STABILITY OF REEB GRAPHS UNDER FUNCTION PERTURBATIONS: THE CASE OF CLOSED CURVES

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ABSTRACT. Reeb graphs provide a method for studying the shape of a manifold by encoding the evolution and arrangement of level sets of a simple Morse function defined on the manifold. Since their introduction in computer graphics they have been gaining popularity as an effective tool for shape analysis and matching. In this context one question deserving attention is whether Reeb graphs are robust against function perturbations. Focusing on 1-dimensional manifolds, we define an editing distance between Reeb graphs of curves, in terms of the cost necessary to transform one graph into another. Our main result is that changes in Morse functions induce smaller changes in the editing distance between Reeb graphs of curves, implying stability of Reeb graphs under function perturbations.

INTRODUCTION

The shape similarity problem has since long been studied by the computer vision community for dealing with shape classification and retrieval tasks. It is now attracting more and more attention also in the computer graphics community where recent improvements in object acquisition and construction of digital models are leading to an increasing accumulation of models in large databases of shapes. Comparison of 2D images is often dealt with considering just the silhouette or contour curve of the studied object, encoding shape properties, such as curvature, in compact representations of shapes, namely, shape descriptors, for the comparison. The same approach is more and more used also in computer graphics where there has been a gradual shift of research interests from methods of representing shapes toward methods of describing shapes of 3D models.

Since [24], Reeb graphs have been gaining popularity as an effective tool for shape analysis and description tasks as a consequence of their ability to extract high-level features from 3D models. Reeb graphs were originally defined by Georges Reeb in 1946 as topological constructs [22]. Given a manifold $\mathscr{M}$ and a generic enough real-valued function f defined on $\mathscr{M}$, the simplicial complex defined by Reeb, conventionally called the Reeb graph of $(\mathscr{M}, f)$, is the quotient space defined by the equivalence relation that identifies the points of $\mathscr{M}$ belonging to the same connected component of level sets of f. Reeb graphs effectively code shapes, both from a topological and a geometrical perspective. While the topology is described by the connectivity of the graph, the geometry can be coded in a variety of different ways, according to the type of applications the Reeb graph is devised for, simply by changing the function f. Different choices of the function yield insights into the manifold from different perspectives. The compactness of the one-dimensional structure, the natural link between the function and the shape, and the possibility of adopting different functions for describing different aspects of shapes and imposing the desired invariance properties, have led to a great interest in the use of Reeb graphs for similarity evaluation. In

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[15], Hilaga *et al.* use Multiresolution Reeb Graphs based on the distribution of geodesic distance between two points as a search key for 3D objects, and the similarity measure constructed in this setting is found to be resistant to noise. In this approach resistance to changes caused by noise essentially relies on the choice of the geodesic distance to build the Reeb graph. In [4], Biasotti *et al.* base the comparison of Extended Reeb Graphs on a relaxed version of the notion of best common subgraph. This approach gives a method for partial shape-matching able to recognize sub-parts of objects, and can be adapted to the context of applications since there is no requirement on the choice of the function f. Both [15] and [4] present algorithms for similarity evaluation.

To the best of our knowledge, mathematical assessment of stability against function perturbations is still an open issue as far as Reeb graphs are concerned. This question deserves attention since it is clear that any data acquisition is subject to perturbations, noise and approximation errors and, if Reeb graphs were not stable, then distinct computational investigations of the same object could produce completely different results. This paper aims to be possibly the first positive answer to this question.

We confine ourselves to consider Reeb graphs of curves. In this setting Reeb graphs are simply cycle graphs with an even number of vertices corresponding alternatively to the maxima and minima of the function. We also equip vertices of Reeb graphs with the value taken by the function at the corresponding critical points.

Our main contribution is the construction of a distance between Reeb graphs of curves such that changes in functions imply smaller changes in the distance. Our distance is based on an adaptation of the well-known notion of editing distance between graphs [25]. We introduce three basic types of editing operations, represented in Table 1, corresponding to the insertion (birth) of a new pair of adjacent points of maximum and minimum, the deletion (death) of such a pair, and the relabelling of the vertices. A cost is associated with each of these operations and our distance is given by the infimum of the costs necessary to transform a graph into another by using these editing operations. Our main result is the global stability of labelled Reeb graphs under function perturbations (Theorem 6.3):

MAIN RESULT. *Let $f,g : S^1 \to \mathbb{R}$ be two simple Morse functions. Then the editing distance between the labelled Reeb graph of (S^1, f) and that of (S^1, g) is always smaller or equal to the C^2-norm of $f - g$.*

The main idea of the proof is to read editing operations in terms of degenerate strata crossings of the space of smooth functions stratified as in [6]. We also obtain a lower bound for our editing distance. Indeed, we find that it can be estimated from below by the natural pseudo-distance between closed curves studied in [13].

The paper is organized as follows. In Section 1, we review some of the standard facts about Morse functions, the C^r topology, the theory of stratification of smooth real valued functions, and Reeb graphs. Section 2 deals with basic properties of labelled Reeb graphs of closed curves. Section 3 is devoted to the definition of the admissible deformations transforming a Reeb graph into another, the cost associated with each kind of deformation, and the definition of an editing distance in terms of this cost. Section 4 is intended to provide a suitable lower bound for our distance, the natural pseudo-distance; this represents a useful tool both to show the well-definiteness of our distance and to compute it in some simple cases. In Sections 5 and 6 it is shown that our distance is both locally and globally upper bounded by the difference, measured in the C^2-norm, between the functions defined on S^1. Eventually, a brief discussion on the results obtained concludes the paper.

1. Preliminary notions

In this section we recall some basic definitions and results about Morse functions and Reeb graphs. Moreover, with the aim of proving stability of Reeb graphs under function perturbations in mind, we recall some concepts concerning the space of smooth real valued functions on a smooth manifold: the C^r topology and the theory of the natural stratification.

Throughout the paper, $\mathscr{M}$ denotes a smooth (i.e. differentiable of class C^∞) compact n-manifold without boundary, and $\mathscr{F}(\mathscr{M},\mathbb{R})$ the set of smooth real functions on $\mathscr{M}$.

1.1. Simple Morse functions.

Let us recall the following concepts from [19].

Let $f \in \mathscr{F}(\mathscr{M},\mathbb{R})$. A point $p \in \mathscr{M}$ is called a *critical point* of f if, choosing a local coordinate system $(x_1,\ldots,x_n)$ in a neighborhood U of p, it holds that

$$\frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0,$$

and it is called a *regular point*, otherwise. Throughout the paper, we set $K(f) = \{p \in \mathscr{M} : p \text{ is a critical point of } f\}$.

If $p \in K(f)$, then the real number $f(p)$ is called a *critical value* of f, and the set $\{q \in \mathscr{M} : q \in f^{-1}(f(p))\}$ is called a *critical level* of f. Otherwise, if $p \notin K(f)$, then $f(p)$ is called a *regular value*. Moreover, a critical point p is called *non-degenerate* if and only if the second derivative matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)$$

is non-singular, i.e. its determinant is not zero.

By the well-known Morse Lemma, in a neighborhood of a non-degenerate critical point p, it is possible to choose a local coordinate system $(x_1,\ldots,x_n)$ such that

$$f = f(p) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2.$$

The number k is uniquely defined for each critical point p and is called the *index* of p. Such an index completely describes the behavior of f at p. For example, $k=0$ means that the corresponding p is a minimum for f; $k=n$ means that p is a maximum; $0<k<n$ means that p is a saddle point for f.

Definition 1.1. A function $f \in \mathscr{F}(\mathscr{M},\mathbb{R})$ is called a *Morse function* if all its critical points are non-degenerate. Moreover, a Morse function is said to be *simple* if each critical level contains exactly one critical point.

It is well-known that every Morse function has only finitely many critical points (which are therefore certainly isolated points). The importance of non-degeneracy is that it is the common situation; indeed, in a sense that will be explained in Subsection 1.3, the occurrence of degenerate critical points is really quite rare.

1.2. The C^r topology on the space of real valued functions.

To topologize $\mathscr{F}(\mathscr{M},\mathbb{R})$, let us recall the definition of C^r-norm, with $0 \le r < \infty$ (see, e. g., [20, 21]). Let $\{U_\alpha\}$ be a finite coordinate covering of $\mathscr{M}$, with coordinate maps $h_\alpha : U_\alpha \to \mathbb{R}^n$, and consider a compact refinement $\{C_\alpha\}$ of $\{U_\alpha\}$ (i.e. $C_\alpha \subseteq U_\alpha$ for each α, and $\bigcup C_\alpha = \mathscr{M}$). For $f \in \mathscr{F}(\mathscr{M},\mathbb{R})$, let us set $f_\alpha = f \circ h_\alpha^{-1} : h_\alpha(C_\alpha) \to \mathbb{R}$. Then the *$C^r$-norm* of f is defined as

$$\|f\|_{C^r} = \max_\alpha \left\{ \max_{u \in h_\alpha(C_\alpha)} |f_\alpha(u)|, \max_{\substack{u \in h_\alpha(C_\alpha)\\ j \in \{1,\ldots,n\}}} \left|\frac{\partial f_\alpha}{\partial u_j}(u)\right|, \ldots, \max_{\substack{u \in h_\alpha(C_\alpha)\\ j_1,\ldots,j_r \in \{1,\ldots,n\}}} \left|\frac{\partial^r f_\alpha}{\partial u_{j_1} \cdots \partial u_{j_r}}(u)\right| \right\}.$$

The above norm defines a topology on $\mathscr{F}(\mathscr{M},\mathbb{R})$, known as the C^r *topology* (or *weak topology*), with $0 \le r < \infty$ (cf. [16, chap. 2]). In the following, we will denote by $B_r(f,\delta)$, $0 \le r < \infty$, the open ball with center f and radius δ in the C^r topology, i.e., $g \in B_r(f,\delta)$ if and only if $\|f-g\|_{C^r} < \delta$. The C^∞ *topology* is simply the union of the C^r topologies on $\mathscr{F}(\mathscr{M},\mathbb{R})$ for every $0 \le r < \infty$.

1.3. **Natural stratification of the space of real valued functions.** Let us endow $\mathscr{F}(\mathscr{M},\mathbb{R})$ with the C^∞ topology, and consider the *natural stratification* of such a space, as exposed by Cerf in [6] (see also [23]). The natural stratification is defined as a sequence of sub-manifolds of $\mathscr{F}(\mathscr{M},\mathbb{R})$, $\mathscr{F}^0, \mathscr{F}^1, \ldots, \mathscr{F}^j, \ldots$, of co-dimension $0,1,\ldots,j,\ldots$, respectively, that constitute a partition of $\mathscr{F}(\mathscr{M},\mathbb{R})$, and such that the disjoint union $\mathscr{F}^0 \cup \mathscr{F}^1 \cup \ldots \cup \mathscr{F}^j$ is open for every j.

Before providing a brief description of the strata, let us recall the following equivalence relation that can be defined on $\mathscr{F}(\mathscr{M},\mathbb{R})$.

Definition 1.2. Two functions $f,g \in \mathscr{F}(\mathscr{M},\mathbb{R})$ are called *topologically equivalent* if there exists a diffeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ and an orientation preserving diffeomorphism $\eta : \mathbb{R} \to \mathbb{R}$ such that $g(\xi(p)) = \eta(f(p))$ for every $p \in \mathscr{M}$.

The above relation is also known as *isotopy* in [6], and *left-right equivalence* in [3].

Let us describe $\mathscr{F}^0$ and $\mathscr{F}^1$, pointing out their main properties that allow us to leave aside the remaining strata.

