THE $\mathscr{U}$-RADIUS AND HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS, AND PRODUCT OF LOGHARMONIC MAPPINGS

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by

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LIST OF SYMBOLS

		Page
$\mathcal{A}$	Class of analytic functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D})$	2
$\mathbb{C}$	Complex plane	1
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	7
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	8
$\mathcal{CV}(\varphi)$	$\left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}$	11
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	8
$\mathbb{D}$	Open unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$	2
$\mathcal{H}(\mathbb{D})$	Class of all analytic functions in $\mathbb{D}$	2
HG	Class of analytic functions φ in $\mathbb{D}$ of the form $\varphi(z) = zh(z)g(z)$, $h, g \in \mathcal{H}(\mathbb{D})$, $h(0) = g(0) = 1$	131
$H_q(n)$	Hankel determinants of functions $f \in \mathcal{A}$	70
Im	Imaginary part of a complex number	13
L_α	$\left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} > 0 \right\}$	9
$L(\alpha, \varphi)$	$\left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\}$	99
M_α	$\left\{ f \in \mathcal{A} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}$	9
$M(\alpha, \varphi)$	$\left\{ f \in \mathcal{A} : (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\}$	105
$\mathcal{P}$	$\{p \in \mathcal{H}(\mathbb{D}) : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ \text{Re}\ p(z) > 0,\ z \in \mathbb{D}\}$	5
$\mathcal{P}(\alpha)$	$\{p \in \mathcal{H}(\mathbb{D}) : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,\ \text{Re}\ p(z) > \alpha,\ z \in \mathbb{D}\}$	5
$\mathcal{P}_{\mathbb{R}}$	Class of all functions $p \in \mathcal{P}$ with real coefficients	14
$\mathcal{P}_{Lh}$	Class of all logharmonic mappings f in $\mathbb{D}$ of the form $f(z) = h(z)\overline{g(z)}$, $h, g \in \mathcal{H}(\mathbb{D})$, $h(0) = g(0) = 1$ satisfying $p(z) = h(z)g(z) \in \mathcal{P}_{\mathbb{R}}$	134
$\mathbb{R}$	Set of all real numbers	3
Re	Real part of a complex number	5

$R_b(\varphi)$	$\left\{f \in \mathcal{A} : 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z)\right\}$	88
$\mathcal{S}$	Class of all normalized univalent functions in $\mathcal{A}$	3
$\mathcal{S}_H$	Class of all normalized univalent and sense-preserving harmonic functions $f = h + \overline{g}$ in the unit disk $\mathbb{D}$	27
$\mathcal{S}_{Lh}$	Class of all normalized univalent logharmonic mappings	33
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	6
$\mathcal{S}_H^0$	The subclass of $\mathcal{S}_H$ consisting of functions $f = h + \overline{g}$ and $g'(0) = 0$	28
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	8
$\mathcal{ST}(\varphi)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}$	11
$\mathcal{ST}(\alpha, \varphi)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec \varphi(z)\right\}$	93
$\mathcal{ST}_\beta$	Class of strongly starlike functions of order β in $\mathcal{A}$	12
$\mathcal{ST}_L$	Class of *lemniscate of Bernoulli starlike* functions in $\mathcal{A}$	12
$\mathcal{ST}_p$	Class of parabolic starlike functions in $\mathcal{A}$	11
$\mathcal{ST}_{Lh}$	Class of all normalized univalent starlike logharmonic mappings	34
$\mathcal{SP}_{Lh}^{\alpha}$	Class of all univalent $\alpha-$spirallike logharmonic mappings	35
$\prec$	Subordinate to	10
T	Class of typically real functions in $\mathcal{A}$	13
TLh	Class of all logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ satisfying $\varphi(z) = zh(z)g(z) \in HG$ is typically real analytic in $\mathbb{D}$	131
$\mathcal{U}$	$\left\{f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1, \quad z \in \mathbb{D}\right\}$	39

JEJARI $\mathscr{U}$ DAN PENENTU HANKEL UNTUK FUNGSI ANALISIS DAN HASIL DARAB PEMETAAN LOGHARMONIK

ABSTRAK

Tesis ini mengkaji tentang ciri-ciri geometrik dan analisis bagi fungsi analisis bernilai kompleks dan pemetaan log-harmonik tertakrif dalam cakera unit terbuka $\mathbb{D}$. Terdapat empat permasalahan penyelidikan yang dikaji. Sebagai permulaan, andaikan $\mathscr{U}$ sebagai kelas yang terdiri daripada fungsi analisis ternormal f yang memenuhi syarat $|(z/f(z))^2 f'(z) - 1| < 1$. Semua fungsi $f \in \mathscr{U}$ adalah univalen. Bagi permasalahan yang pertama, jejari-$\mathscr{U}$ ditentukan untuk beberapa kelas fungsi analisis termasuk kelas fungsi analisis yang memenuhi ketidaksamaan $\operatorname{Re} f(z)/g(z) > 0$, atau $|f(z)/g(z) - 1| < 1$ dalam $\mathbb{D}$, untuk g yang terkandung dalam kelas fungsi analisis tertentu. Bagi kebanyakan kes, jejari-$\mathscr{U}$ yang tepat diperoleh. Konjektur oleh Obradovic dan Ponusamy berkenaan jejari univalen bagi hasil darab yang melibatkan fungsi univalen juga telah dibuktikan. Permasalahan kedua berkaitan dengan penentu Hankel bagi fungsi analisis. Bagi fungsi analisis ternormal f, andaikan $zf'(z)/f(z)$ atau $1 + zf''(z)/f'(z)$ subordinat kepada suatu fungsi analisis φ dalam $\mathbb{D}$. Andaikan juga F sebagai jelmaan punca ke-k, iaitu, $F(z) = z\left[f\left(z^k\right)/z^k\right]^{\frac{1}{k}}$. Batas atas terbaik dalam bentuk pekali bagi fungsi φ yang diberi diperoleh bagi penentu Hankel kedua F, yang f terkandung dalam salah satu kelas di atas. Anggaran bagi penentu Hankel bagi penjelmaan ke-k untuk kelas fungsi α-cembung dan α-cembung secara logaritma juga diperoleh. Dua permasalahan terakhir adalah berkait dengan pemetaan logaritma dalam $\mathbb{D}$. Pertama, bagi pemetaan log-harmonik bak-bintang $f(z) = zh(z)\overline{g(z)}$, syarat cukup didapati bagi $F(z) = f(z)|f(z)|^{2\gamma}$ agar menjadi pemetaan α-bak lingkaren log-harmonik.

