Towards a Complete Design of Linear Functional Observers

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Abstract – *This paper provides a procedure for the design of a reduced order observer of a state linear functional for a linear time-invariant system. The case, defined in [1], where the observer order p is given by the number m of single independent linear functionals to be observed, is called in this paper the minimum case where p = m. The minimum case is revisited and numerically simplified. The aim of this paper is to extend the minimum case to the case where m < p < n-l, named minimal case. A constructive procedure is given to design the linear functional observer.*

Keywords: *Linear functional observers, linear time-invariant systems, generalized inverse*

Nomenclature

Vectors and scalars

Name	Size (or definition)
State $x(t)$	n
Input $u(t)$	γ
Output $y(t)$	l
Observer state $z(t)$	p
Observer output $w(t)$	m
ρ	$l+m+\mu$
ε	$n-l-m$

System Matrices

Name			Size		
A			$n\times n$		
A_1	A_2	A_3	$n\times l$	$n\times m$	$n\times \varepsilon$
A_{11}	A_{12}	A_{13}	$l\times l$	$l\times m$	$l\times \varepsilon$
A_{21}	A_{22}	A_{23}	$m\times l$	$m\times m$	$m\times \varepsilon$
A_{31}	A_{32}	A_{33}	$\varepsilon\times l$	$\varepsilon\times m$	$\varepsilon\times \varepsilon$
B			$n\times \gamma$		
C			$l\times n$		
K			$m\times n$		

Observer Matrices

Name		Size	
D		$p\times p$	
D_1	D_2	$(p-m)\times m$	$m\times m$
D_{11}	D_{12}	$m\times m$	$m\times (p-m)$
D_{21}	D_{22}	$(p-m)\times m$	$(p-m)\times (p-m)$
H		$p\times \gamma$	
E		$p\times l$	
	E_2	$m\times l$	$(p-m)\times l$
P		$m\times p$	
V		$m\times l$	
T		$m\times n$	
	T_2	$m\times n$	$(p-m)\times n$
T_{21}		$(p-m)\times l$	
T_{22}		$(p-m)\times m$	
T_{23}		$(p-m)\times \varepsilon$	

Other Matrices

Name		Size	
Q		$n\times n$	
Σ		$(m+2l)\times n$	
Ψ		$p\times p$	
Φ		$l+m$	
J		$p\times l$	
N_1	M_1	$(2l+m)\times l$	$n\times l$
N_2	M_2	$(2l+m)\times m$	$n\times m$
N_3	M_3	$(2l+m)\times l$	$n\times l$
L		$(l-\mu)\times \rho$	
L_1	Z	$(l-\mu)\times l$	$m\times (2l+m)$
L_2	Z_1	$(l-\mu)\times m$	$m\times \rho$
L_3	Z_2	$(l-\mu)\times \mu$	$m\times (l-\mu)$
Γ_1	Γ_2	$m\times m$	$(p-m)\times m$

I. Introduction

From the seminal Luenberger's paper [2], several attempts have been proposed to design a reduced order observer of a linear functional of the state of a linear time-invariant system (see for instance [1], [3]-[16]). Thus:

- For the state space linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

- For a linear functional of the state, the aim is to design a system

$$\begin{aligned} \dot{z}(t) &= Dz(t) + Hu(t) + Ey(t) \\ w(t) &= Pz(t) + Vy(t) \end{aligned} \tag{2}$$

where for every t, the state $z(t)$, such that:

$$\lim_{t\to\infty}\left(w(t)-Kx(t)\right)=0$$

If this last condition is fulfilled, then (2) is a linear functional observer (LFO). For instance, (2) can be used to implement a state feedback control with a minimal order controller.

From [2], it is known that such an observer exists if and only if the following well-known conditions are satisfied:

$$D \text{ is a Hurwitz matrix} \tag{3}$$

$$TA-DT=EC \tag{4}$$

$$\begin{aligned} K&=PT+VC\\ H&=TB \end{aligned} \tag{5}$$

where T is an unknown constant matrix such that $\lim_{t\to\infty}\left(z(t)-Tx(t)\right)=0$. It can be seen that the last relation gives H. Thus, the design consists in determining the constant matrices of appropriate dimensions D, E, P, V and T and the order of the observer p such that conditions (3), (4) and (5) are verified, keeping in mind that we are looking for a minimal LFO.

Remark 1. In our study we have to define 2 different cases. In the first case, the minimum LFO is the LFO where $p = m$ and $P = I_m$. In the second case, the minimal LFO is the minimal order LFO, where $m < p < n-l$ which exists when existence conditions for minimum LFO are not fulfilled.

In the aforementioned literature we can underline interesting works. [1], [3], [4] propose a procedure to design a LFO in case $p = m$ [7] gives a procedure for a single LFO only. In [17], the authors use an LFO without feedthrough from input to output. Thus, the design problem of a minimal order LFO is not, to our standpoint, completely solved. Indeed, it can be seen that [15] uses the observable lower Hessenberg form of (A, C) which is numerically stable with respect to the observable canonical form. This form is also used in [7], [11], [16]. In the following, for numerical stability sake, we consider that C is full row rank only and the pair (A, C) is not supposed to be in a particular canonical form. Moreover, if a row of K is linearly dependent on the rows of C, the corresponding component of the linear functional (2) can be observed with the output alone. Hence it is supposed in the following that C and K are linearly independent. This property is a very important point of our development.

In [1], a necessary and sufficient condition has been proposed in the case when $p = m$. When the existence conditions are fulfilled, the observer design is based on the use of the Moore-Penrose pseudo-inverse of a matrix, and on an eigenvalue placement procedure to verify the Hurwitz condition for D. Extending the previous work, [14] proposes conditions for the existence of a second order observer for a scalar linear functional of the state. It can be noted that it uses an observable form for the observer to obtain a procedure for the design of a minimal LFO.

