Can Magnetic Forces Do Work?

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Standard lore holds that magnetic forces are incapable of doing mechanical work. More precisely, the claim is that whenever it appears that a magnetic force is doing work, the work is actually being done by another force, with the magnetic force serving only as an indirect mediator. On the other hand, the most familiar instances of magnetic forces acting in everyday life—bar magnets lifting other bar magnets—appear to present manifest evidence of magnetic forces doing work. These sorts of counterexamples are often dismissed as arising from quantum effects that lie outside the classical regime. In this paper, however, we show that quantum theory is not needed to account for these phenomena, and that classical electromagnetism admits a model of elementary magnetic dipoles on which magnetic forces can indeed do work. In order to develop this model, we revisit the foundational principles of the classical theory of electromagnetism, showcase the importance of constraints from relativity, examine the structure of the multipole expansion, and study the connection between the Lorentz force law and conservation of energy and momentum.

I. INTRODUCTION

The question as to whether magnetic forces can do mechanical work presents a marvelous opportunity for exploring basic definitions in analytical mechanics and the fundamental structure of classical electromagnetism. In this paper, which builds off of [1], we show that classically extending Maxwell's theory of electromagnetism to include *elementary* dipoles—meaning dipole moments that are permanent and intrinsic—allows magnetic forces to do work.

We start by carefully reviewing the relevant ingredients of classical mechanics, including the precise definition of mechanical work as well as the Lagrangian formulation and its generalizations. We then turn to a detailed study of electric and magnetic multipole moments in special relativity. Next, extending the work of [2–4], we couple the electromagnetic field to a classical relativistic particle with intrinsic spin and elementary electric and magnetic dipole moments, derive the particle's equations of motion as well as the overall system's energy-momentum tensor and its angular-momentum flux tensor, and show both from the equations of motion and from local conservation of energy and momentum that magnetic forces can do work on the particle if its elementary magnetic dipole moment is nonzero. We also provide a new, classical argument for why a particle's elementary dipole moments must be collinear with its spin axis.

A. Mechanical Preliminaries

Recall that the net force $\mathbf{F}$ on a mechanical object is equal to the instantaneous rate at which the object's momentum $\mathbf{p}$ changes with time t:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \tag{1}$$

Let m be the object's inertial mass, let $\mathbf{X}$ be its position vector, and let $\mathbf{v} \equiv d\mathbf{X}/dt$ be its velocity. In the Newtonian case, the object's momentum is related to its velocity according to

$$\mathbf{p} \equiv m\mathbf{v} \quad \text{[Newtonian]}, \tag{2}$$

meaning that under the assumption that m is constant, (1) becomes Newton's second law,

$$\mathbf{F} = m\mathbf{a}, \tag{3}$$

with $\mathbf{a} \equiv d\mathbf{v}/dt$ the object's acceleration.

The object's kinetic energy is

$$T \equiv \frac{1}{2}m\mathbf{v}^2 = \frac{\mathbf{p}^2}{2m} \quad \text{[Newtonian]}. \tag{4}$$

A simple calculation then shows that the rate of change in the kinetic energy of an object of constant mass m is given by the dot product of the object's velocity $\mathbf{v}$ and the force $\mathbf{F}$:

$$\frac{dT}{dt} = \mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{v} \cdot \mathbf{F} = \frac{d\mathbf{X}}{dt} \cdot \mathbf{F}. \tag{5}$$

B. The Definition of Mechanical Work

By definition, we say that a given force does mechanical work on a classical object if the object moves through space and the vector representing the force has a nonzero component along the object's path.

More precisely, the work W done by the force on the object is the dot product of the force vector $\mathbf{F}$ and the object's incremental displacement vector $d\mathbf{X}$, integrated

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over the total displacement from the object's initial location A to its final location B:

$$W \equiv \int_A^B d\mathbf{X} \cdot \mathbf{F}. \tag{6}$$

Assuming for simplicity that $\mathbf{F}$ is the only force doing work on the object and integrating the relation (5) over the time duration of the object's trajectory, we can use the fundamental theorem of calculus to obtain the work-energy theorem,

$$W = \Delta T, \tag{7}$$

which establishes that the work W done by the force $\mathbf{F}$ on the object translates into an overall change ΔT in the object's kinetic energy T.

As a different question, one may ask whether a given force $\mathbf{F}$ acting on an object arises from some other source of energy, and, if so, what that energy is and where it comes from. The simplest example is provided by a conservative force, which is a force on an object that is a function $\mathbf{F}(\mathbf{X})$ only of the object's instantaneous position $\mathbf{X}$ and with the additional property that any work (6) done by the force only ever depends on the endpoints A and B of whatever arbitrary path the object takes.

Forces that do work need not be conservative, as dissipative forces like friction make clear. On the other hand, conservative forces need not do work, such as a conservative force that acts centripetally on an object and is therefore always perpendicular to the object's motion, meaning that it has an always-vanishing dot product $d\mathbf{X} \cdot \mathbf{F}(\mathbf{X}) = 0$ with the object's incremental displacements $d\mathbf{X}$.

Given a conservative force $\mathbf{F}(\mathbf{X})$, if we replace the upper limit of integration in the definition (6) of W with a variable position $\mathbf{X}$, then the result is a well-defined function of $\mathbf{X}$ that, together with an overall minus sign, defines the object's potential energy $V(\mathbf{X})$ due to that force,

$$V(\mathbf{X}) \equiv -\int^{\mathbf{X}} d\mathbf{X}' \cdot \mathbf{F}(\mathbf{X}'), \tag{8}$$

where we neglect the lower limit of integration because it merely determines an irrelevant additive constant. Taking the gradient of both sides of this definition (8) of $V(\mathbf{X})$, we see that we can express a conservative force as the negative gradient of its corresponding potential energy:

$$\mathbf{F}(\mathbf{X}) = -\nabla V(\mathbf{X}). \tag{9}$$

Once again assuming for simplicity that $\mathbf{F}(\mathbf{X})$ is the only force acting on the object, and combining the integral definition (6) of the work done together with the relationship (9) between the force and its potential energy, we see that the work W done by the force on the object is equal to the overall change ΔV in the object's potential energy:

$$W = -\Delta V. \tag{10}$$

It follows from the work-energy theorem (7), $W = \Delta T$, that the sum of the change ΔT in the object's kinetic energy and the change ΔV in the object's potential energy is zero:

$$\Delta T + \Delta V = \Delta(T + V) = 0. \tag{11}$$

We therefore conclude that there exists an associated conserved total energy E:

$$E = T + V = \text{constant}. \tag{12}$$

Indeed, taking the time derivative of E and using (5) to calculate dT/dt together with the chain rule to calculate dV/dt, we have

$$\begin{aligned} \frac{dE}{dt} &= \frac{dT}{dt} + \frac{dV}{dt} \\ &= \mathbf{v} \cdot \mathbf{F} + \frac{d\mathbf{X}}{dt} \cdot \nabla V \\ &= \mathbf{v} \cdot \mathbf{F} + \mathbf{v} \cdot (-\mathbf{F}) = 0. \end{aligned} \tag{13}$$

C. The Maxwell Equations and the Lorentz Force Law

We next review the fundamentals of the classical theory of electromagnetism, taking the opportunity to establish the various conventions that we will be using in this paper [5].

Working in SI units, we let ϵ_0 and μ_0 respectively denote the permittivity of free space and the permeability of free space. We use $\mathbf{E} = (E_x, E_y, E_z)$ for the electric field, $\mathbf{B} = (B_x, B_y, B_z)$ for the magnetic field, ρ for the volume density of electric charge, and $\mathbf{J} = (J_x, J_y, J_z)$ for the current density or charge flux density, meaning the rate of charge flow per unit time per unit cross-sectional area. We can then write down the four Maxwell equations in their standard form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{14}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{15}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{16}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{17}$$

We will respectively call these the electric Gauss equation, the magnetic Gauss equation, the Faraday equation, and the Ampère equation. The first and fourth equations contain the source functions ρ and $\mathbf{J}$ and are called the inhomogeneous Maxwell equations, whereas the second and third equations do not involve source functions and are called the homogeneous Maxwell equations. Note that ϵ_0, μ_0, and the speed of light c are related by

$$\frac{1}{\sqrt{\epsilon_0 \mu_0}} = c. \tag{18}$$

The Maxwell equations tell us how charged sources generate electric and magnetic fields. The fields, in turn, cause changes to the motion of those charged sources. To provide a precise formulation of this latter statement, one traditionally supplements the Maxwell equations with an additional axiom called the Lorentz force law, whose textbook form expresses the force $\mathbf{F}$ on a particle of charge q and velocity $\mathbf{v}$ due to an external electric field $\mathbf{E}_{\text{ext}}$ and an external magnetic field $\mathbf{B}_{\text{ext}}$ as

$$\mathbf{F} = q\mathbf{E}_{\text{ext}} + q\mathbf{v} \times \mathbf{B}_{\text{ext}}, \tag{19}$$

where the electric and magnetic forces on the particle are therefore given individually by

$$\mathbf{F}_{\text{el}} = q\mathbf{E}_{\text{ext}}, \tag{20}$$

$$\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}_{\text{ext}}. \tag{21}$$

Note that the particle's velocity $\mathbf{v}$ is assumed to be constant here to avoid complications involving radiation and backreactive self-forces.

D. Models of Magnetic Dipoles

We will eventually show that magnetic forces can do work on certain kinds of magnetic dipoles. First, however, we should take a moment to explain why this claim might be in doubt.

According to the usual Ampère model, classical magnetic dipoles are *composite* entities consisting of charged particles—that is, electric monopoles—moving around in current loops. For such a composite magnetic dipole, the textbook Lorentz force law (19) makes clear that magnetic forces cannot do work. The simple reason is that the magnetic force $\mathbf{F}_{\text{mag}}$ on each electric monopole in a given current loop is proportional to the cross product $\mathbf{v} \times \mathbf{B}_{\text{ext}}$ of the particle's velocity $\mathbf{v} \equiv d\mathbf{X}/dt$ and the external magnetic field $\mathbf{B}_{\text{ext}}$, so the magnetic force $\mathbf{F}_{\text{mag}}$ is always perpendicular to the particle's incremental displacements $d\mathbf{X}$. By its definition (6), $W = \int d\mathbf{X} \cdot \mathbf{F}$, work is equal to the dot product of force and incremental displacement, integrated over the full displacement. Because $d\mathbf{X} \cdot \mathbf{F}_{\text{mag}} = 0$, the work done by the magnetic force in this context always vanishes [6].

Notice also that the magnetic force $\mathbf{F}_{\text{mag}} = q\mathbf{v} \times \mathbf{B}_{\text{ext}}$ on electric monopoles is explicitly velocity-dependent, and so cannot represent a conventionally conservative force. By contrast, the electric force $\mathbf{F}_{\text{el}}(\mathbf{X}) = q\mathbf{E}_{\text{ext}}(\mathbf{X})$ due to a time-independent electric field $\mathbf{E}_{\text{ext}}(\mathbf{X})$ depends only on the electric monopole's position $\mathbf{X}$, and the static version of the Faraday equation (16), $\nabla \times \mathbf{E} = 0$, ensures that the electric force $\mathbf{F}_{\text{el}}$ is expressible in terms of a potential energy V as $\mathbf{F} = -\nabla V$, in keeping with (9), so the static electric force on an electric monopole is conservative.

One could, in principle, evade the preceding conclusions about magnetic forces by considering composite magnetic dipoles according to the Gilbert model, in which the magnetic dipoles instead consist of pairs of fundamental magnetic *monopoles*. However, employing the Gilbert model would require generalizing Maxwell's theory of electromagnetism to include magnetic monopoles, as well as generalizing the Lorentz force law accordingly to describe forces acting on them.

On the other hand, experiments indicate that many kinds of particles, including electrons, possess permanent, elementary magnetic dipole moments that do not seem to arise from underlying classical loops of current or as pairs of magnetic monopoles. At a truly fundamental level, these elementary magnetic dipole moments are quantum-mechanical in nature, but, then, so is electric charge, and we obviously still include electric charges as basic sources in Maxwell's classical theory of electromagnetism.

It is therefore worth studying how we might similarly include elementary dipoles as basic sources in a classical extension of Maxwell's theory of electromagnetism, as well as determine from first principles how they should interact with electric and magnetic fields—without assuming the textbook Lorentz force law (19) as one of our starting ingredients. Such an investigation could then be expected to shed light on the specific issues of magnetic forces and work done on elementary dipoles.

Ultimately, we will show that if we are given an external electric field $\mathbf{E}_{\text{ext}}$ and an external magnetic field $\mathbf{B}_{\text{ext}}$, then the following generalization of the Lorentz force law describes the corresponding electromagnetic force $\mathbf{F}$ that acts on a particle with charge q, elementary electric dipole moment $\boldsymbol{\pi}$, and elementary magnetic dipole moment $\boldsymbol{\mu}$ traveling at a constant velocity $\mathbf{v}$ that is slow compared with the speed of light c:

$$\mathbf{F} = q\mathbf{E}_{\text{ext}} + q\mathbf{v} \times \mathbf{B}_{\text{ext}} + \nabla(\boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}}). \tag{22}$$

This formula once again implies that magnetic forces on electric monopoles are proportional to $\mathbf{v} \times \mathbf{B}_{\text{ext}}$ and are therefore incapable of doing work on them. On the other hand, this argument does not hold for the term $\nabla(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}})$ describing the magnetic force on an elementary magnetic dipole, thereby allowing magnetic forces to do work in that case. We will confirm this last statement explicitly by deriving the force law (22) in detail, first from the equations of motion for a particle with elementary electric and magnetic dipole moments coupled to the electromagnetic field, and then again from fundamental principles of local energy and momentum conservation.

E. The Lorentz-Covariant Formulation of Electromagnetism

In order to establish the claimed expression (22) for the appropriate generalization of the Lorentz force law without assuming a composite model for dipoles, we will need to develop a formulation of elementary dipoles within the classical theory of electromagnetism. More broadly, we will see that the Lorentz force law, rather than being a separate postulate of the theory, emerges naturally from constraints provided by relativity as well as by local conservation of energy and momentum.

For these purposes, we will need to review the Lorentz-covariant formulation of classical electromagnetism, once again taking the opportunity to establish our notational conventions [7]. Working always in Cartesian coordinates, we will use Latin indices $i, j, k, l, \ldots$ that each run through the three values x, y, z for three-dimensional vectors and tensors, and we will use Greek indices $\mu, \nu, \rho, \sigma, \ldots$ that each run through the four values t, x, y, z for four-dimensional Lorentz vectors and Lorentz tensors. We have four-dimensional spacetime coordinates

$$\begin{aligned} x^\mu &= (x^t, x^x, x^y, x^z)^\mu \equiv (c\,t, x, y, z)^\mu \\ &= (c\,t, \mathbf{x})^\mu \end{aligned} \tag{23}$$

and four-dimensional spacetime derivatives

$$\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \partial_x, \partial_y, \partial_z)_\mu \\ &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)_\mu \\ &= \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)_\mu, \end{aligned} \tag{24}$$

and we will follow the standard Einstein summation convention in which we implicitly sum all repeated upper-lower index pairs over their full range of values. We will employ the "mostly positive" Minkowski metric,

$$\eta_{\mu\nu} \equiv \eta^{\mu\nu} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}_{\mu\nu}, \tag{25}$$

which means that if we raise or lower a Lorentz index on a Lorentz four-vector v^μ (or, more generally, on a Lorentz tensor $T^{\mu\nu\cdots}{}_{\rho\sigma\cdots}$) according to

$$\left.\begin{aligned} v_\mu &= \eta_{\mu\nu} v^\nu, \\ v^\mu &= \eta^{\mu\nu} v_\nu, \end{aligned}\right\} \tag{26}$$

then raising or lowering a t index entails a change in overall sign, whereas raising or lowering an x, y, or z index has no effect:

$$\left.\begin{aligned} v^t &= -v_t, \\ v^x &= \ \ v_x, \\ v^y &= \ \ v_y, \\ v^z &= \ \ v_z. \end{aligned}\right\} \tag{27}$$

As in [1], we introduce a set of matrices $[\sigma_{\mu\nu}]^\alpha{}_\beta$ called the Lorentz generators,

$$[\sigma_{\mu\nu}]^\alpha{}_\beta = -i\delta^\alpha_\mu \eta_{\nu\beta} + i\eta_{\mu\beta}\delta^\alpha_\nu, \tag{28}$$

which have the commutation relations

$$\begin{aligned} [\sigma_{\mu\nu}, \sigma_{\rho\sigma}] &\equiv \sigma_{\mu\nu}\sigma_{\rho\sigma} - \sigma_{\rho\sigma}\sigma_{\mu\nu} \\ &= i\eta_{\mu\rho}\sigma_{\nu\sigma} - i\eta_{\mu\sigma}\sigma_{\nu\rho} - i\eta_{\nu\rho}\sigma_{\mu\sigma} + i\eta_{\nu\sigma}\sigma_{\mu\rho}, \end{aligned} \tag{29}$$

form a basis for all antisymmetric Lorentz tensors with two indices,

$$A^{\alpha\beta} = -A^{\beta\alpha} = \frac{i}{2} A^{\mu\nu} [\sigma_{\mu\nu}]^{\alpha\beta}, \tag{30}$$

and satisfy the key identities

$$\frac{1}{2}\mathrm{Tr}[\sigma^{\mu\nu}\sigma_{\rho\sigma}] \equiv i[\sigma_{\rho\sigma}]^{\mu\nu} \tag{31}$$

and

$$\frac{1}{2}\mathrm{Tr}[\sigma^{\mu\nu} A] = iA^{\mu\nu}. \tag{32}$$

We can express any Lorentz-transformation matrix Λ_{inf} that differs infinitesimally from the identity as

$$\Lambda_{\mathrm{inf}} = 1 - \frac{i}{2} d\theta^{\mu\nu}\sigma_{\mu\nu}, \tag{33}$$

where $d\theta^{\mu\nu} = -d\theta^{\nu\mu}$ is an antisymmetric array of small parameters given by

$$d\theta^{\mu\nu} = \begin{pmatrix} 0 & d\eta_x & d\eta_y & d\eta_z \\ -d\eta_x & 0 & d\theta_z & -d\theta_y \\ -d\eta_y & -d\theta_z & 0 & d\theta_x \\ -d\eta_z & d\theta_y & -d\theta_x & 0 \end{pmatrix}^{\mu\nu} \tag{34}$$

and describes a passive boost in the direction of the three-vector $d\boldsymbol{\eta} \equiv (d\theta^{tx}, d\theta^{ty}, d\theta^{tz})$ with magnitude $|d\boldsymbol{\eta}|$ together with a passive rotation around the direction of the three-vector $d\boldsymbol{\theta} \equiv (d\theta^{yz}, d\theta^{zx}, d\theta^{xy})$ by an angle $|d\boldsymbol{\theta}|$.

