On the convergence of the hp-BEM with quasi-uniform meshes for the electric field integral equation on polyhedral surfaces *

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Abstract

In this paper the hp-version of the boundary element method is applied to the electric field integral equation on a piecewise plane (open or closed) Lipschitz surface. The underlying meshes are supposed to be quasi-uniform. We use $\mathbf{H}(\mathrm{div})$-conforming discretisations with quadrilateral elements of Raviart-Thomas type and establish quasi-optimal convergence of hp-approximations. Main ingredient of our analysis is a new $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming p-interpolation operator that assumes only $\mathbf{H}^r \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-regularity ($r > 0$) and for which we show quasi-stability with respect to polynomial degrees.

Key words: hp-version with quasi-uniform meshes, electric field integral equation, time-harmonic electro-magnetic scattering, boundary element method
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1 Introduction and formulation of the problem

In this paper we prove convergence of the hp-version of the boundary element method (BEM) for the electric field integral equation (EFIE) on piecewise plane (open or closed) surfaces discretised by quasi-uniform meshes. The EFIE is a boundary integral equation that represents a boundary value problem for the time-harmonic Maxwell's equations in the exterior domain. It models the scattering of electro-magnetic waves at a perfectly conducting body (the scatterer). The solution to the EFIE is the induced electric surface current (a tangential vector field) on the surface of the scatterer, see, e.g., [28]. If the scatterer is a thin object (i.e., its thickness is small in comparison to the wave length), then it can be modelled as an open surface (a sub-manifold with boundary) in $\mathbb{R}^3$. Our analysis covers this theoretically challenging case, which has important applications (e.g., antenna problems).

The basis of our BEM is a variational formulation of the EFIE, called Rumsey's principle. For smooth surfaces, its boundary element discretisation has been studied by Bendali in 1984, see

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[3, 4]. Progress in the numerical analysis of the EFIE on Lipschitz surfaces has been achieved relatively recently and was inspired by the study of traces of functional spaces that govern Maxwell's equations in Lipschitz domains [16]. The main challenges in this analysis concern the solvability and quasi-optimal convergence of approximations and a priori error estimation in the energy norm. In the framework of the h-version of the BEM, i.e., for discretisations with elements of fixed order on refined meshes, these issues were addressed in [15, 24, 18] (for polyhedral surfaces) and in [12] (for open Lipschitz surfaces). We note that in [24, 18, 12] the authors focused on conforming discretisations of Rumsey's principle with Raviart-Thomas (or Brezzi-Douglas-Marini) boundary elements, whereas in [15] a mixed formulation utilising standard (continuous) basis functions was used. In this paper we follow the former approach, called the natural boundary element method for the EFIE.

While in the p-version of the BEM the mesh is fixed and approximations are improved by increasing polynomial degrees, the hp-version combines both mesh refinement and increase of polynomial degrees. In our previous paper [8] we analysed the natural p-BEM for the EFIE on a plane open surface with polygonal boundary. We have proved convergence of the p-version with Raviart-Thomas (RT) parallelogram elements and derived an a priori error estimate which takes into account the strong singular behaviour of the solution at edges and corners of the surface. With the present note we prove the convergence of the natural hp-BEM on polyhedral and piecewise plane open surfaces discretised by quasi-uniform meshes of quadrilateral (in general, curvilinear) elements. We emphasize that RT-spaces are used for both affine and non-affine quadrilateral meshes.

In order to prove convergence of approximations for the EFIE, one usually relies on properties of the continuous and discrete Helmholtz decompositions, and on the proximity in some sense of the discrete decompositions to the continuous one, see [12, 18]. The key property is the orthogonality of decompositions, and the main tool in the analysis is an appropriate interpolation operator (a projector onto the corresponding polynomial space). In [12, 18], $\mathbf{L}^2$-orthogonal discrete decompositions mimicking the Helmholtz decomposition of the energy space were analysed for finite dimensional subspaces based on Raviart-Thomas and Brezzi-Douglas-Marini (BDM) boundary elements. It has been proved that these discrete decompositions are sufficiently close to the continuous one as the mesh parameter h tends to zero (i.e., for the h-version of the BEM). The main tools in the proofs were the standard $\mathbf{H}(\mathrm{div})$-conforming RT or BDM interpolation operators. However, it turns out that a generalisation of that approach to the p- and the hp-versions is not straightforward when sticking to both the $\mathbf{L}^2$-orthogonality of decompositions and classical interpolation operators. In [8], for the p-version, we employed an $\tilde{\mathbf{H}}^{-1/2}$-orthogonality of the Helmholtz decomposition, while using the classical RT interpolation operator. Unfortunately, the extension of this approach to polyhedral surfaces and even to piecewise plane screens does not seem to be easy. In particular, the low regularity of the Laplace-Beltrami operator on polyhedral surfaces is not enough to prove stability (with respect to polynomial degrees) of the classical Raviart-Thomas interpolation operator. That is why in this paper we use an alternative approach: we adhere to the $\mathbf{L}^2$-orthogonality of decompositions but utilise a non-classical $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming interpolation operator (for RT-elements). Our construction of this operator (see Section 5) is much in the spirit of [21], where H^1- and $\mathbf{H}(\mathrm{curl})$-conforming projection-based interpolation operators were introduced and analysed. We, however, employ the $\tilde{H}^{-1/2}$-projection for the divergence term, and thus need the corresponding inner product to be written in an appropriate explicit form. Moreover, the use of an appropriate scaling argument allows to prove the convergence result in the framework of the *hp*-BEM.

We will denote by Γ a piecewise plane (open or closed) Lipschitz surface in $\mathbb{R}^3$. Let us introduce Rumsey's formulation of the electric field integral equation on Γ. For a given wave number $k > 0$ and a scalar function v (resp., tangential vector field $\mathbf{v}$) we define the single layer operator Ψ_k (resp., $\mathbf{\Psi}_k$) by

$$\Psi_k v(x) = \frac{1}{4\pi} \int_\Gamma v(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y, \qquad x \in \mathbb{R}^3 \backslash \Gamma$$

$$\Big(\text{resp.,} \quad \mathbf{\Psi}_k \mathbf{v}(x) = \frac{1}{4\pi} \int_\Gamma \mathbf{v}(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y, \qquad x \in \mathbb{R}^3 \backslash \Gamma\Big).$$

Let $\mathbf{L}_t^2(\Gamma)$ be the space of two-dimensional, tangential, square integrable vector fields on Γ. By ∇_Γ (resp., div_Γ) we denote the surface gradient (resp., surface divergence) acting on scalar functions (resp., tangential vector fields) on Γ. We will need the following space:

$$\mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\|^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}$$

if Γ is a closed surface, and

$$\begin{aligned} \mathbf{X} = \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \quad := \quad & \{\mathbf{u} \in \tilde{\mathbf{H}}_\|^{-1/2}(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in \tilde{H}^{-1/2}(\Gamma) \text{ and} \\ & \langle \mathbf{u}, \nabla_\Gamma v\rangle + \langle \mathrm{div}_\Gamma\, \mathbf{u}, v\rangle = 0 \quad \text{for all} \quad v \in C^\infty(\Gamma)\} \end{aligned}$$

if Γ is an open surface. In the latter definition the brackets $\langle \cdot, \cdot \rangle$ denote dualities associated with $\mathbf{H}_\|^{1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively. For definitions of the space $C^\infty(\Gamma)$ and the Sobolev spaces on Γ see §3.1.

Throughout, we use boldface symbols for vector fields. The spaces (or sets) of vector fields are also denoted in boldface (e.g., $\mathbf{H}^s(\Gamma) = (H^s(\Gamma))^3$), with their norms and inner products being defined in §3.1.

Let $\mathbf{X}'$ be the dual space of $\mathbf{X}$ (with $\mathbf{L}_t^2(\Gamma)$ as pivot space). Now, for a given tangential vector field $\mathbf{f} \in \mathbf{X}'$ ($\mathbf{f}$ represents the excitation by an incident wave), Rumsey's formulation reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \gamma_{\mathrm{tr}}(\Psi_k \mathrm{div}_\Gamma\, \mathbf{u}), \mathrm{div}_\Gamma\, \mathbf{v}\rangle - k^2 \langle \pi_\tau(\mathbf{\Psi}_k \mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{1.1}$$

Here γ_{tr} is the standard trace operator, and π_τ denotes the tangential components trace mapping (see §3.1 for definitions and properties of these operators).

The rest of the paper is organised as follows. In the next section we define the *hp*-version of the BEM for the EFIE and formulate the main result (Theorem 2.1), which states the unique solvability and quasi-optimal convergence of this approximation method. Section 3 gives necessary preliminaries: in §3.1 we recall definitions of functional spaces of scalar functions and vector fields; then in §3.2 we introduce some equivalent norms in the Sobolev spaces H^s and $\tilde{H}^s$ ($s = \pm\frac{1}{2}$) on the reference element and derive expressions for corresponding inner products; some auxiliary lemmas are collected in §3.3. In Section 4 we discuss the continuous Helmholtz decomposition of the energy space $\mathbf{X}$ and its discrete counterpart. Section 5 is devoted to interpolation operators. First, we recall some known operators such as the L^2- and $\tilde{H}^{-1/2}$-projectors, the H^1- and $\mathbf{H}(\mathrm{curl})$-conforming projection-based interpolation operators. Then we introduce an $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming interpolation operator and study its properties. In particular, we prove the quasi-stability of this operator and its commutativity with the $\tilde{H}^{-1/2}$-projector. The $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming interpolation operator and $\mathbf{L}^2$-orthogonal discrete Helmholtz decompositions are the main tools in the proof of Theorem 2.1, which is given in Section 6. We conclude the paper with Section 7, where we comment on extensions and open problems.

Throughout the paper, C denotes a generic positive constant which is independent of h, p, and involved functions, unless stated otherwise.

2 The *hp*-version of the BEM and the main result

For the approximate solution of (1.1) we apply the *hp*-version of the BEM based on Galerkin discretisations with Raviart-Thomas spaces on quasi-uniform meshes. In what follows, $h > 0$ and $p \geq 1$ will always specify the mesh parameter and a polynomial degree, respectively. For any $\Omega \subset \mathbb{R}^n$ we will denote $\rho_\Omega = \sup\{\mathrm{diam}(B);\ B \text{ is a ball in } \Omega\}$. Furthermore, we will denote by $K = (0,1)^2$ the reference square. The sides of K will be denoted by ℓ_i ($i = 1, \ldots, 4$).

Let $\mathcal{T} = \{\Delta_h\}$ be a family of meshes $\Delta_h = \{\Gamma_j;\ j = 1, \ldots, J\}$ on Γ, where the elements Γ_j are open quadrilaterals (in general, curvilinear ones) satisfying the following standard assumptions (below we denote $h_j = \mathrm{diam}(\Gamma_j)$ for any $\Gamma_j \in \Delta_h$):

i) $\bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$; the intersection of any two quadrilaterals $\bar{\Gamma}_j$, $\bar{\Gamma}_k$ ($j \neq k$) is either a common vertex, an entire side, or empty;

ii) The elements are shape regular, i.e., there exists a positive constant C independent of $h = \max_j h_j$ such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{T}$ there holds $h_j \leq C\, \rho_{\Gamma_j}$.

iii) Any element Γ_j is the image of the reference square K, more precisely

$$\bar{\Gamma}_j = T_j(\bar{K}), \quad \mathbf{x} = T_j(\boldsymbol{\xi}),\ \mathbf{x} = (x_1, x_2) \in \bar{\Gamma}_j,\ \boldsymbol{\xi} = (\xi_1, \xi_2) \in \bar{K},$$

where T_j is sufficiently smooth one-to-one mapping with sufficiently smooth inverse $T_j^{-1} : \Gamma_j \to K$. The Jacobian matrix of T_j is denoted by $DT_j(\boldsymbol{\xi})$, it is supposed to be invertible for any $\boldsymbol{\xi} \in K$, and $DT_j^{-1}(\mathbf{x}) = (DT_j(\boldsymbol{\xi}))^{-1}$. We assume that

$$|\det(DT_j(\boldsymbol{\xi}))| \simeq h_j^2 \qquad \text{for any } \boldsymbol{\xi} \in K, \tag{2.1}$$

and there exist positive constants C independent of h such that for $k = 1, 2$ there holds

$$\sup_{\boldsymbol{\xi} \in K} \|D^k T_j(\boldsymbol{\xi})\|_{\mathcal{L}_k(\mathbb{R}^2, \mathbb{R}^2)} \leq C\, h_j^k, \qquad \sup_{\mathbf{x} \in \Gamma_j} \|D^k T_j^{-1}(\mathbf{x})\|_{\mathcal{L}_k(\mathbb{R}^2, \mathbb{R}^2)} \leq C\, h_j^{-1}. \tag{2.2}$$

Here $D^k T_j(\boldsymbol{\xi})$ (resp., $D^k T_j^{-1}(\mathbf{x})$) denotes the k-th (Fréchet) derivative of T_j (resp., T_j^{-1}) and $\|\cdot\|_{\mathcal{L}_k(X,Y)}$ is the operator norm in the space $\mathcal{L}_k(X,Y)$ of continuous k-linear mappings from X^k into Y.

iv) if $\bar\Gamma_j \cap \bar\Gamma_k$ is an entire side ℓ, then denoting by T_j^ℓ (resp., T_k^ℓ) the restriction of T_j (resp., T_k) to the corresponding side $T_j^{-1}(\ell)$ (resp., $T_k^{-1}(\ell)$) of the reference element K, one has $T_j^\ell \equiv T_k^\ell$ as mappings of the unit interval $(0,1)$ onto ℓ.

Remark 2.1 *Assumptions* (2.1), (2.2) *above are always satisfied for affine families of elements (i.e., for meshes of parallelograms). For curvilinear elements these assumptions are satisfied, for instance, if the elements tend to be affine as $h \to 0$ (see* [19, Section 4.3]*). In this case* (2.1), (2.2) *follow from assumption ii) and the smoothness of T_j, provided that h_j is small enough.*

In this paper we consider a family $\mathcal{T}$ of quasi-uniform meshes Δ_h on Γ in the sense that there exists a positive constant C independent of h such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{T}$ there holds $h \le C\, h_j$.

The mapping T_j introduced above is used to associate the scalar function u defined on the real element Γ_j with the function $\hat u$ defined on the reference element K:

$$u = \hat u \circ T_j^{-1} \ \text{ on } \Gamma_j \qquad \text{and} \qquad \hat u = u \circ T_j \ \text{ on } K.$$

Any vector-valued function $\hat{\mathbf{v}}$ defined on K is transformed to the function $\mathbf{v}$ on Γ_j by using the Piola transformation:

$$\mathbf{v} = \mathcal{M}_j(\hat{\mathbf{v}}) = \tfrac{1}{J_j} DT_j \hat{\mathbf{v}} \circ T_j^{-1}, \quad \hat{\mathbf{v}} = \mathcal{M}_j^{-1}(\mathbf{v}) = J_j DT_j^{-1} \mathbf{v} \circ T_j, \tag{2.3}$$

where J_j is the determinant $J_j(\boldsymbol{\xi}) := \det(DT_j(\boldsymbol{\xi}))$.

Let us introduce the needed polynomial sets. By $\mathcal{P}_p(I)$ we denote the set of polynomials of degree $\le p$ on an interval $I \subset \mathbb{R}$, and $\mathcal{P}_p^0(I)$ denotes the subset of $\mathcal{P}_p(I)$ which consists of polynomials vanishing at the end points of I. In particular, these two sets will be used for edges $\ell_i \subset \partial K$.

Further, $\mathcal{P}_{p_1,p_2}(K)$ denotes the set of polynomials on K of degree $\le p_1$ in ξ_1 and degree $\le p_2$ in ξ_2. For $p_1 = p_2 = p$ we denote $\mathcal{P}_p(K) = \mathcal{P}_{p,p}(K)$. The corresponding set of polynomial (scalar) bubble functions on K is denoted by $\mathcal{P}_p^0(K)$.

Let us denote by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ the RT-space of order $p \ge 1$ on K (see, e.g., [11, 29]), i.e.,

$$\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) = \mathcal{P}_{p,p-1}(K) \times \mathcal{P}_{p-1,p}(K).$$

The subset of $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ which consists of vector-valued polynomials with vanishing normal trace on the boundary ∂K (vector bubble-functions) will be denoted by $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$.

