Quality and Advertising in a Dynamic Duopoly

Luca Colombo[#] and Luca Lambertini[§]
Department of Economics, University of Bologna,
Italy; colombo@spbo.unibo.it
Trinity College, University of Dublin,
Ireland; colombol@tcd.ie
§ Department of Economics, University of Bologna,
Italy; lamberti@spbo.unibo.it

Abstract

We investigate a differential duopoly game where each firm, through capital accumulation over time, may invest both in persuasive advertising campaigns aimed at increasing the willingness to pay of consumers and in an R&D process aimed at increasing the level of own product quality. In contrast with the acquired wisdom based on static models, the firm providing the market with the inferior variety may earn higher profits than the rival. More than this, we show that there exists a range of parameters wherein the low quality firm commands monopoly power.
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1 Introduction

In this final chapter, we aim at investigating a dynamic advertising model under vertical differentiation.[1] The scanty literature currently available in this field usually considers advertising as in instrument to convey information about the existence and the characteristics of the advertised good or as a way to increase the stock of goodwill or reputation (Kotowitz and Mathewson, 1979; Conrad, 1985; Ringbeck, 1985). Two exemptions dealing with a full information setting are in Ouardighi and Pasin (2002), extending the well known Lanchester model to account for the interplay between market shares and quality, and in Colombo and Lambertini (2003), assuming a direct relationship between advertising efforts and the rate of change in sales, as in Vidale and Wolfe (1957), under endogenous vertical differentiation.[2] Others have investigated, adopting either static or dynamic approaches, the strategic use of product qualities as firms instruments to build up market shares (Moorthy, 1988; Motta, 1993; Dutta *et al.*, 1995).

In line of principle, the informative role played by advertising is plausible only in a world where the dissemination of information is scarce and where some product characteristics are not perfectly observable by consumers before purchase. Many goods that are actually advertised in modern economies, however, seem to be at odds with this world. Examples abound. Indeed, when the informative gap between producers and consumers is either negligible or, whatever it is, it cannot be reduced by advertising, the informative and signaling role of advertising becomes so marginal that it can not be justified from a theoretical point of view. For instance, the exact formula of Coca Cola is still unknown and, presumably, so it will remain in the foreseeable future. Accordingly, the focus of Coca Cola's advertising campaigns is not information. In this context, the unique theoretical explanation of the fact that huge amounts of money are spent by firms in advertising campaigns, is that either advertising acts so as to increase the marginal willingness to pay of consumers for the advertised good, or the consequence of a prisoners' dilemma generated by the presence of rival firms (like Pepsi) adopting similar strategies. The subject matter of this chapter is the first of the two possibilities.[3] Having said that, a natural question should come into mind: why perfectly informed (and perfectly rational) consumers should be ready to pay more for a good the characteristics of which, included the price vector, are common knowledge? The answer is that even if all the physical characteristics of the good remain unchanged by the fact of being advertised, the psychological responses of consumers are not.

As perfectly recognized by Galbraith (1967, ch. XVIII), advertising in modern

[1] For exhaustive surveys on dynamic advertising, see Sethi (1977); Jørgensen (1982); Feichtinger and Jørgensen (1983); Erickson (1991); Feichtinger *et al.* (1994).

[2] For the formulation of the Lanchester model, see Dockner *et al.* (2000, ch. 11), Case (1979), Sorger (1989) and Erickson (1991).

[3] For the analysis of the first perspective, where advertising aims at attracting additional consumers, see Colombo and Lambertini (2003).

economies is much more concerned with being persuasive rather than informative, where being persuasive means being able to add extra values to the advertised good by conferring to its owner a sense of personal achievement, social recognition, beauty, by diverting his mind from thought, or by being in whatever other manner psychologically rewarding.

In Galbraith's vein, we assume that advertising makes perfectly informed consumers more willing to pay for the good being advertised. We investigate a differential game in order to study the dynamic incentives for oligopolistic firms to invest in such persuasive advertising campaigns coupled with product quality improvements, which are the result of capital accumulation over time. For the sake of expositional homogeneity, we adopt the same structure of consumers' preferences as the one which has been used throughout the book. We characterize both open-loop and closed-loop memory less Nash equilibria, and proceed to a steady state (saddle path) analysis. Our main results can be summarized as follows: in line with the existing static literature on product quality provision in oligopoly (Gabszewicz and Thisse, 1979, 1980; Shaked and Sutton, 1982, 1983; Lehmann-Grube, 1997, *inter alia*), we show that the high quality firm serves more customers and invests more than the low quality firm. However, the resulting ranking of firms' profits may significantly differ from the one we are accustomed with from the aforementioned literature. More precisely, there exists an admissible subset of parameters wherein the low quality firm performs better than the high quality firm in terms of equilibrium profits. Moreover, we prove that it is possible for the low quality firm to become a monopolist, provided that future profits sufficiently matter.

The remainder of this chapter is structured as follows. The model is described in section 2. Section 3 deals with the solution of the game under both open-loop and closed-loop memory less solution concept. Concluding remarks are in section 4.

