NECESSARY CONDITIONS FOR DISCONTINUITIES OF MULTIDIMENSIONAL SIZE FUNCTIONS

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Abstract. Some new results about multidimensional Topological Persistence are presented, proving that the discontinuity points of a k-dimensional size function are necessarily related to the pseudocritical values of the associated measuring function.

Introduction

Topological Persistence is devoted to the study of stable properties of sublevel sets of topological spaces, revealing to be a suitable framework when dealing with applications in the field of Shape Analysis and Comparison. From the beginning of the 90's the research about this subject has been carried out under the name of Size Theory, studying the concept of *size function*, a mathematical tool able to describe the qualitative properties of a shape in a quantitative way. More precisely, the main idea is to model a shape by a topological space $\mathcal{M}$ endowed with a continuous function φ, called *measuring function*. Such a function is chosen according to applications and can be seen as a descriptor of the features considered relevant for shape characterization. Under these assumptions, the size function $\ell_{(\mathcal{M},\varphi)}$ associated to the pair $(\mathcal{M},\varphi)$ is a descriptor of the topological attributes that persist in the sublevel sets of $\mathcal{M}$ induced by the variation of φ. According to this approach, the problem of comparing two shapes can be reduced to the simpler comparison of the related size functions. Since their introduction, these shape descriptors have been widely studied and applied in quite a lot of concrete applications concerning Shape Comparison and Pattern Recognition (cf., e.g., [4, 6, 11, 25, 26, 27]). From a more theoretical point of view, the notion of size function plays an essential role since it is strongly related to the one of *natural pseudodistance*. This is another key tool of Size Theory, defining a (dis)similarity measure between compact and locally connected topological spaces endowed with measuring functions (see [3] for historical references and [12, 14, 15] for a detailed review about the concept of natural pseudodistance). Indeed, size functions provide easily computable lower bounds for the natural pseudodistance (cf. [8, 9, 13]).

Approximately ten years after the introduction of Size Theory, Persistent Homology re-proposed similar ideas from the homological point of view (cf. [18]; for a survey on this topic see [17]). In this context, the notion of size function coincides with the dimension of the 0-th multidimensional persistent homology group.

The study of Topological Persistence is capturing more and more attention in the mathematical community, with particular reference to the multidimensional

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setting (see [17, 22]). When dealing with size functions, the term *multidimensional* means that the measuring functions are vector-valued. However, while the basic properties of a size function ℓ are now clear when it is associated to a measuring function φ taking values in $\mathbb{R}$, very little is known when φ takes values in $\mathbb{R}^k$. More precisely, some questions about the structure of size functions associated with $\mathbb{R}^k$-valued measuring functions need further investigation, with particular reference to the localization of their discontinuities. Indeed, this last research line is essential in the development of efficient algorithms allowing us to apply Topological Persistence to concrete problems in the multidimensional context.

In this paper we start to fill this gap by proving a new result on the discontinuities of the so-called multidimensional size functions, showing that they can be located only at points with at least one pseudocritical coordinate (Theorem 2.8). This is proved by using an approximation technique and the theoretical machinery developed in [2], improving the comprehension of multidimensional Topological Persistence and laying the basis for its computation.

This paper is organized in two sections. In Section 1 the basic results about multidimensional size functions are recalled, while in Section 2 our main theorems are proved.

1. Preliminary Results on Size Theory

The main idea in Size Theory is to study a given shape by performing a geometrical/topological exploration of a suitable topological space $\mathcal{M}$, with respect to some properties expressed by an $\mathbb{R}^k$-valued continuous function $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k)$ defined on $\mathcal{M}$. Following this approach, Size Theory introduces the concept of *size function* as a stable and compact descriptor of the topological changes occurring in the lower level sets $\{P \in \mathcal{M} : \varphi_i(P) \leq t_i, i = 1, \ldots, k\}$ as $\vec{t} = (t_1, \ldots, t_k)$ varies in $\mathbb{R}^k$.

In this section we recall some basic definitions and results about size functions, confining ourselves to those that will be useful in the rest of this paper. For a deeper investigation on these topics, the reader is referred to [2, 3, 21]. For further details about Topological Persistence in the multidimensional setting, see [5, 21].

In proving our new results we need to consider a closed C^1 Riemannian manifold $\mathcal{M}$ endowed with a C^1 function $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k$. However, we prefer to report here the basic concepts of Size Theory in their classical formulation, i.e. by assuming that $\mathcal{M}$ is a non-empty compact and locally connected Hausdorff space and $\vec{\varphi}$ is continuous. We shall come back to the C^1 case later.

In the context of Size Theory, any pair $(\mathcal{M}, \vec{\varphi})$, where $\vec{\varphi} = (\varphi_1, \ldots, \varphi_k) : \mathcal{M} \to \mathbb{R}^k$ is a continuous function, is called a *size pair*. The function $\vec{\varphi}$ is said to be a *k-dimensional measuring function*. The relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^k$ as follows: for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, we write $\vec{x} \preceq \vec{y}$ (resp. $\vec{x} \prec \vec{y}$) if and only if $x_i \leq y_i$ (resp. $x_i < y_i$) for every index $i = 1, \ldots, k$. Furthermore, $\mathbb{R}^k$ is equipped with the usual max-norm: $\|(x_1, x_2, \ldots, x_k)\|_\infty = \max_{1 \leq i \leq k} |x_i|$. Now we are ready to introduce the concept of size function for a size pair $(\mathcal{M}, \vec{\varphi})$. We shall denote the open set $\{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$ by Δ^+, while $\bar{\Delta}^+$ will be the closure of Δ^+. For every k-tuple $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, the set $\mathcal{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$ will be defined as $\{P \in \mathcal{M} : \varphi_i(P) \leq x_i,\ i = 1, \ldots, k\}$.

Definition 1.1. For every k-tuple $\vec{y} = (y_1, \dots, y_k) \in \mathbb{R}^k$, we shall say that two points $P, Q \in \mathcal{M}$ are *$\langle \vec{\varphi} \preceq \vec{y} \rangle$-connected* if and only if a connected subset of $\mathcal{M}\langle \vec{\varphi} \preceq \vec{y} \rangle$ exists, containing P and Q.

Definition 1.2. We shall call the *(k-dimensional) size function* associated with the size pair $(\mathcal{M}, \vec{\varphi})$ the function $\ell_{(\mathcal{M},\vec{\varphi})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y})$ equal to the number of equivalence classes in which the set $\mathcal{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$ is divided by the $\langle \vec{\varphi} \preceq \vec{y} \rangle$-connectedness relation.

Remark 1.3. In other words, $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y})$ is equal to the number of connected components in $\mathcal{M}\langle \vec{\varphi} \preceq \vec{y} \rangle$ containing at least one point of $\mathcal{M}\langle \vec{\varphi} \preceq \vec{x} \rangle$. The finiteness of this number is an easily obtainable consequence of the compactness and local connectedness of $\mathcal{M}$.

In the following, we shall refer to the case of measuring functions taking value in $\mathbb{R}^k$ by using the term "k-dimensional". Before going on, we introduce the following notations: when $\vec{y} \in \mathbb{R}^k$ is fixed, the symbol $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot, \vec{y})$ will be used to denote the function that takes each k-tuple $\vec{x} \prec \vec{y}$ to the value $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \vec{y})$. An analogous convention will hold for the symbol $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x}, \cdot)$.

