On the non existence of cyclical food-consumption patterns in a model of non-addictive eating.

Davide Dragone*

January 22, 2009

Abstract

In a paper previously published in this journal, Levy (2002) [Levy A., 2002, Rational eating: can it lead to overweightness or underweightness? Journal of Health Economics 21, 887–899] presents a model of rational non-addictive eating that is claimed to explain cyclical food-consumption patterns where binges and strict diets alternate. I show that the model admits no oscillation at all, as the unique internal steady state has saddle point stability.

JEL classification: I12

Keywords: Weight, Food consumption

1 Introduction

Levy (2002) analyses rational food-consumption behavior in a model of non-addictive eating by modeling the trade-off between the satisfaction from eating and the increasing probability of dying as weight deviates from a physiologically optimal level. When individuals are able to recognise this trade-off, they choose their path of intertemporal consumption so as to maximise their expected lifetime-utility. Levy finds that the internal steady state corresponds to a condition of overweightness. The steady state is claimed to be an unstable focus, with the consequence that small deviations from it are followed by explosive oscillations, interpreted as binges and strict diets, and possibly leading to a condition of chronic underweightness in a late stage of life. I show that

*Dipartimento di Scienze Economiche, Università di Bologna, Strada Maggiore 45, 40125, Bologna, Italia; tel. +39.051.209.2664; e-mail: davide.dragone@unibo.it. I thank Luca Lambertini and Arsen Palestini for useful suggestions. The usual disclaimer applies.

this conclusion is invalid because the internal steady state has saddle point stability. As a consequence, the model does not admit oscillations over the lifetime but, at most, a pattern where either weight or food-consumption increases in the first stage of life and decreases in the last one.

2 A rational eating model

In this section the model outlined in Levy (2002) is presented. Given an upper bound $T < \infty$ on life expectancy and a physiologically optimal weight $W^* > 0$, the intertemporal expected lifetime-utility is given by the following expression:

$$J = \int_0^T e^{-\rho t} U(c(t)) \Phi((W(t) - W^*)^2) dt, \tag{1}$$

where $\rho > 0$ is an exogenously given intertemporal discount rate, $c(t) \geq 0$ is food-consumption at time t and $U(c(t))$ is an instantaneous utility function such that $U_c > 0$ and $U_{cc} < 0$. The term $\Phi((W(t) - W^*)^2) > 0$ represents the probability of living beyond t and is assumed to be decreasing and concave in the deviation from the optimal weight $W(t) - W^*$.

The equation of motion of weight is the following:

$$\dot{W}(t) = c(t) - \delta W(t) \tag{2}$$

with $\delta > 0$.

Given the initial weight $W(0) = W_0$, the current-value Hamiltonian $\hat{H}$ corresponding to this optimal-control problem is:

$$\hat{H}(t) = e^{-\rho t} U(c(t)) \Phi((W(t) - W^*)^2) + \lambda(t)[c(t) - \delta W(t)] \tag{3}$$

where $\lambda(t)$ is the associated costate variable. The set of necessary conditions for determining the optimal path of consumption and weight over time is (omitting the arguments when no confusion arises):

$$\frac{\partial \hat{H}}{\partial c} = e^{-\rho t} U_c \Phi + \lambda = 0 \tag{4}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial W} = -e^{-\rho t} U \Phi_W + \lambda \delta \tag{5}$$

$$\dot{W} = c - \delta W \tag{6}$$

$$\lambda(T) W(T) = 0. \tag{7}$$

where $\Phi_W \equiv \frac{\partial \Phi(\cdot)}{\partial W}$ and (7) is the transversality condition considered in Levy (2002).

