On the Time Consistency of Equilibria in Additively Separable Differential Games*

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Abstract

The relationship amongst state-redundancy and time consistency of differential games is investigated. A class of state-redundant games is detected, where the state dynamics and the payoff functions of all players are additively separable w.r.t. control variables. We prove that, in this class of games, open-loop Nash and degenerate feedback Stackelberg equilibria coincide, both being subgame perfect. This allows us to bypass the issue of the time inconsistency that typically affects the open-loop Stackelberg solution.

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1 Introduction

The time consistency of equilibria is a crucial and long standing issue in dynamic game theory. In general, open-loop Nash (i.e., simultaneous play) equilibria are only weakly time consistent while open-loop Stackelberg (i.e., sequential play) equilibria are time inconsistent, and there exists a relevant stream of literature investigating special classes of games where these problems do not arise. Starting from Clemhout and Wan (1974), several types of games producing strongly time consistent (or subgame perfect) Nash equilibria under open-loop information have been identified.[1] Attaining time consistency in Stackelberg game is a somewhat more challenging enterprise. After the seminal contributions of Simaan and Cruz (1973a,b), the idea that hierarchical dynamic games yield time inconsistent Stackelberg equilibria has dominated the related literature in economics and, more generally, in the social sciences as a whole. This is made even more troublesome by the fact that appropriate tools for the analytical solution of feedback Stackelberg equilibria are missing.

Here we focus our attention on dynamic games in continuous time, i.e., differential games. Our aim is to characterise a class of games which are additively separable w.r.t. control variables. More precisely, what we label as an additively separable differential game is one where both the state dynamics and the instantaneous payoff functions are additively separable with respect to players' controls. If this property holds, any player's first order condition is independent of the rivals' controls, which in turn entails that instantaneous best response functions are orthogonal to each other. On this basis, we prove two main results:

- If a differential game is additively separable w.r.t. control variables, then its feedback Nash and Stackelberg equilibria coincide.
- Additionally, if the open-loop Nash solution of the same game is subgame perfect, then the feedback Stackelberg equilibrium collapses onto the open-loop Nash one, precisely because the latter also coincides with the feedback Nash equilibrium.

[1]Classes of tractable open-loop differential games have been identified by Dockner *et al.* (1985). Several other contributions illustrate specific games whose open-loop solutions are subgame perfect. For exhaustive surveys of such games, see Mehlmann (1988, ch. 4) and Dockner *et al.* (2000, ch 7).

Note that the above points hold irrespective of whether the open-loop Stackelberg solution is time consistent or not. Accordingly, this makes unnecessary to deal explicitly with the eventual time inconsistency that might well affect the open-loop hierarchical game, as the feedback Stackelberg equilibrium, which is subgame perfect by definition, can be easily characterised by solving the open-loop Nash setup.

We complete the picture by briefly illustrating the applicability of our framework to well known economic examples where additive separability and subgame perfection may (or may not) hold jointly.

The remainder of the paper is organised as follows. The basic structure is laid out in section 2. The time (in)consistency issue is revisited and defined in section 3. Section 4 contains the definition of additive separability, and investigates this property in connection with time consistency of Nash and Stackelberg equilibria. Section 5 illustrates two additively separable differential games where, alternatively, open-loop Nash equilibria are or are not subgame perfect. Concluding remarks are in section 6.

2 The basic setup

Consider an infinite horizon differential game with the following features:

- n is the number of players;
- $x(t) = (x_1(t), \dots, x_m(t)) \in X \subset \mathbb{R}^m$, where X is a compact set, is a vector of state variables;
- $u(t) \in U := U_1 \times \dots \times U_n$, where U_i is a compact set for every $i = 1, \dots, n$, is a vector of control variables; $u_i(t)$ is the control related to the i-th player;
- the i-th player is endowed with the instantaneous payoff $\pi_i(x(t), u(t), t)$ and is supposed to maximize the discounted objective functional:

$$J_i \equiv \int_{t_0}^{\infty} e^{-\rho_i t} \pi_i(x(t), u(t), t) dt \tag{2.1}$$

subject to the kinematic equation:

$$\begin{cases} \dot{x}_s(t) = g_s(x(t), u(t), t) \\ x_s(t_0) = x_{s0} \end{cases}, \tag{2.2}$$

where $g_s(\cdot) \in C^2(X \times U \times [t_0, \infty))$, $s = 1, \ldots, m$ and ρ_i is the constant force of interest for the i-th agent.