1.1. **The particular case $k = 1$.** In this section we will discuss the specific framework of measuring functions taking values in $\mathbb{R}$, namely the 1-dimensional case. Indeed, Size Theory has been extensively developed in this setting (cf. [3]), showing that each 1-dimensional size function admits a compact representation as a formal series of points and lines of $\mathbb{R}^2$ (cf. [20]). Due to this representation, a suitable *matching distance* between 1-dimensional size functions can be easily introduced, proving that these descriptors are stable with respect to such a distance [9]. Moreover, the role of 1-dimensional size functions is crucial in the approach to the k-dimensional case proposed in [2].

Following the notations used in the literature about the case $k = 1$, the symbols $\vec{\varphi}$, $\vec{x}$, $\vec{y}$ will be replaced respectively by φ, x, y.

When dealing with a (1-dimensional) measuring function $\varphi : \mathcal{M} \to \mathbb{R}$, the size function $\ell_{(\mathcal{M},\varphi)}$ associated with $(\mathcal{M}, \varphi)$ gives information about the pairs $(\mathcal{M}\langle \varphi \le x \rangle, \mathcal{M}\langle \varphi \le y \rangle)$, where $\mathcal{M}\langle \varphi \le t \rangle$ is defined by setting $\mathcal{M}\langle \varphi \le t \rangle = \{P \in \mathcal{M} : \varphi(P) \le t\}$ for $t \in \mathbb{R}$. For the sake of clarity we recall here the formal definition of a size function in the 1-dimensional case. Before going on, we observe that for $k = 1$, the domain Δ^+ of a size function reduces to be the open subset of the real plane given by $\{(x, y) \in \mathbb{R}^2 : x < y\}$.

Definition 1.4. For every $y \in \mathbb{R}$, we shall say that two points $P, Q \in \mathcal{M}$ are *$\langle \varphi \le y \rangle$-connected* if and only if a connected subset of $\mathcal{M}\langle \varphi \le y \rangle$ exists, containing P and Q.

Definition 1.5. We shall call the *(1-dimensional) size function* associated with the size pair $(\mathcal{M}, \varphi)$ the function $\ell_{(\mathcal{M},\varphi)} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(\mathcal{M},\varphi)}(x, y)$ equal to the number of equivalence classes in which the set $\mathcal{M}\langle \varphi \le x \rangle$ is divided by the $\langle \varphi \le y \rangle$-connectedness relation.

The example shown in Figure 1 could be helpful in making the previous definition clear. On the left (Figure 1(a)) one can find the considered size pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is the curve depicted by a solid line, and φ is the ordinate function. On the right (Figure 1(b)) the associated 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$ is given. As

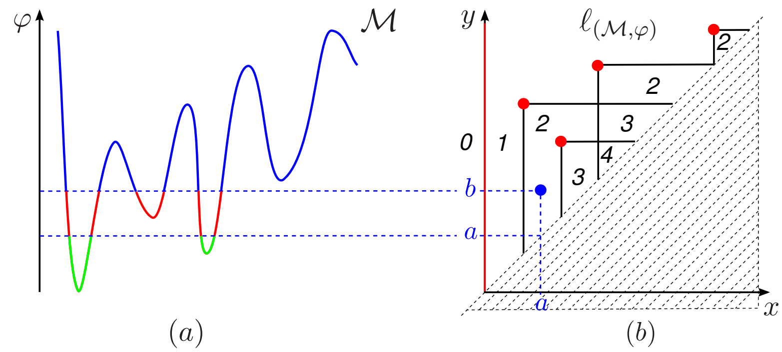

FIGURE 1. (a) The topological spaces $\mathcal{M}$ and the measuring function φ. (b) The related size function $\ell_{(\mathcal{M},\varphi)}$.

can be seen, the domain $\Delta^+ = \{(x,y) \in \mathbb{R}^2 : x < y\}$ is divided into bounded and unbounded regions, in each of which the 1-dimensional size function takes a constant value. The displayed numbers coincide with the values of $\ell_{(\mathcal{M},\varphi)}$ in each region. For example, let us now compute the value of $\ell_{(\mathcal{M},\varphi)}$ at the point (a,b). By applying Remark 1.3 in the case $k=1$, it is sufficient to count how many of the three connected components in the sublevel $\mathcal{M}\langle\varphi \leq b\rangle$ contain at least one point of $\mathcal{M}\langle\varphi \leq a\rangle$. It can be easily verified that $\ell_{(\mathcal{M},\varphi)}(a,b) = 2$.

Remark 1.6. From Definition 1.5 it can be immediately deduced that for every fixed value y the function $\ell_{(\mathcal{M},\varphi)}(\cdot,y)$ is non–decreasing, while for every fixed value x the function $\ell_{(\mathcal{M},\varphi)}(x,\cdot)$ is non–increasing.

Following the 1-dimensional framework, the problem of comparing two size pairs can be easily translated into the simpler one of comparing the related 1-dimensional size functions. In [9], the *matching distance* d_{match} has proven to be the most suitable distance between these descriptors. The definition of d_{match} is based on the observation that 1-dimensional size functions can be compactly described by a formal series of points and lines lying on the real plane, called respectively *proper cornerpoint* and *cornerpoint at infinity* (or *cornerlines*) and defined as follows:

Definition 1.7. For every point $P=(x,y)$ with $x<y$, consider the number $\mu(P)$ defined as the minimum, over all the positive real numbers ε with $x+\varepsilon < y-\varepsilon$, of

$$\ell_{(\mathcal{M},\varphi)}(x+\varepsilon,y-\varepsilon)-\ell_{(\mathcal{M},\varphi)}(x-\varepsilon,y-\varepsilon)-\ell_{(\mathcal{M},\varphi)}(x+\varepsilon,y+\varepsilon)+\ell_{(\mathcal{M},\varphi)}(x-\varepsilon,y+\varepsilon).$$

When this finite number, called *multiplicity of* P, is strictly positive, the point P will be called a *proper cornerpoint* for $\ell_{(\mathcal{M},\varphi)}$.

Definition 1.8. For every line r with equation $x=a$, consider the number $\mu(r)$ defined as the minimum, over all the positive real numbers ε with $a+\varepsilon < 1/\varepsilon$, of

$$\ell_{(\mathcal{M},\varphi)}(a+\varepsilon,1/\varepsilon)-\ell_{(\mathcal{M},\varphi)}(a-\varepsilon,1/\varepsilon).$$

When this finite number, called *multiplicity of* r, is strictly positive, the line r will be called a *cornerpoint at infinity* (or *cornerline*) for $\ell_{(\mathcal{M},\varphi)}$.