The aim and the main motivation of our paper is twofold. Firstly, by considering the property between C and K mentioned before, the minimum case developed by [1] is revisited and some numerical reductions are introduced (minimum case section). Secondly, it is interesting to extend the standpoints proposed in [1]-[14] in case the necessary and sufficient conditions are not fulfilled for a minimum LFO ($p = m$). The case $p > m$ can therefore be called the minimal case and the observer is called minimal LFO (minimal case section).

These points of view induce the following organization for the paper. Firstly, the minimum case is revisited. Secondly, we develop the existence conditions for the minimal case. This leads us to propose a design procedure to get a minimal LFO. In our work, we propose to use the generalized inverse concept which offers more possibilities in the calculus than the Moore-Penrose pseudo-inverse generally used. Some useful results on generalized inverses are developed in [18].

II. The Minimum Case

II.1. The Minimum Case

We have $P = I_m$ and following [1], the condition (5) is written $T=K-VC$. Introducing $J=E-DV$, condition (4) can be written:

$$KA=\begin{bmatrix}J & D & V\end{bmatrix}\Sigma, \text{ with } \Sigma=\begin{bmatrix}C\\ K\\ CA\end{bmatrix}$$

According to [2] this linear system has a solution if and only if:

$$\operatorname{rank}\Sigma=\operatorname{rank}\begin{bmatrix}\Sigma\\ KA\end{bmatrix} \tag{6}$$

When (6) is fulfilled, the solution is:

$$\begin{bmatrix}J & D & V\end{bmatrix}=KA\Sigma^{\{1\}}+Z\left(I_{2l+m}-\Sigma\Sigma^{\{1\}}\right) \tag{7}$$

where $\Sigma^{\{1\}}$ is a generalized inverse of Σ (see [18]). That is to say, it verifies the relationship $\Sigma\Sigma^{\{1\}}\Sigma=\Sigma$ and Z is an arbitrary $\left(m\times(2l+m)\right)$ matrix. With the following partitions:

$$\Sigma^{\{1\}} = \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix}$$
$$I_{2l+m} - \Sigma\Sigma^{\{1\}} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$$

we obtain:

$$J = KAM_1 + ZN_1$$
$$D = KAM_2 + ZN_2$$
$$V = KAM_3 + ZN_3$$

As seen in [1] the following theorem is recalled.

Theorem 1 *[1] The necessary and sufficient conditions for the existence and stability of the minimum (p = m) functional observer (2) for system (1) are condition (6) and the pair* (KAM_2, N_2) *is detectable.*

II.2. The Minimum Case Revisited

The aim of this section is to simplify the Theorem 1 established in [1]. We have to consider the properties mentioned before between C and K.

A full rank decomposition of Σ with rank $(\Sigma) = \rho$ leads to write:

$$FG = \Sigma \tag{8}$$

where F and G are such that $rank(F) = rank(G) = \rho$ with $F \in \Re^{(2l+m)\times\rho}$ and $G \in \Re^{\rho\times n}$.

Firstly, considering Σ, we order the row of C by making a suitable permutation in the output components in order to assure that the μ first rows of CA are linearly independent from each other and linearly independent of $\begin{bmatrix} C \\ K \end{bmatrix}$ too. We can deduce:

$$\rho = l + m + \mu \tag{9}$$

Let $(CA)^*$ be defined as these μ rows of CA. By choosing on the one hand $F = \begin{bmatrix} I_\rho \\ L_{l-\mu,\rho} \end{bmatrix}$ where $L_{l-\mu,\rho}$ will be identified later and on the other hand $G = \begin{bmatrix} C \\ K \\ (CA)^* \end{bmatrix}$, (8) is verified.

We consider a coordinates transformation Q which leads to define a similar matrix such that:

$$\bar{\Sigma} Q = \Sigma \tag{10}$$

with $\bar{\Sigma} \in \Re^{(m+2l)\times n}$. By comparing (8) with (10), Q and $\bar{\Sigma}$ are identified.

Due to the fact that Q must be nonsingular, transformations must be made. If $(l+m+\mu) < n$, $(n-\mu-l-m)$ arbitrary components noted Υ are added to G it can be deduced:

$$Q = \begin{bmatrix} C \\ K \\ (CA)^* \\ \Upsilon \end{bmatrix} \tag{11}$$

As a consequence, by adding a column to F we finally obtain:

$$\bar{\Sigma} = \begin{bmatrix} I_\rho & 0_{\rho,n-\rho} \\ L_{l-\mu,\rho} & 0_{l-\mu,n-\rho} \end{bmatrix} \tag{12}$$

where $L_{l-\mu,\rho}$ is a known matrix obtained by identifying the sub-matrix of:

$$\bar{\Sigma} = \Sigma Q^{-1} = \begin{bmatrix} C \\ K \\ CA \end{bmatrix} \begin{bmatrix} C \\ K \\ (CA)^* \\ \Upsilon \end{bmatrix}^{-1}$$

By using this state variable change, we obtain $\bar{A} = QAQ^{-1}$ and:

$$\bar{C} = CQ^{-1} = \begin{bmatrix} I_l & 0_{l,m} & 0_{l,n-l-m} \end{bmatrix}$$
$$\bar{K} = KQ^{-1} = \begin{bmatrix} 0_{m,l} & I_m & 0_{m,n-l-m} \end{bmatrix} \tag{13}$$

Remark 2. For notational simplicity, if one of the dimensions of the null matrix vanishes or is negative, this matrix must not be considered.