Differentiating (4) w.r.t. time and substituting (5) and the value of λ obtained from condition (4) yields the following:

$$\dot{c} = \frac{1}{\Phi U_{cc}} \left\{ \Phi(\delta + \rho) u_c + \Phi_W \left[U - U_c \dot{W} \right] \right\} \tag{8}$$

or, equivalently (see Levy 2002, eq. 12, p. 891):

$$\frac{U_{cc}}{U_c}\dot{c} + \frac{\Phi_W}{\Phi}\dot{W} - \frac{\Phi_W}{\Phi}\frac{U}{U_c} = \delta + \rho. \tag{9}$$

Substituting (6) in the above expression, the following optimal trajectory of food-consumption obtains:

$$\dot{c} = \frac{1}{\Phi U_{cc}} \left\{ \Phi(\delta + \rho) U_c + \Phi_W \left[U - U_c \left(c - \delta W \right) \right] \right\} \tag{10}$$

that, together with (6), the initial condition W_0 and the transversality condition (7) completely describes the dynamic system of food-consumption and weight.

A steady state (W^{ss}, c^{ss}) must satisfy (6) and (10) with equality. From (6) one obtains that there exists a linear relation between steady state food-consumption and weight:

$$c^{ss} = \delta W^{ss}. \tag{11}$$

Substituting in (10) and equating to zero implies that in steady state the following relation must hold (Levy, 2002, eq. 14, p. 891):

$$\Phi_W(W^{ss})U(c^{ss}) + (\delta + \rho)\Phi(W^{ss})U_c(c^{ss}) = 0 \tag{12}$$

As both $U(c^{ss})$ and $\Phi(W^{ss})$ are strictly positive, the above condition holds if Φ_W is negative, i.e. if the steady state weight corresponds to a condition of overweightness, $W^{ss} > W^*$.

Levy (2002) then proceeds considering specific functional forms for the utility and the survival function and concludes that the optimal control problem admits a steady state that is an unstable focus (i.e. the eigenvalues of the Jacobian matrix computed in the steady states are complex with positive real parts). This implies that deviations from the internal steady state determine explosive oscillations of weight and food-consumption. However this conclusion is invalid because the eigenvalues are real and with opposite signs. As I show in next Section, this is not a consequence of the specific functions that were chosen in the original paper, but a general result that applies whenever the set of necessary conditions (4)-(6) is sufficient to determine the optimal path of food-consumption and weight. In the Appendix I show that this holds *a fortiori* with the specification used in Levy (2002).

3 Saddle point stability of the steady state

In a finite-time horizon and with a linear transition equation, the set of necessary conditions (4)-(6) is sufficient for the maximisation of J if the utility function is differentiable and concave in food-consumption and weight (Mangasarian, 1966). Considering the Hessian matrix associated with the utility function

$$H(U(c,W)) = \begin{bmatrix} U_{cc}\Phi & U_c\Phi_W \\ U_c\Phi_W & U\Phi_{WW} \end{bmatrix}$$

the concavity assumption requires $U_{cc} < 0$ and $\Phi \cdot U \cdot U_{cc}\Phi_{WW} - (U_c\Phi_W)^2 > 0$, which is therefore satisfied if $\Phi_{WW} < 0$, i.e. the survival function is concave. In the following I assume that this is indeed the case at all points in time.

To assess the stability of the steady state implicitly defined by (11) and (12), consider the Jacobian matrix associated to the dynamic system given by (6) and (10), $J = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, whose elements are defined as follows:

$$\begin{aligned}
a_{11} &= \frac{\partial \dot{W}}{\partial W} = -\delta \\
a_{12} &= \frac{\partial \dot{W}}{\partial W} = 1 \\
a_{21} &= \frac{\partial \dot{c}}{\partial W} = \frac{1}{\Phi^2 U_{cc}} \left\{ \left[(c - \delta W)\Phi_W + \delta\Phi \right] U_c - \Phi_W U \right\} \\
&\quad + \frac{\Phi_{WW}}{\Phi \cdot U_{cc}} \left[U - (c - \delta W) U_c \right] \\
a_{22} &= \frac{\partial \dot{W}}{\partial W} = \frac{1}{\Phi \cdot U_{cc}^2} \left\{ (\delta + \rho) \Phi U_{cc}^2 - \left[\Phi_W U + (\delta + \rho)\Phi U_c \right] \right\}
\end{aligned}$$