Then using transformations (2.3), we set

$$\mathbf{X}_{hp} := \{\mathbf{v} \in \mathbf{X}^0;\ \mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}) \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K),\ j = 1, \dots, J\}, \tag{2.4}$$

where the space $\mathbf{X}^0 \subset \mathbf{X}$ is defined in §3.1 ($\mathbf{X}^0 = \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ if Γ is closed and $\mathbf{X}^0 = \mathbf{H}_0(\mathrm{div}_\Gamma, \Gamma)$ if Γ is an open surface). We will denote by $N = N(h,p)$ the dimension of the discrete space $\mathbf{X}_{hp}$. One has $N \simeq h^{-2}$ for fixed p and $N \simeq p^2$ for fixed h.

The hp-version of the Galerkin BEM for the EFIE reads as: *Find* $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *such that*

$$a(\mathbf{u}_{hp}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}_{hp}. \tag{2.5}$$

Let us formulate the result which states the unique solvability of (2.5) and quasi-optimal convergence of the hp-version of the BEM for the EFIE.

Theorem 2.1 *There exists* $N_0 \geq 1$ *such that for any* $\mathbf{f} \in \mathbf{X}'$ *and for arbitrary mesh-degree combination satisfying* $N(h,p) \geq N_0$ *the discrete problem* (2.5) *is uniquely solvable and the hp-version of the Galerkin BEM generated by RT-elements converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_{hp}\|_{\mathbf{X}} \leq C \inf\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}};\ \mathbf{v} \in \mathbf{X}_{hp}\}.$$

Here, $\mathbf{u} \in \mathbf{X}$ *is the solution of* (1.1), $\mathbf{u}_{hp} \in \mathbf{X}_{hp}$ *is the solution of* (2.5), $\|\cdot\|_{\mathbf{X}}$ *denotes the norm in* $\mathbf{X}$*, and* $C > 0$ *is a constant independent of* h *and* p.

The proof of Theorem 2.1 is given in Section 6 below.

3 Preliminaries

3.1 Functional spaces, norms, and inner products

First, let us recall the Sobolev spaces and norms for scalar functions on a Lipschitz domain $\Omega \subset \mathbb{R}^n$, see [25]. To that end we will need the space $C^\infty(\Omega)$ of infinitely differentiable functions in Ω and its subspace $C_0^\infty(\Omega) \subset C^\infty(\Omega)$ which consists of functions with compact support in Ω.

For an integer s, let $H^s(\Omega)$ be the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^{s-1}(\Omega)} + |u|^2_{H^s(\Omega)} \quad (s \geq 1),$$

where

$$|u|^2_{H^s(\Omega)} = \int_\Omega |D^s u(x)|^2\,dx, \quad \text{and} \quad H^0(\Omega) = L^2(\Omega).$$

Here, $|D^s u(x)|^2 = \sum_{|\alpha|=s} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \ldots, x_n)$. For a positive non-integer $s = m + \sigma$ with integer $m \geq 0$ and $0 < \sigma < 1$, the norm in $H^s(\Omega)$ is

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^m(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|^2_{H^s(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\,dx\,dy.$$

The Sobolev spaces $\tilde{H}^s(\Omega)$ for $s \in (0,1)$ and for a bounded Lipschitz domain Ω are defined by interpolation. We use the real K-method of interpolation (see [25]) to define

$$\tilde{H}^s(\Omega) = \Big(L^2(\Omega), H_0^t(\Omega)\Big)_{\frac{s}{t},2} \quad (1/2 < t \le 1,\ 0 < s < t).$$

Here, $H_0^t(\Omega)$ $(0 < t \le 1)$ is the completion of $C_0^\infty(\Omega)$ in $H^t(\Omega)$ and we identify $H_0^1(\Omega)$ and $\tilde{H}^1(\Omega)$. Note that the Sobolev spaces $H^s(\Omega)$ also satisfy the interpolation property

$$H^s(\Omega) = \Big(L^2(\Omega), H^1(\Omega)\Big)_{s,2} \quad (0 < s < 1)$$

with equivalent norms. Furthermore, the semi-norm $|\cdot|_{H^1(\Omega)}$ is a norm in $\tilde{H}^1(\Omega)$ due to the Poincaré inequality. For the $L^2(\Omega)$-norm we will use the notation $\|\cdot\|_{0,\Omega}$.

For $s \in [-1,0)$ the Sobolev spaces and their norms are defined by duality with $L^2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space:

$$H^s(\Omega) = (\tilde{H}^{-s}(\Omega))', \quad \tilde{H}^s(\Omega) = (H^{-s}(\Omega))',$$

$$\|u\|_{H^s(\Omega)} = \sup_{0 \ne v \in \tilde{H}^{-s}(\Omega)} \frac{|\langle u, v\rangle|}{\|v\|_{\tilde{H}^{-s}(\Omega)}}, \quad \|u\|_{\tilde{H}^s(\Omega)} = \sup_{0 \ne v \in H^{-s}(\Omega)} \frac{|\langle u, v\rangle|}{\|v\|_{H^{-s}(\Omega)}}, \tag{3.1}$$

where

$$\langle u, v\rangle = \langle u, v\rangle_{0,\Omega} := \int_\Omega u(x)\bar{v}(x)dx$$

denotes the extension of the $L^2(\Omega)$-inner product by duality (and $\bar{v}$ is the complex conjugate of v).

Now, let Γ be a piecewise smooth (open or closed) Lipschitz surface in $\mathbb{R}^3$. We will assume that Γ has plane faces $\Gamma^{(i)}$ ($i = 1,\dots,\mathcal{I}$; without loss of generality it is assumed that $\mathcal{I} > 1$) and straight edges $e_{ij} = \bar{\Gamma}^{(i)} \cap \bar{\Gamma}^{(j)} \ne \emptyset$ $(i \ne j)$. If Γ is a closed surface, we will denote by Ω the Lipschitz polyhedron bounded by Γ, i.e., $\Gamma = \partial\Omega$. In the case of an open surface Γ, we first introduce a piecewise plane closed Lipschitz surface $\tilde{\Gamma}$ which contains Γ, and then denote by Ω the Lipschitz polyhedron bounded by $\tilde{\Gamma}$, i.e., $\tilde{\Gamma} = \partial\Omega$. For each face $\Gamma^{(i)} \subset \Gamma$ there exists a constant unit normal vector $\boldsymbol{\nu}_i$, which is an outer normal vector to Ω. These vectors are then blended into a unit normal vector $\boldsymbol{\nu}$ defined almost everywhere on Γ. For each pair of indices $i, j = 1,\dots,\mathcal{I}$ such that $\bar{\Gamma}^{(i)} \cap \bar{\Gamma}^{(j)} = e_{ij}$ we consider unit vectors $\boldsymbol{\tau}_{ij}$, $\boldsymbol{\tau}_i^{(j)}$, and $\boldsymbol{\tau}_j^{(i)}$ such that $\boldsymbol{\tau}_{ij} \| e_{ij}$, $\boldsymbol{\tau}_i^{(j)} = \boldsymbol{\tau}_{ij} \times \boldsymbol{\nu}_i$, and $\boldsymbol{\tau}_j^{(i)} = \boldsymbol{\tau}_{ij} \times \boldsymbol{\nu}_j$. Since each $\Gamma^{(i)}$ can be identified with a bounded subset in $\mathbb{R}^2$, the pair $(\boldsymbol{\tau}_i^{(j)}, \boldsymbol{\tau}_{ij})$ is an orthonormal basis of the plane generated by $\Gamma^{(i)}$.

Let Γ be a closed surface. Then $\Gamma = \partial\Omega$ is locally the graph of a Lipschitz function. Since the Sobolev spaces H^s for $|s| \le 1$ are invariant under Lipschitz (i.e., $C^{0,1}$) coordinate transformations, the spaces $H^s(\Gamma)$ with $|s| \le 1$ are defined in the usual way via a partition of unity subordinate to a finite family of local coordinate patches (see [26]). Due to this definition, the properties of Sobolev spaces on Lipschitz domains in $\mathbb{R}^n$ carry over to Sobolev spaces on

Lipschitz surfaces. If Γ is an open surface, then the Sobolev spaces $H^s(\Gamma)$, $\tilde{H}^s(\Gamma)$ for $|s| \le 1$ and $H^s_0(\Gamma)$ for $0 < s \le 1$ are constructed in terms of the Sobolev spaces $H^s(\tilde{\Gamma})$ on a closed Lipschitz surface $\tilde{\Gamma} \supset \Gamma$ (see [26]). Note that the spaces $H^s(\Gamma^{(i)})$ and $\tilde{H}^s(\Gamma^{(i)})$ on each face $\Gamma^{(i)}$ are well-defined for any $s \ge -1$.

For $s > 1$ we define the space $H^s(\Gamma)$ in the following piecewise fashion (hereafter, u_i denotes the restriction of u to the face $\Gamma^{(i)}$):

$$H^s(\Gamma) := \{u \in H^1(\Gamma);\ u_i \in H^s(\Gamma^{(i)}),\ i = 1, \ldots, \mathcal{I}\}.$$

This space is equipped with its natural norm

$$\|u\|_{H^s(\Gamma)} := \left(\|u\|^2_{H^1(\Gamma)} + \sum_{i=1}^{\mathcal{I}} \|u_i\|^2_{H^s(\Gamma^{(i)})} \right)^{\frac{1}{2}}.$$

Besides the above, we will need the following spaces:

$$H^s_*(\Gamma) := \{u \in H^s(\Gamma);\ \langle u, 1\rangle_{0,\Gamma} = 0\},$$

where $s \ge -1$ if Γ is closed, and $s > -\frac{1}{2}$ if Γ is an open surface.

We will denote by $\gamma_{\rm tr}$ the standard trace operator, $\gamma_{\rm tr}(u) = u|_\Gamma$, $u \in C^\infty(\bar{\Omega})$. For $s \in (0,1)$ (resp., $s > 1$), $\gamma_{\rm tr}$ has a unique extension to a continuous operator $H^{s+1/2}(\Omega) \to H^s(\Gamma)$ (resp., $H^{s+1/2}(\Omega) \to H^1(\Gamma)$), see [20, 12]. We will use the notation $C^\infty(\Gamma) = \gamma_{\rm tr}(C^\infty(\bar{\Omega}))$.

Using the introduced Sobolev spaces of scalar functions, we define for $s \ge -1$:

$$\mathbf{H}^s(\Omega) = (H^s(\Omega))^3, \qquad \mathbf{H}^s(\Gamma) = (H^s(\Gamma))^3;$$

$$\mathbf{H}^s(\Gamma^{(i)}) = (H^s(\Gamma^{(i)}))^2, \quad \tilde{\mathbf{H}}^s(\Gamma^{(i)}) = (\tilde{H}^s(\Gamma^{(i)}))^2, \quad 1 \le i \le \mathcal{I}.$$

If Γ is an open surface, then in addition to the above we define the space

$$\tilde{\mathbf{H}}^s(\Gamma) = (\tilde{H}^s(\Gamma))^3, \quad |s| \le 1.$$

The norms and inner products in all these spaces are defined component-wise and usual conventions $\mathbf{H}^0(\Omega) = \mathbf{L}^2(\Omega)$, $\mathbf{H}^0(\Gamma) = \tilde{\mathbf{H}}^0(\Gamma) = \mathbf{L}^2(\Gamma)$, $\mathbf{H}^0(\Gamma^{(i)}) = \tilde{\mathbf{H}}^0(\Gamma^{(i)}) = \mathbf{L}^2(\Gamma^{(i)})$ hold.

Now let us introduce the Sobolev spaces of tangential vector fields defined on Γ (see [13, 14, 16]). We start with the space

$$\mathbf{L}^2_t(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma);\ \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\},$$

which will be identified with the space of two-dimensional, tangential, square integrable vector fields. The norm and inner product in this space will be denoted by $\|\cdot\|_{0,\Gamma}$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{0,\Gamma}$, respectively. The similarity of this notation with the one for scalar functions should not lead to any confusion. Then we define:

$$\mathbf{H}^s_-(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2_t(\Gamma);\ \mathbf{u}_i \in \mathbf{H}^s(\Gamma^{(i)}),\ \ 1 \le i \le \mathcal{I}\}, \quad s \ge 0,$$

$$\|\mathbf{u}\|_{\mathbf{H}^s_-(\Gamma)} := \left(\sum_{i=1}^{\mathcal{I}} \|\mathbf{u}_i\|^2_{\mathbf{H}^s(\Gamma^{(i)})} \right)^{\frac{1}{2}}.$$

Let γ_{tr} be the trace operator (now acting on vector fields), $\gamma_{\mathrm{tr}}(\mathbf{u}) = \mathbf{u}|_\Gamma$, $\gamma_{\mathrm{tr}} : \mathbf{H}^{s+1/2}(\Omega) \to \mathbf{H}^s(\Gamma)$ for $s \in (0,1)$, and let $\gamma_{\mathrm{tr}}^{-1}$ be one of its right inverses. We will use the "tangential components trace" mapping $\pi_\tau : (C^\infty(\bar{\Omega}))^3 \to \mathbf{L}^2_t(\Gamma)$ and the "tangential trace" mapping $\gamma_\tau : (C^\infty(\bar{\Omega}))^3 \to \mathbf{L}^2_t(\Gamma)$, which are defined as $\mathbf{u} \mapsto \boldsymbol{\nu} \times (\mathbf{u} \times \boldsymbol{\nu})|_\Gamma$ and $\mathbf{u} \times \boldsymbol{\nu}|_\Gamma$, respectively. We will also use the notation π_τ (resp., γ_τ) for the composite operator $\pi_\tau \circ \gamma_{\mathrm{tr}}^{-1}$ (resp., $\gamma_\tau \circ \gamma_{\mathrm{tr}}^{-1}$), which acts on traces. Then we define the spaces

$$\mathbf{H}^{1/2}_\parallel(\Gamma) := \pi_\tau(\mathbf{H}^{1/2}(\Gamma)), \qquad \mathbf{H}^{1/2}_\perp(\Gamma) := \gamma_\tau(\mathbf{H}^{1/2}(\Gamma)),$$

endowed with their operator norms

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}^{1/2}_\parallel(\Gamma)} &:= \inf_{\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma)} \{\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma)};\ \pi_\tau(\boldsymbol{\phi}) = \mathbf{u}\}, \\ \|\mathbf{u}\|_{\mathbf{H}^{1/2}_\perp(\Gamma)} &:= \inf_{\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma)} \{\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma)};\ \gamma_\tau(\boldsymbol{\phi}) = \mathbf{u}\}. \end{aligned}$$

It has been shown in [13] that the space $\mathbf{H}^{1/2}_\parallel(\Gamma)$ (resp., $\mathbf{H}^{1/2}_\perp(\Gamma)$) can be characterised as the space of tangential vector fields belonging to $\mathbf{H}^{1/2}_-(\Gamma)$ and satisfying an appropriate "weak continuity" condition for the tangential (resp., normal) component across each edge e_{ij} of Γ.