Let us partition L as:

$$L = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \tag{14}$$

Moreover A is partitioned as:

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 & \bar{A}_3 \end{bmatrix} \tag{15}$$

where:

$$\bar{A}_1 = \begin{bmatrix} \bar{A}_{11} \\ \bar{A}_{21} \\ \bar{A}_{31} \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} \bar{A}_{12} \\ \bar{A}_{22} \\ \bar{A}_{32} \end{bmatrix}, \bar{A}_3 = \begin{bmatrix} \bar{A}_{13} \\ \bar{A}_{23} \\ \bar{A}_{33} \end{bmatrix}$$

From theorem 1 described before, the following corollary can be deduced:

Corollary 1 *A necessary and sufficient condition for the existence of an observer of order m is:*

1. $\text{rank}\left[\bar{A}_{13}\right] = \text{rank}\begin{bmatrix}\bar{A}_{13}\\ \bar{A}_{23}\end{bmatrix}$,
2. pair $\left(\bar{A}_{22}, L_2\right)$ is detectable.

Proof 1 *Starting from (6) and according to the state variable change, (2) has a solution if and only if:*

$$\text{rank}\bar{\Sigma} = \text{rank}\begin{bmatrix}\bar{\Sigma}\\ KA\end{bmatrix} \tag{16}$$

where $\bar{\Sigma}$ is given by (12). Using (13) and (15) we get:

$$\begin{aligned} KA &= \left[\bar{A}_{21} \quad A_{22} \quad \bar{A}_{23}\right] \\ CA &= \left[\bar{A}_{11} \quad A_{12} \quad \bar{A}_{13}\right] \end{aligned} \tag{17}$$

Thus, (16) becomes:

$$\text{rank}\begin{bmatrix} I_l & 0_{l,m} & 0_{l,n-l-m} \\ 0_{m,l} & I_m & 0_{m,n-l-m} \\ \bar{A}_{11} & A_{12} & A_{13} \end{bmatrix} = \text{rank}\begin{bmatrix} I_l & 0_{l,m} & 0_{l,n-l-m} \\ 0_{m,l} & I_m & 0_{m,n-l-m} \\ \bar{A}_{11} & A_{12} & \bar{A}_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

which denotes that then n - ρ last columns of A_{23} must be null or linearly dependent on the n - ρ last columns of A_{13}. Eventually, the existence condition (16) becomes:

$$\text{rank}\left[\bar{A}_{13}\right] = \text{rank}\begin{bmatrix}\bar{A}_{13}\\ \bar{A}_{23}\end{bmatrix} \tag{18}$$

which constitutes condition of corollary 1. Concerning the second condition of this corollary, if (18) is fulfilled. (7) is written in the basis defined in (11), so:

$$\left[J \quad D \quad V\right] = \overline{KA}\bar{\Sigma}^{\{1\}} + Z\left(I_{2l+m} - \bar{\Sigma}\bar{\Sigma}^{\{1\}}\right) \tag{19}$$

To solve (19), we use the expression of KA defined in (17) and as $\bar{\Sigma}$ is defined by (12), we can choose for $\bar{\Sigma}^{\{1\}}$ (see [18]):

$$\bar{\Sigma}^{\{1\}} = \begin{bmatrix} I_\rho & 0_{\rho,l-\mu} \\ 0_{n-\rho,\rho} & 0_{n-\rho,l-\mu} \end{bmatrix} \Rightarrow$$

$$I_{2l+m} - \Sigma\Sigma^{\{1\}} = \begin{bmatrix} 0_{\rho,\rho} & 0_{\rho,l-\mu} \\ -L & I_{l-\mu} \end{bmatrix}$$

The particular choice of $\bar{\Sigma}^{\{1\}}$ leads to major simplifications in the design of the LFO.

Remembering that $\rho = l+m+\mu$, $\bar{\Sigma}^{\{1\}}$ can be partitioned as:

$$\bar{\Sigma}^{\{1\}} = \begin{vmatrix} I_l & 0_{l,m} & 0_{l,\mu} & 0_{l,l-\mu} \\ 0_{m,l} & I_m & 0_{m,\mu} & 0_{m,l-\mu} \\ 0_{\mu,l} & 0_{\mu,m} & I_\mu & 0_{\mu,l-\mu} \\ 0_{n-\rho,l} & 0_{n-\rho,m} & 0_{n-\rho,\mu} & 0_{n-\rho,l-\mu} \end{vmatrix} \tag{20}$$

Using the partition of L defined in (14), we have:

$$I_{2l+m} - \Sigma\Sigma^{\{1\}} = \begin{bmatrix} 0_{\rho,l} & 0_{\rho,m} & 0_{\rho,l-\mu} \\ -L_1 & & I_{l-\mu} \end{bmatrix} \tag{21}$$

This partition induces to write Z as:

$$Z = \left[Z_1 \quad Z_2\right] \tag{22}$$

where $Z_1 \in \Re^{m\times\rho}$ and $Z_2 \in \Re^{m\times(l-\mu)}$. We will see in the following that only Z_2 has to be considered. Eventually, (19) can be expressed as 3 equations:

$$\begin{aligned} J &= \bar{A}_{21} - Z_2 L_1 \\ D &= \bar{A}_{22} - Z_2 L_2 \\ V &= \bar{A}_{23}\begin{bmatrix} I_\mu & 0_{\mu,l-\mu} \\ 0_{n-\rho,\mu} & 0_{n-\rho,l-\mu} \end{bmatrix} + Z_2\left[-L_3\right. \end{aligned} \tag{23}$$

Thanks to (23), if ($\bar{A}_{22}, L_2$) is detectable, Z can be determined by using a standard eigenvalue assignment technique (SEAT) which provides D as a Hurwitz matrix.

