k-dimensional Size Functions for Shape Description and Comparison

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Abstract

This paper advises the use of k-dimensional size functions for comparison and retrieval in the context of multidimensional shapes, where by shape we mean something in two or higher dimensions having a visual appearance. The attractive feature of k-dimensional size functions is that they allow to readily establish a similarity measure between shapes of arbitrary dimension, taking into account different properties expressed by a multivalued real function defined on the shape. This task is achieved through a particular projection of k-dimensional size functions into the 1-dimensional case. Therefore, previous results on the stability for matching purposes become applicable to a wider range of data. We outline the potential of our approach in a series of experiments.

1. Introduction

Shape comparison plays a fundamental role in shape recognition, classification and retrieval, which are very lively research topics for the disciplines of Cognitive Science, Pattern Recognition, Computer Vision and Computer Graphics. Shape models carry a high value with them, and search engines able to match, classify and retrieve multidimensional visual media would be useful to speed-up content design, processing and re-use. Keyword-based annotation is not sufficient to achieve the necessary capability of resource exploration for digital shapes. Therefore, a variety of methods has been proposed in the literature to tackle the problem of *content-based* digital shape analysis and retrieval.

Recently, there has been an increasing interest towards geometrical-topological methods for shape comparison, whose main idea is to perform a topological exploration of the shape according to some quantitative geometric properties provided by a real function defined on the shape [12, 5, 4, 3]. The real function plays the role of a *lens* through which we look at the properties of the shape. A common scenario, however, is to have two or more function defined on the same shape, carrying information on different features of the phenomenon under study. Examples arise in the context of computational biology, in medical environments, as well as in scientific simulations of natural phenomena. Therefore, a great challenge is to develop and define tools to extract knowledge from high-dimensional data, by means of the concurrent analysis of different properties conveyed by different real functions.

In this context, the aim of this paper is to illustrate how recent results in Multidimensional Size Theory [6] can be effectively used to analyze and compare 3D digital shapes (represented by surface or volume models) equipped by multivalued functions instead of scalar functions. In Multidimensional Size Theory, indeed, the approach to shape discrimination and comparison is based on the use of a mathematical descriptor named *k-dimensional size function*. The idea is to compare shape properties that are described by functions taking values in $\mathbb{R}^k$, called *k-dimensional measuring functions*, defined on topological spaces associated to the objects to be studied.

Although the special case of 1-dimensional size functions has already revealed to be useful when applied to the problem of comparing images [7] or 3D-models represented by closed surfaces [2], in this paper we show that k-dimensional size functions show a higher discriminatory power, especially when the dimension of the objects to be studied increase.

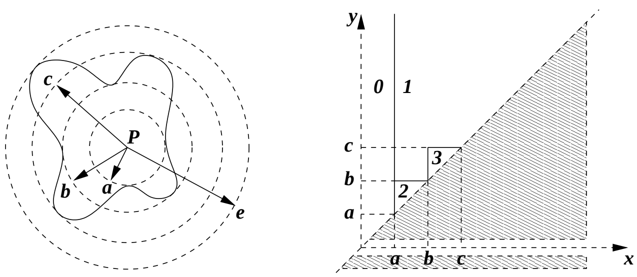
The rest of the paper is organized as follows. After introducing size functions and briefly discussing their properties in the 1-dimensional case in Sec. 2, we summarize the definition of the k-dimensional size functions and how they can be reduced to the 1-dimensional case (Sec. 3). Computational aspects related to the computation of k-dimensional size functions are proposed in Sec. 4. Sec. 5 shows how the effective application to different kind of shapes (triangle meshes and 3D images) makes this framework flexible and independent of the shape representation. Conclusions and suggestions on future developments end the paper.

2. Background: 1-dimensional size functions

This particular case of 1-dimensional size functions (1SF's) has been extensively studied in recent past years, both from the theoretical point of view [9, 12, 13, 15, 18] and the computational one [2, 7, 17, 19], showing quite a lot of interesting properties that turn out to be useful in our approach to the multidimensional problem.

Let us consider a pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a non-empty, compact, locally connected Hausdorff space endowed with a finite number of connected components, and $\varphi : \mathcal{M} \to \mathbb{R}$ is a continuous function. Every such a pair is called a *size pair*, while each function φ is called a *(1-dimensional) measuring function* and its purpose is to encode quantitative properties of the shape $\mathcal{M}$.

