complement of a non-loop remains a non-loop). Also, note that when G has a loop at every vertex, $G \Diamond H$ is isomorphic to the graph $G(H)$ as described in [3] in which the following theorem was proved:

Theorem 2.1 *[3] If the graphs G and H are both 3-e.c., then the graph $G \Diamond H$ is also 3-e.c.*

We devote the remainder of this section to the development and proof of a characterisation of 3-e.c. graphs of the form $G \Diamond H$ where H is 3-e.c. but G is not necessarily

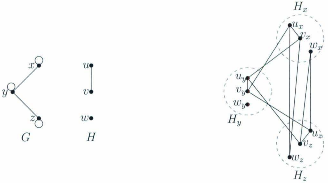

Figure 2.1: $G \diamond H$.

so. This characterisation will help us to find smaller 3-e.c. graphs by simplifying the process of checking when $G \diamond H$ is 3-e.c.

For a graph G, given a set $S \subset V(G)$ and a subset T of S, we say a vertex $x \in V(G) \setminus S$ is a T-solution with respect to S if x is adjacent to every vertex in T and to none in $S \setminus T$. A solution for S is said to exist if there is a T-solution for every $T \in P(S)$ where $P(S)$ denotes the power set of S. Observe that if a solution exists for every n-subset of V, then G is n-e.c.

We say a graph G is weakly n-existentially closed, or n-w.e.c., if for any set S with $|S| = n$ and any $T \subseteq S$, there exists a vertex in $V(G)$ that is adjacent to each vertex in T and to no vertex in $S \setminus T$ or there exists a vertex that is adjacent to each vertex in $S \setminus T$ and to no vertex in T. Such a vertex is called a weak T-solution with respect to S. Note that $K_1 \Diamond H = H$ and it can easily be confirmed that if $|V(G)| \in \{2, 3\}$, then G cannot be 3-w.e.c., so we henceforth assume that $|V(G)| \geq 4$.

For a graph G and a vertex $x \in V(G)$ we define $N[x] = \{y \in V(G) | \ xy \in E(G)\}$, and for a set A of vertices we let $N[A] = \bigcup_{x \in A} N[x]$ and $N'[A] = \bigcap_{x \in A} N[x]$. Also, for a set of vertices $A \subseteq V(G \Diamond H)$ and for each $a \in V(G)$, we let $A_a = \{u_a \in V(H_a) | \text{ there is some } x \in V(H) \text{ such that } u_x \in A\}$.

With these notations, note that a graph G is 3-w.e.c. if and only if for every 3-subset $A \subset V(G)$, the following two items hold

(1) $N'[A] \neq \emptyset$ or $V(G) \setminus N[A] \neq \emptyset$, and

(2) for every vertex $t \in A$, $N[t] \setminus N[A \setminus \{t\}] \neq \emptyset$ or $N'[A \setminus \{t\}] \setminus N[t] \neq \emptyset$.

We are now ready to state and prove a characterisation theorem.

Theorem 2.2 *Let G be a graph with $|V(G)| \geq 4$ and with loops at every vertex of $V(G)$ and let H be a 3-e.c. graph. The graph G is 3-w.e.c. if and only if $G \Diamond H$ is 3-e.c.*

Proof Suppose that H is 3-e.c. and G is 3-w.e.c. In order to show that $G \Diamond H$ is 3-e.c., for an arbitrary set of three vertices $S = \{u_x, v_y, w_z\} \subset V(G \Diamond H)$ we show that there exists a T-solution for each $T \in P(S)$. Note that since $\overline{G \diamond H} = G \diamond \overline{H}$, then if there is a T-solution in $G \diamond H$ for $|T| = 0$ (1, resp.), then there is a T-solution for $|T| = 3$ (2, resp.). To see this, suppose that there is an $\emptyset$-solution in $G \diamond H$, and since $\overline{H}$ is 3-e.c., there is an $\emptyset$-solution in $G \diamond \overline{H}$ and hence in $\overline{G \diamond H}$, too. This implies that there is an S-solution in $G \diamond H$. A similar argument holds for the case $|T| = 1$.

Let $A = \{x, y, z\}$ and $B = \{u, v, w\}$. So $1 \leq |A|, |B| \leq 3$. If $|A| = 1$, then since H is 3-e.c., there exists an S-solution. Now we consider the remaining possibilities for B and A.

Case 1. Suppose that $|A| = 3$. First consider the case $T = \emptyset$. Let a be weak $\emptyset$-solution with respect to A. If $a \in V(G) \setminus N[A]$, then if t is an S_a-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S. If $a \in N'[A]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S.

If $T = \{u_x\}$, then let a be a weak $\{x\}$-solution with respect to A. If $a \in N[x] \setminus N[A \setminus \{x\}]$, then if t is an S_a-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S. If $a \in N'[A \setminus \{x\}] \setminus N[x]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S. Similar arguments hold for $T \in \{\{v_y\}, \{w_z\}\}$.

Case 2. Next suppose that $|A| = 2$. We argue this case in two subcases depending on whether the vertices of S are congruent or incongruent.

Case 2.a. First suppose that the vertices of S are incongruent; $S = \{u_x, v_y, w_y\}$.

We first consider the case $T = \emptyset$. Let a be weak $\emptyset$-solution with respect to A. If $a \in V(G) \setminus N[A]$, then if t is an S_a-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S. If $a \in N'[A]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is an $\emptyset$-solution with respect to S.

If $T = \{u_x\}$, then let a be a weak $\{x\}$-solution with respect to A. If $a \in N[x] \setminus N[y]$, then if t is an S_a-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S. If $a \in N[y] \setminus N[x]$, then if t is an $\emptyset$-solution with respect to S_a, t_a is a $\{u_x\}$-solution with respect to S.

If $T = \{v_y\}$, then let a be a weak $\{x\}$-solution with respect to A. If $a \in N[y] \setminus N[x]$, then if t is a $\{u_a, v_a\}$-solution with respect to S_a, t_a is a $\{v_y\}$-solution with respect to S. If $a \in N[x] \setminus N[y]$, then if t is a $\{w_a\}$-solution with respect to S_a, t_a is a $\{v_y\}$-solution with respect to S. A similar argument holds for $T = \{w_y\}$.

Case 2.b. Now suppose that S contains congruent vertices; $S = \{u_x, u_y, w_y\}$. In this case, the only difference with Case 2.a. is in finding a $\{w_y\}$-solution. Let a be weak $\emptyset$-solution with respect to A. If $a \in V(G) \setminus N[A]$, then if t is a $\{u_a\}$-solution with respect to S_a, t_a is a $\{w_y\}$-solution with respect to S. If $a \in N'[A]$, then if t is a $\{w_a\}$-solution with respect to S_a, t_a is a $\{w_y\}$-solution with respect to S.

Note that for any 3-subset $S \subset V(G \diamond H)$, and any $a \in V(G)$, since H_a is isomorphic to H and hence is 3-e.c., then for each $T' \subseteq S_a$ there exists a T'-solution with respect to S_a. Observe for each case considered in this argument, the solutions found are in $V(G\Diamond H) \setminus S$. As there is a solution for an arbitrary set of three vertices of $G\Diamond H$, we conclude that $G\Diamond H$ is 3-e.c.

To prove the converse implication, suppose that $G\Diamond H$ is 3-e.c. but G is not 3-w.e.c. Assume that $A = \{x, y, z\} \subset G$ for which there is no weak T-solution for some $T \subseteq A$. Let $S = \{u_x, u_y, u_z\}$.

As an initial case, suppose that there is no weak $\emptyset$-solution. If every vertex of G is in the neighbourhood of at least one and at most two of the vertices in A, then every vertex of $G\Diamond H$ is adjacent to at least one and at most two of the vertices in S, and so there is no vertex of $G\Diamond H$ that is an S-solution with respect to S.

Now suppose that there is no weak T-solution for some $T \subseteq A$ with $|T| = 1$. Without loss of generality suppose that $N[x] \setminus N[\{y, z\}] = \emptyset$ and $N[\{y, z\}] \setminus N[x] = \emptyset$. So, any vertex in $N[x]$ is also in $N[\{y, z\}]$ and any vertex in $N[\{y, z\}]$ is also in $N[x]$. These imply that any vertex in $N[x]$ is in $N[y]$ or $N[z]$ and any vertex in $V(G) \setminus N[x]$ is in at most one of $N[y]$ and $N[z]$. Thus any vertex of $G\Diamond H$ that is adjacent to u_x is also adjacent to u_y or to u_z and so there is no $\{u_x\}$-solution with respect to S.

In each case we establish the contradiction that the graph $G\Diamond H$ is not 3-e.c., and the argument is complete. $\blacksquare$

Theorem 2.1 now becomes a corollary of Theorem 2.2.

Proof of Theorem 2.1 Since G is 3-e.c., it is also 3-w.e.c. ■

In general, given graphs G and H such that H is 3-e.c., in order to determine whether or not $G \diamond H$ is 3-e.c., we need to examine the existence of $8\binom{|V(G)||V(H)|}{3}$ T-solutions. However, by applying Theorem 2.2, we only need to examine if G is 3-w.e.c., and hence at most $8\binom{|V(G)|}{3}$ sets would need to be compared with the empty set.

Having shown that the modular product can produce 3-e.c. graph given a 3-w.e.c. graph and a 3-e.c. graph, we now find graphs G that are 3-w.e.c. We focus our attention on cases in which G is either a complete multipartite graph or a strongly regular graph.

2.3 Weakly 3-e.c. Complete Multipartite Graphs

In this section we show that most of the complete multipartite graphs are 3-w.e.c.

Theorem 2.3 *The complete i-partite graph $K_{\ell_1,\ell_2,\ldots,\ell_i}$ with $\ell_j \geq 2$ for all $j \in \{1, 2, \ldots, i\}$ is 3-w.e.c.*

Proof Let X and Y be two distinct parts in the obvious partition of $K_{\ell_1,\ell_2,\ldots,\ell_i}$. Consider a set of vertices $A = \{x, y, z\}$ of $K_{\ell_1,\ell_2,\ldots,\ell_i}$. If all three vertices of A are in the same part, say $A \subseteq X$, any vertex in Y is a weak $\emptyset$-solution with respect to A. Also $x \in N[x] \setminus N[\{y,z\}]$, and similarly $y \in N[y] \setminus N[\{x,z\}]$ and $z \in N[z] \setminus N[\{x,y\}]$, and so there exists a weak T-solution for any $T \subset A$ with $|T| = 1$.

If x is a vertex in a part, say $x \in X$, and y and z are in another part, say $\{y,z\} \subseteq Y$, then $x \in N'[A]$ and so there is a weak $\emptyset$-solution with respect to A. Also note that since $\ell_j \geq 2$, then there exists a vertex $r \in X \setminus \{x\}$, and hence $r \in N'[\{y,z\}] \setminus N[x]$. Also, $z \in N'[\{x,z\}] \setminus N[y]$, and $y \in N'[\{x,y\}] \setminus N[z]$.

It now remains to consider the case when each vertex in A is in a distinct part. Suppose that x', y' and z' are vertices of $V(K_{\ell_1,\ell_2,\ldots,\ell_i}) \setminus A$ and in the same parts as x, y and z respectively. We have $x \in N'[A]$ and so there exists a weak $\emptyset$-solution with respect to A. Also $x' \in N'[\{y,z\}] \setminus N[x]$, $y' \in N'[\{x,z\}] \setminus N[y]$, and $z' \in N'[\{x,y\}] \setminus N[z]$ and so is a weak T-solution for any $T \subset A$ with $|T| = 1$. So, A has a weak solution and the graph $K_{\ell_1,\ell_2,\ldots,\ell_i}$ with $\ell_j \geq 2$ is 3-w.e.c. ■

It follows from Theorem 2.3 that every bipartite graph $K_{\ell,m}$ with $\ell, m \geq 2$ is 3-w.e.c. The only remaining bipartite graphs to consider are of the form $K_{1,m}$ with $m \geq 3$. Let $A = \{x,y,z\} \subset V(K_{1,m})$. We will show that there is a weak solution for A. If all the vertices of A are in the same part, then the argument is similar to the corresponding case in the proof of Theorem 2.3. Now without loss of generality suppose x is the singleton part, and y, z and r are in the part with m vertices. So,

$x \in N'[A]$, $y \in N'[\{x, y\}] \setminus N[z]$, $z \in N'[\{x, z\}] \setminus N[y]$ and $r \in N[x] \setminus N[\{y, z\}]$ and so $K_{1,m}$ is 3-w.e.c.

We have shown that $K_{2,2} \diamond H$ and $K_{1,3} \diamond H$ are 3-e.c. if H is 3-e.c., thereby producing two non-isomorphic 3-e.c. graphs of order $4|V(H)|$. Since the smallest 3-e.c. graph that is known to date has order 28 [27], this order of $4|V(H)|$ is much smaller than $28|V(H)|$ if both graphs were required to be 3-existentially closed (as was required in [3]).

2.4 Weakly 3-e.c. Strongly Regular Graphs

A k-regular graph G in which each pair of adjacent vertices has exactly λ common neighbours, and each pair of non-adjacent vertices has exactly μ common neighbours is called a strongly regular graph; we say that G is a $\mathrm{SRG}(v, k, \lambda, \mu)$ with $v = |V(G)|$. In this section we recognise a few classes of strongly regular graphs that possess the 3-w.e.c. adjacency property.