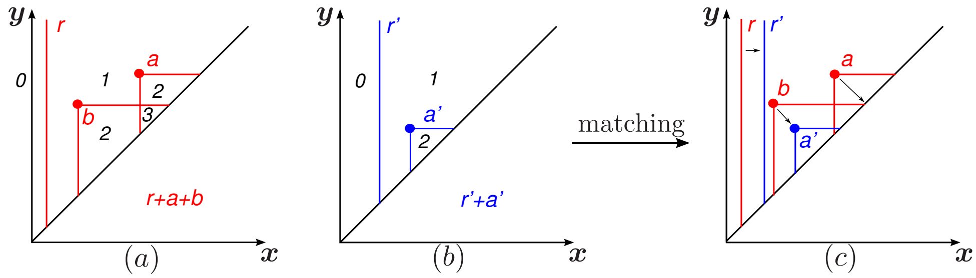

FIGURE 2. (a) Size function corresponding to the formal series $r + a + b$. (b) Size function corresponding to the formal series $r' + a'$. (c) The matching between the two formal series, realizing the matching distance between the two size functions.

The fundamental role of proper cornerpoints and cornerpoints at infinity is explicitly shown in the following Representation Theorem, claiming that their multiplicities completely and univocally determine the values of 1-dimensional size functions.

For the sake of simplicity, each line of equation $x = a$ will be identified to a point at infinity with coordinates (a, ∞).

Theorem 1.9 (Representation Theorem)**.** *For every $\bar{x} < \bar{y} < \infty$, it holds that*

$$\ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y}) = \sum_{\substack{x \leq \bar{x} \\ \bar{y} < y \leq \infty}} \mu((x, y)).$$

Remark 1.10. In plain words, the Representation Theorem 1.9 claims that the value $\ell_{(\mathcal{M},\varphi)}(\bar{x}, \bar{y})$ equals the number of cornerpoints lying above and on the left of $(\bar{x}, \bar{y})$. By means of this theorem we are able to compactly represent 1-dimensional size functions as formal series of cornerpoints and cornerlines (An example is given by Figure 2(a) and Figure 2(b)).

As a first and simple consequence of Theorem 1.9, we have the following result, that will be useful in Section 2 (cf. [20]):

Corollary 1.11. *Each discontinuity point $(\bar{x}, \bar{y})$ for $\ell_{(\mathcal{M},\varphi)}$ is such that either $\bar{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot, \bar{y})$, or $\bar{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\bar{x}, \cdot)$, or both these conditions hold.*

We are now able to introduce the matching distance d_{match}. Before going on, we observe that Theorem 1.9 allows us to reduce the problem of comparing 1-dimensional size functions into the comparison of the related multisets of cornerpoints. Indeed, the matching distance d_{match} can be seen as a measure of the cost of transporting the cornerpoints of a 1-dimensional size function into the cornerpoints of another one, with respect to a functional δ depending on the L_∞-distance between two matched cornerpoints. An example of matching between two formal series is given by Figure 2(c).

Let us now define more formally the matching distance d_{match}. Assume that two 1-dimensional size functions ℓ_1, ℓ_2 are given. Consider the multiset C_1 (respectively C_2) of cornerpoints for ℓ_1 (resp. ℓ_2), counted with their multiplicities and augmented by adding a countable infinity of points of the diagonal $\{(x, y) \in \mathbb{R}^2 :$

$x = y\}$. If we denote by $\bar{\Delta}^*$ the set $\bar{\Delta}^+$ extended by the points at infinity of the kind (a, ∞), i.e. $\bar{\Delta}^* = \bar{\Delta}^+ \cup \{(a, \infty) : a \in \mathbb{R}\}$, the matching distance $d_{match}(\ell_1, \ell_2)$ is then defined as

$$d_{match}(\ell_1, \ell_2) = \min_{\sigma} \max_{P \in C_1} \delta(P, \sigma(P)),$$

where σ varies among all the bijections between C_1 and C_2 and

$$\delta((x, y), (x', y')) = \min \left\{ \max \{|x - x'|, |y - y'|\}, \max \left\{ \frac{y - x}{2}, \frac{y' - x'}{2} \right\} \right\},$$

for every (x, y), $(x', y') \in \bar{\Delta}^*$ and with the convention about ∞ that $\infty - y = y - \infty = \infty$ when $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$.

In plain words, the pseudometric δ measures the pseudodistance between two points (x, y) and (x', y') as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that any two points of the diagonal have vanishing pseudodistance (we recall that a pseudodistance d is just a distance missing the condition $d(X, Y) = 0 \Rightarrow X = Y$, i.e. two distinct elements may have vanishing distance with respect to d).

An application of the matching distance is given by Figure 2(c). As can be seen by this example, different 1-dimensional size functions may in general have a different number of cornerpoints. Therefore d_{match} allows a proper cornerpoint to be matched to a point of the diagonal: this matching can be interpreted as the destruction of a proper cornerpoint. Moreover, we stress that the matching distance is stable with respect to perturbations of the measuring functions, as the following Matching Stability Theorem states:

Theorem 1.12 (Matching Stability Theorem)**.** *If* $(\mathcal{M}, \varphi)$, $(\mathcal{M}, \psi)$ *are two size pairs with* $\max_{P \in \mathcal{M}} |\varphi(P) - \psi(P)| \leq \varepsilon$, *then it holds that* $d_{match}(\ell_{(\mathcal{M},\varphi)}, \ell_{(\mathcal{M},\psi)}) \leq \varepsilon$.

For a proof of the previous theorem and more details about the matching distance the reader is referred to [8, 9] (see also [7] for the analogue of the matching distance in Persistent Homology and its stability).

1.1.1. *Coordinates of cornerpoints and discontinuity points.* Following the related literature (see also [10] for the case of measuring functions with a finite number of critical homological values), it can be easily deduced that, if finite, both the coordinates of a cornerpoint for a 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$ are critical values of the measuring function φ, under the assumption that φ is C^1. However, to the best of our knowledge, this result has never been explicitly proved until now. Therefore, for the sake of completeness we formalize here this statement, that will be used in Section 2:

Theorem 1.13. *Let* $\mathcal{M}$ *be a closed* C^1 *Riemannian manifold, and let* $\varphi : \mathcal{M} \to \mathbb{R}$ *be a* C^1 *measuring function. Then if* $(\bar{x}, \bar{y})$ *is a proper cornerpoint for* $\ell_{(\mathcal{M},\varphi)}$, *it follows that both* $\bar{x}$ *and* $\bar{y}$ *are critical values of* φ. *If* $(\bar{x}, \infty)$ *is a cornerpoint at infinity for* $\ell_{(\mathcal{M},\varphi)}$, *it follows that* $\bar{x}$ *is a critical value of* φ.