Syarat cukup juga diperoleh bagi dua pemetaan logharmonik f_1 dan f_2 yang memastikan hasil darab $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z),\ \ 0 \leq \lambda \leq 1$, adalah pemetaan log-harmonik bak bintang. Beberapa contoh telah dibangunkan daripada hasil darab tersebut. Permasalahan seterusnya melihat pada pemetaan log-harmonik ternormal $f(z) = zh(z)\overline{g(z)}$ dimana $\varphi(z) = zh(z)g(z)$ adalah fungsi analisis nyata biasa dalam $\mathbb{D}$. Pewakilan kamiran bagi pemetaan sedemikian diterbitkan, dan anggaran bagi jejari bak-bintangnya didapati. Anggaran atas terbaik pada lengkok juga ditentukan. Syarat-syarat geometri cukup dan perlu bagi $\varphi(z) = zh(z)g(z)$ untuk menjadi nyata biasa juga dikaji apabila $f(z) = zh(z)\overline{g(z)}$ mempunyai dilatasi dengan pekali nyata.

THE $\mathscr{U}$-RADIUS AND HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS, AND PRODUCT OF LOGHARMONIC MAPPINGS

ABSTRACT

This thesis studies geometric and analytic properties of complex-valued analytic functions and logharmonic mappings in the open unit disk $\mathbb{D}$. It investigates four research problems. As a precursor to the first, let $\mathscr{U}$ be the class consisting of normalized analytic functions f satisfying $|(z/f(z))^2 f'(z) - 1| < 1$. All functions $f \in \mathscr{U}$ are univalent. In the first problem, the $\mathscr{U}$-radius is determined for several classes of analytic functions. These include the classes of functions f satisfying the inequality $\operatorname{Re} f(z)/g(z) > 0$, or $|f(z)/g(z) - 1| < 1$ in $\mathbb{D}$, for g belonging to a certain class of analytic functions. In most instances, the exact $\mathscr{U}$-radius are found. A recent conjecture by Obradović and Ponnusamy concerning the radius of univalence for a product involving univalent functions is also shown to hold true. The second problem deals with the Hankel determinant of analytic functions. For a normalized analytic function f, let $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ be subordinate to a given analytic function φ in $\mathbb{D}$. Further let F be its kth-root transform, that is, $F(z) = z\left[f\left(z^k\right)/z^k\right]^{\frac{1}{k}}$. A bound expressed in terms of the coefficients of the given function φ is obtained for the second Hankel determinant of F, where f belongs to either of the two classes above. Estimates for the Hankel determinant are also found for the kth-root transform of the class of α-convex functions and α-logarithmically convex functions. The final two studied problems studied relate to logharmonic mappings in $\mathbb{D}$. First, for a starlike logharmonic mapping $f(z) = zh(z)\overline{g(z)}$, sufficient conditions are obtained for $F(z) = f(z)|f(z)|^{2\gamma}$ to be $\alpha-$spirallike logharmonic mapping. In addition, sufficient

conditions are determined on two given logharmonic mappings f_1 and f_2 to ensure their product $F(z) = f_1^{\lambda}(z) f_2^{1-\lambda}(z),\ 0 \le \lambda \le 1$, is a univalent starlike logharmonic mapping. Several illustrative examples are constructed from this product. The latter problem looks at normalized logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ where $\varphi(z) = zh(z)g(z)$ is typically real analytic in $\mathbb{D}$. An integral representation for such mappings f is derived, and an estimate found on its radius of starlikeness. An upper estimate on arclength is also determined. Sufficient and necessary geometric conditions for $\varphi(z) = zh(z)g(z)$ to be typically real are also investigated when $f(z) = zh(z)\overline{g(z)}$ has a dilatation with real coefficients.

CHAPTER 1

INTRODUCTION

Geometric function theory is a branch of complex analysis with a long steeped history. It started in the early 20th century. Function theory studies geometric properties of complex-valued functions, and incorporate various tools from analysis.

This introductory chapter presents basic definitions and fundamental results important in the sequel. These are results on analytic functions, as well as on harmonic and log-harmonic mappings. It also serves to provide the motivations for the problems studied in the thesis.

1.1 Analytic Univalent Functions

In this thesis, the complex plane is denoted by $\mathbb{C}$. Further, let

$$\mathbb{D}(z_0, r) := \{z : z \in \mathbb{C}, |z - z_0| < r\}, \quad r > 0,$$

be the neighborhood of z_0. A set D of $\mathbb{C}$ is *open* if for every point $z_0 \in$ D, there is an $r > 0$ such that $\mathbb{D}(z_0, r) \subset$ D. An open set D is *connected* if there is a polygonal path in D joining any pair of points in D.

A *domain* D of $\mathbb{C}$ is an open connected set. A domain D is *simply connected* if the interior to every simple closed curve in D lies completely within D. Geometrically, a simply connected domain is a domain without any holes.

A complex-valued function f defined in D is *differentiable* at a point $z_0 \in$ D if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

exists. A function f defined in D is *analytic* at $z_0 \in$ D if it is differentiable in some neighbourhood of z_0. It is analytic in D if it is analytic at all points in D. It is known in [123, p. 167] that for $z \in \mathbb{D}(z_0, r) \subseteq$ D, an analytic function f in D has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n := \frac{f^{(n)}(z_0)}{n!}.$$

Denote by $\mathcal{H}(\mathbb{D})$ the class of all analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{A}$ denote the class of all normalized analytic functions f in $\mathcal{H}(\mathbb{D})$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function f is *univalent* in D if it is one-to-one in D. Thus f is univalent if it takes different points in D to different values, that is, for any two distinct points z_1 and z_2 with $z_1 \neq z_2$ in D, $f(z_1) \neq f(z_2)$. A function f is called *locally univalent* at z_0 if it is one-to-one in some neighbourhood of z_0. It is known in [38, p. 5] that the condition $f'(z_0) \neq 0$ is necessary and sufficient for local univalence at z_0.

A function that preserves both the magnitude and orientation of angles is said to be *conformal*. For an analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to it being conformal at z_0.

The Riemann mapping theorem is an important theorem in geometric function theory. It states that any simply connected domain which is not the entire complex plane, can be mapped conformally onto $\mathbb{D}$.

Theorem 1.1. (Riemann Mapping Theorem) [38, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique analytic and univalent function f which maps D onto the unit disk $\mathbb{D}$ satisfying $f(\zeta)=0$ and $f'(\zeta)>0$.*

Therefore, the study of conformal mappings on a simply connected domain can be confined to the study of functions that are analytic and univalent on the open unit disk $\mathbb{D}$.

