APPROXIMATING THE 2-DIMENSIONAL MATCHING DISTANCE

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Abstract. Some new approximation results about the 2-dimensional matching distance are presented, leading to the formulation of an algorithm for its computation (up to an arbitrary input error).

Introduction

Shape Description, Comparison and Retrieval are challenging issues in Computer Vision, Computer Graphics and Pattern Recognition. Shape models such as 3D objects represented by surface meshes are characterized by a considerable amount of visual and semantic information. Therefore, they call for powerful feature detection, classification and retrieval methods. Geometric/topological approaches such as Size Theory and Persistent Homology are growing important, as they provide mathematical and computational tools able to code salient shape characteristics [3, 9, 13, 22, 24, 20, 21, 18, 5]. The main idea is to take into account the topological features of a shape with respect to some geometric properties conveyed by a real function defined on the shape itself [1]. This implies representing a shape by a pair (X, φ), with X a topological space and $\varphi : X \to R$ a continuous real-valued measuring function. In the early 90's, Size Functions were introduced, allowing to store quantitatively some qualitative information about (X, φ): the Size Function $\ell_{(X,\varphi)}$ encodes the evolution of the 0-th Betti number in the sublevel sets of X induced by φ. The result is a discrete and proven stable descriptor, made up of a multiset of points in the Euclidean plane, to be compared using a suitable matching distance

Whereas Size Functions, in their original formulation, have successfully been applied to many Pattern Recognition problems [4, 10, 19, 28, 29], it is now clear that more complex shape analysis problems cannot be solved by studying a single real-valued measuring function. In fact, a common scenario in applications is to have data characterized by two or more properties; this happens for example with physical simulations, where several measurements are made about an observed phenomenon, or when data have multidimensional features, such as color in the RGB model. These considerations have recently drawn the attention to the study of a multidimensional setting [1, 2, 6, 8, 20, 24, 25], where the term multidimensional is related to considering measuring functions taking value in R^k. Despite the growing efforts [11, 12], differently from what happens in the 1-dimensional situation, a complete and discrete stable descriptor seems not to be available in the multidimensional setting [1, 7, 14].

The arising computational difficulties have been faced following different strategies, but not completely solved. As a partial solution, in [1] the authors introduced k-dimensional Size Functions and proved that, when $k > 1$, it can be defined a foliation in half planes such that the restriction of a k-dimensional Size Function to these half-planes turns out to be 1-dimensional. This allowed the definition of a stable matching distance between k-dimensional Size Functions. Taking a finite number of half-planes, experiments on the comparison of surfaces and volumetric objects have been performed in the 2- and 3-dimensional settings. Unfortunately, [1] does not make clear how many and what half-planes to choose to have a reasonable approximation of the matching distance, which could require a huge number of calculations.

This contribution aims to solve the problem in the 2-dimensional case. We prove new theoretical results (Proposition 2.1, Proposition 2.3 and Theorem 2.5) about the 2-dimensional matching distance, which allow us to bound the variation of the matching distance values on different half-planes. As a by-product, we develop an algorithm (Subsection 2.1) which takes as input an arbitrary tolerance (representing the maximum error we are disposed to accept in the evaluation of the matching distance between 2-dimensional Size Functions) and gives as output an approximation of the 2-dimensional matching distance up to the input tolerance. Experimental results on 3D objects

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represented by surface meshes will be shown to demonstrate the ability of the algorithm to decimate the number of calculations required to approximate the matching distance (Section 3).

1. Preliminary definitions and results

In what follows, any pair $(X, \vec{\varphi})$, where X is a non-empty, compact and locally connected Hausdorff space, and $\vec{\varphi} = (\varphi_1, \varphi_2) : X \to \mathbb{R}^2$ is a continuous function, will be called a *size pair*. The function $\vec{\varphi}$ will be said to be a 2-*dimensional measuring function*. The following relations $\preceq$ and $\prec$ will be considered in $\mathbb{R}^2$: for $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, we shall write $\vec{u} \preceq \vec{v}$ (resp. $\vec{u} \prec \vec{v}$) if and only if $u_i \leq v_i$ (resp. $u_i < v_i$) for every $i = 1, 2$. Moreover, $\mathbb{R}^2$ will be endowed with the usual max-norm: $\|\vec{u}\|_\infty = \|(u_1, u_2)\|_\infty = \max\{|u_1|, |u_2|\}$.

Now we are ready to introduce the concept of size function for a size pair $(X, \vec{\varphi})$. The open set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \vec{u} \prec \vec{v}\}$ will be denoted by Δ^+, and $\bar{\Delta}^+$ will be the closure of Δ^+. For every pair $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$, we shall define the set $X\langle \vec{\varphi} \preceq \vec{u}\, \rangle$ as $\{x \in X : \varphi_i(x) \leq u_i,\ i = 1, 2\}$.

Definition 1.1. We call the *(2-dimensional) size function* associated with the size pair $(X, \vec{\varphi})$ the function $\ell_{(X,\vec{\varphi})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(X,\vec{\varphi})}(\vec{u}, \vec{v})$ equal to the number of connected components in the set $X\langle \vec{\varphi} \preceq \vec{v}\, \rangle$ containing at least one point of $X\langle \vec{\varphi} \preceq \vec{u}\, \rangle$.

Definition 1.1 can be easily extended to the case of k-dimensional size functions, for any positive integer k. For a deeper investigation about this more general setting, the reader is referred to [1].

1.1. The particular case $k = 1$.

In the present paper, a certain relevance will be given to the 1-dimensional case, i.e. to the case of real-valued measuring functions. Size Theory has been widely developed in this setting [3], proving that each 1-dimensional size function admits a compact representation as a formal series of points and lines of $\mathbb{R}^2$ [23]. As a consequence of this peculiar structure, a suitable *matching distance* between 1-dimensional size functions can be easily introduced, showing the stability of these descriptors with respect to such a distance [15, 17]. All these properties make the concept of 1-dimensional size function central in the approach to the k-dimensional framework [1], and therefore to the 2-dimensional one we consider here.

According to the notations used in the literature about the case $k = 1$, the symbols $\vec{\varphi}$, $\vec{u}$, $\vec{v}$ will be replaced respectively by φ, u, v.

When referring to a real-valued measuring function $\varphi : X \to \mathbb{R}$, the size function $\ell_{(X,\varphi)}$ associated with (X, φ) contains information about the pairs $(X\langle \varphi \leq u\rangle, X\langle \varphi \leq v\rangle)$, where $X\langle \varphi \leq t\rangle$ is defined by setting $X\langle \varphi \leq t\rangle = \{x \in X : \varphi(x) \leq t\}$ for $t \in \mathbb{R}$. Before going on, we observe that for $k = 1$, the domain Δ^+ of a size function reduces to be the open subset of the real plane given by $\{(u, v) \in \mathbb{R}^2 : u < v\}$.

Definition 1.2. We shall call the *(1-dimensional) size function* associated with the size pair (X, φ) the function $\ell_{(X,\varphi)} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{X,\varphi)}(u, v)$ equal to the number of equal to the number of connected components in the set $X\langle \varphi \preceq v\, \rangle$ containing at least one point of $X\langle \varphi \preceq u\, \rangle$.

Figure 1 shows an example of a size pair (X, φ) together with the size function $\ell_{(X,\varphi)}$. On the left one can find (Figure 1(a)) the considered size pair (X, φ), where X is the curve drawn by a solid line, and φ is the ordinate function. On the right (Figure 1(b)) the associated 1-dimensional size function $\ell_{(X,\varphi)}$ is depicted.