Theorem 2.4 *The empty graph G with $|V(G)| \geq 4$ is 3-w.e.c.*

Proof Let $A = \{x, y, z\} \subset V(G)$ and $t \in V(G) \setminus A$. Obviously, $t \in V(G) \setminus N[A]$ which establishes the existence of a weak $\emptyset$-solution. Also $x \in N[x] \setminus N[\{y, z\}]$,

$y \in N[y] \setminus N[\{x, z\}]$, and $z \in N[z] \setminus N[\{x, y\}]$ which establish the existence of a weak T-solution with respect to A for any set $T \subset A$ with $|T|$=1. ■

By Theorem 2.4, in addition to the two 3-e.c. graphs $K_{2,2} \diamond H$ and $K_{1,3} \diamond H$, we obtain $\overline{K_4} \diamond H$ as another 3-e.c. graph on $4|V(H)|$ vertices. We now characterise another family of 3-w.e.c. strongly regular graphs.

Theorem 2.5 *The Petersen graph P, $SRG(10, 3, 0, 1)$, is 3-w.e.c.*

Proof Let $A = \{x, y, z\} \subset V(P)$. First we show there is a weak $\emptyset$-solution with respect to A.

If at least two pairs of the vertices in A are adjacent, then $N'[A] \neq \emptyset$, and so there is a weak $\emptyset$-solution with respect to A.

If only one pair, say x and y, of the vertices in A are adjacent, then x and y have no common neighbour, whereas x and z (resp. y and z) have only one common neighbour. Considering that the degree of each vertex is three, then $|N[A]| = 8$, and since $|V(P)| = 10$, there are two vertices in $V(P) \setminus A$ that are not a neighbour of x, y or z, and hence $V(P) \setminus N[A] \neq \emptyset$.

If there is no pair of adjacent vertices in A, then we deal with two cases. If all the vertices in A share a neighbour, then $N'[A] \neq \emptyset$. Otherwise if $N'[A] = \emptyset$, since P is 3-regular, and since every pair of the vertices of A have a common neighbour,

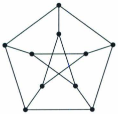

Figure 2.2: The Petersen graph; SRG$(10, 3, 0, 1)$.

then $|N[A]| = 9$ and $V(P) \setminus N[A] \neq \emptyset$. So in any case there is a weak $\emptyset$-solution with respect to A.

Now it only remains to show the existence of a weak T-solution with respect to A for any $T \subset A$ with $|T| = 1$. Without loss of generality we assume that $T = \{x\}$. If x is adjacent to both y and z, then x and y have no common neighbour, and also x and z have no common neighbour. Since $\deg(x) = 3$, then there exists a vertex different from y and z which is adjacent to x and non-adjacent to both y and z and so $N[x] \setminus N[A \setminus \{x\}] \neq \emptyset$.

If x is adjacent to exactly one of y or z, say y, then x and y have no common neighbour, and x and z have only one common neighbour. Again, since $\deg(x) = 3$, then there exists a vertex different from y which is adjacent to x and non-adjacent to both y and z. This implies that $N[x] \setminus N[A \setminus \{x\}] \neq \emptyset$. The case that x is non-adjacent to both y and z can be argued similarly. In each case we find that $N[x] \setminus N[A \setminus \{x\}] \neq \emptyset$

and hence there is a weak T-solution for any $T \subset A$ and so P is 3-w.e.c. ■

Theorem 2.5 can now be generalised.

Theorem 2.6 *If G is a $SRG(v, k, \lambda, \mu)$ such that*

(i) $v \geq \max\{3k - \lambda - 2\mu + 2, 3k - 3\mu + 4\}$ *and*

(ii) $k \geq \max\{2\lambda + 3, \lambda + \mu + 2, 2\mu + 1\}$,

then G is 3-w.e.c.

Proof Let $A = \{x, y, z\} \subseteq V(G)$. We first show that there is a weak $\emptyset$-solution with respect to A.

If at least two pairs of the vertices of A are adjacent, then $N'[A] \neq \emptyset$ and so there is a weak $\emptyset$-solution with respect to A.

If there is only one pair of adjacent vertices in A, say x and y, then x and y have λ common neighbours, whereas x and z (resp. y and z) have μ common neighbours. Considering that the degree of each vertex is k, then by the principle of inclusion and exclusion we have $|N[A]| = 3k - \lambda - 2\mu + 1 + |N'[A]|$. Now if $|N'[A]| \neq \emptyset$, then clearly there is a weak $\emptyset$-solution with respect to A. Otherwise $|N'[A]| = \emptyset$ and $|N[A]| = 3k - \lambda - 2\mu + 1$, and since $v > 3k - \lambda - 2\mu + 1$ by (i) then there is a vertex in $V(G) \setminus A$ that is not a neighbour of x, y, or z; hence $V(G) \setminus N[A] \neq \emptyset$ and there is a weak $\emptyset$-solution with respect to A.

Finally, if there is no pair of adjacent vertices in A, then since G is k-regular and since every pair of the vertices of A have μ common neighbours, then by the principle of inclusion and exclusion $|N[A]| = 3k - 3\mu + 3 + |N'[A]|$. Again, if $|N'[A]| \neq \emptyset$, then there is a weak $\emptyset$-solution with respect to A. Otherwise $|N'[A]| = \emptyset$ and $|N[A]| = 3k - 3\mu + 3$, and since $v > 3k - 3\mu + 3$ by (i) then there exists a vertex in $V(G) \setminus A$ that is non-adjacent to every vertex in A; hence $V(G) \setminus N[A] \neq \emptyset$ which establishes the existence of a weak $\emptyset$-solution with respect to A.

Now it only remains to show that there is a weak T-solution with respect to A for any $T \subset A$ with $|T| = 1$. Without loss of generality let $t = x$. If x is adjacent to both y and z, then x and y (resp. x and z) have λ common neighbours. Since $\deg(x) = k$ and $k > 2\lambda + 2$ by (ii), then there exists a vertex different from y and z which is adjacent to x and non-adjacent to both y and z. This implies that $N[x] \setminus N[\{y, z\}] \neq \emptyset$.

If x is adjacent to one of y or z, say y, then x and y have λ common neighbours, and x and z have μ common neighbours. Again, since $\deg(x) = k$ and $k > \lambda + \mu + 1$ by (ii), then there exists a vertex different from y which is adjacent to x and non-adjacent to both y and z, and hence $N[x] \setminus N[\{y, z\}] \neq \emptyset$. The case that x is non-adjacent to both y and z can be argued similarly.

So, there is a weak solution for A and G is 3-w.e.c. ■

Note that the graphs that satisfy the conditions of Theorem 2.6 tend to be sparse. Examples of such graphs are the Clebsch graph (a SRG$(16,5,0,2)$), the Hoffman-Singleton graph (a SRG$(50,7,0,1)$), the Gewirtz graph (a SRG$(56,10,0,2)$), the M22 graph (a SRG$(77,16,0,4)$), the Brouwer-Haemers graph (a SRG$(81,20,1,6)$), the Higman-Sims graph (a SRG$(100,22,0,6)$), the Local McLaughlin graph (a SRG$(162,56,10,24)$), and the $n \times n$ square rook's graph (a SRG$(n^2, 2n-2, n-2, 2)$) for large enough n. Next we present a family of 3-w.e.c. that are dense.

Theorem 2.7 *If G is a SRG$(v, v-2, v-4, v-2)$ with $v \geq 4$, then G is 3-w.e.c.*

Proof Note that since G is $(v-2)$-regular, for each set of three vertices of G at least two pairs of the vertices are adjacent. Let $A = \{x, y, z\} \subseteq V(G)$ be a set of three vertices, and without loss of generality suppose that x is adjacent to both y and z. Since $\deg(x) = v-2$, there exists a vertex $r \in V(G) \setminus A$ such that $rx \notin E(G)$ and $\{ry, rz\} \subseteq E(G)$. Note that $x \in N'[A]$ and and so there is a weak $\emptyset$-solution with respect to A. It only remains to show that there is a weak T-solution with respect to A for any $T \subset A$ with $|T| = 1$. We will consider two cases depending on whether or not $yz \in E(G)$.

As a first case, suppose that $yz \notin E(G)$. So $y \in N'[\{x,y\}] \setminus N[z]$, $z \in N'[\{x,z\}] \setminus$

$N[y]$ by symmetry, and $r \in N'[\{y,z\}] \setminus N[x]$. Second, suppose that $yz \in E(G)$ and without loss of generality let $t = x$. We have $r \in N'[\{y,z\}] \setminus N[x]$, and by symmetry similar arguments establish the cases $t = y$ and $t = z$. So there is a weak T-solution with respect to A for any $T \subset A$ with $|T| = 1$.

So, there is a weak solution for A and G is 3-w.e.c. ■

Note that for each even $v \geq 4$, SRG$(v, v-2, v-4, v-2)$ is the complement of a perfect matching on v vertices.

2.5 Discussion

Now that we are able to recognise some classes of graphs G that are 3-w.e.c. and hence enabling us to construct new 3-e.c. graphs $G \diamond H$ given that H is 3-e.c., in this section we discuss some graphs G for which G is not 3-w.e.c. In Theorem 2.3 we showed that $K_{2,2}$ is 3-w.e.c. By observing that $K_{2,2}$ is isomorphic to C_4, it is natural to ask which values of m result in 3-w.e.c. C_m. As it happens $m = 4$ is unique in this regard.

Proposition 2.8 *The cycle C_m of order m is 3-w.e.c. if and only if $m = 4$.*

Proof Suppose that we have labelled the vertices of C_m in the clockwise order by $1, 2, \ldots, m$. The graph C_4 is isomorphic to $K_{2,2}$ for which we have shown $K_{2,2}$ is

3-w.e.c. If $m \neq 4$, then for $A = \{1,2,3\}$ there is no weak $\{2\}$-solution with respect to A because $N'[\{1,3\}] \setminus N[2] = \emptyset$ and $N[2] \setminus N[\{1,3\}] = \emptyset$. ■

It is also natural to ask whether it might be possible to use the modular product to obtain graphs that are 4-e.c.

Proposition 2.9 *If G and H are two graphs such that $|V(G)| \geq 2$ and H is 4-e.c., then $G \diamond H$ cannot be 4-e.c.*

Proof Let H be a 4-e.c. graph, and let G be any graph. Consider $S = \{u_x, u_y, v_x, v_y\}$ a set of four vertices of $G \diamond H$ such that $x \neq y$, u_x and u_y are congruent, and v_x and v_y are also congruent. For $T = \{u_x, u_y, v_x\}$ there is no T-solution. ■

Chapter 3

Block Intersection Graphs of Infinite Designs Having Finite Block Size and Index

3.1 Introduction

Recall that the block intersection graph of a design $\mathcal{D}$ is the graph denoted by $G_{\mathcal{D}}$, having vertex set the set of blocks $\mathcal{B}$, and two vertices are adjacent if and only if their corresponding blocks share at least one point of V. As we mentioned in Chapter 1, several results on the n-existential closure property of block intersection graphs of finite designs appear in the literature [26, 33]. In [26] Forbes, Grannell, and Griggs studied the n-e.c. property of block intersection graphs of Steiner triple systems. Subsequently, some results have been found on the n-e.c. property of the block intersection graphs of finite designs with $t = 2$ in general [33]. When v is finite, the block intersection graph of a 2-$(v, 3, 1)$ design is 2-e.c. if and only if $v \geq 13$, and if it is 3-e.c., then v must be 19 or 21 [26]. Also, in [33], it has been shown that $\Xi(G_{\mathcal{D}}) \leq k$ for a finite design $\mathcal{D}$ with $t = 2$, and if $\lambda \geq 2$, then $\Xi(G_{\mathcal{D}}) \leq \lfloor \frac{k+1}{2} \rfloor$.

For infinite designs there has been no work in this area until now. Here, our aim is to investigate the n-e.c. property of the block intersection graphs of infinite t-designs with k and λ finite. We show that the block intersection graph of an infinite t-design $\mathcal{D}$ with k finite, $2 \leq t \leq k$, and $\lambda = 1$ has $\min\{t, \lfloor \frac{k-1}{t-1} \rfloor + 1\}$ as its existential closure number. However, when $2 \leq t \leq k - 1$ and $\lambda \geq 2$, then $2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$.

It follows that the block intersection graphs of infinite designs with a countably infinite number of blocks that are each of finite size k are different from the countably infinite random graph. This can be seen by observing that such block intersection graphs, despite having countably infinitely many vertices, are not n-e.c. for any integer n such that $n \geq \min\{t+1, \lfloor \frac{k-1}{t-1} \rfloor + 2\}$ if $\lambda = 1$ (resp. for any $n \geq \min\{t+1, \lceil \frac{k}{t} \rceil + 1\}$ if $\lambda \geq 2$). In contrast, the countably infinite random graph is e.c.

By comparing our results with those of [33] and [26], we also see that infinite designs behave differently from finite ones, for the block intersection graphs of infinite designs with $2 \leq t \leq k-1$ are guaranteed to be 2-e.c. regardless of the values k and λ. In addition, for the case $\lambda = 1$ we get the exact value of the existential closure number.