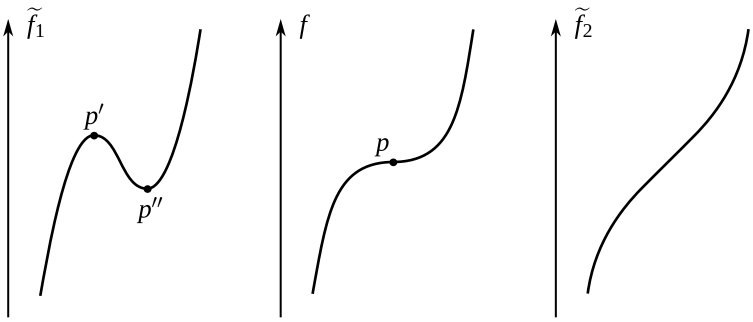
- The stratum $\mathscr{F}^0$ is the set of simple Morse functions.
- The stratum $\mathscr{F}^1$ is the disjoint union of two sets $\mathscr{F}^1_\alpha$ and $\mathscr{F}^1_\beta$ open in $\mathscr{F}^1$, where
  - $\mathscr{F}^1_\alpha$ is the set of functions whose critical levels contain exactly one critical point, and the critical points are all non-degenerate, except exactly one. In a neighborhood of such a point, say p, a local coordinate system $(x_1,\ldots,x_n)$ can be chosen such that
  $$f = f(p) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_{n-1}^2 + x_n^3.$$
  - $\mathscr{F}^1_\beta$ is the set of Morse functions whose critical levels contain at most one critical point, except for one level containing exactly two critical points.

$\mathscr{F}^0$ is dense in the space $\mathscr{F}(\mathscr{M},\mathbb{R})$ endowed with the C^r topology, $2 \le r \le \infty$ (cf. [16, chap. 6, Thm. 1.2]). Therefore, any smooth function can be turned into a simple Morse function by arbitrarily small perturbations. Degenerate critical points can be split into several non-degenerate singularities, with all different critical values (Figure 1 (a)). Moreover, when more than one critical points occur at the same level, they can be moved to close but different levels (Figure 1 (b)).

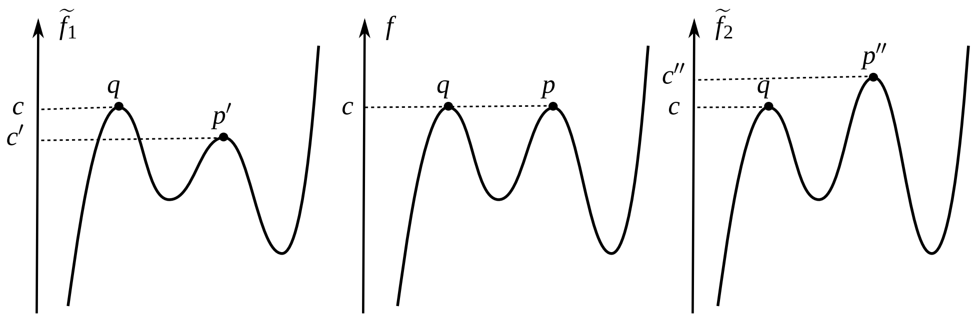
It is well-known that two simple Morse functions are topologically equivalent if and only if they belong to the same arcwise connected component (or *co-cellule*) of $\mathscr{F}^0$ [6, p. 25].

$\mathscr{F}^1$ is a sub-manifold of co-dimension 1 of $\mathscr{F}^0 \cup \mathscr{F}^1$, and the complement of $\mathscr{F}^0 \cup \mathscr{F}^1$ in $\mathscr{F}$ is of co-dimension greater than 1. Consequently, given two functions $f,g \in \mathscr{F}^0$, we can always find $\widehat{f},\widehat{g} \in \mathscr{F}(\mathscr{M},\mathbb{R})$ arbitrarily near to f,g, respectively, for which the path $h(\lambda) = (1-\lambda)\widehat{f} + \lambda\widehat{g}$, with $\lambda \in [0,1]$, is such that

(1) $\widehat{f},\widehat{g} \in \mathscr{F}^0$, and $\widehat{f}$, $\widehat{g}$ are topologically equivalent to f, g, respectively;
(2) $h(\lambda)$ belongs to $\mathscr{F}^0 \cup \mathscr{F}^1$ for every $\lambda \in [0,1]$;
(3) $h(\lambda)$ is transversal to $\mathscr{F}^1$.



(a)



(b)

FIGURE 1. (a) A function $f \in \mathscr{F}^1_\alpha$ admitting a degenerate critical point p (center) can be perturbed into a simple Morse function $\widetilde{f}_1$ with two non-degenerate critical points p', p'' (left), or into a simple Morse function $\widetilde{f}_2$ without critical points around p (right); (b) a function $f \in \mathscr{F}^1_\beta$ (center) can be turned into two simple Morse functions $\widetilde{f}_1, \widetilde{f}_2$, that are not topologically equivalent (left-right).

As a consequence, $h(\lambda)$ belongs to $\mathscr{F}^1$ for at most a finite collection of values λ, and does not traverse strata of co-dimension greater than 1 (see, e.g., [14]).

1.4. **Reeb graph of a manifold.** In this subsection we restate the main results concerning Reeb graphs, starting from the following one shown by Reeb in [22]. Here we consider pairs $(\mathscr{M}, f)$, with $\mathscr{M}$ connected and $f \in \mathscr{F}^0 \subset \mathscr{F}(\mathscr{M}, \mathbb{R})$.

Theorem 1.3. *The quotient space of $\mathscr{M}$ under the equivalence relation "p and q belong to the same connected component of the same level set of f" is a finite and connected simplicial complex of dimension 1.*

This simplicial complex, denoted by Γ_f, is called the *Reeb graph* associated with the pair $(\mathscr{M}, f)$. Its vertex set will be denoted by $V(\Gamma_f)$, and its edge set by $E(\Gamma_f)$. Moreover, if $v_1, v_2 \in V(\Gamma_f)$ are adjacent vertices, i.e., connected by an edge, we will write $e(v_1, v_2) \in E(\Gamma_f)$. Since the vertices of a Reeb graph correspond in a one to one manner to critical points of f on the manifold $\mathscr{M}$ (see, e.g., [5, Lemma 2.1]), we will often identify each $v \in V(\Gamma_f)$ with the corresponding $p \in K(f)$.

Given two topologically equivalent functions $f, g \in \mathscr{F}^0$, it is well-known that the associated Reeb graphs, Γ_f and Γ_g, are isomorphic graphs, i.e., there exists an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$. Beyond that, an even stronger result holds. Two functions

$f, g \in \mathscr{F}^0$ are topologically equivalent if and only if such a bijection Φ also preserves the vertices order, i.e., for every $v, w \in V(\Gamma_f)$, $f(v) < f(w)$ if and only if $g(\Phi(v)) < g(\Phi(w))$.

The preceding result has been used by Arnold in [2] to classify simple Morse functions up to the topological equivalence relation.

2. Labelled Reeb graphs of closed curves

This paper focuses on *Reeb graphs* of closed curves. Hence, the manifold $\mathscr{M}$ that will be considered from now on is S^1, and the function f will be taken in $\mathscr{F}^0 \subset \mathscr{F}(S^1, \mathbb{R})$. The Reeb graph Γ_f associated with (S^1, f) is a cycle graph on an even number of vertices, corresponding, alternatively, to the minima and maxima of f on S^1 [21] (see, for example, Figure 2 $(a)-(b)$). Furthermore, we label the vertices of Γ_f, by equipping each of them with the value of f at the corresponding critical point. We denote such a labelled graph by $(\Gamma_f, f_{|})$, where $f_{|} : V(\Gamma_f) \to \mathbb{R}$ is the restriction of $f : S^1 \to \mathbb{R}$ to $K(f)$. A simple example is displayed in Figure 2 $(a)-(c)$. To facilitate the reader, in all figures of this paper we shall adopt the convention of representing f as the height function, so that $f_{|}(v_a) < f_{|}(v_b)$ if and only if v_a is lower than v_b in the picture.

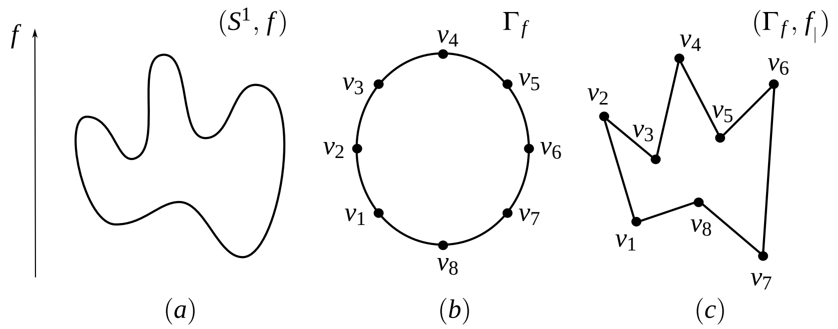


FIGURE 2. (a) A pair (S^1, f), with f the height function; (b) the Reeb graph Γ_f associated with (S^1, f); (c) the labelled Reeb graph $(\Gamma_f, f_{|})$ associated with (S^1, f). Here labels are represented by the heights of the vertices.

The natural definition of isomorphism between labelled Reeb graphs is the following one.

Definition 2.1. We shall say that two labelled Reeb graphs $(\Gamma_f, f_{|}), (\Gamma_g, g_{|})$ are *isomorphic* if there exists an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ such that $f_{|}(v) = g_{|}(\Phi(v))$ for every $v \in V(\Gamma_f)$.

The following Proposition 2.4 provides a necessary and sufficient condition in order that two labelled Reeb graphs are isomorphic. It is based on the next definition of re-parameterization equivalent functions.

Definition 2.2. Let $\mathscr{H}(S^1)$ be the set of homeomorphisms on S^1. We shall say that two functions $f, g \in \mathscr{F}^0 \subset \mathscr{F}(S^1, \mathbb{R})$ are *re-parameterization equivalent* if there exists $\tau \in \mathscr{H}(S^1)$ such that $f(p) = g(\tau(p))$ for every $p \in S^1$.

Lemma 2.3. *Let $(\Gamma_f, f_{|})$ and $(\Gamma_g, g_{|})$ be labelled Reeb graphs associated with (S^1, f) and (S^1, g), respectively. If an edge-preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ exists, then there also exists a piecewise linear $\tau \in \mathscr{H}(S^1)$ such that $\tau_{|V(\Gamma_f)} = \Phi$. If moreover $f_{|} = g_{|} \circ \Phi$, then $f = g \circ \tau$.*

Proof. The proof of the first statement is inspired by [13, Lemma 4.2]. Let us construct τ by extending Φ to S^1 as follows. Let us recall that $V(\Gamma_f) = K(f)$ and $V(\Gamma_g) = K(g)$, and, by abuse of notation, for every pair of adjacent vertices $p', p'' \in V(\Gamma_f)$, let us identify the edge $e(p', p'') \in E(\Gamma_f)$ with the arc of S^1 having endpoints p' and p'', and not containing any other critical point of f. For every $p \in K(f)$, let $\tau(p) = \Phi(p)$. Now, let us define $\tau(p)$ for every $p \in S^1 \setminus K(f)$. Given $p \in S^1 \setminus K(f)$, we observe that there always exist $p', p'' \in V(\Gamma_f)$ such that $p \in e(p', p'')$. Since Φ is edge-preserving, there exists $e(\Phi(p'), \Phi(p'')) = e(\tau(p'), \tau(p'')) \in E(\Gamma_g)$. Hence, we can define $\tau(p)$ as the unique point of $e(\tau(p'), \tau(p''))$ such that, if $f(p) = (1-\lambda_p) f(p') + \lambda_p f(p'')$, with $\lambda_p \in [0,1]$, then $g(\tau(p)) = (1-\lambda_p) g(\tau(p')) + \lambda_p g(\tau(p''))$. Clearly, τ belongs to $\mathscr{H}(S^1)$ and is piecewise linear.