The last point is to see that the detectability condition is independent of the choice of $\Sigma^{\{1\}}$ and this is shown in the appendix.

This standpoint simplifies the design in the minimum case by introducing numerical simplifications in the necessary and sufficient conditions of Theorem 1. All required matrices are obtained without calculation of any generalized inverse.

However when the existence conditions are not fulfilled, we have to study the case where $p > m$. This last case constitutes the next section called minimal case.

III. The Minimal Case

Our philosophy for the minimal case is to find, for a given p, the necessary and sufficient existence conditions for an asymptotic observer. The minimal order observer will be obtained by increasing p until these conditions are fulfilled. To study the minimal case we have to consider conditions (3), (4) and (5), using the frame we have already defined for the minimum case.

III.1. Analysis

In order to simplify the design procedure, we suppose that the observer is written in an observable canonical form (see [2]):

$$P=\begin{bmatrix} I_m & 0_{m,p-m} \end{bmatrix}, D=\begin{bmatrix} D_1 & D_2 \end{bmatrix} \tag{24}$$

with :

$$D_2=\begin{bmatrix} I_{p-m} \\ 0_{m,p-m} \end{bmatrix}$$

$$D_1=\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}, D_2=\begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix}$$
$$E=\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, T=\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \tag{25}$$

We choose to define the above mentioned partitioning in order to treat (4).

Remark 3. D_{12} and D_{22} are a specific partitioning of the p - m last columns of D_2. For a given p, these 2 matrices are known.

Using partitions (24) in (4) yields:

$$T_1A-D_{11}T_1-D_{12}T_2=E_1C \tag{26}$$

$$T_2A-D_{21}T_1-D_{22}T_2=E_2C \tag{27}$$

According to the definition of matrix P, $T_1=K-VC$ and by setting $J_1=E_1-D_{11}V$ and $J_2=E_2-D_{21}V$, (26)-(27) can be written as:

$$KA-D_{12}T_2=VCA+D_{11}K+J_1C$$
$$T_2A-D_{22}T_2=D_{21}K+J_2C$$

or equivalently:

$$\begin{bmatrix} J_1 & D_{11} & V \end{bmatrix}\Sigma=KA-D_{12}T_2 \tag{28}$$

$$\begin{bmatrix} J_2 & D_{21} \end{bmatrix}\Phi=T_2A-D_{22}T_2 \tag{29}$$

where $\Phi=\begin{bmatrix} C \\ K \end{bmatrix}$.

(28)-(29), in the new basis, are defined by:

$$\begin{bmatrix} J_1 & D_{11} & V \end{bmatrix}\overline{\Sigma}=\overline{KA}-D_{12}\overline{T}_2 \tag{30}$$

$$\begin{bmatrix} J_2 & D_{21} \end{bmatrix}\overline{\Phi}=\overline{T}_2\overline{A}-D_{22}\overline{T}_2 \tag{31}$$

Thanks to (13), (31) can be expressed as:

$$\begin{bmatrix} J_2 & D_{21} & 0_{p-m,n-l-m} \end{bmatrix}=\overline{T}_2\overline{A}-D_{22}\overline{T}_2 \tag{32}$$

As seen in the proof of corollary 1, we express $\overline{\Sigma}^{\{1\}}$ by (20) and $I_{2l+m}-\underline{\Sigma\Sigma}^{\{1\}}$ by (21). In addition to the partitioning already defined for L, A and Z respectively in (14), (15) and (22), $\overline{T}_2$ is partitioned as:

$$\overline{T}_2=\begin{bmatrix} \overline{T}_{21} & \overline{T}_{22} & \overline{T}_{23} \end{bmatrix} \tag{33}$$

III.2. Design

To solve (30)-(31), existence conditions must be established.

Existence condition of (30) is established thanks to {1}-inverses properties. Multiplying (30) by $\overline{\Sigma}^{\{1\}}\overline{\Sigma}$ leads to:

$$\left(KA-D_{12}\overline{T}_2\right)\left(I_n-\overline{\Sigma}^{\{1\}}\overline{\Sigma}\right)=0 \tag{34}$$

Due to the particular form of $I_n-\overline{\Sigma}^{\{1\}}\overline{\Sigma}$, (34) leads to:

$$\overline{A}_{23}\begin{bmatrix} 0_{\mu,n-\rho} \\ I_{n-\rho} \end{bmatrix}=D_{12}\overline{T}_{23}\begin{bmatrix} 0_{\mu,n-\rho} \\ I_{n-\rho} \end{bmatrix} \tag{35}$$

(35) constitutes the first relation which constrains $\overline{T}_{23}$ to obtain a solution for (30).

Furthermore, partition of (31) allows to write:

$$0_{p-m,n-l-m}=\overline{T}_2\overline{A}_3-D_{22}\overline{T}_{23} \tag{36}$$

(36) constitutes the second relation which constrains $\overline{T}_{23}$ to obtain a solution for (31).

So, if T_{23} is such that (35) is fulfilled, a solution for (30) is given by:

$$\begin{bmatrix} J_1 & D_{11} & V \end{bmatrix}=$$
$$\left(\overline{KA}-D_{12}\overline{T}_2\right)\overline{\Sigma}^{\{1\}}+Z\left(I_{2l+m}-\overline{\Sigma\Sigma}^{\{1\}}\right) \tag{37}$$

where Z is an arbitrary matrix.