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. An example is the function k given by

$$k(z)=\frac{z}{(1-z)^2}=\frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2-1\right]=\sum_{n=1}^{\infty}nz^n, \qquad z\in\mathbb{D}. \tag{1.2}$$

This function is known as the Koebe function, and it maps $\mathbb{D}$ onto the entire complex plane except for a slit along the half-line $(-\infty,-1/4]$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z)$, $\beta\in\mathbb{R}$, play an important role in the study of the class $\mathcal{S}$. These functions are extremal functions for various problems in the class $\mathcal{S}$.

In 1916, Bieberbach [30] conjectured the coefficients for $f(z)=\sum_{n=1}^{\infty}a_nz^n\in\mathcal{S}$ satisfy $|a_n|\leq n$. This conjecture is known as Bieberbach's conjecture. However, he only proved for the case when $n=2$, and this result is called the Bieberbach theorem.

Theorem 1.2. (Bieberbach theorem) [30] *Let $f\in\mathcal{S}$. Then*

$$|a_2|\leq 2.$$

Equality occurs if and only if f is a rotation of the Koebe function k.

In fact for many years, this conjecture has stood as a challenge to many mathematicians. The problem was resolved only for some initial values of n. Lowner [76] proved the Bieberbach conjecture for the case $n = 3$, followed by Garabedian and Schiffer [47] for $n = 4$. For $n = 6$, it was proved independently by Pederson [119] and Ozawa [116]. Pederson and Schiffer [118] proved the conjecture for $n = 5$. It was not until 1985 that de Branges [36] successfully proved the Bieberbach conjecture.

Theorem 1.3. (de Branges Theorem) [36] *The coefficients of each function* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$ *satisfy* $|a_n| \leq n$ *for* $n = 2, 3, \ldots$. *Equality occurs if and only if* f *is the Koebe function* k *or one of its rotations.*

Bieberbach theorem has significant implications in the theory of univalent functions. These include the well known covering theorem due to Koebe, which states the image of $\mathbb{D}$ under every $f \in \mathcal{S}$ must cover an open disk centered at the origin of radius $1/4$.

Theorem 1.4. (Koebe One-Quarter Theorem) [38, p. 31] *The range of every function of the class* $\mathcal{S}$ *contains the disk* $\{w : |w| < 1/4\}$.

One important consequence of the Bieberbach theorem is the distortion theorem which gives sharp bounds for $|f'(z)|$.

Theorem 1.5. (Distortion Theorem) [38, p. 32] *Let* $f \in \mathcal{S}$. *Then*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad |z| = r < 1.$$

Equality occurs if and only if f *is a suitable rotation of the Koebe function* k.

The growth theorem which results from the distortion theorem provides sharp bounds for $|f(z)|$.

Theorem 1.6. (Growth Theorem) [38, p. 33] *Let* $f \in \mathcal{S}$. *Then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \qquad |z| = r < 1.$$

Equality occurs if and only if f is a suitable rotation of the Koebe function k.

1.2 Subclasses of Analytic Univalent Functions

An important subclass of normalized analytic functions in the open unit disk $\mathbb{D}$ is the class of functions with positive real part.

Definition 1.1. (The class of functions with positive real part) [48, p. 78] *The class $\mathcal{P}$ consists of all analytic functions*

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \qquad z \in \mathbb{D}, \tag{1.3}$$

with

$$\text{Re } p(z) > 0, \qquad z \in \mathbb{D}.$$

An important example of a function in $\mathcal{P}$ is the *Möbius function*

$$m(z) := \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n$$

which maps $\mathbb{D}$ onto the half-plane $\{w : \text{Re } w > 0\}$. The role of this Möbius function m is similar to that of the Koebe function in the class $\mathcal{S}$.

The sharp coefficient bound for functions in the class $\mathcal{P}$ is given in the following result.

Lemma 1.1. (Carathéodory's Lemma) [38, p. 41] *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$. Then the following sharp estimate holds:*

$$|c_n| \leq 2, \quad (n = 1, 2, 3, \ldots).$$

Equality occurs for the Möbius function m or its rotations.

More generally, for $0 \leq \alpha < 1$, let $\mathcal{P}(\alpha)$ denote the class of analytic functions p of the form (1.3) with

$$\operatorname{Re} p(z) > \alpha, \quad z \in \mathbb{D}.$$

The class $\mathcal{P}$ is closely related to several subclasses of univalent functions. These include the important classes of convex and starlike functions. Geometric and analytic relationships between them will soon be made evident.

A set D in $\mathbb{C}$ is called *starlike with respect to an interior point* w_0 in D if the line segment joining w_0 to every other point w in D lies entirely in D. Analytically, this condition is equivalent to

$$(1-t)w_0 + tw \in \mathrm{D}$$

for every $w \in \mathrm{D}$, and $0 \leq t \leq 1$. In the case $w_0 = 0$, the set D is called starlike with respect to the origin, or simply a starlike domain.

Definition 1.2. (Starlike function) [48, p. 108] *A function $f \in \mathcal{A}$ is called a starlike function with respect to w_0 if it maps $\mathbb{D}$ onto a domain that is starlike with respect to w_0. In the particular case that $w_0 = 0$, f is called a starlike function.*

Denote by $\mathcal{ST}$ the subclass of $\mathcal{S}$ consisting of all starlike functions in $\mathbb{D}$. The following theorem gives an analytic description of the class $\mathcal{ST}$.

Theorem 1.7. (Analytical characterization of starlike functions) [38, p. 41] *Let $f \in \mathcal{A}$. Then $f \in \mathcal{ST}$ if and only if*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D}. \tag{1.4}$$

Thus $f \in \mathcal{ST}$ if and only if $zf'/f \in \mathcal{P}$. The Koebe function in (1.2) is an example of starlike function in $\mathbb{D}$. The sharp coefficient bound for $f \in \mathcal{ST}$ is given by the following result.

Theorem 1.8. [89] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{ST}$. *Then*

$$|a_n| \leq n, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the Koebe function k.

A set D in $\mathbb{C}$ is *convex* if it is starlike with respect to each of its points, that is, for every pair of points w_1 and w_2 lying in D, the line segment joining w_1 and w_2 also lies entirely in D. Analytically, this is equivalent to

$$tw_1 + (1-t)w_2 \in \mathrm{D}$$

for every pair $w_1, w_2 \in \mathrm{D}$, and $0 \leq t \leq 1$.

Definition 1.3. (Convex function) [48, p. 107] *A function* $f \in \mathcal{A}$ *is called a convex function if it maps* $\mathbb{D}$ *onto a convex domain.*

Denote by $\mathcal{CV}$ the subclass of $\mathcal{S}$ consisting of all convex functions in $\mathbb{D}$. The following is an analytic description of convex functions.