Given a size pair $(\mathcal{M}, \varphi)$, the *(1-dimensional) size function* $\ell_{(\mathcal{M},\varphi)} : \{(x, y) \in \mathbb{R}^2 : x < y\} \to \mathbb{N}$ can be easily defined by setting $\ell_{(\mathcal{M},\varphi)}(x, y)$ equal to the number of connected components of the lower level set $\mathcal{M}_y = \{P \in \mathcal{M} : \varphi(P) \leq y\}$, containing at least one point of $\mathcal{M}_x$.



(a) (b)

Figure 1. A size pair and the associated 1-dimensional size function.

Figure 1(left) shows an example of a size pair $(\mathcal{M}, \varphi)$, where $\mathcal{M}$ is a closed curve and the chosen measuring function φ is defined as the Euclidean distance from the point P. Figure 1(right) represents the 1-dimensional size function associated to $(\mathcal{M}, \varphi)$. Since φ takes value in $\mathbb{R}$, the domain Δ^+ of $\ell_{(\mathcal{M},\varphi)}$ is the subset of the real plane defined as $\{(x, y) \in \mathbb{R}^2 : x < y\}$. Two of the most interesting properties of 1-dimensional size functions are their resistance to noise (useful especially in applications) and their modularity: in particular, 1SF's inherit their invariance properties directly from the chosen measuring functions. As an example, we observe that it would be possible to apply rotations around P to the closed curve, being sure that no changes occurs in the related 1-dimensional size function.

As can be seen in Figure 1, 1SF's present a typical structure: Δ^+ is divided by solid lines, representing the discontinuity points of the 1-dimensional size function, into triangular regions (that may be bounded or unbounded). In all these regions the value of $\ell_{(\mathcal{M},\varphi)}$ is constant. In the considered example, the values of $\ell_{(\mathcal{M},\varphi)}$ in every triangular region is equal to the numbers displayed. Thanks to this particular structure, each 1SF can be seen as a linear combination (with natural numbers as coefficients) of characteristic functions of triangles. Hence, by taking the formal series of vertices associated to their right angles, called *cornerpoints* for the bounded triangles and *cornerlines* for the unbounded ones, we get a simple and compact representation.

Each distance between formal series naturally produces a distance between 1SF's: The idea is to compare 1-dimensional size functions by measuring the cost of transporting the cornerpoints and cornerlines of a 1-dimensional size function to those of the other one. Two 1SF's represented by formal series can be compared by using different metrics, e.g., the matching distance or the Hausdorff metric.

For details and basic notions about (1-dimensional) size theory see [9, 12, 13, 15, 18].

3. Multidimensional Size Functions

This Section is devoted to the introduction of preliminary definitions generalizing the concept of 1SF's to measuring functions taking values in $\mathbb{R}^k$, and to the presentation of some recent results about k-dimensional size functions (kSF's) [6]. In particular, we will show how a suitable change of variables permits to reduce multidimensional size functions to the 1-dimensional case, making them suitable for computation and use in concrete applications.

3.1. Preliminary definitions

We shall now generalize the definitions given in the previous Section for the 1-dimensional case, and call a *size pair* any pair $(\mathcal{M}, \vec{\varphi})$, where $\mathcal{M}$ is a non-empty, compact, locally connected Hausdorff space endowed with a finite number of connected components, and $\vec{\varphi} : \mathcal{M} \to \mathbb{R}^k$ is a continuous function. The function $\vec{\varphi}$ is said to be a k-*dimensional measuring function*, and can be seen like a descriptor of those features that are considered to be relevant in comparing $(\mathcal{M}, \vec{\varphi})$ with other size pairs.

The following relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^k$: for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, we shall say $\vec{x} \preceq \vec{y}$ (resp. $\vec{x} \prec \vec{y}$) if and only if $x_i \leq y_i$ (resp. $x_i < y_i$) for every index $i = 1, \ldots, k$. According to these notations, for every $\vec{y} \in \mathbb{R}^k$ consider the set $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\rangle = \{P \in \mathcal{M} : \varphi_i(P) \leq y_i,\ i = 1, \ldots, k\}$.