By expressing $\underline{KA}$ using (17) and substituting $\overline{\Sigma}^{\{1\}}$ and $I_{2l+m}-\Sigma\Sigma^{\{1\}}$ in (37), 3 equations are deduced:

$$J_1=\overline{A}_{21}-D_{12}\overline{T}_{21}-Z_2L_1 \quad (38)$$

$$D_{11}=\overline{A}_{22}-D_{12}\overline{T}_{22}-Z_2L_2 \quad (39)$$

$$V=\left(\overline{A}_{23}-D_{12}\overline{T}_{23}\right)\begin{bmatrix} I_\mu & 0_{\mu,l-\mu} \\ 0_{n-\rho,\mu} & 0_{n-\rho,l-\mu}\end{bmatrix} +Z_2\begin{bmatrix}-L_3 & I_{l-\mu}\end{bmatrix} \quad (40)$$

Using the partitions of $\overline{A}$ and $\overline{T}_2$ given respectively in (15) and (33), and by considering that (36) is fulfilled, (31) can be written as:

$$J_2=\overline{T}_2\overline{A}_1-D_{22}\overline{T}_{21} \quad (41)$$

$$D_{21}=\overline{T}_2\overline{A}_2-D_{22}\overline{T}_{22} \quad (42)$$

with D_{11} defined (39) and D_{21} (42), D given in (25) can be written as follows:

$$D=\begin{bmatrix}\overline{A}_{22}+\Gamma_1 & D_{12} \\ \Gamma_2 & D_{22}\end{bmatrix}$$

where:

$$\Gamma_1=-D_{12}\overline{T}_{22}-Z_2L_2 \quad (43)$$

$$\Gamma_2=\overline{T}_2\overline{A}_2-D_{22}\overline{T}_{22} \quad (44)$$

D can also be decomposed into the canonical form allowing the application of a SEAT:

$$D=\Psi+\begin{bmatrix}\Gamma_1 \\ \Gamma_2\end{bmatrix}\begin{bmatrix}I_m & 0_{m,p-m}\end{bmatrix} \quad (45)$$

where:

$$\Psi=\begin{bmatrix}A_{22} & D_{12} \\ 0_{p-m,m} & D_{22}\end{bmatrix} \quad (46)$$

According to remark 2, Ψ is known, so if the pair $\left(\Psi,\begin{bmatrix}I_m & 0_{m,p-m}\end{bmatrix}\right)$ is observable, Γ_1 and Γ_2 can be determined by choosing arbitrary eigenvalues and using a SEAT to fulfill (3). If the pair $\left(\Psi,\begin{bmatrix}I_m & 0_{m,p-m}\end{bmatrix}\right)$ is detectable (3) is fulfilled and Γ_1 and Γ_2 are fixed and not choosen.

III.3. Determination of $\overline{T}_2$ and Z

Once Γ_1 and Γ_2 are calculated, T_2 and Z must be found by solving (35), (36), (43) and (44) using the Kronecker product (see [19]). Let $vec(X)$ be the vector value function of a matrix X, defined by:

$$vec(X)=\begin{bmatrix}x_{11} & \cdots & x_{m1} & x_{12} & \cdots & x_{m2} & & x_{1n} & \cdots & x_{mn}\end{bmatrix}^T$$

where X is defined by:

$$X=\begin{bmatrix}x_{11} & \cdots & x_{1n} \\ & & \\ x_{m1} & & x_{mn}\end{bmatrix}$$

So using the following property, $vec(AXB)=\left(B^T\otimes A\right)(X)$ where $\otimes$ is the Kronecker product, (35), (36), (43) and (44). Eventually, we obtain (system (47)):

$$\begin{aligned}
vec(\Gamma_1)&=\left(I_m\otimes -D_{12}\right)vec\left(\overline{T}_{22}\right)-\left(L_2^T\otimes I_m\right)vec(Z_2)\\
vec(\Gamma_2)&=\left(\overline{A}_2^T\otimes I_{p-m}\right)vec\left(\overline{T}_2\right)-\left(I_m\otimes D_{22}\right)vec\left(\overline{T}_{22}\right)\\
0&=\left(\overline{A}_3^T\otimes I_{p-m}\right)vec\left(\overline{T}_2\right)-\left(I_{n-l-m}\otimes D_{22}\right)vec\left(\overline{T}_{23}\right)\\
&\left(\begin{bmatrix}0_{\mu,n-\rho} \\ I_{n-\rho}\end{bmatrix}^T\otimes D_{12}\right)vec\left(\overline{T}_{23}\right)=vec\left(\overline{A}_{23}\begin{bmatrix}0_{\mu,n-\rho} \\ I_{n-\rho}\end{bmatrix}\right)
\end{aligned}$$

Due to the decomposition of $\overline{T}_2$ (33), it is assumed that:

$$\begin{aligned} vec\left(\overline{T}_{22}\right)&=G_1vec\left(\overline{T}_2\right)\\ vec\left(\overline{T}_{23}\right)&=G_2vec\left(\overline{T}_2\right)\end{aligned} \quad (48)$$

with:

$$G_1=\begin{bmatrix}0_{(p-m)m,(p-m)l} & I_{(p-m)m} & 0_{(p-m)m,(p-m)(n-l-m)}\end{bmatrix}$$

and:

$$G_2=\begin{bmatrix}0_{(p-m)(n-l-m),(p-m)(l+m)} & I_{(p-m)(n-l-m)}\end{bmatrix}$$