2 The Model

Time is continuous and indicated by t. At each $t \in [0, \infty)$, a market for vertically differentiated goods exists. Let this market be supplied by two single-product firms offering goods of quality $q_i(t)$ at a price $p_i(t)$, with $i = H, L$, with $\infty > q_H(t) \geq q_L(t) \geq 0$ at any t. Production exhibits constant returns to scale and, without any loss of generality, we normalize the unit production cost to zero.

On the demand side, each consumer is characterized by a marginal willingness to pay for quality θ, uniformly distributed over the support $[\Theta(t) - 1, \Theta(t)]$, with $\Theta(t) > 1$ at any t. We assume $f(\theta) = 1$, implying that consumers' population is normalized to 1.[4] For the sake of simplicity, we abstract from the presence of switching costs, so customers that, as time goes by, switch from one variety to the

[4]At each point in time, each consumer buys at most one unit of the preferred variety. This rules out the use of second-degree price discrimination.

other, bear any disutility.[5]

The instantaneous net surplus a consumer of type θ draws from the variety characterized by $q_i(t)$ is defined as follows:

$$U_\theta(t) = \begin{cases} \theta q_i(t) - p_i(t) \geq 0 & \text{if he buys variety } i = H, L \\ 0 & \text{if he doesn't buy} \end{cases} \tag{1}$$

In order to derive the expressions of market demands, we compute the threshold value of θ characterising the consumer who is indifferent between buying the high quality good and buying the low quality good:

$$\widehat{\theta}(t) = \frac{p_H(t) - p_L(t)}{q_H(t) - q_L(t)} , \tag{2}$$

and the one which characterizes the consumer who is indifferent between buying the low quality good and not buying at all:

$$\widetilde{\theta}(t) = \frac{p_L(t)}{q_L(t)} . \tag{3}$$

Accordingly, the direct demand system obtains:

$$x_H(t) = \Theta(t) - \widehat{\theta}(t) \; ; \tag{4}$$

$$x_L(t) = \widehat{\theta}(t) - \widetilde{\theta}(t) \; , \tag{5}$$

which can be inverted as long as partial market coverage prevails, i.e. $\widetilde{\theta}(t) > 0$:

$$p_H(t) = q_H(t)\left[\Theta(t) - x_H(t)\right] - q_L(t)\, x_L(t) \; ; \tag{6}$$

$$p_L(t) = q_L(t)\left[\Theta(t) - x_H(t) - x_L(t)\right] \; . \tag{7}$$

Firm i's instantaneous profits write:

$$\pi_H(t) = p_H(t) x_H(t) - \left[a_H(t)\right]^2 - \left[b_H(t)\right]^2 \; ; \tag{8}$$

$$\pi_L(t) = p_L(t) x_L(t) - \left[a_L(t)\right]^2 - \left[b_L(t)\right]^2 \; , \tag{9}$$

where $[a_i(t)]^2$ and $[b_i(t)]^2$ are the instantaneous quadratic costs associated with persuasive advertising campaigns and product quality improvements, respectively; $a_i(t)$ denotes the instantaneous investments in persuasive advertising campaigns made by firm i at time t, and $b_i(t)$ denotes the instantaneous R&D investments in product quality improvements made by firm i at time t.

We assume that the quality of firm i's product evolves over time according to the following dynamics:

$$\frac{dq_i(t)}{dt} \equiv \dot{q}_i = b_i(t) - \delta q_i(t) , \qquad i = \{H, L\} \; , \tag{10}$$

[5] For an exhaustive survey on consumers' switching costs, see Klemperer (1995).

while the upper bound of the support in which the marginal willingness to pay for quality lies evolves over time according to the following dynamics:

$$\frac{d\Theta(t)}{dt} \equiv \dot{\Theta} = a_H(t) + a_L(t) - \delta\Theta(t) \ , \tag{11}$$

where $\delta > 0$ denotes the common depreciation rate.[6] It is worth noting the differences in structure between (10) and (11): while the quality of firm i's product evolves over time independently of firm j's product quality and investment, the dynamics of Θ features non rival properties which are typical of public goods. In this respect, the effects induced by individual advertising campaigns consist in making not only own consumers, but all the consumers in the market more quality-oriented, including those whose willingness to pay for quality is still too low to buy.[7] It is also interesting to compare (10) with its analogous counterpart of last chapter. There, quality depreciation was assumed to be nil for firm L, while here we assume that quality depreciates at the rate δ for both firms.

The objective of firm i consists in maximising the present value of its profits stream over an infinite time horizon w.r.t. controls $a_i(t)$, $b_i(t)$ and $x_i(t)$, under the constraint given by states dynamics:

$$\begin{aligned} &\max_{a_i(t),b_i(t),x_i(t)} \Pi_i(t) = \int_0^\infty \pi_i(t)e^{-\rho t}dt \\ &s.t. \ \frac{dq_i(t)}{dt} \equiv \dot{q}_i = b_i(t) - \delta q_i(t), \quad i = \{H, L\} \\ &\text{and } \frac{d\Theta(t)}{dt} \equiv \dot{\Theta} = a_H(t) + a_L(t) - \delta\Theta(t) \ . \end{aligned} \tag{12}$$

The discount rate $\rho > 0$ is assumed to be constant and common to both firms.