For $s > \frac{1}{2}$ we set

$$\begin{aligned} \mathbf{H}^s_\parallel(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_-(\Gamma);\ \mathbf{u}_i \cdot \boldsymbol{\tau}_{ij} = \mathbf{u}_j \cdot \boldsymbol{\tau}_{ij} \quad \text{at each } e_{ij}\}, \\ \mathbf{H}^s_\perp(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_-(\Gamma);\ \mathbf{u}_i \cdot \boldsymbol{\tau}^{(j)}_i = \mathbf{u}_j \cdot \boldsymbol{\tau}^{(i)}_j \quad \text{at each } e_{ij}\}. \end{aligned}$$

For any $s > \frac{1}{2}$ the spaces $\mathbf{H}^s_\parallel(\Gamma)$ and $\mathbf{H}^s_\perp(\Gamma)$ are closed subspaces of $\mathbf{H}^s_-(\Gamma)$. Finally, for $s \in [0, \frac{1}{2})$ we set

$$\mathbf{H}^s_\parallel(\Gamma) = \mathbf{H}^s_\perp(\Gamma) := \mathbf{H}^s_-(\Gamma).$$

If Γ is an open surface, then we also need to define subspaces of $\mathbf{H}^s_\parallel(\Gamma)$ and $\mathbf{H}^s_\perp(\Gamma)$ incorporating boundary conditions on $\partial\Gamma$ (for tangential and normal components, respectively). In this case, for a given function $\mathbf{u}$ on Γ, we will denote by $\tilde{\mathbf{u}}$ the extension of $\mathbf{u}$ by zero onto a closed Lipschitz polyhedral surface $\tilde{\Gamma} \supset \Gamma$. Then we define the spaces

$$\begin{aligned} \tilde{\mathbf{H}}^s_\parallel(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_\parallel(\Gamma);\ \tilde{\mathbf{u}} \in \mathbf{H}^s_\parallel(\tilde{\Gamma})\}, \quad s \ge 0, \\ \tilde{\mathbf{H}}^s_\perp(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s_\perp(\Gamma);\ \tilde{\mathbf{u}} \in \mathbf{H}^s_\perp(\tilde{\Gamma})\}, \quad s \ge 0, \end{aligned}$$

which are furnished with the norms

$$\|\mathbf{u}\|_{\tilde{\mathbf{H}}^s_\parallel(\Gamma)} := \|\tilde{\mathbf{u}}\|_{\mathbf{H}^s_\parallel(\tilde{\Gamma})}, \quad \|\mathbf{u}\|_{\tilde{\mathbf{H}}^s_\perp(\Gamma)} := \|\tilde{\mathbf{u}}\|_{\mathbf{H}^s_\perp(\tilde{\Gamma})}, \quad s \ge 0.$$

When considering open and closed surfaces at the same time we use the notation $\tilde{\mathbf{H}}^s_{\|}(\Gamma)$, $\tilde{\mathbf{H}}^s_{\perp}(\Gamma)$, etc. also for closed surfaces by assuming that $\tilde{\mathbf{H}}^s_{\|}(\Gamma) = \mathbf{H}^s_{\|}(\Gamma)$, $\tilde{\mathbf{H}}^s_{\perp}(\Gamma) = \mathbf{H}^s_{\perp}(\Gamma)$, etc. in this case. This in particular applies to the following definition of dual spaces. For $s \in [-1, 0)$, the spaces $\mathbf{H}^s_{\|}(\Gamma)$, $\tilde{\mathbf{H}}^s_{\|}(\Gamma)$, $\mathbf{H}^s_{\perp}(\Gamma)$, $\tilde{\mathbf{H}}^s_{\perp}(\Gamma)$, and $\mathbf{H}^s_{-}(\Gamma)$ are defined as the dual spaces of $\tilde{\mathbf{H}}^{-s}_{\|}(\Gamma)$, $\mathbf{H}^{-s}_{\|}(\Gamma)$, $\tilde{\mathbf{H}}^{-s}_{\perp}(\Gamma)$, $\mathbf{H}^{-s}_{\perp}(\Gamma)$, and $\mathbf{H}^{-s}_{-}(\Gamma)$, respectively (with $\mathbf{L}^2_t(\Gamma)$ as pivot space). They are equipped with their natural (dual) norms. Moreover, for any $s \in (-\frac{1}{2}, \frac{1}{2})$ there holds (cf. [22])

$$\tilde{\mathbf{H}}^s_{\|}(\Gamma) = \mathbf{H}^s_{\|}(\Gamma) = \tilde{\mathbf{H}}^s_{\perp}(\Gamma) = \mathbf{H}^s_{\perp}(\Gamma) = \mathbf{H}^s_{-}(\Gamma).$$

Using the above spaces of tangential vector fields, one can define basic differential operators on Γ. The tangential gradient, $\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$, and the tangential vector curl, $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma)$, are defined in the usual way by localisation to each face $\Gamma^{(i)}$ (see [13, 14] for definitions and properties of these operators on both closed and open surfaces). To proceed with extensions of these operators and with their adjoints we need to distinguish between open and closed surfaces.

Let Γ be a closed surface. The adjoint operators of $-\nabla_\Gamma$ and $\mathbf{curl}_\Gamma$ are the surface divergence and the surface scalar curl, respectively:

$$\mathrm{div}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}_*(\Gamma), \quad \mathrm{curl}_\Gamma : \mathbf{L}^2_t(\Gamma) \to H^{-1}_*(\Gamma). \tag{3.2}$$

It has been shown in [14] that ∇_Γ and $\mathbf{curl}_\Gamma$ can be extended to

$$\nabla_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}_{\perp}(\Gamma), \quad \mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}_{\|}(\Gamma).$$

Moreover, they have closed ranges in corresponding spaces. Their adjoint operators

$$\mathrm{div}_\Gamma : \mathbf{H}^{1/2}_{\perp}(\Gamma) \to H^{-1/2}_*(\Gamma), \quad \mathrm{curl}_\Gamma : \mathbf{H}^{1/2}_{\|}(\Gamma) \to H^{-1/2}_*(\Gamma)$$

are linear continuous and surjective.

Finally, the Laplace-Beltrami operator is defined on Γ as follows

$$\Delta_\Gamma\, u = \mathrm{div}_\Gamma(\nabla_\Gamma\, u) = -\,\mathrm{curl}_\Gamma(\mathbf{curl}_\Gamma\, u) \quad \forall u \in H^1(\Gamma). \tag{3.3}$$

One has $\Delta_\Gamma : H^1(\Gamma) \to H^{-1}_*(\Gamma)$, it is linear continuous and invertible.

If Γ is an open surface, then instead of (3.2) there holds

$$\mathrm{div}_\Gamma : \mathbf{L}^2_t(\Gamma) \to \tilde{H}^{-1}(\Gamma), \quad \mathrm{curl}_\Gamma : \mathbf{L}^2_t(\Gamma) \to \tilde{H}^{-1}(\Gamma).$$

The operators ∇_Γ and $\mathbf{curl}_\Gamma$ again can be extended as follows (cf. [14]):

$$\nabla_\Gamma : \tilde{H}^{1/2}(\Gamma) \to \tilde{\mathbf{H}}^{-1/2}_{\perp}(\Gamma), \quad \nabla_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}_{\perp}(\Gamma)$$

and

$$\mathbf{curl}_\Gamma : \tilde{H}^{1/2}(\Gamma) \to \tilde{\mathbf{H}}^{-1/2}_{\|}(\Gamma), \quad \mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}_{\|}(\Gamma);$$

they also have closed ranges in corresponding spaces, and their adjoints

$$\mathrm{div}_\Gamma : \mathbf{H}_\perp^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \quad \mathrm{div}_\Gamma : \tilde{\mathbf{H}}_\perp^{1/2}(\Gamma) \to \tilde{H}^{-1/2}(\Gamma)$$

and

$$\mathrm{curl}_\Gamma : \mathbf{H}_\parallel^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \quad \mathrm{curl}_\Gamma : \tilde{\mathbf{H}}_\parallel^{1/2}(\Gamma) \to \tilde{H}^{-1/2}(\Gamma)$$

are linear continuous and surjective. The Laplace-Beltrami operator $\Delta_\Gamma : H^1(\Gamma) \to \tilde{H}^{-1}(\Gamma)$ is defined as in (3.3).

We will need the following spaces involving Δ_Γ:

$$\begin{aligned} \mathcal{H}(\Gamma) &:= \{u \in H^1(\Gamma)/\mathbb{C};\ \Delta_\Gamma u \in H_*^{-1/2}(\Gamma)\} \quad \text{if } \Gamma \text{ is closed}, \\ \tilde{\mathcal{H}}(\Gamma) &:= \{u \in H^1(\Gamma)/\mathbb{C};\ \Delta_\Gamma u \in \tilde{H}^{-1/2}(\Gamma) \text{ and} \\ &\qquad \langle \nabla_\Gamma u, \nabla_\Gamma v\rangle + \langle \Delta_\Gamma u, v\rangle = 0 \quad \text{for all } v \in H^1(\Gamma)\} \quad \text{if } \Gamma \text{ is an open surface.} \end{aligned}$$

Now we can introduce the spaces which appear when dealing with the EFIE on Γ. First, we set

$$\mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \mathbf{H}_\parallel^s(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in H^s(\Gamma)\}, \quad s \in [-1/2, 0]$$

(here, Γ is either a closed or an open surface). If Γ is an open surface, then we will also use the space

$$\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{u} \in \tilde{\mathbf{H}}_\parallel^s(\Gamma);\ \mathrm{div}_\Gamma\, \mathbf{u} \in \tilde{H}^s(\Gamma)\}, \quad s \in [-1/2, 0].$$

The spaces $\mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma)$ and $\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma)$ are equipped with their graph norms $\|\cdot\|_{\mathbf{H}^s(\mathrm{div}_\Gamma, \Gamma)}$ and $\|\cdot\|_{\tilde{\mathbf{H}}^s(\mathrm{div}_\Gamma, \Gamma)}$, respectively. For $s = 0$ we drop the superscript and for open surfaces also the tilde in the above notation, $\mathbf{H}^0(\mathrm{div}_\Gamma, \Gamma) = \tilde{\mathbf{H}}^0(\mathrm{div}_\Gamma, \Gamma) = \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$.

On open surfaces, one also needs the spaces incorporating homogeneous boundary conditions for the trace of the normal component on $\partial\Gamma$. By $\mathbf{H}_0(\mathrm{div}_\Gamma, \Gamma)$ (resp., $\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$) we denote the subspace of elements $\mathbf{u} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ (resp., $\mathbf{u} \in \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$) such that for all $v \in C^\infty(\Gamma)$ there holds

$$\langle \mathbf{u}, \nabla_\Gamma v\rangle + \langle \mathrm{div}_\Gamma\, \mathbf{u}, v\rangle = 0. \tag{3.4}$$

We note that if $\mathbf{u} \in \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, then identity (3.4) holds for any $v \in H^{3/2}(\Gamma)$ by density. In particular, $\tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ is a closed subspace of $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. To join the notation for open and closed surfaces, we will write

$$\mathbf{X}^0 = \mathbf{H}(\mathrm{div}_\Gamma, \Gamma), \quad \mathbf{X} = \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \quad \text{if } \Gamma \text{ is a closed surface},$$

$$\mathbf{X}^0 = \mathbf{H}_0(\mathrm{div}_\Gamma, \Gamma), \quad \mathbf{X} = \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \quad \text{if } \Gamma \text{ is an open surface.}$$

In both cases the norm in the space $\mathbf{X}$ will be denoted as $\|\cdot\|_{\mathbf{X}}$.

3.2 Some equivalent norms and corresponding inner products

In this subsection we consider the Sobolev spaces H^s and $\tilde{H}^s$ on the reference square K for $s = \pm\frac{1}{2}$. We will derive expressions for norms which are equivalent to those introduced in Section 3.1. We note that all results of this subsection are valid also for K being the equilateral reference triangle. First, let us introduce some notation.

1°. We denote by Ω the cube $\Omega = K \times (0,1)$. Thus $\partial\Omega = \cup_{i=1}^{6}\bar{\Gamma}_i$. Let $K = \Gamma_1 = \{(x_1,x_2,0);\ (x_1,x_2) \in K\}$, $\Gamma_6 = \{(x_1,x_2,1);\ (x_1,x_2) \in K\}$, and denote $\tilde{K} = \partial\Omega\backslash\bar{\Gamma}_6$. Note that $\tilde{K}$ is an open surface. We will use the standard notation for the gradient ∇ and for the Laplace operator Δ, both acting on scalar functions of three variables.

2°. Given $u \in H^{-1/2}(K)$, we denote by $\tilde{u}_K$ the solution of the mixed problem: find $\tilde{u}_K \in H^1(\Omega)$ such that

$$\Delta\tilde{u}_K = 0 \text{ in } \Omega, \quad \tfrac{\partial\tilde{u}_K}{\partial n} = u \text{ on } K, \quad \tilde{u}_K = 0 \text{ on } \partial\Omega\backslash K.$$

If $u \in H^{-1/2}(\tilde{K})$, then we will use the same notation as above with K replaced by $\tilde{K}$.

3°. Given $u \in H^{1/2}(\partial\Omega)$, we denote by $\tilde{\tilde{u}}$ its harmonic extension, i.e., the solution of the Dirichlet problem: find $\tilde{\tilde{u}} \in H^1(\Omega)$ such that

$$\Delta\tilde{\tilde{u}} = 0 \text{ in } \Omega, \quad \tilde{\tilde{u}} = u \text{ on } \partial\Omega. \tag{3.5}$$

4°. Given $u \in \tilde{H}^{1/2}(K)$, we denote by u° the extension of u by zero onto $\partial\Omega$. Thus, $u^\circ \in H^{1/2}(\partial\Omega)$.

We make use of standard definitions for the norm and the semi-norm in $H^1(\Omega)$:

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_{0,\Omega}^2 + |u|_{H^1(\Omega)}^2\right)^{1/2}, \quad |u|_{H^1(\Omega)} = \|\nabla u\|_{0,\Omega}.$$

Since $H^{1/2}(\partial\Omega)$ is the trace space of $H^1(\Omega)$, the norm and the semi-norm in $H^{1/2}(\partial\Omega)$ can be equivalently written as follows

$$\begin{aligned} \|u\|_{H^{1/2}(\partial\Omega)} &\simeq \inf_{\substack{U\in H^1(\Omega)\\ U|_{\partial\Omega}=u}} \|U\|_{H^1(\Omega)}, \\ |u|_{H^{1/2}(\partial\Omega)} &\simeq \inf_{\substack{U\in H^1(\Omega)\\ U|_{\partial\Omega}=u}} |U|_{H^1(\Omega)} = \|\nabla\tilde{\tilde{u}}\|_{0,\Omega}. \end{aligned} \tag{3.6}$$

Now we can define equivalent norms in $\tilde{H}^{1/2}(K)$ and $H^{1/2}(K)$:

$$\|u\|_{\tilde{H}^{1/2}(K)} \simeq |u^\circ|_{H^{1/2}(\partial\Omega)} \simeq \left\|\nabla\widetilde{\widetilde{u^\circ}}\right\|_{0,\Omega}, \tag{3.7}$$

$$\|u\|_{H^{1/2}(K)} \simeq \inf_{\substack{U\in\tilde{H}^{1/2}(\tilde{K})\\ U|_K=u}} \|U\|_{\tilde{H}^{1/2}(\tilde{K})}, \tag{3.8}$$

where $\|\cdot\|_{\tilde{H}^{1/2}(\tilde{K})}$ is defined as in (3.7), because $\tilde{K}$ is an open surface.

From (3.7) one can easily derive the expression for the corresponding $\tilde{H}^{1/2}(K)$-inner product. In fact, applying the parallelogram law twice, integrating by parts, and recalling notations 3°, 4°, we find (see also [21])

$$\begin{aligned}\langle u, v\rangle_{\tilde{H}^{1/2}(K)} &= \left\langle \nabla\widetilde{\widetilde{u^\circ}}, \nabla\widetilde{\widetilde{v^\circ}}\right\rangle_{0,\Omega} = \left\langle \frac{\partial\widetilde{\widetilde{u^\circ}}}{\partial n}, \widetilde{\widetilde{v^\circ}}\right\rangle_{0,\partial\Omega} = \\ &= \left\langle \frac{\partial\widetilde{\widetilde{u^\circ}}}{\partial n}, v\right\rangle_{0,K} = \left\langle u, \frac{\partial\widetilde{\widetilde{v^\circ}}}{\partial n}\right\rangle_{0,K} \quad \forall u, v \in \tilde{H}^{1/2}(K).\end{aligned} \tag{3.9}$$

The space $H^{-1/2}(K)$ is the dual space of $\tilde{H}^{1/2}(K)$. We prove the following result regarding an equivalent norm in $H^{-1/2}(K)$.