Theorem 1.9. (Analytical characterization of convex functions) [38, p. 42] *Let* $f \in \mathcal{A}$. *Then* $f \in \mathcal{CV}$ *if and only if*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \quad z \in \mathbb{D}. \tag{1.5}$$

The function

$$L(z) = \frac{z}{1-z} \tag{1.6}$$

which maps $\mathbb{D}$ onto the half-plane $\{w : \operatorname{Re}\, w > -1/2\}$ is a convex function and belongs to $\mathcal{CV}$. The following result gives sharp coefficient bound for the class $\mathcal{CV}$.

Theorem 1.10. [75] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{CV}$. *Then*

$$|a_n| \leq 1, \qquad (n = 2, 3, \ldots).$$

Equality occurs for all n when f is a rotation of the function L given by (1.6).

In 1915, Alexander [19] showed that there is a close connection between convex and starlike functions.

Theorem 1.11. [19] *Let* $f \in \mathcal{A}$. *Then f is convex in* $\mathbb{D}$ *if and only if* $zf'(z)$ *is starlike in* $\mathbb{D}$.

In 1936, Robertson [129] introduced the classes of *starlike and convex functions of order* α, $0 \leq \alpha < 1$. These are given by

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \ , \ z \in \mathbb{D} \right\},$$

and

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \ , \qquad z \in \mathbb{D} \right\},$$

respectively. In particular, $\mathcal{ST}(0) = \mathcal{ST}$ and $\mathcal{CV}(0) = \mathcal{CV}$. It is clear that

$$\mathcal{ST}(\alpha) \subseteq \mathcal{ST} \quad \text{and} \quad \mathcal{CV}(\alpha) \subseteq \mathcal{CV}.$$

A classical result of Strohhäcker [138] shows that $\mathcal{CV} \subset \mathcal{ST}(1/2)$.

A function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$ if there is a convex function g and a real number θ, $-\pi/2 < \theta < \pi/2$, such that

$$\operatorname{Re} \left(e^{i\theta} \frac{f'(z)}{g'(z)} \right) > 0 \quad z \in \mathbb{D}.$$

This set of functions, denoted by $\mathcal{CCV}$, was introduced by Kaplan [67] in 1952. The subclasses of $\mathcal{S}$, namely convex, starlike and close-to-convex functions are related as follows:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV}.$$

Indeed, a significant result in the theory of univalent functions is the Noshiro-Warschawski theorem. This theorem states that a function $f \in \mathcal{A}$ whose derivative has positive real part in $\mathbb{D}$ is univalent.

Theorem 1.12. (Noshiro-Warschawski Theorem) [103] *If a function f is analytic in a convex domain D and*

$$\operatorname{Re}\left(e^{i\alpha} f'(z)\right) > 0$$

for some real α, then f is univalent in D.

Using the Noshiro-Warschawski theorem, Kaplan [67] proved that every close-to-convex function is univalent, and thus $\mathcal{CCV} \subset \mathcal{S}$.

For $\alpha \geqslant 0$, a function $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$ is said to be an *α-convex* function if and only if

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0, \quad z \in \mathbb{D}.$$

This class of functions, denoted by M_α, was introduced by Mocanu *et al.*[86]. In 1973, Miller *et al.*[84] proved that functions in the class M_α are univalent and starlike in $\mathbb{D}$. They also showed that all α-convex functions are convex for $\alpha \geq 1$. Evidently, M_0 reduces to the class $\mathcal{ST}$ and M_1 reduces to the class $\mathcal{CV}$.

An analytic function $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$ and $1 + zf''(z)f'(z) \neq 0$ is said to be an $\alpha-$*logarithmically convex* function in $\mathbb{D}$ if and only if

$$\operatorname{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\right) > 0, \quad z \in \mathbb{D},$$

where $\alpha \in [0,1]$. This set of functions denoted by L_α was introduced by Lewandowski *et al.* [73]. Darus *et al* [35] proved that functions in this class are starlike. They also obtained bounds for $|a_2|$, $|a_3|$ and $|a_3 - \mu a_2|$, where μ is real. Some extreme coefficient problems are also solved. It is clear that L_0 reduces to the class $\mathcal{CV}$ and L_1 reduces to the class $\mathcal{ST}$.

An analytic function f is *subordinate* to g in $\mathbb{D}$, written $f(z) \prec g(z)$, if there exists an analytic function w in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In terms of subordination, the analytic conditions (1.4) and (1.5) can be written respectively as

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}, \tag{1.7}$$

and

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}. \tag{1.8}$$

This follows because the mapping $m(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane, and thus $\operatorname{Re}(m(z)) > 0$.

Ma and Minda [77] gave a unified presentation of various subclasses of starlike and convex functions by replacing the superordinate function $m(z) = (1+z)/(1-z)$ in (1.7) and (1.8) by a more general analytic univalent function φ which has positive real part in $\mathbb{D}$ and normalized by the conditions $\varphi(0) = 1$, $\varphi'(0) > 0$. Furthermore, it is assumed that $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$, and symmetric with respect to the real axis.

The class of *Ma-Minda starlike functions with respect to* φ, denoted by $\mathcal{ST}(\varphi)$, consists of functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$. This class can be written as

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}. \tag{1.9}$$

Similarly the class of *Ma-Minda convex functions with respect to* φ, denoted by $\mathcal{CV}(\varphi)$, consists of functions $f \in \mathcal{A}$ satisfying the subordination $1 + zf''(z)/f'(z) \prec \varphi(z)$. This class is

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \quad z \in \mathbb{D} \right\}.$$

Note that $f \in \mathcal{CV}(\varphi)$ if and only if $zf' \in \mathcal{ST}(\varphi)$.