Substituting the steady states conditions (11)-(12) allows to simplify a_{21} and a_{22} as follows:

$$\begin{aligned}
a_{21} &= \frac{1}{\Phi^2 U_{cc}} (\delta\Phi U_c - \Phi_W U) + \frac{\Phi_{WW}}{\Phi \cdot U_{cc}} U \\
&= \frac{U}{(\delta + \rho)\Phi^2 U_{cc}} \left[(\delta + \rho)\Phi_{WW}\Phi - (2\delta + \rho)\Phi_W^2 \right] > 0 \\
a_{22} &= \delta + \rho
\end{aligned}$$

The eigenvalues of the Jacobian are

$$e_{1,2} = \frac{1}{2}\left[\rho \pm \sqrt{\rho^2 + 4\delta(\delta+\rho) + 4a_{21}}\right].$$

Given that the discriminant is strictly positive, the two eigenvalues are real and with opposite sign, which implies that the steady state is a saddle point. This is a general property of the model that does not depend on the specification of the utility and survival function.

To visualize the optimal trajectories, it is useful to adopt the functional specifications for the utility and the survival function considered in Levy (2002):

$$U(c) = c^{\beta} \tag{13}$$

$$\Phi(W) = \Phi_0 e^{-\mu(W-W^*)^2} \tag{14}$$

with $\beta, \Phi_0 \in (0,1)$ and $\mu > 0$. Substituting (13)-(14) in (6)-(10) one obtains the following dynamic system

$$\frac{\dot{c}}{c} = \frac{2\mu(W-W^*)\left[(1-\beta)c + \beta\delta W\right] - \beta(\rho+\delta)}{\beta(1-\beta)} \tag{15}$$

$$\dot{W} = c - \delta W \tag{16}$$

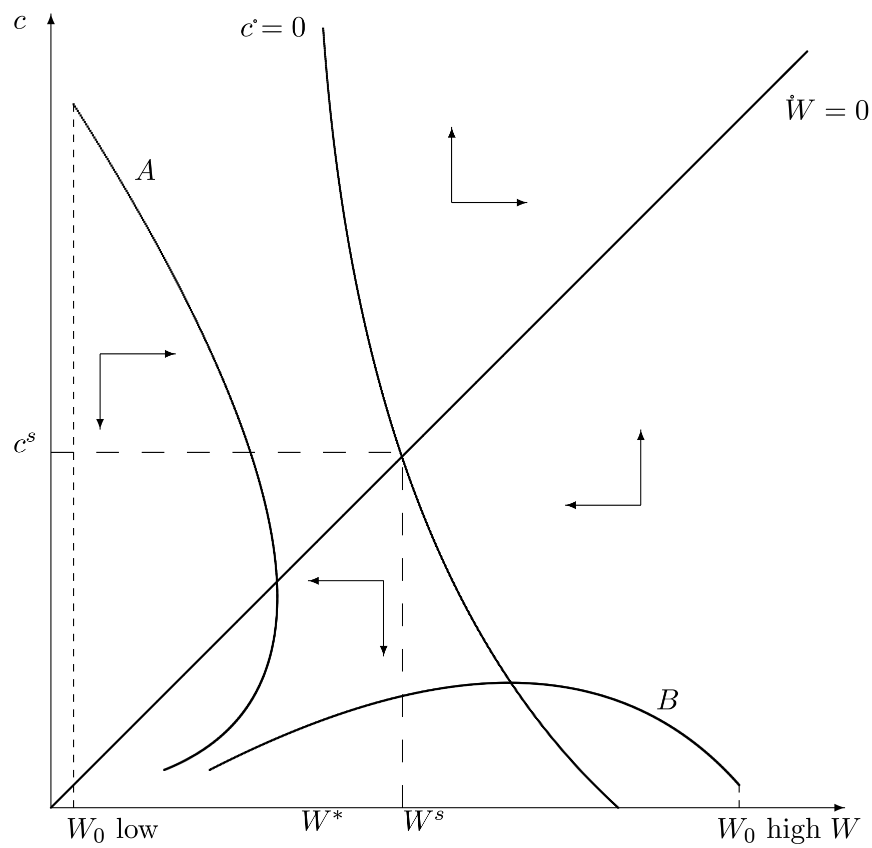
whose associated internal steady state is (Levy, 2002)

$$c^{ss} = \frac{\delta}{2}\left(W^* + \sqrt{W^{*2} + \frac{2\beta\delta(\delta+\rho)}{\delta\mu}}\right) \tag{17}$$

$$W^{ss} = \frac{1}{2}\left(W^* + \sqrt{W^{*2} + \frac{2\beta(\delta+\rho)}{\delta\mu}}\right) > W^*. \tag{18}$$