The Hamiltonian function of each agent shows as follows:

$$H_i(\cdot) = e^{-\rho_i t} \left[\pi_i(x(t), u(t), t) + \lambda_{ii}(t) g_i(x(t), u(t), t) + \sum_{s \neq i} \lambda_{is}(t) g_s(x(t), u(t), t) \right],$$

where $\lambda_{is}(t) = e^{-\rho_i t} \mu_{is}(t)$ is the costate variable associated by player i with state variable x_s; suppose $H_i \in C^2(X \times U \times \mathbb{R}^{n \times m} \times [t_0, \infty))$.

Definition 2.1. *A decision rule $\widetilde{u}_i(\cdot) \in U_i$ for the i-th player is called:*

- *an **open-loop strategy** if it depends on time t and on the given initial condition x_0;*
- *a **closed-loop strategy** if it depends on t, x, x_0 and if it is continuous in t and uniformly Lipschitz in x for each t;*
- *a **feedback strategy** if it depends on t, x and if it is continuous in t and uniformly Lipschitz in x for each t.*

Definition 2.2. *An n-tuple of strategies $(u_1^*, \ldots, u_n^*) \in U$ such that:*

$$J_i(u_1^*, \ldots, u_n^*) \geq J_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_n^*), \quad i = 1, \ldots, n \tag{2.3}$$

is:

1. *an **open-loop Nash equilibrium** if u_i^* is an open-loop strategy for all i, and if (2.3) holds for all open-loop strategies u_i;*
2. *a **closed-loop Nash equilibrium** if u_i^* is a closed-loop strategy for all i, and if (2.3) holds for all closed-loop strategies u_i;*
3. *a **feedback Nash equilibrium** if u_i^* is a feedback strategy for all i, and if (2.3) holds for every possible initial condition (t_0, x_0) of (2.2).*

Every feasible control path $u^*(\cdot) \in U_1 \times \ldots \times U_n$ has a corresponding state trajectory $x^*(t) \in X_1 \times \ldots \times X_m$, which can be deduced from (2.2); alternatively, another approach can be followed:

Definition 2.3. *The* ***optimal value function*** *for the i-th player is:*

$$V^i(x(t),t) = \sup_{u_i \in U_i} \left\{ \int_{t_0}^{\infty} e^{-\rho_i t} \pi_i(x(t),u(t),t)dt \right\}.$$

Every $V^i(x(t),t)$ satisfies the following partial differential **Hamilton-Jacobi-Bellman equation**:

$$-\frac{\partial V^i(x,t)}{\partial t} + \rho_i V^i(x,t) = \max_{u_i \in U_i} \left\{ \pi_i(x,u,t) + \sum_{s=1}^{m} \frac{\partial V^i(x,t)}{\partial x_s} g_s(x,u,t) \right\}. \tag{2.4}$$

The feedback Nash equilibrium can be calculated by solving the n equations (2.4), whenever that is possible.

Reinganum (1982) and Fershtman (1987) identify some classes of differential games for which open-loop and feedback Nash strategies coincide, but in general that is not true. In next Sections we will introduce the main topics on time consistency and subsequently link them to the theory of equilibria.

3 The issue of time consistency

Definition 3.1. *An equilibrium trajectory $\tilde{x}(t)$ is called:*

1. ***weakly time-consistent*** *if its truncated part in the time interval $[T,\infty)$, where $T > t_0$, represents an equilibrium also for any subgame starting in $t = T$, given the vector of initial conditions $\tilde{x}(T)$.*

2. ***strongly time-consistent*** *if its truncated part in the time interval $[T,\infty)$, where $T > t_0$, represents an equilibrium also for any subgame starting in $t = T$, independently on the initial conditions $\tilde{x}(T)$.*

Strong time-consistency corresponds to *subgame perfection*, since it requires the ability on the part of each player to account for the rival's behaviour at any time instant. Weak time-consistency is a milder requirement and does not ensure, in general, that the resulting Nash equilibrium be subgame perfect.

On the other hand, the property of subgame perfection affects feedback strategies as shown by the classical following result:

Proposition 3.1. *Given an n-tuple of feedback strategies $\widehat{u}^*(\cdot) \in U$, if the induced n-tuple of feedback strategies of every subgame starting in T, independently on the initial conditions $\widetilde{x}(T)$, can be played, then $\widehat{u}^*(\cdot)$ is a subgame perfect equilibrium.*

Proof: See Mehlmann (1988, pp. 65-67). ∎

A well-known technique to detect equilibria is based on Pontryagin's maximum principle, by which the first order conditions (FOCs) on the control variables $u_i(\cdot)$ read as follows:

$$\frac{\partial H_i(x^*, u^*)}{\partial u_i} = 0, \quad i = 1, \ldots, n, \tag{3.1}$$

whereas the adjoint equations concerning the costate dynamics are:

$$-\frac{\partial H_i(x^*, u^*)}{\partial x_s} = \frac{\partial \lambda_{is}(t)}{\partial t} - \rho_i \lambda_{is}(t), \quad i = 1, \ldots, n, \quad s = 1, \ldots, m. \tag{3.2}$$