The class of Ma-Minda starlike functions with respect to φ envelops several well-known subclasses of univalent functions by appropriate choices of φ in (1.9). For instance, when φ is given by

$$\varphi_\alpha(z) := \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + 2(1 - \alpha)z^2 + 2(1 - \alpha)z^3 + \cdots,$$

where $0 \le \alpha < 1$, then $\varphi_\alpha(\mathbb{D}) = \{w : \operatorname{Re} w > \alpha\}$. Therefore, the class of starlike functions of order α which satisfies the analytical condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_\alpha(z), \quad z \in \mathbb{D} \right\}.$$

For the choice

$$\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2}\left(\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \cdots,$$

Rønning [131] showed that φ_{PAR} maps $\mathbb{D}$ onto the parabolic region $\{w = u + iv : v^2 < 2u - 1\} = \{w : \operatorname{Re} w > |w - 1|\}$. Consequently, the class $\mathcal{ST}_P$ of *parabolic starlike* functions which satisfies the analytical condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}_P := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_{PAR}, \quad z \in \mathbb{D} \right\}.$$

With the choice

$$\varphi_\beta(z) := \left(\frac{1+z}{1-z}\right)^\beta = 1 + 2\beta z + 2\beta^2 z^2 + \frac{2}{3}\beta(1+2\beta^2)z^3 + \cdots, \quad 0 < \beta \leq 1,$$

it is evident that $\left|\arg \varphi_\beta(z)\right| = \beta \left|\arg\left((1+z)/(1-z)\right)\right| < \beta\pi/2$. Thus the class $\mathcal{ST}_\beta$ of *strongly starlike functions of order* β which satisfies the analytical condition [31]

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}_\beta := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_\beta, \quad z \in \mathbb{D} \right\}.$$

For the choice

$$\varphi_L := \sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 + \cdots,$$

it is clear that $\varphi_L(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$. Therefore, the class $\mathcal{ST}_L$ of *lemniscate of Bernoulli starlike* functions which satisfies the analytical condition [136]

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1, \quad z \in \mathbb{D},$$

can be expressed in the form

$$\mathcal{ST}_L := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_L, \quad z \in \mathbb{D} \right\}.$$

1.3 Analytic Typically Real Functions

An analytic function f is said to be *typically real* in $\mathbb{D}$ if it has real values on the real axis and nonreal values elsewhere. Therefore, typically real function maps the upper unit disk into either the upper half-plane or the lower half-plane, and similarly for the lower unit disk.

Denote by T the class consisting of typically real functions $f \in \mathcal{A}$. This class was introduced and investigated by Rogosinski [130].

For $f \in T$, by definition, f is real whenever z is real, that is, $f(z) = \overline{f(z)}$ for $z = x \in (-1,1)$. Thus $\sum_{n=2}^{\infty}(a_n - \overline{a_n})x^n = 0$, which yields $a_n = \overline{a_n}$. Hence f has real coefficients.

The converse does not hold, as illustrated by the function $f(z) = z + z^2 + 4z^3 \in \mathcal{A}$. It is clear that f has real coefficients. However, f is not typically real because $f(i/2) = -1/4$.

Furthermore, if $f \in T$, then $f'(0) > 0$, and thus near the origin f maps the upper unit disk into the upper half-plane, and the lower unit disk into the lower half-plane. Consequently,

$$(\operatorname{Im} z)(\operatorname{Im} f(z)) > 0, \quad z \in \mathbb{D}\backslash\mathbb{R},$$

when $f \in T$.

Proposition 1.1. *If $f \in \mathcal{S}$ has real coefficients, then $f \in T$.*

Proof. Since f has real coefficients, it follows that f is real whenever z is real. Suppose z is not real. Since f is univalent, it follows that $f(z) \neq f(\bar{z})$. Further f has real coefficients, that is, $f(\bar{z}) = \overline{f(z)}$. Therefore, $f(z) \neq \overline{f(z)}$, and thus $f(z)$ is not real. Hence $f(z)$ is real if and only if z is real, which yields the desired result. □

Note that a typically real function need not be univalent. For instance, let $f(z) = z+z^3$. Then f is not univalent in $\mathbb{D}$ since $f'(i/\sqrt{3}) = 0$. However,

$$\mathrm{Im}(f(z)) = \mathrm{Im}\left((x+iy)(1+x^2-y^2+2ixy)\right) = y(3x^2+1-y^2).$$

Thus

$$\mathrm{Im}(z)\mathrm{Im}(f(z)) = y^2\left(3x^2+(1-y^2)\right) > 0,$$

whenever Im $(z) \neq 0$, and hence f is typically real.

Let $\mathcal{P}_{\mathbb{R}}$ denote the class of all functions in $\mathcal{P}$ with real coefficients. The connection between functions in T and functions in $\mathcal{P}_{\mathbb{R}}$ was established by Rogosinski [130].

Theorem 1.13. [130] *A function $\varphi \in T$ if and only if there exists a function $p \in \mathcal{P}_{\mathbb{R}}$ such that $\varphi(z) = zp(z)/(1-z^2)$.*

1.4 The *k*th-root Transform

Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ with $f(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$. Further, let $k \geq 2$ be a fixed integer. The *kth-root transform* of f is defined by

$$F(z) := \left(f\left(z^k\right)\right)^{\frac{1}{k}} = z\left(\frac{f\left(z^k\right)}{z^k}\right)^{\frac{1}{k}}.$$

The following lemma is required to prove the univalence of the kth-root transform whenever $f \in \mathcal{S}$.

Lemma 1.2. [123, p. 142] *If f is analytic in $\mathbb{D}$ with $0 \notin f(\mathbb{D})$, then there exist an analytic function h in $\mathbb{D}$ and an integer $k \geq 2$ such that $h^k = f$.*

Proof. Since $f(z) \neq 0$ in $\mathbb{D}$, it follows that $f'(z)/f(z)$ is analytic in $\mathbb{D}$. By Cauchy's integral theorem [123, p. 139] , there exists a function $g \in \mathcal{H}(\mathbb{D})$ such that

$$g'(z) = \frac{f'(z)}{f(z)}. \tag{1.10}$$

Let $s(z) = f(z)\exp\{-g(z)\}$. It follows from (1.10) that

$$s'(z) = \big(f'(z) - g'(z)f(z)\big)\exp\{-g(z)\} = 0.$$

Then for a fixed $z_0 \in \mathbb{D}$,

$$s(z) - s(z_0) = \int_{z_0}^{z} s'(\zeta)d\zeta = 0,$$

and thus $f(z)\exp\{-g(z)\} = s(z_0)$. As $s(z_0) \neq 0$ in $\mathbb{D}$, we can let $s(z_0) = \exp\{m\}$ for some m. Then $f(z)\exp\{-g(z)\} = \exp\{m\}$, that is, $f(z) = \exp\{g(z)+m\} = \exp\{G(z)\}$, where $G(z) = g(z) + m$. Hence the proof is completed by taking $h(z) = \exp\{G(z)/k\}$ for every $z \in \mathbb{D}$. □

The following result shows that the kth-root transform preserves univalence.