Lemma 3.1 *For any $u \in H^{-1/2}(K)$ there holds*

$$\|u\|_{H^{-1/2}(K)} \simeq \|\nabla\tilde{u}_K\|_{0,\Omega}. \tag{3.10}$$

The $H^{-1/2}$-inner product corresponding to the norm on the right-hand side of (3.10) *reads as*

$$\langle u, v\rangle_{H^{-1/2}(K)} = \langle u, \tilde{v}_K\rangle_{0,K} = \langle \tilde{u}_K, v\rangle_{0,K} \quad \forall u, v \in H^{-1/2}(K). \tag{3.11}$$

Proof. Using notations $2^\circ - 4^\circ$, we integrate by parts to obtain for any $u \in H^{-1/2}(K)$ and any $v \in \tilde{H}^{1/2}(K)$

$$\left\langle \nabla\tilde{u}_K, \nabla\widetilde{\widetilde{v^\circ}}\right\rangle_{0,\Omega} = \left\langle \frac{\partial\tilde{u}_K}{\partial n}, \widetilde{\widetilde{v^\circ}}\right\rangle_{0,\partial\Omega} = \left\langle \frac{\partial\tilde{u}_K}{\partial n}, \widetilde{\widetilde{v^\circ}}\right\rangle_{0,K} + \left\langle \frac{\partial\tilde{u}_K}{\partial n}, \widetilde{\widetilde{v^\circ}}\right\rangle_{0,\partial\Omega\setminus K} = \langle u, v\rangle_{0,K}.$$

Hence, we find from (3.1) and (3.7)

$$\|u\|_{H^{-1/2}(K)} = \sup_{0\neq v\in\tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla\tilde{u}_K, \nabla\widetilde{\widetilde{v^\circ}}\right\rangle_{0,\Omega}\right|}{\|v\|_{\tilde{H}^{1/2}(K)}} \simeq \sup_{0\neq v\in\tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla\tilde{u}_K, \nabla\widetilde{\widetilde{v^\circ}}\right\rangle_{0,\Omega}\right|}{\left\|\nabla\widetilde{\widetilde{v^\circ}}\right\|_{0,\Omega}}. \tag{3.12}$$

Let $w := \tilde{u}_K|_K$. One has $w \in \tilde{H}^{1/2}(K)$ because $\tilde{u}_K = 0$ on $\partial\Omega\setminus K$. Moreover, $w^\circ = \tilde{u}_K|_{\partial\Omega}$ and, due to the uniqueness of the solution to the Dirichlet problem (3.5), we conclude that $\widetilde{\widetilde{w^\circ}} = \tilde{u}_K$. Therefore,

$$\sup_{0\neq v\in\tilde{H}^{1/2}(K)} \frac{\left|\left\langle \nabla\tilde{u}_K, \nabla\widetilde{\widetilde{v^\circ}}\right\rangle_{0,\Omega}\right|}{\left\|\nabla\widetilde{\widetilde{v^\circ}}\right\|_{0,\Omega}} \geq \frac{\left|\left\langle \nabla\tilde{u}_K, \nabla\widetilde{\widetilde{w^\circ}}\right\rangle_{0,\Omega}\right|}{\left\|\nabla\widetilde{\widetilde{w^\circ}}\right\|_{0,\Omega}} = \|\nabla\tilde{u}_K\|_{0,\Omega}. \tag{3.13}$$

On the other hand, it is easy to see that

$$\sup_{0\neq v\in \tilde H^{1/2}(K)} \frac{\left|\left\langle \nabla\tilde u_K, \nabla\widetilde{\widetilde{v^\circ}}\right\rangle_{0,\Omega}\right|}{\left\|\nabla\widetilde{\widetilde{v^\circ}}\right\|_{0,\Omega}} \le \|\nabla\tilde u_K\|_{0,\Omega}. \tag{3.14}$$

Now (3.10) immediately follows from (3.12)–(3.14).

Using (3.10) together with the parallelogram law we find

$$\langle u, v\rangle_{H^{-1/2}(K)} = \left\langle \nabla\tilde u_K, \nabla\tilde v_K\right\rangle_{0,\Omega} \quad \forall u, v \in H^{-1/2}(K).$$

Hence, integrating by parts and using notation 2°, we derive (3.11). □

The following lemma states an analogous result for the space $\tilde H^{-1/2}(K)$ which is the dual space of $H^{1/2}(K)$.

Lemma 3.2 *For any $u \in \tilde H^{-1/2}(K)$ there holds*

$$\|u\|_{\tilde H^{-1/2}(K)} \simeq \left\|\nabla\widetilde{(u^\circ)}_{\tilde K}\right\|_{0,\Omega}. \tag{3.15}$$

The $\tilde H^{-1/2}$-inner product corresponding to the norm on the right-hand side of (3.15) *reads as*

$$\langle u, v\rangle_{\tilde H^{-1/2}(K)} = \left\langle u, \widetilde{(v^\circ)}_{\tilde K}\right\rangle_{0,K} = \left\langle \widetilde{(u^\circ)}_{\tilde K}, v\right\rangle_{0,K} \quad \forall u, v \in \tilde H^{-1/2}(K). \tag{3.16}$$

Proof. Let $u \in \tilde H^{-1/2}(K)$. Then $u^\circ \in \tilde H^{-1/2}(\tilde K) \subset H^{-1/2}(\tilde K)$. Using (3.1) and (3.8) we have

$$\begin{aligned}
\|u^\circ\|_{H^{-1/2}(\tilde K)} &= \sup_{0\neq w\in \tilde H^{1/2}(\tilde K)} \frac{|\langle u^\circ, w\rangle_{0,\tilde K}|}{\|w\|_{\tilde H^{1/2}(\tilde K)}} = \sup_{0\neq w\in \tilde H^{1/2}(\tilde K)} \frac{|\langle u, w\rangle_{0,K}|}{\|w\|_{\tilde H^{1/2}(\tilde K)}} \\
&= \sup_{0\neq v\in H^{1/2}(K)} \sup_{\substack{V\in \tilde H^{1/2}(\tilde K)\\ V|_K = v}} \frac{|\langle u, V\rangle_{0,K}|}{\|V\|_{\tilde H^{1/2}(\tilde K)}} = \sup_{0\neq v\in H^{1/2}(K)} \frac{|\langle u, v\rangle_{0,K}|}{\inf\limits_{\substack{V\in \tilde H^{1/2}(\tilde K)\\ V|_K = v}} \|V\|_{\tilde H^{1/2}(\tilde K)}} \\
&\simeq \sup_{0\neq v\in H^{1/2}(K)} \frac{|\langle u, v\rangle_{0,K}|}{\|v\|_{H^{1/2}(K)}} = \|u\|_{\tilde H^{-1/2}(K)}.
\end{aligned}$$

Hence, using (3.10) with u replaced by u° and with K replaced by $\tilde K$, we prove (3.15):

$$\|u\|_{\tilde H^{-1/2}(K)} \simeq \|u^\circ\|_{H^{-1/2}(\tilde K)} \simeq \left\|\nabla\widetilde{(u^\circ)}_{\tilde K}\right\|_{0,\Omega} \quad \forall u \in \tilde H^{-1/2}(K).$$

Then, applying the parallelogram law, integrating by parts, and making use of notations 2°, 4°, we derive (3.16). □

Remark 3.1 *The same arguments as above can be used to find equivalent norms and corresponding inner products in the Sobolev spaces on edges $\ell_i \subset \partial K$. In particular, we will need the $\tilde{H}^{1/2}(\ell_i)$-norm and corresponding inner product. Using the notation analogous to 3° and 4°, we have (cf.* (3.7), (3.9)*)*

$$\begin{aligned} \|u\|_{\tilde{H}^{1/2}(\ell_i)} &\simeq \left\| \nabla \widetilde{\widetilde{u^\circ}} \right\|_{0,K} \quad \forall u \in \tilde{H}^{1/2}(\ell_i), \\ \langle u, v \rangle_{\tilde{H}^{1/2}(\ell_i)} &= \left\langle \frac{\partial \widetilde{\widetilde{u^\circ}}}{\partial n}, v \right\rangle_{0,\ell_i} = \left\langle u, \frac{\partial \widetilde{\widetilde{v^\circ}}}{\partial n} \right\rangle_{0,\ell_i} \quad \forall u, v \in \tilde{H}^{1/2}(\ell_i). \end{aligned}$$

The next lemma states the fact that for a constant function v in (3.16) the $\tilde{H}^{-1/2}(K)$-inner product reduces to the $L^2(K)$-inner product.

Lemma 3.3 *For any $u \in \tilde{H}^{-1/2}(K)$ there holds*

$$\langle u, 1 \rangle_{\tilde{H}^{-1/2}(K)} = \langle u, 1 \rangle_{0,K}.$$

Proof. We have by (3.16)

$$\langle u, 1 \rangle_{\tilde{H}^{-1/2}(K)} = \langle u, \varphi|_K \rangle_{0,K}, \tag{3.17}$$

where $\varphi(x)$ ($x = (x_1, x_2, x_3) \in \Omega = K \times (0,1)$) solves the following mixed problem (see (3.16) and notations 1°, 2°, 4°): find $\varphi \in H^1(\Omega)$ such that

$$\Delta\varphi = 0 \text{ in } \Omega, \quad \tfrac{\partial \varphi}{\partial n} = 1 \text{ on } \Gamma_1 = K, \quad \tfrac{\partial \varphi}{\partial n} = 0 \text{ on } \Gamma_i \ (i = 2, \ldots, 5), \quad \varphi = 0 \text{ on } \Gamma_6.$$

It is easy to see that $\varphi = 1 - x_3$. Then $\varphi|_K = \varphi|_{x_3=0} = 1$ and the assertion follows from (3.17). □

3.3 Auxiliary lemmas

The Laplace-Beltrami operator Δ_Γ will be a useful tool in our analysis. The following two lemmas establish its regularity separately on closed and open piecewise plane Lipschitz surfaces. For proofs we refer to [15, Theorem 8] and [12, Proposition 4.11], respectively.

Lemma 3.4 *Let Γ be a closed Lipschitz polyhedral surface, $\psi \in H^s_*(\Gamma)$ for $s \geq -1$, and let $\phi \in H^1(\Gamma)/\mathbb{C}$ be the unique solution to the problem*

$$\langle \nabla_\Gamma \phi, \nabla_\Gamma \tilde{\phi} \rangle = \langle \psi, \tilde{\phi} \rangle \quad \forall \tilde{\phi} \in H^1(\Gamma)/\mathbb{C}.$$

Then $\phi \in H^{1+r}(\Gamma)$ and

$$\|\phi\|_{H^{1+r}(\Gamma)/\mathbb{C}} \leq C \, \|\psi\|_{H^s(\Gamma)}$$

for any $r < \min\{s^*, s+1\}$, where $s^* > 0$ depends on the geometry of Γ in neighbourhoods of its vertices.

Lemma 3.5 *Let Γ be an open piecewise plane Lipschitz surface. Let $\psi \in H^s_*(\Gamma)$, $s > -\frac{1}{2}$ (resp., $\psi \in \tilde{H}^s(\Gamma)$, $-1 \le s \le -\frac{1}{2}$, $\langle \psi, 1 \rangle = 0$) and $\phi \in H^1(\Gamma)/\mathbb{C}$ be the unique solution to the problem*

$$\langle \nabla_\Gamma \phi, \nabla_\Gamma \tilde{\phi} \rangle = \langle \psi, \tilde{\phi} \rangle \quad \forall \tilde{\phi} \in H^1(\Gamma)/\mathbb{C}.$$

Then $\phi \in H^{1+r}(\Gamma)$ and

$$\|\phi\|_{H^{1+r}(\Gamma)/\mathbb{C}} \le C\,\|\psi\|_{H^s(\Gamma)} \quad (\textit{resp.,} \quad \|\phi\|_{H^{1+r}(\Gamma)/\mathbb{C}} \le C\,\|\psi\|_{\tilde{H}^s(\Gamma)})$$

for any $r < \min\{s^*, s+1\}$, where $s^* > 0$ depends on the geometry of Γ in neighbourhoods of all vertices of $\bar{\Gamma}$.

Remark 3.2 *If $s+1 < s^*$ in the above two lemmas for given Γ and s, then both results are valid for $0 \le r \le s+1$.*

In the following lemma we formulate some useful properties of the Piola transform.

Lemma 3.6 *Let K^h and K be two open subsets of $\mathbb{R}^2$ such that $K^h = T(K)$, where T is a sufficiently smooth one-to-one mapping with a sufficiently smooth inverse $T^{-1} : K^h \to K$. Assume that $\operatorname{diam} K^h \simeq \rho_{K^h} \simeq h$, $\operatorname{diam} K \simeq \rho_K \simeq 1$, and the mappings T, T^{-1} satisfy the same relations as in (2.1)–(2.2). Let $\hat{\varphi}$ and $\hat{\mathbf{q}}$ be a scalar function and a vector field, respectively, defined on K, and let $\varphi = \hat{\varphi} \circ T^{-1}$, $\mathbf{q} = \mathcal{M}(\hat{\mathbf{q}})$ be defined on K^h (here, $\mathcal{M}$ is the Piola transform associated with T, see (2.3)). Then*

$$\langle \varphi, \operatorname{div} \mathbf{q} \rangle_{0,K^h} = \langle \hat{\varphi}, \operatorname{div} \hat{\mathbf{q}} \rangle_{0,K}, \tag{3.18}$$

$$\|\mathbf{q}\|_{0,K^h} \simeq \|\hat{\mathbf{q}}\|_{0,K} \tag{3.19}$$

if $\hat{\varphi} \in L^2(K)$ and $\hat{\mathbf{q}} \in \mathbf{H}(\operatorname{div}, K)$. Moreover, for $s \in [0,1]$, there holds

$$\|\hat{\mathbf{q}}\|_{\mathbf{H}^s(K)} \le C\,\|\mathbf{q}\|_{\mathbf{H}^s(K^h)} \tag{3.20}$$

if $\mathbf{q} \in \mathbf{H}^s(K^h)$;

$$\|\operatorname{div} \hat{\mathbf{q}}\|_{\tilde{H}^{-s}(K)} \le C\,h^{1-s}\,\|\operatorname{div} \mathbf{q}\|_{\tilde{H}^{-s}(K^h)} \tag{3.21}$$

if $\operatorname{div} \mathbf{q} \in \tilde{H}^{-s}(K^h)$;

$$\|\operatorname{div} \hat{\mathbf{q}}\|_{H^{-s}(K)} \simeq h^{1-s}\,\|\operatorname{div} \mathbf{q}\|_{H^{-s}(K^h)} \tag{3.22}$$

if $\operatorname{div} \mathbf{q} \in H^{-s}(K^h)$.

Proof. Relations (3.18), (3.19) are well-known (see, e.g., Lemmas 1.5, 1.6 in Chapter III of [11]). If $\mathbf{q} \in \mathbf{H}^1(K^h)$ then (cf. [30])

$$\|\hat{\mathbf{q}}\|_{\mathbf{H}^1(K)} \le C\,\|\mathbf{q}\|_{\mathbf{H}^1(K^h)}. \tag{3.23}$$

Then (3.20) follows from (3.19) and (3.23) by interpolation.

In order to prove (3.21) and (3.22) we use (3.18) and the standard scaling argument for scalar functions. For instance, in the former case we have

$$\begin{aligned}\|\operatorname{div}\hat{\mathbf{q}}\|_{\tilde{H}^{-s}(K)} &= \sup_{0\neq\hat{\varphi}\in H^s(K)} \frac{\langle \operatorname{div}\hat{\mathbf{q}}, \hat{\varphi}\rangle_{0,K}}{\|\hat{\varphi}\|_{H^s(K)}} \\ &\leq C \sup_{0\neq\varphi\in H^s(K^h)} \frac{\langle \operatorname{div}\mathbf{q}, \varphi\rangle_{0,K^h}}{h^{-(1-s)}\,\|\varphi\|_{H^s(K^h)}} = C\,h^{1-s}\,\|\operatorname{div}\mathbf{q}\|_{\tilde{H}^{-s}(K^h)}.\end{aligned}$$

The proof of (3.22) is analogous. □

The following lemma states the inverse inequality for polynomials on K. We refer to [23] for a proof.

Lemma 3.7 *Let $v_p \in \mathcal{P}_p(K)$. Then for any s, $r \in [-1,1]$ with $s \leq r$ there holds*

$$\|v\|_{H^r(K)} \leq C\,p^{2(r-s)}\,\|v\|_{H^s(K)},$$

where C is a positive constant independent of p.

4 Decompositions

The main tool in the analysis of the EFIE is the Helmholtz decomposition of the energy space $\mathbf{X}$. It is used to prove an inf-sup condition for the electric field integral operator and to establish the unique solvability of the EFIE on Γ (see, e.g., [15, 12]). The following statement establishes the Helmholtz decomposition of $\mathbf{X}$ on a (closed or open) Lipschitz polyhedral surface Γ. This result has been proved in [14, Theorems 5.1 and 6.4] (for open surfaces see also [12, Section 2.4]).