Proof. We confine ourselves to prove the former statement, since the proof of the latter is analogous. Our assertion is trivial for a Morse measuring function (see

Theorem 2.2 in [19]). For every real value $\varepsilon > 0$ it is possible to find a Morse measuring function $\varphi_\varepsilon : \mathcal{M} \to \mathbb{R}$ such that $\max_{P\in\mathcal{M}} |\varphi(P) - \varphi_\varepsilon(P)| \le \varepsilon$ and $\max_{P\in\mathcal{M}} \|\nabla\varphi(P) - \nabla\varphi_\varepsilon(P)\| \le \varepsilon$ (cf. [23], Corollary 6.8). Therefore, from the Matching Stability Theorem 1.12 it follows that for every $\varepsilon > 0$ we can find a cornerpoint $(\bar{x}_\varepsilon, \bar{y}_\varepsilon)$ for the size function $\ell_{(\mathcal{M},\varphi_\varepsilon)}$ with $\|(\bar{x},\bar{y}) - (\bar{x}_\varepsilon, \bar{y}_\varepsilon)\|_\infty \le \varepsilon$ and $\bar{x}_\varepsilon, \bar{y}_\varepsilon$ as critical values for φ_ε. Passing to the limit for $\varepsilon \to 0$ we obtain that both $\bar{x}$ and $\bar{y}$ are critical values for φ. □

From the Representation Theorem 1.9 and Theorem 1.13 we can obtain the following corollary, refining Corollary 1.11 in the case C^1 (we skip the easy proof):

Corollary 1.14. *Let $\mathcal{M}$ be a closed C^1 Riemannian manifold, and let $\varphi : \mathcal{M} \to \mathbb{R}$ be a C^1 measuring function. Let also $(\bar{x},\bar{y})$ be a discontinuity point for $\ell_{(\mathcal{M},\varphi)}$. Then at least one of the following statements holds:*

- **(i):** *$\bar{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\cdot,\bar{y})$ and $\bar{x}$ is a critical value for φ;*
- **(ii):** *$\bar{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\varphi)}(\bar{x},\cdot)$ and $\bar{y}$ is a critical value for φ.*

The generalization of Corollary 1.14 in the k-dimensional setting is not so simple and requires some new ideas which are given in Section 2, which also provides our main results.

1.2. **Reduction to the 1-dimensional case.** We are now ready to review the approach to multidimensional Size Theory proposed in [2]. In that work, the authors prove that the case $k > 1$ can be reduced to the 1-dimensional framework by a change of variable and the use of a suitable foliation. In particular, they show that there exists a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ such that the restriction of a k-dimensional size function $\ell_{(\mathcal{M},\vec{\varphi})}$ to each of these half-planes can be seen as a particular 1-dimensional size function. The motivations at the basis of this approach move from the fact that the concepts of proper cornerpoint and cornerpoint at infinity, defined for 1-dimensional size functions, appear not easily generalizable to an arbitrary dimension (namely the case $k > 1$). As a consequence, at a first glance it seems not possible to obtain the multidimensional analogue of the matching distance d_{match} and therefore it is not clear how to generalize the Matching Stability Theorem 1.12. On the other hand, all these problems can be bypassed by means of the results we recall in the rest of this subsection.

Definition 1.15. For every unit vector $\vec{l} = (l_1, \dots, l_k)$ of $\mathbb{R}^k$ such that $l_i > 0$ for $i = 1, \dots, k$, and for every vector $\vec{b} = (b_1, \dots, b_k)$ of $\mathbb{R}^k$ such that $\sum_{i=1}^k b_i = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is *admissible*. We shall denote the set of all admissible pairs in $\mathbb{R}^k \times \mathbb{R}^k$ by Adm_k. Given an admissible pair $(\vec{l}, \vec{b})$, we define the half-plane $\pi_{(\vec{l},\vec{b})}$ of $\mathbb{R}^k \times \mathbb{R}^k$ by the following parametric equations:

$$\begin{cases} \vec{x} = s\vec{l} + \vec{b} \\ \vec{y} = t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

The following proposition implies that the collection of half-planes given in Definition 1.15 is actually a foliation of Δ^+.

Proposition 1.16. *For every $(\vec{x}, \vec{y}) \in \Delta^+$ there exists one and only one admissible pair $(\vec{l}, \vec{b})$ such that $(\vec{x}, \vec{y}) \in \pi_{(\vec{l},\vec{b})}$.*

Now we can show the reduction to the 1-dimensional case.

Theorem 1.17 (Reduction Theorem)**.** *Let $(\vec{l},\vec{b})$ be an admissible pair, and $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}:\mathcal{M}\to\mathbb{R}$ be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P)=\max_{i=1,\dots,k}\left\{\frac{\varphi_i(P)-b_i}{l_i}\right\}\ .$$

Then, for every $(\vec{x},\vec{y})=(s\vec{l}+\vec{b},t\vec{l}+\vec{b})\in\pi_{(\vec{l},\vec{b})}$ the following equality holds:

$$\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y})=\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t)\ .$$

In the following, we shall use the symbol $F^{\vec{\varphi}}_{(\vec{l},\vec{b})}$ in the sense of the Reduction Theorem 1.17.

Remark 1.18. In plain words, the Reduction Theorem 1.17 states that each multidimensional size function corresponds to a 1-dimensional size function on each half-plane of the given foliation. It follows that each multidimensional size function can be represented as a parameterized family of formal series of points and lines, following the description introduced in Subsection 1.1 for the case $k=1$. Indeed, it is possible to associate a formal series $\sigma_{(\vec{l},\vec{b})}$ with each admissible pair $(\vec{l},\vec{b})$, with $\sigma_{(\vec{l},\vec{b})}$ describing the 1-dimensional size function $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$. Therefore, on each half-plane $\pi_{(\vec{l},\vec{b})}$, the matching distance d_{match} and the Matching Stability Theorem 1.12 can be applied. Moreover, the family $\left\{\sigma_{(\vec{l},\vec{b})}:(\vec{l},\vec{b})\in Adm_k\right\}$ turns out to be a complete descriptor for $\ell_{(\mathcal{M},\vec{\varphi})}$, since two multidimensional size functions coincide if and only if the corresponding parameterized families of formal series coincide.

The next result proves the stability of d_{match} with respect to the choice of the half-planes of the foliation: Indeed, the next proposition states that small enough changes in $(\vec{l},\vec{b})$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance.

Proposition 1.19. *If $(\mathcal{M},\vec{\varphi})$ is a size pair, $(\vec{l},\vec{b})\in Adm_k$ and ε is a real number with $0<\varepsilon<\min_{i=1,\dots,k}l_i$, then for every admissible pair $(\vec{l}',\vec{b}')$ with $\|(\vec{l},\vec{b})-(\vec{l}',\vec{b})\|_\infty\le\varepsilon$, it holds that*

$$d_{match}(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})},\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l}',\vec{b}')})})\le\varepsilon\cdot\frac{\max_{P\in\mathcal{M}}\|\vec{\varphi}(P)\|_\infty+\|\vec{l}\|_\infty+\|\vec{b}\|_\infty}{\min_{i=1,\dots,k}\{l_i(l_i-\varepsilon)\}}.$$

Remark 1.20. Analogously, it is possible to prove (cf. [2], Proposition 2) that d_{match} is stable with respect to the chosen measuring function, i.e. that small enough changes in $\vec{\varphi}$ with respect to the max-norm induce small changes of $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ with respect to the matching distance.

Proposition 1.19 and Remark 1.20 guarantee the stability of this approach.

2. Main Results

In this section we are going to prove some new results about the discontinuities of multidimensional size functions. In order to do that, we will confine ourselves to the case of a size pair $(\mathcal{M},\vec{\varphi})$, where $\mathcal{M}$ is a closed C^1 Riemannian m-manifold and $\vec{\varphi}=(\varphi_1,\dots,\varphi_k):\mathcal{M}\to\mathbb{R}^k$ is a C^1 measuring function. From now to Theorem

2.8 we shall assume that an admissible pair $(\vec{l},\vec{b}) \in Adm_k$ is fixed and consider the 1-dimensional size function $\ell_{(\mathcal{M},F)}$, where $F(Q) = \max_{i=1,\dots,k} \frac{\varphi_i(Q)-b_i}{l_i}$. We shall say that F and $\ell_{(\mathcal{M},F)}$ are the (1-dimensional) measuring function and the size function corresponding to the half-plane $\pi_{(\vec{l},\vec{b})}$, respectively.