The electric field $\mathbf{E} = (E_x, E_y, E_z)$ and magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ transform as three-vectors under rotations, but they mix together in a complicated manner under Lorentz boosts. We can correctly capture this transformation behavior by packaging the electric and magnetic fields into an antisymmetric, Lorentz-covariant tensor $F^{\mu\nu}$, called the Faraday tensor, that is defined by

$$F^{\mu\nu} \equiv \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}^{\mu\nu} = -F^{\nu\mu}. \tag{35}$$

Introducing the totally antisymmetric, four-index Levi-Civita symbol,

$$\begin{aligned} \epsilon_{\mu\nu\rho\sigma} &\equiv \begin{cases} +1 & \text{for } \mu\nu\rho\sigma \text{ an even permutation of } txyz, \\ -1 & \text{for } \mu\nu\rho\sigma \text{ an odd permutation of } txyz, \\ 0 & \text{otherwise} \end{cases} \\ &= -\epsilon^{\mu\nu\rho\sigma}, \end{aligned} \tag{36}$$

the dual Faraday tensor $\tilde{F}_{\mu\nu}$ is defined according to

$$\begin{aligned} \tilde{F}_{\mu\nu} &\equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix}_{\mu\nu} \\ &= -\tilde{F}_{\nu\mu}. \end{aligned} \tag{37}$$

We collect the charge density ρ and the current density (or charge flux density) $\mathbf{J}$ into the Lorentz-covariant current density defined by

$$j^{\mu} \equiv (\rho c, J_x, J_y, J_z)^{\mu}, \tag{38}$$

meaning that

$$j^{\mu} = \begin{cases} \text{density of charge} & \text{for } \mu = t, \\ \text{flux density of charge} & \text{for } \mu = x, y, z. \end{cases} \tag{39}$$

The Maxwell equations (14)–(17) are then expressible in Lorentz-covariant form as the pair of tensor equations

$$\partial_{\mu} F^{\mu\nu} = -\mu_0 j^{\nu}, \tag{40}$$

$$\partial_{\mu} \tilde{F}^{\mu\nu} = 0, \tag{41}$$

the first of which encompasses the inhomogeneous Maxwell equations (14) and (17), and the second of which encompasses the homogeneous Maxwell equations (15) and (16). In addition, the second Lorentz-covariant equation (41) is equivalent to the electromagnetic Bianchi identity:

$$\partial^{\mu} F^{\nu\rho} + \partial^{\rho} F^{\mu\nu} + \partial^{\nu} F^{\rho\mu} = 0. \tag{42}$$

Taking the spacetime divergence of the inhomogeneous Maxwell equation (40) yields the equation of local current conservation,

$$\partial_{\mu} j^{\mu} = 0, \tag{43}$$

which, in three-vector notation, becomes the continuity equation for electric charge,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}. \tag{44}$$

This continuity equation also follows from taking the divergence of the Ampère equation (17), using the vector-calculus identity $\nabla \cdot (\nabla \times \mathbf{B}) = 0$, and then invoking the electric Gauss equation (14).

Meanwhile, by the Helmholtz theorem from vector calculus, the homogeneous Maxwell equation (41), $\partial_{\mu} \tilde{F}^{\mu\nu} = 0$, implies the existence of a four-vector field A_{μ}, called the electromagnetic gauge potential, in terms of which we can express the Faraday tensor $F_{\mu\nu}$ as the following antisymmetric pair of spacetime derivatives:

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{45}$$

We give conventional names to the components of the gauge potential A_{μ} according to

$$A_{\mu} = (-\Phi/c, \mathbf{A})_{\mu}, \tag{46}$$

where Φ is called the scalar potential and $\mathbf{A}$ is called the vector potential. A comparison between (45) and the definition (35) of the Faraday tensor $F_{\mu\nu}$ then yields the following relationships between the potentials Φ and $\mathbf{A}$ and the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{47}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{48}$$

The Faraday tensor $F_{\mu\nu}$ is unchanged under gauge transformations, meaning any redefinition of the gauge potential A_{μ} by the addition of the total spacetime derivative of an arbitrary scalar function f:

$$A_{\mu} \mapsto A_{\mu} + \partial_{\mu} f. \tag{49}$$

Translating this gauge transformation into three-vector language, the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are correspondingly invariant under the combined transformation

$$\Phi \mapsto \Phi - \frac{\partial f}{\partial t}, \tag{50}$$

$$\mathbf{A} \mapsto \mathbf{A} + \nabla f, \tag{51}$$

where the minus sign in the first of these two formulas comes from the minus sign in the definition (46) relating A_t and Φ.

Because the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ are unmodified by simultaneously carrying out (50) and (51), gauge transformations have no physical significance for observable quantities. Gauge transformations therefore express a redundancy in the description of electromagnetism when we formulate the theory in terms of potentials.

II. THE LAGRANGIAN FORMULATION AND ITS GENERALIZATIONS

In order to talk fundamentally about momentum, energy, force, and work for systems that go beyond classical particles, such as the electromagnetic field and our model for elementary dipoles, we will find it necessary to employ the Lagrangian formulation of classical dynamics, which we will review here [8].

A. The Lagrangian Formulation for a Classical System

Consider a general classical system with degrees of freedom q_{α} and rates of change $\dot{q}_{\alpha}$ with an action functional $S[q]$ given as the integral of the system's Lagrangian $L(q, \dot{q}, t)$ from an arbitrary initial time t_A to an arbitrary final time t_B:

$$S[q] \equiv \int_{t_A}^{t_B} dt\, L. \tag{52}$$

To say that this action functional encodes the system's dynamics is to say that if we extremize $S[q]$ over all candidate trajectories that share the same initial and final conditions,

$$\delta S[q] = 0, \quad \text{with } q_\alpha(t_A) \text{ and } q_\alpha(t_B) \text{ held fixed for all } \alpha, \tag{53}$$

then the resulting Euler-Lagrange equations

$$\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0 \tag{54}$$

fully capture the system's equations of motion.

We define the system's canonical momenta p_α in terms of the system's Lagrangian L as the partial derivative of L with respect to the corresponding rate of change $\dot{q}_\alpha$:

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha}. \tag{55}$$

Assuming that we can solve these definitions for the rates of change $\dot{q}_\alpha$ as functions of the canonical coordinates q_α and canonical momenta p_α, the system's Hamiltonian $H(q, p, t)$, which roughly describes the system's energy, is then defined as a function of the variables q_α, p_α, and t as the Legendre transformation

$$\begin{aligned} H &\equiv \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha - L, \\ &= \sum_\alpha p_\alpha \dot{q}_\alpha - L. \end{aligned} \tag{56}$$

Employing the chain rule together with the Euler-Lagrange equations, it follows from a straightforward calculation that the time derivative of the Hamiltonian (56) is given by

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}, \tag{57}$$

with the important implication that if the system's Lagrangian has no explicit dependence on the time t, meaning no dependence on t except arising through the degrees of freedom $q_\alpha(t)$ for a given candidate trajectory, then the Hamiltonian is constant in time, $dH/dt = 0$.

The Euler-Lagrange equations (54) are equivalent to the canonical equations of motion:

$$\left.\begin{aligned} \dot{q}_\alpha &= \frac{\partial H}{\partial p_\alpha}, \\ \dot{p}_\alpha &= -\frac{\partial H}{\partial q_\alpha}. \end{aligned}\right\} \tag{58}$$

The canonical equations of motion therefore provide an alternative way to encode the system's dynamics, known as the Hamiltonian formulation.

B. A Pair of Interacting Systems

We will now study a simple example that will turn out to be highly relevant to our work ahead.

In this example, which we will call the xy system, we consider a pair of subsystems, the first of which has a single degree of freedom x and the second of which has a single degree of freedom y. We define the dynamics of the overall xy system by choosing an action functional

$$S[x, y] \equiv \int dt\, L \tag{59}$$

with a Lagrangian defined by

$$L \equiv \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - ay^2 - bxy + c\dot{x}y, \tag{60}$$

where m, M, a, b, and c are constants and where, as usual, dots denote time derivatives. The constants m and M play the role of inertial masses, and a, b, and c can be interpreted as coupling constants.

The Euler-Lagrange equations (54) for x and y respectively then yield the equations of motion

$$m\ddot{x} = -by - c\dot{y}, \tag{61}$$

$$M\ddot{y} = -2ay - bx + c\dot{x}. \tag{62}$$

The physical interpretation of these coupled differential equations is that the right-hand sides describe interaction forces between the two systems. Notice that the force terms involving the constants a and b are conservative in the sense that they can be derived from a potential energy

$$V(x, y) \equiv ay^2 + bxy \tag{63}$$

according to (9):

$$F_x \equiv -\frac{\partial V}{\partial x} = -by, \tag{64}$$

$$F_y \equiv -\frac{\partial V}{\partial y} = -2ay - bx. \tag{65}$$

On the other hand, the force terms involving the constant c depend on the rates of change $\dot{x}$ and $\dot{y}$, and so are manifestly not conservative.

The xy system's canonical momenta are, from (55), given by

$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{x} + cy, \tag{66}$$

$$p_y \equiv \frac{\partial L}{\partial \dot{y}} = M\dot{y}. \tag{67}$$

Solving these equations to obtain $\dot{x}$ and $\dot{y}$ in terms of the canonical variables x, y, p_x, and p_y, we obtain

$$\dot{x} = \frac{p_x - cy}{m}, \tag{68}$$

$$\dot{y} = \frac{p_y}{M}. \tag{69}$$

Then a short calculation of the xy system's Hamiltonian (56) yields the result

$$\begin{aligned} H &\equiv p_x\dot{x} + p_y\dot{y} - L \\ &= \frac{(p_x - cy)^2}{2m} + \frac{p_y^2}{2M} + ay^2 + bxy. \end{aligned} \tag{70}$$

One can verify that the canonical equations of motion (58) derived from this Hamiltonian give back the original equations of motion (61)–(62). Moreover, because the Lagrangian (60) has no explicit time dependence, $\partial L/\partial t = 0$, our formula (57) guarantees that H is constant in time,

$$\frac{dH}{dt} = 0, \tag{71}$$

as one can check explicitly.

Substituting the formulas (68) for $\dot{x}$ and (69) for $\dot{y}$ into the Hamiltonian (70), we can rewrite the Hamiltonian of the xy system as

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + ay^2 + bxy.$$

The first two terms look like Newtonian kinetic energies (4) for the x and y systems individually,

$$T_x \equiv \frac{1}{2}m\dot{x}^2, \tag{72}$$

$$T_y \equiv \frac{1}{2}M\dot{y}^2, \tag{73}$$

and we recognize the final two terms as making up the potential energy defined in (63):

$$V(x, y) = ay^2 + bxy.$$

It is therefore natural to interpret H as the total energy E of the overall xy system,

$$\begin{aligned} E \equiv H &= T_x + T_y + V(x, y) \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + ay^2 + bxy, \end{aligned} \tag{74}$$

where, from (71), this energy is conserved:

$$\frac{dE}{dt} = 0. \tag{75}$$

Observe that we are always free to modify the definition (74) of the total energy E by adding on terms with vanishing time derivative, $d(\cdots)/dt = 0$, as such terms do not alter the conservation equation (75). Notice also that the velocity-dependent interaction term $cx\dot{y}$ in the Lagrangian (60) does not appear in the system's conserved energy.

Crucially, neither the x system nor the y system has a separately conserved energy on its own. Furthermore, although we can derive each of the two equations of motion (61) and (62) individually as the canonical equations of motion (58) for the two individual Hamiltonians defined by

$$H_x \equiv \frac{(p_x - cy)^2}{2m} + bxy, \tag{76}$$

$$H_y \equiv \frac{p_y^2}{2M} + ay^2 + bxy - c\dot{x}y, \tag{77}$$

the *overall* xy system's Hamiltonian (70) is not equal to the sum of the two individual Hamiltonians H_x and H_y, due to a double-counting of the interaction term bxy as well as the appearance of the velocity-dependent interaction term $-c\dot{x}y$:

$$H \neq H_x + H_y. \tag{78}$$

It is therefore up to us to decide whether to interpret the interaction terms ay^2 and bxy as belonging to one of the two individual systems or the other. If, for example, we choose to regard the y system as a "force field" acting on the x system, then it would be natural to regard the interaction terms as part of the energy of the y system, and we would correspondingly define non-conserved energies for the two systems individually as

$$E_x \equiv \frac{1}{2}m\dot{x}^2, \tag{79}$$

$$E_y \equiv \frac{1}{2}M\dot{y}^2 + ay^2 + bxy. \tag{80}$$

In this case, the conserved total energy (74) of the overall xy system is the sum of these two energies:

$$E = E_x + E_y. \tag{81}$$

Notice that in splitting up E in this way, we have effectively taken the energy E_x of the x system to be solely its kinetic energy $T_x \equiv (1/2)m\dot{x}^2$. Additionally, the conservation law (75) for the total energy E immediately implies that the time derivative of either E_x or E_y yields the *opposite* of the rate at which the other system's energy is changing:

$$\frac{dE_x}{dt} = -\frac{dE_y}{dt}. \tag{82}$$

Observe that the left-hand side is given explicitly by

$$\frac{dE_x}{dt} = m\ddot{x}\dot{x} = (\text{force})(\text{speed}),$$

so it precisely represents the rate at which work is being done on the x system.

Looking back at the velocity-dependent interaction term $c\dot{x}y$, notice that we can use the product rule in reverse (that is, "integration by parts" without an integration) to replace it with $-cx\dot{y}$, up to a total time derivative:

$$c\dot{x}y = -cx\dot{y} + \frac{d}{dt}(cxy). \tag{83}$$

By the fundamental theorem of calculus, a total time derivative in a Lagrangian leads to terms in the action functional (52), $S \equiv \int dt\, L$, that depend only on the fixed initial and final conditions and that are therefore constants that do not affect the variational condition (53) or the Euler-Lagrange equations (54). Indeed, one can verify explicitly that the alternative Lagrangian defined by

$$L' \equiv \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 - ay^2 - bxy - cx\dot{y}, \tag{84}$$

which differs from our original Lagrangian (60) by only the total time derivative of cxy,

$$L = L' + \frac{d}{dt}(cxy), \tag{85}$$

leads to precisely the same equations of motion (61) and (62) for the xy system as before. The new Lagrangian L' yields respective canonical momenta

$$p'_x \equiv \frac{\partial L'}{\partial \dot{x}} = m\dot{x}, \tag{86}$$

$$p'_y \equiv \frac{\partial L'}{\partial \dot{y}} = M\dot{y} - cx, \tag{87}$$

and Hamiltonian

$$H' = \frac{p'^2_x}{2m} + \frac{(p'_y + cx)^2}{2M} + ay^2 + bxy, \tag{88}$$

which formally disagree with the canonical momenta (66) and (67) and with the Hamiltonian (70) derived from our original Lagrangian L. However, if we write the Hamiltonians H and H' in terms of $\dot{x}$ and $\dot{y}$, then we see that they actually describe precisely the same conserved total energy (74) for the xy system,

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + ay^2 + bxy,$$

thereby confirming that it does not physically matter whether we use L or L' as the xy system's Lagrangian. In essence, by switching from L to L', we have merely carried out a canonical transformation of the form

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \mapsto \begin{pmatrix} x' \\ p'_x \\ y' \\ p'_y \end{pmatrix} = \begin{pmatrix} x \\ p_x - cy \\ y \\ p_y - cx \end{pmatrix}, \tag{89}$$

but we obviously have not changed the underlying physics.

C. The Lagrangian Formulation for a Relativistic Massive Particle with Spin

As reviewed in [1], one can reformulate the Lagrangian description of a generic classical system in a manifestly covariant language by introducing an arbitrary smooth, strictly monotonic parametrization $t \mapsto t(\lambda)$ in place of the time t, in which case one arrives at the following alternative formula for the system's Lagrangian:

$$S[q,t] = \int d\lambda\, \mathscr{L}(q, \dot{q}, t, \dot{t}). \tag{90}$$

Here dots now denote derivatives with respect to the parameter λ and we have introduced a manifestly covariant Lagrangian according to

$$\mathscr{L}(q, \dot{q}, t, \dot{t}) \equiv \dot{t}\, L(q, \dot{q}/\dot{t}, t). \tag{91}$$

This formalism puts the system's degrees of freedom q_α and the time t on a similar footing, with the system's Hamiltonian H now expressible as the "canonical momentum" conjugate to $-t$.

We are now ready to turn to the Lagrangian formulation for a relativistic particle with spin. We will need to be careful to distinguish between the coordinates x^μ of arbitrary points in spacetime—such as in the arguments of field variables—and the specific coordinates X^μ of our particle's location in spacetime. We will therefore continue to use capital letters for the particle's spacetime coordinates,

$$X^\mu(\lambda) = (cT(\lambda), \mathbf{X}(\lambda))^\mu, \tag{92}$$

where λ is a smooth, strictly monotonic parameter for the particle's four-dimensional worldline.