Definition 3.2. *If along the optimal trajectories of a differential n-player game with Hamiltonians H_i no costate variable depends on any state variable and the following relation holds for every $i = 1, \ldots, n$, $s = 1, \ldots, m$:*

$$\frac{\partial^2 H_i(\cdot)}{\partial u_i \partial x_s} = 0, \tag{3.3}$$

the game is called ***state-redundant****.*

State redundancy occurs when, after solving (3.2) and substituting the found costate variables in (3.1), the resulting expression depends neither on the states nor on their initial values. From Definition 3.2 we deduce that when at least one of the two conditions for state redundancy does not hold, equation (2.4) should be solved, for the open-loop Nash equilibrium is weakly, but not strongly time-consistent. Next proposition (e.g. Cellini *et al.*, 2005) connects state redundancy with time consistency:

Proposition 3.2. *If a differential game is state-redundant, then its open-loop Nash equilibrium is strongly time-consistent.*

Example 3.1. *Consider the following linear state optimal control problem:*

$$\max_{u \in U} J_i \equiv \int_{t_0}^{\infty} e^{-\rho_i t}[A_i(u(t)) + B_i(x(t))]dt \tag{3.4}$$

s.t.

$$\begin{cases} \dot{x}_s(t) = C_s(u(t)) + D_s(x(t)) \\ x_s(t_0) = x_{s0} \end{cases}, \tag{3.5}$$

where

$$B_i(x) = \sum_{s=1}^{m} \beta_{is} x_s, \quad D_s(x) = \sum_{v=1}^{m} \delta_{sv} x_v$$

are linear in all state variables, i.e. B_i *and* D_s *are respectively the* i*-th and the* s*-th rows of the matrices* $(\beta_{is}) \in M_{n,m}(\mathbb{R})$, $(\delta_{sv}) \in M_{m,m}(\mathbb{R})$ *and* $A_i(\cdot)$ *and* $C_s(\cdot)$ *are* C^2 *functions in* u.

The separability of payoffs and dynamics implies the separability of the Hamiltonians $H_i(\cdot)$, *hence* (3.1) *and* (3.2) *respectively become:*

$$\frac{\partial A_i(\cdot)}{\partial u_i} + \sum_{s=1}^{m} \lambda_{is}(t) \frac{\partial C_s(\cdot)}{\partial u_i} = 0; \tag{3.6}$$

$$\dot{\lambda}_{is}(t) - \rho_i \lambda_{is}(t) + \sum_{w=1}^{m} \delta_{ws} \lambda_{iw}(t) + \beta_{is} = 0; \tag{3.7}$$

none of them depends on any state variable, so (3.5) *and* (3.7) *are decoupled: the game is state-redundant.*

The topic of state redundancy was widely discussed by Mehlmann (1988, Ch. 4), which also pointed out subgame perfectness of open-loop equilibria in the particular case of trilinear games.

Now consider the case of a two-player Stackelberg game. Although the structure of such a game is hierarchical, all the previously provided definitions keep the same. We know from Simaan and Cruz (1973a,b) that generally open-loop Stackelberg games yield time-inconsistent equilibria. However, also in this case, by combining solutions of (3.1) and (3.2), one obtains the instantaneous value of the follower's costate variables $\lambda_{ii}^*(t)$ and $\lambda_{is}^*(t)$ associated with (2.2).

Definition 3.3. *If* $\lambda_{ii}^*(t)$ *and* $\lambda_{is}^*(t)$ *do not depend on the leader's control* $u_L(t)$, *then the game is **uncontrollable** by the leader.*

Otherwise, if these expressions contain the leader's control, open-loop Stackelberg strategies are not time-consistent, since the leader controls the follower's costates by manoeuvring the strategy $u_L(t)$ (cfr. Xie, 1997; Dockner *et al.*, 2000).

