Advertising in a Differential Oligopoly Game*

ROBERTO CELLINI •
Università di Catania, Facoltà di Economia,
Dipartimento di Economia e Metodi Quantitativi

and

LUCA LAMBERTINI §
Università di Bologna, Facoltà di Scienze Politiche,
Dipartimento di Scienze Economiche

(October 2001)

Abstract - We illustrate a differential oligopoly game where firms compete *à la* Cournot in homogeneous goods in the market phase, and invest in advertising activities aimed at increasing consumers' reservation price. Such investments produce external effects, characterizing the advertising activity as a public good. We derive the open-loop and the closed-loop Nash equilibria, and show that the properties of the equilibria depend on the curvature of the market demand function. The comparative assessment of these equilibria shows that firms' advertising efforts are larger in the open-loop than in the closed-loop equilibrium. We also show that a cartel involving all firms, setting both quantities and advertising efforts so as to maximize joint profits, may produce a steady state where social welfare is higher than the social welfare levels associated with both the non-cooperative settings.

Keywords: advertising, differential games, capital accumulation, open-loop equilibria, closed-loop equilibria.

JEL Classification: D43, D92, L13, M37.

---

* We thank Vincenzo Denicolò, Emilio Giardina and Gianpaolo Rossini for helpful discussion and suggestions. The usual disclaimer applies.

• Address: Corso Italia 55, 95129 Catania, Italy.
e-mail: cellini@unict.it ; phone: +39-095-375344 (ext. 237); fax: +39-095-370574.

§ Address: Strada Maggiore 45, 40125 Bologna, Italy.
e-mail: lamberti@spbo.unibo.it ; phone: +39-051-2092623; fax +39-051-2092664.

Advertising in a Differential Oligopoly Game

1. Introduction

The existing literature on dynamic models of advertising can be broadly partitioned into two main categories. The first originates from Vidale and Wolfe (1957), and is characterized by a direct relation between the rate of change in sales and the advertising efforts of firms. The second dates back to Nerlove and Arrow (1962), and considers advertising as an instrument to increase the stock of goodwill or reputation, summarising the effects of past and current advertising expenditures carried out by a firm, on the current demand for her goods. A relevant result emerging from the Nerlove-Arrow model is the dynamic version of the well known Dorfman-Steiner (1954) condition, establishing that the advertising investment is proportional to sales.[1]

We present a dynamic model of oligopoly with homogeneous products, where firms compete *à la* Cournot in the market phase, and invest in advertising activities aimed at increasing consumers' reservation price over time. The advertising investment carried out by any firm spills over to the rivals as long as the market demand function – the same for all firms – shifts outwards. Accordingly, this kind of advertising is a public good.

We consider a very general formulation of the demand function, accounting for concavity, lenearity and convexity. Indeed, there emerges that the dynamic properties of the equilibria drastically depend upon the degree of the market demand curvature. In the standard case of linear demand, no sensible equilibria exist. If demand is concave, one (saddle-path) equilibrium emerges. If demand is convex, the equilibrium is unstable.

The determinants of the steady state investment and reservation price are first analyzed in the non-cooperative game, using alternatively the open-loop and the closed-loop equilibria as solution concepts. Then, we analyze (i) the optimal decisions by a cartel made up by all firms in the market, aiming at the maximum joint profits with respect to

[1] For the first class of games, see Leitmann and Schmitendorf (1978), Feichtinger (1983) and Erickson (1985); for the second, see Sethi (1977), Fershtman (1984) and Jørgensen and Zaccour (1999), *inter alia*. For exhaustive surveys, see Jørgensen (1982), Feichtinger, Hartl and Sethi (1994) and Dockner, Jørgensen, Van Long and Sorger (2000, ch. 11).

both output and advertising effort, and (ii) the optimal behavior for a social planner choosing output levels and advertising investments of all firms so as to maximize social welfare. Also in these cases, the dynamic properties of the steady state depend on the curvature degree of the market demand function. Reservation prices, investments and production decisions are compared across the steady states of all the aforementioned regimes. A relevant result is that social welfare in steady state may well be larger under in the cartel equilibrium, than in the non-cooperative Cournot equilibrium (irrespectively of whether the latter is derived under the open-loop or the closed-loop information structure). This is due to the externality effect affecting firms' advertising activities, and the associated inefficiency affecting the Cournot setting.

The structure of the paper is as follows. Section 2 illustrates the basic setup. Sections 3 and 4 solve the model under the open-loop and the closed-loop information structure, respectively. Section 5 compares the steady states derived in section 4 and evaluates them against the existing literature. Section 6 deals with of the optimal cartel behaviour and the first best solution that a social planner would implement. The comparative assessment of all equilibria is in Section 7. Section 8 briefly concludes.

2. The basic model

We consider an oligopoly under full information. At any time $t \in (0, +\infty)$, n firms sell a homogenous good. Marginal production costs are assumed to be identical across firms and normalized to zero for the sake of simplicity.

The market demand curve is borrowed from Anderson and Engers (1992, 1994)[2]:

(1) $$P(t) = [A(t) - Q(t)]^{\frac{1}{a}},$$

[2] The current literature on oligopoly theory usually adopts a linear market demand, with either homogeneous or differentiated goods; for exhaustive surveys, see Tirole (1988), Martin (1993), and Vives (2000) *inter alia*. A relatively scanty attention has been devoted to the analysis of the effects of market demand curvature on firms' strategic behavior; such a problem is studied by Anderson and Engers (1992, 1994) and Lambertini (1996).

where $P(t)$ denotes the price, $Q = \sum_{i=1}^{n} q_i$ the aggregate quantity of the good, $A(t)$ is the reservation price, and $a \in (0,+\infty)$ is a positive parameter determining the curvature of demand. The inverse demand function (1) is convex if $a \in (0,1)$, is concave if $a \in (1,+\infty)$, and is linear if $a = 1$.