Theorem 1.14. *Let $f \in \mathcal{S}$ and $g(z) = (f(z^k))^{1/k}$ be the kth-root transformation of f. Then $g \in \mathcal{S}$. The branch is chosen so that $\big(f(z^k)/z^k\big)^{1/k} = 1$ at $z = 0$.*

Proof. Since $f \in \mathcal{S}$, it follows that $f(z)/z$ is a nonvanishing analytic function. By applying Lemma 1.2, there exist an analytic function h and an integer $k \geq 2$ such that $h^k(z) = f(z)/z$. Let

$$g(z) = z\left(\frac{f(z^k)}{z^k}\right)^{\frac{1}{k}} = zh(z^k).$$

Since $f\left(z^k\right)/z^k = 1 + \sum_{n=2}^{\infty} a_n z^{k(n-1)} := 1 + x$, and

$$\begin{aligned}
(1+x)^{\frac{1}{k}} &= \sum_{n=0}^{\infty} \frac{(-1/k)_n}{n!}(-x)^n \\
&= 1 + \frac{1}{k}x + \frac{\left(\frac{-1}{k}\right)\left(\frac{-1}{k}+1\right)}{2!}x^2 - \frac{\left(\frac{-1}{k}\right)\left(\frac{-1}{k}+1\right)\left(\frac{-1}{k}+2\right)}{3!}x^3 + \cdots \\
&= 1 + \frac{1}{k}x - \frac{(k-1)}{2k^2}x^2 + \frac{(k-1)(2k-1)}{3!k^3}x^3 + \cdots,
\end{aligned}$$

it follows that

$$g(z) = z\left(\frac{f\left(z^k\right)}{z^k}\right)^{\frac{1}{k}} = z\left(1 + \frac{1}{k}\sum_{n=2}^{\infty} a_n z^{k(n-1)} - \frac{(k-1)}{2k^2}\left(\sum_{n=2}^{\infty} a_n z^{k(n-1)}\right)^2 + \cdots\right)$$

$$= z\left(1+\frac{1}{k}\sum_{n=1}^{\infty} a_{n+1}z^{nk} - \frac{(k-1)}{2k^2}\left(\sum_{n=1}^{\infty} a_{n+1}z^{nk}\right)^2 + \cdots\right).$$

Thus g is normalized with $g(0) = 0$ and $g'(0) = 1$.

Suppose $z_1, z_2 \in \mathbb{D}$ such that $g(z_1) = g(z_2)$. Then $g^k(z_1) = g^k(z_2)$, and thus $f(z_1^k) = f(z_2^k)$. The univalence of f in $\mathbb{D}$ implies that $z_1^k = z_2^k$, and hence, there exists $\beta \in \mathbb{C}$, $\beta^k = 1$, such that $z_2 = \beta z_1$. If $\beta = 1$, then $z_2 = z_1$. Assume that $\beta \neq 1$. It follows that

$$g(z_2) = g(\beta z_1) = \beta z_1 h(\beta^k z_1^k) = \beta z_1 h(z_1^k) = \beta g(z_1) = \beta g(z_2),$$

and thus $(1-\beta)g(z_2) = 0$. Since $\beta \neq 1$, it yields that $g(z_2) = 0$, that is, $z_2 = 0$. Furthermore, $g(z_1) = g(z_2)$ implies $z_1 = 0$, and hence $z_2 = z_1 = 0$. This completes the proof. □

The next result shows that the kth-root transform preserves starlikeness.

Theorem 1.15. *Let $g(z) = (f(z^k))^{1/k}$ be the kth-root transformation of f. Then $g \in \mathcal{ST}$ if and only if $f \in \mathcal{ST}$.*

Proof. It is clear that for each $z \in \mathbb{D}$

$$\frac{zg'(z)}{g(z)} = z^k \frac{f'(z^k)}{f(z^k)}.$$

Thus

$$\mathrm{Re}\left(\frac{zg'(z)}{g(z)}\right) = \mathrm{Re}\left(z^k \frac{f'(z^k)}{f(z^k)}\right), \quad z \in \mathbb{D},$$

and hence g is starlike if and only if f is starlike. □

The following result shows that the convexity of the kth-root transform of f implies convexity of f. However, the converse does not hold.

Theorem 1.16. *Let $g(z) = (f(z^k))^{1/k}$ be the kth-root transformation of f and $g \in \mathcal{CV}$. Then $f \in \mathcal{CV}$. However, the converse is false.*

Proof. Evidently,

$$g'(z) = g(z)\left(\frac{z^{k-1}f'(z^k)}{f(z^k)}\right). \tag{1.11}$$

Thus

$$1+\frac{zg''(z)}{g'(z)} = \frac{zg'(z)}{g(z)} + k\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) - k\left(\frac{z^k f'(z^k)}{f(z^k)}\right).$$

From (1.11), the above equality is equivalent to

$$1+\frac{zg''(z)}{g'(z)} = \frac{zg'(z)}{g(z)} + k\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) - k\frac{zg'(z)}{g(z)}. \tag{1.12}$$

It follows that

$$\operatorname{Re}\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) = \frac{1}{k}\left(\operatorname{Re}\left(1+\frac{zg''(z)}{g'(z)}\right) + (k-1)\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)\right),$$

and hence if g is convex, then f is convex.

On the other hand, $f(z) = z/(1-z)$ is a convex function such that for $z \in \mathbb{D}$,

$$\begin{aligned} k\ \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) - (k-1)\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) &= k\ \operatorname{Re}\left(\frac{1+z}{1-z}\right) - (k-1)\operatorname{Re}\left(\frac{1}{1-z}\right) \\ &< k\ \operatorname{Re}\left(\frac{1+z}{1-z}\right) - \frac{(k-1)}{2}. \end{aligned}$$

By taking $k=2$ and $z_0 = \sqrt{3/5}i \in \mathbb{D}$, it is evident that $z_0^2 \in \mathbb{D}$, and

$$2\ \operatorname{Re}\left(1+\frac{z_0^2 f''(z_0^2)}{f'(z_0^2)}\right) - \operatorname{Re}\left(\frac{z_0^2 f'(z_0^2)}{f(z_0^2)}\right) < 2\ \operatorname{Re}\left(\frac{1+z_0^2}{1-z_0^2}\right) - \frac{1}{2} = \frac{1}{2} - \frac{1}{2} = 0. \tag{1.13}$$

Equations (1.11) and (1.12) show that

$$\operatorname{Re}\left(1+\frac{zg''(z)}{g'(z)}\right) = k\ \operatorname{Re}\left(1+\frac{z^k f''(z^k)}{f'(z^k)}\right) - (k-1)\operatorname{Re}\left(\frac{z^k f'(z^k)}{f(z^k)}\right).$$

Thus, it follows from (1.13) that g is not convex. □

The kth-root transform has been widely used in a variety of ways in complex function theory. Bounds for the Fekete-Szegö coefficient functional associated with kth-root transform $(f(z^k))^{1/k}$ of normalized analytic functions f were derived in [20]. Annamalai *et al.* [25] obtained a bound of the Fekete Szegö coefficient functional for the Janowski α-Spirallike functions associated with the kth-root root transformation.