3 The game

Firm i's current value Hamiltonian writes:

$$\mathcal{H}_i(t) = e^{-\rho t} \cdot \left\{ \pi_i(t) + \lambda_{ii}(t)\dot{q}_i + \lambda_{ij}(t)\dot{q}_j + \nu_i(t)\dot{\Theta}(t) \right\} \ , \tag{13}$$

where $\lambda_{ii}(t) = \mu_{ii}(t)e^{\rho t}$, $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ and $\nu_i(t) = \kappa_i(t)e^{\rho t}$; $\mu_{ii}(t)$ and $\mu_{ij}(t)$ are the co-state variable associated to $q_i(t)$ and $\kappa_i(t)$ is the co-state variable associated to $\Theta(t)$. Firms play simultaneously in each point in time. First order conditions (FOCs)

[6] The assumption that parameter δ is the same for both quality and the marginal willingness to pay is not at all crucial, and has been made only to end up with more tractable solutions than otherwise.

[7] A differential game where firms' advertsing campaigns have a public good nature is in Cellini and Lambertini (2003).

on controls are (henceforth, the indication of time and exponential discounting are omitted for brevity):[8]

$$\frac{\partial \mathcal{H}_H}{\partial x_H} = \Theta q_H - 2q_H x_H - q_L x_L = 0 \; ; \tag{14}$$

$$\frac{\partial \mathcal{H}_L}{\partial x_L} = q_L \left(\Theta - x_H - 2x_L\right) = 0 \; ; \tag{15}$$

$$\frac{\partial \mathcal{H}_i}{\partial a_i} = -2a_i + \lambda_{ii} = 0 \; , \; i = H, L \; ; \tag{16}$$

$$\frac{\partial \mathcal{H}_i}{\partial b_i} = -2b_i + \lambda_{ii} = 0 \; , \; i = H, L \; . \tag{17}$$

The above FOCs, in particular equations (14) and (15), imply that, by applying the open-loop solution concept to the present game, we end up with equilibria which are not subgame perfect.[9] As argued in the previous chapter, this can be justified by considering that, in some circumstances, it may be too costly for firms to modify their investment plans on the way. Notice also that FOCs do not contain λ_{ij} because of the assumptions concerning the state equations, characterised by separated dynamics. Therefore, we set $\lambda_{ij} = 0$ for all $t \in [0, \infty)$ and $j \neq i$, and specify only two co-state equations per firm, disregarding the one pertaining to the rival's quality. We first solve the game in the open-loop form. Then, in order to characterize equilibria that are strongly time consistent, we will solve the game according to the closed-loop memoryless solution concept.

3.1 Open-Loop Equilibrium

Under the open-loop solution concept, by definition, feedback effects are not taken into account. The relevant co-state equations write as follows:

$$\begin{aligned} -\frac{\partial \mathcal{H}_i}{\partial q_i} &= \frac{\partial \lambda_{ii}}{\partial t} - \rho \lambda_{ii} \Leftrightarrow \\ \frac{\partial \lambda_{ii}}{\partial t} &= \left(\rho + \delta\right) \lambda_{ii} - \left(\Theta - x_H\right) x_H \end{aligned} \tag{18}$$

$$\begin{aligned} -\frac{\partial \mathcal{H}_i}{\partial \Theta} &= \frac{\partial \nu_i}{\partial t} - \rho \nu_i \Leftrightarrow \\ \frac{\partial \nu_i}{\partial t} &= \left(\rho + \delta\right) \mu_i - q_H x_H \end{aligned} \tag{19}$$

[8] Second order conditions are met throughout the paper. They are omitted for brevity.

[9] See, e.g., Mehlman and Willig (1983), Reinganum (1982), Dockner, Feichtinger and Jørgensen (1985) and Fershtman (1987). For an exhaustive discussion on the coincidence between open-loop and closed-loop memoryless solutions, see Dockner *et al.* (2000, ch.7).

along with the transversality conditions:

$$\lim_{t\to\infty} \mu_i(t) \cdot q_i(t) = 0 \text{ and } \lim_{t\to\infty} \kappa_i(t) \cdot \Theta(t) = 0, \; i = H, L \tag{20}$$

and the initial conditions

$$q_i(0) = q_{i0} > 0, \text{ with } q_{H0} > q_{L0} \text{ and } \Theta(0) = \Theta_0 \geq 1. \tag{21}$$

Solving (14-15) yields the equilibrium output levels for a generic quality pair:

$$x_H^* = \Theta\left(1 - \frac{2q_H}{4q_H - q_L}\right) \; ; \; x_L^* = \frac{\Theta q_H}{4q_H - q_L} \; , \tag{22}$$

with $x_H^* > x_L^* > 0$ for all $q_H > q_L > 0$. If $q_H = q_L$, $p_H^* = p_L^* = 0$ and the allocation of market demand across firms is not determined. Notice that, since output levels do not appear in the state dynamics, the optimal solution obtained through the Hamiltonian coincides with the static ones (see section 2 of the previous chapter).