Proposition 3.3. *If a differential game is uncontrollable by all of its players, then all of its open-loop Stackelberg equilibria are time-consistent.*

Consequently, uncontrollability is a necessary condition to obtain subgame perfect open-loop Stackelberg equilibria, but it is not sufficient, since the occurrence of feedbacks may prevent such a game from generating strongly time-consistent equilibria in open-loop. It follows from Propositions 3.2 and 3.3 (see Cellini *et al.*, 2005) that:

Proposition 3.4. *If a differential game is:*

- *both uncontrollable by all of its players and state-redundant, then all of its open-loop Stackelberg equilibria are time-consistent;*
- *state-redundant but controllable by at least one of the players, then the open-loop Stackelberg equilibrium with that leading player cannot be time-consistent.*

If the game is controllable by the leader, one should calculate the Stackelberg feedback equilibrium, but the necessary instruments are missing. Dockner *et al.* (2000, chapter 5) propose a possible approach involving a state-contingent stationary solution in the unique state variable case, i.e. a linear control of the form $u_L(x) = a + bx$ for the leader and the related maximization with respect to a and b. Consequently the follower's reaction strategy $u_F(\cdot)$ has to depend on the real parameters a and b as well. The resulting Stackelberg equilibrium time consistency is a consequence of the leader's choice of the two numbers. We revisit two oligopoly models (see Cellini and Lambertini, 2002, for the former and Dockner *et al.*, 2000, for the latter) from the viewpoint of state redundancy.

Example 3.2. *Consider a model where n oligopolists produce differentiated commodities and externalities operate through demand functions. The setup of this game involves each firm's maximization of the discounted value of its profit flow:*

$$\textit{max}\ \Pi_i \equiv \int_0^\infty e^{-\rho_i t}[\pi_i(D(t), q_1(t), \ldots, q_n(t), k_1(t), \ldots, k_n(t))]dt \tag{3.8}$$

s.t.

$$\begin{cases} \dot{D}(t) = -\dfrac{K(t)D(t)}{1+K(t)} \\ D(0) = B \end{cases}, \tag{3.9}$$

where the state variable $D(t) \in [0, B]$ *represents the symmetric degree of substitutability between any pair of products or the extent of product differentiation, the control variables are* $q_1(t), \ldots, q_n(t), k_1(t), \ldots, k_n(t)$*, and* $K(t) = \sum_{j=1}^{n} k_j(t)$ *is the industry's overall R&D expenditure. Instantaneous profits are given by*

$$\pi_i(t) = \left[A - Bq_i(t) - D(t) \sum_{j \neq i} q_j(t) - c \right] q_i(t) - k_i(t),$$

where $c \in (0, A)$*, and the output level* $q_i(t)$ *is produced at constant returns to scale. The uniqueness of the state variable remarkably simplifies the expression of the Hamiltonian function and when symmetry among players is assumed, FOCs of the problem yield the following formula for each costate variable* $\lambda_i(t)$*:*

$$\lambda_i(t) = -\frac{[1 + nk(t)]^2}{D(t)}. \tag{3.10}$$

In (3.10) $k(t)$ *is the investment in product differentiation by one firm at a symmetric equilibrium. Since* $\lambda_i(t)$ *depends on* $D(t)$*, this game is not state-redundant. Moreover, it is easy to see that*

$$\frac{\partial^2 \pi_i(t)}{\partial q_i \partial q_j} = -D(t) \neq 0, \forall i, j = 1, \ldots, n; \; i \neq j, \tag{3.11}$$

which implies that the game is not even additively separable, according to the definition that will be provided in Section 4. The ultimate consequences of these observations are two. First, the open-loop Nash equilibrium does not coincide with the closed-loop one, so a subgame perfect equilibrium has to be sought via the Hamilton-Jacobi-Bellman equation approach. Second, if such an approach yielded an explicit solution for the simultaneous-move game, the properties of that solution would not extend to the sequential-move game.

Example 3.3. *Dynamic models of resource extraction (for an exhaustive overview including a rich bibliography*[2]*, see Dockner* et al.*, ch. 12) usually involve an equation of motion with linear-state dynamics and an objective functional not depending on the state variables, so that the game may turn*

[2] In particular, for fishery games that may share the same basic structure with the present example, see Chiarella *et al.* (1984), Dockner and Sorger (1996) and Sorger (1998).

out to be state-redundant. Consider the following problem:

$$max \int_0^\infty e^{-\rho_i t}[\pi_i(x(t), q_1(t), \ldots, q_n(t), t)]dt \tag{3.12}$$

s.t.