As can be seen, the domain $\Delta^+ = \{(u, v) \in \mathbb{R}^2 : u < v\}$ is divided into regions. Each one is labeled by a number, coinciding with the constant value that $\ell_{(X,\varphi)}$ takes in the interior of that region. For example, let us compute the value of $\ell_{(X,\varphi)}$ at the point (c, d). By applying Definition 1.2, it is sufficient to count how many of the three connected components in the sublevel $X\langle \varphi \leq d\rangle$ contain at least one point of $X\langle \varphi \leq c\rangle$. It can be easily checked that $\ell_{(X,\varphi)}(c, d) = 2$.

Due to its typical structure, it has been proved that the information conveyed by a 1-dimensional size function can be combinatorially stored in a formal series of points and lines [23]. Roughly speaking, this can be done by observing that each 1-dimensional size function can be seen as a linear combination (with natural numbers as coefficients) of characteristic functions associated to triangles, possibly unbounded, laying on the domain Δ^+. Indeed, the bounded triangles are of the form $\{(u, v) \in \Delta^+ : \alpha \leq u < v < \beta\}$, while the unbounded ones are of the form $\{(u, v) \in \Delta^+ : \eta \leq u < v)\}$. Hence, a simple and compact representation can be provided if one takes the formal series obtained by associating a triangular set given by $\{(u, v) \in \Delta^+ : \alpha \leq u < v < \beta\}$ to the point (α, β), and a triangular set given by $\{(u, v) \in \Delta^+ : \eta \leq u < v)\}$ to the point at infinity $(\eta, +\infty)$. The points of a formal series having finite coordinates are called *proper cornerpoints*, while the ones with a coordinate at infinity are said to be *cornerpoints*

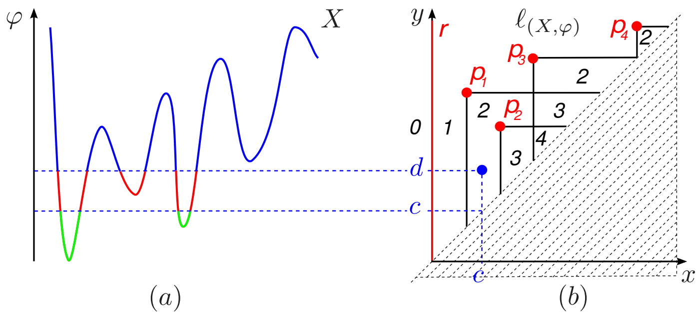

FIGURE 1. (a) The topological space X and the measuring function φ. (b) The associated size function $\ell_{(X,\varphi)}$.

at infinity or *cornerlines*. For example, the size function $\ell_{(X,\varphi)}$ shown in Figure 1(b) admits the representation by formal series given by $r + p_1 + p_2 + p_3 + p_4$, where r is the only cornerpoint at infinity, with coordinates $(0, +\infty)$.

According to the 1-dimensional setting, the problem of comparing two size pairs can be easily translated into the simpler one of comparing sets of points, via the representation by formal series of the associated 1-dimensional size functions. In [15, 17], the *matching distance* d_{match} has proven to be a suitable distance between these descriptors. Roughly speaking, the matching distance d_{match} can be seen as a measure of the cost of transporting the cornerpoints of a 1-dimensional size function into the cornerpoints of another one, with respect to a functional δ depending on the L_∞-distance between two matched cornerpoints and on their L_∞-distance from the diagonal $\{(u, v) \in \mathbb{R}^2 : u = v\}$. An application of d_{match} is shown in Figure 2(c).

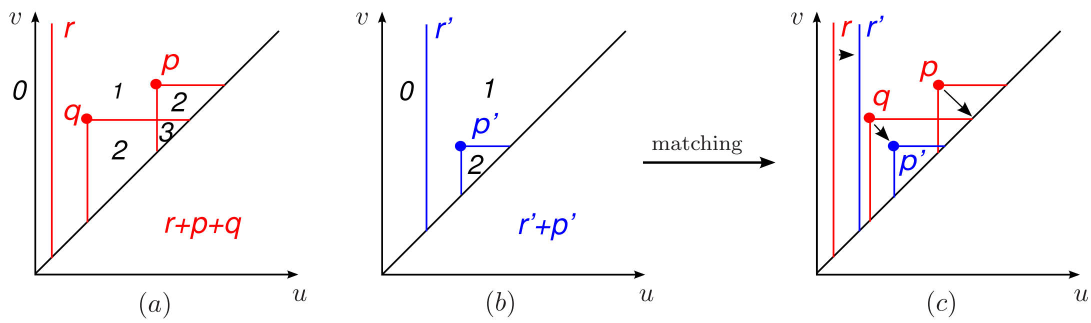

FIGURE 2. (a) The size function corresponding to the formal series $r + p + q$. (b) The size function corresponding to the formal series $r' + p'$. (c) The matching between the two formal series, realizing the matching distance between the two size functions.

Let us now define more formally the matching distance d_{match}. Assume that two 1-dimensional size functions ℓ_1, ℓ_2 are given. Consider the multiset C_1 (respectively C_2) of cornerpoints for ℓ_1 (resp. ℓ_2), counted with their multiplicities and augmented by adding the points of the diagonal $\{(u, v) \in \mathbb{R}^2 : u = v\}$ counted with infinite multiplicity. If we denote by $\bar{\Delta}^*$ the set $\bar{\Delta}^+$ extended by the points at infinity of the kind (a, ∞), i.e. $\bar{\Delta}^* = \bar{\Delta}^+ \cup \{(a, \infty) : a \in \mathbb{R}\}$, the matching distance $d_{match}(\ell_1, \ell_2)$ is then defined as

$$d_{match}(\ell_1, \ell_2) = \min_{\sigma} \max_{P \in C_1} \delta(P, \sigma(P)),$$

where σ varies among all the bijections between C_1 and C_2 and

$$\delta((u, v), (u', v')) = \min \left\{ \max \{|u - u'|, |v - v'|\}, \max \left\{ \frac{v - u}{2}, \frac{v' - u'}{2} \right\} \right\},$$

for every (u, v), $(u', v') \in \bar{\Delta}^*$ and with the convention about ∞ that $\infty - v = v - \infty = \infty$ when $v \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$.

In plain words, the pseudometric δ measures the pseudodistance between two points (u, v) and (u', v') as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that any two points of the diagonal have vanishing pseudodistance.

As can be seen in Figure 2, different 1-dimensional size functions may in general have a different number of cornerpoints. Therefore d_{match} allows a proper cornerpoint to be matched to a point of the diagonal: this matching can be interpreted as the deletion of a proper cornerpoint. Moreover, we stress that the matching distance is stable with respect to perturbations of the measuring functions. Indeed, in [15, 17] the following Matching Stability Theorem has been proved:

Theorem 1.3 (Matching Stability Theorem). *If (X, φ), (X, ψ) are two size pairs with $\max_{P \in X} |\varphi(P) - \psi(P)| \leq \varepsilon$, then it holds that $d_{match}(\ell_{(X,\varphi)}, \ell_{(X,\psi)}) \leq \varepsilon$.*

For a formal definition and further details about the matching distance the reader is referred to [16, 17].

1.2. **Reduction to the 1-dimensional case.** In what follows, we shall adopt the methodology first introduced in [1] and able to reduce the framework of 2-dimensional size functions to the case $k = 1$, by a change of variable and the use of a suitable foliation. More precisely, in [1] the authors prove that, when $k > 1$, a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ can be given, such that the restriction of a k-dimensional size function $\ell_{(X,\vec{\varphi})}$ to each of these half-planes turns out to be a particular 1-dimensional size function.