By placing (48) into (47) we get:

$$S_1vec(Z_2)=S_2 \quad (49)$$

$$H_1vec\left(\overline{T}_2\right)=H_2 \quad (50)$$

with:

$$H_1 = \begin{bmatrix} \left(\left(\overline{A}_2^T \otimes I_{p-m}\right) - \left(I_m \otimes D_{22}\right)G_1\right) \\ \left(\left(\overline{A}_3^T \otimes I_{p-m}\right) - \left(I_{n-l-m} \otimes D_{22}\right)G_2\right) \\ \left(\begin{bmatrix} 0_{\mu,n-\rho} \\ I_{n-\rho} \end{bmatrix}^T \otimes D_{12}\right)G_2 \end{bmatrix}$$

$$\begin{bmatrix} vec\left(\Gamma_2\right) \\ 0_{(n-l-m)(p-m).1} \\ vec\left(\overline{A}_{23}\begin{bmatrix} 0_{\mu,n-\rho} \\ I_{n-\rho} \end{bmatrix}\right) \end{bmatrix}$$

$$S_1 = L_2^T \otimes I_m$$

$$S_2 = \left(I_m \otimes -D_{12}\right)G_1 vec\left(\overline{T}_2\right) - vec\left(\Gamma_1\right)$$

A solution for T_2 exists if and only if:

$$rank\left(H_1\right) = rank\left(\begin{bmatrix} H_1 & H_2 \end{bmatrix}\right) \quad (51)$$

and is given by:

$$vec\left(T_2\right) = H_1^{\{1\}} H_2 + \left(I_n - H_1^{\{1\}} H_1\right)Y \quad (52)$$

where Y is an arbitrary matrix that can be taken equal to zero as shown in the presented example.

Once T_2 is determined, a solution for Z_2 exists if and only if:

$$rank\left(S_1\right) = rank\left(\begin{bmatrix} S_1 & S_2 \end{bmatrix}\right) \quad (53)$$

and a solution is given by:

$$vec\left(Z_2\right) = S_1^{\{1\}} S_2 + \left(I_n - S_1^{\{1\}} S_1\right)W \quad (54)$$

where W is an arbitrary matrix. As mentionned above for Y, W can be taken equal to zero.

The previous development leads to propose the following theorem:

Theorem 2. The necessary and sufficient conditions for the existence and stability of the minimal functional observer (2) of order p, for system (1) are:

1. $\left(\Psi, \begin{bmatrix} I_m & 0_{m,p-m} \end{bmatrix}\right)$ is detectable,
2. (51) and (53) are fulfilled.

IV. Design Procedure

The following procedure is proposed:

Minimum LFO

1. Set $p = m$
2. Compute Q with (11) and deduce $\overline{\Sigma}$, $\overline{A}$ and L
3. *Existence condition:*
   if (18) is true then go to *step 4*
   else go to *step 8*
4. If $(\overline{A}_{22}, L_2)$ is observable, then go to *step 5*
   else if $(\overline{A}_{22}, L_2)$ is detectable, then go to *step 6*
   else go to *step 8*
5. Choose eigenvalues for D and calculate Z with a SEAT
6. Compute D, V and J which are defined in (19) and compute $E = J + DV$ and $\overline{T} = \overline{K} - V\overline{C}$
7. End: a minimum LFO is designed

Minimal LFO

8. Increase p : $p = p+1$
   if $p < n - l$, then go to *step 9*
   else the reduced order observer has to be considered
9. Compute Ψ with (46)
10. If $(\Psi, [I_m \quad 0])$ is observable, then go to *step 11*
    else if $(\Psi, [I_m \quad 0])$ is detectable, then go to *step 12*
    else go to *step 8*
11. Choose observer eigenvalue, get Γ_1 and Γ_2 by taking $Y = W = 0$. Calculate D with (45)
12. Existence condition
    If (51) and (53) are fulfilled, calculate $\overline{T}_2$ and Z_2 with (52) and (54)
    else go to *step 8*
13. Compute Z, J with (21-38) and V with (40).
    Compute E and $\overline{T}$ with $E = \begin{bmatrix} J_1 + D_{11}V \\ J_2 + D_{21}V \end{bmatrix}$ and $\overline{T} = \begin{bmatrix} \overline{K} - V\overline{C} \\ \overline{T}_2 \end{bmatrix}$
14. End: a minimal LFO is designed

V. Numerical Examples

V.1. Case 1: A Minimum Order Observer Exists

Considering the following system:

$$A = \begin{array}{cccccc} -1 & 0 & 0.1 & & -1 & \\ 2 & 1 & 0 & 0.5 & 1 & 2 \\ -1 & 0 & 0 & 0 & 2 & -1 \\ & 0.5 & 1 & 0 & 0.1 & 0.2 \\ 2 & -0.1 & 0.2 & 0.3 & 0.9 & 0 \\ 0 & 0 & 1 & 2 & 0.1 & 0.5 \end{array}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \end{bmatrix},$$

and $K = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

From this system we get $n = 6$, $l = 3$ and $m = 2$. Using the design procedure we get:

1. Minimum case: $p = 2$
2. A change in the state variable (Q) in order to compute C in the form (12) is given by:

$$Q = \begin{bmatrix} C \\ K \\ (CA)_1 \end{bmatrix} = \begin{array}{cccccc} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ -1 & 0 & 0.1 & & -1 & \end{array}$$

where $(CA)_1$ ($\mu = 1$) is the first line of CA linearly independent of $\begin{bmatrix} C \\ K \end{bmatrix}$. So, the matrix A is such that:

$$\bar{A} = QAQ^{-1} = \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \\ 4 & 1 & -0.2 & -1.5 & 3 & 2 \\ -2 & 0 & 0.1 & 1 & 1 & -1 \\ 1.2 & 0.5 & 0.98 & -0.2 & 0.3 & 0.2 \\ 2 & -0.1 & 0.2 & 0.3 & 0.9 & 0 \\ -0.5 & 0.6 & 1.74 & 1.1 & 0.1 & -0.4 \end{array}$$

Following (9) as $p = l + m + \mu = 6 = n$, (18) is always fulfilled.