In the following sections we are going to investigate the n-existential closure property of infinite designs with k and λ finite. Note that the block set of any 1-$(v, k, 1)$ design is a partition of its point set and hence the block intersection graph of the design must have existential closure number 0 (since no two blocks share a point). Thus, in our results throughout this chapter we will often assume that $(t, \lambda) \neq (1, 1)$. Also, when $t = k$ and $\lambda \geq 2$, there are repeated blocks and $\mathcal{D}$ is not a simple design, so we also assume that if $\lambda \geq 2$ then $t < k$.

The results of this chapter have been published in *Journal of Combinatorial Designs* [34].

3.2 When $\lambda = 1$

In this section we consider the existential closure of the block intersection graphs of infinite designs with k finite and $\lambda = 1$. We show that if $2 \leq t \leq k$, then $\Xi(G_{\mathcal{D}}) = \min\{t, \lfloor \frac{k-1}{t-1} \rfloor + 1\}$. We begin with the following lemma:

Lemma 3.1 *Let $\mathcal{D}$ be an infinite t-$(v, k, 1)$ design with k finite. Let n be a positive integer such that $(n-1)(t-1) \leq (k-1)$. If $\mathcal{S} = \{B_1, B_2, \ldots, B_n\}$ is an n-set of blocks of $\mathcal{D}$, then for each $i \in \{1, 2, \ldots, n\}$, there exists a point x_i such that $x_i \in B_i$ and $x_i \notin B_j$ for $1 \leq j \leq n$ and $j \neq i$.*

Proof We will show that by the assumptions of the lemma, every block $B_i \in \mathcal{S}$ contains a point x_i that does not belong to any other blocks of $\mathcal{S} \setminus \{B_i\}$. Every t elements occur in exactly one block, and so the intersection of any two blocks has at most $(t-1)$ elements; i.e., $|B_i \cap B_j| \leq (t-1)$ where $1 \leq j \leq n$ and $j \neq i$. Note that $|\mathcal{S} \setminus \{B_i\}| = n - 1$ and $(n-1)(t-1) \leq (k-1)$. As a result, the number of elements in common between B_i and the blocks in $\mathcal{S} \setminus \{B_i\}$ is at most $(t-1)(n-1)$ which is at most $(k-1)$. Since B_i is of size k, there exists an element, say x_i, such that $x_i \in B_i$ and $x_i \notin B_j$ for $1 \leq j \leq n$ and $j \neq i$. The case $t = 1$ gives a partition of the point set as blocks, and obviously the lemma holds. ■

Note that Lemma 3.1 holds for finite designs as well. Moreover, note that Lemma 3.1 is stronger than establishing the existence of a system of distinct representative for $\mathcal{S}$ which is defined as follows.

Definition 3.2 *A system of distinct representative of a collection of sets $A_1, A_2, \ldots, A_m$ is a collection of distinct elements $x_1, x_2, \ldots, x_m$ such that $x_i \in A_i$ for each i.*

Theorem 3.3 *If $\mathcal{D}$ is an infinite t-$(v, k, 1)$ design with k finite and $t \geq 2$ and n is an integer such that $1 \leq n \leq t \leq k$ and $(n-1)(t-1) \leq (k-1)$, then $\Xi(G_{\mathcal{D}}) \geq n$.*

Proof Let n be such an integer and we show that $G_{\mathcal{D}}$ is n-e.c. Let $\mathcal{S} = \{B_1, B_2, \ldots, B_n\}$ be a set of n blocks of $\mathcal{D}$ and $\mathcal{T}$ be an m-subset of $\mathcal{S}$ such that $0 \leq m \leq n$. Without loss of generality, let $\mathcal{T} = \{B_1, B_2, \ldots, B_m\}$ and $\mathcal{T}^c = \{B_{m+1}, B_{m+2}, \ldots, B_n\}$. We are going to show that there exists a block not in $\mathcal{S}$ that intersects every block in $\mathcal{T}$ and is disjoint from every block in $\mathcal{T}^c$.

By Lemma 3.1, for every block in $\mathcal{T}$ we can find a point that belongs to it but does not belong to the other blocks in $\mathcal{S}$. For each $i \in \{1, 2, \ldots, m\}$ fix $x_i \in B_i$ such that $x_i \notin B_j$ for $1 \leq j \leq n$ and $j \neq i$. Let $X = \{x_1, x_2, \ldots, x_m\}$, and observe that since the elements of X are distinct, $|X| = |\mathcal{T}|$.

As a first case, suppose $\mathcal{T} = \mathcal{S}$, and so $|X| = n$. If $|X| \leq (t-1)$, then add $t - |X|$ additional distinct points of $V \setminus \bigcup_{i=1}^{n} B_i$ to X to get a t-set of points. By the definition of a t-design, there is a unique block, say $\widehat{B}$, that contains these t points and hence intersects all the blocks in $\mathcal{T}$.

For a second case, suppose $0 \leq |\mathcal{T}| \leq (n-1)$. In this case we have $0 \leq |X| \leq (t-1)$. Add to X an additional $t - |X|$ distinct points of $V \setminus \bigcup_{i=1}^{n} B_i$ until we have a t-set $X_1 = \{x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_t\}$. Since X_1 is a t-set, by the definition of a t-design, there is a unique block, say $B^{(1)}$, containing X_1. If no points of the blocks in $\mathcal{T}^c$ are in $B^{(1)}$, then $\widehat{B} = B^{(1)}$ and we are done; i.e., we have found a block which intersects every block in $\mathcal{T}$ and is disjoint from all blocks in $\mathcal{T}^c$. Otherwise, there is a point of $B^{(1)}$, say z_1, in some block in $\mathcal{T}^c$. Now let $X_2 = (X_1 \setminus \{x_t\}) \cup \{x_{t+1}\}$ where x_{t+1} is a point selected from $V \setminus ((\bigcup_{i=1}^n B_i) \cup B^{(1)})$. Such a point exists since v is infinite and $(\bigcup_{i=1}^n B_i) \cup B^{(1)}$ contains a finite number of points. Since X_2 is a t-set of points, there is a unique block $B^{(2)}$ containing X_2. Observe that $z_1 \notin B^{(2)}$, for if $z_1 \in B^{(2)}$ then $B^{(2)}$ would contain the t-set $\{x_1, x_2, \ldots, x_{t-1}, z_1\}$, and since $\lambda = 1$, it follows that $B^{(1)} = B^{(2)}$; but $B^{(1)} \neq B^{(2)}$ for $x_{t+1} \in B^{(2)} \setminus B^{(1)}$. The block $B^{(2)}$ intersects all blocks in $\mathcal{T}$, and if it does not intersect any block of $\mathcal{T}^c$, then $\widehat{B} = B^{(2)}$ and we are done. Otherwise we take an iterative approach in which we suppose each of $B^{(1)}, B^{(2)}, \ldots, B^{(\ell)}$ have been constructed. In general, if $B^{(\ell)}$ intersects a block of $\mathcal{T}^c$ then proceed to find a next block $B^{(\ell+1)}$ which is the unique block containing $X_{\ell+1} = (X_\ell \setminus \{x_{t+\ell-2}\}) \cup \{x_{t+\ell-1}\}$ where $x_{t+\ell-1} \in V \setminus ((\bigcup_{i=1}^n B_i) \cup (\bigcup_{j=1}^{\ell} B^{(j)}))$. Note that if block $B^{(j)}$ for $1 \leq j \leq \ell$ intersects some block of $\mathcal{T}^c$ at point z_j, then $B^{(\ell+1)} \cap \{z_1, z_2, \ldots, z_\ell\} = \emptyset$; i.e., in each iteration we avoid at least one point of the blocks in $\mathcal{T}^c$. Since there are at most $k(n-m)$ points in the blocks of $\mathcal{T}^c$, whereas v is infinite, and since each time that we get a new block we avoid at least one point of the blocks in $\mathcal{T}^c$, then after a finite number of iterations, say p, we will get a block $\widehat{B} = B^{(p)}$ intersecting all the blocks of $\mathcal{T}$ and none of $\mathcal{T}^c$.

Note that in both cases above, $\widehat{B} \notin \mathcal{S}$. To justify this, we consider two cases. First, if $|X| \leq 1$, then in each iteration, since $t \geq 2$, the set X_j contains $t - |X| \geq 1$ distinct points of $V \setminus \bigcup_{i=1}^{n} B_i$, and any block $B^{(j)}$ (and consequently $\widehat{B}$) containing X_j will be distinct from each block B_i for $1 \leq i \leq n$. Second, if $|X| \geq 2$ (which implies $|\mathcal{T}| \geq 2$), then X contains at least two distinct points, one from each block in $\mathcal{T}$. Since $X \subset \widehat{B}$, $\widehat{B}$ will contain at least two points, one from each block in $\mathcal{T}$; however, any block in $\mathcal{S}$ contains at most one point of X, because each block in $\mathcal{T}$ intersects X in exactly one point, and the blocks in $\mathcal{T}^c$ are disjoint from X. ■

By Theorem 3.3, it follows that any infinite 3-$(v, 5, 1)$ design has 3-e.c. block intersection graph. In general, when $k = (t-1)^2 + 1$, we get the maximum bound of t on n for the block intersection graph of an infinite design to be n-e.c.

We now proceed to establish upper bounds on the existential closure number, beginning with two results that hold for all $\lambda \geq 1$.

Lemma 3.4 *If $\mathcal{D}$ is an infinite t-(v, k, λ) design with k and λ finite, then for any fixed positive integer m there exists an m-set of mutually pairwise disjoint blocks.*

Proof Given an infinite design $\mathcal{D}$ with k and λ finite, and given the positive integer m, our goal is to construct an m-set, $\mathcal{S} = \{B_1, B_2, \ldots, B_m\}$, of pairwise disjoint blocks of the design $\mathcal{D}$. Let $B_1 = \{p_1, p_2, \ldots, p_k\}$ be an arbitrary block of $\mathcal{D}$, and let

$\mathcal{C} = \{B_1\}$. We will now proceed to find B_2.

Let $X_2 = \{x_1, x_2, \ldots, x_t\}$ be an arbitrary t-subset of $V \setminus B_1$, and consider all the λ blocks containing X_2, and add them to the set $\mathcal{C}$. If one of them is disjoint from B_1, then let it be B_2. Otherwise, if z_1 is a point of B_1 which appears in some such blocks, then add all the λ blocks containing $(X_2 \setminus \{x_t\}) \cup \{z_1\}$ to $\mathcal{C}$. Then, consider $X_3 = (X_2 \setminus \{x_t\}) \cup \{x_{t+1}\}$ where again $x_{t+1} \in V \setminus \bigcup_{B \in \mathcal{C}} B$. Now consider all the λ blocks containing X_3, and add all of them to the set $\mathcal{C}$. If any of them is disjoint from B_1, then let it be B_2; otherwise if some of them intersect B_1 at a point, say z_2, then add all the λ blocks containing $(X_3 \setminus \{x_{t+1}\}) \cup \{z_2\}$ to the set $\mathcal{C}$, and consider the t-set $X_4 = (X_3 \setminus \{x_{t+1}\}) \cup \{x_{t+2}\}$ where $x_{t+2} \in V \setminus \bigcup_{B \in \mathcal{C}} B$. After at most k iterations (the number of the elements of B_1) we will get a block B_2 disjoint from B_1.

Now, given two disjoint blocks B_1 and B_2, we are going to find a third block B_3 which is disjoint from B_1 and B_2 by a similar approach. We let $Y_3 = \{y_1, y_1, \ldots, y_t\}$ be a t-set of points of $V \setminus \bigcup_{B \in \mathcal{C}} B$ and consider all λ blocks containing it and add them to $\mathcal{C}$. If one of them is disjoint from both B_1 and B_2, then let it be B_3. Otherwise, if $z_1{}' \in B_1 \cup B_2$ is a point that appears in some of these blocks, then add to $\mathcal{C}$ all blocks containing $(Y_3 \setminus \{y_t\}) \cup \{z_1{}'\}$ and let $Y_4 = (Y_3 \setminus \{y_t\}) \cup \{y_{t+1}\}$ where $y_{t+1} \in V \setminus \bigcup_{B \in \mathcal{C}} B$. The procedure now continues in a manner similar to how we found B_2. It is possible in at most $2k$ iterations to find B_3. Continue this process to get an m-set of pairwise disjoint blocks $\mathcal{S} = \{B_1, B_2, \ldots, B_m\}$. Since finding each B_i is possible in at most $(i-1)k$ iterations where $2 \leq i \leq m$, and given that m is finite, whereas the number of points is infinite, then after a finite number of iterations we obtain the set $\mathcal{S}$. ■

Theorem 3.5 *If $\mathcal{D}$ is an infinite t-(v, k, λ) design with k and λ finite, then $\Xi(G_{\mathcal{D}}) \leq t$.*

Proof To prove the theorem, we show that the block intersection graph of an infinite t-(v, k, λ) design $\mathcal{D}$ with k and λ finite is not n-e.c. for any $n \geq (t+1)$.