In this setting, the *k-dimensional size function* of the size pair $(\mathcal{M}, \vec{\varphi})$ is the function $\ell_{(\mathcal{M},\vec{\varphi})} : \{(\vec{x}, \vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\} \to \mathbb{N}$ taking each point $(\vec{x}, \vec{y})$ of the domain into the number of the connected components in $\mathcal{M}\langle\vec{\varphi} \preceq \vec{y}\rangle$ containing at least one point of $\mathcal{M}\langle\vec{\varphi} \preceq \vec{x}\rangle$.

3.2 Reduction to the 1-dimensional case

When dealing with k-dimensional size functions, we have to face some problems: (i) the analogues for cornerpoints and cornerlines (see Section 2) seem not to exist, meaning that we are not able to represent kSF's by formal series; (ii) a direct approach to the multidimensional case implies working in subsets of $\mathbb{R}^k \times \mathbb{R}^k$: In this case, the absence of a compact representation for k-dimensional size functions involves great efforts from a computational point of view. All these problems can be by-passed by means of a suitable change of variables, introduced in [6], that allows us to reduce k-dimensional size functions to the 1-dimensional case. Indeed, it has been demonstrated that there exists a parameterized family of half-planes in $\mathbb{R}^k \times \mathbb{R}^k$ such that the restriction of $\ell_{(\mathcal{M},\vec{\varphi})}$ to each of these planes can be seen as a particular 1-dimensional size function.

Let $(\mathcal{M},\vec{\varphi})$ be a size pair, with $\vec{\varphi} = (\varphi_1,\ldots,\varphi_k) : \mathcal{M} \to \mathbb{R}^k$. We shall call *admissible pair* any pair $(\vec{l},\vec{b}) \in \mathbb{R}^k \times \mathbb{R}^k$ with $\vec{l}$ unit vector such that $l_i > 0$, $i = 1,\ldots,k$, and $\sum_{i=1}^k b_i = 0$. The set of all admissible pairs will be denoted by Adm_k. In this setting, consider the foliation of the open set $\Delta^+ = \{(\vec{x},\vec{y}) \in \mathbb{R}^k \times \mathbb{R}^k : \vec{x} \prec \vec{y}\}$ given by the parameterized family of half-planes $\{\pi_{(\vec{l},\vec{b})}\}_{(\vec{l},\vec{b})\in Adm_k}$ defined by the parametric equations:

$$\begin{cases} \vec{x} = s\vec{l} + \vec{b} \\ \vec{y} = t\vec{l} + \vec{b} \end{cases}$$

with $s,t \in \mathbb{R}$, $s < t$. Under these assumptions, in [6] the following result has been proved:

Theorem 1. *Let $(\vec{l},\vec{b})$ be an admissible pair, and $F^{\vec{\varphi}}_{(\vec{l},\vec{b})} : \mathcal{M} \to \mathbb{R}$ be defined by setting*

$$F^{\vec{\varphi}}_{(\vec{l},\vec{b})}(P) = \max_{i=1,\ldots,k} \left\{ \frac{\varphi_i(P) - b_i}{l_i} \right\} .$$

Then, for every $(\vec{x},\vec{y}) = (s\vec{l}+\vec{b}, t\vec{l}+\vec{b}) \in \pi_{(\vec{l},\vec{b})}$ the following equality holds: $\ell_{(\mathcal{M},\vec{\varphi})}(\vec{x},\vec{y}) = \ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}(s,t)$.

In other words, Theorem 1 states that a foliation of Δ^+ in half-planes can be given, such that the restriction of a k-dimensional size function to these half-planes turns out to be a classical size function in two scalar variables. This result implies that each size function, with respect to a k-dimensional measuring function, can be completely and compactly described by a parameterized family of discrete descriptors: Indeed, we can associate a formal series $\sigma_{(\vec{l},\vec{b})}$ (see Section 2) with each admissible pair $(\vec{l},\vec{b})$, with $\sigma_{(\vec{l},\vec{b})}$ describing the 1-dimensional size function $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$.