As for the second statement, it is sufficient to observe that, if $f_| = g_| \circ \Phi$, since $\tau(p) = \Phi(p)$ for every $p \in K(f)$, then clearly $f_|(p) = g_|(\tau(p))$ for every $p \in K(f)$. Moreover, for every $p \in S^1 \setminus K(f)$, by the construction of τ, it holds that $g(\tau(p)) = (1-\lambda_p) g(\Phi(p')) + \lambda_p g(\Phi(p'')) = (1-\lambda_p) f(p') + \lambda_p f(p'') = f(p)$. In conclusion, $f(p) = g(\tau(p))$ for every $p \in S^1$, and, hence, f, g are re-parameterization equivalent.

□

Proposition 2.4 (Uniqueness theorem)**.** *Let $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$ be labelled Reeb graphs associated with (S^1, f) and (S^1, g), respectively. Then $(\Gamma_f, f_|)$ is isomorphic to $(\Gamma_g, g_|)$ if and only if f and g are re-parameterization equivalent.*

Proof. The direct statement is a trivial consequence of Lemma 2.3.

As for the converse statement, it is sufficient to observe that any $\tau \in \mathscr{H}(S^1)$ such that $f = g \circ \tau$, as well as its inverse τ^{-1}, takes the minima of f to the minima of g and the maxima of f to the maxima of g. Hence, $\Phi : V(\Gamma_f) \to V(\Gamma_g)$, with $\Phi = \tau_{|V(\Gamma_f)}$, is an edge preserving bijection such that $f_| = g_| \circ \Phi$. □

As a consequence of Proposition 2.4, two labelled Reeb graphs isomorphic in the sense of Definition 2.1 will always be identified, and in such case we will simply write $(\Gamma_f, f_|) = (\Gamma_g, g_|)$.

The following Proposition 2.5 ensures that, for every cycle graph with an appropriate vertices labelling, there exists a unique (up to re-parameterization) pair (S^1, f), with $f \in \mathscr{F}^0$, having such a graph as the associated labelled Reeb graph.

Proposition 2.5 (Realization theorem)**.** *Let (G, ℓ) be a labelled graph, where G is a cycle graph on an even number of vertices, and $\ell : V(G) \to \mathbb{R}$ is an injective function such that, for any vertex v_2 adjacent (that is connected by an edge) to the vertices v_1 and v_3, either both $\ell(v_1)$ and $\ell(v_3)$ are smaller than $\ell(v_2)$, or both $\ell(v_1)$ and $\ell(v_3)$ are greater than $\ell(v_2)$. Then there exists a simple Morse function $f : S^1 \to \mathbb{R}$ such that $(\Gamma_f, f_{|K(f)}) = (G, \ell)$.*

Proof. It is evident. □

By virtue of the above Uniqueness and Realization theorems (Propositions 2.4 and 2.5), for conciseness, when a labelled Reeb graph will be introduced in the sequel, the associated pair will be often omitted.

3. EDITING DISTANCE BETWEEN LABELLED REEB GRAPHS

We now define the editing deformations admissible to transform a labelled Reeb graph of a closed curve into another. We introduce at first elementary deformations and then the deformations obtained by their composition. Next, we associate a cost with each type of deformation, and define a distance between labelled Reeb graphs in terms of such a cost.

Definition 3.1. Let $(\Gamma_f, f_|)$ be a labelled Reeb graph with $2n$ vertices, $n \geq 1$. We call an *elementary deformation* of $(\Gamma_f, f_|)$ any of the following transformations:

(B) (Birth): Assume $e(v_1, v_2) \in E(\Gamma_f)$ with $f_|(v_1) < f_|(v_2)$. Then $(\Gamma_f, f_|)$ is transformed into a labelled graph (G, ℓ) according to the following rule: G is the new graph on $2n+2$ vertices, obtained deleting the edge $e(v_1, v_2)$ and inserting two new vertices u_1, u_2 and the edges $e(v_1,u_1), e(u_1,u_2), e(u_2,v_2)$; moreover, $\ell : V(G) \to \mathbb{R}$ is defined by extending $f_|$ from $V(\Gamma_f)$ to $V(G) = V(\Gamma_f) \cup \{u_1, u_2\}$ in such a way that $\ell_{|V(\Gamma_f)} \equiv f_|$, and $f_|(v_1) < \ell(u_2) < \ell(u_1) < f_|(v_2)$.

(D) (Death): Assume $n \geq 2$, and $e(v_1,u_1), e(u_1,u_2), e(u_2,v_2) \in E(\Gamma_f)$, with $f_|(v_1) < f_|(u_2) < f_|(u_1) < f_|(v_2)$. Then $(\Gamma_f, f_|)$ is transformed into a labelled graph (G, ℓ) according to the following rule: G is the new graph on $2n-2$ vertices, obtained deleting u_1, u_2 and the edges $e(v_1,u_1)$, $e(u_1,u_2)$, $e(u_2,v_2)$, and inserting an edge $e(v_1,v_2)$; moreover, $\ell : V(G) \to \mathbb{R}$ is defined as the restriction of $f_|$ to $V(\Gamma_f) \setminus \{u_1,u_2\}$.

(R) (Relabelling): $(\Gamma_f, f_|)$ is transformed into a labelled graph (G, ℓ) according to the following rule: $G = \Gamma_f$, and for any vertex v_2 adjacent to the vertices v_1 and v_3 (possibly $v_1 \equiv v_3$ for $n = 1$), if both $f_|(v_1)$ and $f_|(v_3)$ are smaller (greater, respectively) than $f_|(v_2)$, then both $\ell(v_1)$ and $\ell(v_3)$ are smaller (greater, respectively) than $\ell(v_2)$; moreover, for every $v \neq w$, $\ell(v) \neq \ell(w)$.

We shall denote by $T(\Gamma_f, f_|)$ the result of the elementary deformation T applied to $(\Gamma_f, f_|)$.

Table 1 schematically illustrates the elementary deformations described in Definition 3.1.

Proposition 3.2. *Let T be an elementary deformation of $(\Gamma_f, f_|)$, and let $(G, \ell) = T(\Gamma_f, f_|)$. Then (G, ℓ) is a Reeb graph $(\Gamma_g, g_|)$ associated with a pair (S^1, g), and $g \in \mathscr{F}^0$ is unique up to re-parameterization equivalence.*

Proof. The claim follows from Propositions 2.5 and 2.4. □

As a consequence of the above result, from now on, we will directly write $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$.

Moreover, since the previous Proposition 3.2 shows that an elementary deformation of a labelled Reeb graph is still a labelled Reeb graph, we can also apply elementary deformations iteratively. This fact is used in the next Definition 3.3.

Given an elementary deformation T of $(\Gamma_f, f_|)$ and an elementary deformation S of $T(\Gamma_f, f_|)$, the juxtaposition ST means applying first T and then S.

Definition 3.3. We shall call *deformation* of $(\Gamma_f, f_|)$ any finite ordered sequence $T = (T_1, T_2, \ldots, T_r)$ of elementary deformations such that T_1 is an elementary deformation of $(\Gamma_f, f_|)$, T_2 is an elementary deformation of $T_1(\Gamma_f, f_|)$, ..., T_r is an elementary deformation of $T_{r-1}T_{r-2}\cdots T_1(\Gamma_f, f_|)$. We shall denote by $T(\Gamma_f, f_|)$ the result of the deformation T applied to $(\Gamma_f, f_|)$.

Let us define the cost of a deformation.

Definition 3.4. Let T be an elementary deformation transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$.

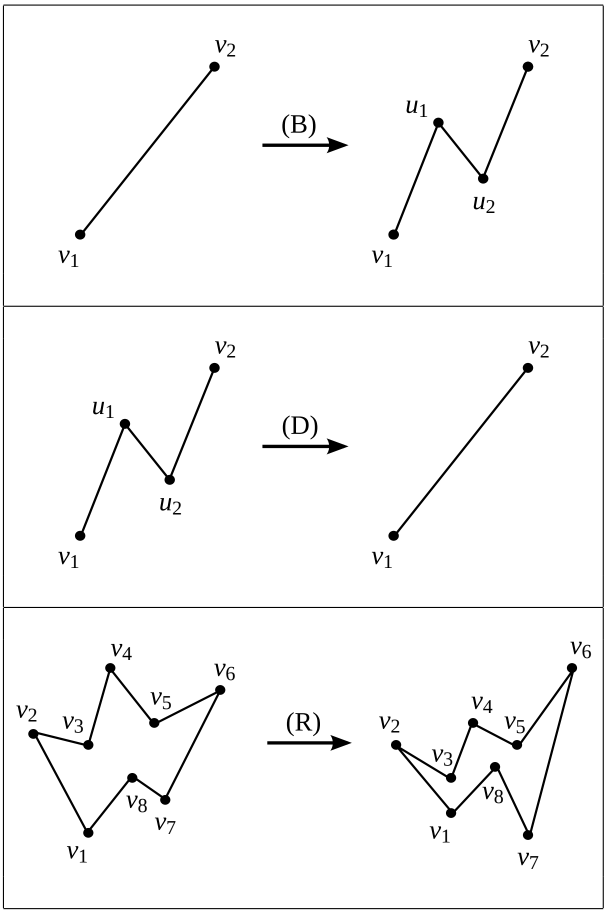


TABLE 1. The upper two figures schematically show the elementary deformations of type (B) and (D), respectively; the third figure shows an example of elementary deformation of type (R).

- If T is of type (B) inserting the vertices $u_1, u_2 \in V(\Gamma_g)$, then we define the associated cost as
$$c(T) = \frac{|g_{|}(u_1) - g_{|}(u_2)|}{2};$$
- If T is of type (D) deleting the vertices $u_1, u_2 \in V(\Gamma_f)$, then we define the associated cost as
$$c(T) = \frac{|f_{|}(u_1) - f_{|}(u_2)|}{2};$$
- If T is of type (R) relabelling the vertices $v \in V(\Gamma_f) = V(\Gamma_g)$, then we define the associated cost as
$$c(T) = \max_{v \in V(\Gamma_f)} |f_{|}(v) - g_{|}(v)|.$$

Moreover, if $T = (T_1, \dots, T_r)$ is a deformation such that $T_r \cdots T_1(\Gamma_f, f_|) = (\Gamma_g, g_|)$, we define the associated cost as $c(T) = \sum_{i=1}^{r} c(T_i)$.

We now introduce the concept of inverse deformation.

Definition 3.5. Let T be a deformation such that $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$. Then we denote by T^{-1}, and call it the *inverse* of T, the deformation such that $T^{-1}(\Gamma_g, g_|) = (\Gamma_f, f_|)$ defined as follows:

- If T is elementary of type (B) inserting two vertices, then T^{-1} is of type (D) deleting the same vertices;
- If T is elementary of type (D) deleting two vertices, then T^{-1} is of type (B) inserting the same vertices, with the same labels;
- If T is elementary of type (R) relabelling vertices of $V(\Gamma_f)$, then T^{-1} is again of type (R) relabelling these vertices in the inverse way;
- If $T = (T_1, \dots, T_r)$, then $T^{-1} = (T_r^{-1}, \dots, T_1^{-1})$.