Let $\mathcal{S}_1 = \{B_1, B_2, \ldots, B_t\}$ be a t-set of pairwise disjoint blocks as stated in Lemma 3.4. We are going to construct a block $\widehat{B}$ such that there is no block intersecting all the blocks in the $(t+1)$-set of blocks $\mathcal{S} = \mathcal{S}_1 \cup \{\widehat{B}\}$ (and hence $G_{\mathcal{D}}$ will fail to be $(t+1)$-e.c.). For $G_{\mathcal{D}}$ to be $(t+1)$-e.c., there should be a block, say B^*, intersecting all blocks in $\mathcal{S}$. So, the block B^* should contain at least one point of each block B_i for $1 \leq i \leq t$. Since the blocks B_i are pairwise disjoint for $1 \leq i \leq t$, B^* must contain t distinct points, one from each block B_i.

Let $\mathcal{A}$ be the set of all blocks containing at least one point of each block B_i. $|\mathcal{A}|$ is at most λk^t, and hence the number of points in the blocks of $\mathcal{A}$ is at most λk^{t+1}. Now, take any t distinct points of $V \setminus \bigcup_{B \in \mathcal{A}} B$. If at least one of the λ blocks containing these t points is disjoint from all the blocks in $\mathcal{A}$, then let it be the block $\widehat{B}$. Otherwise, by using a similar argument as in the proof of Theorem 3.3 where we were trying to find a block disjoint from all the blocks in $\mathcal{T}^c$, in at most λk^{t+1} iterations we can construct a block $\widehat{B}$ which is disjoint from all the blocks in $\mathcal{A}$.

To have a block intersecting all B_i for $1 \leq i \leq t$, we have to pick one of those in $\mathcal{A}$, but every block in $\mathcal{A}$ is disjoint from $\widehat{B}$, so there is no block intersecting all blocks in $\mathcal{S}$, and hence $G_{\mathcal{D}}$ is not $(t+1)$-e.c. as desired.

Since $G_{\mathcal{D}}$ is not $(t+1)$-e.c., by Theorem 1.4 it cannot be n-e.c. for any $n \geq (t+2)$ as well. Therefore $G_{\mathcal{D}}$ is not n-e.c. for any $n \geq (t+1)$ and hence $\Xi(G_{\mathcal{D}}) \leq t$. ■

In addition to Theorem 3.5, we also have the following upper bound on existential closure number.

Theorem 3.6 *For the block intersection graph of an infinite t-$(v, k, 1)$ design $\mathcal{D}$ with k finite and $t \geq 2$, $\Xi(G_{\mathcal{D}}) \leq (n-1)$ for any integer n such that $(n-1)(t-1) \geq k$.*

Proof We will show that if $(n-1)(t-1) \geq k$ then $G_{\mathcal{D}}$ is not n-e.c. by showing that for the least such n, there is a set of n blocks $\mathcal{S} = \{B_1, B_2, \ldots, B_n\}$ such that there is no block intersecting one of them, but disjoint from the others.

Suppose n is the least integer such that that $(n-1)(t-1) \geq k$, so $(n-1)(t-1) = k + s$ where $0 \leq s \leq (t-2)$. Having this and by the fact that n is the least integer for which $(n-1)(t-1) \geq k$ holds, we let $(n-2)(t-1) + s' = k$ where $1 \leq s' \leq (t-1)$. Now let B_1 be an arbitrary block of $\mathcal{D}$ and then fix a partition of

$B_1 = X_1 \cup X_2 \cup \cdots \cup X_{n-2} \cup X_{n-1}$ such that $|X_i| = t-1$ for $1 \leq i \leq n-2$ and $|X_{n-1}| = s'$. We also let $y_1, y_2, \ldots, y_{n-2}, z_1, \ldots, z_{t-s'}$ be arbitrary distinct points in $V \setminus B_1$. Now we are going to construct blocks B_{i+1} for $1 \leq i \leq (n-1)$ such that there is no block which intersects B_1 and is disjoint from B_{i+1} for $1 \leq i \leq (n-1)$. Let B_{i+1} for $1 \leq i \leq (n-2)$ be the block containing the t-set of points $X_i \cup \{y_i\}$, and B_n be the block containing the t-set $X_{n-1} \cup \{z_1, \ldots, z_{t-s'}\}$. Obviously, any block intersecting B_1 intersects at least one of B_{i+1} for $1 \leq i \leq (n-1)$. ■

At this point, we have determined the existential closure number for the block intersection graph of infinite t-designs $(v, k, 1)$ with $2 \leq t \leq k$ and k finite as follows:

Proposition 3.7 *For the block intersection graph of an infinite t-$(v,k,1)$ design $\mathcal{D}$ with k finite and $2 \leq t \leq k$, $\Xi(G_{\mathcal{D}}) = \min\{t, \lfloor \frac{k-1}{t-1} \rfloor + 1\}$.*

Proof It is a direct consequence of Theorems 3.3, 3.5, and 3.6. ■

3.3 When $\lambda \geq 2$

In this section we investigate the graphs arising from the designs with k finite and $\lambda \geq 2$. We show that for the block intersection graphs $G_{\mathcal{D}}$ of such infinite designs $2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$ when $2 \leq t \leq k-1$, and the block intersection graph of an infinite 1-(v, k, λ) design has existential closure number 1.

Theorem 3.8 *If $\mathcal{D}$ is an infinite t-(v,k,λ) design with k finite, $2 \leq t \leq k-1$, and $\lambda \geq 2$ and finite, then $\Xi(G_{\mathcal{D}}) \geq 2$.*

Proof We show that $G_{\mathcal{D}}$ is 2-e.c. The idea of the proof is similar to that of Theorem 3.3. Suppose $\mathcal{S} = \{B_1, B_2\}$ is a 2-set of blocks of $\mathcal{D}$. Let $x_1 \in B_1 \setminus B_2$ and $x_2 \in B_2 \setminus B_1$. Since we are dealing with simple designs, such x_1 and x_2 exist.

If $\mathcal{T} = \mathcal{S}$, then do as in Theorem 3.3 to get a t-set containing $X = \{x_1, x_2\}$. As there are λ blocks containing each t-set, choose one of them to be the block $\widehat{B}$ intersecting B_1 and B_2. The block $\widehat{B}$ is distinct from B_1 and B_2 because it does contain both x_1 and x_2; however, B_1 and B_2 each contain only one of them.

If $\mathcal{T} = \{B_1\}$, let $X = \{x_1\}$ and construct the set X_1 to be a t-set containing x_1 and $(t-1)$ more points from $V \setminus (B_1 \cup B_2)$ as in the proof of Theorem 3.3, and proceed in a similar fashion. But, since $\lambda \geq 2$, whenever the set X_j has been constructed, there are λ blocks containing X_j. If at least one such block is disjoint from B_2, then we are done. Otherwise continue to find a block $\widehat{B}$ intersecting B_1 and disjoint from B_2 after at most k iterations (the number of elements of B_2, because in each iteration we ignore all λ blocks containing X_j and we do not meet them twice) which is finite. Evidently $\widehat{B} \notin \mathcal{S}$ for it does contain at least one (exactly $t - |X| = t-1$ points which is at least one) point of $V \setminus (B_1 \cup B_2)$.

The case where $\mathcal{T} = \emptyset$ is similar to the same case in the proof of Theorem 3.3. ■

Theorem 3.9 *Let $\mathcal{D}$ be an infinite t-(v, k, λ) design with k finite, $\lambda \geq 2$ and finite, and $1 \leq t \leq k-1$. Then $\Xi(G_\mathcal{D}) \leq (n-1)$ for any n such that $(n-1)t \geq k$.*

Proof We show that $G_\mathcal{D}$ is not n-e.c. for any n such that $(n-1)t \geq k$. The argument is similar to the proof of Theorem 3.6, except that here every t elements occur in more than one block, and hence for the block $B_1 = X_1 \cup X_2 \cup \cdots \cup X_{n-2} \cup X_{n-1}$ we assume that $|X_i| = t$ for $1 \leq i \leq n-2$ and $|X_{n-1}| = s'$ where $1 \leq s' \leq t$. Since $\lambda \geq 2$ there are other blocks containing each set X_i. Let $z_1, z_2 \ldots, z_{t-s'}$ be arbitrary distinct points in $V \setminus B_1$. For $1 \leq i \leq (n-2)$ let B_{i+1} be any block containing the t-set of points X_i, and B_n be any block containing the t-set $X_{n-1} \cup \{z_1, \ldots, z_{t-s'}\}$. Again, any block intersecting B_1 intersects at least one of B_{i+1} for $1 \leq i \leq (n-1)$.

It is possible that we get $B_j = B_{j'}$ for some $2 \leq j, j' \leq n$. In this case, because of the block repetitions, we get a set $\mathcal{S}$ with cardinality less than n, say n', for which there is no block intersecting B_1 and disjoint from the blocks in $\mathcal{S} \setminus \{B_1\}$. As a result, $G_\mathcal{D}$ is not n'-e.c. and hence is not n-e.c. as desired. ■

Proposition 3.10 *Let $\mathcal{D}$ be an infinite t-(v, k, λ) design with k finite, $\lambda \geq 2$, and $1 \leq t \leq k-1$. Then $\Xi(G_\mathcal{D}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$.*

Proof It is a direct consequence of Theorems 3.5, and 3.9. ■

Now that we have an upper bound for the existential closure number, we will see that when λ is finite, for such block intersection graphs $\Xi(G_{\mathcal{D}}) \geq 2$ for $2 \leq t \leq k-1$, although when $\lambda \geq 2$, the block intersection graph is not necessarily 3-e.c. The following corollary is a direct consequence of Theorems 3.3 and 3.8, since $(t-1) \leq (k-1)$.

Corollary 3.11 *If $\mathcal{D}$ is an infinite t-(v, k, λ) design with k and λ finite and $2 \leq t \leq k-1$, then $\Xi(G_{\mathcal{D}}) \geq 2$.*

So, the block intersection graph of an infinite t-design with $2 \leq t \leq k-1$ is guaranteed to be 2-e.c. when k and λ are finite regardless of the value of λ. We now show that the property of being 3-e.c. does not share this ubiquity.

Proposition 3.12 *Let $\mathcal{D}$ be an infinite t-(v, k, λ) design with k finite, $\lambda \geq 2$, and $2 \leq t \leq k-1$. Then $G_{\mathcal{D}}$ is not necessarily 3-e.c and $2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$.*

Proof It is sufficient to prove that $G_{\mathcal{D}}$ is not necessarily 3-e.c. To see this, consider a design with the block B having a partition $B = X_1 \cup X_2$ such that X_1 and X_2 are nonempty. Now consider two possible blocks $B_1 = X_1 \cup Y_1$, and $B_2 = X_2 \cup Y_2$ where $|Y_i| = k - |X_i|$ for $i = 1, 2$. This is possible because of the fact that since $\lambda \geq 2$, then intersection of every two blocks can have any number of points less than k. Now if $\mathcal{S} = \{B, B_1, B_2\}$, then there is no block intersecting B and disjoint from B_1, B_2. So

$G_{\mathcal{D}}$ is not necessarily 3-e.c and by Corollary 3.11 and Proposition 3.10 we conclude that $2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$. ■

For $\lambda \geq 2$ we can show that there exist infinite t-designs with block intersection graphs that are not 3-e.c., however $\min\{t, \lceil \frac{k}{t} \rceil\} \geq 3$ holds. According to [15], for each finite t and k such that $t < k$, large sets of infinite Steiner systems exist. A large set of Steiner systems is a partition of the k-subsets of the point set so that each partition is a Steiner system.

As an example consider an infinite 3-$(\aleph_0, 7, \lambda)$ with $\lambda \geq 2$. By the results of Theorem 3.8 and Proposition 3.10, we already know that the block intersection graph of such a design is 2-e.c. and is not 4-e.c., but we do not yet know whether it is n = 3-e.c. or not. Consider a large set of infinite Steiner systems, say $\mathcal{L}_{7,3}$, on a countably infinite set V, for $k = 7$, and $t = 3$. For $\lambda = 2$, we construct an example of an infinite design which is the union of two distinct Steiner systems $\mathcal{D}, \mathcal{D}_2$ of $\mathcal{L}_{7,3}$ whose block intersection graph is not 3-e.c. We let $\mathcal{D}$ be a Steiner system having blocks B, B_1 such that $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $B \cap B_1 = \{1\}$ (note that $r = v$, so there are infinitely many blocks having point 1 beside B). Also we let $\mathcal{D}_2$ be a Steiner system having block $B_2 = \{2, 3, 4, 5, 6, 7, 8\}$ (since in a Steiner system $\lambda = 1$, B and B_2 cannot appear together in the same design and hence $\mathcal{D} \neq \mathcal{D}_2$). Now the union of $\mathcal{D}$ and $\mathcal{D}_2$ is an infinite 3-$(\aleph_0, 7, \lambda)$ design with $\lambda = 2$ whose block intersection graph is not 3-e.c. as there is no block intersecting B and disjoint from B_1 and B_2.