The main result of this section is stated in Theorem 2.8, showing a necessary condition for a point $(\vec{x},\vec{y}) \in \Delta^+$ to be a discontinuity point for the size function $\ell_{(\mathcal{M},\vec{\varphi})}$. For the sake of clarity, we will now provide a sketch of the arguments that will lead us to the proof of our main result.

Theorem 2.8 is a generalization in the k-dimensional setting of Corollary 1.14, stating that each discontinuity point for a 1-dimensional size function $\ell_{(\mathcal{M},\varphi)}$, related to a C^1 measuring function φ, is such that at least one of its coordinates is a critical value for φ. We recall that Corollary 1.14 directly descends from the Representation Theorem 1.9 and from Theorem 1.13, according to which each finite coordinate of a cornerpoint for $\ell_{(\mathcal{M},\varphi)}$ has to be a critical value for φ. Our first goal is to prove that a modified version of this last statement holds for the 1-dimensional size function $\ell_{(\mathcal{M},F)}$ corresponding to the half-plane $\pi_{(\vec{l},\vec{b})}$. The reason of such an adaptation is that the 1-dimensional measuring function F is not C^1, and therefore we need to generalize the concepts of critical point and critical value by introducing the definitions of $(\vec{l},\vec{b})$-*pseudocritical point* and $(\vec{l},\vec{b})$-*pseudocritical value* for $\vec{\varphi}$ (Definition 2.1). These notions, together with an approximation in C^0 of the function F by C^1 functions, are used to prove that each finite coordinate of a cornerpoint for $\ell_{(\mathcal{M},F)}$ has to be an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$ (Theorem 2.2). Next, we show (Proposition 2.3) that a correspondence exists between the discontinuity points of $\ell_{(M,F)}$ and the ones of $\ell_{(M,\vec{\varphi})}$. Theorem 2.2 and Proposition 2.3 lead us to the relation (Theorem 2.5) between the discontinuity points for $\ell_{(\mathcal{M},\vec{\varphi})}$, lying on the half-plane $\pi_{(\vec{l},\vec{b})}$, and the $(\vec{l},\vec{b})$-pseudocritical values for $\vec{\varphi}$. Finally, we refine this last result in Theorem 2.8, by providing a necessary condition for discontinuities of $\ell_{(\mathcal{M},\vec{\varphi})}$ that does not depend on the half-planes of the foliation. This can be done by introducing the concepts of *pseudocritical point* and *pseudocritical value* for an $\mathbb{R}^k$-valued C^1 function (Definition 2.6), and considering a suitable projection $\rho : \mathbb{R}^k \to \mathbb{R}^h$.

Before going on, we need the following definition:

Definition 2.1. For every $Q \in \mathcal{M}$, set $I_Q = \left\{ i \in \{1,\dots,k\} : \frac{\varphi_i(Q)-b_i}{l_i} = F(Q) \right\}$. We shall say that Q is an $(\vec{l},\vec{b})$-*pseudocritical point for* $\vec{\varphi}$ if the convex hull of the gradients $\nabla\varphi_i(Q)$, $i \in I_Q$, contains the null vector, i.e. for every $i \in I_Q$ there exists a real value λ_i such that $\sum_{i\in I_Q} \lambda_i \nabla\varphi_i(Q) = \mathbf{0}$, with $0 \le \lambda_i \le 1$ and $\sum_{i\in I_Q}\lambda_i = 1$. If Q is an $(\vec{l},\vec{b})$-pseudocritical point for $\vec{\varphi}$, the value $F(Q)$ will be said an $(\vec{l},\vec{b})$-*pseudocritical value for* $\vec{\varphi}$.

We can now state our first result.

Theorem 2.2. *If* (σ,τ) *is a proper cornerpoint of* $\ell_{(\mathcal{M},F)}$, *then both* σ *and* τ *are* $(\vec{l},\vec{b})$-*pseudocritical values for* $\vec{\varphi}$. *If* (σ,∞) *is a cornerpoint at infinity of* $\ell_{(\mathcal{M},F)}$, *then* σ *is an* $(\vec{l},\vec{b})$-*pseudocritical value for* $\vec{\varphi}$.

Proof. We confine ourselves to prove the former statement, since the proof of the latter is analogous. The idea is to show that our thesis holds for a C^1 function approximating the measuring function $F : \mathcal{M} \to \mathbb{R}$ in C^0, and verify that this property passes to the limit. Let us now set $\Phi_i(Q) = \frac{\varphi_i(Q)-b_i}{l_i}$ and choose $c \in \mathbb{R}$ such that $\min_{Q\in\mathcal{M}} \Phi_i(Q) > -c$, for every $i = 1, \dots, k$. Consider the function sequence (F_p), $p \in \mathbb{N}^+ = \mathbb{N}\backslash\{0\}$, where $F_p : \mathcal{M} \to \mathbb{R}$ and $F_p(Q) = \left(\sum_{i=1}^k (\Phi_i(Q) + c)^p\right)^{\frac{1}{p}} - c$: Such a sequence converges uniformly to the function F. Indeed, for every $Q \in \mathcal{M}$ and for every index p we have that

$$
\begin{aligned}
|F(Q) - F_p(Q)| &= \left|\max_i \Phi_i(Q) - \left(\left(\sum_{i=1}^k (\Phi_i(Q) + c)^p\right)^{\frac{1}{p}} - c\right)\right| = \\
&= \left|\max_i\{\Phi_i(Q) + c\} - \left(\sum_{i=1}^k (\Phi_i(Q) + c)^p\right)^{\frac{1}{p}}\right| = \\
&= \left(\sum_{i=1}^k (\Phi_i(Q) + c)^p\right)^{\frac{1}{p}} - \max_i\{\Phi_i(Q) + c\} \leq \\
&\leq \max_i\{\Phi_i(Q) + c\} \cdot (k^{\frac{1}{p}} - 1).
\end{aligned}
$$