Proposition 3.6. *For every deformation T such that $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$, $c(T^{-1}) = c(T)$.*

Proof. Trivial. □

We prove that, for every two labelled Reeb graphs, a finite number of elementary deformations always allows us to transform any of them into the other one. We recall that we identify labelled Reeb graphs that are isomorphic according to Definition 2.1. We first need a lemma, stating that in any labelled Reeb graph with at least four vertices we can find two adjacent vertices that can be deleted.

Lemma 3.7. *Let $(\Gamma_f, f_|)$ be a labelled Reeb graph with at least four vertices. Then there exist $e(v_1,u_1), e(u_1,u_2), e(u_2,v_2) \in E(\Gamma_f)$, with $f_|(v_1) < f_|(u_2) < f_|(u_1) < f_|(v_2)$.*

Proof. Let $V(\Gamma_f) = \{a_0, b_0, a_1, b_1, \dots a_{m-1}, b_{m-1}\}$, $m \geq 2$. In the following, we convene that, for $k \in \mathbb{Z}$, a_k and b_k are equal to $a_{(k \mod m)}$ and $b_{(k \mod m)}$, respectively. We assume that $E(\Gamma_f) = \{e(a_i, b_i) : i \geq 0\} \cup \{e(b_i, a_{i+1}) : i \geq 0\}$, and $f_|(a_i) < f_|(b_i)$ for every i. From the definition of labelled Reeb graph associated with a pair (S^1, f), it follows that $f_|(b_i) > f_|(a_{i+1})$, $f_|(a_i) \neq f_|(a_{i+1})$, $f_|(b_i) \neq f_|(b_{i+1})$, for every i.

The claim can be restated saying that there is at least one index i such that either (I) $f_|(a_i) < f_|(a_{i+1})$ and $f_|(b_i) < f_|(b_{i+1})$ or (II) $f_|(a_{i+1}) < f_|(a_i)$ and $f_|(b_i) < f_|(b_{i-1})$ hold. We prove this statement by contradiction, assuming that for every $i \geq 0$ neither (I) nor (II) hold. Since (I) does not hold, either $f_|(a_0) > f_|(a_1)$ or $f_|(b_0) > f_|(b_1)$ or both. Let us consider the case when $f_|(b_0) > f_|(b_1)$. Since (II) does not hold either, it follows that $f_|(a_2) > f_|(a_1)$. Recalling that (I) does not hold, we obtain $f_|(b_1) > f_|(b_2)$. Iterating the same argument, we deduce that $f_|(b_i) > f_|(b_i + 1)$ for every $i \geq 0$, contradicting the fact that $b_m = b_0$. An analogous proof works when we consider the case $f_|(a_0) > f_|(a_1)$. □

Proposition 3.8. *Let $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$ be two labelled Reeb graphs. Then the set of all the deformations T such that $T(\Gamma_f, f_|) = (\Gamma_g, g_|)$ is non-empty. This set of deformations will be denoted by $\mathscr{T}((\Gamma_f, f_|), (\Gamma_g, g_|))$.*

Proof. If $(\Gamma_f, f_|) = (\Gamma_g, g_|)$, then it is sufficient to take the elementary deformation T of type (R) transforming $(\Gamma_f, f_|)$ into itself. Otherwise, if $(\Gamma_f, f_|) \neq (\Gamma_g, g_|)$ and Γ_f has at least four vertices, by Lemma 3.7, we can apply a finite sequence of elementary deformations of type (D) to $(\Gamma_f, f_|)$, so that in the resulting labelled Reeb graph $(\Gamma_h, h_|)$, Γ_h has only two vertices, say u, v, with $h_|(u) < h_|(v)$. If $(\Gamma_g, g_|)$ has also at least four vertices, by Lemma 3.7, there exists a finite sequence of elementary deformations of type (D) to $(\Gamma_g, g_|)$, say $S = (S_1, \dots, S_p)$, so that in the resulting labelled Reeb graph $(\Gamma_{h'}, h'_|)$, $\Gamma_{h'}$ has only two vertices, say u', v', with $h'_|(u') < h'_|(v')$. So, we can apply to $(\Gamma_h, h_|)$ an elementary deformation of type (R) so to obtain $(\Gamma_{h'}, h'_|)$. Finally, by Definition 3.5, we can apply to $(\Gamma_{h'}, h'_|)$ the finite sequence of elementary inverse deformations of type (B), $S^{-1} = (S_p^{-1}, \dots, S_1^{-1})$, in order to obtain $(\Gamma_g, g_|)$. For $(\Gamma_f, f_|)$ or $(\Gamma_g, g_|)$ with only two vertices, the same proof applies without need of deformations of type (D) or (B), respectively. □

A simple example explaining the above proof is given in Figure 3.

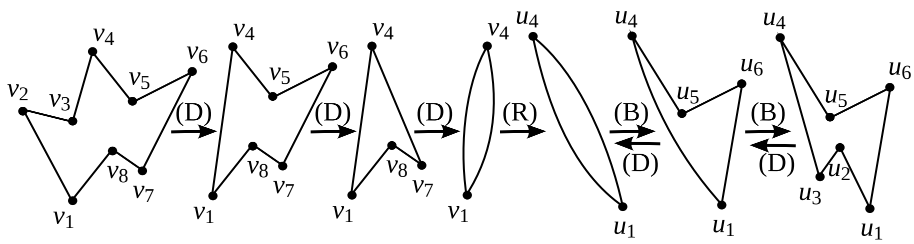


FIGURE 3. The leftmost labelled Reeb graph is transformed into the rightmost one applying first three elementary deformations of type (D), then one elementary deformation of type (R), and finally two elementary deformations of type (B).

We point out that the deformation constructed in the proof of Proposition 3.8 is not necessarily the cheapest one, as can be seen in Example 2.

We now introduce an editing distance between labelled Reeb graphs, in terms of the cost necessary to transform one graph into another.

Theorem 3.9. *For every two labelled Reeb graphs $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$, we set*

$$d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \inf_{T \in \mathscr{T}((\Gamma_f, f_|), (\Gamma_g, g_|))} c(T).$$

Then d is a distance.

The proof of the above theorem will be postponed to the end of the following section. Indeed, even if the properties of symmetry and triangular inequality can be easily verified, the property of the positive definiteness of d is not straightforward because the set of all possible deformations transforming $(\Gamma_f, f_|)$ to $(\Gamma_g, g_|)$ is not finite. In order to prove the positive definiteness of d, we will need a further result concerning the connection between the editing distance between two labelled Reeb graphs, $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$, and the natural pseudo-distance between the associated pairs (S^1, f), (S^1, g).

4. A LOWER BOUND FOR THE EDITING DISTANCE

Now we provide a suitable lower bound for our editing distance by means of the *natural pseudo-distance*.

The natural pseudo-distance is a measure of the dissimilarity between two pairs (X, φ), (Y, ψ), with X and Y compact, homeomorphic topological spaces and $\varphi : X \to \mathbb{R}$, $\psi : Y \to \mathbb{R}$ continuous functions. Roughly speaking, it is defined as the infimum of the variation of the values of φ and ψ, when we move from X to Y through homeomorphisms (see [11, 12, 13] for more details).

Such a lower bound is useful for achieving two different results. The first result, as mentioned in the preceding section, concerns the proof of Theorem 3.9, i.e., that d is a distance (see Corollary 4.2). The second one is related to an immediate question that can arise looking at the definition of d: Is it always possible to effectively compute the cheapest deformation transforming a labelled Reeb graph into another, since the number of such deformations is not finite? By using the natural pseudo-distance, we can estimate from below the value of d, and, in certain simple cases, knowing the value of the natural pseudo-distance allows us to determine the value of d (see, e.g., Examples 1–2).

The following Theorem 4.1 states that the natural pseudo-distance computed between the pairs (S^1,f) and (S^1,g) is a lower bound for the editing distance between the associated labelled Reeb graphs.

Theorem 4.1. *Let $(\Gamma_f, f_|)$, $(\Gamma_g, g_|)$ be labelled Reeb graphs associated with (S^1,f) and (S^1,g), respectively. Then $d((\Gamma_f, f_|),(\Gamma_g, g_|)) \geq \inf_{\tau\in\mathscr{H}(S^1)} \|f-g\circ\tau\|_{C^0}$.*

Proof. Let us prove that, for every $T\in\mathscr{T}((\Gamma_f, f_|),(\Gamma_g, g_|))$, $c(T)\geq \inf_{\tau\in\mathscr{H}(S^1)} \|f-g\circ\tau\|_{C^0}$.

First of all, assume that T is an elementary deformation transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$. For conciseness, slightly abusing notations, we will identify arcs of S^1 having as endpoints two critical points $p',p''\in V(\Gamma_f)$, and not containing other critical points of f, with the edges $e(p',p'')\in E(\Gamma_f)$.

(1) Let T be of type (R) relabelling vertices of $V(\Gamma_f)$. Since, by Definition 3.1 (R), $\Gamma_f=\Gamma_g$, we can always apply Lemma 2.3, considering Φ as the identity map, to obtain a piecewise linear $\tau\in\mathscr{H}(S^1)$ such that $\tau(p)=p$ for every $p\in K(f)$. As far as non-critical points are concerned, following the proof of Lemma 2.3, for every $p\in S^1\setminus K(f)$, $\tau(p)$ is defined as that point on S^1 such that, if $p\in e(p',p'')\in E(\Gamma_f)$, with $f(p)=(1-\lambda_p)f(p')+\lambda_p f(p'')$, $\lambda_p\in[0,1]$, then $\tau(p)\in e(p',p'')$ with $g(\tau(p))=(1-\lambda_p)g(p')+\lambda_p g(p'')$. Therefore, by substituting to $f(p)$ and $g(\tau(p))$ the above expressions, we see that $\max_{p\in S^1}|f(p)-g(\tau(p))|=\max_{p\in V(\Gamma_f)}|f_|(p)-g_|(p)|=c(T)$.

(2) Let T be of type (D) deleting $q_1,q_2\in V(\Gamma_f)$, the edges $e(p_1,q_1)$, $e(q_1,q_2)$, $e(q_2,p_2)$, and inserting the edge $e(p_1,p_2)$. Thus, for every $p\in K(f)\backslash\{q_1,q_2\}$, $f(p)=g(p)$. It is not restrictive to assume that $f(p_1)<f(q_2)<f(q_1)<f(p_2)$. Then we can define a sequence (τ_n) of piecewise linear homeomorphisms on S^1 approximating this elementary deformation. Let $\tau_n(p)=p$ for every every $p\in V(\Gamma_f)\backslash\{q_1,q_2\}=V(\Gamma_g)$ and $n\in\mathbb{N}$. Moreover, let $\overline{q}$ be the point of $e(p_1,p_2)\in E(\Gamma_g)$ such that $g(\overline{q})=\frac{f(q_1)+f(q_2)}{2}$ (such a point $\overline{q}$ exists because $g(p_1)=f(p_1)<f(q_2)<f(q_1)<f(p_2)=g(p_2)$ and it is unique because we are assuming that no critical points of g occur in the considered arc). Let us fix a positive real number $c<\min\{g(p_2)-g(\overline{q}),g(\overline{q})-g(p_1)\}$. For every $n\in\mathbb{N}$, let us define $\tau_n(q_1)$ (resp. $\tau_n(q_2)$) as the only point on S^1 belonging to the arc with endpoints $p_1,\overline{q}$ (resp. $\overline{q},p_2$) contained in $e(p_1,p_2)$, such that $g(\tau_n(q_1))=g(\overline{q})-\frac{c}{n}$ (resp. $g(\tau_n(q_2))=g(\overline{q})+\frac{c}{n}$) as shown in Figure 4. Now, let us linearly extend τ_n to all S^1 in the following way. For every $p\in S^1\setminus K(f)$, if p belongs to the arc with endpoints $p',p''\in K(f)$ not containing any other critical point, and is such that $f(p)=(1-\lambda_p)f(p')+\lambda_p f(p'')$, $\lambda_p\in[0,1]$, then $\tau_n(p)$ belongs to the arc

with endpoints $\tau_n(p'), \tau_n(p'')$ not containing any other critical point, and is such that $g(\tau_n(p)) = (1-\lambda_p)g(\tau_n(p')) + \lambda_p g(\tau_n(p''))$. Hence, τ_n is piecewise linear for every $n \in \mathbb{N}$, and $\lim_{n\to\infty} \max_{p \in S^1} |f(p) - g(\tau_n(p))| = \lim_{n\to\infty} \max_{p \in V(\Gamma_f)} |f(p) - g(\tau_n(p))| = \lim_{n\to\infty} \max\{f(q_1) - g(\tau_n(q_1)), f(q_2) - g(\tau_n(q_2))\} = |f(q_1) - g(\overline{q})| = \frac{f_|(q_1) - f_|(q_2)}{2} = c(T)$.