3. With Q, $\bar{\Sigma}$ is such that:

$$\bar{\Sigma} = \begin{bmatrix} CQ^{-1} \\ KQ^{-1} \\ CAQ^{-1} \end{bmatrix}$$

$$\bar{\Sigma} = \left| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \\ 4 & & -0.2 & -1.5 & 3 & 2 \\ -2 & 0 & 0.1 & & & -1 \end{array} \right.$$

$$L = \begin{bmatrix} 4 & 1 & -0.2 & -1.5 & 3 & 2 \\ -2 & 0 & 0.1 & 1 & 1 & -1 \end{bmatrix}$$

By partitioning the matrix L, we obtain

$$L_1 = \begin{bmatrix} 4 & 1 & -0.2 \\ -2 & 0 & 0.1 \end{bmatrix}, L_2 = \begin{bmatrix} -1.5 & 3 \\ 1 & 1 \end{bmatrix}$$

and $L_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

4. Following the partition for A defined in (15), we have:

$$\bar{A}_{22} = \begin{bmatrix} -0.2 & 0.3 \\ 0.3 & 0.9 \end{bmatrix}$$

$(\bar{A}_{22}, L_2)$ is observable: A_{22} is a (2×2) matrix and $\text{rank}\left(\begin{bmatrix} L_2 \\ L_2\bar{A}_{22} \end{bmatrix}\right) = 2$.

5. Eigenvalues for D are chosen in -1 and -2, it follows that:

$$Z = \begin{bmatrix} -0.1111 & 0.6333 \\ 0.5778 & 1.1667 \end{bmatrix}$$

6. D, V, J, E and T are such that:

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$J = \begin{bmatrix} 2.7111 & 0.6111 & 0.9144 \\ 2.0222 & -0.6778 & 0.1989 \end{bmatrix}$$

$$V = \begin{bmatrix} 1.0556 & 0 & 0 \\ 0.0111 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1.6556 & 0.6111 & 0.9144 \\ 2 & -0.6778 & 0.1989 \end{bmatrix}$$

$$T = \begin{bmatrix} -1.0556 & 0 & 0 & 1 & 0 & 0 \\ -0.0111 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

V.2. Case 2: A Minimum Order Observer Exists

Considering the following system:

$A =$

	0	2	0.5
1	−0.5	1	0
−3		−4	−0.5
	0.5	3	
1.8	0.1	3.8	
−1	1.5	1	1
2	−1	0.5	−0.25
3	0	−2	−1
−1	0.25	−0.5	0.6
	1	0	−2

−2		0.1	0.2	0	1
2				2	0
−1	−5	−2.1	−2.2	−4	−1
−4	0.5	−0.8	−0.6	−2	2
−4.4	0	−0.1	2	−0.4	2
−10	−1	−2.8	−2.6	−6	2
1.2	1.4	0	0	−2	1
−0.25	0.1	2	−1	0	1
0.3	0.1	1	−1	3	0
	0	−1.3	1.2	0.8	−1

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $K = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \end{bmatrix}$

Using the design procedure, we obtain:

1. Minimum case: $p = 2$
2. A change in the state variable (Q) in order to compute C in the form (12) is given by:

$$Q = \begin{vmatrix} C \\ K \\ (CA)_1 \\ (CA)_2 \\ X \end{vmatrix}$$

1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0		0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	0	0		0	0	0	0
	0	2	0.5	−2		0.1	0.2	0	1
	−0.5	1	0	2			1	2	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0		0	0

as the two first lines of CA are linearly independent of $\begin{bmatrix} C \\ K \end{bmatrix}$, $\mu = 2$. So, matrices $\overline{A}$ and $\overline{\Sigma}$ are such that:

$\overline{\Sigma} =$

1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0		0	0
0	0	0	0		−2	−1	−2	0	0
0	0	0	0	2	−0.5	2	−1	0	0

$\overline{A} =$

0	0	0	0	0
0	0	0	0	0
0	0	0	0	
0	0	0	0	2
0	0	0	0	0
0	0	0	0	0
2.2	1.05	0.75	−1.8750	2.07
−1	0.5	−8.5	−1.05	2.55
2	−1.5	−0.5	−0.75	5.2
2	0	−4	−1.5	1.75

0	1	0	0	0
0	0	1	0	0
−2	−1	−2	0	0
−0.5	2	−1	0	0
−1.8	2	−0.2	−0.1	1.8
0	2	−3	0	0
−0.09	−2.7	−6.8	−0.93	−2.76
−7.9	8	−3.9	1.6	−1.8
1.4		−1	0.9	0.8
−0.9		0	1.9	−1.2

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 & 2 & -0.5 & 2 & -1 \end{bmatrix}$$

As rank $\left[\bar{A}_{13}\right] = 2$ and rank $\begin{bmatrix} \bar{A}_{13} \\ \bar{A}_{23} \end{bmatrix} = 3$, (18) is not fulfilled, then the minimal order observer does not exist, so go to step 8.