This approach finds motivations in the fact that generalizing to an arbitrary dimension the concepts of proper cornerpoint and cornerpoint at infinity seems not to be trivial. We recall that these notions, defined in the case of 1-dimensional size functions, play a central role in the representation by formal series. Consequently, at a first glance it seems not possible to provide the multidimensional analogue of the matching distance d_{match} and therefore it is not clear how to obtain stability under perturbations of the measuring functions. On the other hand, all these problems can be overcome via the results we are going to introduce. We shall present them under the assumption $k = 2$, and referring the reader to [1] for the generalization to any positive integer $k > 1$.

First of all, we need to define the half-planes collection foliating Δ^+. Before going on, we observe that the foliation we are going to introduce is differently parameterized with respect to the one given in [1, Def. 7]. On the other hand, it has been proved that our choice does not affect the following definitions and results (see [12] for details). Moreover, it allows us to simplify some technicalities in the rest of the present work.

Definition 1.4. For every vector $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$, and for every vector $\vec{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2$, such that $\beta_1 + \beta_2 = 0$, the pair $(\vec{\lambda}, \vec{\beta})$ will be said *linearly admissible*. We shall denote by $Ladm_2$ the set of all linearly admissible pairs in $\mathbb{R}^2 \times \mathbb{R}^2$. For every linearly admissible pair, let us define the half-plane $\pi_{(\vec{\lambda},\vec{\beta})}$ of $\mathbb{R}^2 \times \mathbb{R}^2$ by the parametric equations $\vec{u} = \sigma\vec{\lambda} + \vec{\beta}$, $\vec{v} = \tau\vec{\lambda} + \vec{\beta}$, with $\sigma, \tau \in \mathbb{R}$ and $\sigma < \tau$.

From now on, we shall denote by Π_{Ladm_2} the collection $\left\{\pi_{(\vec{\lambda},\vec{\beta})} : (\vec{\lambda}, \vec{\beta}) \in Ladm_2\right\}$.

The next proposition shows that Π_{Ladm_2} is actually a foliation of Δ^+.

Proposition 1.5. *For every $(\vec{u}, \vec{v}) \in \Delta^+$ there exists one and only one linearly admissible pair $(\vec{\lambda}, \vec{\beta})$ such that $(\vec{u}, \vec{v}) \in \pi_{(\vec{\lambda},\vec{\beta})}$. Moreover, such a bijection continuously depends on $(\vec{u}, \vec{v})$.*

Proof. The claim immediately follows by taking, for $i = 1, 2$,

$$\lambda_i = \frac{v_i - u_i}{\sum_{j=1}^{2}(v_j - u_j)} , \quad \beta_i = \frac{u_i \sum_{j=1}^{2} v_j - v_i \sum_{j=1}^{2} u_j}{\sum_{j=1}^{2}(v_j - u_j)}.$$

Therefore, $\vec{u} = \sigma\vec{\lambda} + \vec{\beta}$, $\vec{v} = \tau\vec{\lambda} + \vec{\beta}$, with $\sigma = u_1 + u_2$ and $\tau = v_1 + v_2$. □

We can now state the result allowing us to reduce the 2-dimensional setting to the case $k = 1$.

Theorem 1.6 (Reduction Theorem). *Let $(\vec{\lambda}, \vec{\beta}) \in Ladm_2$, and let $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})} : X \to \mathbb{R}$ be the function defined by setting*

$$F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x) = \max\left\{\frac{\varphi_1(x) - \beta_1}{\lambda_1}, \frac{\varphi_2(x) - \beta_2}{\lambda_2}\right\}.$$

Then, for every $(\vec{u}, \vec{v}) = (\sigma\vec{\lambda} + \vec{\beta}, \tau\vec{\lambda} + \vec{\beta}) \in \pi_{(\vec{\lambda},\vec{\beta})}$ it holds that $\ell_{(X,\vec{\varphi})}(\vec{u}, \vec{v}) = \ell_{(X,F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})})}(\sigma, \tau)$.

We skip the proof of Theorem 1.6 since it can be immediately obtained from the one of [1, Thm. 3].

In the following, we shall use the symbol $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}$ in the sense of the Reduction Theorem 1.6.

Roughly speaking, the Reduction Theorem 1.6 states that, on each half-plane of Π_{Ladm_2}, the restriction of a given 2-dimensional size function coincides with a particular size function in two scalar variables, i.e. a 1-dimensional one. A first important consequence is the possibility of representing a 2-dimensional size function $\ell_{(X,\vec{\varphi})}$ by a collection of formal series of points and lines, following the machinery described in Example 1.1 for the case $k=1$. Therefore, for every $\pi_{(\vec{\lambda},\vec{\beta})} \in \Pi_{Ladm_2}$ the matching distance between 1-dimensional size functions can be applied, showing that it is stable with respect to perturbations of the multidimensional measuring functions and to the choice of the leaves of the foliation [1, Prop. 2 and 3]. These stability properties lead to the following definition of a distance between 2-dimensional size functions (see also [1, Def. 8]).

Definition 1.7. Let $(X,\vec{\varphi})$ and $(Y,\vec{\psi})$ be two size pairs, with $\vec{\varphi},\vec{\psi}$ vector-valued in $\mathbb{R}^2$. The *2-dimensional matching distance* $D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ is the (extended) distance defined by setting

$$D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})}) = \sup_{(\vec{\lambda},\vec{\beta})\in Ladm_2} \min_{i=1,2} \lambda_i \cdot d_{match}(\ell_{(X,F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})})},\ell_{(Y,F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})})}).$$

Remark 1.8. It can be proved [12] that the distance defined in Definition 1.7 coincides with the 2-dimensional analogous of the one given in [1, Def. 8].

The term “extended” in Definition 1.7 refers to the fact that, if the spaces X and Y are not assumed to be homeomorphic, the multidimensional matching distance D_{match} still verifies all the properties of a distance, except for the fact that it may take the value $+\infty$.

Moreover, let us observe that choosing a non-empty and finite subset $A \subseteq Ladm_2$, and substituting $\sup_{(\vec{\lambda},\vec{\beta})\in Ladm_2}$ with $\max_{(\vec{\lambda},\vec{\beta})\in A}$ in Definition 1.7, we obtain a computable pseudodistance between 2-dimensional size function, that is stable and effectively computable.

From now on, for every $(\vec{\lambda},\vec{\beta}) \in Ladm_2$ the symbol $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ will be used to denote the value $\min_{i=1,2}\lambda_i \cdot d_{match}(\ell_{(X,F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})})},\ell_{(Y,F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})})})$.

2. Computation of the multidimensional matching distance

This section is devoted to provide an algorithm for computing an approximation of the multidimensional matching distance D_{match} between two multidimensional size functions $\ell_{(X,\vec{\varphi})}$ and $\ell_{(Y,\vec{\psi})}$.

Before proceeding, let us recall the general ideas leading to its formulation.

By Definition 1.7 it follows that, in general, a direct computation of $D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ is not possible, since we should calculate the value $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ for an infinite number of pairs $(\vec{\lambda},\vec{\beta})$. On the other hand, as stressed in Remark 1.8, if we choose a non-empty and finite subset $A \subseteq Ladm_2$, and substitute $\sup_{(\vec{\lambda},\vec{\beta})\in Ladm_2}$ with $\max_{(\vec{l},\vec{b})\in A}$ in Definition 1.7, we get an easily computable pseudo-distance, say $\tilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$, to be used in concrete applications.