Denoting by $d(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})})$ the (matching) distance [9] between the 1-dimensional size functions $\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}$ and $\ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}$ induced by the related representations by formal series, the distance between two k-dimensional size functions $\ell_{(\mathcal{M},\vec{\varphi})}$, $\ell_{(\mathcal{N},\vec{\psi})}$ can be defined as

$$D(\ell_{(\mathcal{M},\vec{\varphi})}, \ell_{(\mathcal{N},\vec{\psi})}) = \sup_{(\vec{l},\vec{b})\in Adm_k} \min_{i=1,\ldots,k} l_i \cdot d(\ell_{(\mathcal{M},F^{\vec{\varphi}}_{(\vec{l},\vec{b})})}, \ell_{(\mathcal{N},F^{\vec{\psi}}_{(\vec{l},\vec{b})})}).$$

If we choose a non-empty and finite subset $A \subseteq Adm_k$, and we substitute $\sup_{(\vec{l},\vec{b})\in Adm_k}$ with $\max_{(\vec{l},\vec{b})\in A}$ in the definition of $D(\ell_{(\mathcal{M},\vec{\varphi})}, \ell_{(\mathcal{N},\vec{\psi})})$, then we obtain a distance between k-dimensional size function, which is computable and suitable for applications.

4. Computational aspects

From the computational point of view, the reduction of k-dimensional size functions to the 1-dimensional case allows us to use the existing framework for computing 1SF's. In this discrete setting, the counterpart of a size pair is given by a *size graph* (G,φ), where $G = (V(G), E(G))$ is a finite graph, with $V(G)$ and $E(G)$ the set of vertices and edges respectively, and $\varphi : V(G) \to \mathbb{R}$ is a measuring function labeling the nodes of the graph [8].

In this paper, we deal with models represented by triangle meshes and black-and-white voxel images. In the first case, the size graph is made of the vertices and the edges of the triangle mesh. In the latter case, $V(G)$ corresponds to the image voxels and $E(G)$ represents the 18-neighborhood connectivity.

Once the size graph has been built, the computational complexity for computing k-dimensional size functions on a single half-plane of a given foliation is $O(n \log n + m \cdot \alpha(2m+n,n))$, where n and m are the number of vertices and edges in the size graph, respectively, and α is the inverse of the Ackermann function [8].

Statistics related to the time required to compute the descriptor for a single half-plane of a given foliation and the storage size are reported in Table 1. The values show that size functions are fast to compute and easy to store, for both vector and raster models. These results have been obtained on a 1.5GH Pentium 4, RAM 1M.

5. Experiments

The aim of this Section is to analyze the potential of the proposed approach for comparing and retrieving 2- and 3-dimensional data, using both vectorial (i.e., triangle meshes) and raster (voxel images) representations.

Model	$\mid V \mid$	$\mid E \mid$	Time	Descriptor
Human1	5772	11540	0.088s	$<$1k
Human2	5775	11546	0.089s	$<$1k
Teddy	12831	25658	0.131s	$<$1k
Plier	14844	95699	0.256s	$<$1k
Cup	121329	760717	0.680s	$<$1k
Chair	30487	197759	2.316s	$<$1k

Table 1. Statistics relating the dimension of a model (in terms of number of vertices and edges) with the time to compute multidimensional size functions on a single half-plane of a foliation, and with the storage size of the descriptor. The first three models are triangle meshes, the latter three are voxel images.

To perform our tests we have considered the database of 280 triangle meshes classified in 14 classes of 20 models used in [1] and the McGill 3D Shape Benchmark [16] that offers about 420 volume models, classified in 19 classes.

5.1 Triangle meshes

In order to compare and retrieve the triangle meshes in the first database [1], we have defined a bi-dimensional measuring function $\vec{\varphi} = (\varphi_1, \varphi_2)$. Here φ_1 is a normalized Euclidean distance from the barycentre of the model, and φ_2 is a normalization of the the averaged geodesic distance proposed in [14]. More formally,

$$\varphi_1(v) = 1 - \frac{\mid v - B \mid_E}{\varphi_{1M}},$$

where B denotes the barycentre of the mesh and $\varphi_{1M} = \max_{v_i \in V} \mid v_i - B \mid_E$; and

$$\varphi_2(v) = 1 - \frac{\sum_i g(v, b_i) \cdot area(b_i)}{\varphi_{2M}},$$

where g represents the geodesic distance, $\{b_i\} = \{b_0, \ldots, b_k\}$ is an almost uniform sampling of the vertices of the mesh, $area(b_i)$ is the area of the neighborhood of b_i and $\varphi_{2M} = \max_{v_j \in V} \sum_i g(v_j, b_i) \cdot area(b_i)$.