1.5 The Second Hankel Determinant

For positive integers q and n, the Hankel determinant $H_q(n)$ for an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.14}$$

Hankel determinants play an important role in the study of singularities. For instance, Dienes [37, p.333] showed that if the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has at most p poles and no other singularities on the circumference of its circle of convergence, then $\lim_{n\to\infty}\left|\sqrt[n]{H_p(n)}\right| = 1$. Furthermore, Hankel determinants are useful in the study of a function of bounded characteristic. For example, Cantor [32] proved that if the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a ratio of two bounded analytic functions in $\mathbb{D}$, then

$\lim_{q\to\infty} H_{q+1}(n) = 0$.

The growth rate of Hankel determinant $H_q(n)$ as $n \to \infty$ was obtained by Pommerenke [121]. Various authors [92, 94] and [100] have investigated the growth rate of Hankel determinant $H_q(n)$ for a certain subclass of analytic functions by essentially following Pommerenke's method.

Pommerenke [122] proved that Hankel determinants of univalent functions satisfy

$$|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1,2,\ldots,\ q = 2,3,\ldots),$$

where $\beta > 1/4000$ and K depends only on q.

Hankel determinants have also been discussed for several subclasses of analytic functions by many authors. For instance, in the works by Ehrenborg [42], Layman [71], Noor [95, 96, 97, 98, 99], Noor and Al-Bany [101] and Noor [102]. The Hankel determinant of meromorphic functions was obtained in [142]. Various properties of these determinants can be found in [141, Chapter 4].

It is evident that $H_2(1) = a_3 - a_2^2$ is the Fekete-Szegö coefficient functional for $f \in \mathcal{A}$. Interestingly the determinant also satisfies $H_2(1) = S_f(0)/6$, where S_f is the Schwarzian derivative of f defined in [33] by $S_f = (f''/f')' - (f''/f')^2/2$. Ali *et al.* [20] investigated the Fekete-Szegö coefficient functional for the kth-root transform of functions belonging to several classes defined via subordination.

In recent years, several authors have investigated bounds for the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ of functions belonging to various subclasses of univalent and multivalent functions. For example, Elhosh obtained bounds for the second Hankel determinant of univalent functions and close-to-convex functions respectively in [43, 44]. In addition, Halim *et al.* [53, 63] and [64] obtained bounds for the second

Hankel determinant for certain subclasses of analytic functions. Singh [134] established a bound for the second Hankel determinant for analytic functions with respect to other points. Moreover, Lee *et al.* [72] investigated bounds for the second Hankel determinant for functions belonging to subclasses of Ma-Minda starlike and convex functions and two other related classes defined by subordination.

Hayami and Owa [55, 57] obtained a bound for the generalized functional $|a_n a_{n+2} - \mu a_{n+1}^2|$ by using the Hankel determinant $H_2(n)$ for all $n \geq 1$ and some real number μ for several subclasses of $\mathcal{A}$. These authors [54] also studied a bound for the functional $|a_{p+2} - \mu a_{p+1}^2|$ for p-valent analytic functions. They also obtained a bound for the functional $|a_{p+1}a_{p+3} - \mu a_{p+2}^2|$ for p-valent analytic functions in [56]. Similar study of finding bounds for other classes of p-valent analytic functions was discussed in [140].

1.6 Radius Problems

Another active topic of investigation in the theory of univalent functions is the radius problem. Although not all analytic functions $f \in \mathcal{A}$ are univalent in the unit disk, for z near to the origin, the behavior of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is similar to the identity map. Therefore, f maps a sufficiently small disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ univalently onto some domain. The radius of the largest disk in $\mathbb{D}$ where f is univalent is called the radius of univalence for f. For instance, the function $f(z) = z + 3z^2 \in \mathcal{A}$ is not locally univalent at $z_0 = -1/6$ since $f'(-1/6) = 0$. However, the Noshiro-Warschawski result shows that the function f is univalent in the disk $|z| < 1/6$. Thus *the radius of univalence* for the function $f(z) = z + 3z^2$ is $r_0 = 1/6$.

Similarly, every univalent function $f \in \mathcal{A}$ is not necessarily starlike in the unit disk. However, we can find a sufficiently small disk $\mathbb{D}_r$ such that f maps $\mathbb{D}_r$ onto a starlike domain. The radius of the largest disk with this property is called the radius of starlikeness for f. Let $\mathcal{F} \subset \mathcal{A}$ be a set of analytic functions. The radius of the largest disk in $\mathbb{D}$ such that every function $f \in \mathcal{F}$ maps the disk onto a starlike domain is called *the radius of starlikeness* for the class $\mathcal{F}$. If $r_{\mathcal{ST}}$ is the radius of starlikeness for the class $\mathcal{F}$, then equivalently $r^{-1}f(rz) \in \mathcal{ST}$ for $r \leq r_{\mathcal{ST}}$, and $f \in \mathcal{F}$. In particular, if $\mathcal{F} = \mathcal{S}$, then the radius of starlikeness for the class $\mathcal{S}$ is $r_{\mathcal{ST}} = \tanh(\pi/4) \approx 0.65579$ [52].

Analogously, the radius of the largest disk in $\mathbb{D}$ such that every function $f \in \mathcal{F}$ maps the disk onto a convex domain is called *the radius of convexity* for the class $\mathcal{F}$. If $r_{\mathcal{CV}}$ is the radius of convexity for the class $\mathcal{F}$, then equivalently $r^{-1}f(rz) \in \mathcal{CV}$ for $r \leq r_{\mathcal{CV}}$, and $f \in \mathcal{F}$. It is known [90] that the *radius of convexity* for the class $\mathcal{S}$ is $r_{\mathcal{CV}} = 2 - \sqrt{3} \approx 0.26795$.

Let $\mathcal{F} \subset \mathcal{A}$ be a set of analytic functions, and let $\mathcal{U}$ be the class of functions $f \in \mathcal{A}$ satisfying $|(z/f(z))^2 f'(z) - 1| < 1$ for $z \in \mathbb{D}$. Then every analytic function $f \in \mathcal{F}$ is not necessarily in the class $\mathcal{U}$. However, we can find a sufficiently small disk $\mathbb{D}_r$ such that f satisfies $|(z/f(z))^2 f'(z) - 1| < 1$ in the disk $\mathbb{D}_r$. The radius of the largest disk in $\mathbb{D}$ such that every function $f \in \mathcal{F}$ satisfies $r^{-1}f(rz) \in \mathcal{U}$ is called the $\mathcal{U}$-radius for the class $\mathcal{F}$ and denoted by $r_{\mathcal{U}}$.