Let us now consider a proper cornerpoint $\bar{C}$ of the size function $\ell_{(\mathcal{M},F)}$. By the Matching Stability Theorem 1.12 it follows that it is possible to find a large enough p and a proper cornerpoint C_p of the 1-dimensional size function $\ell_{(\mathcal{M},F_p)}$ (associated with the size pair $(\mathcal{M}, F_p)$) such that C_p is arbitrarily close to $\bar{C}$. Since C_p is a proper cornerpoint of $\ell_{(\mathcal{M},F_p)}$, it follows from Theorem 1.13 that its coordinates are critical values of the C^1 function F_p. By focusing the attention on the abscissa of C_p (analogous considerations hold for the ordinate of C_p) it follows that there exists $Q_p \in \mathcal{M}$ with $x(C_p) = F_p(Q_p)$ and (in respect to local coordinates $x_1, \dots, x_m$ of the m-manifold $\mathcal{M}$)

$$
\begin{aligned}
0 &= \frac{\partial F_p}{\partial x_1}(Q_p) = \left(\sum_{i=1}^k (\Phi_i(Q_p) + c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{i=1}^k (\Phi_i(Q_p) + c)^{p-1} \cdot \frac{\partial \Phi_i}{\partial x_1}(Q_p)\right) \\
&\vdots \\
0 &= \frac{\partial F_p}{\partial x_m}(Q_p) = \left(\sum_{i=1}^k (\Phi_i(Q_p) + c)^p\right)^{\frac{1-p}{p}} \cdot \left(\sum_{i=1}^k (\Phi_i(Q_p) + c)^{p-1} \cdot \frac{\partial \Phi_i}{\partial x_m}(Q_p)\right) .
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\sum_{i=1}^k \ (\Phi_i(Q_p) + c)^{p-1} \cdot \frac{\partial \Phi_i}{\partial x_1}(Q_p) = 0 \\
&\vdots \\
&\sum_{i=1}^k \ (\Phi_i(Q_p) + c)^{p-1} \cdot \frac{\partial \Phi_i}{\partial x_m}(Q_p) = 0 .
\end{aligned}
$$

Therefore, by setting

$$
\boldsymbol{v}_p = (v_p^1, \dots, v_p^k) = \left((\Phi_1(Q_p) + c)^{p-1}, \dots, (\Phi_k(Q_p) + c)^{p-1}\right),
$$

we can write ${}^t\mathcal{J}(Q_p)\cdot{}^t\boldsymbol{v}_p = \mathbf{0}$, where $J(Q_p)$ is the Jacobian matrix of $\vec{\Phi} = (\Phi_1, \dots, \Phi_k)$ computed at the point Q_p. By the compactness of $\mathcal{M}$, we can assume (possibly by extracting a subsequence) that (Q_p) converges to a point $\bar{Q}$. Let us define $\boldsymbol{u}_p = \frac{\boldsymbol{v}_p}{\|\boldsymbol{v}_p\|_\infty}$. By compactness (recall that $\|\boldsymbol{u}_p\|_\infty = 1$) we can also assume (possibly by considering a subsequence) that the sequence $(\boldsymbol{u}_p)$ converges to a vector $\bar{\boldsymbol{u}} = (\bar{u}^1, \dots, \bar{u}^k)$, where $\bar{u}^i = \lim_{p\to\infty} \frac{v_p^i}{\|\boldsymbol{v}_p\|_\infty}$ and $\|\bar{\boldsymbol{u}}\|_\infty = 1$. Obviously ${}^t\mathcal{J}(Q_p)\cdot{}^t\boldsymbol{u}_p = \mathbf{0}$ and hence we have

$$
{}^t\mathcal{J}(\bar{Q})\cdot{}^t\bar{\boldsymbol{u}} = \mathbf{0}. \tag{2.1}
$$

Since for every index p and for every $i = 1, \dots, k$ the relation $0 < u_p^i \le 1$ holds, for each $i = 1, \dots, k$ the condition $0 \le \bar{u}^i = \lim_{p\to\infty} u_p^i \le 1$ is satisfied. Let us now recall that $F(\bar{Q}) = \max_i \Phi_i(\bar{Q})$, by definition, and consider the set $I_{\bar{Q}} = \{i \in \{1, \dots, k\} : \Phi_i(\bar{Q}) = F(\bar{Q})\} = \{i_1, \dots, i_h\}$. For every $r \notin I_{\bar{Q}}$ the component $\bar{u}^r$ is equal to 0, since $0 \le u_p^r = \left(\frac{\Phi_r(Q_p)+c}{\max_i\{\Phi_r(Q_p)+c\}}\right)^{p-1}$ and $\lim_{p\to\infty} \frac{\Phi_r(Q_p)+c}{\max_i\{\Phi_r(Q_p)+c\}} = \frac{\Phi_r(\bar{Q})+c}{F(\bar{Q})+c}$, which is strictly less than 1 for $\Phi_r(\bar{Q}) < F(\bar{Q})$. Hence we have $\bar{\boldsymbol{u}} = \bar{u}^{i_1}\cdot\boldsymbol{e}_{i_1} + \dots + \bar{u}^{i_h}\cdot\boldsymbol{e}_{i_h}$, where $\boldsymbol{e}_i$ is the i^{th} vector of the standard basis of $\mathbb{R}^k$. Thus, from equality (2.1) we have $\sum_{j=1}^h \bar{u}^{i_j}\cdot\frac{\partial\Phi_{i_j}}{\partial x_1}(\bar{Q}) = 0, \dots, \sum_{j=1}^h \bar{u}^{i_j}\cdot\frac{\partial\Phi_{i_j}}{\partial x_m}(\bar{Q}) = 0$, that is $\sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}}\cdot\frac{\partial\varphi_{i_j}}{\partial x_1}(\bar{Q}) = 0, \dots, \sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}}\cdot\frac{\partial\varphi_{i_j}}{\partial x_m}(\bar{Q}) = 0$, since $\Phi_i = \frac{\varphi - b_i}{l_i}$. Hence, $\sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}}\nabla\varphi_{i_j}(\bar{Q}) = \mathbf{0}$. By recalling that $\bar{u}^{i_j} \ge 0$, $l_{i_j} > 0$ and $\bar{\boldsymbol{u}}$ is a non–vanishing vector, it follows immediately that $\sum_{j=1}^h \frac{\bar{u}^{i_j}}{l_{i_j}} > 0$ and therefore the convex hull of the gradients $\nabla\varphi_{i_1}(\bar{Q}), \dots, \nabla\varphi_{i_h}(\bar{Q})$ contains the null vector. Thus, $\bar{Q}$ is an $(\vec{l}, \vec{b})$-pseudocritical point for $\vec{\varphi}$ and hence $F(\bar{Q})$ is an $(\vec{l}, \vec{b})$-pseudocritical value for $\vec{\varphi}$. Moreover, from the uniform convergence of the sequence (F_p) to F and from the continuity of the function F, we have (recall that $\bar{C} = \lim_{p\to\infty} C_p$)

$$
x(\bar{C}) = \lim_{p\to\infty} x(C_p) = \lim_{p\to\infty} F_p(Q_p) = F(\bar{Q}).
$$

In other words, the abscissa $x(\bar{C})$ of a proper cornerpoint of $\ell_{(\mathcal{M},F)}$ is the image of an $(\vec{l}, \vec{b})$-pseudocritical point $\bar{Q}$ through F, i.e. an $(\vec{l}, \vec{b})$-pseudocritical value for $\vec{\varphi}$. An analogous reasoning holds for the ordinate $y(\bar{C})$ of a proper cornerpoint. □

Our next result shows that each discontinuity of $\ell_{(\mathcal{M},\vec{\varphi})}$ corresponds to a discontinuity of the 1-dimensional size function associated with a suitable half-plane of the foliation.