$$\begin{cases} \dot{x}(t) = \delta x(t) - \sum_{j=1}^n q_j(t) \\ x(0) = x_0 \end{cases} , \tag{3.13}$$

where $q_i(t) \geq 0$ *is the i-th firm's strategy, whereas* $x(t)$ *represents the resource stock's dynamics and* $\delta > 0$ *is the regeneration rate. If the utility function is* $\pi_i(\cdot) = P(q_1 + q_2 + \ldots + q_n)q_i$*, where* $P(\cdot)$ *is the inverse demand function, for instance* $P(Q) = Q^{-1/\xi}$*, where* $\xi > 0$ *is the constant elasticity of demand, the i-th player's Hamiltonian reads as follows:*

$$H_i(\cdot) = e^{-\rho_i t} \left[\frac{q_i(t)}{(\sum_{k=1}^n q_k(t))^{1/\xi}} + \lambda_i(t) \left(\delta x(t) - \sum_{j=1}^n q_j(t) \right) \right].$$

The adjoint equations involving multipliers are:

$$\dot{\lambda}_i(t) - (\rho_i - \delta)\lambda_i(t) = 0, \tag{3.14}$$

so no dependence occurs between optimal state and costate variables; (3.3) *holds too, therefore the game is state-redundant.*

Suppose that $n = 2$ *and that this game is played sequentially: if we call* $\widetilde{q}_j(t)$ *the leader's open-loop strategy, the follower's response* $\widetilde{q}_{-j}(\cdot)$ *necessarily depends on* $\widetilde{q}_j(t)$*, because of the linearity in the dynamics and the presence of both controls in both FOCs. Since the leader controls the game, time consistency cannot hold for the open-loop Stackelberg equilibrium.*

The previous two examples highlight the fact that some specific hypotheses on the form of the differential game under examination is necessary to entail time consistency of equilibria. In the remainder, we intend to determine a suitable hierarchical game structure whose properties be such that open-loop Nash and Stackelberg equilibria can turn out to be time-consistent.

4 Additive separability of games and time consistency

In this Section, we aim at exploiting the properties connected to the additive separability of the Hamiltonians to further characterise the issue of time consistency of equilibria for a specific class of differential games.

Consider a Stackelberg game where a population of n players is divided in two groups, respectively formed by l and $n-l$ agents. Call:

- $u_L = (u_{1L}, \ldots, u_{lL})$ the control variable vector of the first group of players (the leaders);
- $u_F = (u_{1F}, \ldots, u_{(n-l)F})$ the control variable vector of the second group of players (the followers);
- $x(t) = (x_1(t), \ldots, x_m(t))$ the usual state variable vector,

all of them belonging to suitable compact control and state sets. The capital letters L and F have been chosen for a descriptive reason: if the game is played hierarchically, the first l agents can be thought of as the leaders' group, whereas the remaining $n-l$ players represent the followers' group.

The timing of moves we are envisaging is the following: the leaders incorporate the followers' FOCs into their optimum problems before picking their own optimal controls; inside each of the two groups moves are simultaneous.

Definition 4.1. *We call an* ***additively separable game*** *a differential game such that:*

1. *the i-th player is endowed with a payoff $\pi_i(\cdot)$ which is additively separable in controls, i.e.:*

$$\pi_i(x(t), u_L(t), u_F(t), t) = \alpha_i(x(t), u_L(t), t) + \beta_i(x(t), u_F(t), t),$$

for all $i = 1, \ldots, n$, where $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are C^2 functions with respect to all variables;

2. *the i-th player is supposed to maximize the objective function:*

$$J_i \equiv \int_{t_0}^{\infty} e^{-\rho_i t} \pi_i(x(t), u_L(t), u_F(t), t) dt \tag{4.1}$$

s.t.

$$\begin{cases} \dot{x}_s(t) = L_s(x(t), u_L(t), t) + F_s(x(t), u_F(t), t) \\ x_s(t_0) = x_{s0} \end{cases}, \tag{4.2}$$

$s = 1, \ldots, m$, where $L_s(\cdot)$ and $F_s(\cdot)$ are C^2 functions with respect to all variables.

In an additively separable differential game both dynamics and payoffs are additively separable with respect to both group's controls,[3] so that the i-th player's Hamiltonian can be expressed as follows:

$$H_i(\cdot) = e^{-\rho_i t}\left[\alpha_i(x,u_L,t) + \beta_i(x,u_F,t) + \lambda_{ii}(t)(L_i(x,u_L,t) + F_i(x,u_F,t)) + \right.$$
$$\left. + \sum_{s\neq i} \lambda_{is}(t)(L_s(x,u_L,t) + F_s(x,u_F,t))\right].$$

Proposition 4.1. *Along every optimal trajectory of an additively separable game with Hamiltonians $H_i(\cdot)$, we have:*

$$\frac{\partial^2 H_i(\cdot)}{\partial u_{jL}\partial u_{kF}} = 0, \; i = 1,\dots,n, \quad j = 1,\dots,l, \quad k = 1,\dots,n-l.$$