The foliation of $\mathbb{R}^2 \times \mathbb{R}^2$ chosen in our experiments is the following one:

$$\begin{cases} x_1 = s\cos\theta + b \\ x_2 = s\sin\theta - b \\ y_1 = t\cos\theta + b \\ y_2 = t\sin\theta - b. \end{cases}$$

In Figure 2 we depict the 2-dimensional size functions obtained combining φ_1 and φ_2, restricted to three different half-planes of the foliation above. From left to right, for each model, we represent the size functions obtained from the following pairs of parameter vectors: $\vec{l_1} = (\frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{\sqrt{3}-1}{2\sqrt{2}})$, $\vec{b_1} = (0,0)$, $\vec{l_2} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\vec{b_2} = (0,0)$ and $\vec{l_3} = (\frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}})$, $\vec{b_3} = (0,0)$. Vectors $\vec{l_i}$ correspond to values of the angle θ of $\frac{\pi}{12}$, $\frac{\pi}{4}$, and $\frac{11\pi}{12}$ respectively. Each row in the Figure depicts, from left to right, a model and its 1SF associated to the three half-planes defined above.

Observing the first two rows, we can notice how the same structure in size functions corresponds to the similarity between shapes, resulting in a high discriminatory power. Indeed, the size functions of the two human models are each other very similar, while those of the third object are quite different.

Notice also how size functions homogeneously evolve over the half-planes of the foliation. Indeed the shape information conveyed by the multivalued measuring functions is distributed over the different half-planes. This means that the similarity (or dissimilarity) between objects can be evaluated by concurrently analyzing different shape properties. In other words, what we expect is that the k-dimensional size functions of similar objects are close one to the other over the whole foliation, thus guaranteeing also robustness with respect to small changes and perturbations of the model.

5.2 Voxel images

As a first approach, in order to describe the raster models in [16], we have chosen a 3-dimensional measuring function that respect the grid orientation of the voxels. In other words, we discriminate the models with respect to their spatial extent. Thus, denoting $B = (B_x, B_y, B_z)$ the coordinates of the center of mass of the model, for each voxel $v = (v_x, v_y, v_z)$ we have considered the 3D measuring function $\vec{\varphi} = (\varphi_x, \varphi_y, \varphi_z)$, where:

$$\begin{cases} \varphi_x = - \mid v_x - B_x \mid_E \\ \varphi_y = - \mid v_y - B_y \mid_E \\ \varphi_z = - \mid v_z - B_z \mid_E \,. \end{cases}$$

The aim of Table 2 is to show the stability of the distance defined in Section 3.2. The distances between six different objects in our database (two spiders, two cups and two manufactured models) are reported. These results are obtained using $\vec{l} = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ and $\vec{b} = (0,0,0)$ as naive parameters to define an half-plane of $\mathbb{R}^3 \times \mathbb{R}^3$. As expected, the comparison framework satisfies the identity property, guaranteeing that a model has a null distance from itself. In addition, the distance between two models in the same class is significantly smaller than the distance between objects belonging to different classes (e.g. a spider and a manufactured model).

	0.00	1.59	7.77	7.77	23.54	23.99
	1.59	0.00	7.77	7.77	24.71	25.15
	7.77	7.77	0.00	3.42	22.63	23.08
	7.77	7.77	3.42	0.00	20.07	20.51
	23.54	24.71	22.63	20.07	0.00	1.25
	23.99	25.15	23.08	20.51	1.25	0.00

Table 2. Matching distances between six different models in our database over a single plane of a foliation. Computing 420 $\times$ 420 comparisons between the models in the database requires 9.68s.

Finally, we show how the synergy of more shape properties, analyzed by means of multidimensional measuring functions, better characterizes the elements of a class. Figure 3 exhibits what happens when φ_x, φ_y, φ_z and $\vec{\varphi} = \varphi_{xyz}$ are considered, either φ_x, φ_y, φ_z alone as 1-dimensional measuring functions (first three columns), or combined in a 3-dimensional measuring function $\vec{\varphi}$ (last column). In addition, we compare the performance of $\vec{\varphi}$ with respect to a 1-dimensional measuring function which is independent of the spatial embedding, namely the distance from the barycentre (φ_1). In the columns of Figure 3 we rank the firstly retrieved items in the 3D image database [16], when the query is the model on the top row. It can be seen that the performance of $\vec{\varphi}$ improves the retrieval results, diminishing the number of false positives. These results show that the k-dimensional size functions are promising, and we foresee their usage for retrieval of multidimensional digital shapes.