If we think of $\tilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ as an approximation of $D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$, it is reasonable to guess that the larger the set $A \subseteq Ladm_2$ is , the smaller the difference between the two values can be. On the other hand, the smaller the set A is, the faster the computation of $\tilde{D}_{match}$ is. In this context, we implement an algorithm in order to find a set A representing a compromise between these two situations. Additionally, A is such that, given an arbitrary real value $\varepsilon > 0$ as input tolerance, the output $\widetilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})}) = \max_{(\vec{\lambda},\vec{\beta})\in A} d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ satisfies the inequality $|D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})}) - \widetilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})| \le \varepsilon$.

Consider the size pairs $(X,\vec{\varphi})$ and $(Y,\vec{\psi})$, and assume X,Y homeomorphic, so that $D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})}) < +\infty$. The first step toward the formulation of our algorithm is to estimate the changing of $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$, the pair $(\vec{\lambda},\vec{\beta})$ varying in $Ladm_2$, i.e. when moving from one leaf to another in the half-planes foliation of Δ^+.

Let us start by observing that $Ladm_2 = \{(\vec{\lambda},\vec{\beta}) = (\lambda_1,\lambda_2,b_1,b_2) \in \mathbb{R}^2\times\mathbb{R}^2 : \lambda_2 = 1-\lambda_1, \beta_2 = -\beta_1, \lambda_1 \in (0,1)\} = \{(a,1-a,b,-b) \in \mathbb{R}^2\times\mathbb{R}^2 : a \in (0,1)\}$. In what follows, for every $\vec{\lambda} = (a,1-a)$ with $a \in (0,1)$, we shall denote by $\mu(\vec{\lambda})$ the value $\min\{a,1-a\}$.

The next result allows us to avoid the study of $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ in a large part of $Ladm_2$. Before stating it, set $C=\max\left\{\|\vec{\varphi}\|_\infty,\|\vec{\psi}\|_\infty\right\}$, with $\|\vec{\varphi}\|_\infty=\max_{x\in X}\max_i|\varphi_i(x)|$ and $\|\vec{\psi}\|_\infty=\max_{y\in Y}\max_i|\psi_i(y)|$, and consider the set $Ladm_2^*=\{(a,1-a,b,-b)\in Ladm_2:|b|<C\}$.

Proposition 2.1. *Let $(\vec{\lambda},\vec{\beta})\in Ladm_2\setminus Ladm_2^*$, with $(\vec{\lambda},\vec{\beta})=(a,1-a,b,-b)$. Then it follows that*

$$d_{(\vec{\lambda},\vec{b})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})=\begin{cases}\frac{\mu(\vec{\lambda})}{a}\cdot d_{match}(\ell_{(X,\varphi_1)},\ell_{(Y,\psi_1)}), & \text{if } b\le -C;\\ \frac{\mu(\vec{\lambda})}{1-a}\cdot d_{match}(\ell_{(X,\varphi_2)},\ell_{(Y,\psi_2)}), & \text{if } b\ge C.\end{cases}$$

Proof. It is sufficient to observe that $b\le -C$ implies $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x)=\frac{\varphi_1(x)-b}{a}$ for every $x\in X$, and $F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}(y)=\frac{\psi_1(y)-b}{a}$ for every $y\in Y$, while $b\ge C$ implies $F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x)=\frac{\varphi_2(x)+b}{1-a}$ for every $x\in X$ and $F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}(y)=\frac{\psi_2(y)+b}{1-a}$ for every $y\in Y$. From the definition of the 1-dimensional matching distance d_{match} (see also [12, Proposition 2.10]) the claim easily follows. □

Remark 2.2. The maximum value for $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ over the set $\{(\vec{\lambda},\vec{b})=(a,1-a,b,-b)\in Ladm_2:b\le -C\}$ is assumed when $a\le\frac{1}{2}$, and it equals to $d_{match}(\ell_{(X,\varphi_1)},\ell_{(Y,\psi_1)})$. Analogously, the maximum value for $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ over $\{(\vec{\lambda},\vec{\beta})\in Ladm_2:b\ge C\}$ is assumed when $a\ge\frac{1}{2}$, and it equals to $d_{match}(\ell_{(X,\varphi_2)},\ell_{(Y,\psi_2)})$.

According to Proposition 2.1 and Remark 2.2, in order to know the values $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ when $(\vec{\lambda},\vec{\beta})\in Ladm_2\setminus Ladm_2^*$, it is sufficient to consider just two suitable points of that region, e.g., the points of coordinates $(\frac{1}{2},\frac{1}{2},C,-C)$ and $(\frac{1}{2},\frac{1}{2},-C,C)$. It only remains to study the changing of $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ over the set $Ladm_2^*$. To this aim, we need the next result.

Proposition 2.3. *Assume $(\vec{\lambda},\vec{\beta})\in Ladm_2^*$ and $(\vec{\lambda}',\vec{\beta}')\in Ladm_2$, with $\|(\vec{\lambda},\vec{\beta})-(\vec{\lambda}',\vec{\beta}')\|_\infty\le\delta$ and $\delta\le\frac{1}{4}$. Then it follows that*

$$\left|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\right|\le\delta\cdot(32C+2).$$

Proof. Let $(\vec{\lambda},\vec{\beta})=(a,1-a,b,-b)$ and $(\vec{\lambda}',\vec{\beta}')=(a',1-a',b',-b')$. We start by observing that, from the definition of d_{match} (see also [12, Proposition 2.10]), we can write $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})=d_{match}\left(\ell_{(X,\mu(\vec{\lambda})\cdot F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})})},\ell_{(Y,\mu(\vec{\lambda})\cdot F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})})}\right)$ for every considered $(\vec{\lambda},\vec{\beta})$. Therefore it holds that

$$\left|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\right|= \tag{2.1}$$

$$=\left|d_{match}\left(\ell_{(X,\mu(\vec{\lambda})\cdot F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})})},\ell_{(Y,\mu(\vec{\lambda})\cdot F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})})}\right)-d_{match}\left(\ell_{(X,\mu(\vec{\lambda}')\cdot F^{\vec{\varphi}}_{(\vec{\lambda}',\vec{\beta}')})},\ell_{(Y,\mu(\vec{\lambda}')\cdot F^{\vec{\psi}}_{(\vec{\lambda}',\vec{\beta}')})}\right)\right|\le \tag{2.2}$$

$$\le d_{match}\left(\ell_{(X,\mu(\vec{\lambda})\cdot F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})})},\ell_{(X,\mu(\vec{\lambda}')\cdot F^{\vec{\varphi}}_{(\vec{\lambda}',\vec{\beta}')})}\right)+d_{match}\left(\ell_{(Y,\mu(\vec{\lambda})\cdot F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})})},\ell_{(Y,\mu(\vec{\lambda}')\cdot F^{\vec{\psi}}_{(\vec{\lambda}',\vec{\beta}')})}\right)\le \tag{2.3}$$

$$\le\max_{x\in X}\left|\mu(\vec{\lambda})\cdot F^{\vec{\varphi}}_{(\vec{\lambda},\vec{\beta})}(x)-\mu(\vec{\lambda}')\cdot F^{\vec{\varphi}}_{(\vec{\lambda}',\vec{\beta}')}(x)\right|+\max_{y\in Y}\left|\mu(\vec{\lambda})\cdot F^{\vec{\psi}}_{(\vec{\lambda},\vec{\beta})}(y)-\mu(\vec{\lambda}')\cdot F^{\vec{\psi}}_{(\vec{\lambda}',\vec{\beta}')}(y)\right|= \tag{2.4}$$