We will assume that the particle has a positive mass $m > 0$, a future-directed four-momentum p^μ whose temporal component $p^t > 0$ encodes the particle's relativistic kinetic energy E and whose spatial components $\mathbf{p} = (p_x, p_y, p_z)$ encode the particle's relativistic three-momentum,

$$p^\mu \equiv (E/c, \mathbf{p})^\mu, \tag{93}$$

and an intrinsic spin that is encoded in an antisymmetric spin tensor $S^{\mu\nu} = -S^{\nu\mu}$ whose independent components define a pair of three-vectors

$$\mathbf{S} \equiv (S^{yz}, S^{zx}, S^{xy}), \tag{94}$$

$$\tilde{\mathbf{S}} \equiv (S^{tx}, S^{ty}, S^{tz}). \tag{95}$$

The particle's Pauli-Lubanski pseudovector is then given by

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} p_\nu S_{\rho\sigma}. \tag{96}$$

As explained in detail in [1], a massive particle with positive energy $E = p^t c > 0$ is a classical system whose phase space provides an irreducible representation (or, more precisely, a transitive group action or homogeneous space) of the Poincaré group characterized by the fixed scalar quantities

$$p^2 \equiv p_\mu p^\mu \equiv -m^2c^2, \tag{97}$$

$$W^2 \equiv W_\mu W^\mu \equiv w^2, \tag{98}$$

$$\frac{1}{2}S^2 \equiv \frac{1}{2}S_{\mu\nu}S^{\mu\nu} \equiv s^2 = \mathbf{S}^2 - \tilde{\mathbf{S}}^2, \tag{99}$$

as well as the fixed pseudoscalar quantity

$$\frac{1}{8}\epsilon_{\mu\nu\rho\sigma}S^{\mu\nu}S^{\rho\sigma} \equiv \tilde{s}^2 = \mathbf{S}\cdot\tilde{\mathbf{S}}. \tag{100}$$

The constancy of the quantities (97)–(100) is a fundamental feature of the particle's phase space whether or not interactions are present, and leads to several self-consistency conditions, the most important of which is that the contraction of the particle's four-momentum with its spin tensor must vanish [9]:

$$p_\mu S^{\mu\nu} = 0. \tag{101}$$

As shown in [1], we can use the following manifestly covariant action functional of the form (90) for the case in which the particle is free from interactions:

$$\begin{aligned} S_{\text{particle}}[X, \Lambda] &= \int d\lambda\, \mathscr{L}_{\text{particle}} \\ &= \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2}\text{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) \\ &= \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2} S_{\mu\nu}\dot{\theta}^{\mu\nu} \right). \end{aligned} \tag{102}$$

The degrees of freedom in this description are the particle's spacetime coordinates $X^\mu(\lambda)$ and a variable Lorentz-transformation matrix $\Lambda^\mu{}_\nu(\lambda)$. The particle's four-momentum $p^\mu(\lambda)$ and its spin tensor $S^{\mu\nu}(\lambda)$ are given respectively in terms of fixed reference values p_0^μ and $S_0^{\mu\nu}$ [10] according to

$$p^\mu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda) p_0^\nu, \tag{103}$$

$$\begin{aligned} S^{\mu\nu}(\lambda) &\equiv \Lambda^\mu{}_\rho(\lambda) S_0^{\rho\sigma} (\Lambda^{\mathrm{T}})_\sigma{}^\nu(\lambda) \\ &= -\frac{i}{2}\text{Tr}[\sigma^{\mu\nu}\Lambda(\lambda) S_0 \Lambda^{-1}(\lambda)]. \end{aligned} \tag{104}$$

Note that neither $p^\mu(\lambda)$ nor $S^{\mu\nu}(\lambda)$ depends on the particle's spacetime degrees of freedom $X^\mu(\lambda)$ before the equations of motion are imposed. Here, again, $[\sigma_{\mu\nu}]^\alpha{}_\beta$ are the Lorentz generators (28), and we can use (33) to express the derivative of $\Lambda(\lambda)$ with respect to the world-line parameter λ in terms of the rates of change $\dot{\theta}^{\mu\nu}$ in the corresponding boost and angular parameters as

$$\dot{\Lambda}(\lambda) = -\frac{i}{2}\dot{\theta}^{\mu\nu}(\lambda)\sigma_{\mu\nu}\Lambda(\lambda). \tag{105}$$

As in [1], we take the reference value of the particle's four-momentum to be

$$p_0^\mu \equiv (mc, \mathbf{0})^\mu = mc\,\delta_t^\mu, \tag{106}$$

in which case the particle's four-momentum (103) is given for general states by

$$p^\mu(\lambda) = mc\,\Lambda^\mu{}_t(\lambda). \tag{107}$$

The self-consistency condition (101) then tells us that the reference value $S_0^{\mu\nu}$ of the particle's spin tensor satisfies

$$mc\, S_0^{t\nu} = 0, \tag{108}$$

and therefore has the general form

$$S_0^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_{0,z} & -S_{0,y} \\ 0 & -S_{0,z} & 0 & S_{0,x} \\ 0 & S_{0,y} & -S_{0,x} & 0 \end{pmatrix}^{\mu\nu}. \tag{109}$$

D. The Limit of Vanishing Spin

We now specialize momentarily to the case of a free particle without spin, $S^{\mu\nu} = 0$. In principle, we can then solve the condition $p^2 \equiv -m^2c^2$ from (97) for $p^t \equiv E/c$ to obtain the mass-shell relation

$$E^2 = \mathbf{p}^2c^2 + m^2c^4. \tag{110}$$

Setting our parameter $\lambda \equiv t$ to be the physical time coordinate and switching back to the traditional, non-covariant Lagrangian formulation, we end up with the Hamiltonian

$$H = E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}. \tag{111}$$

The canonical equations of motion (58) derived from this Hamiltonian then imply that the components of the particle's three-velocity,

$$\mathbf{v} \equiv \frac{d\mathbf{X}}{dt} = (v_x, v_y, v_z), \tag{112}$$

are given by

$$v_i \equiv \frac{dX^i}{dt} = \frac{\partial H}{\partial p^i}, \tag{113}$$

which yields the following relationship between the particle's three-velocity $\mathbf{v}$, its three-momentum $\mathbf{p}$, and its energy E:

$$\mathbf{v} = \frac{\mathbf{p}c^2}{E} = \frac{\mathbf{p}c^2}{\sqrt{\mathbf{p}^2c^2 + m^2c^4}}. \tag{114}$$

Solving for $\mathbf{p}$ in terms of $\mathbf{v}$ gives the formula

$$\mathbf{p} = \gamma m \mathbf{v}, \tag{115}$$

where the Lorentz factor γ is defined by

$$\gamma \equiv \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}. \tag{116}$$

Using γ, we can also express the particle's relativistic energy E as

$$E = \gamma mc^2, \tag{117}$$

and so we find that the four-momentum (93) takes the form

$$p^\mu = (E/c, \mathbf{p})^\mu = (\gamma mc, \gamma m\mathbf{v})^\mu = mu^\mu. \tag{118}$$

Here u^μ is the particle's normalized four-velocity,

$$u^\mu \equiv (\gamma c, \gamma\mathbf{v})^\mu = \gamma\frac{dX^\mu}{dt}, \tag{119}$$

where by "normalized," we mean that u^μ satisfies the normalization condition

$$u^2 \equiv u_\mu u^\mu = -c^2. \tag{120}$$

It then follows from a straightforward calculation that the particle's action functional (102) reduces to the non-covariant form

$$\begin{aligned} S_{\text{particle}}[\mathbf{X}] &= \int dt\, p_\mu \frac{dX^\mu}{dt} = -mc^2 \int dt/\gamma \\ &= -mc^2 \int dt\, \sqrt{1-\mathbf{v}^2/c^2}. \end{aligned} \tag{121}$$

By another calculation, one can also show that in the non-relativistic limit, $\mathbf{v}^2 \ll c^2$, (121) reduces to the action functional of a Newtonian particle with Lagrangian $(1/2)m\mathbf{v}^2 - mc^2$, describing a particle with kinetic energy $(1/2)m\mathbf{v}^2$ and "intrinsic potential energy" mc^2.

By definition, the squared proper-time interval $d\tau^2$ is the infinitesimal squared arc length of the particle's worldline, up to a factor of $-c^2$, so

$$\begin{aligned} -c^2 d\tau^2 &= \eta_{\mu\nu} dX^\mu dX^\nu \\ &= \eta_{\mu\nu}(c\,dT, d\mathbf{X})^\mu (c\,dT, d\mathbf{X})^\nu \\ &= -c^2 dT^2 + d\mathbf{X}^2 \\ &= -c^2 dt^2 (1-\mathbf{v}^2/c^2). \end{aligned}$$

We therefore obtain the familiar time-dilation formula

$$d\tau = \frac{dt}{\gamma}, \tag{122}$$

so we can write the particle's normalized four-velocity (119) as

$$u^\mu = \frac{dX^\mu}{d\tau}, \tag{123}$$

and we can compactly express the formula (121) for the particle's action functional as the particle's Lorentz-invariant, integrated proper time $\int d\tau$, up to a proportionality factor of $-mc^2$:

$$S_{\text{particle}}[\mathbf{X}] \equiv -mc^2 \int d\tau. \tag{124}$$

It is important to note that if a particle with intrinsic spin $S^{\mu\nu} \neq 0$ and elementary dipole moments is interacting with a nonvanishing electromagnetic field, then the particle's four-momentum will not necessarily take the familiar form (118), $p^\mu = mu^\mu$, that holds for a free particle, as we will show explicitly later.

E. The Dynamics of a Relativistic Massive Particle with Spin

Once again allowing the particle to have a nonzero spin tensor, $S^{\mu\nu} \neq 0$, we can vary the particle's action functional (102),

$$S_{\text{particle}}[X, \Lambda] = \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right),$$

to obtain the particle's equations of motion, in accordance with the extremization condition (53).

Extremizing the particle's action functional with respect to its spacetime coordinates X^μ yields

$$\dot{p}^\mu = 0. \tag{125}$$

This equation of motion implies that the particle's energy and momentum are constant in time, as would be expected for an isolated particle that is not subject to any forces.

On the other hand, as shown in [1], extremizing the particle's action functional (102) with respect to the variable Lorentz-transformation matrix $\Lambda^\mu{}_\nu(\lambda)$ yields

$$\dot{J}^{\mu\nu} = \dot{L}^{\mu\nu} + \dot{S}^{\mu\nu} = 0, \tag{126}$$

where $J^{\mu\nu} = -J^{\nu\mu}$ is the particle's antisymmetric total angular-momentum tensor,

$$J^{\mu\nu} \equiv L^{\mu\nu} + S^{\mu\nu}, \tag{127}$$

and $L^{\mu\nu} = -L^{\nu\mu}$ is the particle's antisymmetric orbital angular-momentum tensor,

$$L^{\mu\nu} \equiv X^\mu p^\nu - X^\nu p^\mu. \tag{128}$$

The equation of motion (126) tells us that the particle's total angular-momentum tensor is conserved, as would be expected in the absence of external torques.

Using (118), which tell us that the four-momentum of a massive free particle is related to its four-velocity according to $p^\mu = mu^\mu \propto \dot{X}^\mu$, it follows that the particle's orbital angular-momentum tensor $L^{\mu\nu}$ is constant by itself,

$$\dot{L}^{\mu\nu} = 0, \tag{129}$$

so the particle's spin tensor is likewise separately conserved,

$$\dot{S}^{\mu\nu} = 0. \tag{130}$$

F. The Lagrangian Formulation of Classical Field Theories and Electromagnetism

The Lagrangian formulation naturally accommodates the case of a classical field theory with local field degrees of freedom $\varphi_\alpha(x)$ and an action functional $S[\varphi]$ defined in terms of a Lagrangian density $\mathcal{L}(\varphi, \partial\phi, x)$ as

$$S[\varphi] = \int dt \int d^3x\, \mathcal{L}, \tag{131}$$

where d^3x denotes the usual three-dimensional volume element,

$$d^3x \equiv dx\, dy\, dz. \tag{132}$$

The extremization condition (53) on the action functional $S[\varphi]$ yields a field-theoretic generalization of the Euler-Lagrange equations (54) given by:

$$\frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)} \right) = 0. \tag{133}$$

We now turn to the specific case of electromagnetism. If we temporarily assume the absence of electromagnetic sources, meaning that we take the four-dimensional current density (38) to be zero,

$$j^\mu \equiv (\rho c, \mathbf{J})^\mu = 0, \tag{134}$$

then we can encode the Maxwell equations (14)–(17) in a Lagrangian formulation using the Lorentz-invariant, translation-invariant, gauge-invariant Lagrangian density

$$\mathcal{L}_{\text{field}} \equiv -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \tag{135}$$

with a corresponding action functional defined by

$$\begin{aligned} S_{\text{field}}[A] &\equiv \int dt \int d^3x\, \mathcal{L}_{\text{field}} \\ &= \int dt \int d^3x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right), \end{aligned} \tag{136}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ from (45) and where we regard the gauge potential A_μ as constituting the Maxwell theory's underlying degrees of freedom. Indeed, the field-theoretic Euler-Lagrange equations (133) yield

$$\begin{aligned} &\frac{\partial \mathcal{L}_{\text{field}}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\text{field}}}{\partial(\partial_\mu A_\nu)} \right) \\ &\qquad = 0 - \partial_\mu \left(-\frac{1}{\mu_0} F^{\mu\nu} \right) = 0, \end{aligned}$$

which immediately gives us the inhomogeneous Maxwell equation (40) with vanishing current density $j^\nu = 0$:

$$\partial_\mu F^{\mu\nu} = 0.$$

The homogeneous Maxwell equation (41), on the other hand, follows immediately from the relation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

III. ELEMENTARY MULTIPOLES

A. The Multipole Expansion of the Current Density

For our first step toward modeling electromagnetic multipoles—meaning not just electric monopoles, but also electric and magnetic dipoles, electric and magnetic quadrupoles, and higher multipoles—we express the Lorentz-covariant current density j^ν from (38) as a series expansion of local terms with increasingly many spacetime derivatives ∂_μ, where the requirements of Lorentz covariance dictate the schematic structure

$$j^\nu = (\cdots)^\nu + \partial_\mu (\cdots)^{\mu\nu} + \partial_\mu \partial_\rho (\cdots)^{\mu\rho\nu} + \cdots . \tag{137}$$

As we will see, the series (137) represents a multipole expansion

$$j^\nu = j^\nu_{\text{e}} + j^\nu_{\text{d}} + j^\nu_{\text{q}} + \cdots , \tag{138}$$

where each term $j^\nu_{\text{e}}, j^\nu_{\text{d}}, j^\nu_{\text{q}}$ has a specific physical interpretation.

- The four-vector j^ν_{e} represents the overall contribution to j^ν from charged sources whose spatial densities involve no derivatives. We will show that j^ν_{e} describes the distribution of electric monopoles throughout physical space.

- The four-vector j^ν_{d} represents the net contribution from all charged sources whose spatial densities involve a single spacetime divergence. Lorentz covariance implies that j^ν_{d} is expressible in terms of a tensor field $M^{\mu\nu}$ according to

$$j^\nu_{\text{d}} = \partial_\mu M^{\mu\nu}. \tag{139}$$

  We will show later that j^ν_{d} represents the spatial distribution of electric and magnetic dipoles, so we will call $M^{\mu\nu}$ the dipole-density tensor.

- Similarly, the four-vector j^ν_{q} is given in terms of a pair of spacetime divergences of a tensor field $N^{\mu\rho\nu}$,

$$j^\nu_{\text{q}} = \partial_\mu \partial_\rho N^{\mu\rho\nu}, \tag{140}$$

  and represents the spatial distribution of electric and magnetic quadrupoles.

- Subsequent terms in the series (138) represent still-higher multipoles and involve incrementally more spacetime divergences.

We can now write the schematic multipole expansion (137) in the more concrete form

$$\begin{aligned} j^\nu &= j^\nu_{\text{e}} + j^\nu_{\text{d}} + j^\nu_{\text{q}} + \cdots \\ &= j^\nu_{\text{e}} + \partial_\mu M^{\mu\nu} + \partial_\mu \partial_\rho N^{\mu\rho\nu} + \cdots . \end{aligned} \tag{141}$$

To ensure individual local conservation laws for each category of elementary multipole, we take the tensors $M^{\mu\nu}, N^{\mu\sigma\nu}, \ldots$ to obey the antisymmetry conditions

$$M^{\mu\nu} = -M^{\nu\mu}, \tag{142}$$

$$N^{\mu\rho\nu} = -N^{\nu\rho\mu} = -N^{\mu\nu\rho}, \tag{143}$$

and so on. It then follows immediately from the symmetry $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ of mixed partial derivatives that the current density for each kind of multipole separately obeys its own local conservation equation, so that

$$\partial_\nu j_{\mathrm{e}}^\nu = 0, \tag{144}$$
$$\partial_\nu j_{\mathrm{d}}^\nu = 0, \tag{145}$$
$$\partial_\nu j_{\mathrm{q}}^\nu = 0, \tag{146}$$

and so forth.

Note that the local conservation law (145) for the dipole current density j_{d}^ν is not related to the fact that the elementary dipole moments of our particles are permanent. Nor does one need to invoke quantum mechanics and quantization of angular momentum to explain their permanence, either. In our model, the intrinsic spin and the associated elementary dipole moments of a classical particle are invariant features of the particle in the sense sense that the rest mass of the particle is permanent. As detailed in [1], the invariance of a classical particle's rest mass and the invariance of its intrinsic spin follow from group-theoretic considerations in constructing the particle's phase space (or, in the analogous quantum case, the particle's Hilbert space). That is, the particle's phase space simply lacks the degrees of freedom that would be necessary to allow the rest mass or the intrinsic spin of the particle to be able to change.

Notice that we can recast the multipole expansion (141) as

$$j^\nu = j_{\mathrm{e}}^\nu + \partial_\mu (M^{\mu\nu} + \partial_\rho N^{\mu\rho\nu} + \cdots). \tag{147}$$

Introducing the multipole-density tensor,

$$Q^{\mu\nu} \equiv M^{\mu\nu} + \partial_\rho N^{\mu\rho\nu} + \cdots , \tag{148}$$

which is antisymmetric on its two indices,

$$Q^{\mu\nu} = -Q^{\nu\mu}, \tag{149}$$

we can therefore write the multipole expansion for j^ν more compactly as

$$j^\nu = j_{\mathrm{e}}^\nu + \partial_\mu Q^{\mu\nu}. \tag{150}$$

B. The Auxiliary Faraday Tensor

Correspondingly, we define the antisymmetric auxiliary Faraday tensor $H^{\mu\nu} = -H^{\nu\mu}$ to absorb all source contributions from dipoles and higher multipoles:

$$\begin{aligned} H^{\mu\nu} &\equiv \frac{1}{\mu_0} F^{\mu\nu} + Q^{\mu\nu} \\ &= \frac{1}{\mu_0} F^{\mu\nu} + M^{\mu\nu} + \partial_\rho N^{\mu\rho\nu} + \cdots . \end{aligned} \tag{151}$$

We can then re-cast the inhomogeneous Maxwell equation (40), $\partial_\mu F^{\mu\nu} = -\mu_0 j^\nu$, in the alternative form

$$\partial_\mu H^{\mu\nu} = -j_{\mathrm{e}}^\nu, \tag{152}$$

where again j_{e}^ν represents contributions to the current density that arise solely from electric monopoles,

$$j_{\mathrm{e}}^\nu = (\rho_{\mathrm{e}} c, \mathbf{J}_{\mathrm{e}})^\nu. \tag{153}$$

The auxiliary Faraday tensor $H^{\mu\nu}$ can be expressed in terms of the electric displacement field $\mathbf{D} \equiv (H^{tx}/c, H^{ty}/c, H^{tz}/c)$ and the auxiliary magnetic field $\mathbf{H} \equiv (H^{yz}, H^{zx}, H^{xy})$ according to

$$H^{\mu\nu} \equiv \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & H_z & -H_y \\ -cD_y & -H_z & 0 & H_x \\ -cD_z & H_y & -H_x & 0 \end{pmatrix}^{\mu\nu}. \tag{154}$$

These definitions permit us to write the auxiliary version (152) of the inhomogeneous Maxwell equation in three-vector form as the pair of equations

$$\nabla \cdot \mathbf{D} = \rho_{\mathrm{e}}, \tag{155}$$
$$\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{e}} + \frac{\partial \mathbf{D}}{\partial t}. \tag{156}$$

These two equations can be used in place of the three-vector inhomogeneous Maxwell equations (14) (the electric Gauss equation) and (17) (the Ampère equation).