It is easy to verify that the price elasticity of demand is, in absolute value, $e_{Q,P} = a(A-Q)/Q$, while the consumers' surplus evaluated at Q=A, i.e., the total area below the inverse demand curve, is $CS_{Q=A} = aA^{(1+a)/a} / (1+a)$, which also measures social welfare at the perfectly competitive equilibrium. This means that the curvature parameter a affects (i) the price elasticity; (ii) market size; and (iii) total surplus. Moreover, for a given a, the higher is A, the larger is market size; likewise, given A, the higher is a, the larger is market size.

Firms are able to increase $A(t)$ over time through investment in advertising campaigns. Formally, the following differential equation describing the dynamics of variable $A(t)$ amounts to saying that the market size increases as result of the sum of advertisement activities of all firms, and, furthermore, it is subject to a constant depreciation rate d :

(2) $$\frac{dA(t)}{dt} = \sum_{i=1}^{n} k_i(t) - dA(t) .$$

This type of advertising is indeed a pure public good, in that the effort carried out by any firm benefits all firms alike (see Fershtman, 1984; Fershtman and Nitzan, 1991); accordingly, it is sometimes referred to as *cooperative*, with the implicit *caveat* that firms do not necessarily cooperate in the sense of joint profit maximisation.[3]

The advertisement activity of each firm entails a quadratic cost $c_i(k_i(t)) = b \cdot (k_i(t))^2$, with b>0. Of course, the spillover effect characterising the advertising activity trivially entails that the individually optimal advertising effort is undersized from the standpoint of the whole population of firms.

[3] This labelling dates back to Friedman (1983); see also Martin (1993, ch. 6). For a model where advertising is both cooperative and predatory, see Piga (2000, pp. 517-21).

The dynamic problem can be summarized as follows: the objective of each firm is to achieve the maximum present value of the flows of future profits, taking into consideration that the price depends on the production decisions of all firms, the costs are represented by the individual investment costs in advertisement (which spills over to all rivals), and that a positive discounting rate r , common to all firms, applies to future costs and benefits. Formally, each individual firm faces the problem:

$$(3)\qquad \begin{aligned} &Max\ J_i = \int_0^{+\infty} e^{-rt} \cdot \left\{ [A(t) - q_i(t) - Q_{-i}(t)]^{\frac{1}{a}} q_i(t) - b(k_i(t))^2 \right\} dt \\ &s.t. \qquad \frac{dA(t)}{dt} = k_i(t) + K_{-i}(t) - dA(t) \end{aligned}$$

where $Q_{-i} = \sum_{j \neq i} q_j$ and $K_{-i} = \sum_{j \neq i} k_j$.

The choice variables of firm i are $q_i(t)$ and $k_i(t)$. $A(t)$ is the state variable, common to all agents, subject to the initial condition $A(0)=A_0>0$.

The solution of this dynamic optimization problem by player i is based on the Hamiltonian function H_i. Let $m_i(t)$ be the co-state variable associated to the state variable $A(t)$ by player i, and let $l_i(t) = m_i(t)e^{rt}$ be the current-value co-state variable. The Hamiltonian function is therefore:

$$(4)\qquad H_i = \left\{ [A(t) - q_i(t) - Q_{-i}(t)]^{\frac{1}{a}} q_i(t) - b(k_i(t))^2 + l_i(t)[k_i(t) + K_{-i}(t) - dA(t)] \right\} e^{-rt} .$$

The next sections solve the fully non-cooperative problem, according to the open-loop Nash equilibrium concept, and the closed-loop memoryless Nash equilibrium concept, respectively. The difference between these two concepts rests on the possibility of modifying the plans, once the game has started. In particular, under the former solution concept, firms precommit their decisions on the control variables to a given time path: they design the optimal plan at the initial time and then stick to it forever. Under the latter solution concept, firms do not precommit on any path and their decisions at any instant t depend on all the preceding history, and specifically on the value of the state variable at that instant.

The closed-loop solution is strongly time consistent and therefore subgame perfect, while the open-loop solution is only weakly time-consistent, i.e., it is not sub-game perfect.[4]

3. The open-loop Nash equilibrium

The first order (necessary) conditions to characterize the open-loop Nash equilibrium are:[5]

$$(5)\quad \begin{cases} \dfrac{\partial H_i(t)}{\partial q_i(t)} = 0 \\ \dfrac{\partial H_i(t)}{\partial k_i(t)} = 0 \\ -\dfrac{\partial H_i(t)}{\partial A(t)} = \dfrac{d\,l_i(t)}{dt} - r\,l_i(t) \end{cases}$$

along with the initial condition and the transversality condition $\lim_{t\to+\infty} A(t)\,l_i(t)e^{-rt} = 0$.

The first condition of system (5) turns out to be:

$$(6)\quad \frac{\partial H_i(t)}{\partial q_i(t)} = -\frac{1}{a}\left[A(t) - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{a}-1} + \left[A(t) - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{a}} = 0\,,$$

from which the Cournot reaction curve can be obtained. Under the symmetry condition $q_i(t)=q(t)$, which implies $Q_{-i}(t)=(n-1)q$, the optimal quantity to be produced by each firm is:

[4] Another strongly time-consistent (and therefore subgame perfect) solution concept is the feedback equilibrium using Bellman's equation. The feedback equilibrium is a closed-loop equilibrium, while the opposite is not true in general. For a clear exposition of the difference among these equilibrium solutions see Basar and Olsder (1982, pp. 318-327, and chapter 6, in particular Proposition 6.1). There exist classes of games where the closed-loop and the open-loop solutions coincide (see Mehlmann, 1988, ch. 4; Reinganum, 1982; Fershtman, 1987; Fershtman, Kamien and Muller, 1992; Dockner, Jørgensen, Van Long and Sorger, 2000, ch. 7, Cellini and Lambertini, 2001).