More generally, suppose we would like to construct an infinite t-(v, k, λ) design with $\lambda \geq 2$, and $\min\{t, \lceil \frac{k}{t} \rceil\} \geq 3$ whose block intersection graph is not 3-e.c. We consider a large set of infinite Steiner systems $\mathcal{L}_{k,t}$. Let $\mathcal{D} \in \mathcal{L}_{k,t}$ be a Steiner system having blocks $B = \{x_1, x_2, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_k\}$, and $B_1 = \{x_1, y_2, \ldots, y_{t-1}, y_t, y_{t+1}, \ldots, y_k\}$. Also let $\mathcal{D}_2 \in \mathcal{L}_{k,t}$ be a Steiner system having block $B_2 = \{z_1, x_2, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots, x_k\}$ (note that $\mathcal{D} \neq \mathcal{D}_2$). Now consider the union of $\mathcal{D}, \mathcal{D}_2$, and $\lambda - 2$ more distinct Steiner systems of the large set other than $\mathcal{D}, \mathcal{D}_2$. Of course, this is an infinite t-design with index λ, and since there is no block intersecting B and disjoint from B_1 and B_2, its block intersection graph is not 3-e.c.

Chapter 4

Block Intersection Graphs of Infinite Designs Having Infinite Block Size

4.1 Introduction

In this chapter, we consider the n-existential closure property of block intersection graphs of infinite t-(v, k, λ) designs with finite values of t and λ and infinite block size. In Chapter 3 we considered the case when the block size is finite; for such an infinite design $\mathcal{D}$ it was shown that $\Xi(G_{\mathcal{D}}) = \min\{t, \lfloor \frac{k-1}{t-1} \rfloor + 1\}$ if $\lambda = 1$ and $2 \leq t \leq k$, and

$2 \leq \Xi(G_{\mathcal{D}}) \leq \min\{t, \lceil \frac{k}{t} \rceil\}$ if $\lambda \geq 2$ and $2 \leq t \leq k-1$.

Throughout this chapter we will be working in Zermelo-Fraenkel set theory with the axiom of choice. The reader is referred to [30] for basic facts about set theory and cardinal arithmetic. Also note that the block set of any 1-$(v, k, 1)$ design is a partition of its point set and hence the block intersection graph of the design must have existential closure number 0 (since no two blocks share a point). Thus, in our results throughout this chapter we will often assume that $(t, \lambda) \neq (1, 1)$. The main results of this chapter are as follows.

Theorem 4.1 *Let v be an infinite cardinal and let t and λ be positive integers such that $(t, \lambda) \neq (1, 1)$. Then, for each non-negative integer n, there exists a t-(v, v, λ) design $\mathcal{D}$ such that $\Xi(G_{\mathcal{D}}) = n$. Furthermore, there exists a t-(v, v, λ) design $\mathcal{D}'$ such that $G_{\mathcal{D}'}$ is n-e.c. for each non-negative integer n.*

Theorem 4.2 *Let v and k be infinite cardinals with $k < v$ and let t and λ be positive integers such that $(t, \lambda) \neq (1, 1)$. Then there exists a t-(v, k, λ) design $\mathcal{D}$ with $\Xi(G_{\mathcal{D}}) = n$ if and only if*

- *$n = t$ when $\lambda = 1$ or $t = 1$;*
- *$2 \leq n \leq t$ when $t, \lambda \geq 2$.*

Furthermore, if $\mathcal{D}$ is a t-(v, k, λ) design then $\Xi(G_{\mathcal{D}}) = \min\{\ell, t\}$, where ℓ is the smallest cardinal such that there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$.

Theorem 4.1 is proved in Section 4.2, where we consider infinite t-(v, k, λ) designs with $k = v$. The case $k < v$ is considered separately in Section 4.3, where we prove Theorem 4.2.

The results of this chapter have been published in the *Journal of Combinatorial Designs* [29].

4.2 When $k = v$

Our main goal in this section is to prove Theorem 4.1. In Lemma 4.5 we establish the existence of the required designs with block intersection graphs having existential closure number n for some non-negative integer n, and in Lemma 4.6 we establish the existence of the required designs with block intersection graphs which are n-e.c. for each non-negative integer n. We will make use of the following well-known result (for a proof see [20], for example).

Lemma 4.3 *Let a be an infinite cardinal and let A be an a-set. Then, for each* *positive integer i, there are exactly a i-subsets of A.*

Corollary 4.4 *Let a be an infinite cardinal and let A be an a-set. Then, for each positive integer i, there are exactly a ordered pairs (X, Y) of disjoint subsets of A such that $|X| + |Y| = i$. Also, there are exactly a ordered pairs (X, Y) of disjoint subsets of A such that $|X| + |Y|$ is finite.*

Proof Let A be an a-set and, for each positive integer i, let P_i be the set of ordered pairs (X, Y) of disjoint subsets of A such that $|X|+|Y| = i$. Since, by Lemma 4.3, the number of i-subsets of A is a for each positive integer i, it follows that $|P_i| = 2^i a = a$ for each positive integer i. Let P be the set of ordered pairs (X, Y) of disjoint subsets of A such that $|X| + |Y|$ is finite. Then $P = (\emptyset, \emptyset) \cup \bigcup_{i\in\mathbb{Z}^+} P_i$ and it follows that $|P| = 1 + \aleph_0 a = a$. ■

Lemma 4.5 *Let v be an infinite cardinal and let t and λ be positive integers such that $(t, \lambda) \neq (1, 1)$. Then, for each non-negative integer n, there exists a t-(v, v, λ) design $\mathcal{D}$ such that $\Xi(G_{\mathcal{D}}) = n$.*

Proof We will show that for a fixed non-negative integer n there exists a t-(v, v, λ) design whose block intersection graph is n-e.c. but not $(n + 1)$-e.c. Let V be a point set with $|V| = v$. Throughout this proof when we refer to a block this will imply that it is a v-subset of V. By Lemma 4.3 there are v t-subsets of V and hence we can write the set of all t-subsets of V as $\{T_\alpha\}_{\alpha<v}$, a family of sets indexed by the set of ordinals less than v. Let $\mathcal{B}_0$ be a set of $n+1$ pairwise disjoint blocks such that there are v points of V which are in no block in $\mathcal{B}_0$. Let $U_0 = \bigcup_{B\in\mathcal{B}_0} B$. Since $|V \setminus U_0| = v$ we can find v pairwise disjoint v-subsets of $V \setminus U_0$, $\{S'_\alpha\}_{\alpha<v}$ say. For each $\alpha < v$ let $S_\alpha = S'_\alpha \setminus (\bigcup_{\beta<\alpha} T_\beta)$ and observe that $|S_\alpha| = v$ for each $\alpha < v$ (note that, since v is a cardinal, $|\alpha| < v$ and hence $|\bigcup_{\beta<\alpha} T_\beta| \leq t|\alpha| < v$).

We claim that there is a transfinite sequence $\{\mathcal{B}_\alpha\}_{\alpha<v}$ of sets of blocks such that, for each ordinal $\alpha < v$, $\mathcal{B}_\alpha$ satisfies

(i) for each ordinal $\beta < \alpha$, $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$;

(ii) any two blocks in $\mathcal{B}_\alpha$ intersect in at most t points;

(iii) no block of $\mathcal{B}_\alpha$ contains any of the points in $\bigcup_{\alpha<\beta<v} S_\beta$;

(iv) each t-subset of V is a subset of at most λ blocks of $\mathcal{B}_\alpha$ and, for each ordinal $\beta < \alpha$, T_β is a subset of exactly λ blocks of $\mathcal{B}_\alpha$;

(v) for each ordinal $\beta < \alpha$ and for any pair of disjoint sets $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X} \subseteq \mathcal{B}_\beta$, $\mathcal{Y} \subseteq \mathcal{B}_\beta$ and $|\mathcal{X}| + |\mathcal{Y}| = n$, there exists a block in $\mathcal{B}_\alpha \setminus (\mathcal{X} \cup \mathcal{Y})$ which intersects each block in $\mathcal{X}$ and is disjoint from each block in $\mathcal{Y}$;

(vi) each block in $\mathcal{B}_\alpha \setminus \mathcal{B}_0$ intersects at least one block in $\mathcal{B}_0$; and

(vii) $|\mathcal{B}_\alpha|$ is finite if α is finite and $|\mathcal{B}_\alpha| \leq |\alpha|$ if α is infinite.

Note that, for each ordinal $\alpha < v$, $|\alpha| < v$ since v is a cardinal and hence (vii) implies that $|\mathcal{B}_\alpha| < v$.

If such a sequence exists, then the structure $\mathcal{D}$ with point set V and block set $\bigcup_{\alpha<v} \mathcal{B}_\alpha$ is a t-(v, v, λ) design such that $\Xi(G_\mathcal{D}) = n$. To see this, note that (ii) implies that no block of $\mathcal{D}$ is a subset of another, that (iii) implies that for each block of $\mathcal{D}$ there are v points of V not in that block and that (iv) implies that every t-subset of V is a subset of exactly λ blocks of $\mathcal{D}$. Thus $\mathcal{D}$ is a t-(v, v, λ) design. Furthermore, (v) implies that the block intersection graph of $\mathcal{D}$ is n-e.c. and (vi) implies that it is not $(n+1)$-e.c., and hence $\Xi(G_\mathcal{D}) = n$. Thus it only remains to show that the sequence $\{\mathcal{B}_\alpha\}_{\alpha<v}$ exists. We will do so by transfinite induction.

Note that $\mathcal{B}_0$ satisfies (i)-(vii). Now we assume that, for some ordinal γ with $1 \leq \gamma < v$, we have constructed a sequence $\{\mathcal{B}_\alpha\}_{\alpha<\gamma}$ such that $\mathcal{B}_\alpha$ satisfies (i)-(vii) for each ordinal $\alpha < \gamma$, and we will demonstrate how to construct a set of blocks $\mathcal{B}_\gamma$ which satisfies (i)-(vii) for $\alpha = \gamma$.

If γ is a limit ordinal then let $\mathcal{B}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{B}_\alpha$. Using the fact that, for each ordinal $\alpha < \gamma$, $\mathcal{B}_\alpha$ satisfies (i)-(vii), it is routine to check that $\mathcal{B}_\gamma$ satisfies (i)-(vii) for $\alpha = \gamma$ (to see that (vii) holds, note that $|\mathcal{B}_\gamma| = \sup\{|\mathcal{B}_\alpha| : \alpha < \gamma\}$).

If γ is a successor ordinal, then we construct $\mathcal{B}_\gamma$ from $\mathcal{B}_{\gamma-1}$ in the following way.

1. Add all the blocks of $\mathcal{B}_{\gamma-1}$ to $\mathcal{B}_\gamma$.

2. Let P be the set of all the ordered pairs $(\mathcal{X}, \mathcal{Y})$ of disjoint subsets of $\mathcal{B}_{\gamma-1}$ such that $|\mathcal{X}| + |\mathcal{Y}| = n$. Note that $|P| < v$ since $|P|$ is finite if $|\mathcal{B}_{\gamma-1}|$ is finite and, by Corollary 4.4, $|P| = |\mathcal{B}_{\gamma-1}| < v$ if $|\mathcal{B}_{\gamma-1}|$ is infinite. For each block $B \in \mathcal{B}_{\gamma-1}$, take $|P|+\lambda$ distinct points of B each of which is in no other block of $\mathcal{B}_{\gamma-1}$ (these exist since $|B| = v$, $|\mathcal{B}_{\gamma-1}| < v$ and every other block in $\mathcal{B}_{\gamma-1}$ intersects B in at most t points) and place them in one-to-one correspondence with the elements of $P \cup \{1, 2, \ldots, \lambda\}$. Also, take $|P| + \lambda$ pairwise disjoint v-subsets of S_γ and place them in one-to-one correspondence with the elements of $P \cup \{1, 2, \ldots, \lambda\}$. For each $(\mathcal{X}, \mathcal{Y}) \in P$, where $\mathcal{X} = \{X_1, X_2, \ldots, X_s\}$ say, do the following.

   - If $\mathcal{B}_0 \cap \mathcal{X} \neq \emptyset$, then add to $\mathcal{B}_\gamma$ the block $\{x_1, x_2, \ldots, x_s\} \cup S$, where x_i is the point of X_i corresponding to $(\mathcal{X}, \mathcal{Y})$ for each $i \in \{1, 2, \ldots, s\}$ and S is the v-subset of S_γ corresponding to $(\mathcal{X}, \mathcal{Y})$.

   - If $\mathcal{B}_0 \cap \mathcal{X} = \emptyset$, then add to $\mathcal{B}_\gamma$ the block $\{x_1, x_2, \ldots, x_s\} \cup \{x^*\} \cup S$, where x_i is the point of X_i corresponding to $(\mathcal{X}, \mathcal{Y})$ for each $i \in \{1, 2, \ldots, s\}$, x^* is the point of some block in $\mathcal{B}_0 \setminus \mathcal{Y}$ corresponding to $(\mathcal{X}, \mathcal{Y})$, and S is the v-subset of S_γ corresponding to $(\mathcal{X}, \mathcal{Y})$.

Note that the block corresponding to the pair $(\mathcal{X}, \mathcal{Y})$ intersects each block in $\mathcal{X}$

in exactly one point and is disjoint from each block in $\mathcal{Y}$, and that each block added in this step intersects each other block in $\mathcal{B}_\gamma$ in at most one point. If $t \geq 2$, then, for each t-subset T of V, the number of blocks in $\mathcal{B}_\gamma$ which are supersets of T either remains the same through this step or increases from zero to one. If $t = 1$, then $\lambda \geq 2$ and, for each point x of V, the number of blocks in $\mathcal{B}_\gamma$ which contain x either remains the same through this step, increases from zero to one, or increases from one to two.