From (16) and (17) we obtain:

$$\lambda_{ii} = 2a_i \Rightarrow \dot{\lambda}_{ii} = 2\dot{a}_i; \tag{23}$$

$$\mu_{ii} = 2b_i \Rightarrow \dot{\mu}_{ii} = 2\dot{b}_i. \tag{24}$$

Now, by plugging (23) into (18) and (24) into (19), and using (22), we derive the dynamics of firm H's investments:

$$\dot{a}_H = a_H(\delta + \rho) + \frac{\Theta q_H (q_L - 2q_H)}{8q_H - 2q_L} \; ; \tag{25}$$

$$\dot{b}_H = b_H(\delta + \rho) + \frac{\Theta^2 q_H (q_L - 2q_H)}{(q_L - 4q_H)^2} \; , \tag{26}$$

and those referred to firm L:

$$\dot{a}_L = a_L(\delta + \rho) - \frac{\Theta q_H q_L}{8q_H - 2q_L} \; ; \tag{27}$$

$$\dot{b}_L = b_L(\delta + \rho) - \frac{\Theta^2 q_H^2}{2(q_L - 4q_H)^2} \; . \tag{28}$$

The dynamic system formed by (25), (26), (27) and (28), together with the state equations (10) and (11), yields the following admissible steady state point:[10]

$$a_H^{OL} = 4.6629\delta^2(\delta + \rho) \; ; \; a_L^{OL} = 0.80002\delta^2(\delta + \rho) \; ; \tag{29}$$

$$b_H^{OL} = 3.7071\delta^2(\delta + \rho) \; ; \; b_L^{OL} = 1.0858\delta^2(\delta + \rho) \; ; \tag{30}$$

$$\Theta^{OL} = 5.4629\delta(\delta + \rho) \; ; \; q_H^{OL} = 3.7071\delta(\delta + \rho) \; ; \; q_L^{OL} = 1.0858\delta(\delta + \rho) \; . \tag{31}$$

[10]We find four equilibria, only the one reported in the text being admissible. Among the other three critical points, two are non real and one does not respect the condition $q_H > q_L$.

Proposition 1 *The steady state defined by* $\{a_H^{OL}, a_H^{OL}, b_H^{OL}, b_L^{OL}, \Theta^{OL}, q_H^{OL}, q_L^{OL}\}$ *is a saddle point equilibrium.*

Proof. See the Appendix ■

By a direct comparison between equilibrium qualities and investments, we can write:

Lemma 2 *Under the open-loop solution concept, firm H invests more than firm L both in advertising and in quality improvement.*

Concerning equilibrium output levels, we have:

$$x_H^{OL} = 2.5156\delta\left(\delta+\rho\right) \; ; \; x_L^{OL} = 1.4736\delta\left(\delta+\rho\right) \; , \tag{32}$$

which are admissible iff $x_H^{OL} + x_L^{OL} < 1$. This condition is satisfied iff:

$$\rho < \overline{\rho} = -1.0027 \times 10^{-4} \frac{-2500 + 9973\delta^2}{\delta} \; . \tag{33}$$

Equilibrium prices are:

$$p_H^{OL} = 9.3259\delta^2\left(\delta+\rho\right)^2 \; ; \; p_L^{OL} = 1.6001\delta^2\left(\delta+\rho\right)^2 . \tag{34}$$

Lemma 3 *Under the open-loop solution concept, firm H attains a larger market share and charges a higher market price than firm L.*

We are now in a position to assess the relative performance of firms in terms of equilibrium profits:

$$\pi_H^{OL} = 23.46\delta^3\left(\delta+\rho\right)^3 - 35.485\delta^4\left(\delta+\rho\right)^2 \; ; \tag{35}$$

$$\pi_L^{OL} = 2.357\,9\delta^3\left(\delta+\rho\right)^3 - 1.819\delta^4\left(\delta+\rho\right)^2 \; . \tag{36}$$

The sustainability of either the monopoly or the duopoly regime depends upon the non-negativity of profits, which, in turns, depends upon intertemporal parameters. In this respect, our main result is as follows:

Proposition 4 *Provided that* $\rho < \overline{\rho}$*, ensuring that partial market coverage prevails, under the open-loop solution concept the following holds:*

$$(i)\;\rho < 0.51257\delta \Rightarrow \pi_L^{OL} > 0, \pi_H^{OL} < 0$$

$$(ii)\;\rho \in (0.51257\delta, 0.59539\delta] \Rightarrow \pi_L^{OL} > \pi_H^{OL} > 0$$

$$(iii)\;\rho > 0.59539\delta \Rightarrow \pi_H^{OL} > \pi_L^{OL} > 0.$$

The following figure illustrates the above Proposition:

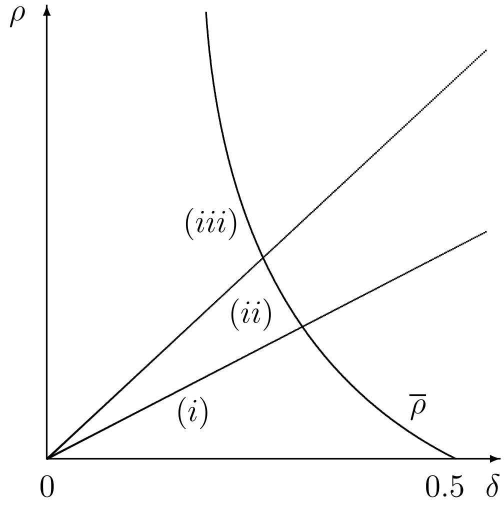

Figure 1 : Parameter Space

In region (i), only the low quality firm survives. In region (ii), the market is served by both firms; yet, contrary to the conventional wisdom coming from the existing static literature, the low quality firm outperforms the high quality firm in terms of equilibrium profits. Finally, in region (iii), we find the traditional (or *quasi*-static) result on the distribution of profits in a setting where quality improvements require capital accumulation to take place and consumers' preferences are affected by firms' advertising campaigns carried out over time. This is in accordance with the fact that, as the discount rate becomes higher, firms become increasingly myopic, and the dynamic game is perceived as closer to the static one.

3.2 Closed-Loop Memory Less Equilibrium

Now, we write firm i's co-state equations according to the closed-loop memory less solution concept:

$$-\frac{\partial \mathcal{H}_i}{\partial q_i} - \frac{\partial \mathcal{H}_i}{\partial x_j}\frac{\partial x_j^*}{\partial q_i} - \frac{\partial \mathcal{H}_i}{\partial a_j}\frac{\partial a_j^*}{\partial q_i} - \frac{\partial \mathcal{H}_i}{\partial b_j}\frac{\partial b_j^*}{\partial q_i} = \frac{\partial \lambda_{ii}}{\partial t} - \rho\lambda_{ii} \; ; \tag{37}$$

$$-\frac{\partial \mathcal{H}_i}{\partial \Theta} - \frac{\partial \mathcal{H}_i}{\partial x_j}\frac{\partial x_j^*}{\partial \Theta} - \frac{\partial \mathcal{H}_i}{\partial a_j}\frac{\partial a_j^*}{\partial \Theta} - \frac{\partial \mathcal{H}_i}{\partial b_j}\frac{\partial b_j^*}{\partial \Theta} = \frac{\partial \nu_i}{\partial t} - \rho\nu_i \; , \tag{38}$$

where starred variables indicate that partial derivatives are obtained using FOCs, along with the same transversality and initial conditions as in open-loop. First, notice that:

$$\frac{\partial a_j^*}{\partial q_i} = \frac{\partial b_j^*}{\partial q_i} = 0 \; ; \; \frac{\partial a_j^*}{\partial \Theta} = \frac{\partial b_j^*}{\partial \Theta} = 0 \; , \tag{39}$$

meaning that the only feedbacks that have to be taken into account are those regarding optimal output levels:

$$\frac{\partial \mathcal{H}_i}{\partial x_j} = q_L x_i \; ; \; \frac{\partial x_H^*}{\partial q_L} = -\frac{x_L}{2q_H} \; ; \; \frac{\partial x_L^*}{\partial q_H} = 0 \; ; \; \frac{\partial x_j^*}{\partial \Theta} = \frac{1}{2} \; . \tag{40}$$

Now, by plugging (23) into (37) and (24) into (38), and using equilibrium sales (22), we derive the dynamics of firm H's investments:

$$\dot{a}_H = a_H (\delta + \rho) + \Theta \left(\frac{q_L}{4} - \frac{q_H^2}{4q_H - q_L} \right) \; ; \tag{41}$$

$$\dot{b}_H = b_H (\delta + \rho) + \frac{\Theta^2 q_H (q_L - 2q_H)}{(q_L - 4q_H)^2} \; , \tag{42}$$

and those referred to firm L:

$$\dot{a}_L = a_L (\delta + \rho) - \frac{\Theta q_H q_L}{4 (4q_H - q_L)} \; ; \tag{43}$$

$$\dot{b}_L = b_L (\delta + \rho) - \frac{\Theta^2 q_H (2q_H + q_L)}{4 (q_L - 4q_H)^2} \; . \tag{44}$$

The dynamic system formed by (41), (42), (43) and (44), together with the state equations (10) and (11), yields the following admissible steady state point:[11]

$$a_H^{CL} = 5.478\delta^2 (\delta + \rho) \; ; \; a_L^{CL} = 0.73208\delta^2 (\delta + \rho) \; ; \tag{45}$$

$$b_H^{CL} = 4.7735\delta^2 (\delta + \rho) \; ; \; b_L^{CL} = 1.7166\delta^2 (\delta + \rho) \; ; \tag{46}$$

$$\Theta^{CL} = 6.21\delta (\delta + \rho) \; ; \; q_H^{CL} = 4.7735\delta (\delta + \rho) \; ; \; q_L^{CL} = 1.7166\delta (\delta + \rho) \; . \tag{47}$$