[5] Second order conditions are omitted throughout the paper, for the following reason. When demand is concave, they are always met in that the problem is concave and single-peaked; when demand is either linear or convex, second order conditions are irrelevant in that there exists no stable equilibrium, as it will become clear in the remainder of the analysis.

(6') $$q(t) = \frac{aA(t)}{an+1}.$$

The second condition of system (5) is

(7) $$\frac{\partial H_i(t)}{\partial k_i(t)} = -2bk_i(t) + l_i(t) = 0,$$

from which we can easily derive $l_i(t) = 2bk_i(t)$ and, by differentiating w.r.t. time:

(7') $$\frac{dk_i(t)}{dt} = \frac{1}{2b} \cdot \frac{d l_i(t)}{dt}.$$

Finally, from the third condition of system (5) we derive:

(8) $$\frac{d l_i(t)}{dt} = (r + d)\, l_i(t) - \frac{q_i(t)}{a}\left[A(t) - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{a}-1}.$$

Considering simultaneously conditions (7') and (8) we obtain:

(9) $$\frac{dk_i(t)}{dt} = (r + d)k_i(t) - \frac{q_i(t)}{2ba}\left[A(t) - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{a}-1}.$$

Now we are in a position able to completely characterize the dynamic system under optimal decision rules by part of Cournot firms, and under symmetry conditions $q_i(t)=q(t)$, $k_i(t)=k(t)$:

(10) $$\begin{cases} \dfrac{dA(t)}{dt} = nk(t) - dA(t) \\ \dfrac{dk(t)}{dt} = (r + d)k(t) - \dfrac{1}{2b}\left[\dfrac{A(t)}{an+1}\right]^{\frac{1}{a}} \end{cases}$$

A phase diagram can be easily drawn to represent this dynamic system. In steady state, $\mathrm{d}A(t)/\mathrm{d}t$=0 implies $k = dA/n$ while $\mathrm{d}k(t)/\mathrm{d}t$=0 implies $k = A^{1/a}/[2b(r+d)(an+1)^{1/a}]$. In the space ($A$,$k$), the former relation corresponds to a straight line with positive slope equal to d/n, while the latter corresponds to a curve which may be convex (if $a \in (0,1)$), concave (if $a \in (1,+\infty)$), or linear (if $a = 1$). The three panels of Figure 1 show the alternative cases. The linear case is shown under the hypothesis that the locus $\mathrm{d}A(t)/\mathrm{d}t$=0 lies above the locus $\mathrm{d}k(t)/\mathrm{d}t$=0, but the opposite case could be equally possible, as well as the case that the two loci coincide. The arrows are drawn taking into account that above (below) the line $\mathrm{d}A(t)/\mathrm{d}t$=0, the state variable A increases (decreases), and above (below) the curve $\mathrm{d}k(t)/\mathrm{d}t$=0, variable k increases (decreases). The dotted line identifies the saddle path.

The dynamic system admits one meaningful steady state solution, besides the trivial one (k*=0, A*=0) :

$$(10) \qquad A^* = \left(\frac{1}{an+1}\right)^{\frac{1}{a-1}} \cdot \left(\frac{n}{2bd(r+d)}\right)^{\frac{a}{a-1}} \quad ; \quad k^* = dA^*/n\,.$$

This solution does not exist in the particular case where $a = 1$, as it clear in the graphical representation. Correspondingly, it is immediate to find the amount of the individual production level in steady state: $q^* = aA^*/(an+1)$. The steady state values of the relevant variables are also reported in Table 1 (Section 7), in order to ease comparisons across different regimes.

INSERT FIGURE 1 HERE

The linearization of the dynamic system is as follows:

(10')
$$\begin{bmatrix} \dfrac{dA(t)}{dt} \\ \\ \dfrac{dk(t)}{dt} \end{bmatrix} = \Omega \begin{bmatrix} A(t) \\ \\ k(t) \end{bmatrix} \quad , \quad \Omega = \begin{bmatrix} -d & n \\ \\ -\dfrac{A^{(1-a)/a}}{2ab(an+1)^{1/a}} & (r+d) \end{bmatrix}.$$

The Jacobian matrixΩ, evaluated at the steady state point, has the following trace and determinant:

$$tr(\Omega) = r$$
$$\det(\Omega) = (r+d)\frac{d(1-a)}{a}$$

It is straightforward to verify that the trace is always positive, while the determinant is positive, negative or nil, if 0<a <1, a >1 or a =1, respectively. Thus, the following holds:

Proposition 1. *The steady state of the non-cooperative game, where*

$$A^* = \left(\frac{1}{an+1}\right)^{\frac{1}{a-1}} \cdot \left(\frac{n}{2bd(r+d)}\right)^{\frac{a}{a-1}} \quad ; \quad k^* = dA^*/n \, ; \quad q^* = \frac{aA^*}{an+1}$$

(i) is unstable when the market demand curve is convex, (ii) does not exist when the market demand is linear, and (iii) is a saddle when the market demand is concave.

Proposition 1 states that there exists no economically meaningful saddle point when $\alpha \leq 1$. The intuition behind this result is as follows. When market demand is either linear or convex, the market size (i.e., the area below the demand curve) is too small to justify firms' investment in a public good. The saddle point where A=0 can obviously be disregarded.