6. Conclusions

In this paper we have sketched how recent results in Multidimensional Size Theory can be effectively used to analyze and compare 3D digital shapes (represented by surface or volume models) equipped by multivalued functions instead of scalar functions. The added value of this approach relies in the fact that it provides a modular framework based on the idea of describing shapes by geometrical-topological properties of different real functions. We are currently working at the development of more measuring functions, in order to analyze different kind of shape features of 3-dimensional models.

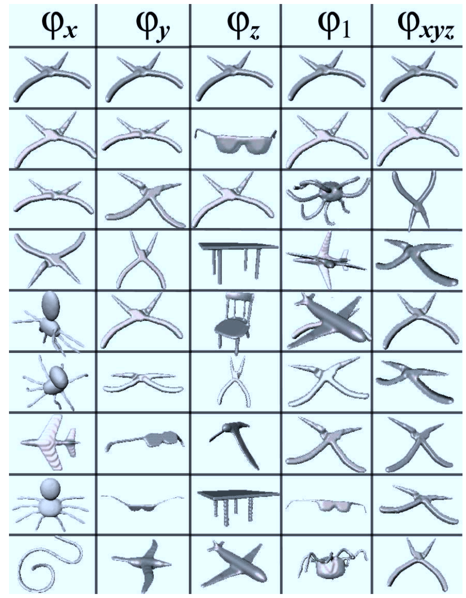


Figure 3. Top retrieval results when four single measuring functions and the 3-dimensional size function that combines φ_x, φ_y and φ_z are used. Results are depicted in every column in increasing order of distance from the first model.

Moreover, the first experimental results encourage us to further investigate the application of this modular theoretical framework to higher dimensional and also time-dependent data.

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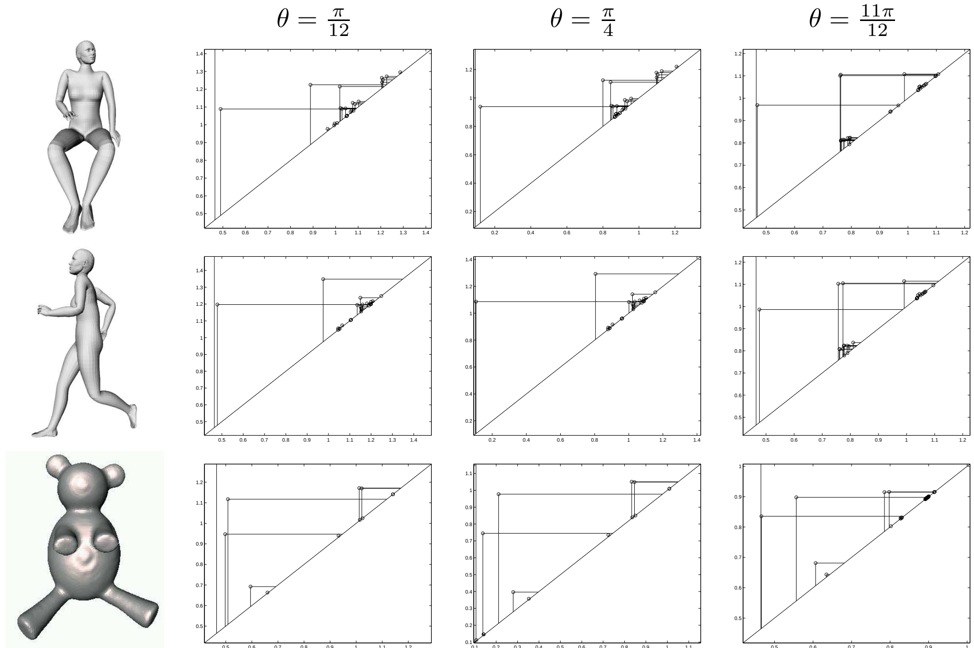

Figure 2. 1-dimensional size functions of three models over the half-planes of $\mathbb{R}^2 \times \mathbb{R}^2$ defined by the parameters ($\theta = \frac{\pi}{12}$, $\theta = \frac{\pi}{4}$, $\theta = \frac{11\pi}{12}$) and $\vec{b} = (0, 0)$.

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