Proof: It follows trivially from Definition 4.1:

$$\frac{\partial^2 \alpha_i(\cdot)}{\partial u_{jL}\partial u_{kF}} = 0, \quad \frac{\partial^2 \beta_i(\cdot)}{\partial u_{jL}\partial u_{kF}} = 0, \quad \frac{\partial^2 L_s(\cdot)}{\partial u_{jL}\partial u_{kF}} = 0, \quad \frac{\partial^2 (F_s(\cdot))}{\partial u_{jL}\partial u_{kF}} = 0,$$

for all $i = 1,\dots,n, \quad, s = 1,\dots,m, \quad j = 1,\dots,l, \quad k = 1,\dots,n-l$, so the mixed second partial derivatives of all Hamiltonians vanish when they are calculated with respect to any two control variables belonging to different groups. ∎

Now we proceed to compare both open-loop and feedback Nash as well as Stackelberg equilibria of such a game.

The open-loop Stackelberg equilibrium can be found by considering the FOCs for the leaders and separately the ones for the followers of such a game:

$$\frac{\partial H_j(\cdot)}{\partial u_{jL}} = 0 \iff \frac{\partial \alpha_j(\cdot)}{\partial u_{jL}} + \sum_{s=1}^{m} \lambda_{js}(t)\frac{\partial L_j(\cdot)}{\partial u_{sL}} = 0, \;\; j = 1,\dots,l. \tag{4.3}$$

$$\frac{\partial H_k(\cdot)}{\partial u_{kF}} = 0 \iff \frac{\partial \beta_k(\cdot)}{\partial u_{kF}} + \sum_{s=1}^{m} \lambda_{ks}(t)\frac{\partial F_k(\cdot)}{\partial u_{sF}} = 0, \;\; k = 1,\dots,n-l. \tag{4.4}$$

[3]Dockner *et al.* (1985, p. 188) define a class of differential games which are state-control separated w.r.t. state dynamics and objectives, so that $\partial^2 H_i(\cdot)/\partial u_i \partial x_k = 0$ for all i,k. This clearly differs from our separability requirement w.r.t. controls only.

On the other hand, the feedback Stackelberg equilibrium should be determined by solving the Hamilton-Jacobi-Bellman equation for every follower:

$$-\frac{\partial V^k(x,t)}{\partial t} + \rho_k V^k(x,t) = \max_{u_F} \{\alpha_k(x(t), u_L(t), t) + \beta_k(x(t), u_F(t), t) +$$

$$+\sum_{s=1}^{m} \frac{\partial V^k(x,t)}{\partial x_s}[L_s(x(t), u_L(t), t) + F_s(x(t), u_F(t), t)]\Bigg\}, \ k = 1, \ldots, n-l. \tag{4.5}$$

and subsequently by substituting the optimal control thus found in the same expression for the leader. The property of additive separability allows us to formulate what follows:

Proposition 4.2. *If $u^* = (u_L^*, u_F^*)$ is an optimal control vector for the Hamilton-Jacobi-Bellman equations of an additively separable game played hierarchically, then it is an optimal control vector for the same equations of the game played simultaneously as well.*

Proof: Suppose that $u_F^*(\cdot)$ is an optimal control vector for (4.5). The necessary conditions for maximization are given by:

$$\frac{\partial \left(\beta_k(x(t), u_F(t), t) + \sum_{s=1}^{m} \frac{\partial V^k(x,t)}{\partial x_s}[F_s(x(t), u_F(t), t)]\right)}{\partial u_{kF}} = 0, \tag{4.6}$$

Since

$$\frac{\partial u_{kF}^*}{\partial u_{jL}} = 0, \quad j = 1, \ldots, l, \quad k = 1, \ldots, n-l, \tag{4.7}$$

the substitution of $u_F^*(\cdot)$ in the Hamilton-Jacobi-Bellman equations for the leader does not affect the conditions for maximization, which become:

$$\frac{\partial \left(\alpha_j(x(t), u_L(t), t) + \sum_{s=1}^{m} \frac{\partial V^j(x,t)}{\partial x_s}[L_s(x(t), u_L(t), t)]\right)}{\partial u_{jL}} = 0, \tag{4.8}$$

and which are exactly the same appearing in the feedback Nash case. Therefore, no change occurs either in (4.6) or in (4.8) when the game is played simultaneously, so the resulting optimal controls remain the same. ■

Note that the assumption of additive separability of instantaneous payoff functions and state equations w.r.t. controls entails that

$$\frac{\partial u_z^*}{\partial u_w} = \frac{\partial u_w^*}{\partial u_z} = 0$$

irrespective of whether firm z and firm w act as a leader or a follower or play simultaneously. This means that any firm's instantaneous best reply function is flat w.r.t. the rivals' controls at any time t. Put it in other terms, $u_z^*(u_w)$ is a constant w.r.t. u_w, i.e., u_z^* and u_w^* are orthogonal. Hence, the optimal control of any firm at each generic point in time during this game is dictated by a dominant strategy, on which basis we can state:

Remark 4.1. *Additive separability w.r.t. controls yields a feedback solution in dominant strategies which is unaffected by the order of moves.*

The implication is that Nash and Stackelberg feedback solutions are observationally equivalent and therefore, *ex post*, one could not tell whether the vector of optimal controls is the outcome of sequential rather than simultaneous play.