The transversality condition (7) is satisfied if, at time $T < \infty$, either the costate variable or the optimal weight is zero. According to condition (4), the former case would imply $\Phi((W(T) - W^*)^2) = 0$, i.e. the probability of living beyond T should be nil (see Levy, 2002, eq. 5b, p. 890). As Levy (2002) considers an exponential survival function, $\Phi(W) = \Phi_0 e^{-\mu(W-W^*)^2} > 0$, this is unfeasible and the transversality condition is satisfied only when $W(T) = 0$ (which corresponds to no food-consumption at time T). The trajectory leading to the $(0,0)$ corner depends on the initial condition, as shown in Fig. 1. For example, if W_0 is large enough the optimal path of food-consumption is non monotonic during the lifetime and strictly decreasing in weight, as it is optimal to begin with a low food-consumption level, to increase it for some time and finally decrease it in the late stage of life (path B). If, instead, the initial weight is low, $W_0 < W^{ss}$, then consumption must steadily decrease over time, while weight initially increases and finally decreases when time approaches T (path A).

Figure 1: Phase diagram in the (W, c) space. Path A begins with a low initial weight, path B with a high initial weight



As a final remark, note that the model implies that, if $T \to \infty$ and $W_0 > W^*$, the individual should steadily increase consumption until the stationary weight is reached. This occurs because the transition equation implies that the rate of weight-decrease depends on the level of weight, so that, for the same level of food-consumption, an overweight person loses more weight than an underweight person just for the fact of being overweight. On the contrary, if $W_0 < W^*$, an underweight person must consume more food than the steady state level if the stationary weight is to be reached.

References

[1] Levy, A. (2002), "Rational eating: can it lead to overweightness or underweightness?", *Journal of Health Economics*, **21**, 887–899.

[2] Mangasarian, O.L. (1966). "Sufficiency conditions for the optimal control of non-linear systems", *SIAM Journal on Control*, **4**, 139-152

4 Appendix

Given the functional forms (13)-(14) proposed in Levy (2002), observe that the survival function Φ is concave for $W \in \left(W^* - \sqrt{\frac{1}{2\mu}}, W^* + \sqrt{\frac{1}{2\mu}}\right)$. W^{ss} stays in this interval if either (i) $\delta > \frac{\beta}{1-\beta}\rho$, or (ii) $\delta < \frac{\beta}{1-\beta}\rho$ and $\mu > \frac{1}{2}\left[\frac{\beta\rho-(1-\beta)\delta}{\delta W^*}\right]^2$. Assuming that concavity holds, then the set of necessary conditions (4)-(7) is sufficient for identifying optimal trajectories.

The elements of the Jacobian matrix, computed in the steady state, are:

$$
\begin{aligned}
a_{11} &= -\delta \\
a_{12} &= 1 \\
a_{21} &= \frac{\delta}{\beta(1-\beta)}\left\{\beta(1+\beta)(\delta+\rho) + W^*\left[\delta\mu W^* + \sqrt{\delta\mu\left(2\beta(\delta+\rho)+\delta\mu W^{*2}\right)}\right]\right\} \\
a_{22} &= \delta+\rho
\end{aligned}
$$

The corresponding eigenvalues are:

$$e_{1,2} = \frac{1}{2}\left[\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}(J)^2 - 4\det(J)}\right] \tag{19}$$

Since $\operatorname{tr}(J) = a_{11} + a_{22} = \rho > 0$ and $\det(J) = -\frac{2\delta\beta(\delta+\rho)+\delta W^*(\delta\mu W^*+\Xi)}{b(1-b)} < 0$, where $\Xi \equiv \sqrt{2\delta\mu\beta(\delta+\rho)+\delta^2\mu^2 W^{*2}}$, the discriminant of (19) is strictly positive and therefore the two eigenvalues must be real and with opposite sign. This qualifies the steady state as a saddle point.