Focusing our attention on the case of concave market demand (i.e., on the saddle path equilibrium of the dynamic game), the determinants of the steady state variables, as shown by eq. (10), are as follows:

$$\frac{\partial A^*}{\partial b} < 0, \quad \frac{\partial A^*}{\partial d} < 0, \quad \frac{\partial A^*}{\partial r} < 0, \quad \frac{\partial A^*}{\partial a} < 0,$$
$$\frac{\partial k^*}{\partial b} < 0, \quad \frac{\partial k^*}{\partial d} < 0, \quad \frac{\partial k^*}{\partial r} < 0, \quad \frac{\partial k^*}{\partial a} < 0 .$$

The economic meaning is easy to interpret. First of all, there exists a direct relationship between the steady state levels of A and k. The larger is the steady state value of the market size, the larger is the investment effort required to neutralize the depreciation of A and to remain in steady state. Second, the larger is the cost of investment effort (captured by parameter b), the smaller the steady state optimal level of investment effort. Similarly, the higher the depreciation rate, the lower the optimal steady state level of market size and investment efforts. Furthermore, the higher the discounting factor, the less important are future profits, and then the lower the optimal market size in the steady state. Finally, the larger is a, i.e., the wider is the market (*ceteris paribus*), the less convenient is to invest in demand-increasing advertising.

The effect of the number of firms, n, upon the optimal steady state values is not clearcut, because two opposite effects are in operation. On the one hand, the larger is n, the harsher is market competition, which makes it more convenient to have a larger market size. On the other hand, the larger is n, the larger is the spillover effect coming from the investments operated by other firms.

4. The closed-loop Nash equilibrium

In this section, we investigate the closed-loop solution of the differential game, that is, the optimal solution when each player takes into account that – at any point in time – the control variables of other players affect the state variable(s), requiring a revision of optimal production and investment plans. In particular, here we adopt the so-called "memoryless" closed-loop solution concept, according to which every player is required to know only the current value of the state variable, and the knowledge of the complete past history is not necessary to compute the optimum. This kind of closed-loop solution is strongly time consistent. Analytically, the first order conditions to characterize the closed-loop memoryless Nash equilibrium are:

$$(12)\quad \begin{cases} \dfrac{\partial H_i(t)}{\partial q_i(t)} = 0 \\ \dfrac{\partial H_i(t)}{\partial k_i(t)} = 0 \\ -\dfrac{\partial H_i(t)}{\partial A(t)} - \sum_{j\neq i} \dfrac{\partial H_i(t)}{\partial q_j(t)} \dfrac{\partial q_j{}^*(t)}{\partial A(t)} - \sum_{j\neq i} \dfrac{\partial H_i(t)}{\partial k_j(t)} \dfrac{\partial k_j{}^*(t)}{\partial A(t)} = \dfrac{d\,l_i(t)}{dt} - r\,l_i(t) \end{cases}$$

where H_i denotes the Hamiltonian function of player i, as defined by equation (4).

The first and second conditions of system (12) are the same as in the open-loop solution represented in system (5), while the third condition takes into account the interaction between the other players' controls and the current level of the state variable. As a consequence, equations (6) and (7), yielding the reaction curves, hold also in this problem, while the co-state equation is now:

$$(13)\quad \frac{d\,l_i(t)}{dt} = (r+d)\,l_i(t) - \frac{(a+1)q_i(t)}{a(an+1)}\left[A(t) - q_i(t) - Q_{-i}(t)\right]^{\frac{1}{a}-1}.$$

Of course, the conditions derived from system (12) have to be considered along with the usual initial conditions and the transversality conditions.

Under the symmetry assumptions $q_i(t)=q(t)$ and $k_i(t)=k(t)$, and following the same procedure as in the open-loop problem, equations (6), (7) and (13) lead to the following dynamic system in $A(t)$ and $k(t)$:

$$(14)\quad \begin{cases} \dfrac{dA(t)}{dt} = nk(t) - dA(t) \\ \dfrac{dk(t)}{dt} = (r+d)k(t) - \dfrac{(a+1)}{2b(an+1)}\left[\dfrac{A(t)}{an+1}\right]^{\frac{1}{a}} \end{cases}$$

or, in matrix form:

$$(14') \quad \begin{bmatrix} \frac{dA(t)}{dt} \\ \\ \frac{dk(t)}{dt} \end{bmatrix} = \Psi \begin{bmatrix} A(t) \\ \\ k(t) \end{bmatrix} \quad , \quad \Psi = \begin{bmatrix} -d & n \\ \\ -\frac{(a+1)A^{(1-a)/a}}{2ab(an+1)^{(1+a)/a}} & (r+d) \end{bmatrix}$$

Also in this case, the trace of the Jacobian matrix is always positive, while the determinant –evaluated at the steady state value of A and k – is positive, nil or negative, according to the value of parameter a. Consequently, the properties of the steady state again depend on the curvature of the market demand function.

More specifically, the locus $\mathrm{d}A(t)/\mathrm{d}t=0$ coincides with its counterpart in the open-loop solution, while the locus $\mathrm{d}k(t)/\mathrm{d}t=0$ under the closed-loop decision rule is a curve with equation $k=[(a+1)A^{1/a}]/[2b(r+d)(an+1)^{(1+(1/a))}]$. This can be straight, convex or concave, according to the value of a, as in the open-loop solution, except that it always lies below its counterpart in the open-loop solution.