Basically, in an additively separable game, optimal controls u_L come from (4.3), whereas (4.4) yield u_F, and no dependence occurs amongst the two groups of controls, i.e.:

$$\frac{\partial u_{jL}^*}{\partial u_{kF}} = 0, \qquad \frac{\partial u_{kF}^*}{\partial u_{jL}} = 0, \quad j = 1, \ldots, l, \quad k = 1, \ldots, n-l.$$

Moreover, when such a game is a Stackelberg one, then it turns out to be uncontrollable by all players. We can link these considerations to the issue of time consistency by proving:

Proposition 4.3. *If an additively separable game is state-redundant, then its open-loop Nash and feedback Stackelberg equilibria coincide and they are all subgame perfect.*

Proof: Proposition 4.2 implies that feedback Nash and feedback Stackelberg equilibria coincide. Proposition 3.3 ensures time-consistency for an uncontrollable hierarchical game. Finally, Proposition 3.2 ensures subgame perfection for open-loop Nash equilibria. ∎

In the next examples, we shall briefly discuss the state-redundancy and additive separability properties of two models of differential oligopolistic competition.

5 Comparison among equilibria in two additively separable games

Here we provide two examples of additively separable differential games based upon two well known oligopoly models. The first example refers to a dynamic Cournot game with sticky prices (Simaan and Takayama, 1978; Fershtman and Kamien, 1987), which is additively separable but not state redundant,[4] while the second example refers to an R&D race (Reinganum, 1982) which is both additively separable and state redundant.

Example 5.1. *Fershtman and Kamien (1987) analyze a model of simultaneous duopolistic competition with an homogeneous good and price stickiness, i.e., a property for which the price does not adjust instantaneously to the level given by the demand function for a given output level.*

Call $u_1(t)$ and $u_2(t)$ the output levels of the two firms and $p(t)$ the price, whose evolution is subject to the Cauchy problem:

$$\begin{cases} \dot{p}(t) = s[a - (u_1(t) + u_2(t)) - p(t)] \\ p(0) = p_0 \end{cases} , \tag{5.1}$$

where s is the speed of convergence to its level on the demand function and a is a positive constant. Both firms aim to maximize:

$$J_i^{FK} \equiv \int_0^\infty e^{-\rho t}\left[p(t)u_i(t) - cu_i(t) - \frac{1}{2}u_i^2(t)\right] dt, \tag{5.2}$$

where $c > 0$ is the marginal cost and ρ is the interest rate, common to both firms.

They show that, as the speed of price adjustment increases, the price at the open-loop Nash equilibrium approaches the static Cournot price, whereas the one at the closed-loop equilibrium approaches a price lower than that. This game is additively separable, since the Hamiltonians are both separable in the players' controls, but it is not state-redundant, as we can deduce from the FOCs and the adjoint equations:

$$p(t) - u_i(t) - c - \lambda_i(t)s = 0, \tag{5.3}$$

[4] For the comparative analysis of open-loop, closed-loop memoryless and feedback solutions of the Cournot oligopoly with sticky prices and n firms, see Cellini and Lambertini (2004).

$$\dot{\lambda}_i(t) - (\rho + s)\lambda_i(t) + u_i(t) = 0, \tag{5.4}$$

together with the transversality conditions:

$$\lim_{t \to +\infty} e^{-\rho t} \lambda_i(t) = 0, \quad i = 1, 2. \tag{5.5}$$

Solving (5.3) *and* (5.4) *yields the optimal value of the costate variables:*

$$\lambda_i(t) = \int_t^{\infty} e^{-(2s+\rho)(\tau - t)}[p(\tau) - c]d\tau \tag{5.6}$$