3. Let a be the number of blocks already in $\mathcal{B}_\gamma$ which are supersets of $T_{\gamma-1}$. If $a = \lambda$ then do nothing. If $a < \lambda$ then do the following.

    ◦ If $T_{\gamma-1} \cap U_0 \neq \emptyset$, then, for each $i \in \{1, 2, \ldots, \lambda - a\}$, add to $\mathcal{B}_\gamma$ the block $T_{\gamma-1} \cup S$ where S is the v-subset of S_γ corresponding to i.

    ◦ If $T_{\gamma-1} \cap U_0 = \emptyset$, then, for each $i \in \{1, 2, \ldots, \lambda - a\}$, add to $\mathcal{B}_\gamma$ the block $T_{\gamma-1} \cup \{x^*\} \cup S$ where x^* is the point of some block in $\mathcal{B}_0$ corresponding to i and S is the v-subset of S_γ corresponding to i.

Note that each block added in this step intersects each other block in $\mathcal{B}_\gamma$ in at most t points. If $t \geq 2$ then, for each t-subset T of V other than $T_{\gamma-1}$, the number of blocks in $\mathcal{B}_\gamma$ which are supersets of T either remains the same through this step or increases from zero to one. If $t = 1$ then, for each point x

of V other than the point in $T_{\gamma-1}$, the number of blocks in $\mathcal{B}_\gamma$ which contain x either remains the same through this step, increases from zero to one, or increases from one to two. Furthermore, through this step the number of blocks in $\mathcal{B}_\gamma$ which are supersets of $T_{\gamma-1}$ either remains at λ or increases to λ.

Using the fact that $\mathcal{B}_{\gamma-1}$ satisfies (i)-(vii) for $\alpha = \gamma - 1$, it is routine to check from the construction that $\mathcal{B}_\gamma$ satisfies (i)-(vii) for $\alpha = \gamma$ (to see that (vii) holds, note that $|\mathcal{B}_\gamma| \leq |\mathcal{B}_{\gamma-1}| + |P| + \lambda$, that $|P|$ is finite if $|\mathcal{B}_{\gamma-1}|$ is finite, and that $|P| = |\mathcal{B}_{\gamma-1}|$ if $|\mathcal{B}_{\gamma-1}|$ is infinite). Thus the required sequence $\{\mathcal{B}_\alpha\}_{\alpha<v}$ does indeed exist and the proof is complete. ■

Lemma 4.6 *Let v be an infinite cardinal and let t and λ be positive integers such that $(t, \lambda) \neq (1, 1)$. Then there exists a t-(v, v, λ) design whose block intersection graph is n-e.c. for all non-negative integers n.*

Proof We can construct such a design by following an argument similar to the argument in the proof of Lemma 4.5 with the following exceptions. Firstly, we let $\mathcal{B}_0 = \emptyset$. Secondly, for each ordinal $\alpha < v$, $\mathcal{B}_\alpha$ should, rather than (i)-(vii), satisfy (i)-(iv), (vii) and

(v′) for each ordinal $\beta < \alpha$ and for any pair of disjoint sets $(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{X} \subset \mathcal{B}_\beta$, $\mathcal{Y} \subset \mathcal{B}_\beta$, and $|\mathcal{X}|+|\mathcal{Y}|$ is finite, there exists a block in $\mathcal{B}_\alpha \backslash (\mathcal{X} \cup \mathcal{Y})$ which intersects each block in $\mathcal{X}$ and is disjoint from each block in $\mathcal{Y}$.

Thirdly, in order to construct $\mathcal{B}_\gamma$ from $\mathcal{B}_{\gamma-1}$ when γ is a successor ordinal, we replace steps 2 and 3 with the following.

2′. Let P be the set of all the ordered pairs $(\mathcal{X}, \mathcal{Y})$ of disjoint subsets of $\mathcal{B}_{\gamma-1}$ such that $|\mathcal{X}| + |\mathcal{Y}|$ is finite. Note that $|P| < v$ since $|P|$ is finite if $|\mathcal{B}_{\gamma-1}|$ is finite and, by Corollary 4.4, $|P| = |\mathcal{B}_{\gamma-1}| < v$ if $|\mathcal{B}_{\gamma-1}|$ is infinite. For each block $B \in \mathcal{B}_{\gamma-1}$, take $|P|$ distinct points of B each of which is in no other block of $\mathcal{B}_{\gamma-1}$ (these exist since $|B| = v$, $|\mathcal{B}_{\gamma-1}| < v$ and every other block in $\mathcal{B}_{\gamma-1}$ intersects B in at most t points) and place them in one-to-one correspondence with the elements of P. Also, take $|P| + \lambda$ pairwise disjoint v-subsets of S_γ and place them in one-to-one correspondence with the elements of $P \cup \{1, 2, \ldots, \lambda\}$. For each $(\mathcal{X}, \mathcal{Y}) \in P$, where $\mathcal{X} = \{X_1, X_2, \ldots, X_s\}$ say, add to $\mathcal{B}_\gamma$ the block $\{x_1, x_2, \ldots, x_s\} \cup S$, where x_i is the point of X_i corresponding to $(\mathcal{X}, \mathcal{Y})$ for each $i \in \{1, 2, \ldots, s\}$ and S is the v-subset of S_γ corresponding to $(\mathcal{X}, \mathcal{Y})$.

3′. Let a be the number of blocks already in $\mathcal{B}_\gamma$ which are supersets of $T_{\gamma-1}$. If $a = \lambda$ then do nothing. If $a < \lambda$ then, for each $i \in \{1, 2, \ldots, \lambda - a\}$, add to $\mathcal{B}_\gamma$ the block $T_{\gamma-1} \cup S$ where S is the v-subset of S_γ corresponding to i.

■

Observe that when $v = \aleph_0$, Lemma 4.6 implies that there exists a t-$(\aleph_0, \aleph_0, \lambda)$ design whose block intersection graph is the Rado graph (it is well known that the Rado graph is the only graph on $\aleph_0$ vertices which is n-e.c. for all non-negative integers n).

Proof of Theorem 4.1 This follows immediately from Lemmas 4.5 and 4.6. ■

Now that we have our main results of this section, we present examples of two designs which are more naturally constructed than those constructed in the proof of Lemma 4.5 and whose block intersection graphs have existential closure numbers 1 and 2, respectively.

Example 4.7 *Let $\mathcal{D}_1$ be the 2-$(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design whose point set is $\mathbb{R}^2$, and whose block set consists of all lines in $\mathbb{R}^2$. Then $\Xi(G_{\mathcal{D}_1}) = 1$.*

Proof Clearly $G_{\mathcal{D}_1}$ is 1-e.c. as, for each line L, there is a line distinct from L intersecting L and there is a line distinct from L parallel to L. However, $G_{\mathcal{D}_1}$ is not 2-e.c. as for a set of two parallel lines, there is no line outside of the set intersecting one and disjoint from the other. Thus $\Xi(G_{\mathcal{D}_1}) = 1$. ■

Example 4.8 *Let $\mathcal{D}_2$ be the 2-$(\aleph_0, \aleph_0, 1)$ design whose point set is $\mathbb{Z}^2$, and whose block set is $\{L \cap \mathbb{Z}^2 : L \in \mathcal{L}\}$ where $\mathcal{L}$ is the set of all lines in $\mathbb{R}^2$ which contain* *at least one point in* $\mathbb{Z}^2$ *and which are either vertical or have rational slope. Then* $\Xi(G_{\mathcal{D}_2}) = 2$.

Proof We first show that $G_{\mathcal{D}_2}$ is not 3-e.c. For each $i \in \{1, 2, 3\}$, let L_i be the line $\{(i, y) : y \in \mathbb{R}\}$ and let $B_i = L_i \cap \mathbb{Z}^2$. Any block of $\mathcal{D}_2$ which intersects B_1 and B_2 also intersects B_3, and hence there is no block intersecting B_1 and B_2 and disjoint from B_3. Thus $G_{\mathcal{D}_2}$ is not 3-e.c.

We now prove that $G_{\mathcal{D}_2}$ is 2-e.c. Let $B_1 = L_1 \cap \mathbb{Z}^2$ and $B_2 = L_2 \cap \mathbb{Z}^2$ be two distinct blocks of $\mathcal{D}_2$, where $L_1, L_2 \in \mathcal{L}$. Without loss of generality it suffices to find a block of $\mathcal{D}_2$ distinct from B_1 and B_2 which intersects B_1 and B_2, a block of $\mathcal{D}_2$ distinct from B_1 and B_2 which intersects B_1 and is disjoint from B_2, and a block of $\mathcal{D}_2$ distinct from B_1 and B_2 which is disjoint from both B_1 and B_2. Let L' be a line which passes through a point of $(L_1 \cap \mathbb{Z}^2) \setminus L_2$ and a point of $(L_2 \cap \mathbb{Z}^2) \setminus L_1$. It can be seen that $L' \cap \mathbb{Z}^2$ is a block of $\mathcal{D}_2$ which intersects B_1 and B_2. Let L'' be a line which passes through a point of $(L_1 \cap \mathbb{Z}^2) \setminus L_2$ and a point of $(L_2 \cap \mathbb{Q}^2) \setminus (\mathbb{Z}^2 \cup L_1)$. It can be seen that $L'' \cap \mathbb{Z}^2$ is a block of $\mathcal{D}_2$ which intersects B_1 and is disjoint from B_2. Let L''' be a line in $\mathcal{L}$ which passes through a point (x, y) of $(L_1 \cap (\mathbb{Q} \setminus \mathbb{Z})^2) \setminus L_2$ and a point of $L_2 \setminus (\mathbb{Z}^2 \cup L_1)$ (if L_2 is not vertical then the line through (x, y) and $(\lceil x \rceil, y^*)$ for a sufficiently large integer y^* suffices, and if L_2 is vertical then the line through (x, y) and $(x^*, \lceil y \rceil)$ for a sufficiently large integer x^* suffices). It can be seen that $L''' \cap \mathbb{Z}^2$ is a block of $\mathcal{D}_2$ which is disjoint from both B_1 and B_2. ■

We conclude this section with a result linking the existence of infinite 2-$(v, v, 1)$ designs whose block intersection graphs have existential closure number 0 to the existence of infinite 2-$(v, v, 1)$ designs whose block set can be partitioned into sets such that each set is a partition of the point set. It is tempting to call infinite designs with this latter property resolvable, but we refrain from doing so pending an investigation of whether this is in fact the best definition of resolvability for infinite designs in general. We first require the following lemma.

Lemma 4.9 *Let v be an infinite cardinal, let t be an integer such that $t \geq 2$, and let $\mathcal{D}$ be a t-$(v, v, 1)$ design. Suppose that there is a partition of the block set of $\mathcal{D}$ into sets such that each set is a partition of the point set. Then there are v sets in the partition and there are v blocks in each set.*

Proof Let x be a point of $\mathcal{D}$. By Corollary 3.1 of [20], x is contained in exactly v blocks of $\mathcal{D}$ and hence, since each set of the partition contains exactly one block which contains x, there must be v sets in the partition. Now suppose for a contradiction that one set $\mathcal{C}$ of the partition contains fewer than v blocks. Since $\mathcal{D}$ is a t-design with $\lambda = 1$, a block of $\mathcal{D}$ not in $\mathcal{C}$ can intersect each block in $\mathcal{C}$ in at most $t-1$ points, and hence must contain fewer than v points. This is a contradiction. ■

Theorem 4.10 *Let v be an infinite cardinal. Then there exists a 2-$(v, v, 1)$ design $\mathcal{D}$ with $\Xi(G_{\mathcal{D}}) = 0$ if and only if there exists a 2-$(v, v, 1)$ design whose block set can be partitioned into sets such that each set is a partition of the point set.*

Proof Suppose that $\mathcal{D}$ is a 2-$(v, v, 1)$ design with $\Xi(G_{\mathcal{D}}) = 0$. Let V be the point set of $\mathcal{D}$ and let $\mathcal{B}$ be the block set of $\mathcal{D}$. Clearly, for each block B in $\mathcal{B}$ there is another block in $\mathcal{B}$ which intersects B. So, since $\Xi(G_{\mathcal{D}}) = 0$, there must exist a block B^* in $\mathcal{B}$ such that every other block in $\mathcal{B}$ intersects B^*.

For each $x \in B^*$, let $\mathcal{B}_x$ be the set of blocks in $\mathcal{B} \setminus \{B^*\}$ which contain x. By Corollary 3.1 of [20] each point in V occurs in v blocks in $\mathcal{B}$ and hence $|\mathcal{B}_x| = v$ for each $x \in B^*$. Since $\mathcal{D}$ is a 2-design with $\lambda = 1$, $\mathcal{P}_x = \{B \setminus B^* : B \in \mathcal{B}_x\}$ is a partition of $V \setminus B^*$ for each $x \in B^*$, and $\{\mathcal{P}_x : x \in B^*\}$ is a partition of $\{B \setminus B^* : B \in \mathcal{B} \setminus \{B^*\}\}$.

Let $\mathcal{D}'$ be the structure with point set $V \setminus B^*$ and block set $\{B \setminus B^* : B \in \mathcal{B} \setminus \{B^*\}\}$. It is easy to confirm that $\mathcal{D}'$ is a 2-$(v, v, 1)$ design and we have seen that $\{\mathcal{P}_x : x \in B^*\}$ is a partition of its block set into sets such that each set is a partition of the point set.