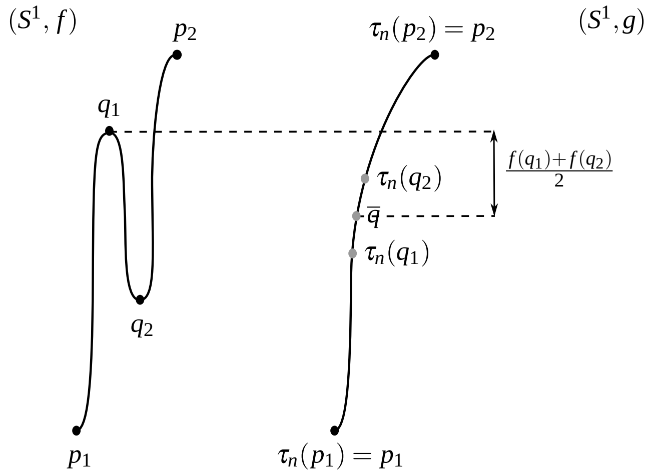


FIGURE 4. The construction of the homomorphism τ_n as described in step (2) of the proof of Theorem 4.1. The arc $e(p_1, q_1)$ ($e(q_1, q_2)$, and $e(q_2, p_2)$, respectively) is piecewise linearly taken to the arc having $\tau_n(p_1), \tau_n(q_1)$ ($\tau_n(q_1), \tau_n(q_2)$ and $\tau_n(q_2), \tau_n(p_2)$, respectively) as endpoints.

(3) Let T be of type (B) deleting $e(p_1, p_2) \in E(\Gamma_f)$, and inserting two vertices q_1, q_2 and the edges $e(p_1, q_1)$, $e(q_1, q_2)$, $e(q_2, p_2)$. Then we can apply the same proof as (2), by considering the inverse deformation T^{-1} that, by Definition 3.5, is of type (D) and, by Proposition 3.6, has the same cost of T.

Therefore, observing that in (1), the piecewise linear τ can be clearly replaced by a sequence (τ_n), with $\tau_n = \tau$ for every $n \in \mathbb{N}$, we can assert that, for every elementary deformation T, there exists a sequence of piecewise linear homeomorphisms on S^1, (τ_n), such that $c(T) = \lim_{n\to\infty} \|f - g \circ \tau_n\|_{C^0} \geq \inf_{\tau \in \mathscr{H}(S^1)} \|f - g \circ \tau\|_{C^0}$.

Now, let $T = (T_1, \ldots, T_r) \in \mathscr{T}((\Gamma_f, f_|), (\Gamma_g, g_|))$ and prove that, also in this case, $c(T) \geq \inf_{\tau \in \mathscr{H}(S^1)} \|f - g \circ \tau\|_{C^0}$. Let us set $T_i \cdots T_1(\Gamma_f, f_|) = (\Gamma_{f^{(i)}}, f_|^{(i)})$, $f = f^{(0)}$, $g = f^{(r)}$. For $i = 1, \ldots, r$, let $(\tau_n^{(i)})_n$ be a sequence of piecewise linear homeomorphisms on S^1 for which it holds that $c(T_i) = \lim_{n\to\infty} \|f^{(i-1)} - f^{(i)} \circ \tau_n^{(i)}\|_{C^0}$, and let $(\tau_n^{(0)})_n$ be the constant sequence such that $\tau_n^{(0)} = Id$ for every $n \in \mathbb{N}$. Then

$$\begin{aligned} c(T) &= \sum_{i=1}^{r} c(T_i) = \lim_{n\to\infty} \| f^{(0)} - f^{(1)} \circ \tau_n^{(1)} \|_{C^0} + \sum_{i=1}^{r-1} \lim_{n\to\infty} \| f^{(i)} - f^{(i+1)} \circ \tau_n^{(i+1)} \|_{C^0} \\ &= \lim_{n\to\infty} \| f^{(0)} - f^{(1)} \circ \tau_n^{(1)} \|_{C^0} \\ &\quad + \sum_{i=1}^{r-1} \lim_{n\to\infty} \| f^{(i)} \circ \tau_n^{(i)} \circ \ldots \circ \tau_n^{(0)} - f^{(i+1)} \circ \tau_n^{(i+1)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)} \|_{C^0} \\ &\geq \lim_{r\to\infty} \| f^{(0)} - f^{(r)} \circ \tau_n^{(r)} \circ \tau_n^{(r-1)} \circ \cdots \circ \tau_n^{(0)} \|_{C^0} \geq \inf_{\tau \in \mathscr{H}(S^1)} \| f - g \circ \tau \|_{C^0}, \end{aligned}$$

where the third equality is obtained by observing that

$$f^{(i)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)} - f^{(i+1)} \circ \tau_n^{(i+1)} \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)} = (f^{(i)} - f^{(i+1)} \circ \tau_n^{(i+1)}) \circ \tau_n^{(i)} \circ \cdots \circ \tau_n^{(0)}$$

for every $i \in \{1, \ldots, r-1\}$, and that $\| \cdot \|_{C^0}$ is invariant under re-parameterization; the first inequality is consequent to the triangular inequality. □

Corollary 4.2. *If* $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = 0$ *then* $(\Gamma_f, f_|) = (\Gamma_g, g_|)$.

Proof. From Theorem 4.1, $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = 0$ implies that $\inf_{\tau \in \mathscr{H}(S^1)} \| f - g \circ \tau \|_{C^0} = 0$. In [7] it has been proved that when $\inf_{\tau \in \mathscr{H}(X,Y)} \| f - g \circ \tau \|_{C^0} = 0$, with X, Y two closed curves of class at least C^2, a homeomorphism $\overline{\tau} \in \mathscr{H}(X,Y)$ exists such that $f = g \circ \overline{\tau}$. Therefore, the claim follows from Proposition 2.4. □

Proof of Theorem 3.9. The positive definiteness of d has been proved in Corollary 4.2; the symmetry is a consequence of Proposition 3.6; the triangular inequality can be easily verified in the standard way. □

Now we describe two simple examples showing how it is possible to compute the editing distance between two labelled Reeb graphs, $(\Gamma_f, f_|), (\Gamma_g, g_|)$, by exploiting the knowledge of the natural pseudo-distance value between the associated pairs $(S^1, f), (S^1, g)$. In particular, Example 1 provides a situation in which the infimum cost over all the deformations transforming $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$ is actually a minimum. In Example 2 this infimum is obtained by applying a passage to the limit.

Example 1. Let us consider the two pairs $(S^1, f), (S^1, g)$ depicted in Figure 5, with $f, g \in \mathscr{F}^0$. We now show that $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \frac{1}{2}(f(q_1) - f(p_1))$. Indeed, in this case, the natural pseudo-distance between (S^1, f) and (S^1, g) is equal to $\frac{1}{2}(f(q_1) - f(p_1))$ (cf. [13]). Therefore, by Theorem 4.1, it follows that $d((\Gamma_f, f_|), (\Gamma_g, g_|)) \geq \frac{1}{2}(f(q_1) - f(p_1))$. On the other hand, the deformation T of type (D) that deletes the vertices $p_1, q_1 \in V(\Gamma_f)$, the edges $e(p, q_1), e(q_1, p_1), e(p_1, q)$ and inserts the edge $e(p, q)$ transforms $(\Gamma_f, f_|)$ into $(\Gamma_g, g_|)$ with cost $c(T) = \frac{1}{2}(f(q_1) - f(p_1))$. Hence $d((\Gamma_f, f_|), (\Gamma_g, g_|)) = \frac{1}{2}(f(q_1) - f(p_1))$.

Example 2. Let us consider now the two pairs $(S^1, f), (S^1, g)$ illustrated in Figure 6. Let $f(q_1) - f(p_1) = f(q_2) - f(p_2) = a$. Then, clearly, $\inf_{\tau \in \mathscr{H}(S^1)} \| f - g \circ \tau \|_{C^0} = \frac{a}{2}$. Let us show that the editing distance between $(\Gamma_f, f_|)$ and $(\Gamma_g, g_|)$ is $\frac{a}{2}$, too. For every $0 < \varepsilon < \frac{a}{2}$, we can apply to $(\Gamma_f, f_|)$ a deformation of type (R), that relabels p_1, p_2, q_1, q_2 in such a way that $f(p_i)$ is increased of $\frac{a}{2} - \varepsilon$, and $f(q_i)$ is decreased of $\frac{a}{2} - \varepsilon$ for $i = 1, 2$, composed with two deformations of type (D) that delete p_i with q_i, $i = 1, 2$. Thus, since the total

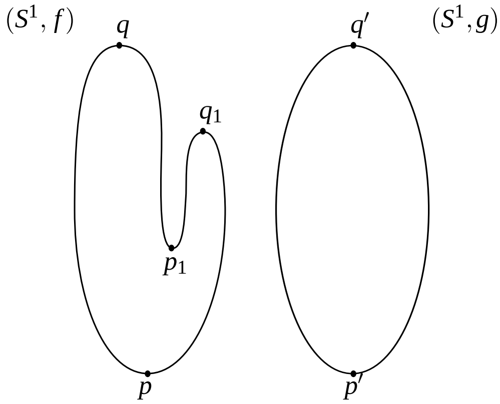


FIGURE 5. The pairs considered in Example 1. In this case $d((\Gamma_f, f_|),(\Gamma_g, g_|)) = \inf_{\tau\in\mathscr{H}(S^1)} \|f - g\circ\tau\|_{C^0} = \frac{1}{2}(f(q_1) - f(p_1))$

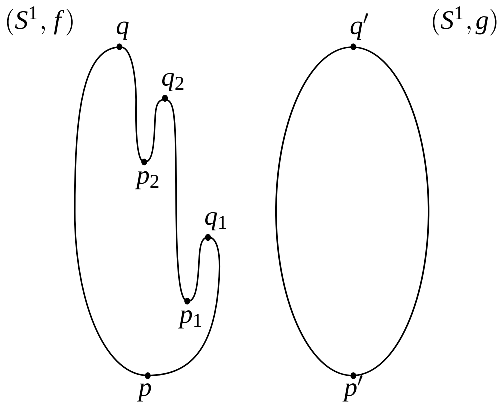


FIGURE 6. The pairs considered in Example 2. Even in this case $d((\Gamma_f, f_|),(\Gamma_g, g_|)) = \inf_{\tau\in\mathscr{H}(S^1)} \|f - g\circ\tau\|_{C^0} = \frac{1}{2}(f(q_1) - f(p_1))$

cost is equal to $\frac{a}{2} - \varepsilon + 2\varepsilon$, by the arbitrariness of ε, it holds that $d((\Gamma_f, f_|),(\Gamma_g, g_|)) \le \frac{a}{2}$. Applying Theorem 4.1, we deduce that $d((\Gamma_f, f_|),(\Gamma_g, g_|)) = \frac{a}{2}$.