Theorem 4.1 *Let*

$$\begin{aligned}\mathbf{W} &= \{\mathbf{w}\in\mathbf{X};\ \operatorname{div}_\Gamma \mathbf{w} = 0\}, \\ \mathbf{V} &= \{\mathbf{v}\in\mathbf{X};\ \langle \mathbf{v},\mathbf{w}\rangle = 0\ \ \forall \mathbf{w}\in\mathbf{W}\cap\mathbf{L}^2_t(\Gamma)\}.\end{aligned}$$

Then there holds

$$\mathbf{X} = \mathbf{V}\oplus\mathbf{W}. \tag{4.1}$$

Furthermore, $\mathbf{V}$ *and* $\mathbf{W}$ *are closed subspaces of* $\mathbf{X}$*, and they can be written as*

$$\mathbf{V} = \nabla_\Gamma \mathcal{H}(\Gamma), \qquad \mathbf{W} = \mathbf{curl}_\Gamma\,(H^{1/2}(\Gamma)/\mathbb{C})$$

if Γ is a closed surface, and

$$\mathbf{V} = \nabla_\Gamma \tilde{\mathcal{H}}(\Gamma), \qquad \mathbf{W} = \mathbf{curl}_\Gamma\,\tilde{H}^{1/2}(\Gamma)$$

if Γ is an open surface.

In this paper we discretise the EFIE by the *hp*-version of the Galerkin BEM based on the sequence of the RT-subspaces $\mathbf{X}_{hp} \subset \mathbf{X}$ (see (2.4), (2.5)). To prove the well-posedness of (2.5) (see Theorem 2.1) we follow [12, 18] and consider $\mathbf{L}^2_t(\Gamma)$-orthogonal discrete decompositions of $\mathbf{X}_{hp}$ mimicking the Helmholtz decomposition of $\mathbf{X}$:

$$\mathbf{X}_{hp} = \mathbf{V}_{hp} \oplus \mathbf{W}_{hp}, \tag{4.2}$$

where

$$\begin{aligned} \mathbf{W}_{hp} &:= \{\mathbf{w} \in \mathbf{X}_{hp};\ \mathrm{div}_\Gamma\, \mathbf{w} = 0\}, \\ \mathbf{V}_{hp} &:= \{\mathbf{v} \in \mathbf{X}_{hp};\ \langle \mathbf{v}, \mathbf{w}\rangle = 0\ \ \forall \mathbf{w} \in \mathbf{W}_{hp}\}. \end{aligned} \tag{4.3}$$

It is easy to see that $\mathbf{W}_{hp} \subset \mathbf{W}$, however, in general, $\mathbf{V}_{hp} \not\subset \mathbf{V}$. That is why the discrete inf-sup condition (and thus, the unique solvability of (2.5) and quasi-optimal convergence of the BEM) cannot be deduced by standard arguments, which are usually applied to conforming Galerkin discretisations of coercive variational problems.

Sufficient conditions to establish the well-posedness of the Galerkin BEM applied to problem (1.1) were found in [12]. It turns out that it is enough to prove that discrete decompositions (4.2) are in some sense close to the Helmholtz decomposition (4.1) of $\mathbf{X}$ when the dimension of the discrete space tends to infinity. The abstract formulation of this approach is given in the next theorem (here, we quote [18, Theorem 4.1], see also Proposition 4.1, Corollary 4.2, and Theorem 4.5 in [12]).

Theorem 4.2 *Let $\{\mathbf{X}_n\}_n$ be a sequence of closed subspaces $\mathbf{X}_n \subset \mathbf{X}$ with decompositions $\mathbf{X}_n = \mathbf{V}_n \oplus \mathbf{W}_n$ which are stable with respect to complex conjugation and which satisfy the following assumptions:*

(A1) *the family $\{\mathbf{X}_n\}_n$ is dense in the space $\mathbf{X}$, namely*

$$\overline{\bigcup_n \mathbf{X}_n} = \mathbf{X};$$

(A2) *the spaces $\mathbf{V}_n$ and $\mathbf{W}_n$ are such that $\mathbf{W}_n \subset \mathbf{W}$ and*

$$\sup_{\mathbf{v}_n \in \mathbf{V}_n \setminus \{\mathbf{0}\}} \inf_{\mathbf{v} \in \mathbf{V}} \frac{\|\mathbf{v}_n - \mathbf{v}\|_{\mathbf{X}}}{\|\mathbf{v}_n\|_{\mathbf{X}}} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.4}$$

Then there exists n_0 such that for all $\mathbf{f} \in \mathbf{X}'$ and $n \geq n_0$ the Galerkin system

$$a(\mathbf{u}_n, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}_n$$

has a unique solution $\mathbf{u}_n \in \mathbf{X}_n$ which converges quasi-optimally, i.e.,

$$\|\mathbf{u} - \mathbf{u}_n\|_{\mathbf{X}} \leq C \inf\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{X}};\ \mathbf{v} \in \mathbf{X}_n\},$$

where $\mathbf{u} \in \mathbf{X}$ is the solution of (1.1).

It has been proved in [12, 18] that $\mathbf{L}_t^2(\Gamma)$-orthogonal discrete decompositions mimicking the Helmholtz decomposition of the space $\mathbf{X}$ satisfy assumptions (A1), (A2) of Theorem 4.2 with respect to the mesh parameter h, i.e., in the framework of the h-version of the BEM for the EFIE. To prove this result for the hp-version on quasi-uniform meshes, we will need an $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming p-interpolation operator, which is introduced and analysed in the next section.

5 Interpolation operators

The main purpose of this section is to introduce and analyse a new $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming p-interpolation operator. This operator is necessary to deal with low regular vector fields, such as gradients of solutions to boundary value problems for the Laplace-Beltrami operator on polyhedral surfaces (see the regularity results of Lemmas 3.4 and 3.5). We will construct the $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming interpolation operator by employing the $\tilde{H}^{-1/2}$-projection for the divergence term. For this operator we then prove quasi-stability with respect to polynomial degrees and commutativity with the $\tilde{H}^{-1/2}$-projector.

In this section we use standard differential operators div, curl and ∇, **curl** acting on 2D vector fields and scalar functions, respectively. First, let us recall some known interpolation operators acting on scalar functions and vector fields on K. Let $\Pi_p^0 : L^2(K) \to \mathcal{P}_p(K)$ be the standard L^2-projection onto the set of polynomials $\mathcal{P}_p(K)$. We will also use the $\tilde{H}^{-1/2}$-projector onto $\mathcal{P}_p(K)$ denoted by $\Pi_p^{-1/2} : \tilde{H}^{-1/2}(K) \to \mathcal{P}_p(K)$ and satisfying

$$\langle u - \Pi_p^{-1/2} u, v \rangle_{-\frac{1}{2},K} = 0 \quad \forall v \in \mathcal{P}_p(K).$$

Here and below $\langle \cdot, \cdot \rangle_{-\frac{1}{2},K}$ denotes the $\tilde{H}^{-1/2}(K)$-inner product (see (3.16)).

In [21] two projection-based interpolation operators have been introduced and analysed. These are the H^1-conforming interpolation operator $\Pi_p^1 : H^{1+r}(K) \to \mathcal{P}_p(K)$ and the $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator $\Pi_p^{\mathrm{curl}} : \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{curl}, K) \to \boldsymbol{\mathcal{P}}_p^{\mathrm{Ned}}(K)$ (here, $r > 0$ in both cases and $\boldsymbol{\mathcal{P}}_p^{\mathrm{Ned}}(K) = \mathcal{P}_{p-1,p}(K) \times \mathcal{P}_{p,p-1}(K)$ is the $\mathbf{H}(\mathrm{curl})$-conforming (first) Nédélec space of degree p). Later, in [9] these operators were employed to prove the discrete compactness property for hp adaptive rectangular edge finite elements. Due to the isomorphism of the curl and the div operator in 2D (and, as a consequence, the isomorphism of the Nédélec elements of the first type and the RT elements), one can use the results of [21, 9] related to the operator Π_p^{curl} in the $\mathbf{H}(\mathrm{div})$-settings.

We will denote by $\Pi_p^{\mathrm{div},0}$ the corresponding $\mathbf{H}(\mathrm{div})$-conforming projection-based interpolation operator. Then for $r > 0$ the following diagram commutes (see Proposition 3 in [21]):

$$\begin{array}{ccccc} H^{1+r}(K) & \xrightarrow{\mathbf{curl}} & \mathbf{H}^r(K) \cap \mathbf{H}(\mathrm{div}, K) & \xrightarrow{\mathrm{div}} & L^2(K) \\ \downarrow \Pi_p^1 & & \downarrow \Pi_p^{\mathrm{div},0} & & \downarrow \Pi_{p-1}^0 \\ \mathcal{P}_p(K) & \xrightarrow{\mathbf{curl}} & \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{p-1}(K). \end{array} \tag{5.1}$$

Furthermore, we use the above mentioned isomorphisms to reformulate the following two results from [9].

Lemma 5.1 [9, Theorem 4] *Let* $\mathbf{A}_p = \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ *and* $\mathbf{B}_p = \mathbf{curl}\,\mathcal{P}_p^0(K) \oplus \nabla\,\mathrm{div}\,\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$. *Then the following stability condition holds*

$$\inf_{\mathbf{a}\in\mathbf{A}_p} \sup_{\mathbf{b}\in\mathbf{B}_p} \frac{\langle \mathbf{a},\mathbf{b}\rangle_{0,K}}{\|\mathbf{a}\|_{0,K}\,\|\mathbf{b}\|_{0,K}} = \left(\frac{2(2p+1)}{(p+1)(p+2)}\right)^{1/2} = O(p^{-1/2}).$$

This lemma and the definition of the interpolation operator $\Pi_p^{\mathrm{div},0}$ imply the following $\mathbf{L}^2$-stability result for the p-version.

Lemma 5.2 [9, Theorem 7] *Let* $\mathbf{u} \in \mathbf{H}^r(K)\cap\mathbf{H}_0(\mathrm{div},K)$*,* $r > 0$*, be a curl-free bubble function on* K*, and let* $\mathbf{u}_p^{\mathrm{div}} := \Pi_p^{\mathrm{div},0}\mathbf{u}$*. Then* $\mathbf{u}_p^{\mathrm{div}} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ *is discrete curl-free (i.e.,* $\langle \mathbf{u}_p^{\mathrm{div}}, \mathbf{curl}\,\phi\rangle_{0,K} = 0$ *for any* $\phi \in \mathcal{P}_p^0(K)$*) and there holds*

$$\|\mathbf{u} - \mathbf{u}_p^{\mathrm{div}}\|_{0,K} \le C\,p^{1/2} \inf_{\mathbf{q}_p\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)} \|\mathbf{u} - \mathbf{q}_p\|_{0,K}.$$

In the next lemma we estimate the error of the best $\mathbf{L}^2$-approximation of low regular vector bubble functions by RT-elements of degree p.

Lemma 5.3 *Let* $\mathbf{u} \in \mathbf{H}^r(K) \cap \mathbf{H}_0(\mathrm{div},K)$*,* $r \in [0,1]$*, be a general bubble function on* K*. Then*

$$\inf_{\mathbf{q}_p\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)} \|\mathbf{u} - \mathbf{q}_p\|_{0,K} \le C\,p^{-r}\,\|\mathbf{u}\|_{\mathbf{H}^r(K)}.$$

Proof. For $r = 1$, the statement follows from [9, Lemma 8]. For $r = 0$ it is trivial, because

$$\inf_{\mathbf{q}_p\in\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)} \|\mathbf{u} - \mathbf{q}_p\|_{0,K} \le \|\mathbf{u}\|_{0,K}.$$

Then we obtain the whole range $r \in [0,1]$ via interpolation. □

We will use the above two lemmas to prove the following auxiliary result.

Lemma 5.4 *Let* $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ *be such that* $\langle \mathbf{v}_p, \mathbf{w}_p\rangle_{0,K} = 0$ *for any* $\mathbf{w}_p \in \mathbf{W}_p(K) := \{\mathbf{w} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K);\ \mathrm{div}\,\mathbf{w} = 0\}$*. Then there exists* $\mathbf{v} \in \nabla\tilde{\mathcal{H}}(K)$ *such that*

$$\|\mathbf{v}\|_{0,K} \le C\,\|\mathrm{div}\,\mathbf{v}_p\|_{\tilde{H}^{-1/2}(K)} \tag{5.2}$$

and

$$\|\mathbf{v} - \mathbf{v}_p\|_{\mathbf{H}(\mathrm{div},K)} \le C\,p^{\varepsilon_0}\,\|\mathrm{div}\,\mathbf{v}_p\|_{\tilde{H}^{-1/2}(K)} \tag{5.3}$$

for any $\varepsilon_0 > 0$ *with* $C = C(\varepsilon_0) > 0$.

Proof. For given $\mathbf{v}_p$, we solve the Neumann problem to find $f \in H^1(K)/\mathbb{C}$ such that

$$\langle \nabla f, \nabla g \rangle_{0,K} = -\langle \operatorname{div} \mathbf{v}_p, g \rangle_{0,K} \quad \forall g \in H^1(K)/\mathbb{C}. \tag{5.4}$$

Then we set $\mathbf{v} := \nabla f$. One has

$$\operatorname{div} \mathbf{v} = \Delta f = \operatorname{div} \mathbf{v}_p \in \mathcal{P}_{p-1}(K). \tag{5.5}$$

Hence, $f \in \tilde{\mathcal{H}}(K)$, $\mathbf{v} \in \nabla \tilde{\mathcal{H}}(K)$, and (5.2) holds, because $\Delta : H^1(K)/\mathbb{C} \to \tilde{H}^{-1}(K)$ is an isomorphism.

Note that $\operatorname{div} \mathbf{v}_p \in H_*^{-1/2+\varepsilon}(K)$ for any $\varepsilon > 0$. Therefore, the standard regularity result for problem (5.4) reads as (see, e.g., [22] and cf. Lemma 3.5 and Remark 3.2): $f \in H^{1+r}(K)$ and

$$\|f\|_{H^{1+r}(K)/\mathbb{C}} \leq C \, \|\operatorname{div} \mathbf{v}_p\|_{H^{-1/2+\varepsilon}(K)}$$

for any $0 < r \leq \frac{1}{2} + \varepsilon$ and for arbitrary $\varepsilon \in (0,1]$. Then, using the continuity of the gradient as a mapping $H^{1+r}(K) \to \mathbf{H}^r(K)$, we have

$$\|\mathbf{v}\|_{\mathbf{H}^r(K)} \leq C \, \|f\|_{H^{1+r}(K)/\mathbb{C}} \leq C \, \|\operatorname{div} \mathbf{v}_p\|_{H^{-1/2+\varepsilon}(K)}, \quad 0 < r \leq 1/2 + \varepsilon, \;\; \varepsilon \in (0,1]. \tag{5.6}$$

Since $\mathbf{v} \in \mathbf{H}^r(K) \cap \mathbf{H}_0(\operatorname{div}, K)$, we can apply the interpolation operator $\Pi_p^{\operatorname{div},0}$ to define $\mathbf{v}_p^{\operatorname{div}} := \Pi_p^{\operatorname{div},0} \mathbf{v} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$. Recalling that $\Pi_p^{\operatorname{div},0}$ commutes with the L^2-projector (see (5.1)) and using (5.5), we find that

$$\operatorname{div} \mathbf{v}_p^{\operatorname{div}} = \operatorname{div} \mathbf{v}_p = \operatorname{div} \mathbf{v}.$$

Hence, $(\mathbf{v}_p - \mathbf{v}_p^{\operatorname{div}}) \in \mathbf{W}_p(K)$. This fact implies the relations

$$\langle \mathbf{v}, \mathbf{v}_p - \mathbf{v}_p^{\operatorname{div}} \rangle_{0,K} = \langle \nabla f, \mathbf{v}_p - \mathbf{v}_p^{\operatorname{div}} \rangle_{0,K} = 0 \quad \text{and} \quad \langle \mathbf{v}_p, \mathbf{v}_p - \mathbf{v}_p^{\operatorname{div}} \rangle_{0,K} = 0,$$

where the latter equation holds by assumptions on $\mathbf{v}_p$. Therefore,

$$\|\mathbf{v} - \mathbf{v}_p\|_{0,K} \leq \|\mathbf{v} - \mathbf{v}_p^{\operatorname{div}}\|_{0,K}. \tag{5.7}$$