In the other direction, suppose that there exists a 2-$(v, v, 1)$ design whose block set can be partitioned into sets such that each set is a partition of the point set. In view of the result of Lemma 4.9 it can be seen that the procedure above can be reversed to obtain a 2-$(v, v, 1)$ design whose block intersection graph has existential closure number 0. ■

It is easy to see that the block sets of the designs given in Examples 4.7 and 4.8 can be partitioned into sets such that each set is a partition of the point set (in Example 4.7 partition the lines according to their slope, and in Example 4.8 partition the blocks according to the slope of their corresponding line). Thus the construction in the proof of Theorem 4.10 can be applied to Examples 4.7 and 4.8 to obtain examples of a 2-$(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design and a 2-$(\aleph_0, \aleph_0, 1)$ design whose block intersection graphs have existential closure number 0.

4.3 When $k < v$

In this section we will prove Theorem 4.2. We start with a lemma which will prove useful throughout the section.

Lemma 4.11 *Let v and k be infinite cardinals with $k < v$, let t and λ be positive integers, and let $\mathcal{D}$ be a t-(v, k, λ) design. Let S and S' be disjoint subsets of the point set of $\mathcal{D}$ such that $|S| \leq t-1$ and $|S'| < v$. Then there is a block of $\mathcal{D}$ which is a superset of S and which is disjoint from S'.*

Proof Since $|S| \leq t-1$ and $|S'| < v$, it is easy to see that there is a set $S^\dagger$ of points of $\mathcal{D}$ such that $S \subseteq S^\dagger$, $S^\dagger \cap S' = \emptyset$ and $|S^\dagger| = t-1$. Let $\mathcal{X}$ be the set of all blocks of

$\mathcal{D}$ which are supersets of $S^\dagger$ and let $\mathcal{Y}$ be the set of all blocks of $\mathcal{D}$ which are supersets of $S^\dagger$ and contain at least one point in S'. It suffices to show that $|\mathcal{Y}| < |\mathcal{X}|$.

Clearly there are exactly v t-sets of points of $\mathcal{D}$ which are supersets of $S^\dagger$. Thus, since the only blocks of $\mathcal{D}$ which can be supersets of these t-sets are those in $\mathcal{X}$, since by Lemma 4.3 a block of $\mathcal{D}$ can be a superset of at most k of these t-sets, and since each of these t-sets is a subset of exactly λ blocks of $\mathcal{D}$, it follows that $k|\mathcal{X}| \geq \lambda v$ and hence that $|\mathcal{X}| \geq v$ (to be more precise, $|\mathcal{X}| = v$ since $\mathcal{D}$ has v blocks).

Clearly there are $|S'|$ t-subsets of the point set of $\mathcal{D}$ that are supersets of $S^\dagger$ and also contain a point of S'. Thus, since each block in $\mathcal{Y}$ is a superset of at least one of these t-sets and since each of these t-sets is a subset of exactly λ blocks of $\mathcal{D}$, it follows that $|\mathcal{Y}| \leq \lambda|S'| < v$. So $|\mathcal{Y}| < |\mathcal{X}|$ and the lemma holds. ■

We will often make use of the special case of Lemma 4.11 where $S = \emptyset$. We will also make use of the following lemma which is an easy consequence of Lemma 4.11.

Lemma 4.12 *Let v and k be infinite cardinals with $k < v$, let t and λ be positive integers, and let $\mathcal{D}$ be a t-(v, k, λ) design. Then, for each positive integer n, $\mathcal{D}$ has n pairwise disjoint blocks.*

Proof We proceed by induction on n. The result is trivial for $n = 1$. If $\mathcal{D}$ has m pairwise disjoint blocks $B_1, B_2, \ldots, B_m$ for some positive integer m then, by applying

Lemma 4.11 with $S = \emptyset$ and $S' = B_1 \cup B_2 \cup \cdots \cup B_m$, we can find a block B_{m+1} of $\mathcal{D}$ which is disjoint from $B_1 \cup B_2 \cup \cdots \cup B_m$. Then $B_1, B_2, \ldots, B_{m+1}$ are $m+1$ pairwise disjoint blocks of $\mathcal{D}$. ■

We now give a characterisation of the existential closure number of the block intersection graph of an infinite design with k infinite and $k < v$.

Lemma 4.13 *Let v and k be infinite cardinals with $k < v$, let t and λ be positive integers such that $(t, \lambda) \neq (1, 1)$, and let $\mathcal{D}$ be a t-(v, k, λ) design. Then $\Xi(G_{\mathcal{D}}) = \min\{\ell, t\}$ where ℓ is the smallest cardinal such that there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$.*

Proof Let $m = \min\{\ell, t\}$. We first show that $G_{\mathcal{D}}$ is m-e.c. Let $\mathcal{A}$ and $\mathcal{A}'$ be two disjoint sets of blocks of $\mathcal{D}$ such that $|\mathcal{A}| + |\mathcal{A}'| = m$. It suffices to find a block of $\mathcal{D}$ not in $\mathcal{A} \cup \mathcal{A}'$ that intersects each block in $\mathcal{A}$ and is disjoint from each block in $\mathcal{A}'$.

If $\mathcal{A} = \emptyset$, then $|\mathcal{A}'| = m$ and we can find a block of $\mathcal{D}$ disjoint from each block in $\mathcal{A}'$ by applying Lemma 4.11 with $S = \emptyset$ and $S' = \bigcup_{B \in \mathcal{A}'} B$ (note that $|S'| \leq mk < v$). If $\mathcal{A}' = \emptyset$, then $|\mathcal{A}| = m$ and if we take a t-set T of points of $\mathcal{D}$ which intersects each block in $\mathcal{A}$ (one exists since $m \leq t$) then there is a block B of $\mathcal{D}$ that is a superset of T and hence intersects each block in $\mathcal{A}$. Thus we can assume that $1 \leq |\mathcal{A}|, |\mathcal{A}'| \leq m-1$. Let $U' = \bigcup_{B \in \mathcal{A}'} B$ and note that $|U'| = k < v$. Since no block of $\mathcal{D}$ is a subset of the union of at most $m-1$ others, every block in $\mathcal{A}$ contains a point which is not in U'. Thus, since $|\mathcal{A}| \leq m-1 \leq t-1$, it follows that there is a set U of at most $t-1$ points of $\mathcal{D}$ that intersects each block in $\mathcal{A}$ and is disjoint from U'. By applying Lemma 4.11 with $S = U$ and $S' = U'$, we can find a block of $\mathcal{D}$ which intersects each block in $\mathcal{A}$ and is disjoint from each block in $\mathcal{A}'$.

Now we show that $G_{\mathcal{D}}$ is not $(m+1)$-e.c. If $m = \ell$ then let $B^*, B_1, \ldots, B_\ell$ be blocks of $\mathcal{D}$ such that $B^* \subseteq B_1 \cup B_2 \cup \cdots \cup B_\ell$. Clearly there is no block of $\mathcal{D}$ that intersects B^* and is disjoint from each block in $\{B_1, B_2, \ldots, B_\ell\}$ and hence $G_{\mathcal{D}}$ is not $(\ell+1)$-e.c. Thus we can assume that $m = t$. By Lemma 4.12 there exists a set $\{B_1, B_2, \ldots, B_t\}$ of t pairwise-disjoint blocks of $\mathcal{D}$. Let $\mathcal{C}$ be the set of all blocks of $\mathcal{D}$ that intersect each block in $\{B_1, B_2, \ldots, B_t\}$. By Lemma 4.3 there are $|B_1 \cup B_2 \cup \cdots \cup B_t| = k$ t-subsets of $B_1 \cup B_2 \cup \cdots \cup B_t$. Thus, since each block in $\mathcal{C}$ is a superset of at least one of these t-sets and since each of these t-sets is a subset of exactly λ blocks of $\mathcal{D}$, it can be seen that $|\mathcal{C}| \leq \lambda k = k$. By applying Lemma 4.11 with $S = \emptyset$ and $S' = \bigcup_{B \in \mathcal{C}} B$ (note that $|S'| \leq |\mathcal{C}|k \leq k < v$), we can find a block $B^\dagger$ of $\mathcal{D}$, that is disjoint from $\bigcup_{B \in \mathcal{C}} B$. By the definition of $\mathcal{C}$ there is no block of $\mathcal{D}$ that intersects each block in $\{B_1, B_2, \ldots, B_t, B^\dagger\}$ and hence $G_{\mathcal{D}}$ is not $(t+1)$-e.c. ■

Corollary 4.14 *Let v and k be infinite cardinals with $k < v$, let t be an integer such that $t \geq 2$, and let $\mathcal{D}$ be a t-$(v, k, 1)$ design. Then $\Xi(G_{\mathcal{D}}) = t$.*

Proof Since $\mathcal{D}$ is a t-design with $\lambda = 1$, the intersection of any two blocks of $\mathcal{D}$ has size at most $t - 1$ and hence there cannot be finitely many blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$. The result now follows from Lemma 4.13. ∎

Corollary 4.15 *Let v and k be infinite cardinals with $k < v$, let t and λ be positive integers such that $\lambda \geq 2$, and let $\mathcal{D}$ be a t-(v, k, λ) design. If $t = 1$ then $\Xi(G_{\mathcal{D}}) = 1$, and if $t \geq 2$ then $2 \leq \Xi(G_{\mathcal{D}}) \leq t$.*

Proof By the definition of an infinite design, no block of $\mathcal{D}$ is a superset of another block of $\mathcal{D}$. The result now follows by Lemma 4.13. ∎

Lemma 4.13 and Corollaries 4.14 and 4.15 establish the non-existence results of Theorem 4.2. It only remains to establish the existence results.

Lemma 4.16 *Let v and k be infinite cardinals with $k < v$, let t and λ be positive integers such that $t, \lambda \geq 2$. Then, for each $\ell \in \{2, 3, \ldots, t\}$, there exists a t-(v, k, λ) design such that ℓ is the smallest cardinal for which there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$.*

Proof We will show that for a fixed $\ell \in \{2, 3, \ldots, t\}$ there exists a t-(v, k, λ) design $\mathcal{D}$ such that ℓ is the smallest cardinal for which there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$. Let V be a point set with $|V| = v$. Throughout this proof when we refer to a block this will imply that it is a k-subset of V. By Lemma 4.3 there are v t-subsets of V and hence we can write the set of all t-subsets of V as $\{T_\alpha\}_{\alpha<v}$.

Let $\mathcal{B}_0 = \{B^*, B_1, B_2, \ldots, B_\ell\}$ be a set of blocks such that $B^* \subseteq B_1 \cup B_2 \cup \cdots \cup B_\ell$, $|B^* \cap B_i| = k$ for each $i \in \{1, 2, \ldots, \ell\}$, $|B_i \setminus B^*| = |B^* \setminus B_i| = k$ for each $i \in \{1, 2, \ldots, \ell\}$, and $B_i \cap B_j = \emptyset$ for all $i, j \in \{1, 2, \ldots, \ell\}$ with $i \neq j$. It is easy to construct such a set of blocks.

We claim that there is a transfinite sequence $\{\mathcal{B}_\alpha\}_{\alpha<v}$ of sets of blocks such that, for each ordinal $\alpha < v$, $\mathcal{B}_\alpha$ satisfies

(i) for each ordinal $\beta < \alpha$, $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$;

(ii) any two blocks in $\mathcal{B}_\alpha$ intersect in at most t points unless one is B^* and the other is in $\{B_1, B_2, \ldots, B_\ell\}$;

(iii) each t-subset of V is a subset of at most λ blocks of $\mathcal{B}_\alpha$ and, for each ordinal $\beta < \alpha$, T_β is a subset of exactly λ blocks of $\mathcal{B}_\alpha$;

(iv) $|\mathcal{B}_\alpha|$ is finite if α is finite and $|\mathcal{B}_\alpha| \leq |\alpha|$ if α is infinite.

Note that, for each ordinal $\alpha < v$, $|\alpha| < v$ since v is a cardinal and hence (iv) implies that $|\mathcal{B}_\alpha| < v$.

If such a sequence exists, then the structure $\mathcal{D}$ with point set V and block set $\bigcup_{\alpha<v} \mathcal{B}_\alpha$ is a t-(v,k,λ) design such that ℓ is the smallest cardinal for which there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$. To see this, note that (ii) together with the definition of $\mathcal{B}_0$ implies that no block of $\mathcal{D}$ is a subset of another, that for each block of $\mathcal{D}$ there are v points of V not in that block since $v-k=v$, and that (iii) implies that every t-subset of V is a subset of exactly λ blocks of $\mathcal{D}$. Thus $\mathcal{D}$ is a t-(v,k,λ) design. Furthermore, $B^* \subseteq B_1 \cup B_2 \cup \cdots \cup B_\ell$ and (ii) together with the definition of $\mathcal{B}_0$ implies that no block of $\mathcal{D}$ is a subset of the union of $\ell-1$ others (note that (ii) implies that only a block in $\mathcal{B}_0$ could possibly be a subset of $\ell-1$ others and by (ii) and the definition of $\mathcal{B}_0$ this is not the case). Thus ℓ is the smallest cardinal for which there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$. So it only remains to show that the sequence $\{\mathcal{B}_\alpha\}_{\alpha<v}$ exists. We will do so by transfinite induction.