Hence, it is clear from (5.3) *and* (5.6) *that the optimal quantity of firm i and its shadow price depend on the state* $p(t)$ *and consequently state redundancy does not hold. In the game analyzed by Fershtman and Kamien (1987) the open-loop Nash equilibrium and the closed-loop one do not coincide, and hence the former is not strongly time consistent. Fershtman and Kamien proved that the open-loop Nash equilibrium does not coincide with the closed-loop equilibrium, which is subgame perfect, differently from the former one. The open-loop equilibrium cannot be strongly time-consistent, but for Proposition 4.2 simultaneous-move feedback equilibrium strategies coincide with the sequential-move feedback equilibrium strategies.*

Example 5.2. *Reinganum (1982) investigates the subgame perfect Nash equilibrium of an exponential game of R&D with n firms describing the race for an innovation consisting either in a new product or a new technology. Such a model is presented and widely discussed by Mehlmann (1988, ch. 5) and Dockner* et al. *(2000, p. 277). We restrict the analysis to the case* $n = 2$.

This game is stochastic since the flow of innovation investments by the two firms ends up when one of them reaches complete innovation at a random date $\tau_i \in [0, T]$. *The probability distribution of the random variable* τ_i *is given by:*

$$F_i(t) = Pr\{\tau_i \leq t\},$$

which is related to the R&D investment of the i-th firm by the dynamics:

$$\begin{cases} \dot{F}_i(t) = \nu u_i(t)[1 - F_i(t)] \\ F_i(0) = 0 \end{cases}, \tag{5.7}$$

where $\nu > 0$ *and* $u_i(t) \geq 0$ *is the control variable representing R&D effort. If we call* V_W *the prize awarded to the winning player and* V_L *the one awarded* *to the loser, u_i and F_i the control and state for the i-th firm and u_{-i} and F_{-i} for the other one, the i-th expected profit flow to be maximized is:*

$$\int_0^T \left\{ V_W \dot{F}_i(t)[1 - F_{-i}(t)] + V_L \dot{F}_i(t)[1 - F_i(t)] - \frac{\Pi_{k=1}^2 (1 - F_k(t))}{2e^{\rho t}} u_i^2(t) \right\} dt =$$

$$= \int_0^T [1 - F_1(t)][1 - F_2(t)] \left[\nu V_W u_i(t) + \nu V_L u_{-i}(t) - \frac{u_i^2(t)}{2e^{\rho t}} \right] dt.$$

By posing $y(t) \equiv [1-F_1(t)][1-F_2(t)]$, we can obtain a single state kinematics which is common to both firms:

$$\dot{y}(t) = -\nu y(t)(u_1(t) + u_2(t)), \tag{5.8}$$

yielding the i-th firm's Hamiltonian as follows:

$$H_i(y(t), u_1(t), u_2(t) = y(t) \left[\nu V_W u_i(t) + \nu V_L u_{-i}(t) - \frac{u_i^2(t)}{2e^{\rho t}} - \nu \lambda_i(t) \sum_{k=1}^2 u_k(t) \right], \tag{5.9}$$

where $y(t)$ represents the aggregate stock of knowledge in the industry. It can be immediately checked that the game is additively separable. It is state redundant too, as we can deduce from the form of its adjoint equations:

$$\dot{\lambda}_i(t) - \rho \lambda_i(t) = - \left[\nu V_W u_i(t) + \nu V_L u_{-i}(t) - \frac{u_i^2(t)}{2e^{\rho t}} - \nu \lambda_i(t) \sum_{k=1}^2 u_k(t) \right] \tag{5.10}$$

and from the fact that after solving (5.10) *and plugging $\lambda_i(t)$ into the FOC associated to the i-th firm:*

$$\frac{\partial H_i(\cdot)}{\partial u_i} = y(t) \left[\nu V_W - \frac{u_i(t)}{e^{\rho t}} - \nu \lambda_i(t) \right] = 0, \tag{5.11}$$

we obtain an expression not depending on the state variable $y(t)$, which cannot identically vanish since the initial condition of (5.8) *is $y(0) = 1$ because of the performed substitution.*

Proposition 4.3 can therefore be applied to this model, consequently its open-loop Nash equilibrium collapse on its feedback Stackelberg equilibrium and they are both subgame perfect.

6 Concluding remarks

We have revisited the issue of time (in)consistency of differential games, showing that additive separability w.r.t. controls, in combination with state-redundancy, implies that the feedback Stackelberg equilibrium and the open-loop Nash equilibrium coincide. This is due to the fact that additive separability entails the coincidence between feedback Stackelberg and Nash equilibria, the latter collapsing onto the open-loop Nash solution due to state redundancy. This offers, at least in the class of games we have identified, a way out of two well known problems: (i) the time inconsistency issue that usually obtains in open-loop Stackelberg games; and (ii) the lack of mathematical tools for the analytical solution of feedback Stackelberg games.

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