As a consequence, under the closed-loop solution, the steady state solution of the dynamic system, i.e., the intersection between the two loci (obviously disregarding the origin), occurs in the correspondence of higher (respectively, lower) levels of both k and A than in the open-loop equilibrium, when $a<1$ ($a>1$). This is evident from the analytical solution of the steady state equilibrium under the closed-loop decision rule, which gives:

$$(15) \qquad A^* = \frac{[n(a+1)]^{\frac{a}{a-1}}}{(an+1)^{\frac{a+1}{a-1}}} \cdot \left(\frac{1}{2bd(r+d)} \right)^{\frac{a}{a-1}} \quad ; \quad k^* = dA^*/n$$

Qualitatively speaking, the dynamic properties of the closed-loop steady state are the same as stated in Proposition 1 for the open-loop equilibrium. This prompts for a comparative evaluation of the two equilibria, which we carry out in the next section.

5. Steady states under open-loop Nash equilibrium and closed-loop Nash equilibrium

A quick comparison between the steady state levels of the highest reservation price under the Nash open-loop equilibrium (A^*_{OL}) and the Nash closed-loop equilibrium (A^*_{CL}), as given by equations (10) and (15) respectively, permits us to write:

$$(15') \qquad A^*_{CL} = \left(\frac{a+1}{an+1} \right)^{\frac{a}{a-1}} \cdot A^*_{OL}$$

where it is evident that the closed-loop steady state equilibrium value of the highest reservation price is larger under the closed-loop solution concept than under the open-loop one if and only if $a<1$. In the more interesting case where $a>1$, the saddle equilibria are such that the steady state equilibrium value of the reservation price is smaller under the close-loop solution than under open-loop one. A graphical representation of the case with $a \in (1,+\infty)$ is given in figure 2, where the locus pertaining to the closed-loop memoryless information structure is denoted by CL.

Our main result may be summarized as follows:

Proposition 2. *When the market demand function is concave, and the equilibrium is a saddle, the steady state levels of A and k under the closed-loop decision rule are smaller than the steady state levels of A and k under the open-loop decision rule.*

Otherwise, when the market demand function is convex, and the equilibrium is unstable, the steady state levels of A and k under the closed loop rule are larger than the steady state levels of A and k under the open-loop rule. Finally, when demand is linear, only the degenerate steady state with zero investment and nil market size does exist.

This result is seemingly in contrast with the acquired wisdom. In fact, the available models of dynamic oligopoly suggest that the closed-loop solution entails larger steady state levels of investment (e.g., investment in production capacity or R&D) than the capacity associated with the steady state under the open-loop Nash solution concept. The reason is that, under the closed-loop solution concept, all players take into account the simultaneous action of rivals, and react by investing more than they would do on the basis of the plans designed at the beginning of the game according to the open-loop Nash solution (see, e.g., Fudenberg and Tirole, 1983, 1991 ch. 13); Reynolds, 1987, 1991). The interpretation of these results relies upon the attempt at pre-emptimg rivals by acquiring a large capacity or by investing a large amount of resources in R&D activities. By contrast, in the present model, advertising investments produce a reciprocal spillover, which firms take into account when playing the closed-loop equilibrium. Therefore, subgame perfection entails lower advertising efforts (and therefore lower spillover effects to the benefit of rivals) as compared to what emerges at the weakly time consistent solution.[6]

6. Full cartelisation and social planning

In this section, we study the model under two alternative hypotheses: (i) all the firms in the market build a cartel where all decision variables are set cooperatively, in order to maximize the present value of the joint profits. We label this case as *full cartelisation*; (ii) the decisions concerning investment and output levels are taken in order to maximize the present value of the social welfare, i.e., as if the market were ruled by a social planner aiming at the maximizing total surplus. We will refer to this case as *social planning*.

In the case of a cartel, the dynamic problem can be summarized as follows. We have to find the optimal plans of each of the n (symmetric) firms concerning individual production q and investment k, in order to achieve the maximum present value of the flows of future joint profits. Formally, the problem is

$$(16)\quad \begin{aligned} & Max\ J = \int_0^{+\infty} e^{-rt} \cdot \left\{ n\left[A(t) - nq(t)\right]^{\frac{1}{a}} q(t) - nb(k_i(t))^2 \right\} dt \\ & s.t. \qquad \frac{dA(t)}{dt} = nk(t) - dA(t) \end{aligned}$$

The corresponding Hamiltonian function H^{FC} is:

$$(17)\quad H^{FC} = \left\{ \left[A(t) - nq(t)\right]^{\frac{1}{a}} nq(t) - nb(k_i(t))^2 + l\left[nk(t) - dA(t)\right] \right\} e^{-rt}.$$

---

[6] An analogous result is obtained by Bertuzzi and Lambertini (2001) in a model of horizontal product differentiation where firms' advertising campaigns are aimed at increasing the density of consumers along Hotelling's linear city (see also Piga, 1998).

The first order conditions are:

$$\text{(18)}\quad \begin{cases} \dfrac{\partial H^{FC}(t)}{\partial q(t)} = 0 \quad \Rightarrow \quad -\dfrac{n^2 q(t)}{a}\left[A(t) - nq(t)\right]^{\frac{1}{a}-1} + n\left[A(t) - nq(t)\right]^{\frac{1}{a}} = 0 \\ \dfrac{\partial H^{FC}(t)}{\partial k(t)} = 0 \quad \Rightarrow \quad l(t) = 2bk(t) \\ -\dfrac{\partial H^{FC}(t)}{\partial A(t)} = \dfrac{d\,l(t)}{dt} - r\,l(t) \end{cases}$$

along with the initial condition $A(0)=A_0$ and the transversality condition $\lim_{t\to+\infty} A(t)\,l\,(t)e^{-rt} = 0$.