Note that $\mathcal{B}_0$ satisfies (i)-(iv) (to see that (iii) holds recall that $\lambda \geq 2$). Now we assume that, for some ordinal γ with $1 \leq \gamma < v$, we have constructed a sequence $\{\mathcal{B}_\alpha\}_{\alpha<\gamma}$ such that $\mathcal{B}_\alpha$ satisfies (i)-(iv) for each ordinal $\alpha < \gamma$, and we will demonstrate how to construct a set of blocks $\mathcal{B}_\gamma$ which satisfies (i)-(iv) for $\alpha = \gamma$.

If γ is a limit ordinal then let $\mathcal{B}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{B}_\alpha$. Using the fact that, for each ordinal $\alpha < \gamma$, $\mathcal{B}_\alpha$ satisfies (i)-(iv), it is routine to check that $\mathcal{B}_\gamma$ satisfies (i)-(iv) for $\alpha = \gamma$

(to see that (iv) holds, note that $|\mathcal{B}_\gamma| = \sup\{|\mathcal{B}_\alpha| : \alpha < \gamma\}$).

If γ is a successor ordinal, then we construct $\mathcal{B}_\gamma$ from $\mathcal{B}_{\gamma-1}$ in the following way. Let a be the number of blocks in $\mathcal{B}_{\gamma-1}$ which are supersets of $T_{\gamma-1}$. If $a = \lambda$, then let $\mathcal{B}_\gamma = \mathcal{B}_{\gamma-1}$. If $a < \lambda$, then form $\mathcal{B}_\gamma$ by adding to $\mathcal{B}_{\gamma-1}$ the blocks $T_{\gamma-1} \cup S_1, T_{\gamma-1} \cup S_2, \ldots, T_{\gamma-1} \cup S_{\lambda-a}$ where $S_1, S_2, \ldots, S_{\lambda-a}$ are pairwise disjoint k-subsets of V such that no point of $S_1 \cup S_2 \cup \cdots \cup S_{\lambda-a}$ is in a block of $\mathcal{B}_{\gamma-1}$ (note that such sets must exist since at most $k|\mathcal{B}_{\gamma-1}|$ points of V are contained in the union of the blocks in $\mathcal{B}_{\gamma-1}$ and $v - k|\mathcal{B}_{\gamma-1}| = v$ by (iv)). Note that each of these blocks intersects each other block in $\mathcal{B}_\gamma$ in at most t points. For each t-subset T of V other than $T_{\gamma-1}$, the number of blocks in $\mathcal{B}_\gamma$ which are supersets of T either remains the same through this process or increases from zero to one. Furthermore, through this process the number of blocks in $\mathcal{B}_\gamma$ which are supersets of $T_{\gamma-1}$ either remains at λ or increases to λ. Using the fact that $\mathcal{B}_{\gamma-1}$ satisfies (i)-(iv) for $\alpha = \gamma - 1$, it is routine to check from the construction that $\mathcal{B}_\gamma$ satisfies (i)-(iv) for $\alpha = \gamma$. Thus the required sequence $\{\mathcal{B}_\alpha\}_{\alpha<v}$ does indeed exist and the proof is complete. ■

Proof of Theorem 4.2 Firstly, by Lemma 4.13, if $\mathcal{D}$ is a t-(v, k, λ) design then $\Xi(G_\mathcal{D}) = \min\{\ell, t\}$, where ℓ is the smallest cardinal for which there are ℓ blocks of $\mathcal{D}$ whose union is a superset of another block of $\mathcal{D}$. By Corollaries 4.14 and 4.15, if there exists a t-(v, k, λ) design $\mathcal{D}$ with $\Xi(G_\mathcal{D}) = n$, then $n = t$ when $\lambda = 1$ or

$t = 1$, and $2 \leq n \leq t$ when $t, \lambda \geq 2$. If $t = 1$ then it is easy to see that there exists a t-(v, k, λ) design and if $\lambda = 1$ then there exists a t-(v, k, λ) design by Proposition 7.1 of [20]. From what we have already proved, such designs necessarily have block intersection graphs with existential closure number t. If $t, \lambda \geq 2$, then, by Lemma 4.16 (and by what we have already proved), for each $n \in \{2, 3, \ldots, t\}$ there exists a t-(v, k, λ) design whose block intersection graph has existential closure number n. ■

Chapter 5

Conclusion and Future Work

In this thesis, we have studied the n-existential closure property of graphs by presenting the background in Chapter 1 and our contributions in Chapters 2, 3 and 4. We mainly have produced new families of 3-e.c. graphs using a binary graph operation and studied the existential closure property of block intersection graphs of infinite designs. In this chapter we present some open problems that we have encountered and which may provide directions for future research.

5.1 Graph Operations

In Chapter 2 we studied producing n-e.c. graphs using binary graph operations whereby we constructed new families of 3-e.c. graphs using the modular graph product. Although this graph operation was studied briefly before, the advantage of our approach is that only one of the graphs in the operation needs to be 3-e.c. Here we present some problems that need further investigations.

Problem 5.1 *Other than graph complementation, no graph operation has yet been found that preserves the n-e.c. property for $n \geq 4$. Find an n-e.c. preserving (binary) operation for $n = 4$ and then for higher values of n.*

Problem 5.2 *Produce n-e.c. graphs using graph operations such that none of the graphs in the operation needs to be n-e.c.*

5.2 Minimum Orders

As was mentioned in Chapter 1, almost all graphs are n-e.c. by Theorem 1.3. This theorem implies that there are many examples of n-e.c. graphs on large graph orders. Also, most of the n-e.c. graphs known to date are of large orders. This motivates research on finding n-e.c. graphs on small orders.

Problem 5.3 *It is known that* $24 \leq m_{ec}(3) \leq 28$. *Find* $m_{ec}(3)$.

Problem 5.4 *Improve the upper bound on* $m_{ec}(n)$ *for* $n \geq 4$. *Find* $m_{ec}(n)$ *or bounds on it for* $n = 4$ *and then for* $n \geq 5$.

5.3 Block Intersection Graphs of Designs

In Chapters 3 and 4 we studied the n-existential closure property of block intersection graphs of infinite designs. Some open research areas are as follows.

Problem 5.5 *Conduct research on infinite designs with the property that their block set can be partitioned into sets such that each set is a partition of the point set. Is "resolvability" an appropriate expression for such designs?*

Problem 5.6 *In our investigations, we noticed that there is a lack of examples of simple infinite* t-(v, k, λ) *designs with* k, λ *finite such that* $\lambda \geq 2$. *Find explicit examples of such designs.*

Problem 5.7 *Investigate the* n*-existential closure property of block intersection graphs of infinite designs with* t *or* λ *infinite.*

Problem 5.8 *Investigate other properties of the infinite graphs which are the block intersection graphs of infinite designs.*

Bibliography

[1] W. Ananchuena, On the adjacency properties of generalised Paley graphs, *Australas. J. Combin.*, **24**, 2001, 129-147.

[2] W. Ananchuena and L. Caccetta, Cubic and quadruple Paley graphs with the n-e.c. property, *Discrete Math.*, **306(22)**, 2006, 2954-2961.

[3] C. Baker, A. Bonato and J. Brown, Graphs with the 3-e.c. adjacency property constructed from affine planes, *J. Combin. Math. Combin. Comput.*, **46**, 2003, 65-83.

[4] C.A. Baker, A. Bonato, J.M.N. Brown and T. Szőnyi, Graphs with n-e.c. adjacency property constructed from affine planes, *Discrete Math.*, **208**, 2008, 901-912.

[5] C.A. Baker, A. Bonato, N.A. McKay and P. Prałat, Graphs with the n-e.c. adjacency property constructed from resolvable designs, *J. Combin. Des.*, **17**,

2009, 294-306.

[6] P. Beth, D. Jungnickel and H. Lenz, 1986, Design Theory, 2nd ed., Cambridge, 1999.

[7] A. Blass, G. Exoo and F. Harary, Paley graphs satisfy all first-order adjacency axioms, *J. Graph Theory*, **5**, 1981, 435-439.

[8] B. Bollobás, A. Thomason, Graphs which contain all small graphs, *European J. Combin.*, **2**, 1981, 13-15.

[9] A. Bonato, The search for n-e.c. graphs, *Contributions to Discrete Mathematics*, **4**, 2009, 40-53.

[10] A. Bonato and K. Cameron, On an adjacency property of almost all graphs, *Discrete Math.*, **231**, 2001, 103-119.

[11] A. Bonato and K. Cameron, On an adjacency property of almost all tournaments, *Discrete Math.*, **306**, 2006, 2327-2335.

[12] A. Bonato and A. Costea, Adjacency properties of graphs and a conjecture of Erdős, preprint.

[13] A. Bonato, P. Gordinowicz and P. Prałat, Bounds and constructions for n-e.c. tournaments, *Contributions to Discrete Mathematics*, **5**, 2010, 52-66.

[14] A. Bonato, W.H. Holzmann and H. Kharaghani, Hadamard matrices and strongly regular graphs with 3-e.c. adjacency property, *Electron. J. Combin.*, **8**, 2001, no. 1, Research Paper 1, 9pp.

[15] P.J. Cameron, Note on large sets of infinite Steiner systems, *J. Combin. Des.*, **3**, 1995, 307-311.

[16] P.J. Cameron, Permutation Groups, London Math. Soc. Student Texts **45**, Cambridge University Press, Cambridge, 1999.

[17] P.J. Cameron, The random graph, In: The Mathematics of Paul Erdős, II, *Algorithms and Combinatorics*, **14**, Springer, Berlin, 1997, 333-351.

[18] P.J. Cameron, The random graph revisited, *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, Progr. Math., **201**, Basel: Birkhäuser, 2001, 267-274.

[19] P.J. Cameron and D. Stark, A prolific construction of strongly regular graphs with the n-e.c. property, *Electron. J. Combin.*, **9**, 2002, no. 1, Research Paper 31, 12 pp.

[20] P.J. Cameron and B.S. Webb, What is an infinite design?, *J. Combin. Des.*, **10**, 2002, 79-91.

[21] C.J. Colbourn, A.D. Forbes, M.J. Grannell, T.S. Griggs, P. Kaski, P.R.J. Östergård, D.A. Pike and O. Pottonen, Properties of the Steiner triple systems of order 19, *Electron. J. Combin.*, **17(1)**, 2010, Research Paper 98, 30 pp.

[22] P. Erdős and A. Rényi, Asymmetric graphs, *Acta Math. Acad Sci. Hungar*, **14**, 1963, 295-316.

[23] T.G. Faticoni, The mathematics of infinity: a guide to great ideas, John Wiley & Sons Inc., 2006.

[24] R.A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, *Annals of Eugenics*, **10**, 1940, 52-75.

[25] D.G. Fon-Der-Flaass, New prolific constructions of strongly regular graphs, *Adv. Geom.*, **2(3)**, 2002, 301-306.

[26] A.D. Forbes, M.J. Grannell and T.S. Griggs, Steiner triple systems and existentially closed graphs, *Electron. J. Combin.*, **12**, 2005, Research Paper 42, 11 pp.

[27] P. Gordinowicz and P. Prałat, The search for the smallest 3-e.c. graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, **74**, 2010, 129-142.

[28] R.L. Graham and J.H. Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.*, **14**, 1971, 45-48.

[29] D. Horsley, D.A. Pike and A. Sanaei, Existential closure of block intersection graphs of infinite designs having infinite block size, *J. of Combin. Des.*, **19**, 2011, 317-327.

[30] T. Jech, Set Theory, Springer-Verlag, Berlin (2003).

[31] P. Kaski and P.R.J. Östergård, The Steiner triple systems of order 19, *Math. Comp.*, **73**, 2004, no. 248, 2075-2092.

[32] A. Kisielewicz and W. Peisert, Pseudo-random properties of self-complementary symmetric graphs, *J. Graph Theory*, **47**, 2004, 310-316.

[33] N.A. McKay and D.A. Pike, Existentially closed BIBD block-intersection graphs, *Electron. J. Combin.*, **14**, 2007, Research Paper 70, 10 pp.

[34] D.A. Pike and A. Sanaei, Existential closure of block intersection graphs of infinite designs having finite block size and index, *J. Combin. Des.*, **9**, 2011, 85-94.

[35] D.A. Pike and A. Sanaei, Modular product and existential closure, *Australas. J. Combin.*, in press.

[36] R. Rado, Universal graphs and universal functions, *Acta Arith.*, **9**, 1964, 331-340.

[37] J.W. Raymond, J. Gardiner and P. Willett, RASCAL: calculation of graph similarity using maximum common edge subgraphs, *The Computer Journal*, **45** (6), 2002, 631-644.

[38] L.A. Vinh, A construction of 3-e.c. graphs using quadrances, *Australas. J. Combin.*, **51**, 2011, 3-6.

[39] V.G. Vizing, Reduction of the problem of isomorphism and isomorphic entrance to the task of finding the nondensity of a graph, *Proc. 3rd All-Union Conf. Problems of Theoretical Cybernetics*, 1974, p. 124.

[40] A.A. Zykov, Fundamentals of Graph Theory, BSC Associates, Moscow, Idaho, 1990.