The formulation of electromagnetism in terms of this pair of alternative three-vector equations is particularly suited to the study of "macroscopic" electromagnetic fields in charged matter. In that case, the overall current density j^ν is regarded as a coarse-grained spatial average over appropriately large regions of the physical material in question, with the result that electromagnetic multipoles arise, in part, *emergently* from the averaging process. Indeed, in textbooks, the equations (155) and (156) are conventionally derived from this sort of averaging.

In this paper, by contrast, we have obtained these equations by expanding our fundamental charged sources as a series (137) in spacetime derivatives and imposing Lorentz covariance. In this way, we are expressly allowing for the possibility of *elementary* electromagnetic multipoles.

C. The Lorentz-Invariant Pointlike Volume Density

If we wish, we can regard our elementary electric monopoles as providing a classical model of electrons and other elementary particles, and our elementary magnetic dipoles as providing a classical model of their magnetic dipole moments. In order to study the behavior of pointlike electric monopoles and elementary multipoles in detail, we will need to review the formalism of Dirac delta functions in three and four dimensions.

Consider a product of three delta functions describing an abstract volume density sharply localized at a spatial point $\mathbf{x}' = (x', y', z')$:

$$\delta^3(\mathbf{x} - \mathbf{x}') \equiv \delta(x - x')\, \delta(y - y')\, \delta(z - z'). \tag{157}$$

The defining feature of this three-dimensional delta function is that its integral $\int d^3x\,(\cdots) \equiv \int dx\,dy\,dz\,(\cdots)$ over any spatial volume $\mathcal{V}$ containing the point $\mathbf{x}' = (x', y', z')$ yields the number 1, whereas its integral over any spatial volume not containing the point $\mathbf{x}'$ yields 0:

$$\int_{\mathcal{V}} d^3x\,\delta^3(\mathbf{x}-\mathbf{x}') = \begin{cases} 1 & \text{if } \mathcal{V} \text{ contains } \mathbf{x}', \\ 0 & \text{if } \mathcal{V} \text{ does not contain } \mathbf{x}'. \end{cases} \tag{158}$$

We can extend this construction to four-dimensional spacetime. An isolated event with coordinates $x'^{\mu} = (c\,t', x', y', z')^{\mu}$ in spacetime corresponds to a product of four delta functions,

$$\begin{aligned} &\delta^4(x - x') \\ &\qquad \equiv \delta(c\,t - c\,t')\,\delta(x - x')\,\delta(y - y')\,\delta(z - z'), \end{aligned} \tag{159}$$

with the defining feature that its integral $\int d^4x\,(\cdots) \equiv \int c\,dt\,dx\,dy\,dz\,(\cdots)$ over any four-dimensional region $\mathcal{M}$ of spacetime yields the number 1 or 0 depending on whether that region contains the spacetime point labeled by x'^{μ}:

$$\begin{aligned} &\int_{\mathcal{M}} d^4x\,\delta^4(x - x') \\ &\qquad = \begin{cases} 1 & \text{if } \mathcal{M} \text{ contains } x'^{\mu}, \\ 0 & \text{if } \mathcal{M} \text{ does not contain } x'^{\mu}. \end{cases} \end{aligned} \tag{160}$$

Under an arbitrary Lorentz transformation $x^{\mu} \mapsto \Lambda^{\mu}{}_{\nu} x^{\nu}$, the four-dimensional integration measure $d^4x \equiv c\,dt\,dx\,dy\,dz$ incurs a trivial Jacobian factor of $|\det \Lambda| = 1$, and is therefore invariant. The defining condition (160) then implies that the four-dimensional delta function $\delta^4(x - x')$ is likewise invariant under Lorentz transformations.

Generalizing from an isolated spacetime event to the worldline trajectory of a particle, we replace $x'^{\mu} = (c\,t', x', y', z')^{\mu}$ with appropriate coordinate functions $X^{\mu}(\lambda) = (c\,T(\lambda), X(\lambda), Y(\lambda), Z(\lambda))^{\mu}$ of a smooth, strictly monotonic parameter λ. Our Lorentz-invariant four-dimensional delta function (159) becomes

$$\begin{aligned} &\delta^4(x - X) \\ &\qquad \equiv \delta(c\,t - c\,T)\,\delta(x - X)\,\delta(y - Y)\,\delta(z - Z). \end{aligned} \tag{161}$$

Infinitesimal durations of the particle's Lorentz-invariant proper time τ are related to corresponding intervals of the coordinate time t according to the usual formula (122) for time dilation,

$$d\tau = \frac{dt}{\gamma},$$

where again γ is the particle's Lorentz factor defined as in (116) according to

$$\gamma \equiv \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}.$$

The integral of the product of the Lorentz-invariant quantity dt/γ and the Lorentz-invariant delta function $\delta^4(x - X(\lambda))$ over the particle's four-dimensional worldline is manifestly Lorentz invariant:

$$\int \frac{dt}{\gamma}\,\delta^4(x - X).$$

Evaluating this worldline integral explicitly yields a Lorentz-invariant version of the three-dimensional delta function $\delta^3(\mathbf{x} - \mathbf{X}(\lambda))$:

$$\begin{aligned} &\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \\ &\qquad = \frac{1}{\gamma}\delta(x - X)\,\delta(y - Y)\,\delta(z - Z). \end{aligned} \tag{162}$$

Because the special combination of $1/\gamma$ and $\delta^3(\mathbf{x}-\mathbf{X}(\lambda))$ appearing in this formula maintains its form under Lorentz transformations, it represents the appropriate Lorentz-invariant generalization of a pointlike volume density. We can also understand the Lorentz invariance of (162) from the fact that under coordinate changes, $\delta^3(\mathbf{x} - \mathbf{X}(\lambda))$ transforms like the inverse of a three-dimensional volume element d^3x, and because d^3x experiences Lorentz contractions by $1/\gamma$, the three-dimensional delta function $\delta^3(\mathbf{x}-\mathbf{X}(\lambda))$ *grows* by a factor of γ, which is then compensated by the $1/\gamma$ appearing in (162).

Notice that in the limiting case $\mathbf{X}(\lambda) \to \mathbf{x}'$ and $\mathbf{v} \to 0$ in which the particle is at rest, we have $1/\gamma \to 1$. In this limit, (162) therefore reduces to the static three-dimensional delta function $\delta^3(\mathbf{x} - \mathbf{x}')$ that we originally introduced in (157).

D. Electric Monopoles

We now have the tools necessary to model various pointlike sources more precisely. To start, we consider a pointlike electric monopole of charge q at rest at a location $\mathbf{x}' = (x', y', z')$. The electric monopole has charge density

$$\rho_{\mathrm{e}}(\mathbf{x}) = q\,\delta^3(\mathbf{x} - \mathbf{x}') \tag{163}$$

and vanishing current density

$$\mathbf{J}_{\mathrm{e}}(\mathbf{x}) = 0. \tag{164}$$

An elementary calculation using the Maxwell equations (14)–(17) shows that the resulting electric field for all $\mathbf{x} \neq \mathbf{x}'$ is directed outward from the point $\mathbf{x}'$, with an inverse-square dependence on the distance $|\mathbf{x} - \mathbf{x}'|$ from $\mathbf{x}'$, whereas the magnetic field vanishes:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x} - \mathbf{x}'|^2}\,\mathbf{e}_{\mathbf{x}-\mathbf{x}'}, \tag{165}$$

$$\mathbf{B} = 0. \tag{166}$$

Here $\mathbf{e}_{\mathbf{x}-\mathbf{x}'}$ is a unit vector pointing in the direction from the source point $\mathbf{x}'$ to the field point $\mathbf{x}$:

$$\mathbf{e}_{\mathbf{x}-\mathbf{x}'} \equiv \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}. \tag{167}$$

We can therefore conclude that this source distribution describes an electric monopole at rest at $\mathbf{x}'$, as claimed.

The electric monopole has Lorentz-covariant current density

$$\begin{aligned} j_{\mathrm{e}}^{\nu} &= (\rho_{\mathrm{e}} c, \mathbf{J}_{\mathrm{e}})^{\nu} \\ &= (q\,\delta^3(\mathbf{x}-\mathbf{x}')\,c, \mathbf{0})^{\nu} \\ &= q\,(c,\mathbf{0})^{\nu}\,\delta^3(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{168}$$

Identifying $u^{\nu}_{\text{rest}} = (c,\mathbf{0})^{\nu}$ as the electric monopole's normalized ($u^2_{\text{rest}} = -c^2$) four-velocity (119) in its own rest frame, and recalling our formula (162) for the Lorentz-invariant generalization of a three-dimensional delta function, we can immediately write down the Lorentz-covariant current density of a pointlike electric monopole of charge q moving along a trajectory $\mathbf{X}(t) = (X(t), Y(t), Z(t))$:

$$j_{\mathrm{e}}^{\nu}(\mathbf{x},t) = q u^{\nu}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}). \tag{169}$$

Notice that q is a Lorentz scalar, u^{ν} is a Lorentz four-vector, and the combination of $1/\gamma$ together with the three-dimensional delta function $\delta^3(\mathbf{x}-\mathbf{X}(t))$ is Lorentz invariant, so (169) is indeed a Lorentz four-vector, as required.

Using the formula (119) for the electric monopole's normalized four-velocity when it is in motion at a three-velocity $\mathbf{v} = (v_x, v_y, v_z)$,

$$u^{\nu} = (\gamma c, \gamma\mathbf{v})^{\nu} = \gamma\frac{dX^{\nu}}{dt}, \tag{170}$$

where the derivative of X^{ν} is taken with respect to the coordinate time t, we see that the factors of γ in (169) cancel out and thus our formula for the current density becomes

$$\begin{aligned} j_{\mathrm{e}}^{\nu}(\mathbf{x},t) &= (qc\,\delta^3(\mathbf{x}-\mathbf{X}), q\mathbf{v}\,\delta^3(\mathbf{x}-\mathbf{X})) \\ &= q\frac{dX^{\nu}}{dt}\delta^3(\mathbf{x}-\mathbf{X}), \end{aligned} \tag{171}$$

meaning that the charge density and current density are given respectively by

$$\rho_{\mathrm{e}}(\mathbf{x},t) = q\,\delta^3(\mathbf{x}-\mathbf{X}), \tag{172}$$

$$\mathbf{J}_{\mathrm{e}}(\mathbf{x},t) = q\mathbf{v}\,\delta^3(\mathbf{x}-\mathbf{X}). \tag{173}$$

In particular, these two functions are related according to

$$\mathbf{J}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v}. \tag{174}$$

E. The Dipole-Density Tensor

In contrast with the case of electric monopoles, we will see that the formula (174) does not hold for elementary dipoles and higher multipoles, a fact that will turn out to have important implications for magnetic forces and mechanical work.

We will be particularly interested in studying elementary dipoles. To begin, we give names to the various components of the dipole-density tensor $M^{\mu\nu}$ appearing in our expression (139), $j_{\mathrm{d}}^{\nu} = \partial_{\mu}M^{\mu\nu}$, for the dipole-current density. Remembering from (142) that the dipole-density tensor is antisymmetric on its two indices, $M^{\mu\nu} = -M^{\nu\mu}$, we name its components according to

$$M^{\mu\nu} = \begin{pmatrix} 0 & cP_x & cP_y & cP_z \\ -cP_x & 0 & -M_z & M_y \\ -cP_y & M_z & 0 & -M_x \\ -cP_z & -M_y & M_x & 0 \end{pmatrix}^{\mu\nu}. \tag{175}$$

Here $\mathbf{P} = (M^{tx}/c, M^{ty}/c, M^{tz}/c)$ defines a three-vector field called the polarization, which we will see describes the volume density of electric dipoles, and $\mathbf{M} = (M^{yz}, M^{zx}, M^{xy})$ defines a three-vector field called the magnetization, which describes the volume density of magnetic dipoles. (The component combinations that define $\mathbf{P}$ and $\mathbf{M}$ transform as three-vectors under rotations, but transform as parts of the full antisymmetric tensor $M^{\mu\nu}$ under Lorentz boosts.) In terms of the electric displacement field $\mathbf{D}$ and the auxiliary magnetic field $\mathbf{H}$ introduced in (154), we have

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \tag{176}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{177}$$

Defining a charge density ρ_{d} and three-vector current density $\mathbf{J}_{\mathrm{d}} = (J_{\mathrm{d},x}, J_{\mathrm{d},y}, J_{\mathrm{d},z})$ from the components of the Lorentz-covariant dipole-current density j_{d}^{ν} according to

$$j_{\mathrm{d}}^{\nu} = (\rho_{\mathrm{d}} c, \mathbf{J}_{\mathrm{d}})^{\mu}, \tag{178}$$

it follows from a straightforward calculation starting with (139), $j_{\mathrm{d}}^{\nu} = \partial_{\mu}M^{\mu\nu}$, that ρ_{d} and $\mathbf{J}_{\mathrm{d}}$ are related to the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ according to the pair of equations

$$\rho_{\mathrm{d}} = -\nabla\cdot\mathbf{P}, \tag{179}$$

$$\mathbf{J}_{\mathrm{d}} = \frac{\partial\mathbf{P}}{\partial t} + \nabla\times\mathbf{M}. \tag{180}$$

Notice that these two formulas imply that ρ_{d} and $\mathbf{J}_{\mathrm{d}}$ automatically satisfy the continuity equation

$$\frac{\partial\rho_{\mathrm{d}}}{\partial t} = -\nabla\cdot\mathbf{J}_{\mathrm{d}}, \tag{181}$$

as was ultimately ensured by the local conservation equation (145). Observe also that ρ_{d} and $\mathbf{J}_{\mathrm{d}}$ are not related by a formula analogous to the equation (174), $\mathbf{J}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v}$, that held for the case of electric monopoles.

F. Composite Dipoles

We can provide an intuitive explanation for why the formulas (179) for ρ_{d} and (180) for $\mathbf{J}_{\mathrm{d}}$ indeed describe dipoles, as claimed. For this purpose, we momentarily put aside the case of *elementary* dipoles and consider instead a *composite* electric dipole consisting more fundamentally of a pair of electric monopoles with respective charges $q > 0$ and $-q < 0$ located respectively at positions $x = d > 0$ and $x = 0$ on the x axis. The charge density is then

$$\rho(x, y, z) = (+q)\,\delta(x-d)\,\delta(y)\,\delta(z) + (-q)\,\delta(x)\,\delta(y)\,\delta(z).$$

Letting $\mathbf{d} = (d, 0, 0)$ denote the spatial displacement vector extending from the negative electric monopole to the positive electric monopole, we define the system's electric dipole moment by

$$\boldsymbol{\pi} \equiv q\mathbf{d}. \tag{182}$$

Taking the limit $d \to 0$ with $\boldsymbol{\pi} \equiv q\mathbf{d}$ held fixed at finite magnitude and direction, we can write our expression for the charge density as

$$\begin{aligned}\rho(x, y, z) &= qd\,(-\delta'(x))\,\delta(y)\,\delta(z) \\ &= -q\mathbf{d}\cdot\nabla(\delta(x)\,\delta(y)\,\delta(z)) \\ &= -\nabla\cdot(\boldsymbol{\pi}\,\delta^3(\mathbf{x})),\end{aligned} \tag{183}$$

which replicates (179), $\rho_{\mathrm{d}} = -\nabla\cdot\mathbf{P}$, for a polarization $\mathbf{P}$ defined as the pointlike density $\boldsymbol{\pi}\,\delta^3(\mathbf{x})$ corresponding to the dipole moment $\boldsymbol{\pi} = q\mathbf{d}$ of the pair of electric point charges.

Under Lorentz boosts, the polarization transforms as part of the antisymmetric Lorentz tensor $M_{\mu\nu}$ in (175), whose form then dictates the formula (180) for the current density $\mathbf{J}_{\mathrm{d}}$, which we can alternatively understand by analogy with *composite* electric dipoles consisting of time-dependent pairs of electric monopoles and *composite* magnetic dipoles consisting of circulating loops of electric current.

G. Elementary Dipoles

We can also study the case of a pointlike *elementary* dipole at rest at $\mathbf{x}' = (x', y', z')$. We define the particle's elementary electric dipole moment $\boldsymbol{\pi}$ and elementary magnetic dipole moment $\boldsymbol{\mu}$ in terms of the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$ in the delta-function limit as

$$\mathbf{P}(\mathbf{x}) = \boldsymbol{\pi}\,\delta^3(\mathbf{x}-\mathbf{x}'), \tag{184}$$

$$\mathbf{M}(\mathbf{x}) = \boldsymbol{\mu}\,\delta^3(\mathbf{x}-\mathbf{x}'). \tag{185}$$

From (179), $\rho_{\mathrm{d}} = -\nabla\cdot\mathbf{P}$, the corresponding charge density is precisely as in (183) from the composite case,

$$\begin{aligned}\rho_{\mathrm{d}}(\mathbf{x}) &= -\nabla\cdot(\boldsymbol{\pi}\,\delta^3(\mathbf{x}-\mathbf{x}')) \\ &= -\boldsymbol{\pi}\cdot\nabla\delta^3(\mathbf{x}-\mathbf{x}'),\end{aligned} \tag{186}$$

and from (180), the current density $\mathbf{J}_{\mathrm{d}}$ is

$$\begin{aligned}\mathbf{J}_{\mathrm{d}}(\mathbf{x}) &= \nabla\times(\boldsymbol{\mu}\,\delta^3(\mathbf{x}-\mathbf{x}')) \\ &= -\boldsymbol{\mu}\times\nabla\delta^3(\mathbf{x}-\mathbf{x}').\end{aligned} \tag{187}$$

Another elementary calculation using the Maxwell equations (14)–(17) shows that the resulting electric field and magnetic field for all $\mathbf{x} \neq \mathbf{x}'$ have the standard inverse-cube dependence characteristic of dipoles,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{\pi}\cdot\mathbf{e}_{\mathbf{x}-\mathbf{x}'})\mathbf{e}_{\mathbf{x}-\mathbf{x}'} - \boldsymbol{\pi}}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{\boldsymbol{\pi}}{3\epsilon_0}\delta^3(\mathbf{x}-\mathbf{x}'), \tag{188}$$

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\,\frac{3(\boldsymbol{\mu}\cdot\mathbf{e}_{\mathbf{x}-\mathbf{x}'})\mathbf{e}_{\mathbf{x}-\mathbf{x}'} - \boldsymbol{\mu}}{|\mathbf{x}-\mathbf{x}'|^3} + \frac{2\mu_0\boldsymbol{\mu}}{3}\delta^3(\mathbf{x}-\mathbf{x}'), \tag{189}$$

where the unit vector $\mathbf{e}_{\mathbf{x}-\mathbf{x}'}$, defined in (167), is directed from the source point $\mathbf{x}'$ toward the field point $\mathbf{x}$, and where the delta-function contact terms ensure agreement with the homogeneous Maxwell equations (15) and (16). We conclude that this source distribution indeed describes an elementary dipole at rest at $\mathbf{x}'$.