5. LOCAL STABILITY

This section is intended to show that labelled Reeb graphs of closed curves are stable under small function perturbations with respect to our editing distance (see Theorem 5.5). The main tool we will use is provided by Theorem 5.3, that ensures the stability of simple Morse function critical values. This latter result can be deduced by the homological properties of the lower level sets of a simple Morse function f on a manifold $\mathscr{M}$, and its validity does not depend on the dimension of $\mathscr{M}$. Therefore, it will be given for any smooth compact manifold without boundary.

For every $f \in \mathscr{F}(\mathscr{M}, \mathbb{R})$, and for every $a \in \mathbb{R}$, let us denote by f^a the lower level set $f^{-1}(-\infty, a] = \{p \in \mathscr{M} : f(p) \le a\}$. Let us recall the existing link between the topology of a pair of lower level sets (f^b, f^a), with $a, b \in \mathbb{R}$, $a < b$, regular values of f, and the critical points of f lying between a and b. The following statements hold (cf. [19]):

(St. 1) If the interval $f^{-1}([a,b])$ contains no critical points, then f^a is a deformation retract of f^b, so that the inclusion map $f^a \to f^b$ is a homotopy equivalence.

(St. 2) If $f^{-1}([a,b])$ contains exactly one critical point of index $\bar{k}$, then, denoting by G the homology coefficient group, it holds that

$$H_k(f^b, f^a) = \begin{cases} G, & \text{if } k = \bar{k} \\ 0, & \text{otherwise.} \end{cases}$$

In the remainder of this section we require f to be a simple Morse function. Accordingly, it makes sense to use the terminology *critical value of index k* to indicate a critical value that is the image of a critical point of index k.

Lemma 5.1. *Let $f \in \mathscr{F}^0 \subset \mathscr{F}(\mathscr{M}, \mathbb{R})$, and let $a, b \in \mathbb{R}$, $a < b$, be regular values of f. If there exists $\bar{k} \in \mathbb{Z}$ such that $H_{\bar{k}}(f^b, f^a) \neq 0$, then $[a,b]$ contains at least one critical value of index $\bar{k}$.*

Proof. From *(St. 1)*, the absence of critical values in $[a,b]$ implies that the homomorphism induced by inclusion $\imath_k : H_k(f^a) \to H_k(f^b)$ is an isomorphism for each $k \in \mathbb{Z}$. Consequently, by using the long exact sequence of the pair:

$$\cdots \longrightarrow H_k(f^a) \xrightarrow{i_k} H_k(f^b) \xrightarrow{j_k} H_k(f^b, f^a) \xrightarrow{\partial_k} H_{k-1}(f^a) \xrightarrow{i_{k-1}} H_{k-1}(f^b) \longrightarrow \cdots,$$

it is easily seen that, for every $k \in \mathbb{Z}$, the surjectivity of i_k and the injectivity of i_{k-1} imply the triviality of $H_k(f^b, f^a)$. This proves that if there exists $\bar{k} \in \mathbb{Z}$ such that $H_{\bar{k}}(f^b, f^a) \neq 0$, then $[a,b]$ contains at least one critical value of f. That the index of at least one of the critical values of f contained in $[a,b]$ is exactly $\bar{k}$ is consequent to the sub-additivity property of the rank of the relative homology groups and to *(St. 2)*. In fact, let $c_1, \ldots, c_m$ be the critical values of f belonging to $[a,b]$, and let $s_0, \ldots, s_m$ be $m+1$ regular values such that $a = s_0 < c_1 < s_1 < c_2 < \ldots < s_{m-1} < c_m < s_m = b$. Since it holds that $\text{rank} H_{\bar{k}}(f^b, f^a) \leq \sum_{i=1}^{m} \text{rank} H_{\bar{k}}(f^{s_i}, f^{s_{i-1}})$, and by hypothesis $\text{rank} H_{\bar{k}}(f^b, f^a) \geq 1$, there exists at least one index $i \in \{1, \ldots, m\}$ such that $H_{\bar{k}}(f^{s_i}, f^{s_{i-1}}) \neq 0$. Now, applying *(St. 2)* with a replaced by s_{i-1} and b replaced by s_i, we deduce that c_i is a critical value of f of index $\bar{k}$. □

The above statements *(St. 1-2)*, Lemma 5.1, together with the following lemma, that is a reformulation of Lemma 4.1 in [17], provide the tools for proving the stability of critical values under small function perturbations (Theorem 5.3).

Lemma 5.2. *Let $X_1, X_2, X_3, X'_1, X'_2, X'_3$ be topological spaces such that $X_1 \subseteq X_2 \subseteq X_3 \subseteq X'_1 \subseteq X'_2 \subseteq X'_3$. Let $H_k(X_3, X_1) = 0$, $H_k(X'_3, X'_1) = 0$ for every $k \in \mathbb{Z}$. Then the homomorphism induced by inclusion $H_k(X'_1, X_1) \to H_k(X'_2, X_2)$ is injective for every $k \in \mathbb{Z}$.*

Theorem 5.3 (Stability of critical values)**.** *Let $f \in \mathscr{F}^0 \subset \mathscr{F}(\mathscr{M}, \mathbb{R})$ and let c be a critical value of index $\bar{k}$ of f. Then there exists a real number $\delta(f,c) > 0$ such that each $g \in \mathscr{F}^0$ verifying $\|f - g\|_{C^0} \leq \delta$, $0 \leq \delta \leq \delta(f,c)$, admits at least one critical value of index $\bar{k}$ in $[c - \delta, c + \delta]$.*

Proof. Since f is Morse, we can choose a real number $\delta(f,c) > 0$ such that $[c - 3 \cdot \delta(f,c), c + 3 \cdot \delta(f,c)]$ does not contain any critical value of f besides c. Let $0 \leq \delta \leq$

$\delta(f,c)$, and let g be a simple Morse function such that $\|f-g\|_{C^0} \le \delta$. If $\delta = 0$, then the claim immediately follows. Let $\delta > 0$. Then, for every $n \in \mathbb{N}$,

$$f^{c-\delta\cdot\frac{2n+1}{n}} \subseteq g^{c-\delta\cdot\frac{n+1}{n}} \subseteq f^{c-\delta/n} \subseteq f^{c+\delta/n} \subseteq g^{c+\delta\cdot\frac{n+1}{n}} \subseteq f^{c+\delta\cdot\frac{2n+1}{n}}.$$

Since $[c-\delta\cdot\frac{2n+1}{n}, c-\delta/n]$ and $[c+\delta/n, c+\delta\cdot\frac{2n+1}{n}]$ do not contain any critical value of f for every $n \in \mathbb{N}$, both $H_k(f^{c-\delta/n}, f^{c-\delta\cdot\frac{2n+1}{n}})$ and $H_k(f^{c+\delta\cdot\frac{2n+1}{n}}, f^{c+\delta/n})$ are trivial for every $k \in \mathbb{Z}$, and $n \in \mathbb{N}$. Consequently, from Lemma 5.2, the homomorphism induced by inclusion $H_k(f^{c+\delta/n}, f^{c-\delta\cdot\frac{2n+1}{n}}) \to H_k(g^{c+\delta\cdot\frac{n+1}{n}}, g^{c-\delta\cdot\frac{n+1}{n}})$ is injective for each $k \in \mathbb{Z}$, and $n \in \mathbb{N}$. Moreover, since, for every $n \in \mathbb{N}$, $[c-\delta\cdot\frac{2n+1}{n}, c+\delta/n]$ contains c, that is a critical value of index $\bar{k}$ of f, from *(St. 2)*, it holds that $H_{\bar{k}}(f^{c+\delta/n}, f^{c-\delta\cdot\frac{2n+1}{n}}) \neq 0$ for every $n \in \mathbb{N}$. This fact, together with the injectivity of the above map, implies that also $H_{\bar{k}}(g^{c+\delta\cdot\frac{n+1}{n}}, g^{c-\delta\cdot\frac{n+1}{n}}) \neq 0$ for every $n \in \mathbb{N}$. So, by Lemma 5.1, for every $n \in \mathbb{N}$, there exists at least one critical value c'_n of index $\bar{k}$ of g with $c'_n \in (c-\delta\cdot\frac{n+1}{n}, c+\delta\cdot\frac{n+1}{n})$. By contradiction, let us suppose that $[c-\delta, c+\delta]$ contains no critical values of index $\bar{k}$ of g. Then, since g is Morse, there would exist a sufficiently small real number $\varepsilon > 0$ such that $(c-\delta-\varepsilon, c+\delta+\varepsilon)$ does not contain critical values of index $\bar{k}$ of g either, giving an absurd. □

We now prove the local stability of labelled Reeb graphs of closed curves. We need a lemma that holds for manifolds of arbitrary dimension. The global stability will be exposed in the next section.

Lemma 5.4. *Let $f \in \mathscr{F}^0 \subset \mathscr{F}(\mathscr{M},\mathbb{R})$. Then there exists a positive real number $\delta(f)$ such that, for every δ, $0 \le \delta \le \delta(f)$, and for every $g \in \mathscr{F}^0$, with $\|f-g\|_{C^2} \le \delta$, an edge and vertices order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ exists for which $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \delta$.*

Proof. Let $p_1, \ldots, p_n$ be the critical points of f, and $c_1, \ldots, c_n$ the respective critical values, with $c_i < c_{i+1}$ for $i = 1 \ldots, n-1$. Since $\mathscr{F}^0$ is open in $\mathscr{F}(\mathscr{M},\mathbb{R})$, endowed with the C^2 topology, there always exists a sufficiently small $\delta(f) > 0$, such that the closed ball with center f and radius $\delta(f)$, $\overline{B_2(f,\delta(f))}$, is contained in $\mathscr{F}^0$. Moreover, $\delta(f)$ can be chosen so small that, for every $i = 1, \ldots, n-1$, the intervals $[c_i - \delta(f), c_i + \delta(f)]$ and $[c_{i+1} - \delta(f), c_{i+1} + \delta(f)]$ are disjoint.