3. With $p = 3$, the system is such that $p < n - l$.
4. Ψ is given by (46):

$$\Psi = \begin{bmatrix} 0 & -1.8 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

5. $\left(\Psi, \begin{bmatrix} I_2 & 0_{2.1} \end{bmatrix}\right)$ is observable.
6. By choosing observer eigenvalues in -1, -2 and -3, we get Γ_1 and Γ_2:

$$\Gamma_1 = \begin{bmatrix} -3 & 1.8 \\ 0 & -3 \end{bmatrix}$$

$$\Gamma_2 = \begin{bmatrix} -2 & 0 \end{bmatrix}$$

so we get:

$$D = \begin{bmatrix} -3 & 0 & 1 \\ 0 & -3 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

7. T_2 and Z_2 exist as (51) and (53) are fulfilled:

$$\bar{T}_2 = [2.3802 \quad 2.1561 \quad -3.8728 \quad -2.7875 \quad 5.0277 \\ -1.7078 \quad 2.6248 \quad -0.2414 \quad -0.1 \quad 1.8]$$

$$Z_2 = \begin{bmatrix} 0.3423 & -1.185 \\ -1.7143 & 0.8571 \end{bmatrix}$$

8. Matrices J, V, E and T are respectively defined by:

$$J = \begin{bmatrix} -2.3802 & -2.1561 & 3.8728 & 2\ 7875 \\ 0 & 0 & 0 & 0 \\ 9.4159 & 2.7853 & -3.1298 & -7.2930 \end{bmatrix}$$

$$V = \begin{bmatrix} 2.0876 & -0.459 & 0.3423 & -1.185 \\ -1.4286 & -5.5714 & -1.7143 & 0.8571 \end{bmatrix}$$

$$E = \begin{bmatrix} -8.6432 & -0.7792 & 2.8458 & 6.3426 \\ 4.2857 & 16.7143 & 5.1429 & -2.5714 \\ 5.2406 & 3.7033 & -3.8145 & -4.9229 \end{bmatrix}$$

$$T = \begin{bmatrix} -2\ 08 & 0.45 & -0.34 & 1.18 & & 0 & 0 & 0 & 0 & 0 \\ 1.42 & 5.57 & 1.71 & -0.85 & 0 & & 0 & 0 & 0 & 0 \\ 2.38 & 2.15 & -3.87 & -2.78 & 5.02 & -1.70 & 2.62 & -0.24 & -0.1 & 1\ 8 \end{bmatrix}$$

VI. Conclusion

We have shown that the procedure established by [1] can be simplified in the case where $p = m$. Derived from theorem 1, corollary 1 is proposed and summarizes our results. Moreover, when it is not possible to design a LFO in this case, we propose a procedure which enables to design a minimal LFO by increasing order p. Theorem 2 gives necessary and sufficient conditions in this latter case.

Appendix

To verify that the detectability condition is independent of the choice of $\Sigma^{\{1\}}$, we consider two different choices for $\Sigma^{\{1\}}$ denoted $\Sigma^{\{1\}'}$ and $\Sigma^{\{1\}''}$. Supposing that we get two different solutions for (7): $\begin{bmatrix} J' & D' & V' \end{bmatrix}$ and $\begin{bmatrix} J'' & D'' & V'' \end{bmatrix}$ respectively, we have:

$$\begin{bmatrix} J' & D' & V' \end{bmatrix} = KA\Sigma^{\{1\}'} + Z'\left(I_{2l+m} - \Sigma\Sigma^{\{1\}'}\right)$$
$$\begin{bmatrix} J'' & D'' & V'' \end{bmatrix} = KA\Sigma^{\{1\}''} + Z''\left(I_{2l+m} - \Sigma\Sigma^{\{1\}''}\right)$$

$\Sigma^{\{1\}'}$ and $\Sigma^{\{1\}''}$ verify:

$$\Sigma^{\{1\}'} = \Sigma^{\{1\}''} + Y - \Sigma^{\{1\}''}\Sigma Y \Sigma\Sigma^{\{1\}''}$$

where Y is an arbitrary matrix with appropriate dimensions:

$$\begin{bmatrix} J' & D' & V' \end{bmatrix} = KA\left(\Sigma^{\{1\}''} + Y - \Sigma^{\{1\}''}\Sigma Y \Sigma\Sigma^{\{1\}''}\right)$$
$$+ Z'\left(I_{2l+m} - \Sigma\left(\Sigma^{\{1\}''} + Y - \Sigma^{\{1\}''}\Sigma Y \Sigma\Sigma^{\{1\}''}\right)\right)$$
$$= KA\Sigma^{\{1\}''} + KAY - KA\Sigma^{(1)''}\Sigma Y \Sigma\Sigma^{\{1\}''} +$$
$$Z' - Z'\Sigma\Sigma^{\{1\}''} + -Z'\Sigma Y + Z'\Sigma\Sigma^{\{1\}''}\Sigma Y \Sigma\Sigma^{\{1\}''}$$

For compatibility reasons $KA\Sigma^{\{1\}''}\Sigma = KA$, so:

$$\begin{bmatrix} J' & D' & V' \end{bmatrix} = KA\Sigma^{\{1\}''} + KAY\left(I_{2l+m} - \Sigma\Sigma^{\{1\}''}\right) +$$
$$Z'\left(I_{2l+m} - \Sigma\Sigma^{\{1\}''}\right) - Z'\Sigma Y\left(I_{2l+m} - \Sigma\Sigma^{\{1\}''}\right)$$
$$= KA\Sigma^{\{1\}''} + \left(KAY + Z' - Z'\Sigma Y\right)\left(I_{2l+m} - \Sigma\Sigma^{\{1\}''}\right)$$

If we choose $Z'' = KAY + Z' - Z'\Sigma Y$, we get $\begin{bmatrix} J' & D & V' \end{bmatrix} = \begin{bmatrix} J'' & D'' & V'' \end{bmatrix}$. If we suppose now that an arbitrary pole placement on D is possible by computing a matrix Z', it will always be possible to arbitrarily assign the pole of D'' by substituting Z'' for $KAY + Z' - Z'\Sigma Y$.

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