The first condition of system (18) yields:

$$\text{(19)}\quad q(t) = \frac{aA(t)}{an+n},$$

which is clearly smaller as compared to its counterpart in the case of Cournot oligopoly.

By the usual manipulation of the remaining conditions of the system we obtain:

$$\text{(20)}\quad \frac{dk(t)}{dt} = (r+d)k(t) - \frac{1}{2b}\left(\frac{1}{1+a}\right)^{1/a} \cdot [A(t)]^{1/a},$$

and we can characterize the dynamic system under the cartel rules. In particular, in steady state, $\mathrm{d}A(t)/\mathrm{d}t=0$ implies that $k = dA/n$ and $\mathrm{d}k(t)/\mathrm{d}t=0$ implies $k = A^{1/a}/[2b(r+d)(a+1)^{1/a}]$. In the space (A,k), the optimal investment function corresponds to a curve which may be convex (if $a \in (0,1)$), concave (if $a \in (1,+\infty)$), or linear (if $a = 1$). In any case, this curve lies above its counterpart under the open-loop decision rule, and – *a fortiori* – above the relevant curve obtained under the closed-loop decision rule in the non-cooperative game. As a consequence, a steady state point exists also in the cartel case and its dynamic properties are the same as in the Cournot cases. In particular, this steady state is a saddle under the case $a \in (1,+\infty)$. Moreover, it is also easy to verify what follows:

Proposition 3. *For all* $\alpha \in (1,+\infty)$*, the cartel equilibrium is a saddle point, and the steady state levels of A and k under full cartelisation are both larger than the steady state levels of A and k in the open-loop and closed-loop Cournot-Nash equilibria.*

Needless to say, if $\alpha \in (0,1)$ steady state values of A and k under the chartel case are both smaller than the respective variables under Cournot behaviour, while the steady state does not exist if $\alpha = 1$.

A very similar procedure can be adopted to find the decision of a social planner aiming at the maximum present value of the social welfare flows. Social welfare at time t is defined as the sum of the consumer surplus and net profits of firms at that time. Under the symmetry hypothesis, consumers' surplus at time t and aggregate profits are respectively:

(21) $$CS(t) = \int_0^{q(t)} [A(t) - s(t)]^{\frac{1}{\alpha}} ds(t) = \frac{\alpha}{1-\alpha}\left[A(t)^{\frac{\alpha+1}{\alpha}} - \left(A(t) - Q(t)\right)^{\frac{\alpha+1}{\alpha}}\right]$$

(22) $$n\pi(t) = nq(t)[A(t) - Q(t)]^{1/\alpha} - nb(k(t))^2 .$$

We assume that the planner discounts future welfare flows at the same constant rate $r > 0$ as firms, so that his dynamic problem is:

(23)
$$Max\ SW = \int_0^{+\infty} e^{-rt} \cdot \left\{\frac{\alpha}{1-\alpha}\left[A(t)^{\frac{\alpha+1}{\alpha}} - \left(A(t) - Q(t)\right)^{\frac{\alpha+1}{\alpha}}\right] + nq(t)[A(t) - Q(t)]^{1/\alpha} - nb(k_i(t))^2\right\} dt$$
$$s.t. \quad \frac{dA(t)}{dt} = nk(t) - \delta A(t)$$

Solving the planner's problem, we find that $dk(t)/dt=0$ implies $k = A^{1/\alpha} / [2b(r+\delta)]$, which is a locus that lies always above *all* its counterparts in the previous games, in the space (A,k), both when it is concave, and when it is convex. The intersection with the straight line dA(t)/dt=0 gives the steady state (see, once again, figure 2, where the line in the case of social planning is labeled as SP). Therefore, we have:

Proposition 4. *For all* $a \in (1,+\infty)$, *the socially optimal advertising effort in steady state is higher than in any other regime.*

Once again, if $a \in (0,1)$ steady state values of A and k under the social optimum case are both smaller than the respective variables under Cournot behaviour, while the steady state does not exist if $a = 1$.

7. Comparisons across different regimes

Table 1 gathers the steady state values of the relevant variables under the different considered regimes. To ease the exposition, we define:

$$b = \left(\frac{n}{2bd(r+d)}\right)^{\frac{a}{a-1}}.$$

Table 1. - Relevant steady state variable values across regimes.

	Reservation price	Investment	Production
Open-loop oligopoly	$A^*_{OL} = \left(\frac{1}{an+1}\right)^{\frac{1}{a-1}} \cdot b$	$k^*_{OL} = \frac{d}{n}\left(\frac{1}{an+1}\right)^{\frac{1}{a-1}} \cdot b$	$q^*_{OL} = \left(\frac{1}{an+1}\right)^{\frac{a}{a-1}} \cdot ab$
Closed-loop oligopoly	$A^*_{CL} = \frac{(a+1)^{\frac{a}{a-1}}}{(an+1)^{\frac{a+1}{a-1}}} \cdot b$	$k^*_{CL} = \frac{d(a+1)^{\frac{a}{a-1}}}{n(an+1)^{\frac{a+1}{a-1}}} \cdot b$	$q^*_{CL} = \frac{(a+1)^{\frac{a}{a-1}}}{(an+1)^{\frac{2a}{a-1}}} \cdot ab$
Full Cartelisation	$A^*_{FC} = \left(\frac{1}{a+1}\right)^{\frac{1}{a-1}} \cdot b$	$k^*_{FC} = \frac{d}{n}\left(\frac{1}{an+1}\right)^{\frac{1}{a-1}} \cdot b$	$q^*_{FC} = \frac{1}{n}\left(\frac{1}{a+1}\right)^{\frac{a}{a-1}} ab$
Social planning	$A^*_{SP} = b$	$k^*_{SP} = \frac{d}{n} b$	$q^*_{SP} = \frac{1}{n} b$

In the remainder of this section, we confine our attention to the interesting case where saddle paths lead to the steady state, i.e., we confine to the case $a \in (1,+\infty)$, and we proceed to compare the different steady states. The analogous exercise under the case $a \in (0,1)$ is trivial and it is omitted for the sake of brevity.