IV. CLASSICAL ELECTROMAGNETISM WITH ELEMENTARY DIPOLES

Now that we have introduced sources into classical electromagnetism—namely, electric monopoles, elementary dipoles, and higher multipoles—we will need to determine the resulting dynamics. We will start by characterizing the electromagnetic properties of elementary dipoles before moving on to the Lagrangian formulation of the theory.

A. Dynamical Elementary Dipoles

Recall from its definition (148) that the multipole-density tensor $Q^{\mu\nu} = -Q^{\nu\mu}$ is given in terms of the tensors $M^{\mu\nu}, N^{\mu\rho\nu}, \dots$ respectively describing the densities of dipoles, quadrupoles, and higher multipoles by

$$Q^{\mu\nu} \equiv M^{\mu\nu} + \partial_\rho N^{\mu\rho\nu} + \cdots.$$

For a pointlike charged particle with position $\mathbf{X}$, recall that the electric-monopole current density j^ν_{e} is given in terms of the Lorentz four-vector qu^ν and the Lorentz-invariant, three-dimensional delta function (162) according to (169),

$$j^\nu_{\mathrm{e}}(\mathbf{x}, t) = qu^\nu\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}).$$

Similarly, the density tensors $M^{\mu\nu}, N^{\mu\rho\nu}, \dots$ for such a particle are given in terms of Lorentz tensors $m^{\mu\nu}, n^{\mu\rho\nu}, \ldots$ and the Lorentz-invariant, three-dimensional delta function according to

$$M^{\mu\nu} = m^{\mu\nu}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \tag{190}$$

$$N^{\mu\rho\nu} = n^{\mu\rho\nu}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \tag{191}$$

and so forth, meaning that the particle's overall multipole-density tensor would be

$$Q^{\mu\nu} = \frac{1}{\gamma}(m^{\mu\nu} + n^{\mu\rho\nu}\partial_\rho + \cdots)\delta^3(\mathbf{x}-\mathbf{X}). \tag{192}$$

To simplify our work ahead, we will assume that the particle does not have elementary quadrupole and higher multipole moments [11]. In that case, $Q^{\mu\nu}$ reduces to the dipole-density tensor (190),

$$Q^{\mu\nu} = M^{\mu\nu} = m^{\mu\nu}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \tag{193}$$

where we will call the antisymmetric tensor $m^{\mu\nu} = -m^{\nu\mu}$ the particle's elementary dipole tensor.

Mimicking our formula (175) relating the dipole-density tensor $M_{\mu\nu}$ to the polarization $\mathbf{P}$ and magnetization $\mathbf{M}$, we define the particle's elementary electric-dipole moment as $\boldsymbol{\pi} \equiv (m^{tx}/c, m^{ty}/c, m^{tz}/c)$ and its elementary magnetic-dipole moment as $\boldsymbol{\mu} \equiv (m^{yz}, m^{zx}, m^{xy})$, so that these three-vectors are related to the particle's elementary dipole tensor $m^{\mu\nu}$ according to

$$m^{\mu\nu} \equiv \begin{pmatrix} 0 & c\pi_x & c\pi_y & c\pi_z \\ -c\pi_x & 0 & -\mu_z & \mu_y \\ -c\pi_y & \mu_z & 0 & -\mu_x \\ -c\pi_z & -\mu_y & \mu_x & 0 \end{pmatrix}^{\mu\nu}. \tag{194}$$

In the particle's reference state, for which its four-momentum p_0^μ is (106) and its spin tensor $S_0^{\mu\nu}$ is (109), we can introduce a pair of purely spacelike four-vectors defined by

$$\pi_0^\mu \equiv (0, \boldsymbol{\pi}_0)^\mu, \tag{195}$$

$$\mu_0^\mu \equiv (0, \boldsymbol{\mu}_0)^\mu. \tag{196}$$

As in [12], we can then write the particle's elementary dipole tensor in general as

$$m^{\mu\nu} = \pi^{\mu\nu} + \mu^{\mu\nu}, \tag{197}$$

with

$$\pi^{\mu\nu} \equiv \frac{1}{mc}(p^\mu\pi^\nu - p^\nu\pi^\mu), \tag{198}$$

$$\mu^{\mu\nu} \equiv \frac{1}{mc}\epsilon^{\mu\nu\rho\sigma}p_\rho\mu_\sigma, \tag{199}$$

where $\pi^\nu(\lambda)$ and $\mu^\mu(\lambda)$ are related to their reference values π_0^ν and μ_0^ν and to the particle's variable Lorentz-transformation matrix $\Lambda^\mu{}_\nu(\lambda)$ according to

$$\pi^\mu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda)\pi_0^\nu, \tag{200}$$

$$\mu^\mu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda)\mu_0^\nu. \tag{201}$$

B. The Maxwell Action Functional with Sources

If our particle carries an electric-monopole charge q in addition to its elementary dipole tensor $m^{\mu\nu}$, then coupling the particle to the electromagnetic field leads immediately to the following generalization of the particle's action functional (102) and the Maxwell action functional (136), and thereby provides a classical extension of Maxwell's original theory of electromagnetism:

$$\begin{aligned} S[X,\Lambda,A] &\equiv S_{\text{particle}}[X,\Lambda] + S_{\text{field}}[A] + S_{\text{int}}[X,\Lambda,A] \\ &= \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}]\right) && (S_{\text{particle}}) \\ &\quad + \int dt \int d^3x \left(-\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}\right) && (S_{\text{field}}) \\ &\quad + \int dt \int d^3x\, j^\nu A_\nu && (S_{\text{int}}). \end{aligned} \tag{202}$$

Here we have included an important new contribution $S_{\text{int}}[X,\Lambda,A]$ that describes interactions between the particle and the electromagnetic field:

$$S_{\text{int}}[X,\Lambda,A] \equiv \int dt \int d^3x\, j^\nu A_\nu. \tag{203}$$

The terms in the action functional (202) that contain a dependence on the field degrees of freedom A_μ have the standard form (131), $S \equiv \int dt \int d^3x\, \mathcal{L}$, for a Lagrangian density $\mathcal{L}$ given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}} \\ &= -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} && (\mathcal{L}_{\text{field}}) \\ &\quad + j^\nu A_\nu && (\mathcal{L}_{\text{int}}). \end{aligned} \tag{204}$$

Using this Lagrangian density, the field-theoretic Euler-Lagrange equations (133) yield

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) \\ &\qquad = j^\nu - \partial_\mu\left(-\frac{1}{\mu_0}F^{\mu\nu}\right) = 0, \end{aligned}$$

thereby giving us the inhomogeneous Maxwell equation (40),

$$\partial_\mu F^{\mu\nu} = -\mu_0 j^\nu.$$

As was true for the free electromagnetic field, the homogeneous Maxwell equation (41) is already encoded into the formula $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ from (45):

$$\partial_\mu \tilde{F}^{\mu\nu} = 0.$$

The interaction term $j^\nu A_\nu$ appearing in the action functional (202) may not look gauge invariant, but under a gauge transformation (49),

$$A_\nu \mapsto A_\nu + \partial_\nu f,$$

the interaction term changes according to

$$j^\nu A_\nu \mapsto j^\nu A_\nu + j^\nu(\partial_\nu f).$$

Using the product rule in reverse (again, "integration by parts" without an integration), to move the spacetime derivative from f to j^μ at the cost of a minus sign, we end up with

$$j^\nu A_\nu - (\partial_\nu j^\nu) f + \begin{pmatrix}\text{total spacetime}\\ \text{divergence}\end{pmatrix}.$$

The second term vanishes by local current conservation (43), $\partial_\nu j^\nu = 0$, when the system's equations of motion are imposed, and the total spacetime divergence disappears from the action functional by the four-dimensional divergence theorem, under the assumption that our fields go to zero sufficiently rapidly at infinity. The action functional (202) is therefore effectively unchanged by gauge transformations, as required.

Before we can discuss the equations of motion for the particle, or the total energy and momentum of the overall system consisting of the particle together with the electromagnetic field, we will need to begin by recalling the multipole expansion (138):

$$\begin{aligned} j^\nu &= j_{\mathrm{e}}^\nu + j_{\mathrm{d}}^\nu + j_{\mathrm{q}}^\nu + \cdots \\ &= j_{\mathrm{e}}^\nu + \partial_\mu M^{\mu\nu} + \partial_\mu \partial_\rho N^{\mu\rho\nu} + \cdots . \end{aligned}$$

Dropping quadrupole and higher multipole moments, in line with our assumptions about the particle, this expansion truncates to just its electric-monopole and elementary-dipole terms:

$$j^\nu = j_{\mathrm{e}}^\nu + \partial_\mu M^{\mu\nu}. \tag{205}$$

Substituting this expression into the interaction term $j^\nu A_\nu$ yields

$$j^\nu A_\nu = j_{\mathrm{e}}^\nu A_\nu + (\partial_\mu M^{\mu\nu}) A_\nu,$$

so the overall system's action functional (202) becomes

$$\begin{aligned} S[X,\Lambda,A] &\equiv S_{\text{particle}}[X,\Lambda] + S_{\text{field}}[A] + S_{\text{int}}[X,\Lambda,A] \\ &= \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) && (S_{\text{particle}}) \\ &\quad + \int dt \int d^3x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right) && (S_{\text{field}}) \\ &\quad + \int dt \int d^3x \, (j_{\mathrm{e}}^\nu A_\nu + (\partial_\mu M^{\mu\nu}) A_\nu) && (S_{\text{int}}). \end{aligned} \tag{206}$$

Recalling the Lagrangian (60) for our xy system consisting of a pair of systems with degrees of freedom x and y,

$$L \equiv \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M \dot{y}^2 - a y^2 - b x y + c \dot{x} y,$$

we have an analogy in which the x system plays the role of our relativistic particle and the y system plays the role of the electromagnetic field, with the following detailed correspondences:

$$\left.\begin{aligned} \frac{1}{2} m \dot{x}^2 &\iff p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}], \\ \frac{1}{2} M \dot{y}^2 - a y^2 &\iff \int d^3x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right), \\ -bxy &\iff \int d^3x \, (j_{\mathrm{e}}^\nu A_\nu), \\ c\dot{x}y &\iff \int d^3x \, (\partial_\mu M^{\mu\nu}) A_\nu. \end{aligned}\right\} \tag{207}$$

We will find it useful to refer back to this analogy on several more occasions in our work ahead.

At the cost of a minus sign and an irrelevant additive total spacetime divergence, we are free to use the product rule in reverse to rewrite the final interaction term $(\partial_\mu M^{\mu\nu}) A_\nu$ in the integrand of the action functional (206) as

$$(\partial_\mu M^{\mu\nu}) A_\nu = -M^{\mu\nu}(\partial_\mu A_\nu) + \begin{pmatrix}\text{total spacetime}\\ \text{divergence}\end{pmatrix}.$$

Taking advantage of the antisymmetry of the dipole-density tensor $M^{\mu\nu}$, we can write the first term as

$$-M^{\mu\nu}(\partial_\mu A_\nu) = -\frac{1}{2} M^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu).$$

Remembering again the formula (45) relating the Faraday tensor $F_{\mu\nu}$ to the gauge potential A_μ, we have

$$-\frac{1}{2} M^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{1}{2} M^{\mu\nu} F_{\mu\nu}.$$

We can therefore write the overall system's action functional (202) in the alternative but physically equivalent form

$$\begin{aligned} S[X,\Lambda,A] &\equiv S_{\text{particle}}[X,\Lambda] + S_{\text{field}}[A] + S_{\text{int}}[X,\Lambda,A] \\ &= \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) && (S_{\text{particle}}) \\ &\quad + \int dt \int d^3x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right) && (S_{\text{field}}) \\ &\quad + \int dt \int d^3x \left(j_{\mathrm{e}}^\nu A_\nu - \frac{1}{2} M^{\mu\nu} F_{\mu\nu} \right) && (S_{\text{int}}). \end{aligned} \tag{208}$$

This last step of using the product rule in reverse to replace $(\partial_\mu M^{\mu\nu}) A_\nu$ with $-(1/2) M^{\mu\nu} F_{\mu\nu}$ is analogous to our use of the product rule in reverse in (83) to replace $c\dot{x}y$ with $-cx\dot{y}$ for the xy system. As was true in that example, this manipulation has no physical consequences, but we will find that our calculations ahead will be easier if we use (208) rather than (206) as our system's action functional, as the former ends up requiring fewer computations that explicitly involve delta functions.

C. The Action Functional for a Charged Particle with an Elementary Dipole Moment

Gathering together all the terms in the action functional (208) that involve the particle's degrees of freedom $X^\mu(\lambda) = (cT(\lambda), \mathbf{X}(\lambda))^\mu$ and $\Lambda^\mu{}_\nu(\lambda)$, we obtain

$$S_{\text{particle+int}}[X, \Lambda, A] = \int d\lambda \left(p_\mu \dot{X}^\mu + \frac{1}{2}\text{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) + \int dt \int d^3x\, j_{\text{e}}^\nu A_\nu + \int dt \int d^3x \left(-\frac{1}{2} \right) M^{\mu\nu} F_{\mu\nu}. \tag{209}$$

Before we can compute the system's Euler-Lagrangian equations, we will need to replace the integrals $\int dt \int d^3x\,(\cdots)$ over time and space with appropriate integrals $\int d\lambda\,(\cdots)$ over the particle's worldline parameter λ, and we will need to make the particle's worldline degrees of freedom $X^\mu(\lambda)$ and $\Lambda^\mu{}_\nu(\lambda)$ more manifest.

Under the assumption that our particle has charge q, the electric-monopole current density is (169),

$$\begin{aligned} j_{\text{e}}^\nu &= q\frac{dX^\nu(t)}{dt}\delta^3(\mathbf{x} - \mathbf{X}(t)) \\ &= q\int dT\, \frac{dX^\nu(T)}{dt}\delta(t-T)\,\delta^3(\mathbf{x} - \mathbf{X}(T)) \\ &= q\int d\lambda\, \frac{dT(\lambda)}{d\lambda}\frac{dX^\nu(T(\lambda))}{dt}\delta(t-T(\lambda))\,\delta^3(\mathbf{x} - \mathbf{X}(T(\lambda))) \\ &= \int d\lambda\, q\frac{dX^\nu(T(\lambda))}{d\lambda}\delta(t-T(\lambda))\,\delta^3(\mathbf{x} - \mathbf{X}(T(\lambda))), \end{aligned}$$

which we can write more succinctly as

$$j_{\text{e}}^\nu = \int d\lambda\, q\dot{X}^\nu \delta(t-T)\delta^3(\mathbf{x} - \mathbf{X}),$$

where, as usual, dots denote derivatives with respect to the particle's worldline parameter λ. We can therefore express the first interaction term in the particle's action functional (209) as

$$\int dt \int d^3x\, j_{\text{e}}^\nu A_\nu = \int d\lambda\, q\dot{X}^\nu A_\nu. \tag{210}$$

Similarly, we can write the dipole-density tensor $M^{\mu\nu}$ in terms of the particle's elementary dipole tensor $m^{\mu\nu}$ and the Lorentz-invariant three-dimensional delta function (162) as in (190),

$$\begin{aligned} M^{\mu\nu} &= m^{\mu\nu}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \\ &= \int d\lambda\, \frac{dT}{d\lambda} m^{\mu\nu}\frac{1}{\gamma}\delta(t-T)\delta^3(\mathbf{x} - \mathbf{X}). \end{aligned} \tag{211}$$

Combining the factor of $dT/d\lambda$ with the reciprocal Lorentz factor $1/\gamma$ to obtain

$$\begin{aligned} \frac{dT}{d\lambda}\frac{1}{\gamma} &= \frac{dT}{d\lambda}\sqrt{1 - \left(\frac{d\mathbf{X}}{dT}\right)^2 / c^2} \\ &\sqrt{\left(\frac{dT}{d\lambda}\right)^2 - \left(\frac{d\mathbf{X}}{d\lambda}\right)^2 / c^2} \\ &= \frac{1}{c}\sqrt{-\dot{X}^2}, \end{aligned}$$

the second interaction term becomes

$$\int dt \int d^3x \left(-\frac{1}{2} \right) M^{\mu\nu} F_{\mu\nu} = \int d\lambda \left(-\frac{1}{2c} \right) \sqrt{-\dot{X}^2}\, m^{\mu\nu} F_{\mu\nu}. \tag{212}$$

Putting everything together, we see that the particle's action functional is of the manifestly covariant form described in [1],

$$S_{\text{particle+int}}[X, \Lambda, A] = \int d\lambda\, \mathscr{L}_{\text{particle+int}}, \tag{213}$$

for a manifestly covariant Lagrangian defined by

$$\mathscr{L}_{\text{particle+int}} \equiv p_\mu \dot{X}^\mu + \frac{1}{2}\text{Tr}[S\dot{\Lambda}\Lambda^{-1}] + q\dot{X}^\nu A_\nu - \frac{1}{2c}\sqrt{-\dot{X}^2} m^{\mu\nu} F_{\mu\nu}. \tag{214}$$

D. The Dynamics of the Canonical Momentum of an Elementary Dipole

Now we are ready to calculate the particle's canonical momenta and its equations of motion. As we proceed, we will need to keep in mind that $A_\nu = A_\nu(X(\lambda))$ and $F_{\mu\nu} = F_{\mu\nu}(X(\lambda))$ depend on the particle's spacetime degrees of freedom $X^\mu(\lambda)$, as well as remember from (107) that $p^\mu(\lambda) = mc\,\Lambda^\mu{}_t(\lambda)$ does not depend on $X^\mu(\lambda)$ before the equations of motion have been imposed.