Fixed such a $\delta(f)$, for every real number δ, with $0 \le \delta \le \delta(f)$, and for every $g \in \mathscr{F}^0$ such that $\|f-g\|_{C^2} \le \delta$, f and g belong to the same arcwise connected component of $\mathscr{F}^0$ endowed with the C^∞ topology, and, therefore, are topologically equivalent functions. Consequently, there exists an edge and vertices order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ (see Subsection 1.4). Let us prove that Φ is such that $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \delta$. Since f and g are topologically equivalent, it follows that g has exactly n critical points, $p'_1, \ldots, p'_n$. Let $c'_1 = g(p'_1), \ldots, c'_n = g(p'_n)$. We can assume $c'_i < c'_{i+1}$, for $i = 1, \ldots, n-1$. The assumption $\|f-g\|_{C^2} \le \delta$ implies that $\|f-g\|_{C^0} \le \delta$. Therefore, by the previous Theorem 5.3, for every critical value c_i of f, there exists at least one critical value of g of the same index of c_i belonging to $[c_i - \delta, c_i + \delta]$. Moreover, since $[c_i - \delta, c_i + \delta] \cap [c_{i+1} - \delta, c_{i+1} + \delta] = \emptyset$ for every $i = 1, \ldots, n-1$, it follows that $c'_i \in [c_i - \delta, c_i + \delta]$ for every $i = 1, \ldots, n$. Hence, since Φ preserves the order of the vertices, necessarily $\Phi(p_i) = p'_i$, yielding that $\max_{v \in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| = \max_{p_i \in K(f)} |f_|(p_i) - g_|(\Phi(p_i))| = \max_{1 \le i \le n} |c_i - c'_i| \le \delta$. □

Theorem 5.5 (Local stability)**.** *Let $f \in \mathscr{F}^0 \subset \mathscr{F}(S^1,\mathbb{R})$. Then there exists a positive real number $\delta(f)$ such that, for every δ, $0 \le \delta \le \delta(f)$, and for every $g \in \mathscr{F}^0$, with $\|f-g\|_{C^2} \le \delta$, it holds that $d((\Gamma_f, f_|),(\Gamma_g,g_|)) \le \delta$.*

Proof. By Lemma 5.4, an edge and vertices order preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ exists for which $\max_{v\in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \delta$. Necessarily Φ takes minima into minima and maxima into maxima. Therefore, $(\Gamma_f, g_| \circ \Phi) = T(\Gamma_f, f_|)$, with T an elementary deformation of type (R), relabelling vertices of $V(\Gamma_f)$, having cost $c(T) = \max_{v\in V(\Gamma_f)} |f_|(v) - g_|(\Phi(v))| \le \delta$. Moreover, let us observe that $(\Gamma_f, g_| \circ \Phi)$ is isomorphic to $(\Gamma_g, g_|)$ as labelled Reeb graph (see Definition 2.1). Thus, $d((\Gamma_f, f_|),(\Gamma_g,g_|)) = d((\Gamma_f, f_|),(\Gamma_f, g_| \circ \Phi)) = \inf_{T\in\mathscr{T}((\Gamma_f,f_|),(\Gamma_g,g_|))} c(T) \le \delta$. □

6. GLOBAL STABILITY

This section is devoted to proving that Reeb graphs of closed curves are stable under arbitrary function perturbations. More precisely, it will be shown that arbitrary changes in simple Morse functions imply smaller changes in the editing distance between Reeb graphs. The proof is by steps: the following Proposition 6.1 shows such a stability property when the functions defined on S^1 belong to the same arcwise connected component of $\mathscr{F}^0$; Proposition 6.2 proves the same result in the case that the linear convex combination of two simple Morse functions traverses the stratum $\mathscr{F}^1$ at most in one point; Theorem 6.3 extends the result to two arbitrary functions in $\mathscr{F}^0$.

Proposition 6.1. *Let $f,g \in \mathscr{F}^0$ and let us consider the path $h : [0,1] \to \mathscr{F}(S^1,\mathbb{R})$ defined by $h(\lambda) = (1-\lambda)f + \lambda g$. If $h(\lambda) \in \mathscr{F}^0$ for every $\lambda \in [0,1]$, then $d((\Gamma_f, f_|),(\Gamma_g,g_|)) \le \|f-g\|_{C^2}$.*

Proof. Let $\delta(h(\lambda)) > 0$ be the fixed real number playing the same role of $\delta(f)$ in Theorem 5.5, after replacing f by $h(\lambda)$. For conciseness, let us denote it by $\delta(\lambda)$, and $\|f-g\|_{C^2}$ by a. If $a = 0$, the claim trivially follows. If $a > 0$, let C be the open covering of $[0,1]$ constituted of open intervals $I_\lambda = \left(\lambda - \frac{\delta(\lambda)}{2a}, \lambda + \frac{\delta(\lambda)}{2a}\right)$. Let C' be a finite minimal (i.e. such that, for every i, $I_{\lambda_i} \not\subseteq \bigcup_{j\neq i} I_{\lambda_j}$) sub-covering of C, with $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ the middle points of its intervals. Since C' is minimal, for every $i \in \{1,\ldots,n-1\}$, $I_{\lambda_i} \cap I_{\lambda_{i+1}}$ is non-empty. This implies that

$$\lambda_{i+1} - \lambda_i < \frac{\delta(\lambda_i)}{2a} + \frac{\delta(\lambda_{i+1})}{2a} \le \frac{\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}}{a}. \tag{6.1}$$

Moreover, by the definition of h and the linearity of derivatives, it can be deduced that

$$\|h(\lambda_{i+1}) - h(\lambda_i)\|_{C^2} = (\lambda_{i+1} - \lambda_i) \cdot \|f-g\|_{C^2}. \tag{6.2}$$

Now, substituting (6.1) in (6.2), we obtain

$$\|h(\lambda_{i+1}) - h(\lambda_i)\|_{C^2} < \frac{\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}}{a} \cdot \|f-g\|_{C^2} = \max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}.$$

Let $(\Gamma_{h(\lambda_j)}, h(\lambda_j)_|)$ be the labelled Reeb graphs associated with $(S^1, h(\lambda_j))$, $j = 1,\ldots,n$. Let $i \in \{1,\ldots,n-1\}$. If $\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\} = \delta(\lambda_i)$, then using Theorem 5.5, with f replaced by $h(\lambda_i)$, g by $h(\lambda_{i+1})$ and δ by $\|h(\lambda_{i+1}) - h(\lambda_i)\|_{C^2}$, it holds that

$$d((\Gamma_{h(\lambda_i)}, h(\lambda_i)_|),(\Gamma_{h(\lambda_{i+1})}, h(\lambda_{i+1})_|)) \le \|h(\lambda_{i+1}) - h(\lambda_i)\|_{C^2}. \tag{6.3}$$

The same inequality holds when $\max\{\delta(\lambda_i),\delta(\lambda_{i+1})\}=\delta(\lambda_{i+1})$, as can be analogously checked.

Now, setting $\lambda_0=0$, $\lambda_{n+1}=1$, it can be verified that (6.3) also holds for $i=0,n$. Consequently, since $\Gamma_f=\Gamma_{h(\lambda_0)}$, and $\Gamma_g=\Gamma_{h(\lambda_{n+1})}$, we have

$$\begin{aligned} d((\Gamma_f,f_|),(\Gamma_g,g_|)) &\leq \sum_{i=0}^{n} d((\Gamma_{h(\lambda_i)},h(\lambda_i)_|),(\Gamma_{h(\lambda_{i+1})},h(\lambda_{i+1})_|)) \leq \sum_{i=0}^{n}\|h(\lambda_{i+1})-h(\lambda_i)\|_{C^2} \\ &= \sum_{i=0}^{n}(\lambda_{i+1}-\lambda_i)\cdot\|f-g\|_{C^2}=\|f-g\|_{C^2}, \end{aligned}$$

where the first inequality is due to the triangular inequality, the second one to (6.3), the first equality holds because of (6.2), the second one because $\sum_{i=0}^{n}(\lambda_{i+1}-\lambda_i)=1$. □

Proposition 6.2. *Let $f,g\in\mathscr{F}^0$ and let us consider the path $h:[0,1]\to\mathscr{F}(S^1,\mathbb{R})$ defined by $h(\lambda)=(1-\lambda)f+\lambda g$. If $h(\lambda)\in\mathscr{F}^0$ for every $\lambda\in[0,1]\setminus\{\overline{\lambda}\}$, with $0<\overline{\lambda}<1$, and h transversely intersects $\mathscr{F}^1$ at $\overline{\lambda}$, then $d((\Gamma_f,f_|),(\Gamma_g,g_|))\leq\|f-g\|_{C^2}$.*

Proof. We begin proving the following claim.

Claim. For every $\delta>0$ there exist two real numbers $\lambda',\lambda''\in[0,1]$, with $\lambda'<\overline{\lambda}<\lambda''$, such that $d((\Gamma_{h(\lambda')},h(\lambda')_|),(\Gamma_{h(\lambda'')},h(\lambda'')_|))\leq\delta$.

To prove this claim, let us first assume that $h(\overline{\lambda})$ belongs to $\mathscr{F}^1_\alpha$. To simplify the notation, we denote $h(\overline{\lambda})$ simply by $\overline{h}$. Let $\overline{p}$ be the sole degenerate critical point for $\overline{h}$. It is well known that there exists a suitable local coordinate system x around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h}=\overline{h}(\overline{p})+x^3$ (see Subsection 1.3 and Figure 1 (a) with $\overline{h}$ replaced by f).

Let us take a smooth function $\omega:S^1\to\mathbb{R}$ whose support is contained in the coordinate chart around $\overline{p}$ in which $\overline{h}=\overline{h}(\overline{p})+x^3$; moreover, let us assume that ω is equal to 1 in a neighborhood of $\overline{p}$, and decreases moving from $\overline{p}$. Let us consider the family of smooth functions $\overline{h}_t$ obtained by locally modifying $\overline{h}$ near $\overline{p}$ as follows: $\overline{h}_t=\overline{h}+t\cdot\omega\cdot x$. There exists $\bar{t}>0$ sufficiently small such that (i) for $0<t\leq\bar{t}$, $\overline{h}_t$ has no critical points in the support of ω and is equal to $\overline{h}$ everywhere else (see Figure 1 (a) with $\overline{h}_t$ replaced by $\widetilde{f}_2$), and (ii) for $-\bar{t}\leq t<0$, $\overline{h}_t$ has exactly two critical points in the support of ω whose values difference tends to vanish as t tends to 0, and $\overline{h}_t$ is equal to $\overline{h}$ everywhere else (see [6] and Figure 1 (a) with $\overline{h}_t$ replaced by $\widetilde{f}_1$).

Since $\overline{h}_t$ is a universal deformation of $\overline{h}=h(\overline{\lambda})$, and h intersect $\mathscr{F}^1$ transversely at $\overline{\lambda}$, either the maps $h(\lambda)$ with $\lambda<\overline{\lambda}$ are topologically equivalent to h_t with $t>0$ or to h_t with $t<0$ (cf. [6, 18, 23]). Analogously for the maps $h(\lambda)$ with $\lambda>\overline{\lambda}$. Let us assume that $h(\lambda)$ is topologically equivalent to h_t with $t<0$ when $\lambda<\overline{\lambda}$, while $h(\lambda)$ is topologically equivalent to h_t with $t>0$ when $\lambda>\overline{\lambda}$. Hence, for every $\delta>0$, there exist λ', with $0\leq\lambda'<\overline{\lambda}$, and λ'', with $\overline{\lambda}<\lambda''\leq 1$, such that $h(\lambda')$ and $h(\lambda'')$ have the same critical points, with the same values, except for two critical points of $h(\lambda')$, whose values difference is smaller than δ, that are non-critical for $h(\lambda'')$. Therefore, $(\Gamma_{h(\lambda')},h(\lambda')_|)$ can be transformed into $(\Gamma_{h(\lambda'')},h(\lambda'')_|)$ by an elementary deformation of type (D) whose cost is not greater than δ. In the case when $h(\lambda)$ is topologically equivalent to h_t with $t>0$ when $\lambda<\overline{\lambda}$, while $h(\lambda)$ is topologically equivalent to h_t with $t<0$ when $\lambda>\overline{\lambda}$, the claim can be proved similarly, applying an elementary deformation of type (B).