$$\begin{aligned}=&\max_{x\in X}\left|\max\left\{\mu(\vec{\lambda})\cdot\frac{\varphi_1(x)-b}{a},\mu(\vec{\lambda})\cdot\frac{\varphi_2(x)+b}{1-a}\right\}-\max\left\{\mu(\vec{\lambda}')\cdot\frac{\varphi_1(x)-b'}{a'},\mu(\vec{\lambda}')\cdot\frac{\varphi_2(x)+b'}{1-a'}\right\}\right|+\\ &+\max_{y\in Y}\left|\max\left\{\mu(\vec{\lambda})\cdot\frac{\psi_1(y)-b}{a},\mu(\vec{\lambda})\cdot\frac{\psi_2(y)+b}{1-a}\right\}-\max\left\{\mu(\vec{\lambda}')\cdot\frac{\psi_1(y)-b'}{a'},\mu(\vec{\lambda}')\cdot\frac{\psi_2(y)+b'}{1-a'}\right\}\right|\le\end{aligned} \tag{2.5}$$

$$\begin{aligned}\le&\max_{x\in X}\max\left\{\left|\mu(\vec{\lambda})\cdot\frac{\varphi_1(x)-b}{a}-\mu(\vec{\lambda}')\cdot\frac{\varphi_1(x)-b'}{a'}\right|,\left|\mu(\vec{\lambda})\cdot\frac{\varphi_2(x)+b}{1-a}-\mu(\vec{\lambda}')\cdot\frac{\varphi_2(x)+b'}{1-a'}\right|\right\}+\\ &+\max_{y\in Y}\max\left\{\left|\mu(\vec{\lambda})\cdot\frac{\psi_1(y)-b}{a}-\mu(\vec{\lambda}')\cdot\frac{\psi_1(y)-b'}{a'}\right|,\left|\mu(\vec{\lambda})\cdot\frac{\psi_2(y)+b}{1-a}-\mu(\vec{\lambda}')\cdot\frac{\psi_2(y)+b'}{1-a'}\right|\right\},\end{aligned} \tag{2.6}$$

where inequality (2.4) is a consequence of the Matching Stability Theorem 1.3, and inequality (2.6) comes from the relation $|\max(u_1,u_2)-\max(v_1,v_2)|\le\max(|u_1-v_1|,|u_2-v_2|)$, for every $u_1,u_2,v_1,v_2\in\mathbb{R}$ (see Appendix).

Let us now distinguish three cases: $a \le \frac{1}{4}$, $a \ge \frac{3}{4}$ and $\frac{1}{4} < a < \frac{3}{4}$.

If $\underline{a \le \frac{1}{4}}$, since $\delta \le \frac{1}{4}$ it follows that $a' \le \frac{1}{2}$. Hence, $\mu(\vec{\lambda}) = a$, $\mu(\vec{\lambda}') = a'$ and in the first part of inequality (2.6) we have

$$\max_{x\in X}\max\left\{\left|\mu(\vec{\lambda}\,)\cdot\frac{\varphi_1(x)-b}{a}-\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b'}{a'}\right|,\left|\mu(\vec{\lambda}\,)\cdot\frac{\varphi_2(x)+b}{1-a}-\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_2(x)+b'}{1-a'}\right|\right\}= \tag{2.7}$$

$$=\max_{x\in X}\max\left\{|\varphi_1(x)-b-\varphi_1(x)+b'|\,,\left|a\cdot\frac{\varphi_2(x)+b}{1-a}-a'\cdot\frac{\varphi_2(x)+b'}{1-a'}\right|\right\}= \tag{2.8}$$

$$=\max\left\{|b-b'|\,,\max_{x\in X}\left|a\cdot\frac{\varphi_2(x)+b}{1-a}-a\cdot\frac{\varphi_2(x)+b'}{1-a}+a\cdot\frac{\varphi_2(x)+b'}{1-a}-a'\cdot\frac{\varphi_2(x)+b'}{1-a'}\right|\right\}= \tag{2.9}$$

$$=\max\left\{|b'-b|\,,\max_{x\in X}\left|\frac{a}{1-a}\cdot(b-b')+(\varphi_2(x)+b')\cdot(\frac{a}{1-a}-\frac{a'}{1-a'})\right|\right\}\le \tag{2.10}$$

$$\le\max\left\{|b'-b|\,,\frac{a}{1-a}\cdot|b-b'|+\max_{x\in X}|\varphi_2(x)+b'|\cdot\frac{|a-a'|}{(1-a)(1-a')}\right\}\le \tag{2.11}$$

$$\le\max\left\{\delta,\frac{\delta}{3}+2C\cdot\delta\cdot\frac{8}{3}\right\}=\max\left\{\delta,\frac{\delta}{3}\cdot(16C+1)\right\}, \tag{2.12}$$

where the inequality in (2.12) holds since $1-a \ge \frac{3}{4}$ e $1-a' \ge \frac{1}{2}$.

Analogously, in the second part of inequality (2.6) we obtain

$$\max_{y\in Y}\max\left\{\left|\mu(\vec{\lambda}\,)\cdot\frac{\psi_1(y)-b}{a}-\mu(\vec{\lambda}'\,)\cdot\frac{\psi_1(y)-b'}{a'}\right|,\left|\mu(\vec{\lambda}\,)\cdot\frac{\psi_2(y)+b}{1-a}-\mu(\vec{\lambda}'\,)\cdot\frac{\psi_2(y)+b'}{1-a'}\right|\right\}\le \tag{2.13}$$

$$\le\max\left\{\delta,\frac{\delta}{3}\cdot(16C+1)\right\}, \tag{2.14}$$

and hence, when $a \le \frac{1}{4}$ and under the hypothesis $\delta \le \frac{1}{4}$, it follows that

$$\left|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\right|\le 2\max\left\{\delta,\frac{\delta}{3}\cdot(16C+1)\right\}. \tag{2.15}$$

If $\underline{a \ge \frac{3}{4}}$, similar arguments lead to the inequality

$$\left|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\right|\le 2\max\left\{\delta,\frac{\delta}{3}\cdot(16C+1)\right\}. \tag{2.16}$$

It only remains to consider when $\underline{\frac{1}{4} < a < \frac{3}{4}}$. In this case, in the first part of inequality (2.6) we have

$$\left|\mu(\vec{\lambda}\,)\cdot\frac{\varphi_1(x)-b}{a}-\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b'}{a'}\right|= \tag{2.17}$$

$$=\left|\mu(\vec{\lambda}\,)\cdot\frac{\varphi_1(x)-b}{a}-\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b}{a}+\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b}{a}+\right. \tag{2.18}$$

$$\left.-\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b}{a'}+\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b}{a'}-\mu(\vec{\lambda}'\,)\cdot\frac{\varphi_1(x)-b'}{a'}\right|= \tag{2.19}$$

$$=\left|(\mu(\vec{\lambda}\,)-\mu(\vec{\lambda}'\,))\cdot\frac{\varphi_1(x)-b}{a}+\mu(\vec{\lambda}'\,)\cdot(\varphi_1(x)-b)\cdot(\frac{1}{a}-\frac{1}{a'})+\frac{\mu(\vec{\lambda}'\,)}{a'}\cdot(b'-b)\right|\le \tag{2.20}$$

$$\le\left|(\mu(\vec{\lambda}\,)-\mu(\vec{\lambda}'\,))\cdot\frac{\varphi_1(x)-b}{a}\right|+\left|\mu(\vec{\lambda}'\,)\cdot(\varphi_1(x)-b)\cdot(\frac{1}{a}-\frac{1}{a'})\right|+\left|\frac{\mu(\vec{\lambda}'\,)}{a'}\cdot(b'-b)\right|\le \tag{2.21}$$