Since $\mathbf{v}$ is curl-free, we apply Lemmas 5.2, 5.3 and then use inequality (5.6). As a result, we obtain

$$\begin{aligned} \|\mathbf{v} - \mathbf{v}_p^{\operatorname{div}}\|_{0,K} &\leq C \, p^{1/2} \inf_{\mathbf{q}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)} \|\mathbf{v} - \mathbf{q}_p\|_{0,K} \\ &\leq C \, p^{1/2-r} \, \|\mathbf{v}\|_{\mathbf{H}^r(K)} \leq C p^{1/2-r} \|\operatorname{div} \mathbf{v}_p\|_{H^{-1/2+\varepsilon}(K)} \end{aligned} \tag{5.8}$$

for any $\varepsilon \in (0,1]$. Then making use of the inverse inequality (see Lemma 3.7) we estimate

$$\|\operatorname{div} \mathbf{v}_p\|_{H^{-1/2+\varepsilon}(K)} \leq C p^{2\varepsilon} \|\operatorname{div} \mathbf{v}_p\|_{H^{-1/2}(K)} \leq C p^{2\varepsilon} \|\operatorname{div} \mathbf{v}_p\|_{\tilde{H}^{-1/2}(K)}. \tag{5.9}$$

Now we again use (5.5) and then put together (5.7)–(5.9). We obtain

$$\|\mathbf{v}-\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div},K)} = \|\mathbf{v}-\mathbf{v}_p\|_{0,K} \le Cp^{1/2+2\varepsilon-r}\|\mathrm{div}\,\mathbf{v}_p\|_{\tilde{H}^{-1/2}(K)}. \tag{5.10}$$

Given an arbitrary $\varepsilon_0 > 0$, we select $\varepsilon = \min\{\varepsilon_0, 1\}$. Then (5.3) follows from (5.10) by taking $r = \frac{1}{2} + \varepsilon$. □

$\tilde{\mathbf{H}}^{-1/2}$(div)-conforming interpolation operator. Now we proceed to the main goal of this section. Given a vector field $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$ with $r > 0$, we construct an interpolant $\mathbf{u}^p = \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$. In particular, $\mathbf{u}^p$ is defined as the sum of three terms:

$$\mathbf{u}^p = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p. \tag{5.11}$$

The definition of $\mathbf{u}_1$ and $\mathbf{u}_2^p$ follows the construction from [21]. Let $\mathbf{u}_1$ be a lowest order interpolant defined as

$$\mathbf{u}_1 = \sum_{i=1}^{4}\Big(\int_{\ell_i}\mathbf{u}\cdot\mathbf{n}\,d\sigma\Big)\,\boldsymbol{\phi}_i, \tag{5.12}$$

where $\mathbf{n}$ denotes the outward normal unit vector to K, and $\boldsymbol{\phi}_i$ $(i = 1, \ldots, 4)$ are the standard basis functions for $\boldsymbol{\mathcal{P}}_1^{\mathrm{RT}}(K)$, defined by

$$\boldsymbol{\phi}_i\cdot\mathbf{n} = \begin{cases} 1 & \text{on } \ell_i, \\ 0 & \text{on } \partial K\backslash\ell_i. \end{cases}$$

For any edge $\ell_i \subset \partial K$ one has

$$\int_{\ell_i}(\mathbf{u}-\mathbf{u}_1)\cdot\mathbf{n}\,d\sigma = 0. \tag{5.13}$$

Hence, there exists a function ψ, defined on the boundary ∂K, such that

$$\frac{\partial\psi}{\partial\sigma} = (\mathbf{u}-\mathbf{u}_1)\cdot\mathbf{n}, \quad \psi = 0 \ \text{ at all vertices.} \tag{5.14}$$

Then we define $\psi_2^{\ell_i} \in \mathcal{P}_p^0(\ell_i)$ by projection

$$\langle \psi|_{\ell_i} - \psi_2^{\ell_i}, \phi\rangle_{\tilde{H}^{1/2}(\ell_i)} = 0 \quad \forall \phi \in \mathcal{P}_p^0(\ell_i) \tag{5.15}$$

(see Remark 3.1 for the expression of $\langle\cdot,\cdot\rangle_{\tilde{H}^{1/2}(\ell_i)}$). Extending $\psi_2^{\ell_i}$ by zero from ℓ_i onto ∂K (and keeping its notation), we denote by $\psi_{2,p}^{\ell_i} \in \mathcal{P}_p(K)$ a polynomial extension of $\psi_2^{\ell_i}$ from ∂K onto K, i.e.,

$$\psi_{2,p}^{\ell_i} \in \mathcal{P}_p(K), \quad \psi_{2,p}^{\ell_i}|_{\ell_i} = \psi_2^{\ell_i}, \quad \psi_{2,p}^{\ell_i}|_{\partial K\backslash\ell_i} = 0. \tag{5.16}$$

Then we set

$$\mathbf{u}_2^p = \sum_{i=1}^{4}\mathbf{u}_{2,\ell_i}^p, \quad \text{where } \ \mathbf{u}_{2,\ell_i}^p = \mathbf{curl}\,\psi_{2,p}^{\ell_i}. \tag{5.17}$$

The interior interpolant $\mathbf{u}_3^p$ is a vector bubble function living in $\boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ and satisfying the following system of equations:

$$\begin{aligned}
&\langle \operatorname{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p)), \operatorname{div} \mathbf{v}\rangle_{-1/2,K} \\
&\quad = \langle \operatorname{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_3^p)), \operatorname{div} \mathbf{v}\rangle_{-1/2,K} = 0 \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K), \qquad (5.18)
\end{aligned}$$

$$\langle \mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p), \mathbf{curl}\, \phi\rangle_{0,K} = 0 \quad \forall \phi \in \mathcal{P}_p^0(K). \qquad (5.19)$$

We note that any polynomial extension satisfying (5.16) can be used for the construction of the edge interpolant $\mathbf{u}_2^p$. Nevertheless, the interpolant $\mathbf{u}^p = \mathbf{u}_1 + \mathbf{u}_2^p + \mathbf{u}_3^p$ is uniquely defined, which follows from (5.17)–(5.19). It is also easy to see that $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ preserves polynomial vector fields, i.e., $\Pi_p^{\mathrm{div},-\frac{1}{2}} \mathbf{v}_p = \mathbf{v}_p$ for any $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$.

Remark 5.1 *In contrast to the L^2-inner product employed to define the $\mathbf{H}(\mathrm{curl})$-conforming interpolation operator (and its $\mathbf{H}(\mathrm{div})$-conforming counterpart) in* [21]*, we use the $\tilde{H}^{-1/2}$-inner product in* (5.18)*.*

Proposition 5.1 *For $r > 0$ the operator*

$$\Pi_p^{\mathrm{div},-\frac{1}{2}} : \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K) \to \mathbf{L}^2(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$$

is well defined and bounded. For arbitrarily small $\varepsilon_0 > 0$ the norm of this operator satisfies

$$\left\| \Pi_p^{\mathrm{div},-\frac{1}{2}} \right\|_{\mathcal{L}} \le C\, p^{\varepsilon_0},$$

where $C > 0$ is independent of p but depends on ε_0 and r, and $\|\cdot\|_{\mathcal{L}}$ denotes the operator norm in the space $\mathcal{L}\Big(\mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K), \mathbf{L}^2(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)\Big)$.

Proof. Let $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$, $r > 0$. We will study each term on the right-hand side of (5.11). Throughout the proof we denote by s a small parameter such that $0 < s < \min\{\frac{1}{2}, r\}$ for given $r > 0$.

Step 1. Fixing an edge $\ell_i \subset \partial K$ and using a function

$$\phi_i \in H^{1-s}(K), \quad \phi_i = \begin{cases} 1 & \text{on } \ell_i, \\ 0 & \text{on } \partial K \backslash \ell_i \end{cases}$$

as a test function, we integrate by parts to obtain

$$\begin{aligned}
\int_{\ell_i} \mathbf{u}\cdot\mathbf{n}\, d\sigma &= \int_{\partial K} (\mathbf{u}\cdot\mathbf{n})\, \phi_i\, d\sigma = \int_K (\operatorname{div} \mathbf{u})\, \phi_i\, dx + \int_K \mathbf{u}\cdot\nabla\phi_i\, dx \\
&\le \|\operatorname{div} \mathbf{u}\|_{\tilde{H}^{-1+s}(K)} \|\phi_i\|_{H^{1-s}(K)} + \|\mathbf{u}\|_{\mathbf{H}^s(K)} \|\nabla\phi_i\|_{\mathbf{H}^{-s}(K)} \\
&\le C(\phi_i, s) \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\operatorname{div} \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right).
\end{aligned}$$

Note that if $\operatorname{div} \mathbf{u} \in H^{-1+s}(K)$ then an extension to $\operatorname{div} \mathbf{u} \in \tilde{H}^{-1+s}(K)$ exists but is not unique. However, by assumption $\operatorname{div} \mathbf{u} \in \tilde{H}^{-1/2}(K) \subset \tilde{H}^{-1+s}(K)$, which is a unique extension (see [27] for details). Thus, $\mathbf{u}_1$ in (5.12) is well defined. Moreover, since $\mathbf{u}_1$ is a lowest order interpolant, we find by the equivalence of norms in finite-dimensional spaces that

$$\|\mathbf{u}_1\|_{\mathbf{H}(\mathrm{div},K)} \le C \sum_{i=1}^{4} \Big| \int_{\ell_i} \mathbf{u} \cdot \mathbf{n}\, d\sigma \Big| \le C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\operatorname{div} \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right). \tag{5.20}$$

Let use denote by $\gamma_{\mathrm{tr}}^{-1}$ a right inverse of γ_{tr} with $\gamma_{\mathrm{tr}}^{-1} : H^{1/2-s}(\partial K) \to H^{1-s}(K)$. Taking an arbitrary $v \in H^{1/2-s}(\partial K)$ we integrate by parts similarly as above to estimate

$$\begin{aligned} &\int_{\partial K} (\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}\, v\, d\sigma \\ &\qquad \le \|\operatorname{div}(\mathbf{u} - \mathbf{u}_1)\|_{\tilde{H}^{-1+s}(K)} \|\gamma_{\mathrm{tr}}^{-1} v\|_{H^{1-s}(K)} + \|\mathbf{u} - \mathbf{u}_1\|_{\mathbf{H}^s(K)} \|\nabla(\gamma_{\mathrm{tr}}^{-1} v)\|_{\mathbf{H}^{-s}(K)} \\ &\qquad \le C \left(\|\mathbf{u} - \mathbf{u}_1\|_{\mathbf{H}^s(K)} + \|\operatorname{div}(\mathbf{u} - \mathbf{u}_1)\|_{\tilde{H}^{-1+s}(K)} \right) \|v\|_{H^{1/2-s}(\partial K)}. \end{aligned}$$

Hence, $(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n} \in H^{-1/2+s}(\partial K)$ and, due to the finite dimensionality of $\mathbf{u}_1$, we obtain by using estimate (5.20):

$$\begin{aligned} \|(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}\|_{H^{-1/2+s}(\partial K)} &= \sup_{0 \ne v \in H^{1/2-s}(\partial K)} \frac{|\int_{\partial K} (\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}\, v\, d\sigma|}{\|v\|_{H^{1/2-s}(\partial K)}} \\ &\le C \left(\|\mathbf{u} - \mathbf{u}_1\|_{\mathbf{H}^s(K)} + \|\operatorname{div}(\mathbf{u} - \mathbf{u}_1)\|_{\tilde{H}^{-1+s}(K)} \right) \\ &\le C \left(\|\mathbf{u}\|_{\mathbf{H}^s(K)} + \|\operatorname{div} \mathbf{u}\|_{\tilde{H}^{-1+s}(K)} + \|\mathbf{u}_1\|_{\mathbf{H}(\mathrm{div},K)} \right) \\ &\le C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\operatorname{div} \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right). \end{aligned} \tag{5.21}$$

Step 2. From the construction of $\mathbf{u}_1$ and from the result of Step 1 we conclude that

$$(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n} \in H^{-1/2+s}(\partial K), \qquad \int_{\partial K} (\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}\, d\sigma = 0.$$

Therefore, due to the isomorphism $\frac{\partial}{\partial \sigma} : H^{1/2+s}(\partial K)/\mathbb{C} \to H_*^{-1/2+s}(\partial K)$ (see [21, Lemma 2]), the function ψ in (5.14) is well defined, $\psi \in H^{1/2+s}(\partial K)$, $\psi|_{\ell_i} \in \tilde{H}^{1/2}(\ell_i)$ for any edge $\ell_i \subset \partial K$, and

$$\sum_{i=1}^{4} \|\psi|_{\ell_i}\|_{\tilde{H}^{1/2}(\ell_i)} \le C \sum_{i=1}^{4} \|\psi|_{\ell_i}\|_{H_0^{1/2+s}(\ell_i)} \le C \|\psi\|_{H^{1/2+s}(\partial K)} \le C \|(\mathbf{u} - \mathbf{u}_1) \cdot \mathbf{n}\|_{H^{-1/2+s}(\partial K)}. \tag{5.22}$$

Hence, (5.15) is uniquely solvable and

$$\|\psi_2^{\ell_i}\|_{\tilde{H}^{1/2}(\ell_i)} \le C\, \|\psi|_{\ell_i}\|_{\tilde{H}^{1/2}(\ell_i)}. \tag{5.23}$$

Furthermore, applying the polynomial extension result of Babuška and Suri [2], we find the desired polynomial $\psi_{2,p}^{\ell_i} \in \mathcal{P}_p(K)$ (see (5.16)) satisfying

$$\|\psi_{2,p}^{\ell_i}\|_{H^1(K)} \le C\, \|\psi_2^{\ell_i}\|_{\tilde{H}^{1/2}(\ell_i)}. \tag{5.24}$$

Thus, $\mathbf{u}_2^p$ in (5.17) is well defined. Putting together (5.22)–(5.24) we find

$$\begin{aligned}\|\mathbf{u}_2^p\|_{0,K} &\le C \sum_{i=1}^{4} \|\mathbf{curl}\, \psi_{2,p}^{\ell_i}\|_{0,K} \le C \sum_{i=1}^{4} \|\psi_{2,p}^{\ell_i}\|_{H^1(K)} \\ &\le C \sum_{i=1}^{4} \|\psi_2^{\ell_i}\|_{\tilde{H}^{1/2}(\ell_i)} \le C \sum_{i=1}^{4} \|\psi|_{\ell_i}\|_{\tilde{H}^{1/2}(\ell_i)} \le C\, \|(\mathbf{u}-\mathbf{u}_1)\cdot \mathbf{n}\|_{H^{-1/2+s}(\partial K)}.\end{aligned}$$

Hence, making use of (5.21), we obtain

$$\|\mathbf{u}_2^p\|_{\mathbf{H}(\mathrm{div},K)} = \|\mathbf{u}_2^p\|_{0,K} \le C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)}\right). \tag{5.25}$$

Step 3. The vector bubble function $\mathbf{u}_3^p$ is uniquely defined by (5.18)–(5.19). To estimate the norms of $\mathbf{u}_p^3$ and $\mathrm{div}\, \mathbf{u}_p^3$ we use the discrete Helmholtz decomposition (4.2) restricted to $\mathbf{X}_p(K) \equiv \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$

$$\mathbf{u}_3^p = \mathbf{v}_p + \mathbf{curl}\, \phi_p, \tag{5.26}$$

where $\phi_p \in \mathcal{P}_p^0(K)$ and $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K)$ is such that $\langle \mathbf{v}_p, \mathbf{w}_p\rangle_{0,K} = 0$ for all $\mathbf{w}_p \in \mathbf{W}_p(K)$ (see Lemma 5.4 for the definition of $\mathbf{W}_p(K)$).