Proposition 5 *The steady state defined by* $\{a_H^{CL}, a_H^{CL}, b_H^{CL}, b_L^{CL}, \Theta^{CL}, q_H^{CL}, q_L^{CL}\}$ *is saddle point equilibrium.*

Proof. See the Appendix ■

By a direct comparison between equilibrium qualities and investments we can write:

Lemma 6 *Under the closed-loop memoryless solution concept, firm H invests more than firm L both in advertising and in quality improvement.*

[11] As for the open-loop case, we find four equilibria, only the one reported in the text being admissible.

Equilibrium output and price levels turn out to be:

$$x_H^{CL} = 2.7983\delta(\delta+\rho) \; ; \; x_L^{CL} = 1.7059\delta(\delta+\rho) \; ; \tag{48}$$

$$p_H^{CL} = 13.357\delta^2(\delta+\rho)^2 \; ; \; p_L^{CL} = 2.9282\delta^2(\delta+\rho)^2 \; , \tag{49}$$

provided that $x_H^{CL} + x_L^{CL} < 1$. Partial market coverage prevails iff:

$$\rho < \overline{\overline{\rho}} = -2.0 \times 10^{-10} \frac{-1.1101 \times 10^9 + 5.0 \times 10^9 \delta^2}{\delta} \; . \tag{50}$$

Lemma 7 *Under the closed-loop memoryless solution concept, firm H attains a larger market share and charges a higher market price than firm L.*

As in the previous case, we may assess the relative performance of firms in terms of equilibrium profits:

$$\pi_H^{CL} = 37.377\delta^3(\delta+\rho)^3 - 52.795\delta^4(\delta+\rho)^2 \; ; \tag{51}$$

$$\pi_L^{CL} = 4.9952\delta^3(\delta+\rho)^3 - 3.4827\delta^4(\delta+\rho)^2 \; . \tag{52}$$

From a direct comparison between (51) and (52) we obtain:

Proposition 8 *Provided that $\rho < \overline{\overline{\rho}}$, ensuring that partial market coverage prevails, under the closed-loop memoryless solution concept the following holds:*

$$(i)\ \rho < 0.4125\delta \Rightarrow \pi_L^{CL} > 0, \pi_H^{CL} < 0$$

$$(ii)\ \rho \in (0.4125\delta, 0.52282\delta] \Rightarrow \pi_L^{CL} > \pi_H^{CL} > 0$$

$$(iii)\ \rho > 0.52282\delta \Rightarrow \pi_H^{CL} > \pi_L^{CL} > 0.$$

The above Proposition is qualitatively equivalent to Proposition 4. Therefore, also in the closed-loop case, there exists a range of parameters wherein the low quality firm performs better than the high quality firm in terms of equilibrium profits. Furthermore, in contrast with the so-called *finiteness property* (Shaked and Sutton, 1983), the firm providing the market with the inferior variety may become a natural monopolist.[12]

As to the comparison between open- and closed-loop equilibria, it is self-evident from (31) and (47) that the marginal willingness to pay and the level of product quality for both varieties are higher in the closed-loop case. Moreover,

[12]In the differential game model at stake, contrary to what we have seen in the previous chapter, the so-called *4/7 rule* (Choi and Shin, 1992) never obtains, under either solution concept. Of course, this is due to the fact that firms behave *à la* Cournot. It is worth noting, however, that in Choi and Shin's model the quantity-setting behaviour would entail product homogeneity, with both firms supplying the highest feasible quality.

Corollary 9 *The degree of vertical differentiation is larger at the closed-loop equilibrium than at the open-loop one.*

To see this it suffices to carry out a straightforward computation, from which we get $q_L^{OL}/q_H^{OL} = 0.2929$ and $q_L^{CL}/q_H^{CL} = 0.35961$. So, the 'closed-loop motive', whereby firms explicitly take into consideration the rivals' behaviour as the game unravels over time, translates here into a wider product variety.

However, unlike the conventional wisdom according to which firms make greater investment efforts under closed-loop than under open-loop strategies (see Reynolds, 1987, *inter alia*), the firm providing the inferior quality makes a lower effort in the advertising activity when feedback effects are taken into account. This result depends on the fact that the benefits from any increase in consumers' marginal willingness to pay cannot be internalised, and spill over to the other firm. Indeed, the high quality firm invests much more in closed- than in open-loop yielding greater positive externalities to the rival. Equilibrium sales and market prices are always higher at the closed-loop equilibrium. A non-trivial result is that equilibrium profits are also higher in the closed-loop case than in the open-loop one. This is due to the fact that higher efforts (at least in quality supply) are adequately counterbalanced by larger revenues brought about by the increase in the marginal willingness to pay.