Let us now prove the claim when $\overline{h} = h(\overline{\lambda})$ belongs to $\mathscr{F}^1_\beta$. Let us denote by $\overline{p}$ and $\overline{q}$ the critical points of $\overline{h}$ such that $\overline{h}(\overline{p}) = \overline{h}(\overline{q})$. Since $\overline{p}$ is non-degenerate there exists a suitable local coordinate system x around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h} = \overline{h}(\overline{p}) + x^2$ (see Figure 1 (b) with $\overline{h}$ replaced by f). Let us take ω as before, whose support is contained in such a coordinate chart. Let us locally modify $\overline{h}$ near $\overline{p}$ as follows: $\overline{h}_t = \overline{h} + t\cdot\omega$. There exists $\bar{t} > 0$ sufficiently small such that for $|t| \leq \bar{t}$, $\overline{h}_t$ has exactly the same critical points as $\overline{h}$. As for critical values, they are the same as well, apart from the value taken at $\overline{p}$: $\overline{h}_t(\overline{p}) < \overline{h}(\overline{p})$, for $-\bar{t} \leq t < 0$ (see Figure 1 (b) with $\overline{h}_t$ replaced by $\widetilde{f}_1$), while $\overline{h}_t(\overline{p}) > \overline{h}(\overline{p})$, for $0 < t \leq \bar{t}$ (see Figure 1 (b) with $\overline{h}_t$ replaced by $\widetilde{f}_2$), and $\overline{h}_t(\overline{p})$ tends to $\overline{h}(\overline{p})$ as t tends to 0 (cf. [6]). Since $\overline{h}_t$ is a universal deformation of $\overline{h} = h(\overline{\lambda})$, and h intersect $\mathscr{F}^1$ transversely at $\overline{\lambda}$, we deduce that for every $\delta > 0$ there exist λ', with $0 \leq \lambda' < \overline{\lambda}$ and λ'', with $\overline{\lambda} < \lambda'' \leq 1$, such that $(\Gamma_{h(\lambda')}, h(\lambda')_{|})$ can be transformed into $(\Gamma_{h(\lambda'')}, h(\lambda'')_{|})$ by an elementary deformation of type (R) whose cost is not greater than δ. Therefore the initial claim is proved.

Let us now estimate $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|}))$. By the claim, for every $\delta > 0$, there exist $0 < \lambda' < \lambda'' < 1$ such that, applying the triangular inequality,

$$\begin{aligned} d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) &\leq d((\Gamma_f, f_{|}), (\Gamma_{h(\lambda')}, h(\lambda')_{|})) + d((\Gamma_{h(\lambda')}, h(\lambda')_{|}), (\Gamma_{h(\lambda'')}, h(\lambda'')_{|})) \\ &\quad + d((\Gamma_{h(\lambda'')}, h(\lambda'')_{|}), (\Gamma_g, g_{|})) \\ &\leq d((\Gamma_f, f_{|}), (\Gamma_{h(\lambda')}, h(\lambda')_{|})) + d((\Gamma_{h(\lambda'')}, h(\lambda'')_{|}), (\Gamma_g, g_{|})) + \delta. \end{aligned}$$

By Proposition 6.1,

$$d((\Gamma_f, f_{|}), (\Gamma_{h(\lambda')}, h(\lambda')_{|})) \leq \|f - h(\lambda')\|_{C^2} = \lambda' \cdot \|f - g\|_{C^2},$$

and

$$d((\Gamma_{h(\lambda'')}, h(\lambda'')_{|}), (\Gamma_g, g_{|})) \leq \|h(\lambda'') - g\|_{C^2} = (1 - \lambda'') \cdot \|f - g\|_{C^2}.$$

Hence, $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \leq \|f - g\|_{C^2} + \delta$, yielding the conclusion by the arbitrariness of δ. □

Theorem 6.3 (Global stability)**.** *Let* $f, g \in \mathscr{F}^0$. *Then* $d((\Gamma_f, f_{|}), (\Gamma_g, g_{|})) \leq \|f - g\|_{C^2}$.

Proof. For every sufficiently small $\delta > 0$ such that $B_2(f,\delta), B_2(g,\delta) \subset \mathscr{F}^0$, there exist $\widehat{f} \in B_2(f,\delta)$ and $\widehat{g} \in B_2(g,\delta)$ such that the path $h : [0,1] \to \mathscr{F}(S^1,\mathbb{R})$, with $h(\lambda) = (1-\lambda)\widehat{f} + \lambda\widehat{g}$, belongs to $\mathscr{F}^0$ for every $\lambda \in [0,1]$, except for at most a finite number n of values $0 < \mu_1 < \mu_2 < \ldots < \mu_n < 1$ at which h transversely intersects $\mathscr{F}^1$. If $n = 0$ ($n = 1$, respectively), then the claim immediately follows from Proposition 6.1 (Proposition 6.2, respectively). If $n > 1$, let $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{2n-1} < 1$, with $\lambda_{2i-1} = \mu_i$ for $i = 1, \ldots, n$. Then $h(\lambda_{2i-1}) \in \mathscr{F}^1$ for $i = 1, \ldots, n$, $h(\lambda_{2i}) \in \mathscr{F}^0$ for $i = 1, \ldots, n-1$. Set $\lambda_0 = 0$ so that $\widehat{f} = h(\lambda_0)$, and $\lambda_{2n} = 1$ so that $\widehat{g} = h(\lambda_{2n})$ (a schematization of this path can be visualized in Figure 7). Then, by Proposition 6.2, we have

$$d((\Gamma_{h(\lambda_{2i})}, h(\lambda_{2i})_{|}), (\Gamma_{h(\lambda_{2i+2})}, h(\lambda_{2i+2})_{|})) \leq \|h(\lambda_{2i}) - h(\lambda_{2i+2})\|_{C^2}$$

for every $i = 0, \ldots, n-1$. Therefore

$$\begin{aligned} d((\Gamma_{\widehat{f}}, \widehat{f}_{|}), (\Gamma_{\widehat{g}}, \widehat{g}_{|})) &\leq \sum_{i=0}^{n-1} d((\Gamma_{h(\lambda_{2i})}, h(\lambda_{2i})_{|}), (\Gamma_{h(\lambda_{2i+2})}, h(\lambda_{2i+2})_{|})) \\ &\leq \sum_{i=0}^{n-1} \|h(\lambda_{2i}) - h(\lambda_{2i+2})\|_{C^2} \leq \|\widehat{f} - \widehat{g}\|_{C^2}. \end{aligned}$$

Then, recalling that $\widehat{f} \in B_2(f,\delta)$ means $\|\widehat{f}-f\|_{C^2} \le \delta$, and $B_2(f,\delta) \subset \mathscr{F}^0$ implies that $(1-\lambda)f+\lambda\widehat{f} \in \mathscr{F}^0$ for every $\lambda \in [0,1]$, we can apply Proposition 6.1 to state that $d((\Gamma_f, f_|),(\Gamma_{\widehat{f}}, \widehat{f}_|)) \le \delta$. It is analogous for g and $\widehat{g}$. Thus, from the triangular inequality, we have

$$\begin{aligned} d((\Gamma_f, f_|),(\Gamma_g, g_|)) &\le d((\Gamma_f, f_|),(\Gamma_{\widehat{f}}, \widehat{f}_|)) + d((\Gamma_{\widehat{f}}, \widehat{f}_|),(\Gamma_{\widehat{g}}, \widehat{g}_|)) + d((\Gamma_{\widehat{g}}, \widehat{g}_|),(\Gamma_g, g_|)) \\ &\le 2\delta + \|\widehat{f}-\widehat{g}\|_{C^2}. \end{aligned}$$

Now, since by the triangular inequality, $\|\widehat{f}-\widehat{g}\|_{C^2} \le \|\widehat{f}-f\|_{C^2} + \|f-g\|_{C^2} + \|g-\widehat{g}\|_{C^2}$, with $\|\widehat{f}-f\|_{C^2} \le \delta$, and $\|g-\widehat{g}\|_{C^2} \le \delta$, it follows that $d((\Gamma_f, f_|),(\Gamma_g, g_|)) \le 4\delta + \|f-g\|_{C^2}$. Finally, because of the arbitrariness of δ, we can let δ tend to zero and obtain the claim. □

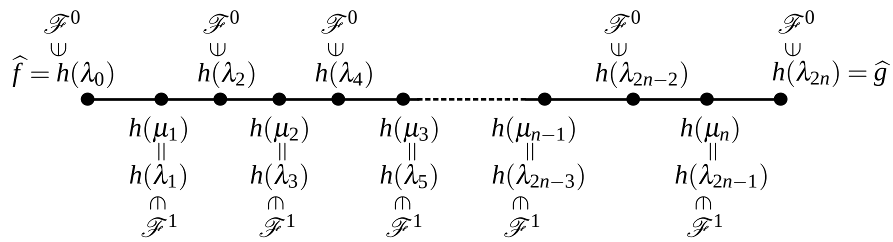


FIGURE 7. The linear path used in the proof of Theorem 6.3.

7. DISCUSSION

In this paper, we have considered Reeb graphs of curves and have shown that they stably represent topological properties of smooth functions. Precisely, we have constructed an editing distance between Reeb graphs of closed curves endowed with smooth functions f and g, that is bounded from below by the natural pseudo-distance between (S^1, f) and (S^1, g), and from above by the C^2-norm of $f-g$.

This paper is meant as a first step toward the study of stability of Reeb graphs of surfaces. While the general technique we use to prove our main result, as well as many intermediate results, could be easily generalized to surfaces, the definition of the editing distance would need to be appropriately modified. This requires us to classify the possible degeneracies of Reeb graphs of surfaces. Moreover, our proof of the metric properties of the editing distance exploits some particular properties of curves that are no longer valid for surfaces.

Furthermore, other shape descriptors consisting of graphs constructed out of Morse theory, such as the Morse Connection Graph introduced in [9] and further developed in [1], could possibly benefit of some of the results proved in this paper.

However, some questions remain unanswered also in the case of curves. In the examples shown in this paper, the editing distance coincides with the natural pseudo-distance. Is this always the case? Moreover, looking at the analogous results proved in [8, 10] about the stability of persistent homology groups, another shape descriptor used both in computer vision and computer graphics for shape comparison, we may notice that the C^0-norm rather than the C^2-norm is used to evaluate function changes. So another open question, strictly related to the previous one, is whether it would be possible to improve our result in this sense. Other open questions are concerned with applications of the Main Result (Theorem 6.3) to measure shape dissimilarity coping well with noisy data. On one hand, the result ensures the stability of Reeb graphs against noise, while, on the other, we may wonder how likely it is that noise encountered in real data is small with respect to the C^2-norm. Indeed, it is easy to conceive examples where perturbations that could be seen as noise do not correspond to a small value of the C^2-norm. For example, the functions represented in Figure 8 belong to a sequence of functions (f_n) all having the same C^2-norm although they tend to 0 with respect to the C^0-norm. However, one could argue that in a discrete setting, at a fixed resolution, sequences of functions as in Figure 8 cannot be found. Moreover, this problem would be overcome if the editing distance coincides with the natural pseudo-distance.

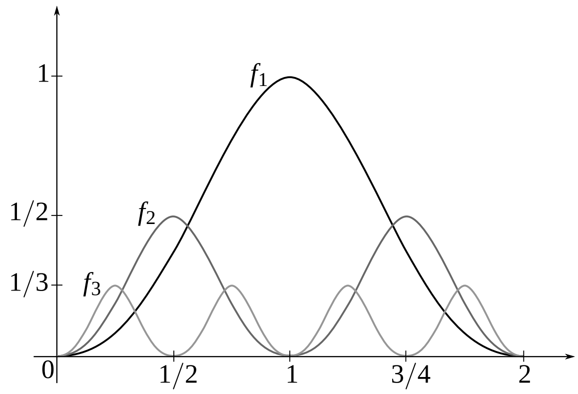


FIGURE 8. The graphs of three functions having the same C^2-norm.

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