$$\le\delta\cdot\left(\frac{|\varphi_1(x)-b|}{a}+\frac{\mu(\vec{\lambda}'\,)}{a\cdot a'}|\varphi_1(x)-b|+\frac{\mu(\vec{\lambda}'\,)}{a'}\right)\le\delta\cdot\left(\frac{2|\varphi_1(x)-b|}{a}+1\right)\le\delta\cdot(16C+1), \tag{2.22}$$

where the second inequality in (2.22) holds since $\frac{\mu(\vec{\lambda}'\,)}{\alpha'} \le 1$. Similarly, in inequality (2.6) we also obtain

$$\left|\mu(\vec{\lambda})\cdot\frac{\varphi_2(x)+b}{1-a}-\mu(\vec{\lambda}')\cdot\frac{\varphi_2(x)+b'}{1-a'}\right|\le\delta\cdot(16C+1), \tag{2.23}$$

and hence

$$\max_{x\in X}\max\left\{\left|\mu(\vec{\lambda})\cdot\frac{\varphi_1(x)-b}{a}-\mu(\vec{\lambda}')\cdot\frac{\varphi_1(x)-b'}{a'}\right|,\left|\mu(\vec{\lambda})\cdot\frac{\varphi_2(x)+b}{1-a}-\mu(\vec{\lambda}')\cdot\frac{\varphi_2(x)+b'}{1-a'}\right|\right\}\le\delta\cdot(16C+1).$$

Similar reasonings allows us to estimate the second addend in inequality (2.6) leading to

$$\max_{y\in Y}\max\left\{\left|\mu(\vec{\lambda})\cdot\frac{\psi_1(y)-b}{a}-\mu(\vec{\lambda}')\cdot\frac{\psi_1(y)-b'}{a'}\right|,\left|\mu(\vec{\lambda})\cdot\frac{\psi_2(y)+b}{1-a}-\mu(\vec{\lambda}')\cdot\frac{\psi_2(y)+b'}{1-a'}\right|\right\}\le\delta\cdot(16C+1).$$

therefore, when $\frac{1}{4}<a<\frac{3}{4}$, it holds that

$$\left|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\right|\le\delta\cdot(32C+2). \tag{2.24}$$

By comparing the bounds for the three cases $a\le\frac{1}{4}$, $a\ge\frac{3}{4}$ and $\frac{1}{4}<a<\frac{3}{4}$, we obtain $2\max\left\{\delta,\frac{\delta}{3}\cdot(16C+1)\right\}\le\delta\cdot(32C+2)$ and hence the claim is proved. □

Remark 2.4. It can be proved that $d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\le 2C$ for every $(\vec{\lambda},\vec{\beta})\in Ladm_2$ (this is a trivial consequence of [1, Thm. 4]), implying that the inequality $|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})|\le 4C$ is always verified. Since $\delta\cdot(32C+2)\le 4C\Leftrightarrow\delta\le\frac{4C}{32C+2}\xrightarrow[C\to\infty]{}\frac{1}{8}$, it follows that, in applications, Proposition 2.3 is significant only if we assume such a restriction for δ.

The previous Proposition 2.1 and 2.3 can be merged together to obtain the following more general result.

Theorem 2.5. *Assume $(\vec{\lambda},\vec{\beta}),(\vec{\lambda}',\vec{\beta}')\in Ladm_2$, with $\|(\vec{\lambda},\vec{\beta})-(\vec{\lambda}',\vec{\beta}')\|_\infty\le\delta$ and $\delta\le\frac{1}{4}$. Then it follows that*

$$\left|d_{(\vec{\lambda},\vec{\beta})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-d_{(\vec{\lambda}',\vec{\beta}')}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})\right|\le\delta\cdot(32C+2).$$

2.1. **Algorithm.** The results proved in Proposition 2.1, Proposition 2.3 and Theorem 2.5 can be exploited in the development of an algorithm able to compute an approximation of $D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$. More precisely, the idea is to fix as input a tolerance ε, i.e. the maximum error we are disposed to accept in the computation, and to let the algorithm run until the output $\widetilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$, representing an approximation of $D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$, satisfies the inequality $|D_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})-\widetilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})|\le\varepsilon$.

We now describe the algorithm in details.

First of all let us observe that the set $Ladm_2=\{(a,1-a,b,-b):a\in(0,1)\}$ can be identified with the subset on $\mathbb{R}^2$ given by $\{(a,b):a\in(0,1)\}=(0,1)\times\mathbb{R}$. To be more precise, there exists a bijective correspondence between $Ladm_2^*=\{(a,1-a,b,-b)\in Ladm_2:|b|<C\}$ and the set $(0,1)\times(-C,C)$, as well as between $Ladm_2\setminus Ladm_2^*$ and the set $(0,1)\times\mathbb{R}\setminus(0,1)\times(-C,C)$.

The algorithm works as follows.

(1) Fix an input tolerance $\varepsilon>0$ and set $\delta=\frac{1}{8}$ (cf. Remark 2.4);
(2) For every $i,j\in\mathbb{Z}$ such that $0\le i\le\frac{1}{2\delta}-1$, $-\frac{1}{2\delta}([C]+1)\le j\le\frac{1}{2\delta}([C]+1)$, with $[C]$ floor function at C, set $P_{ij}=(u_{ij},v_{ij})=(\delta(2i+1),2\delta j)$, and consider the covering of the set $(0,1)\times(-C,C)$ given by $\mathcal{Q}=\{Q_\delta(P_{ij})\}$, where $Q_\delta(P_{ij})=\{P=(u,v)\in\mathbb{R}^2:\|P_{ij}-P\|_\infty\le\delta\}$ (see Figure 3). Set $\mathcal{P}=\{P_{ij}\}$;
(3) Consider two further points A and B in $\mathbb{R}^2$, whose coordinates are $(\frac{1}{2},-(C+\frac{1}{2}))$ and $(\frac{1}{2},C+\frac{1}{2})$ respectively (see once more Figure 3);
(4) For every $P_{ij}\in\mathcal{P}$, consider the associated linearly admissible pair $(\vec{\lambda}_{ij},\vec{\beta}_{ij})=(u_{ij},1-u_{ij},v_{ij},-v_{ij})$ and compute the value $D_{P_{ij}}=d_{(\vec{\lambda}_{ij},\vec{\beta}_{ij})}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$. Analogously for A and B, obtaining D_A and D_B;
(5) Compute $\bar{D}=\max_{P_{ij}\in\mathcal{P}\cup\{A,B\}}D_{P_{ij}}$. If $\delta\cdot(32C+2)<\varepsilon$, set $\bar{D}=\widetilde{D}_{match}(\ell_{(X,\vec{\varphi})},\ell_{(Y,\vec{\psi})})$ and the algorithms ends. Otherwise, delete all points $P_{ij}\in\mathcal{P}$ such that $\bar{D}-D_{P_{ij}}>\delta\cdot(32+C)$, and the associated $Q_\delta(P_{ij})\in\mathcal{Q}$;
(6) Subdivide each square still in $\mathcal{Q}$ into 4 equal squares, substitute the points in $\mathcal{C}$ with the new square centers, change δ with $\frac{\delta}{2}$ and restart from step 4.