Proposition 2.3. *A point $(\vec{x}, \vec{y}) = (s\cdot\vec{l}+\vec{b}, t\cdot\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$ if and only if (s,t) is a discontinuity point for $\ell_{(\mathcal{M},F)}$.*

Proof. Obviously, if (s,t) is a discontinuity point for $\ell_{(\mathcal{M},F)}$, then $(\vec{x}, \vec{y}) = (s\cdot\vec{l}+\vec{b}, t\cdot\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$, because of the Reduction Theorem 1.17. In order to prove the inverse implication, we shall verify the contrapositive statement, i.e. if (s,t) is not a discontinuity point for $\ell_{(\mathcal{M},F)}$, then $(s\cdot\vec{l}+\vec{b}, t\cdot\vec{l}+\vec{b})$ is not a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. Indeed, if (s,t) is not a discontinuity point for $\ell_{(\mathcal{M},F)}$, then $\ell_{(\mathcal{M},F)}$ is locally constant at (s,t) (recall that each size function is natural–valued). Therefore it will be possible to choose a real number $\eta > 0$ such that

$$\ell_{(\mathcal{M},F)}(s-\eta, t+\eta) = \ell_{(\mathcal{M},F)}(s+\eta, t-\eta). \tag{2.2}$$

Before proceeding in our proof, we need the following result:

Lemma 2.4. *Let $(\mathcal{M}, \psi)$, $(\mathcal{M}, \psi')$ be two size pairs, with $\psi, \psi' : \mathcal{M} \to \mathbb{R}$. If $d_{match}\left(\ell_{(\mathcal{M},\psi)}, \ell_{(\mathcal{M},\psi')}\right) \leq 2\varepsilon$, then it holds that*

$$\ell_{(\mathcal{M},\psi)}(s-\varepsilon, t+\varepsilon) \leq \ell_{(\mathcal{M},\psi')}(s+\varepsilon, t-\varepsilon),$$

for every (s, t) with $s + \varepsilon < t - \varepsilon$.

Proof of Lemma 2.4. Let Δ^* be the open set given by $\Delta^+ \cup \{(a, \infty) : a \in \mathbb{R}\}$. For every (s, t) with $s < t$, let us define the set $L_{(s,t)} = \{(\sigma, \tau) \in \Delta^* : \sigma \leq s, \tau > t\}$. By the Representation Theorem 1.9 we have that $\ell_{(\mathcal{M},\psi)}(s-\varepsilon, t+\varepsilon)$ equals the number of proper cornerpoints and cornerpoints at infinity for $\ell_{(\mathcal{M},\psi)}$ belonging to the set $L_{(s-\varepsilon,t+\varepsilon)}$. Since $d_{match}\left(\ell_{(\mathcal{M},\psi)}, \ell_{(\mathcal{M},\psi')}\right) \leq 2\varepsilon$, the number of proper cornerpoints and cornerpoints at infinity for $\ell_{(\mathcal{M},\psi')}$ in the set $L_{(s+\varepsilon,t-\varepsilon)}$ is not less than $\ell_{(\mathcal{M},\psi)}(s-\varepsilon, t+\varepsilon)$. The reason is that the change from ψ to ψ' does not move the cornerpoints more than 2ε, with respect to the max-norm, because of the Matching Stability Theorem 1.12. By applying the Representation Theorem 1.9 once again to $\ell_{(\mathcal{M},\psi')}$, we get our thesis. $\square$

Let us go back to the proof of Proposition 2.3. By Proposition 1.19, we can then consider a real value $\varepsilon = \varepsilon(\eta)$ with $0 < \varepsilon < \min_{i=1,\ldots,k} l_i$ such that for every admissible pair $(\vec{l}', \vec{b}')$ with $\left\|(\vec{l}, \vec{b}) - (\vec{l}', \vec{b}')\right\|_\infty \leq \varepsilon$, the relation $d_{match}(\ell_{(\mathcal{M},F)}, \ell_{(\mathcal{M},F')}) \leq \frac{\eta}{2}$ holds, where $\ell_{(\mathcal{M},F')}$ is the 1-dimensional size function corresponding to the half-plane $\pi_{(\vec{l}',\vec{b}')}$. By applying Lemma 2.4 twice and the monotonicity of $\ell_{(\mathcal{M},F')}$ in each variable (cf. Remark 1.6), we get the inequalities

$$\begin{aligned}\ell_{(\mathcal{M},F)}(s-\eta, t+\eta) &\leq \ell_{(\mathcal{M},F')}(s-\frac{\eta}{2}, t+\frac{\eta}{2}) \leq \\ &\leq \ell_{(\mathcal{M},F')}(s+\frac{\eta}{2}, t-\frac{\eta}{2}) \leq \ell_{(\mathcal{M},F)}(s+\eta, t-\eta).\end{aligned} \tag{2.3}$$

Because of equality (2.2) we have that the inequalities (2.3) imply

$$\begin{aligned}\ell_{(\mathcal{M},F)}(s-\eta, t+\eta) &= \ell_{(\mathcal{M},F')}(s-\frac{\eta}{2}, t+\frac{\eta}{2}) = \\ &= \ell_{(\mathcal{M},F')}(s+\frac{\eta}{2}, t-\frac{\eta}{2}) = \ell_{(\mathcal{M},F)}(s+\eta, t-\eta).\end{aligned} \tag{2.4}$$

Therefore, once again because of the monotonicity of $\ell_{(\mathcal{M},F')}$ in each variable, for every (s', t') with $\|(s, t) - (s', t')\|_\infty \leq \frac{\eta}{2}$ and for every $(\vec{l}', \vec{b}')$ with $\|(\vec{l}, \vec{b}) - (\vec{l}', \vec{b}')\|_\infty \leq \varepsilon$ the equality $\ell_{(\mathcal{M},F')}(s', t') = \ell_{(\mathcal{M},F)}(s, t)$ holds. By applying the Reduction Theorem 1.17 we get $\ell_{(\mathcal{M},\vec{\varphi})}(s' \cdot \vec{l}' + \vec{b}', t' \cdot \vec{l}' + \vec{b}') = \ell_{(\mathcal{M},\vec{\varphi})}(s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b})$. In other words, $\ell_{(\mathcal{M},\vec{\varphi})}$ is locally constant at the point $(\vec{x}, \vec{y})$, and hence $(\vec{x}, \vec{y})$ is not a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. $\square$

The following theorem associates the discontinuities of a multidimensional size function to the $(\vec{l}, \vec{b})$-pseudocritical values of $\vec{\varphi}$.