We can easily draw the graphical representation of the dynamic system under the four different regimes under consideration: the open-loop Cournot-Nash equilibrium (labeled by OL), the closed-loop Cournot-Nash equilibrium (CL), full cartelisation (FC) and social planning (SP). In all cases, the locus $\mathrm{d}A(t)/\mathrm{d}t=0$ is represented by the straight line $k = dA/n$, while the locus $\mathrm{d}k(\mathrm{t})/\mathrm{dt}$ is a concave curve; we have shown that the position of such curves in the space (A,k) is clear-cut. In particular, the relevant curve under the closed-loop oligopoly lies below the curve corresponding to the open-loop oligopoly, that – in turn – lies below the curve associated to the cartel case, that – finally – lies below the curve of the social planning problem.

INSERT FIGURE 2 HERE

The graphical representation, as well as the analytical study based on Table 1, show that reservation prices and investment in steady state are such that $A^*_{CL}<A^*_{OL}<A^*_{FC}<A^*_{\mathrm{SP}}$ and $k^*_{CL}<k^*_{OL}<k^*_{FC}<k^*_{\mathrm{SP}}$. As to the amount of the individual firm's production in steady state, the order is not univocal. Indeed, the following chain of inequalities turns out to hold: $q^*_{CL}<q^*_{OL}<q^*_{\mathrm{SP}}$, while the production under the cartel is smaller than the production of the planner, but it may fall in different positions with respect to the levels of production associated with the Cournot-Nash equilibria. In particular, three cases are possible:

$$(21) \quad \begin{cases} q^*_{CL} < q^*_{OL} < q^*_{FC} \quad iff \quad 0 < n < \left(\dfrac{an+1}{a+1}\right)^{\frac{a}{a-1}} \\ q^*_{CL} < q^*_{FC} < q^*_{OL} \quad iff \quad \left(\dfrac{an+1}{a+1}\right)^{\frac{a}{a-1}} < n < \left(\dfrac{an+1}{a+1}\right)^{\frac{2a}{a-1}} \\ q^*_{FC} < q^*_{CL} < q^*_{OL} \quad iff \quad \left(\dfrac{an+1}{a+1}\right)^{\frac{2a}{a-1}} < n \end{cases}$$

Accordingly, the ranking of the social welfare levels across regimes is not unique. Put differently, the social welfare level is not a monotonic function of A. Indeed, a larger value of A implies larger consumer surplus and firms' operative profits, but it also requires a larger investment. Nevertheless, it is possible that the ranking of social welfare levels in steady state replicates the ranking of the equilibrium values of A. In such a case, the maximum social level is (obviously) attained under social planning, followed by the allocation chosen by the cartel, and then by the allocations associated with the open-loop Cournot-Nash equilibrium and the closed-loop Cournot-Nash equilibrium.[7] It is also worth stressing that, in this dynamic game, the full cartelisation among firms is not necessarily detrimental to social welfare, as compared to the either of the non-cooperative games played *à la* Cournot. The point that collusion may have positive effects, able to more than compensate consumers for the negative effects of collusive prices has been already made, in different models. Recently, Fershtman and Pakes (2000) and Pakes (2001) show that collusion may affect the variety, cost, and quality of the products marketed by an industry, and this may affect the welfare as do the price effects of collusion. Our present model adds the effect of adverting into the list of factors leading to the conclusion that - in a dynamic framework - society is not necessarily better off when cartels are forbidden. This amounts to saying that there exist relevant circumstances where the *per se rule* might profitably be replaced by a more elastic *rule of reason*.

8. Conclusions

We have analyzed a differential oligopoly game, where firms invest in order to increase the consumers' reservation prices in a market where the demand function is non-linear.

The consideration of non-linear market demand has allowed us to highlight that the curvature of the market demand affects the stability properties of the steady state equilibria. In particular, in the present model, when the market demand is linear no meaningful steady state exists, besides the trivial situation where reservation price and

[7] This can be easily verified by numerical calculations, which are available from the authors upon request.

investment are zero. When the market demand is concave or convex, a meaningful steady state does exist. It is a saddle in the former case, while it is unstable in the latter.

We have also shown that steady state investments and output levels state are sensitive to the solution concept. In particular, when the market demand is convex, the equilibrium investments by firms (and the associated consumer reservation price) in steady state are larger under the closed-loop decision rule than under the open-loop one. In this case, however, the steady state is unstable. In the case where the market demand function is concave, and the steady state is a saddle, the closed-loop equilibrium investments (and the reservation price) are smaller than the optimal investments (and reservation price) emerging at the open-loop equilibrium. This case is the most interesting one, not only because a saddle path leads the system to the steady state equilibrium, but especially because the conclusion contradicts the widespread idea that closed-loop equilibrium entails larger investments on the part of firms, as compared to the open-loop equilibrium. This is not true in the present model, because advertising investments produce a reciprocal spillover which firms take into account when paying according to the closed-loop Nash equilibrium.