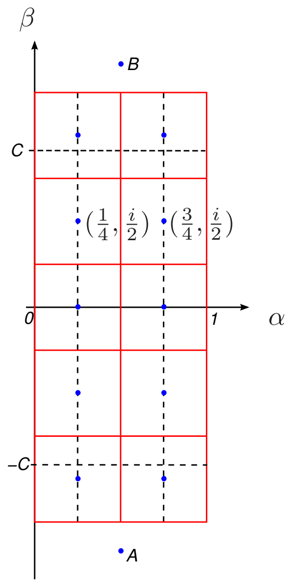

FIGURE 3. The starting covering for the set $(0, 1) \times (-C, C)$ described in the algorithm at step 2.

3. EXPERIMENTS

Figures 4,5,6,7,8 show some examples of the 2-dimensional matching distance between models taken from the SHREC 2007 database [26]. The 2-dimensional measuring function is $\vec{\varphi} = (\varphi_1, \varphi_2)$, with φ_1 the integral geodesic distance [27] and φ_2 the distance from the principal vector $\vec{v}$ defined in [1]. The values of $\vec{\varphi}$ are normalized so that they range in the interval $[0, 1]$. This implies the constant C is equal to 1.

We fix an input error ϵ equal to 5% of the constant C, that is, $\epsilon = 0.050000$. Six iterations are required for the threshold $t = \delta \cdot (32C + 2)$ to became less than ϵ.

Each plot in Figures 4,5,6,7,8 shows the values of the 2-dimensional matching distance outside and inside $Ladm_2^*$. In the color coding, red corresponds to higher values, whereas blue corresponds to lower values.

If the computation were done using a single tiling strategy of $Ladm_2^*$, without the point cancellation procedure introduced above, a total amount of 830072 of distances would be required. We show in the examples that the number of distances actually computed is much lower, up to 3% of the original number.

It should be noted that, whereas the drop in the number of computations is evident when comparing objects belonging to different categories (Figures 4,5,6), the improvement is less noticeable if same-class objects are compared (Figures 7,8).

APPENDIX

Let us prove that $(*)$ $|\max(u_1, u_2) - \max(v_1, v_2)| \leq \max(|u_1 - v_1|, |u_2 - v_2|)$. To this aim, we distinguish 6 different cases.

(1) If $u_1 \geq u_2$ and $v_1 \geq v_2$ then inequality $(*)$ trivially follows;
(2) If $u_1 < u_2$ and $v_1 < v_2$ then inequality $(*)$ trivially follows;
(3) If $u_1 \geq u_2$ and $v_1 < v_2$, with $u_1 \geq v_2$, then it follows that $u_1 \geq v_2 > v_1$, hence $|u_1 - v_2| < |a - v_1|$;
(4) If $u_1 \geq u_2$ and $v_1 < v_2$, with $u_1 < v_2$, then it follows that $v_2 > u_1 \geq u_2$, hence $|u_1 - v_2| < |u_2 - v_2|$;
(5) If $u_1 < u_2$ and $v_1 \geq v_2$, with $v_1 \geq u_2$, then it follows that $v_1 \geq u_2 > u_1$, hence $|u_2 - v_1| < |u_1 - v_1|$;
(6) If $u_1 < u_2$ and $v_1 \geq v_2$, with $v_1 < u_2$, then it follows that $u_2 > v_1 \geq v_2$, hence $|u_2 - v_1| < |u_2 - v_2|$.

REFERENCES

[1] S. Biasotti, A. Cerri, P. Frosini, D. Giorgi, and C. Landi. Multidimensional size functions for shape comparison. *J. Math. Imaging Vis.*, 32(2):161–179, 2008.

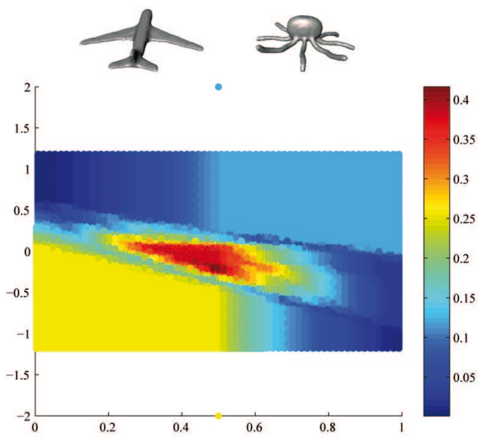

FIGURE 4. 2-dimensional matching distance between an airplane and an octopus models, shown on top of the plot. We fix an input error ϵ equal to 5% of the constant C, that is, $\epsilon = 0.050000$, being $C = 1$. Outside $Ladm_2^*$, the distance takes values 0.178103 and 0.244962. Inside $Ladm_2^*$, after 6 iterations we get the maximum value 0.460318. According to our strategy of point camcellation, we get the approximated value by computing the distance on 33912 points out of a total amount of 830072, i.e. on 4% of points.

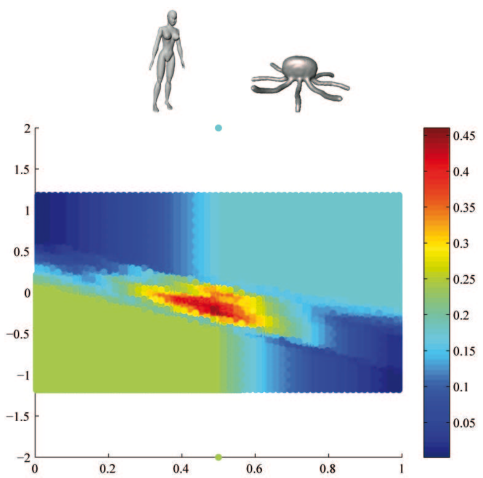

FIGURE 5. 2-dimensional matching distance between a human and an octopus models, shown on top of the plot. We fix an input error ϵ equal to 5% of the constant C, that is, $\epsilon = 0.050000$, being $C = 1$. Outside $Ladm_2^*$, the distance takes values 0.118791 and 0.257325. Inside $Ladm_2^*$, after 6 iterations we get the maximum value 0.416100. According to our strategy of point camcellation, we get the approximated value by computing the distance on 48192 points out of a total amount of 830072, i.e. on 6% of points.

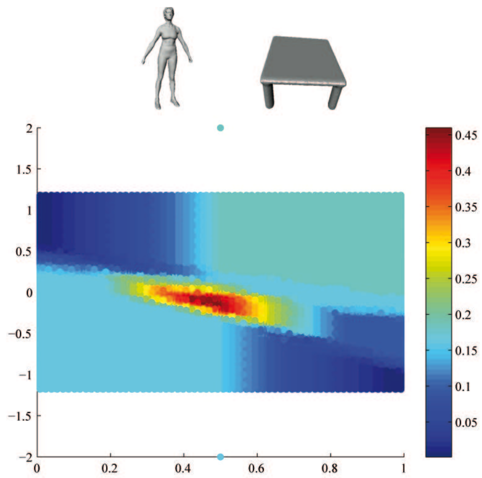

FIGURE 6. 2-dimensional matching distance between a human and a table models, shown on top of the plot. We fix an input error ϵ equal to 5% of the constant C, that is, $\epsilon = 0.050000$, being $C = 1$. Outside $Ladm_2^*$, the distance takes values 0.187161 and 0.159492. Inside $Ladm_2^*$, after 6 iterations we get the maximum value 0.459919. According to our strategy of point camcellation, we get the approximated value by computing the distance on 24564 points out of a total amount of 830072, i.e. on 3% of points.