Theorem 2.5. *Let $(\vec{x},\vec{y}) \in \Delta^+$ be a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$, with $(\vec{x},\vec{y}) = (s\cdot\vec{l}+\vec{b}, t\cdot\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$. Then it follows that either $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ and s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$, or $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$ and t is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$, or both the previous conditions hold.*

Proof. By Proposition 2.3 we have that (s,t) is a discontinuity point for $\ell_{(\mathcal{M},F)}$, and from Corollary 1.11 it follows that either s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$ or t is a discontinuity point for $\ell_{(\mathcal{M},F)}(s,\cdot)$, or both these conditions hold. Let us now suppose that s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$. Since $\ell_{(\mathcal{M},F)}(\cdot,t)$ is monotonic, then there exists an arbitrarily small real value $\varepsilon > 0$ such that $\ell_{(\mathcal{M},F)}(s-\varepsilon,t) \neq \ell_{(\mathcal{M},F)}(s+\varepsilon,t)$. Moreover, the following equalities hold because of the Reduction Theorem 1.17:

$$\begin{aligned}\ell_{(M,F)}(s-\varepsilon,t) &= \ell_{(M,\vec{\varphi})}((s-\varepsilon)\cdot\vec{l}+\vec{b}, t\cdot\vec{l}+\vec{b}) = \ell_{(M,\vec{\varphi})}(\vec{x}-\varepsilon\cdot\vec{l},\vec{y})\\ \ell_{(M,F)}(s+\varepsilon,t) &= \ell_{(M,\vec{\varphi})}((s+\varepsilon)\cdot\vec{l}+\vec{b}, t\cdot\vec{l}+\vec{b}) = \ell_{(M,\vec{\varphi})}(\vec{x}+\varepsilon\cdot\vec{l},\vec{y}).\end{aligned}$$

By setting $\vec{\varepsilon} = \varepsilon\cdot\vec{l}$, we get $\ell_{(M,\vec{\varphi})}(\vec{x}-\vec{\varepsilon},\vec{y}) \neq \ell_{(M,\vec{\varphi})}(\vec{x}+\vec{\varepsilon},\vec{y})$. Therefore $\vec{x}$ is a discontinuity point for $\ell_{(M,\vec{\varphi})}(\cdot,\vec{y})$. Moreover, since s is a discontinuity point for $\ell_{(\mathcal{M},F)}(\cdot,t)$, from the Representation Theorem 1.9 it follows that s is the abscissa of a cornerpoint (possibly at infinity), and hence by Theorem 2.2 we have that s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$. Analogously we can examine the case that t is a discontinuity point for $\ell_{(\mathcal{M},F)}(s,\cdot)$, and get our statement. □

Before giving our final result, we need the following definition.

Definition 2.6. Let $\vec{\chi} : \mathcal{M} \to \mathbb{R}^h$ be a C^1 function. A point $P \in \mathcal{M}$ is said to be a *pseudocritical point for* $\vec{\chi}$ if the convex hull of the gradients $\nabla\chi_i(P)$, $i = 1,\ldots,h$, contains the null vector, i.e. there exist $\lambda_1,\ldots,\lambda_h \in \mathbb{R}$ such that $\sum_{i=i}^h \lambda_i\cdot\nabla\chi_i(P) = \mathbf{0}$, with $0 \le \lambda_i \le 1$ and $\sum_{i=1}^h \lambda_i = 1$. If P is a pseudocritical point of $\vec{\chi}$, then $\vec{\chi}(P)$ will be called a *pseudocritical value for* $\vec{\chi}$.

Remark 2.7. Definition 2.6 corresponds to the Fritz John necessary condition for optimality in Nonlinear Programming [1]. We shall use the term "pseudocritical" just for the sake of conciseness. For further references see [24]. The concept of the pseudocritical point is strongly related also to the one of Jacobi Set (cf. [16]).

In the following, we shall say that $\rho : \mathbb{R}^k \to \mathbb{R}^h$ is a *projection* if there exist h indices $i_1,\ldots,i_h$ such that $\rho((x_1,\ldots,x_k)) = (x_{i_1},\ldots,x_{i_h})$, for every $\vec{x} = (x_1,\ldots,x_k) \in \mathbb{R}^k$.

We are now ready to give the main result of this paper.

Theorem 2.8. *Let $(\vec{x},\vec{y}) \in \Delta^+$ be a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}$. Then at least one of the following statements holds:*

- **(i):** *$\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ and a projection ρ exists such that $\rho(\vec{x})$ is a pseudocritical value for $\rho\circ\vec{\varphi}$;*
- **(ii):** *$\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$ and a projection ρ exists such that $\rho(\vec{y})$ is a pseudocritical value for $\rho\circ\vec{\varphi}$.*

Proof. By Theorem 2.5 we have that either $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$, or $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$, or both these conditions hold. Let us now confine ourselves to assume that $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$ and prove that a projection ρ exists such that $\rho(\vec{x})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$. The proof in the case that $\vec{y}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\cdot)$ proceeds in quite a similar way. Consider the half-plane $\pi_{(\vec{l},\vec{b})}$ of the foliation containing the point $(\vec{x},\vec{y})$, and the pair (s,t) such that $(\vec{x},\vec{y}) = (s \cdot \vec{l} + \vec{b}, t \cdot \vec{l} + \vec{b})$. Since $\vec{x}$ is a discontinuity point for $\ell_{(\mathcal{M},\vec{\varphi})}(\cdot,\vec{y})$, by applying once more Theorem 2.5 we obtain that s is an $(\vec{l},\vec{b})$-pseudocritical value for $\vec{\varphi}$. Therefore, by Definition 2.1 there exist a point $P \in \mathcal{M}$ and some indices $i_1,\dots,i_h$ with $1 \le h \le k$, such that $s = F(P) = \frac{\varphi_{i_1}(P)-b_{i_1}}{l_{i_1}} = \cdots = \frac{\varphi_{i_h}(P)-b_{i_h}}{l_{i_h}}$ and $\sum_{j=1}^{h} \lambda_j \cdot \nabla\vec{\varphi}_{i_j}(P) = \mathbf{0}$, with $0 \le \lambda_j \le 1$ for $j = 1,\dots,h$, and $\sum_{j=1}^{h} \lambda_j = 1$. Let us now consider the projection $\rho : \mathbb{R}^k \to \mathbb{R}^h$ defined by setting $\rho(\vec{x}) = (x_{i_1},\dots,x_{i_h})$. Since $(\vec{x},\vec{y}) = (x_1,\dots,x_k,y_1,\dots,y_k) = (s\cdot l_1+b_1,\dots,s\cdot l_k+b_k,t\cdot l_1+b_1,\dots,t\cdot l_k+b_k)$, we observe that $x_{i_j} = \left(\frac{\varphi_{i_j}(P)-b_{i_j}}{l_{i_j}}\right) \cdot l_{i_j} + b_{i_j} = \varphi_{i_j}(P)$, for every $j = 1,\dots,h$. Therefore it follows that $\rho(\vec{x})$ is a pseudocritical value for $\rho \circ \vec{\varphi}$. □

Remark 2.9. We stress that Theorem 2.8 improves the result obtained in Theorem 2.5, providing a necessary condition for discontinuities of multidimensional size functions that does not depend on the foliation of the domain Δ^+.

Conclusions and future work

In this paper we have proved that a discontinuity point for a multidimensional size function has at least one pseudocritical coordinate, under the hypothesis that the considered measuring function is C^1. This result is a first step in the development of the theory for $\mathbb{R}^k$-valued measuring functions. Indeed, the localization of the unique points where k-dimensional size functions can be discontinuous allows us to better understand Topological Persistence and opens the way to the formulation of effective algorithms for its computation. On the other hand, several interesting problems need further investigation. First of all, more study is possible on how to generalize our results when dealing with less regular measuring functions, with particular reference to the piecewise C^1 case. Furthermore, it is worth noting that our framework could be applicable also to the study of discontinuities in persistent algebraic topology, including Persistent Homology Groups and Size Homotopy Groups. However, some difficulties could derive from the present lack of the analogue of Theorem 1.12 for those structures, i.e. a stability result in the case of *continuous* (possibly not tame [7]) measuring functions. These last research lines appear to be promising, both from the theoretical and the applicative point of view.

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