From (5.18) one has by using the result of Step 1

$$\begin{aligned}\|\mathrm{div}\, \mathbf{u}_3^p\|_{\tilde{H}^{-1/2}(K)} &\le C\, \|\mathrm{div}(\mathbf{u}-\mathbf{u}_1)\|_{\tilde{H}^{-1/2}(K)} \le C \left(\|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)} + |\mathrm{div}\, \mathbf{u}_1|\right) \\ &\le C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)}\right). \end{aligned} \tag{5.27}$$

Then we apply Lemma 5.4: there exists $\mathbf{v} \in \nabla\tilde{\mathcal{H}}(K)$ and an arbitrarily small $\varepsilon_0 > 0$ such that

$$\|\mathbf{v}_p\|_{0,K} \le \|\mathbf{v}-\mathbf{v}_p\|_{0,K} + \|\mathbf{v}\|_{0,K} \le C\, p^{\varepsilon_0}\, \|\mathrm{div}\, \mathbf{v}_p\|_{\tilde{H}^{-1/2}(K)}.$$

Hence, recalling that $\mathrm{div}\, \mathbf{v}_p = \mathrm{div}\, \mathbf{u}_3^p$, we obtain

$$\|\mathbf{v}_p\|_{0,K} \le C\, p^{\varepsilon_0}\, \|\mathrm{div}\, \mathbf{u}_3^p\|_{\tilde{H}^{-1/2}(K)}. \tag{5.28}$$

Since $\langle \mathbf{v}_p, \mathbf{curl}\, \phi_p\rangle_{0,K} = 0$, we estimate the norm of $\mathbf{curl}\, \phi_p$ by using (5.19) and by employing the results of the first two steps:

$$\|\mathbf{curl}\, \phi_p\|_{0,K} \le \|\mathbf{u}-\mathbf{u}_1-\mathbf{u}_2^p\|_{0,K} \le C \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\mathrm{div}\, \mathbf{u}\|_{\tilde{H}^{-1/2}(K)}\right). \tag{5.29}$$

Combining (5.27)–(5.29) and applying the triangle inequality we obtain by making use of decomposition (5.26)

$$\|\mathbf{u}_3^p\|_{0,K} + \|\operatorname{div}\mathbf{u}_3^p\|_{\tilde{H}^{-1/2}(K)} \le C\, p^{\varepsilon_0} \left(\|\mathbf{u}\|_{\mathbf{H}^r(K)} + \|\operatorname{div}\mathbf{u}\|_{\tilde{H}^{-1/2}(K)} \right).$$

To finish the proof it remains to combine the results of the three individual steps and to apply the triangle inequality to decomposition (5.11). □

Proposition 5.1 proves a quasi-stability of the interpolation operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$. The following proposition states the commuting diagram property for $\Pi_p^{\mathrm{div},-\frac{1}{2}}$.

Proposition 5.2 *For any $\mathbf{u} \in \mathbf{H}^r(K) \cap \tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, K)$, $r > 0$, there holds*

$$\operatorname{div}\left(\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}\right) = \Pi_{p-1}^{-1/2}(\operatorname{div}\mathbf{u}). \tag{5.30}$$

Proof. For any $\varphi \in \mathcal{P}_{p-1}(K)$ there exists $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ such that $\operatorname{div}\mathbf{v}_p = \varphi$. Therefore, decomposing $\Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}$ as in (5.11), we need to show that for all $\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ there holds

$$\left\langle \operatorname{div}\left(\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}\right), \operatorname{div}\mathbf{v}_p \right\rangle_{-1/2,K} = \langle \operatorname{div}(\mathbf{u} - (\mathbf{u}_1 + \mathbf{u}_3^p)), \operatorname{div}\mathbf{v}_p\rangle_{-1/2,K} = 0. \tag{5.31}$$

Let us also decompose $\mathbf{v}_p = \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{v}_p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT}}(K)$ as in (5.11):

$$\mathbf{v}_p = \mathbf{v}_1 + \mathbf{v}_2^p + \mathbf{v}_3^p, \quad \operatorname{div}\mathbf{v}_1 = \mathrm{const}, \quad \operatorname{div}\mathbf{v}_2^p = 0, \quad \mathbf{v}_3^p \in \boldsymbol{\mathcal{P}}_p^{\mathrm{RT},0}(K).$$

Then, recalling (5.18), applying Lemma 3.3, and integrating by parts, we prove (5.31):

$$\begin{aligned}
&\left\langle \operatorname{div}\left(\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}\right), \operatorname{div}\mathbf{v}_p \right\rangle_{-1/2,K} \\
&\quad = \langle \operatorname{div}(\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^p), \mathrm{const}\rangle_{-1/2,K} + \left\langle \operatorname{div}\left(\mathbf{u} - \Pi_p^{\mathrm{div},-\frac{1}{2}}\mathbf{u}\right), \operatorname{div}\mathbf{v}_3^p \right\rangle_{-1/2,K} \\
&\quad = \langle \operatorname{div}(\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^p), \mathrm{const}\rangle_{0,K} = \mathrm{const} \int_{\partial K} (\mathbf{u} - \mathbf{u}_1 - \mathbf{u}_3^p)\cdot\mathbf{n}\, d\sigma = 0.
\end{aligned}$$

For the last step we used the fact that $\mathbf{u}_3^p \cdot \mathbf{n}|_{\partial K} = 0$ and then applied (5.13). □

6 Proof of Theorem 2.1

In this section we prove Theorem 2.1 relying on the abstract convergence result of Theorem 4.2. Since the discrete decomposition (4.2) is stable with respect to complex conjugation, one needs to check that assumptions (A1) and (A2) are satisfied. First, we note that the family $\{\mathbf{X}_{hp}\}$ of RT-spaces is dense in $\mathbf{X}^0$. Since the injection $\mathbf{X}^0 \subset \mathbf{X}$ is dense as well (see, e.g., [17, Lemma 2.4]), we conclude that the family $\{\mathbf{X}_{hp}\}$ satisfies assumption (A1) of Theorem 4.2.

It was mentioned in Section 4 that $\mathbf{W}_{hp} \subset \mathbf{W}$ by construction. Thus, it remains to prove that the subspace $\mathbf{V}_{hp}$ defined by (4.3) satisfies assumption (4.4). In particular, we will show below that there exists a sequence $\{\delta_{hp}\}$, $\delta_{hp} \to 0$, such that for any given $\mathbf{v}_{hp} \in \mathbf{V}_{hp}$ there exists $\mathbf{v} \in \mathbf{V}$ satisfying

$$\|\mathbf{v}_{hp} - \mathbf{v}\|_{\mathbf{X}} \le \delta_{hp} \|\mathbf{v}_{hp}\|_{\mathbf{X}}. \tag{6.1}$$

We prove (6.1) for a closed (resp., open) surface Γ. The proof consists of 5 steps.

Step 1: Construction of v. Given $\mathbf{v}_{hp} \in \mathbf{V}_{hp}$, we solve the following problem to find $f \in H^1(\Gamma)/\mathbb{C}$ such that

$$\langle \nabla_\Gamma f, \nabla_\Gamma g \rangle = -\langle \mathrm{div}_\Gamma \mathbf{v}_{hp}, g \rangle \quad \forall g \in H^1(\Gamma)/\mathbb{C}. \tag{6.2}$$

We set $\mathbf{v} := \nabla_\Gamma f$. Then

$$\mathrm{div}_\Gamma \mathbf{v} = \mathrm{div}_\Gamma \mathbf{v}_{hp} \in L^2(\Gamma), \tag{6.3}$$

and, due to Theorem 4.1, there holds $\mathbf{v} \in \mathbf{V}$. Note that $\mathrm{div}_\Gamma \mathbf{v}_{hp} \in H^{-1/2}(\Gamma)$ (resp., $\mathrm{div}_\Gamma \mathbf{v}_{hp} \in \tilde{H}^{-1/2}(\Gamma)$) and $\langle \mathrm{div}_\Gamma \mathbf{v}_{hp}, 1 \rangle = 0$. Therefore, the regularity result for problem (6.2) reads as (cf. Lemma 3.4 (resp., Lemma 3.5)): $f \in H^{1+r}(\Gamma)$ with $r = \min\{s^*, \frac{1}{2}\} - \varepsilon_1$, where $s^* > 0$ depends on the geometry of Γ, $\varepsilon_1 > 0$ is arbitrarily small. Moreover, using the continuity of ∇_Γ, we conclude that $\mathbf{v} \in \mathbf{H}^r_-(\Gamma)$ and

$$\|\mathbf{v}\|_{\mathbf{H}^r_-(\Gamma)} \le C \, \|f\|_{H^{1+r}(\Gamma)/\mathbb{C}} \le C \, \|\mathrm{div}_\Gamma \mathbf{v}_{hp}\|_{\tilde{H}^{-1/2}(\Gamma)}, \quad r = \min\{s^*, \tfrac{1}{2}\} - \varepsilon_1 \tag{6.4}$$

(here and below we use the convention $\tilde{H}^{-1/2}(\Gamma) = H^{-1/2}(\Gamma)$ if Γ is closed).

Step 2: Bounding $\|\mathbf{v}_{hp} - \mathbf{v}\|_{\mathbf{X}}$ by $\|\mathbf{v} - \mathbf{v}_{hp}\|_{0,\Gamma}$. In view of (6.3) we can estimate the norm on the left-hand side of (6.1) as

$$\begin{aligned} \|\mathbf{v}_{hp} - \mathbf{v}\|_{\mathbf{X}} &= \|\mathbf{v}_{hp} - \mathbf{v}\|_{\tilde{\mathbf{H}}_\|^{-1/2}(\Gamma)} \le C \, \|\mathbf{v}_{hp} - \mathbf{v}\|_{\mathbf{H}_-^{-1/2+\varepsilon_1}(\Gamma)} \\ &= C \sup_{\mathbf{w} \in \mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma) \setminus \{\mathbf{0}\}} \frac{\langle \mathbf{v} - \mathbf{v}_{hp}, \mathbf{w} \rangle}{\|\mathbf{w}\|_{\mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma)}} \end{aligned} \tag{6.5}$$

with the same $\varepsilon_1 > 0$ as in (6.4). Let $\mathbf{w} \in \mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma)$. Then, by [14, Theorem 3.4] (resp., [14, Theorem 6.1]), there exists a unique pair w_1, $w_2 \in H^1(\Gamma)/\mathbb{C}$ (resp., $w_1 \in H^1(\Gamma)/\mathbb{C}$, $w_2 \in H^1_0(\Gamma)$) such that

$$\mathbf{w} = \nabla_\Gamma w_1 + \mathbf{curl}_\Gamma w_2. \tag{6.6}$$

Moreover, w_2 is the solution of the problem

$$-\Delta_\Gamma w_2 = \mathrm{curl}_\Gamma \mathbf{curl}_\Gamma w_2 = \mathrm{curl}_\Gamma \mathbf{w}.$$

Since $\mathrm{curl}_\Gamma \mathbf{w} \in H^{-1/2-\varepsilon_1}(\Gamma)$ and $\langle \mathrm{curl}_\Gamma \mathbf{w}, 1\rangle = 0$ (resp., $\mathrm{curl}_\Gamma \mathbf{w} \in H^{-1/2-\varepsilon_1}(\Gamma)$), we apply Lemma 3.4 (resp., the Dirichlet analog of Lemma 3.5) to prove that $w_2 \in H^{1+r}(\Gamma)$ (with the same r as in (6.4)) and there holds

$$\begin{gathered} \|w_2\|_{H^{1+r}(\Gamma)/\mathbb{C}} \le C\,\|\mathrm{curl}_\Gamma \mathbf{w}\|_{H^{-1/2-\varepsilon_1}(\Gamma)} \le C\,\|\mathbf{w}\|_{\mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma)} \\ \text{(resp.,} \quad \|w_2\|_{H^{1+r}(\Gamma)} \le C\,\|\mathbf{w}\|_{\mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma)}). \end{gathered} \tag{6.7}$$

Therefore, $w_2 \in H^{1+r}(\Gamma_j)$, $r > 0$, for any element Γ_j, and one can apply the H^1-conforming interpolation operator Π_p^1 to find a continuous piecewise polynomial w_2^{hp} defined on Γ such that $\hat{w}_{2,j}^{hp} := w_2^{hp}|_{\Gamma_j} \circ T_j = \Pi_p^1 \hat{w}_{2,j} \in \mathcal{P}_p(K)$ (here $\hat{w}_{2,j} := w_2|_{\Gamma_j} \circ T_j$). To estimate the H^1-semi-norm of the error $(w_2 - w_2^{hp})$, we apply the standard scaling argument (cf. [19, Theorem 4.3.2]) and the approximation result for Π_p^1 on the reference element (see [21, Theorem 2]):

$$\begin{aligned} |w_2 - w_2^{hp}|_{H^1(\Gamma_j)} &\le C\left\|\hat{w}_{2,j} - \Pi_p^1 \hat{w}_{2,j}\right\|_{H^1(K)} \\ &= C\left\|\hat{w}_{2,j} - \hat{\varphi}_p - \Pi_p^1(\hat{w}_{2,j} - \hat{\varphi}_p)\right\|_{H^1(K)} \qquad (\forall \hat{\varphi}_p \in \mathcal{P}_p(K)) \\ &\le C\,p^{-(r-\varepsilon_2)} \inf_{\hat{\varphi}_p \in \mathcal{P}_p(K)} \|\hat{w}_{2,j} - \hat{\varphi}_p\|_{H^{1+r}(K)}, \quad 0 < \varepsilon_2 < r < 1/2. \end{aligned} \tag{6.8}$$

Let $s = 1, 2$. Using Theorem 3.1.1 of [19] and the scaling argument, one has

$$\begin{aligned} \inf_{\hat{\varphi}_p \in \mathcal{P}_p(K)} \|\hat{w}_{2,j} - \hat{\varphi}_p\|_{H^s(K)} &\le \inf_{\hat{\varphi} \in \mathcal{P}_{s-1}(K)} \|\hat{w}_{2,j} - \hat{\varphi}\|_{H^s(K)} \\ &\le C\,|\hat{w}_{2,j}|_{H^s(K)} \le C\,h^{s-1}\,\|w_2\|_{H^s(\Gamma_j)}. \end{aligned}$$

Therefore, by interpolation,

$$\inf_{\hat{\varphi}_p \in \mathcal{P}_p(K)} \|\hat{w}_{2,j} - \hat{\varphi}_p\|_{H^{1+r}(K)} \le C\,h^r\,\|w_2\|_{H^{1+r}(\Gamma_j)},$$

and from (6.8) we conclude that

$$|w_2 - w_2^{hp}|_{H^1(\Gamma_j)} \le C\,h^r\,p^{-(r-\varepsilon_2)}\,\|w_2\|_{H^{1+r}(\Gamma_j)}.$$

Hence, for a closed (resp., open) surface Γ we obtain

$$\begin{gathered} |w_2 - w_2^{hp}|_{H^1(\Gamma)} = \left(\sum_j |w_2 - w_2^{hp}|^2_{H^1(\Gamma_j)}\right)^{1/2} \le C\,h^r\,p^{-(r-\varepsilon_2)}\,\|w_2\|_{H^{1+r}(\Gamma)/\mathbb{C}} \\ \text{(resp.,} \quad |w_2 - w_2^{hp}|_{H^1(\Gamma)} \le C\,h^r\,p^{-(r-\varepsilon_2)}\,\|w_2\|_{H^{1+r}(\Gamma)}). \end{gathered} \tag{6.9}$$

In addition, for the case of an open surface, we note that w_2^{hp} vanishes on $\partial\Gamma$. Then recalling the commuting diagram property for Π_p^1 (see (5.1)) and the definition of $\mathbf{X}_{hp}$ (see (2.4)), we conclude that $\mathbf{curl}_\Gamma\, w_2^{hp} \in \mathbf{X}_{hp} \subset \mathbf{X}$. Moreover, $\mathbf{curl}_\Gamma\, w_2^{hp} \in \mathbf{W}_{hp} \subset (\mathbf{W} \cap \mathbf{L}_t^2(\Gamma))$. This fact together with the $\mathbf{L}_t^2(\Gamma)$-orthogonalities $\mathbf{V} \perp (\mathbf{W} \cap \mathbf{L}_t^2(\Gamma))$ and $\mathbf{V}_{hp} \perp \mathbf{W}_{hp}$ implies

$$\langle \mathbf{v} - \mathbf{v}_{hp}, \mathbf{curl}_\Gamma\, w_2^{hp} \rangle = 0. \tag{6.10}$$

Now we use (6.3), (6.6), (6.7), (6.9), (6.10) (and (3.4), if Γ is an open surface) to prove for any $\mathbf{w} \in \mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma)$

$$\begin{aligned} \langle \mathbf{v} - \mathbf{v}_{hp}, \mathbf{w} \rangle &= \langle \mathbf{v} - \mathbf{v}_{hp}, \nabla_\Gamma\, w_1 \rangle + \langle \mathbf{v} - \mathbf{v}_{hp}, \mathbf{curl}_\Gamma\, w_2 \rangle \\ &= -\langle \mathrm{div}_\Gamma (\mathbf{v} - \mathbf{v}_{hp}), w_1 \rangle + \langle \mathbf{v} - \mathbf{v}_{hp}, \mathbf{curl}_\Gamma (w_2 - w_2^{hp}) \rangle \\ &\le \|\mathbf{v} - \mathbf{v}_{hp}\|_{0,\Gamma}\, |w_2 - w_2^{hp}|_{H^1(\Gamma)} \le C\, h^r\, p^{-(r-\varepsilon_2)}\, \|\mathbf{v} - \mathbf{v}_{hp}\|_{0,\Gamma}\, \|\mathbf{w}\|_{\mathbf{H}_-^{1/2-\varepsilon_1}(\Gamma)}. \end{aligned}$$

Using this estimate in (6.5) we find

$$\|\mathbf{v}_{hp} - \mathbf{v}\|_{\mathbf{X}} \le C\, h^r\, p^{-(r-\varepsilon_2)}\, \|\mathbf{v} - \mathbf{v}_{hp}\|_{0,\Gamma}, \tag{6.11}$$

where $r = \min\{s^*, \frac{1}{2}\} - \varepsilon_1$, $\varepsilon_1 > 0$, $\varepsilon_2 \in (0, r)$ (ε_1, ε_2 can be arbitrarily small).