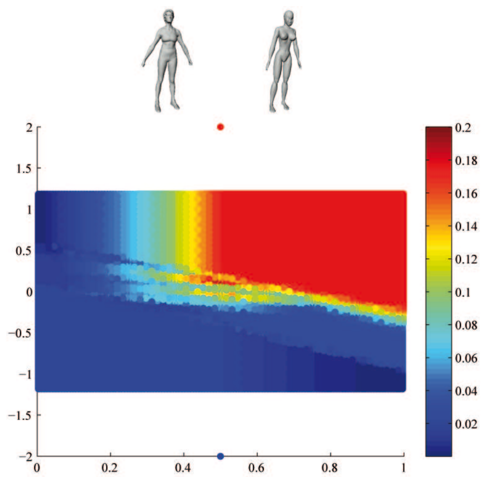

FIGURE 7. 2-dimensional matching distance between two human models, shown on top of the plot. We fix an input error ϵ equal to 5% of the constant C, that is, $\epsilon = 0.050000$, being $C = 1$. Outside $Ladm_2^*$, the distance takes values 0.176755 and 0.019991. Inside $Ladm_2^*$, after 6 iterations we get the maximum value 0.200139. According to our strategy of point camcellation, we get the approximated value by computing the distance on 300144 points out of a total amount of 830072, i.e. on 36% of points.

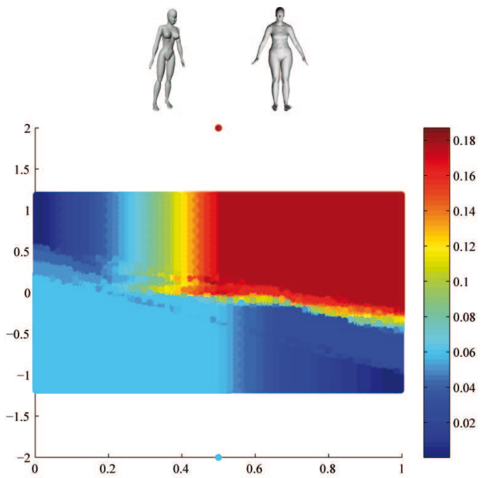

FIGURE 8. 2-dimensional matching distance between two human models, shown on top of the plot. We fix an input error ϵ equal to 5% of the constant C, that is, $\epsilon = 0.050000$, being $C = 1$. Outside $Ladm_2^*$, the distance takes values 0.176755 and 0.060072. Inside $Ladm_2^*$, after 6 iterations we get the maximum value 0.186972. According to our strategy of point camcellation, we get the approximated value by computing the distance on 371908 points out of a total amount of 830072, i.e. on 45% of points.

[2] S. Biasotti, A. Cerri, and D. Giorgi. k-dimensional size functions for shape description and comparison. In R. Cucchiara, editor, *ICIAP*, pages 795–800. IEEE Computer Society, 2007.
[3] S. Biasotti, L. De Floriani, B. Falcidieno, P. Frosini, D. Giorgi, C. Landi, L. Papaleo, and M. Spagnuolo. Describing shapes by geometrical-topological properties of real functions. *ACM Comput. Surv.*, 40(4):1–87, 2008.
[4] S. Biasotti, D. Giorgi, M. Spagnuolo, and B. Falcidieno. Size functions for comparing 3D models. *Pattern Recogn.*, 41(9):2855–2873, 2008.
[5] P. Bubenik and P. Kim. A statistical approach to persistent homology. *HHA*, 9(2):337–362, 2007.
[6] G. Carlsson. Topology and data. *Bull. Amer. Math. Soc.*, 46(2):255–308, 2009.
[7] G. Carlsson, G. Singh, and A. Zomorodian. Computing multidimensional persistence. In Y. Dong, D.-Z. Du, and O. H. Ibarra, editors, *ISAAC*, volume 5878 of *Lecture Notes in Computer Science*, pages 730–739. Springer, 2009.
[8] G. Carlsson and A. Zomorodian. The theory of multidimensional persistence. In J. Erickson, editor, *Symposium on Computational Geometry*, pages 184–193. ACM, 2007.
[9] G. Carlsson, A. Zomorodian, A. Collins, and L. J. Guibas. Persistence barcodes for shapes. *IJSM*, 11(2):149–187, 2005.
[10] A. Cerri, M. Ferri, and D. Giorgi. Retrieval of trademark images by means of size functions. *Graph. Models*, 68(5):451–471, 2006.
[11] A. Cerri and P. Frosini. Necessary conditions for discontinuities of multidimensional size functions. *CoRR*, abs/0811.1868, 2008.
[12] A. Cerri and P. Frosini. Re-parameterization invariance of the multidimensional matching distance in topological persistence. Preprint, 2009.
[13] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Stability of persistence diagrams. In J. S. B. Mitchell and G. Rote, editors, *Symposium on Computational Geometry*, pages 263–271. ACM, 2005.
[14] D. Cohen-Steiner, H. Edelsbrunner, and D. Morozov. Vines and vineyards by updating persistence in linear time. In *SCG '06: Proceedings of the twenty-second annual symposium on Computational geometry*, pages 119–126, New York, NY, USA, 2006. ACM.
[15] M. d'Amico, P. Frosini, and C. Landi. optimal matching between reduced size functions. Technical Report no. 35, DISMI, Università di Modena e Reggio Emilia, 2003.
[16] M. d'Amico, P. Frosini, and C. Landi. Using matching distance in size theory: A survey. *Int. J. Imag. Syst. Tech.*, 16(5):154–161, 2006.
[17] M. d'Amico, P. Frosini, and C. Landi. Natural pseudo-distance and optimal matching between reduced size functions. *Acta. Appl. Math.*, published online–in press.
[18] V. de Silva and R. Ghrist. Coverage in sensor networks via persistent homology. *Algebr. Geom. Topol.*, 7:339–358, 2007.
[19] F. Dibos, P. Frosini, and D. Pasquignon. The use of size functions for comparison of shapes through differential invariants. *J. Math. Imaging Vis.*, 21(2):107–118, 2004.

[20] H. Edelsbrunner and J. Harer. Persistent homology—a survey. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 257–282. Amer. Math. Soc., Providence, RI, 2008.

[21] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological persistence and simplification. *Discrete Comput. Geom.*, 28(4):511–533, 2002.

[22] P. Frosini and C. Landi. Size theory as a topological tool for computer vision. *Pattern Recogn. and Image Anal.*, 9:596–603, 1999.

[23] P. Frosini and C. Landi. Size functions and formal series. *Appl. Algebra Engrg. Comm. Comput.*, 12(4):327–349, 2001.

[24] P. Frosini and M. Mulazzani. Size homotopy groups for computation of natural size distances. *Bulletin of the Belg. Math. Soc.*, 6(3):455–464, 1999.

[25] R. Ghrist. Barcodes: the persistent topology of data. *Bull. Amer. Math. Soc. (N.S.)*, 45(1):61–75 (electronic), 2008.

[26] D. Giorgi, S. Biasotti, and L. Paraboschi. Watertight models track, 2007.

[27] M. Hilaga, Y. Shinagawa, T. Kohmura, and T. L. Kunii. Topology matching for fully automatic similarity estimation of 3D shapes. In *SIGGRAPH '01: Proceedings of the* 28^{th} *Annual Conference on Computer Graphics and Interactive Techniques*, pages 203–212, Los Angeles, CA, 2001. ACM Press.

[28] C. Uras and A. Verri. Computing size functions from edge maps. *Int. J. Comput. Vision*, 23(2):169–183, 1997.

[29] A. Verri, C. Uras, P. Frosini, and M. Ferri. On the use of size functions for shape analysis. *Biol. Cybern.*, 70:99–107, 1993.

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