Following the manifestly covariant formalism presented in [1], the covariant canonical four-momentum conjugate to $X^\mu(\lambda)$ is given by

$$\begin{aligned} p_{\text{can},\mu} &\equiv \frac{\partial \mathscr{L}_{\text{particle+int}}}{\partial \dot{X}^\mu} \\ &= p_\mu + qA_\mu + \frac{1}{2c}\frac{\dot{X}_\mu}{\sqrt{-\dot{X}^2}} m^{\rho\sigma} F_{\rho\sigma}. \end{aligned} \tag{215}$$

Using the chain rule to write $d/d\lambda = \dot{X}^\nu \partial_\nu$ as needed, the covariant Euler-Lagrange equation for $X^\mu(\lambda)$,

$$\frac{\partial \mathscr{L}_{\text{particle+int}}}{\partial X^\mu} - \frac{d}{d\lambda}\left(\frac{\partial \mathscr{L}_{\text{particle+int}}}{\partial \dot{X}^\mu} \right) = 0, \tag{216}$$

yields the following equation of motion for the particle's four-momentum p^μ:

$$\dot{p}^\mu = -q\dot{X}_\nu F^{\nu\mu} - \frac{1}{2}\sqrt{-\dot{X}^2}m^{\rho\sigma}\partial^\mu F_{\rho\sigma} - \frac{1}{2c}\frac{d}{d\lambda}\left(\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}m^{\rho\sigma}F_{\rho\sigma}\right). \qquad (217)$$

This equation simplifies if we choose our worldline parameter λ to be the particle's proper time τ, in which case

$$\sqrt{-\dot{X}^2} \mapsto \sqrt{-(dX/d\tau)^2} = c. \qquad (218)$$

The particle's normalized four-velocity (119) then takes the form (123),

$$u^\mu = \frac{dX^\mu}{d\tau},$$

and so the equation of motion (217) becomes

$$\begin{aligned}\frac{dp^\mu}{d\tau} &= -qu_\nu F^{\nu\mu} - \frac{1}{2}m^{\rho\sigma}\partial^\mu F_{\rho\sigma} - \frac{1}{2c^2}\frac{d}{d\tau}(u^\mu m^{\rho\sigma}F_{\rho\sigma}) \\ &= -qu_\nu F^{\nu\mu} - \frac{1}{2}m^{\rho\sigma}(\eta^{\mu\nu} + u^\mu u^\nu)\partial_\nu F_{\rho\sigma} \\ &\quad - \frac{1}{2c^2}\frac{d}{d\tau}(u^\mu m^{\rho\sigma})F_{\rho\sigma}, \end{aligned} \qquad (219)$$

as obtained in [3, 4, 13].

E. The Non-Relativistic Limit with Time-Independent External Fields

We now examine the equation of motion (219) in the non-relativistic limit, in which the particle's proper time τ reduces to the coordinate time t and the particle's four-velocity u^ν reduces to a four-vector consisting of the speed of light c and the particle's three-dimensional velocity $\mathbf{v}$:

$$\begin{aligned}\tau &\approx t, \\ u^\nu &\approx (c, \mathbf{v})^\nu. \end{aligned} \qquad (220)$$

We will assume that the particle's velocity $\mathbf{v}$ changes slowly enough that we can neglect radiative effects. We will accordingly drop contributions to the electromagnetic fields from the particle itself, so that $\mathbf{E} \mapsto \mathbf{E}_{\text{ext}}$ and $\mathbf{B} \mapsto \mathbf{B}_{\text{ext}}$, where we will also assume that these external fields are time-independent (but not necessarily uniform in space) in the given inertial reference frame [14]. Making use of the tensor-contraction identity

$$m^{\rho\sigma}(\cdots)F_{\rho\sigma} = -2((\cdots)\mathbf{E})\cdot\boldsymbol{\pi} - 2((\cdots)\mathbf{B})\cdot\boldsymbol{\mu}, \qquad (221)$$

where $(\cdots)$ represents numerical quantities or derivative operators and where the particle's elementary dipole moments $\boldsymbol{\pi}$ and $\boldsymbol{\mu}$ are defined in terms of $m^{\rho\sigma}$ according to (194), the equation of motion (219) then reduces to the pair of three-dimensional equations

$$\frac{dE}{dt} \approx \mathbf{v}\cdot(q\mathbf{E}_{\text{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}} + \boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}})), \qquad (222)$$

$$\frac{d\mathbf{p}}{dt} \approx q(\mathbf{E}_{\text{ext}} + \mathbf{v}\times\mathbf{B}_{\text{ext}}) + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}} + \boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}}), \qquad (223)$$

where E is the particle's kinetic energy and $\mathbf{p}$ is its three-dimensional momentum, with

$$p^\mu = (E/c, \mathbf{p})^\mu \approx (mc^2 + (1/2)m\mathbf{v}^2, \mathbf{p})^\mu. \qquad (224)$$

Note that although the final term in the four-dimensional equation of motion (219) for $dp^\nu/d\tau$ includes an explicit factor of $1/c^2$, it is not a purely relativistic correction, and is necessary for getting the correct relativistic formula for dE/dt above.

F. The Generalized Lorentz Force Law for Elementary Dipoles

From the second of these two dynamical equations, (223), we can identify the electromagnetic force on the particle as

$$\mathbf{F} = q\mathbf{E}_{\text{ext}} + q\mathbf{v}\times\mathbf{B}_{\text{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}}), \quad (225)$$

which agrees with our claimed generalization (22) of the Lorentz force law.

Notice that the magnetic field participates in the dipole terms $\nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}})$ of this force law on an equal footing with the electric field. Furthermore, if the particle moves at a constant velocity $\mathbf{v}$ through incremental spatial displacements $d\mathbf{X} = \mathbf{v}dt$ over infinitesimal time intervals dt, then the work (6) done by the electromagnetic field on the particle as it travels from an initial location A to a final location B is

$$\begin{aligned} W &= \int_A^B d\mathbf{X}\cdot\mathbf{F} = \int_A^B dt\,\mathbf{v}\cdot\mathbf{F} \\ &= \int_A^B dt\,\mathbf{v}\cdot(q\mathbf{E}_{\text{ext}} + q\mathbf{v}\times\mathbf{B}_{\text{ext}}) \\ &\qquad + \int_A^B dt\,\mathbf{v}\cdot(\nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}})) \\ &= \int_A^B dt\,(q\mathbf{v}\cdot\mathbf{E}_{\text{ext}}) + \Delta(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}}) + \Delta(\boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}}), \end{aligned} \qquad (226)$$

where Δ denotes a total change over the particle's full displacement from A to B. We see right away that magnetic forces do not do work on *electric monopoles*, but are entirely capable of doing work on *elementary magnetic dipoles*.

Moreover, the *rate* at which electromagnetic forces do work on the particle is

$$\begin{aligned}\frac{dW}{dt} &= \frac{d}{dt}\int^{t} d\mathbf{X}\cdot\mathbf{F} = \frac{d}{dt}\int^{t} dt\,\mathbf{v}\cdot\mathbf{F}\\ &= \mathbf{v}\cdot\mathbf{F}\\ &= \mathbf{v}\cdot(q\mathbf{E}_{\text{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}} + \boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}})),\end{aligned} \tag{227}$$

where we have dropped the $q\mathbf{v}\times\mathbf{B}_{\text{ext}}$ term because its dot product with $\mathbf{v}$ vanishes. The formula (227) precisely agrees with our non-relativistic equation of motion (222) for the rate dE/dt at which the particle's kinetic energy is changing, so the work being done on the particle by the electromagnetic field is translating directly into the particle's kinetic energy.

Under our assumption that the external fields are all constant in time in the given inertial reference frame, the formula (47) relating the electric field to the scalar potential Φ and the vector potential $\mathbf{A}$ implies that the electric field is determined by the gradient of the scalar potential according to

$$\mathbf{E}_{\text{ext}} = -\nabla\Phi_{\text{ext}}. \tag{228}$$

The electromagnetic force (225) on the particle is therefore conservative in the sense of (9),

$$\mathbf{F} = -\nabla V,$$

where the potential energy V in the present case is given by

$$V = q\Phi_{\text{ext}} - \boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}} - \boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}}. \tag{229}$$

The work (226) done by the electromagnetic field on the particle then simplifies to

$$W = -\Delta V,$$

in accordance with the general relationship (10) between the work W done on a mechanical object and the object's corresponding potential energy V.

G. The Dynamics of the Intrinsic Spin

Next, we will use the particle's action functional (213) to calculate the equation of motion for the particle's spin tensor $S^{\mu\nu}(\lambda)$. Varying the action functional with respect to the variable Lorentz-transformation matrix $\Lambda^{\mu}{}_{\nu}(\lambda)$, we obtain

$$\delta S_{\text{particle+int}} = \int d\lambda\left(\delta p^{\mu}\dot{X}_{\mu} + \frac{1}{2}\mathrm{Tr}[\delta(S\dot{\Lambda}\Lambda^{-1})] - \frac{1}{2c}\sqrt{-\dot{X}^2}\,\delta m^{\mu\nu}F_{\mu\nu}\right). \tag{230}$$

As in [1], the first two terms yield

$$\begin{aligned}\delta p^{\mu}\dot{X}_{\mu} &= \frac{1}{2}(-\dot{X}_{\rho}p_{\sigma} + \dot{X}_{\sigma}p_{\rho})\delta\theta^{\rho\sigma},\\ \frac{1}{2}\mathrm{Tr}[\delta(S\dot{\Lambda}\Lambda^{-1})] &= \frac{1}{2}S_{\rho\sigma}\frac{d}{d\lambda}\delta\theta^{\rho\sigma},\end{aligned}$$

where $\delta\theta^{\rho\sigma}$ is an array of small boost and rotation parameters corresponding to the infinitesimal variation in $\Lambda^{\mu}{}_{\nu}(\lambda)$. Meanwhile, using the commutation relations (29) satisfied by the Lorentz generators, together with [15]

$$\begin{aligned}\delta m^{\mu\nu} &= -\frac{1}{4}\mathrm{Tr}[m(\sigma^{\mu\nu}\sigma^{\rho\sigma} - \sigma^{\rho\sigma}\sigma^{\mu\nu})]\delta\theta_{\rho\sigma}\\ &= \frac{1}{2}(-m^{\nu\rho}\eta^{\mu\sigma} - m^{\mu\sigma}\eta^{\nu\rho} + m^{\nu\sigma}\eta^{\mu\rho} + m^{\mu\rho}\eta^{\nu\sigma})\delta\theta_{\rho\sigma},\end{aligned} \tag{231}$$

the third term in the varied action functional (230) gives

$$\begin{aligned}&-\frac{1}{2c}\sqrt{-\dot{X}^2}\,\delta m^{\mu\nu}F_{\mu\nu}\\ &\quad= -\frac{1}{2c}\sqrt{-\dot{X}^2}(m^{\rho\mu}F^{\sigma}{}_{\mu} - m^{\sigma\mu}F^{\rho}{}_{\mu})\delta\theta_{\rho\sigma}.\end{aligned} \tag{232}$$

Putting everything together and setting the overall variation (230) in the particle's action functional to zero in accordance with the usual extremization condition (53), we obtain

$$\begin{aligned}&\delta S_{\text{particle+int}}\\ &= \int d\lambda\,\frac{1}{2}\bigg(-(\dot{X}^{\rho}p^{\sigma} - \dot{X}^{\sigma}p^{\rho}) - \dot{S}^{\rho\sigma}\\ &\qquad - \frac{1}{c}\sqrt{-\dot{X}^2}(m^{\rho\mu}F^{\sigma}{}_{\mu} - m^{\sigma\mu}F^{\rho}{}_{\mu})\bigg)\delta\theta_{\rho\sigma} = 0,\end{aligned}$$

where we have dropped the total derivative $d(S^{\rho\sigma}\delta\theta_{\rho\sigma})/d\lambda$ from the middle term. We therefore find the following equation of motion for the particle's spin tensor $S^{\mu\nu}$:

$$\begin{aligned}\dot{S}^{\mu\nu} &= -(\dot{X}^{\mu}p^{\nu} - \dot{X}^{\nu}p^{\mu})\\ &\qquad - \frac{1}{c}\sqrt{-\dot{X}^2}(m^{\mu\rho}F^{\nu}{}_{\rho} - m^{\nu\rho}F^{\mu}{}_{\rho}).\end{aligned} \tag{233}$$

Once again, we simplify this equation by choosing the worldline parameter λ to be the particle's proper time, so that from (218), we have

$$\sqrt{-\dot{X}^2} \mapsto c,$$

and from (123), we have

$$\dot{X}^{\mu} \mapsto u^{\mu}.$$

The equation of motion for $S^{\mu\nu}$ then becomes

$$\frac{dS^{\mu\nu}}{d\tau} = -(u^{\mu}p^{\nu} - u^{\nu}p^{\mu}) - (m^{\mu\rho}F^{\nu}{}_{\rho} - m^{\nu\rho}F^{\mu}{}_{\rho}), \tag{234}$$

which generalizes the results of [2–4].

The particle's orbital angular-momentum tensor is defined as in (128) by

$$L^{\mu\nu} \equiv X^\mu p^\nu - X^\nu p^\mu,$$

with $\mathbf{L} \equiv (L^{yz}, L^{zx}, L^{xy}) = \mathbf{X} \times \mathbf{p}$ the particle's orbital angular-momentum pseudovector. Using

$$\frac{dL^{\mu\nu}}{d\tau} = u^\mu p^\nu - u^\nu p^\mu + X^\mu \frac{dp^\nu}{d\tau} - X^\nu \frac{dp^\mu}{d\tau}, \qquad (235)$$

it follows from a straightforward calculation that if we ignore self-field effects, then the non-relativistic limit of the spin tensor's equation of motion (234) is

$$\frac{d(\mathbf{L}+\mathbf{S})}{dt} \approx \mathbf{X} \times \frac{d\mathbf{p}}{dt} + \boldsymbol{\pi} \times \mathbf{E}_{\text{ext}} + \boldsymbol{\mu} \times \mathbf{B}_{\text{ext}}, \qquad (236)$$

which describes a net torque on the particle given by the sum of orbital and dipole contributions.

H. Self-Consistency Conditions

Now that we have obtained the particle's equations of motion, we will need to ensure that they are compatible with the fundamental structure of the particle's phase space—specifically, that they are consistent with the constancy of the invariant quantities m^2, w^2, s^2, and $\tilde{s}^2$ defined (97)–(100), as well as with the condition $p_\mu S^{\mu\nu} = 0$ from (101).

We will start by examining the condition $p_\mu S^{\mu\nu} = 0$. Taking its derivative with respect to the proper time τ, we find

$$\frac{dp_\mu}{d\tau} S^{\mu\nu} + p_\mu \frac{dS^{\mu\nu}}{d\tau} = 0,$$

which yields an equation of the form

$$p^\mu = m_{\text{eff}} u^\mu + b^\mu. \qquad (237)$$

Here the coefficient function $m_{\text{eff}}(\lambda)$ is defined by

$$m_{\text{eff}} \equiv -\frac{m^2 c^2}{p \cdot u}, \qquad (238)$$

and we naturally identify it as the particle's effective inertial mass. The four-vector $b^\mu(\lambda)$, which represents the discrepancy between $p^\mu(\lambda)$ and $m_{\text{eff}}(\lambda) u^\mu(\lambda)$, is defined by

$$b^\mu \equiv \frac{1}{p \cdot u} \left(\frac{dp_\nu}{d\tau} S^{\nu\mu} - p_\nu (m^{\nu\rho} F^\mu{}_\rho - m^{\mu\rho} F^\nu{}_\rho) \right). \qquad (239)$$

Following [4], we regard (237) as an *implicit formula* for the particle's four-velocity u^μ.

Combining the condition $p_\mu S^{\mu\nu} = 0$ with the definition (239) of b^μ, we see that b^μ has vanishing Lorentz dot product with the particle's four-momentum p^μ:

$$b \cdot p = 0. \qquad (240)$$

Contracting both sides of (237) with p_μ then yields (97), $p^2 = -m^2 c^2$, thereby ensuring that p^2 is constant, as required:

$$\frac{d}{dt}(p^2) = 0. \qquad (241)$$

If the electromagnetic field is zero, $F_{\mu\nu} = 0$, then it follows from a straightforward calculation that $b^\mu = 0$ and $m_{\text{eff}} = m$, so the particle's four-momentum p^μ is parallel to its four-velocity and with m playing the role of the proportionality constant:

$$p^\mu = m u^\mu \quad (F_{\mu\nu} = 0). \qquad (242)$$

On the other hand, for nonzero electromagnetic field, $F_{\mu\nu} \neq 0$, the terms in the definition (239) of b^μ go like $1/c^2$, so the discrepancy four-vector b^μ is a relativistic correction. It follows that $m_{\text{eff}} - m$ is likewise a relativistic correction of order $1/c^2$, so

$$p^\mu = m u^\mu + (\text{terms of order } 1/c^2). \qquad (243)$$

One key implication of these results is that when discussing work done by electromagnetic forces on the particle in the non-relativistic limit, as in (222)–(223), there is no ambiguity over whether we should identify $E = p^t c$ or $u^t m c^2$ as the particle's "true" relativistic kinetic energy. Indeed, in the non-relativistic limit, they agree:

$$E = p^t c \approx u^t m c^2 \approx mc^2 + \frac{1}{2} m \mathbf{v}^2. \qquad (244)$$

Next, we study the invariant spin-squared scalar s^2 defined in (99). Invoking the spin tensor's equation of motion (234) together with the condition (101), $p_\mu S^{\mu\nu} = 0$, we have

$$\begin{aligned} \frac{d}{d\tau}(s^2) &= \frac{d}{d\tau}\left(\frac{1}{2} S_{\mu\nu} S^{\mu\nu}\right) = 2 S_{\mu\nu} \frac{dS^{\mu\nu}}{d\tau} \\ &= (S^\rho{}_\mu m^{\mu\sigma} - S^\sigma{}_\mu m^{\mu\rho}) F_{\rho\sigma}. \end{aligned} \qquad (245)$$

The scalar quantity s^2 is therefore constant along the particle's worldline for generic states of the electromagnetic field only if the quantity in parentheses above vanishes, meaning that

$$S^\rho{}_\mu m^{\mu\sigma} = S^\sigma{}_\mu m^{\mu\rho}. \qquad (246)$$

This equality implies that in the particle's reference state, the reference values (195)–(196) of the particle's three-dimensional elementary electric and magnetic dipole moments must both have vanishing cross products with the reference value $\mathbf{S}_0$ of the particle's spin three-vector:

$$\left.\begin{aligned} \boldsymbol{\pi}_0 \times \mathbf{S}_0 &= 0, \\ \boldsymbol{\mu}_0 \times \mathbf{S}_0 &= 0. \end{aligned}\right\} \qquad (247)$$

Hence, at the level of the particle's underlying kinematics, the particle's elementary electric and magnetic dipole moments must be collinear with its spin three-vector:

$$\left.\begin{aligned} \boldsymbol{\pi}_0 &= \frac{1}{c} \Xi \, \mathbf{S}_0, \\ \boldsymbol{\mu}_0 &= \Gamma \, \mathbf{S}_0. \end{aligned}\right\} \qquad (248)$$

Here Ξ is a pseudoscalar constant and Γ is a scalar constant, the latter of which is called the particle's gyromagnetic ratio, and the factor of $1/c$ appearing in the formula for $\boldsymbol{\pi}_0$ compensates for the factors of c appearing in (194).