In general, for two families $\mathcal{G}$ and $\mathcal{F}$ of $\mathcal{A}$, the *$\mathcal{G}$-radius* for the class $\mathcal{F}$, denoted by $R_{\mathcal{G}(\mathcal{F})}$, is the largest number R such that $r^{-1}f(rz) \in \mathcal{G}$ for $0 < r \leq R$, and $f \in \mathcal{F}$.

The radius of close-to-convexity for the class $\mathcal{S}$ was determined by Krzyż [69]. Several authors have investigated the problem of finding the radius constants for subclasses of $\mathcal{A}$. For instance, Ali *et al.* [23] obtained radius constants for several classes of analytic functions on the unit disk $\mathbb{D}$ which includes the radius of starlikeness of positive order, radius of parabolic starlikeness, radius of Bernoulli lemniscate starlikeness, and radius of uniform convexity. Some results of radius problems have also been derived by Goodman [48, Chapter 13].

1.7 Harmonic Mappings

Let D be a domain in $\mathbb{R}^2$. A real-valued function $u : \mathrm{D} \longrightarrow \mathbb{R}$ is called *harmonic* if all its second partial derivatives exist and are continuous in D, and satisfies the Laplacian equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A complex-valued function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ in a domain D is harmonic if the two coordinate functions u and v are real harmonic in D. Thus a complex-valued harmonic function f satisfies Laplacian equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Since $z = x + iy$, it follows that

$$x = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \frac{z-\bar{z}}{2i}.$$

By using the chain rule, it is evident that

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \tag{1.15}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1.16}$$

Consequently, for a complex valued function $w = u + iv$ with continuous partial derivatives, it is clear that

$$\begin{aligned}\frac{\partial \overline{w}}{\partial z} &= \frac{1}{2}\big((u_x - iv_x) - i(u_y - iv_y)\big) = \frac{1}{2}\big((u_x - v_y) - i(u_y + v_x)\big) \\ &= \overline{\frac{1}{2}\big((u_x - v_y) + i(u_y + v_x)\big)} = \overline{\left(\frac{\partial w}{\partial \overline{z}}\right)},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \overline{w}}{\partial \overline{z}} &= \frac{1}{2}\big((u_x - iv_x) + i(u_y - iv_y)\big) = \frac{1}{2}\big((u_x + v_y) + i(u_y - v_x)\big) \\ &= \overline{\frac{1}{2}\big((u_x + v_y) - i(u_y - v_x)\big)} = \overline{\left(\frac{\partial w}{\partial z}\right)}.\end{aligned}$$

Since

$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}\big((u_x + iv_x) + i(u_y + iv_y)\big) = \frac{1}{2}\big((u_x - v_y) + i(u_y + v_x)\big),$$

it follows from the Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$, that a function f is analytic in a domain D if and only if $f_{\overline{z}} = 0$, that is, f is independent of $\overline{z}$.

A direct calculation shows that the Laplacian of f becomes

$$\begin{aligned}\Delta f = f_{xx} + f_{yy} = f_{xx} - if_{yx} + if_{xy} + f_{yy} &= (f_x - if_y)_x + i(f_x - if_y)_y \\ &= 4\left(\frac{f_x - if_y}{2}\right)_{\overline{z}} = 4f_{z\overline{z}}.\end{aligned}$$

Thus f is harmonic if and only if f_z is analytic.

Proposition 1.2. *Let f be a harmonic function in a domain* D. *Then the composition $f \circ \psi$ is harmonic in Ω for any analytic function $\psi : \Omega \longrightarrow$* D.

Proof. Setting

$$F(z) = (f \circ \psi)(z) = f(\psi(z)) = f(w),$$

then

$$F_z(z) = \frac{\partial f(w)}{\partial w}\frac{\partial w(z)}{\partial z} + \frac{\partial f(w)}{\partial \overline{w}}\frac{\partial \overline{w(z)}}{\partial z}.$$

Since w is analytic, it follows that $\partial\overline{w(z)}/\partial z = \overline{(\partial w(z)/\partial \overline{z})} = 0$. Thus

$$F_z(z) = \frac{\partial f(w)}{\partial w}\frac{\partial w(z)}{\partial z}.$$

Also, f is harmonic. Then

$$\begin{aligned} F_{z\overline{z}}(z) &= \frac{\partial}{\partial \overline{z}}\left(\frac{\partial f(w)}{\partial w}\frac{\partial w(z)}{\partial z}\right) = \left(\frac{\partial^2 f(w)}{\partial w^2}\frac{\partial w(z)}{\partial \overline{z}} + \frac{\partial^2 f(w)}{\partial \overline{w}\partial w}\frac{\partial \overline{w(z)}}{\partial \overline{z}}\right)\frac{\partial w}{\partial z} + \frac{\partial f(w)}{\partial w}\left(\frac{\partial^2 w(z)}{\partial \overline{z}\partial z}\right) \\ &= \frac{\partial^2 f(w)}{\partial \overline{w}\partial w}\frac{\partial w(z)}{\partial z}\frac{\partial \overline{w(z)}}{\partial \overline{z}} = f_{w\overline{w}}(w)\frac{\partial w(z)}{\partial z}\overline{\left(\frac{\partial w(z)}{\partial z}\right)} = f_{w\overline{w}}(w)\left|\frac{\partial w(z)}{\partial z}\right|^2 = 0. \end{aligned}$$

Hence the composition $f \circ \psi$ is harmonic. □

However, if f is a harmonic function and ψ is analytic, then $\psi \circ f$ need not be harmonic. For instance, $F(z) = \psi \circ f = (z + \overline{z}/2)^2$, where $f(z) = z + \overline{z}/2$ and $\psi = z^2$. It is evident that $f_{z\overline{z}} = 0$, but $F_{z\overline{z}} = 1$. Thus $\psi \circ f$ is not harmonic.

A mapping is said to be sense-preserving if it preserves the orientation, or sense of the angle between two curves. A sense-preserving mapping does not necessarily preserve the magnitude of the angle between the intersecting curves.

The Jacobian of a function $f(z) = u(x,y) + iv(x,y)$ at a point z is given by

$$J_f(z) := \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

If f is an analytic function, then f satisfies the Cauchy Riemann equations, and thus its Jacobian has the following form

$$J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2.$$