4 Concluding Remarks

We have investigated a differential duopoly game where each firm, through capital accumulation over time, may invest both in persuasive advertising campaigns aimed at increasing the willingness to pay of consumers and in an R&D process aimed at increasing the level of own product quality. The willingness to pay of consumers and the levels of product qualities have been treated as state variables evolving (in continuous time) in response to the interplay between firms' investments and decay rates. Unlike multi-stage games, differential games are particularly suitable to shed light on the nature of investments, which is inherently a dynamic one. To the best of our knowledge, the model presented in this chapter represents the first attempt to capture formally the dynamic incentives for oligopolistic firms to devote resources to advertising campaigns and quality improvements, jointly.

The main result we have obtained is that, in contrast with the acquired wisdom based on static models, the firm providing the market with the inferior variety may earn higher profits than the rival. Furthermore, we have shown that there exists a range of parameters wherein the low quality firm commands monopoly power. The rationale for these results is that, while in a static setting there are only instantaneous production costs, and we know from the existing static literature that it is always more profitable to produce the superior variety, in our dynamic setting things are much more involved: quality production and improvement require firms to make increasing efforts in each point in time, due to the assumption on dynamic decreasing returns w.r.t. the investment technology. Indeed, in the long run, it may become too costly for firms not only to produce but also to maintain a high quality level. This occurs when the discount rate is very small compared to the decay rate, i.e. future profits matter almost as present ones. When, instead, future profits are highly discounted, we come back to a *quasi*-static world where the relevant time horizon is perceived as a very short one due to the myopic attitude of firms, thus confirming the conventional static wisdom.

Appendix

Proof of Proposition 1.

We are interested in the dynamic system formed by (25), (26), (27), (28) and the state equations (10), (11). To verify that the steady state point defined in Proposition 1 is stable along a saddle path, we consider the following 7×7 matrix:

$$\Omega^{OL} = \begin{bmatrix} \frac{\partial\dot{\Theta}}{\partial\Theta} & \frac{\partial\dot{\Theta}}{\partial q_H} & \frac{\partial\dot{\Theta}}{\partial q_L} & \frac{\partial\dot{\Theta}}{\partial a_H} & \frac{\partial\dot{\Theta}}{\partial a_L} & \frac{\partial\dot{\Theta}}{\partial b_H} & \frac{\partial\dot{\Theta}}{\partial b_L} \\ \frac{\partial\dot{q}_H}{\partial\Theta} & \frac{\partial\dot{q}_H}{\partial q_H} & \frac{\partial\dot{q}_H}{\partial q_L} & \frac{\partial\dot{q}_H}{\partial a_H} & \frac{\partial\dot{q}_H}{\partial a_L} & \frac{\partial\dot{q}_H}{\partial b_H} & \frac{\partial\dot{q}_H}{\partial b_L} \\ \frac{\partial\dot{q}_L}{\partial\Theta} & \frac{\partial\dot{q}_L}{\partial q_H} & \frac{\partial\dot{q}_L}{\partial q_L} & \frac{\partial\dot{q}_L}{\partial a_H} & \frac{\partial\dot{q}_L}{\partial a_L} & \frac{\partial\dot{q}_L}{\partial b_H} & \frac{\partial\dot{q}_L}{\partial b_L} \\ \frac{\partial\dot{a}_H}{\partial\Theta} & \frac{\partial\dot{a}_H}{\partial q_H} & \frac{\partial\dot{a}_H}{\partial q_L} & \frac{\partial\dot{a}_H}{\partial a_H} & \frac{\partial\dot{a}_H}{\partial a_L} & \frac{\partial\dot{a}_H}{\partial b_H} & \frac{\partial\dot{a}_H}{\partial b_L} \\ \frac{\partial\dot{a}_L}{\partial\Theta} & \frac{\partial\dot{a}_L}{\partial q_H} & \frac{\partial\dot{a}_L}{\partial q_L} & \frac{\partial\dot{a}_L}{\partial a_H} & \frac{\partial\dot{a}_L}{\partial a_L} & \frac{\partial\dot{a}_L}{\partial b_H} & \frac{\partial\dot{a}_L}{\partial b_L} \\ \frac{\partial\dot{b}_H}{\partial\Theta} & \frac{\partial\dot{b}_H}{\partial q_H} & \frac{\partial\dot{b}_H}{\partial q_L} & \frac{\partial\dot{b}_H}{\partial a_H} & \frac{\partial\dot{b}_H}{\partial a_L} & \frac{\partial\dot{b}_H}{\partial b_H} & \frac{\partial\dot{b}_H}{\partial b_L} \\ \frac{\partial\dot{b}_L}{\partial\Theta} & \frac{\partial\dot{b}_L}{\partial q_H} & \frac{\partial\dot{b}_L}{\partial q_L} & \frac{\partial\dot{b}_L}{\partial a_H} & \frac{\partial\dot{b}_L}{\partial a_L} & \frac{\partial\dot{b}_L}{\partial b_H} & \frac{\partial\dot{b}_L}{\partial b_L} \end{bmatrix}_{q_i^{OL},\Theta^{OL}}$$

By computing the seven eigenvalues, we find that at least one is negative over the entire admissible range of parameters:

$$\underline{\xi} = 0.5\rho - 0.3218\sqrt{8\delta^2 + 8\delta\rho + 2.4142\rho^2} < 0$$

while at least one is positive:

$$\overline{\xi} = \delta + \rho > 0$$

Therefore, the steady state is a saddle point equilibrium.