Equilibrium allocations under Cournot oligopoly have been compared to the optimal allocation under full cartelisation and to the optimal behaviour of a social planner aiming at the maximization of discounted social welfare. In this respect, the main result is that the social welfare level associated to the steady state allocation of the cartel may well be larger than the social welfare level generated by the fully non-cooperative Cournot oligopoly, both in the open-loop and in the closed-loop case. This is due to the fact that a full externality characterizes the advertising campaign, and the cartel invests considerably more than the population of non-cooperative Cournot firms. The benefits from this fact is larger than the damage deriving from the restriction of quantity under the cartel agreement.

REFERENCES

Anderson, S.P. and M. Engers (1992), "Stackelberg vs Cournot Oligopoly Equilibrium", *International Journal of Industrial Organization*, **10**, 127-35.

Anderson, S.P. and M. Engers (1994), "Strategic Investment and Timing of Entry", *International Economic Review*, **35**, 833-53.

Basar, T. and G. J. Olsder (1982, 1995 2nd), *Dynamic Noncooperative Game Theory*, San Diego, Academic Press.

Bertuzzi, G. and L. Lambertini (2001), "Advertising in a Differential Game of Spatial Competition", working paper no. 400, Dipartimento di Scienze Economiche, Università di Bologna.

Cellini, R. and L. Lambertini (2001), "Differential Oligopoly Games where the Open-Loop Memoryless and Open-Loop Equilibria Coincide", working paper no. 402, Dipartimento di Scienze Economiche, Università di Bologna.

Dorfman, R. and P.O. Steiner (1954), "Optimal Advertising and Optimal Quality", *American Economic Review*, **44**, 826-36.

Dockner, E.J, S. Jørgensen, N. Van Long and G. Sorger (2000), *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

Erickson, G.M. (1985), "A Model of Advertising Competition", *Journal of Marketing Research*, **22**, 297-304.

Feichtinger, G. (1983), "The Nash Solution of an Advertising Differential Game: Generalization of a Model by Leitmann and Schmitendorf", *IEEE Transactions on Automatic Control*, **28**, 1044-48.

Feichtinger, G., R.F. Hartl and S.P. Sethi (1994), "Dynamic Optimal Control Models in Advertising: Recent Developments", *Management Science*, **40**, 195-226.

Fershtman, C. (1984), "Goodwill and Market Shares in Oligopoly", *Economica*, **51**, 271-82.

Fershtman, C. and S. Nitzan (1991), "Dynamic Voluntary Provision of Public Goods", *European Economic Review*, **35**, 1057-67.

Fershtman, C. and A. Pakes (2000), "A Dynamic Oligopoly with Collusion and Price Wars", *Rand Journal of Economics*, **31**, 207-36.

Friedman, J.W. (1983), "Advertising and Oligopolistic Equilibrium", *Bell Journal of Economics*, **14**, 464-73.

Fudenberg, D. and J. Tirole (1983), “Capital as a Commitment: Strategic Investment to Deter Mobility”, *Journal of Economic Theory*, **31**, 227-50.

Fudenberg, D. and J. Tirole (1991), *Game Theory*, Cambridge, MA, MIT Press.

Jørgensen, S. (1982), “A Survey of Some Differential Games in Advertising”, *Journal of Economic Dynamics and Control*, **4**, 341-69.

Jørgensen, S. and G. Zaccour (1999), “Equilibrium Pricing and Advertising Strategies in a Marketing Channel”, *Journal of Optimization Theory and Applications*, **102**, 111-25.

Lambertini, L. (1996), “Cartel Stability and the Curvature of Market Demand”, *Bulletin of Economic Research*, **48**, 329-34.

Leitmann, G. and W.E. Schmitendorf (1978), “Profit Maximization through Advertising: A Nonzero Sum Differential Game Approach”, *IEEE Transactions on Automatic Control*, **23**, 646-50.

Martin, S. (1993), *Advanced Industrial Economics*, Oxford, Blackwell.

Mehlmann, A. (1988), *Applied Differential Games*, New York, Plenum Press.

Nerlove, M. and K.J. Arrow (1962), “Optimal Advertising Policy under Dynamic Conditions”, *Economica*, **29**, 129-42.

Pakes A. (2001), “A Framework for Applied Analysis in Dynamic I.O. ”, Plenary Lecture at the 28th Annual Conference of the European Association for Research in Industrial Economics, Dublin, August 30 – September 2, 2001.

Piga, C. (1998), “A Dynamic Model of Advertising and Product Differentiation”, *Review of Industrial Organization*, **13**, 509-22.

Piga, C. (2000), “Competition in a Duopoly with Sticky Price and Advertising”, *International Journal of Industrial Organization*, **18**, 595-614.

Reinganum, J. (1982), “A Class of Differential Games for Which the Closed Loop and Open Loop Nash Equilibria Coincide”, *Journal of Optimization Theory and Applications*, **36**, 253-62.

Reynolds, S.S. (1987), “Capacity Investment, Preemption and Commitment in an Infinite Horizon Model”, *International Economic Review*, **28**, 69-88.

Reynolds, S.S. (1991), “Dynamic Oligopoly with Capacity Adjustment Costs”, *Journal of Economic Dynamics and Control*, **15**, 491-514.

Sethi, S.P. (1977), “Optimal Advertising for the Nerlove-Arrow Model Under a Budget Constraint”, *Operations Research Quarterly*, **28**, 683-93.

Tirole, J. (1988), *The Theory of Industrial Organization*, Cambridge, MA, MIT Press.

Vidale, M.L. and H.B. Wolfe (1957), "An Operations Research Study of Sales Response to Advertising", *Operations Research*, **5**, 370-81.

Vives, X. (2000), *Oligopoly Pricing: Old Ideas and New Tools*, Cambridge, MA, MIT Press.