The conditions (248) make physical sense, because if the particle had elementary dipole moments that were not parallel or antiparallel to the particle's spin axis, then electromagnetic torques acting on the particle's elementary dipole moments would be capable of "speeding up" or "slowing down" the particle's total spin, thereby contravening the invariance of s^2 [16].

Finally, one can readily show that

$$w^2 = m^2c^2s^2, \tag{249}$$

$$\tilde{s}^2 = 0. \tag{250}$$

Hence, w^2 and $\tilde{s}^2$ are likewise constant, as required:

$$\frac{d}{d\tau}(w^2) = 0, \tag{251}$$

$$\frac{d}{d\tau}(\tilde{s}^2) = 0. \tag{252}$$

V. CONSERVATION LAWS AND THEIR IMPLICATIONS

To provide a crucial set of consistency checks on our results so far, we now proceed to replicate them from the perspective of local conservation laws. We will begin by discussing Noether's theorem, which we will use to construct tensors that encode conserved notions of energy, momentum, and angular momentum. After calculating these tensors for the electromagnetic field coupled to a relativistic charged particle with elementary electric and magnetic dipole moments, we will show explicitly that the exchange of relevant conserved quantities precisely accounts for the generalized Lorentz force law and the work done by the field on the particle.

A. Conservation Laws and Noether's Theorem

In its various versions, Noether's theorem establishes a correspondence between the symmetries of a physical system's dynamics and the quantities that are conserved when the system evolves according to its equations of motion. We will present and prove one version of the theorem whose details will end up being particularly relevant to our elementary-dipole model.

To begin, we consider a continuous symmetry of our system's dynamics, meaning a transformation $q_\alpha \mapsto q'_\alpha$ of the system's degrees of freedom that can be performed by an arbitrarily small amount and that leaves the system's Euler-Lagrange equations (54) unchanged. More precisely, a continuous symmetry has the following ingredients.

- The transformation rule can be expressed in infinitesimal form as

$$\begin{aligned} q_\alpha \mapsto q'_\alpha = q_\alpha + \delta_\epsilon q_\alpha, \\ \delta_\epsilon q_\alpha = \sum_b g_{q_\alpha,b}\epsilon_b, \end{aligned} \tag{253}$$

where the coefficients $g_{q_\alpha,b}$ depend on the degrees of freedom and where the parameters ϵ_b are constants that are assumed to be small but are otherwise arbitrary.

- The system's Lagrangian L does not depend explicitly on the parameters ϵ_b,

$$\frac{\partial L}{\partial \epsilon_b} = 0, \tag{254}$$

meaning that any possible dependence of L on the parameters ϵ_b arises solely through the degrees of freedom q_α.

- The Lagrangian is invariant under the given transformation rule, up to a possible total time derivative:

$$\begin{aligned} &L \mapsto L + \delta_\epsilon L, \\ &\delta_\epsilon L = \frac{d}{dt}\left(\sum_b f_b\epsilon_b\right) = \sum_b \frac{df_b}{dt}\epsilon_b. \end{aligned} \tag{255}$$

The functions f_b here are zero in the simplest cases.

The condition (255) ensures that the system's action functional $S \equiv \int dt\, L$ changes by at most boundary terms that give no contribution when we apply the extremization condition (53) to obtain the system's Euler-Lagrange equations.

It is important to keep in mind that in order for the transformation (253) to qualify as a symmetry of the dynamics, the condition (255) on the Lagrangian must hold *before* applying the system's equations of motion. Note also that identifying the correct functions f_b is a crucial step, as we will see when we use Noether's theorem to derive both the conserved energy-momentum and the conserved angular momentum for the electromagnetic field coupled to an elementary dipole.

To prove the theorem and derive an explicit formula for the associated conserved quantities, we begin by applying the chain rule to the variation $\delta_\epsilon L$ of the Lagrangian appearing on the left-hand side of (255):

$$\begin{aligned} &\delta_\epsilon L - \sum_b \frac{df_b}{dt}\epsilon_b \\ &\quad = \sum_\alpha \frac{\partial L}{\partial q_\alpha}\delta_\epsilon q_\alpha + \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}\delta_\epsilon \dot{q}_\alpha \\ &\qquad + \sum_b \frac{\partial L}{\partial \epsilon_b}\epsilon_b - \sum_b \frac{df_b}{dt}\epsilon_b = 0. \end{aligned}$$

Invoking the transformation formula (253) together with the requirement (254) that the Lagrangian has no explicit dependence on the transformation parameters ϵ_b, we have

$$\sum_b \left(\sum_\alpha \frac{\partial L}{\partial q_\alpha} g_{q_\alpha,b}\epsilon_b + \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \dot{g}_{q_\alpha,b}\epsilon_b - \frac{df_b}{dt}\epsilon_b \right) = 0.$$

Using the product rule in reverse on the second term, we obtain

$$\sum_b \sum_\alpha \left(\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} \right) g_{q_\alpha,b}\epsilon_b + \sum_b \frac{d}{dt}\left(\sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha,b} - f_b \right)\epsilon_b = 0.$$

If we now consider a trajectory $q_\alpha(t)$ that satisfies the system's Euler-Lagrangian equations (54), then the first term above vanishes and we are left with

$$\sum_b \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha,b} - f_b \right)\epsilon_b = 0.$$

This equation must hold for arbitrary values of the parameters ϵ_b, so we conclude that the quantity Q_b defined as the terms in parentheses for each value of b is individually conserved.

We have thereby proved Noether's theorem, and obtained an explicit formula for the conserved quantities Q_b corresponding to the given continuous symmetry:

$$Q_b \equiv \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha,b} - f_b, \quad \frac{dQ}{dt} = 0. \tag{256}$$

Two important examples merit discussion.

- If the Lagrangian $L(q, \dot{q}, t)$ of the system is invariant under constant translations along the coordinates,

  $$q_\alpha \mapsto q'_\alpha \equiv q_\alpha + \epsilon_\alpha, \tag{257}$$

  so that

  $$\delta_\epsilon q_\alpha = \epsilon_\alpha = \sum_\beta g_{q_\alpha,\beta}\epsilon_\beta, \quad g_{q_\alpha,\beta} = \delta_{\alpha\beta}, \tag{258}$$

  with

  $$\delta_\epsilon L = 0, \tag{259}$$

  then the functions in (255) vanish, $f_\beta = 0$, and the conserved quantities (256) are just the canonical momenta (55):

  $$Q_\beta = \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha,\beta} = p_\beta. \tag{260}$$

- On the other hand, consider the time translation $t \mapsto t' \equiv t+\epsilon$ in which we shift t by a small constant ϵ. We require that the values $q'_\alpha(t')$ of the system's transformed degrees of freedom at the new time $t' \equiv t+\epsilon$ agree with their original values $q_\alpha(t)$ at the time t, so that

  $$q_\alpha(t) \mapsto q'_\alpha(t') = q_\alpha(t). \tag{261}$$

  Equivalently, the values $q'_\alpha(t)$ of the system's transformed degrees of freedom at the original time t agree with their values $q_\alpha(t-\epsilon)$ at the earlier time $t-\epsilon$,

  $$q_\alpha(t) \mapsto q'_\alpha(t) = q_\alpha(t-\epsilon). \tag{262}$$

  Then, by the chain rule, the system's degrees of freedom q_α and the Lagrangian L both transform by total time derivatives:

  $$\delta_\epsilon q_\alpha = -\dot{q}_\alpha\epsilon, \quad g_{q_\alpha} = -\dot{q}_\alpha, \tag{263}$$

  $$\delta_\epsilon L = -\frac{dL}{dt}\epsilon. \tag{264}$$

  If the Lagrangian $L(q, \dot{q})$ has no explicit dependence on the time t, meaning no dependence on t outside of the degrees of freedom q_α and their rates of change $\dot{q}_\alpha$, then

  $$\frac{\partial L}{\partial \epsilon} \equiv \frac{\partial L}{\partial t} = 0,$$

  so all the conditions of Noether's theorem are satisfied with $f \equiv -L$, and the associated conserved quantity is just the system's Hamiltonian (56), up to an overall minus sign:

  $$Q = \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha} - f = -\sum_\alpha p_\alpha \dot{q}_\alpha + L = -H. \tag{265}$$

Noether's theorem (256) generalizes naturally to the manifestly covariant Lagrangian framework described in [1], with the time t replaced by a more general smooth, strictly monotonic parameter λ and with the Lagrangian L replaced by the manifestly covariant Lagrangian $\mathscr{L} = (dt/d\lambda)L$, as in (91).

B. Energy-Momentum Tensors for Classical Field Theories

Noether's theorem (256) is a powerful tool for studying the possible conservation laws for various classical systems, including classical field theories.

Given a classical field theory with local field degrees of freedom $\varphi_\alpha(x)$ and an action functional (131),

$$S[\varphi] = \int dt \int d^3x\, \mathcal{L}(\varphi, \partial\varphi, x),$$

we will start by considering the infinitesimal transformation $x^\mu \mapsto x'^\mu \equiv x^\mu + \epsilon^\mu$ in which we translate the spacetime coordinates x^μ by a small constant four-vector ϵ^μ. We will then require that the transformed values $\varphi'_\alpha(x')$ of the field degrees of freedom at the new spacetime point $x'^\mu = x^\mu + \epsilon^\mu$ are equal to their values $\varphi_\alpha(x)$ at the original spacetime point x^μ:

$$\varphi'_\alpha(x') = \varphi_\alpha(x). \tag{266}$$

Replacing x'^μ with x^μ and replacing x^μ with $x^\mu - \epsilon^\mu$, and using the chain rule, we obtain the following infinitesimal transformation rule for the field degrees of freedom:

$$\begin{aligned}\varphi_\alpha(x) \mapsto \varphi'_\alpha(x) &\equiv \varphi_\alpha(x-\epsilon)\\ &= \varphi_\alpha(x) - \partial_\mu\varphi_\alpha(x)\epsilon^\mu.\end{aligned} \tag{267}$$

That is, the infinitesimal changes in the field degrees of freedom are given by

$$\delta_\epsilon\varphi_\alpha = -\partial_\mu\varphi_\alpha\,\epsilon^\mu, \quad g_{\varphi_\alpha,\mu} = -\partial_\mu\varphi_\alpha. \tag{268}$$

If the Lagrangian density $\mathcal{L}(\varphi, \partial\varphi)$ has no explicit dependence on the spacetime coordinates x^μ, meaning no dependence on x^μ apart from any dependence arising through φ_α and $\partial_\mu\varphi_\alpha$, then all the conditions of Noether's theorem will be satisfied if we can determine the corresponding functions f_μ appearing in (255). Assuming that the fields go to zero sufficiently rapidly at spatial infinity so that we can neglect boundary terms, we have from the chain rule that

$$\begin{aligned}\delta_\epsilon L &= \int d^3x\,(-\partial_\mu\mathcal{L}\,\epsilon^\mu) = -\int d^3x\,\partial_t\mathcal{L}\,\epsilon^t\\ &= -\frac{1}{c}\frac{d}{dt}\int d^3x\,\mathcal{L}\,\epsilon^t\\ &= \frac{df_\mu}{dt}\epsilon^\mu,\end{aligned}$$

for

$$f_\mu = -\int d^3x\,\frac{1}{c}\delta^t_\nu\mathcal{L}. \tag{269}$$

From Noether's theorem (256), we therefore obtain the following collection of conserved quantities:

$$\begin{aligned}Q_\nu &= \int d^3x\left(\sum_\alpha \frac{\partial\mathcal{L}}{\partial(\partial\varphi_\alpha/\partial t)}g_{\varphi_\alpha,\nu}\right) - f_\nu\\ &= \frac{1}{c}\int d^3x\left(-\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi_\alpha)}\partial_\nu\varphi_\alpha + \delta^t_\nu\mathcal{L}\right).\end{aligned} \tag{270}$$

Introducing a unit timelike four-vector $n_\mu \equiv (-1, \mathbf{0})_\mu$ that is orthogonal to the three-dimensional spatial hypersurface of integration, we can write the conserved quantities (270) more covariantly as

$$Q_\nu = \frac{1}{c}\int d^3x\,(-n_\mu)\left(-\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\partial_\nu\varphi_\alpha + \delta^\mu_\nu\mathcal{L}\right). \tag{271}$$

The conservation law $dQ_\nu/dt = 0$ then corresponds to the vanishing of the difference between three-dimensional integrations (271) on two adjacent spatial hypersurfaces separated by an infinitesimal amount of time dt. Hence, by the four-dimensional divergence theorem, and under the assumption that the fields go to zero sufficiently rapidly at spatial infinity, the equation $dQ_\nu/dt = 0$ implies that the quantity in parentheses in (271) has vanishing spacetime divergence:

$$\partial_\mu\left(-\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\partial_\nu\varphi_\alpha + \delta^\mu_\nu\mathcal{L}\right) = 0. \tag{272}$$

Raising the ν index using the Minkowski metric tensor, we define the quantity in parentheses as the system's canonical energy-momentum tensor:

$$T^{\mu\nu}_{\text{can}} \equiv -\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\partial^\nu\varphi_\alpha + \eta^{\mu\nu}\mathcal{L}. \tag{273}$$

This tensor satisfies the local conservation law

$$\partial_\mu T^{\mu\nu}_{\text{can}} = 0 \tag{274}$$

and naturally generalizes the Hamiltonian (265) to a local, Lorentz-covariant density of energy and momentum.

Notice that Noether's theorem does not determine $T^{\mu\nu}_{\text{can}}$ uniquely, because we are free to add terms to the definition (273) that have vanishing spacetime divergence without affecting the local energy-momentum conservation law (274):

$$T^{\mu\nu} \equiv T^{\mu\nu}_{\text{can}} + (\cdots)^{\mu\nu}, \qquad \partial_\mu(\cdots)^{\mu\nu} = 0. \tag{275}$$

That is, this redefined energy-momentum tensor $T^{\mu\nu}$ continues to satisfy the equation

$$\partial_\mu T^{\mu\nu} = 0. \tag{276}$$

The addition of terms as in (275) may be necessary to ensure that the energy-momentum tensors $T^{\mu\nu}$ for certain field theories have particular properties, like gauge invariance. However, even when such a redefinition (275) provides a better description of a system's underlying physics, the canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}}$ may still be more convenient for certain calculations, as we will see in our work ahead.

The first index μ on $T^{\mu\nu}$ determines whether we are referring to a volume density or to a flux density, the latter representing a rate of flow per unit time per unit cross-sectional area, so we will refer to μ as the flux index of $T^{\mu\nu}$. The second index ν tells us whether the physical quantity in question is energy or momentum, so we will refer to ν as the four-momentum index of $T^{\mu\nu}$. In analogy with (39) for the charge-current density j^μ, we therefore have the schematic formula

$$T^{\mu\nu} = \begin{cases}\text{density of (momentum)}^\nu & \text{for } \mu = t,\\ \text{flux density of (momentum)}^\nu & \text{for } \mu = x, y, z.\end{cases} \tag{277}$$

More concretely, the individual components of $T^{\mu\nu}$ have the following physical interpretations.

- The three-dimensional scalar
$$u = T^{tt} \tag{278}$$
represents the volume density of the field's mass-energy.

- The three-dimensional vector
$$\mathbf{S} = c\,(T^{xt}, T^{yt}, T^{zt}) \tag{279}$$
represents the flux density of the field's energy, meaning the rate of energy flow per unit time per unit cross-sectional area.

- The three-dimensional vector
$$\mathbf{g} = \frac{1}{c}(T^{tx}, T^{ty}, T^{tz}) \tag{280}$$
represents the field's momentum density.

- The three-dimensional tensor
$$\mathbb{T}_{ij} = -T^{ij}, \tag{281}$$
called the field's stress tensor, represents the field's momentum flux densities, with the (i, j) component representing the flux density of the jth component of momentum in the ith direction. The diagonal components $\mathbb{T}_{xx}, \mathbb{T}_{yy}, \mathbb{T}_{zz}$ encode the pressures in each of the three Cartesian directions, and the off-diagonal components $\mathbb{T}_{xy}, \mathbb{T}_{xz}, \mathbb{T}_{yx}, \mathbb{T}_{yz}, \mathbb{T}_{zx}, \mathbb{T}_{zy}$ encode shearing effects.

If we introduce terms into the action functional (131) that describe interactions between the field and source systems, such as mechanical particles, then these source systems will generically exchange energy and momentum with the field in the form of work and forces. Because these flows of energy and momentum imply that the field can gain or lose energy and momentum, they appear as violations of the local conservation equation (276), $\partial_\mu T^{\mu\nu} = 0$, that would have otherwise held for the field alone.

Specifically, any energy entering or leaving the field corresponds to violations of the $\nu = t$ component of (276) that describe the rate at which work is done by the field on sources. Any momentum entering or leaving the field corresponds to violations of the $\nu = x, y, z$ components of (276) that describe forces due to the field on sources.

We can capture all these violations in terms of a new four-vector f^ν that is related to the spacetime divergence of the field's energy-momentum tensor $T^{\mu\nu}$ according to the local four-force law
$$f^\nu = -\partial_\mu T^{\mu\nu}. \tag{282}$$

Letting $\partial w/\partial t$ denote the power density on sources, meaning the rate at which the field does work on sources per unit volume, and letting $\mathbf{f} = (f_x, f_y, f_z)$ denote the field's force density on sources, the preceding analysis implies that
$$f^\nu \equiv \left(\frac{1}{c}\frac{\partial w}{\partial t}, f_x, f_y, f_z\right)^\nu, \tag{283}$$
and so we naturally refer to f^ν as the field's four-force density. (Four-forces are also called Minkowski forces.)