Proof of Proposition 5.

We are interested in the dynamic system formed by (41), (42), (43), (44) and the state equations (10), (11). To verify that the steady state point defined in Proposition 5 is stable along a saddle path, we consider the following 7×7 matrix:

$$\Omega^{CL} = \begin{bmatrix} \frac{\partial\dot{\Theta}}{\partial\Theta} & \frac{\partial\dot{\Theta}}{\partial q_H} & \frac{\partial\dot{\Theta}}{\partial q_L} & \frac{\partial\dot{\Theta}}{\partial a_H} & \frac{\partial\dot{\Theta}}{\partial a_L} & \frac{\partial\dot{\Theta}}{\partial b_H} & \frac{\partial\dot{\Theta}}{\partial b_L} \\ \frac{\partial\dot{q}_H}{\partial\Theta} & \frac{\partial\dot{q}_H}{\partial q_H} & \frac{\partial\dot{q}_H}{\partial q_L} & \frac{\partial\dot{q}_H}{\partial a_H} & \frac{\partial\dot{q}_H}{\partial a_L} & \frac{\partial\dot{q}_H}{\partial b_H} & \frac{\partial\dot{q}_H}{\partial b_L} \\ \frac{\partial\dot{q}_L}{\partial\Theta} & \frac{\partial\dot{q}_L}{\partial q_H} & \frac{\partial\dot{q}_L}{\partial q_L} & \frac{\partial\dot{q}_L}{\partial a_H} & \frac{\partial\dot{q}_L}{\partial a_L} & \frac{\partial\dot{q}_L}{\partial b_H} & \frac{\partial\dot{q}_L}{\partial b_L} \\ \frac{\partial\dot{a}_H}{\partial\Theta} & \frac{\partial\dot{a}_H}{\partial q_H} & \frac{\partial\dot{a}_H}{\partial q_L} & \frac{\partial\dot{a}_H}{\partial a_H} & \frac{\partial\dot{a}_H}{\partial a_L} & \frac{\partial\dot{a}_H}{\partial b_H} & \frac{\partial\dot{a}_H}{\partial b_L} \\ \frac{\partial\dot{a}_L}{\partial\Theta} & \frac{\partial\dot{a}_L}{\partial q_H} & \frac{\partial\dot{a}_L}{\partial q_L} & \frac{\partial\dot{a}_L}{\partial a_H} & \frac{\partial\dot{a}_L}{\partial a_L} & \frac{\partial\dot{a}_L}{\partial b_H} & \frac{\partial\dot{a}_L}{\partial b_L} \\ \frac{\partial\dot{b}_H}{\partial\Theta} & \frac{\partial\dot{b}_H}{\partial q_H} & \frac{\partial\dot{b}_H}{\partial q_L} & \frac{\partial\dot{b}_H}{\partial a_H} & \frac{\partial\dot{b}_H}{\partial a_L} & \frac{\partial\dot{b}_H}{\partial b_H} & \frac{\partial\dot{b}_H}{\partial b_L} \\ \frac{\partial\dot{b}_L}{\partial\Theta} & \frac{\partial\dot{b}_L}{\partial q_H} & \frac{\partial\dot{b}_L}{\partial q_L} & \frac{\partial\dot{b}_L}{\partial a_H} & \frac{\partial\dot{b}_L}{\partial a_L} & \frac{\partial\dot{b}_L}{\partial b_H} & \frac{\partial\dot{b}_L}{\partial b_L} \end{bmatrix}_{q_i^{CL},\Theta^{CL}}$$

$$= \begin{bmatrix} -\delta & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & -\delta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\delta & 0 & 0 & 0 & 1 \\ -0.882\delta(\delta+\rho) & -1.537\delta(\delta+\rho) & 1.084\delta(\delta+\rho) & \delta+\rho & 0 & 0 & 0 \\ -0.118\delta(\delta+\rho) & 0.0151\delta(\delta+\rho) & -0.468\delta(\delta+\rho) & 0 & \delta+\rho & \delta+\rho & \delta+\rho \\ -1.537\delta(\delta+\rho) & -0.0217\delta(\delta+\rho) & 0.0602\delta(\delta+\rho) & 0 & 0 & \delta+\rho & 0 \\ -0.553\delta(\delta+\rho) & 0.126\delta(\delta+\rho) & -0.350\delta(\delta+\rho) & 0 & 0 & 0 & \delta+\rho \end{bmatrix}$$

Again, computing the seven eigenvalues we find that at least one is negative and at least one is positive over the entire admissible parameter range. Therefore, the steady state equilibrium is a saddle point. We omit the expressions of eigenvalues since they are cumbersome. Anyway, they are available upon request.

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