Step 3: Bounding $\|\mathbf{v} - \mathbf{v}_{hp}\|_{0,\Gamma}$ by $\|\mathbf{v} - \Pi_p^{\mathrm{div},-1/2} \mathbf{v}\|_{0,\Gamma}$. We recall that $\mathbf{v} \in \mathbf{H}^r_-(\Gamma) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$, $r \in (0, \frac{1}{2})$. Therefore, $\mathbf{v}|_{\Gamma_j} \in \mathbf{H}^r(\Gamma_j) \cap \mathbf{H}(\mathrm{div}_\Gamma, \Gamma_j)$ for each element Γ_j and we can define $\mathbf{v}_{hp}^{\mathrm{div}} \in \mathbf{X}_{hp}$ such that

$$\mathcal{M}_j^{-1}\Big(\mathbf{v}_{hp}^{\mathrm{div}}|_{\Gamma_j}\Big) = \Pi_p^{\mathrm{div},-\frac{1}{2}}\Big(\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j})\Big),$$

where $\mathcal{M}_j$ is the Piola transform (see (2.3)).

Using commutativity (5.30) and the fact that $\mathrm{div}_\Gamma\, \mathbf{v} = \mathrm{div}_\Gamma\, \mathbf{v}_{hp}$ is a piecewise polynomial on Γ, we find

$$\mathrm{div}_\Gamma\, \mathbf{v}_{hp}^{\mathrm{div}} = \mathrm{div}_\Gamma\, \mathbf{v} = \mathrm{div}_\Gamma\, \mathbf{v}_{hp}.$$

Hence, $(\mathbf{v}_{hp} - \mathbf{v}_{hp}^{\mathrm{div}}) \in \mathbf{W}_{hp} \subset (\mathbf{W} \cap \mathbf{L}_t^2(\Gamma))$, and using again the orthogonalities $\mathbf{V} \perp (\mathbf{W} \cap \mathbf{L}_t^2(\Gamma))$, $\mathbf{V}_{hp} \perp \mathbf{W}_{hp}$ we have

$$\langle \mathbf{v} - \mathbf{v}_{hp}, \mathbf{v}_{hp} - \mathbf{v}_{hp}^{\mathrm{div}} \rangle = 0.$$

Therefore,

$$\|\mathbf{v} - \mathbf{v}_{hp}\|_{0,\Gamma} \le \|\mathbf{v} - \mathbf{v}_{hp}^{\mathrm{div}}\|_{0,\Gamma}. \tag{6.12}$$

Step 4: Estimating $\|\mathbf{v} - \mathbf{v}_{hp}^{\mathrm{div}}\|_{0,\Gamma}$. Using (3.19) and the quasi-stability of $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ (see Proposition 5.1), we estimate

$$\begin{aligned} \|\mathbf{v}_{hp}^{\mathrm{div}}|_{\Gamma_j}\|_{0,\Gamma_j} &= \Big\|\mathcal{M}_j\Big(\Pi_p^{\mathrm{div},-\frac{1}{2}}(\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}))\Big)\Big\|_{0,\Gamma_j} \le C\, \Big\|\Pi_p^{\mathrm{div},-\frac{1}{2}}(\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}))\Big\|_{0,K} \\ &\le C\, p^{\varepsilon_0}\, \Big(\|\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j})\|_{\mathbf{H}^r(K)} + \|\mathrm{div}(\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}))\|_{\tilde{H}^{-1/2}(K)}\Big) \\ &\le C\, p^{\varepsilon_0}\, \Big(\|\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j})\|_{\mathbf{H}^r(K)} + \|\mathrm{div}(\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}))\|_{H^{-1/2+\varepsilon_3}(K)}\Big), \end{aligned}$$

where $\varepsilon_0, \varepsilon_3 > 0$ are arbitrarily small. Hence, recalling that $\mathrm{div}(\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j})) = \mathrm{div}(\mathcal{M}_j^{-1}(\mathbf{v}_{hp}|_{\Gamma_j}))$ is a polynomial on K, we make use of the inverse inequality (see Lemma 3.7) and then apply (3.20), (3.21):

$$\begin{aligned}\|\mathbf{v}_{hp}^{\mathrm{div}}|_{\Gamma_j}\|_{0,\Gamma_j} &\leq C\, p^{\varepsilon_0+2\varepsilon_3} \left(\|\mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j})\|_{\mathbf{H}^r(K)} + \|\mathrm{div}(\mathcal{M}_j^{-1}(\mathbf{v}_{hp}|_{\Gamma_j}))\|_{H^{-1/2}(K)}\right)\\ &\leq C\, p^{\varepsilon_0+2\varepsilon_3} \left(\|\mathbf{v}\|_{\mathbf{H}^r(\Gamma_j)} + h^{1/2}\, \|\mathrm{div}_\Gamma\, \mathbf{v}_{hp}\|_{H^{-1/2}(\Gamma_j)}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{v} - \mathbf{v}_{hp}^{\mathrm{div}}\|_{0,\Gamma} &\leq \|\mathbf{v}\|_{0,\Gamma} + \left(\sum_j \|\mathbf{v}_{hp}^{\mathrm{div}}|_{\Gamma_j}\|_{0,\Gamma_j}^2\right)^{1/2}\\ &\leq \|\mathbf{v}\|_{0,\Gamma} + C\, p^{\varepsilon_0+2\varepsilon_3}\left(\sum_j \left(\|\mathbf{v}\|_{\mathbf{H}^r(\Gamma_j)}^2 + h\, \|\mathrm{div}_\Gamma\, \mathbf{v}_{hp}\|_{H^{-1/2}(\Gamma_j)}^2\right)\right)^{1/2}\\ &\leq C\, p^{\varepsilon_0+2\varepsilon_3}\left(\|\mathbf{v}\|_{\mathbf{H}^r_-(\Gamma)} + h^{1/2}\, \|\mathrm{div}_\Gamma\, \mathbf{v}_{hp}\|_{H^{-1/2}(\Gamma)}\right). \qquad (6.13)\end{aligned}$$

Step 5: Conclusion. First, we select all small parameters ε_k $(k = 0, 1, 2, 3)$ such that $\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 = \varepsilon$ for any given $\varepsilon \in (0, \min\{s^*, \frac{1}{2}\})$. Then putting together the results of all previous steps, i.e., estimates (6.4) and (6.11)–(6.13), we obtain

$$\|\mathbf{v}_{hp} - \mathbf{v}\|_{\mathbf{X}} \leq C \left(\tfrac{h}{p}\right)^{\min\{s^*,\frac{1}{2}\}-\varepsilon} \|\mathrm{div}_\Gamma\, \mathbf{v}_{hp}\|_{\tilde{H}^{-1/2}(\Gamma)} \leq C \left(\tfrac{h}{p}\right)^{\min\{s^*,\frac{1}{2}\}-\varepsilon} \|\mathbf{v}_{hp}\|_{\mathbf{X}}.$$

This proves (6.1). Therefore, the subspace $\mathbf{V}_{hp}$ satisfies (4.4). Thus we have shown that the discrete decomposition (4.2) verifies assumption (A2) of Theorem 4.2 in the framework of the *hp*-version of the BEM with quasi-uniform meshes, and the proof is finished.

7 Concluding remarks

The main result of this paper – the convergence of the *hp*-BEM with quasi-uniform meshes (and thus the convergence of the *h*- and the *p*- versions as particular cases) for the EFIE – is proved for a sequence of Raviart-Thomas spaces on quadrilateral elements, which can be parallelograms, convex quadrilaterals or curvilinear ones. To that end it was essential to introduce and analyse a new $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div})$-conforming interpolation operator $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ on the reference element. To show the stability of $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ (with respect to polynomial degrees) we relied, in particular, on the discrete inf-sup condition (see Lemma 5.1), which was established in [9].

The case of triangular elements. We note that Lemma 5.1 is not available for the reference triangle. However, the corresponding result has been conjectured and numerically evidenced in [10] for $\mathbf{H}(\mathrm{curl})$-conforming Nédélec elements of the second type. In our $\mathbf{H}(\mathrm{div})$-conforming settings it means that the conjectured result of Lemma 5.1 is numerically justified for Brezzi-Douglas-Marini spaces on the reference triangle (this is due to the isomorphism of the curl and the div operator in 2D). Since all remaining arguments in the construction of $\Pi_p^{\mathrm{div},-\frac{1}{2}}$ and in the proofs of Propositions 5.1, 5.2 and Theorem 2.1 are valid for BDM-spaces on triangles (cf. also [21]), we conclude that the main result of the paper holds for these spaces, provided that the discrete inf-sup condition discussed above is true.

Regularity results and error analysis. In this paper we do not present the regularity result for the problem under consideration, neither perform an a priori error analysis for the *hp*-BEM. However, we note that in [8] we derived explicit formulas for typical singularities inherent to the solution of the EFIE on piecewise smooth (open or closed) Lipschitz surfaces. The *p*-approximation analysis of these singularities (including the least regular ones) was performed in [8] on a plane open surface. This analysis relied on our results for the Laplacian, see [6, 7, 5], by using continuity properties of the surface (vector) curl operator. Since those results and properties (in corresponding spaces) are valid for polyhedral and piecewise plane open surfaces, the proof of an a priori error estimate for the *p*-BEM from [8] carries over without essential modifications to the more general case considered in the present paper. We stress that this *p*-approximation result holds only for affine families of meshes. The error analysis of the *hp*-version on quasi-uniform meshes presents more difficulties. In particular, the involved Sobolev spaces of negative order are not scalable under affine transformations. Therefore, this analysis is not a trivial extension of our *p*-approximation results.

Non-affine quadrilateral meshes. Although the main convergence result of the paper holds for non-affine quadrilateral meshes, it is not clear if the RT-spaces in this case would provide optimal approximations in the energy norm for the EFIE. For instance, in the *h*-version of the finite element method the degradation of convergence rates (in the $\mathbf{H}(\mathrm{div})$-norm) was observed for RT-elements on general convex quadrilaterals (see [1]). An alternative family of finite elements on non-affine meshes was introduced in [1] and was shown to provide optimal *h*-approximation order in $\mathbf{H}(\mathrm{div})$. These Arnold-Boffi-Falk (ABF) elements can also be used in the BEM for the EFIE. However, the solvability, convergence, and a priori error estimation of the *h*-, *p*-, or *hp*-BEM with ABF-elements are open problems.

References

[1] D. N. Arnold, D. Boffi, and R. S. Falk, *Quadrilateral $H(\mathrm{div})$ finite elements*, SIAM J. Numer. Anal., 42 (2005), pp. 2429–2451.

[2] I. Babuška and M. Suri, *The h-p version of the finite element method with quasiuniform meshes*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 199–238.

[3] A. Bendali, *Numerical analysis of the exterior boundary value problem for time-harmonic maxwell equations by a boundary finite element method, Part 1: The continuous problem*, Math. Comp., 43 (1984), pp. 29–46.

[4] ——, *Numerical analysis of the exterior boundary value problem for time-harmonic maxwell equations by a boundary finite element method, Part 2: The discrete problem*, Math. Comp., 43 (1984), pp. 47–68.

[5] A. Bespalov, *A note on the polynomial approximation of vertex singularities in the boundary element method in three dimensions*, Brunel University Research Archive (BURA), Brunel University, UK, 2008. http://hdl.handle.net/2438/1654 (to appear in J. Integral Equations Appl.).

[6] A. Bespalov and N. Heuer, *The p-version of the boundary element method for hypersingular operators on piecewise plane open surfaces*, Numer. Math., 100 (2005), pp. 185–209.

[7] ——, *The p-version of the boundary element method for weakly singular operators on piecewise plane open surfaces*, Numer. Math., 106 (2007), pp. 69–97.

[8] ——, *Natural p-BEM for the electric field integral equation on screens*, Report 08/2, BICOM, Brunel University, UK, 2008 (to appear in IMA J. Numer. Anal.).

[9] D. Boffi, M. Costabel, M. Dauge, and L. Demkowicz, *Discrete compactness for the hp version of rectangular edge finite elements*, SIAM J. Numer. Anal., 44 (2006), pp. 979–1004.

[10] D. Boffi, L. Demkowicz, and M. Costabel, *Discrete compactness for the p and hp 2D edge finite elements*, Math. Models Methods Appl. Sci., 13 (2003), pp. 1673–1687.

[11] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, no. 15 in Springer Series in Computational Mathematics, Springer-Verlag, New York, 1991.

[12] A. Buffa and S. H. Christiansen, *The electric field integral equation on Lipschitz screens: definitions and numerical approximation*, Numer. Mat., 94 (2003), pp. 229–267.

[13] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations, Part I: An integration by parts formula in Lipschitz polyhedra*, Math. Methods Appl. Sci., 24 (2001), pp. 9–30.

[14] ——, *On traces for functional spaces related to Maxwell's equations, Part II: Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., 24 (2001), pp. 31–48.

[15] A. Buffa, M. Costabel, and C. Schwab, *Boundary element methods for Maxwell's equations on non-smooth domains*, Numer. Mat., 92 (2002), pp. 679–710.

[16] A. Buffa, M. Costabel, and D. Sheen, *On traces for H(curl, Ω) in Lipschitz domains*, J. Math. Anal. Appl., 276 (2002), pp. 845–867.

[17] A. Buffa and R. Hiptmair, *A coercive combined field integral equation for electromagnetic scattering*, SIAM J. Numer. Anal., 42 (2004), pp. 621–640.

[18] A. Buffa, R. Hiptmair, T. von Petersdorff, and C. Schwab, *Boundary element methods for Maxwell transmission problems in Lipschitz domains*, Numer. Math., 95 (2003), pp. 459–485.

[19] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[20] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[21] L. Demkowicz and I. Babuška, *p interpolation error estimates for edge finite elements of variable order in two dimensions*, SIAM J. Numer. Anal., 41 (2003), pp. 1195–1208.

[22] P. Grisvard, *Singularities in Boundary Value Problems*, no. 22 in Research Notes in Applied Mathematics, Masson, Paris, 1992.

[23] N. Heuer, *Additive Schwarz method for the p-version of the boundary element method for the single layer potential operator on a plane screen*, Numer. Math., 88 (2001), pp. 485–511.

[24] R. Hiptmair and C. Schwab, *Natural boundary element methods for the electric field integral equation on polyhedra*, SIAM J. Numer. Anal., 40 (2002), pp. 66–86.

[25] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, New York, 1972.

[26] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[27] S. E. Mikhailov, *About traces, extensions, and co-normal derivative operators on Lipschitz domains*, in Integral Methods in Science and Engineering, Birkhäuser Boston, Boston, MA, 2008, pp. 149–160.

[28] J.-C. Nédélec, *Acoustic and Electromagnetic Equations*, no. 144 in Applied Mathematical Sciences, Springer, New York, 2001.

[29] R. E. Roberts and J.-M. Thomas, *Mixed and hybrid methods*, in Handbook of Numerical Analysis. Vol. II, P. G. Ciarlet and J. L. Lions, eds., Amsterdam, 1991, North-Holland, pp. 523–639.

[30] J.-M. Thomas, *Sur l'analyse numérique des méthodes d'eléments finis hybrides et mixtes*, PhD thesis, Université Pierre et Marie Curie, Paris, France, 1977.