Given a knowledge of a field's energy-momentum tensor, the local four-force equation (282) provides a very general way to derive force laws on source particles. In particular, we will see in the example of the electromagnetic field that (282) will end up yielding the Lorentz force law in the more general form (22) that includes forces on elementary dipoles.

C. Angular-Momentum Flux Tensors for Classical Field Theories

We can also use Noether's theorem to determine the local conservation law corresponding to Lorentz invariance. Under Lorentz transformations, the spacetime coordinates x^μ transform as
$$x^\mu \mapsto x'^\mu \equiv \Lambda^\mu{}_\nu x^\nu. \tag{284}$$
We require that the new values $\varphi'_\alpha(x')$ of the field degrees of freedom at x'^μ are related to their values $\varphi_\alpha(x)$ according to a general rule of the form
$$\varphi_\alpha(x) \mapsto \varphi'_\alpha(x') \equiv (F(\Lambda)\varphi)_\alpha(x), \tag{285}$$
where $F(\Lambda)$ captures the possibility that the field index α has a nontrivial behavior under Lorentz transformations. Equivalently, replacing $x'^\mu = (\Lambda x)^\mu$ with x^μ and replacing x^μ with $(\Lambda^{-1}x)^\mu$, we have
$$\varphi'_\alpha(x) = (F(\Lambda)\varphi)_\alpha(\Lambda^{-1}x). \tag{286}$$

Specializing now to an infinitesimal Lorentz transformation (33), chosen to be an active transformation by replacing $-d\theta^{\mu\nu} \mapsto +\epsilon^{\mu\nu}$, we have
$$\Lambda_{\text{inf}} = 1 + \frac{i}{2}\epsilon^{\mu\nu}\sigma_{\mu\nu}, \tag{287}$$
and the field degrees of freedom transform as
$$\begin{aligned}\varphi'_\alpha(x) &= (F(1 + (i/2)\epsilon^{\mu\nu}\sigma_{\mu\nu})\varphi)_\alpha(x - (i/2)\epsilon^{\rho\sigma}\sigma_{\rho\sigma})\\ &= \varphi_\alpha(x) - \partial_\mu\varphi(x)\frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\nu x^\nu + \frac{1}{2}(\Delta_{\rho\sigma}\varphi)_\alpha(x)\epsilon^{\rho\sigma},\end{aligned} \tag{288}$$
where the final term represents the infinitesimal changes in the fields at fixed x:
$$\frac{1}{2}(\Delta_{\rho\sigma}\varphi)_\alpha(x)\epsilon^{\rho\sigma} \equiv (F(1 + (i/2)\epsilon^{\mu\nu}\sigma_{\mu\nu})\varphi)_\alpha(x) - \varphi_\alpha(x). \tag{289}$$

Dropping factors of 1/2 to avoid double-counting independent variables, we can therefore identify

$$g_{\varphi_\alpha,\rho\sigma} = -\partial_\mu\varphi\, i[\sigma_{\rho\sigma}]^\mu{}_\nu x^\nu + (\Delta_{\rho\sigma}\varphi)_\alpha. \tag{290}$$

If the field theory's Lagrangian density is Lorentz invariant, then all we have left to do is determine the functions $f_{\rho\sigma}$ appearing in (255). We find

$$\begin{aligned}\delta_\epsilon L &= \int d^3x\,(\partial_\mu\mathcal{L})\bigg(-\frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\nu x^\nu\bigg)\\ &= \frac{1}{2}\frac{df_{\rho\sigma}}{dt}\epsilon^{\rho\sigma},\end{aligned}$$

with

$$f_{\rho\sigma} = -\int d^3x\,\frac{1}{c}\delta^t_\nu\mathcal{L}\, i[\sigma_{\rho\sigma}]^\mu{}_\nu x^\nu. \tag{291}$$

Thus, according to Noether's theorem (256), we end up with the following conserved quantities:

$$\begin{aligned}Q_{\nu\rho} = \frac{1}{c}\int d^3x\,(-n_\mu)&\\ \times\bigg(-\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\partial_\sigma\varphi_\alpha + \delta^\mu_\sigma\mathcal{L}\bigg)& i[\sigma_{\nu\rho}]^\sigma{}_\lambda x^\lambda\\ + \frac{1}{c}\int d^3x\,(-n_\mu)\bigg(\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}\bigg)&(\Delta_{\nu\rho}\varphi)_\alpha,\end{aligned} \tag{292}$$

where, again, $n_\mu \equiv (-1,\mathbf{0})_\mu$ is a unit timelike four-vector that is orthogonal to the three-dimensional spatial hypersurface of integration. Raising the ν and ρ indices, and recalling the definition (273) of the field's canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}}$ together with the formula (28) for the Lorentz generators $[\sigma_{\mu\nu}]^\alpha{}_\beta$, we can write these conserved quantities as

$$Q^{\nu\rho} = -\int d^3x\,(-n_\mu)\mathcal{J}^{\mu\nu\rho}_{\text{can}}, \tag{293}$$

where

$$\mathcal{J}^{\mu\nu\rho}_{\text{can}} \equiv \mathcal{L}^{\mu\nu\rho} + \mathcal{S}^{\mu\nu\rho} = -\mathcal{J}^{\mu\rho\nu}_{\text{can}} \tag{294}$$

and

$$\mathcal{L}^{\mu\nu\rho} \equiv x^\nu\frac{1}{c}T^{\mu\rho}_{\text{can}} - x^\rho\frac{1}{c}T^{\mu\nu}_{\text{can}} = -\mathcal{L}^{\mu\rho\nu}, \tag{295}$$

$$\mathcal{S}^{\mu\nu\rho} \equiv -\frac{1}{c}\sum_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\alpha)}(\Delta^{\nu\rho}\varphi)_\alpha = -\mathcal{S}^{\mu\rho\nu} \tag{296}$$

are all antisymmetric on their final two indices, and where $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ is locally conserved:

$$\partial_\mu\mathcal{J}^{\mu\nu\rho}_{\text{can}} = 0. \tag{297}$$

The tensor $\mathcal{L}^{\mu\nu\rho}$ generalizes the mechanical definition $\mathbf{L} = \mathbf{X}\times\mathbf{p}$ of orbital angular momentum for particles, whereas the tensor $\mathcal{S}^{\mu\nu\rho}$ represents intrinsic spin angular momentum in the field itself, so $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ is called the canonical total angular-momentum flux tensor.

The local conservation laws (274) for $T^{\mu\nu}_{\text{can}}$ and (297) for $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ together imply that the spacetime divergence of the field's spin flux tensor $\mathcal{S}^{\mu\nu\rho}$ characterizes the lack of symmetry in the two indices of the field's canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}}$:

$$T^{\nu\rho}_{\text{can}} - T^{\rho\nu}_{\text{can}} = -c\,\partial_\mu\mathcal{S}^{\mu\nu\rho}. \tag{298}$$

As reviewed in [17], we can use this relation to construct a symmetric energy-momentum tensor and simplify the formula (294) for the canonical total angular-momentum flux tensor. We start by defining the Belinfante-Rosenfeld tensor,

$$\mathcal{B}^{\mu\rho\nu} \equiv \frac{c}{2}(\mathcal{S}^{\mu\nu\rho} + \mathcal{S}^{\nu\mu\rho} + \mathcal{S}^{\rho\mu\nu}), \tag{299}$$

which is antisymmetric on its first two indices,

$$\mathcal{B}^{\mu\rho\nu} = -\mathcal{B}^{\rho\mu\nu}, \tag{300}$$

is *asymmetric* on its first and last indices according to

$$\mathcal{B}^{\mu\rho\nu} = \mathcal{B}^{\nu\rho\mu} + c\mathcal{S}^{\rho\mu\nu}, \tag{301}$$

and has the property that its spacetime divergence $\partial_\rho\mathcal{B}^{\mu\rho\nu}$ on its second index is automatically locally conserved,

$$\partial_\mu(\partial_\rho\mathcal{B}^{\mu\rho\nu}) = 0. \tag{302}$$

The redefined energy-momentum tensor

$$T^{\mu\nu} \equiv T^{\mu\nu}_{\text{can}} + \partial_\rho\mathcal{B}^{\mu\rho\nu} \tag{303}$$

then continues to satisfy the local conservation equation (274),

$$\partial_\mu T^{\mu\nu} = 0,$$

is symmetric on its two indices,

$$T^{\mu\nu} = T^{\nu\mu}, \tag{304}$$

and, assuming that the fields go to zero sufficiently rapidly at spatial infinity, $T^{t\nu}$ has the same integrated value over all of three-dimensional space as $T^{t\nu}_{\text{can}}$,

$$\int d^3x\,T^{t\nu} = \int d^3x\,T^{t\nu}_{\text{can}}. \tag{305}$$

Moreover, the new total angular-momentum flux tensor defined by

$$\mathcal{J}^{\mu\nu\rho} \equiv x^\nu\frac{1}{c}T^{\mu\rho} - x^\rho\frac{1}{c}T^{\mu\nu} \tag{306}$$

differs from the canonical total angular-momentum flux tensor $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ by a term that is antisymmetric on its final two indices and has vanishing spacetime divergence, so $\mathcal{J}^{\mu\nu\rho}$ is still locally conserved:

$$\partial_\mu \mathcal{J}^{\mu\nu\rho} = 0. \tag{307}$$

The tensor $\mathcal{J}^{\mu\nu\rho}$ also has the same integrated value over all of three-dimensional space as $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$, so we are free to use $\mathcal{J}^{\mu\nu\rho}$ instead of $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ to describe the field's total angular momentum.

If we include terms in the field's action functional (131) that describe interactions with source systems, then the spacetime divergence $\partial_\mu \mathcal{J}^{\mu\nu\rho}$ characterizes the degree to which the angular momentum of the field is locally conserved, and satisfies the equation

$$-c\partial_\mu \mathcal{J}^{\mu\nu\rho} = x^\nu f^\rho - x^\rho f^\nu, \tag{308}$$

where $f^\nu = -\partial_\mu T^{\mu\nu}$ is the four-force density from (282). The terms $x^\nu f^\rho - x^\rho f^\nu$, which generalize the mechanical definition $\boldsymbol{\tau} = \mathbf{X} \times \mathbf{F}$ of torque, describe the density of torques exerted by the field on the source system. If this torque density vanishes, then we get back the local conservation law (307),

$$\partial_\mu \mathcal{J}^{\mu\nu\rho} = 0,$$

thereby implying that the field's angular momentum is locally conserved.

As an aside, notice the formal resemblance between the decomposition (303) of the redefined energy-momentum tensor,

$$T^{\mu\nu} \equiv T^{\mu\nu}_{\text{can}} + \partial_\rho \mathcal{B}^{\mu\rho\nu},$$

and the first two terms of the series expansion (141) for the current density j^ν,

$$j^\nu = j^\nu_{\text{e}} + \partial_\mu M^{\mu\nu} + \cdots .$$

We see that the spacetime-divergence term in $T^{\mu\nu}$ representing the intrinsic spin of the classical field is analogous to the spacetime-divergence term in j^ν representing the contribution from electric and magnetic dipoles.

Observe also that if we use the energy-momentum tensor (303), which is symmetric on its two indices, $T^{\mu\nu} = T^{\nu\mu}$, then

$$(T^{xt}, T^{yt}, T^{zt}) = (T^{tx}, T^{ty}, T^{tz}),$$

so we have the following simple relationship between the field's energy flux density (279) and the field's momentum density (280):

$$\mathbf{S} = \mathbf{g}c^2. \tag{309}$$

If we consider spatially compact distributions of the field propagating at an overall velocity $\mathbf{v}$, then integrating this formula over three-dimensional space yields a relationship between the total field energy E and the total field momentum $\mathbf{p}$,

$$\mathbf{v}E = \mathbf{p}c^2,$$

or, equivalently,

$$\mathbf{v} = \frac{\mathbf{p}c^2}{E},$$

which we first saw in our formula (114) for relativistic particles.

D. Local Conservation of Energy and Momentum for the Free Electromagnetic Field

For the electromagnetic field in the absence of charges and currents, meaning that $j^\mu = (\rho c, \mathbf{J})^\mu = 0$, the action functional is (136),

$$\begin{aligned} S_{\text{field}}[A] &\equiv \int dt \int d^3x\, \mathcal{L}_{\text{field}} \\ &= \int dt \int d^3x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right). \end{aligned}$$

Thus, the definition (273) of the electromagnetic field's canonical energy-momentum tensor yields

$$\begin{aligned} T^{\mu\nu}_{\text{can}} &\equiv -\frac{\partial \mathcal{L}_{\text{field}}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho + \eta^{\mu\nu} \mathcal{L}_{\text{field}} \\ &= \frac{1}{\mu_0} F^{\mu\rho} \partial^\nu A_\rho - \eta^{\mu\nu} \frac{1}{4\mu_0} F^{\rho\sigma} F_{\rho\sigma}. \end{aligned} \tag{310}$$

As a consequence of the invariance of the dynamics under constant translations in time and space, Noether's theorem guarantees that this canonical energy-momentum tensor satisfies the local conservation law (274),

$$\partial_\mu T^{\mu\nu}_{\text{can}} = 0. \tag{311}$$

However, $T^{\mu\nu}_{\text{can}}$ is not invariant under gauge transformations (49), due to the explicit appearance of the gauge potential A_ρ in its first term,

$$\frac{1}{\mu_0} F^{\mu\rho} \partial^\nu A_\rho. \tag{312}$$

Notice that we could remedy this issue by adding on a new term

$$T^{\mu\nu}_{\text{add}} \equiv -\frac{1}{\mu_0} F^{\mu\rho} \partial_\rho A^\nu, \tag{313}$$

which would have the effect of converting the non-gauge-invariant term (312) into the manifestly gauge-invariant combination

$$\frac{1}{\mu_0} F^{\mu\rho} (\partial^\nu A_\rho - \partial_\rho A^\nu) = \frac{1}{\mu_0} F^{\mu\rho} F^\nu{}_\rho. \tag{314}$$

Invoking the inhomogeneous Maxwell equation (40) in the absence of sources, $\partial_\rho F^{\mu\rho} = 0$, we can write $T^{\mu\nu}_{\text{add}}$ alternatively as a total spacetime divergence:

$$T^{\mu\nu}_{\text{add}} = \partial_\rho \left(-\frac{1}{\mu_0} F^{\mu\rho} A^\nu \right). \tag{315}$$

It follows immediately from the antisymmetry of the indices μ and ρ on $F^{\mu\rho}$ that this proposed new term has vanishing spacetime divergence,

$$\partial_\mu T^{\mu\nu}_{\mathrm{add}} = \partial_\mu \partial_\rho \biggl(-\frac{1}{\mu_0} F^{\mu\rho} A^\nu \biggr) = 0, \tag{316}$$

so adding it to the canonical energy-momentum tensor $T^{\mu\nu}_{\mathrm{can}}$ would have no effect on the local conservation equation (311). Furthermore, if we integrate the energy-momentum volume density $T^{t\nu}_{\mathrm{add}}$ over three-dimensional space, then because $F^{tt} = 0$, we end up with the integral of a total three-dimensional divergence that vanishes under the assumption that our fields go to zero sufficiently rapidly at spatial infinity:

$$\begin{aligned} \int d^3x\, T^{t\nu}_{\mathrm{add}} &= \int d^3x\, \partial_\rho \biggl(-\frac{1}{\mu_0} F^{t\rho} A^\nu \biggr) \\ &= \int d^3x\, \nabla \cdot (\cdots) = 0. \end{aligned}$$

Hence, adding $T^{\mu\nu}_{\mathrm{add}}$ to $T^{\mu\nu}_{\mathrm{can}}$ does not alter the field's overall energy and momentum.

The sum $T^{\mu\nu}_{\mathrm{can}} + T^{\mu\nu}_{\mathrm{add}}$ gives us the physical (and gauge-invariant) electromagnetic energy-momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\mathrm{can}} - \frac{1}{\mu_0} F^{\mu\rho} \partial_\rho A^\nu \\ &= \frac{1}{\mu_0} F^{\mu\rho} F^\nu{}_\rho - \eta^{\mu\nu} \frac{1}{4\mu_0} F^{\rho\sigma} F_{\rho\sigma}. \end{aligned} \tag{317}$$

By construction, in the absence of charged sources, this energy-momentum continues to satisfy the local conservation law (274),

$$\partial_\mu T^{\mu\nu} = 0, \tag{318}$$

and its individual components describe the density and flux of electromagnetic energy and momentum throughout three-dimensional space.

- The electromagnetic energy density is

$$u = T^{tt} = \frac{1}{2} \biggl(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \biggr). \tag{319}$$

- The electromagnetic energy flux density is

$$\mathbf{S} = c(T^{xt}, T^{yt}, T^{zt}) = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}, \tag{320}$$

  which is also known as the Poynting vector.

- The electromagnetic momentum density is

$$\mathbf{g} = \frac{1}{c} (T^{tx}, T^{ty}, T^{tz}) = \epsilon_0 \mathbf{E} \times \mathbf{B}. \tag{321}$$

- The electromagnetic momentum flux density is given by the Maxwell stress tensor,

$$\begin{aligned} \mathbb{T} &= -\begin{pmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{pmatrix} \\ &= \epsilon_0 \mathbf{E}\mathbf{E} + \frac{1}{\mu_0} \mathbf{B}\mathbf{B} - \mathbb{I} \frac{1}{2} \biggl(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0} \mathbf{B}^2 \biggr), \end{aligned} \tag{322}$$

  where $\mathbb{I}$ is the identity tensor.

E. Local Conservation of Energy and Momentum for the Electromagnetic Field Coupled to an Elementary Dipole

When we couple the electromagnetic field to a charged particle with elementary dipole moments, the energy and momentum of the field become mixed together with those of the particle. As a result, in order to study local conservation of energy and momentum for the overall system, we will need to look again at the full action functional (208), which we can use (210) and (212) to write as

$$\begin{aligned} S[X, \Lambda, A] &= \int dt \int d^3x\, \mathcal{L} \\ &= \int d\lambda \left(p_\nu \dot{X}^\nu + \frac{1}{2} \mathrm{Tr}[S \dot{\Lambda} \Lambda^{-1}] \right) \\ &\quad + \int dt \int d^3x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} \right) \\ &\quad + \int d\lambda\, q \dot{X}^\nu A_\nu \\ &\quad - \frac{1}{2c} \int d\lambda \sqrt{-\dot{X}^2} m^{\mu\nu} F_{\mu\nu}. \end{aligned} \tag{323}$$

Our plan will be to use the symmetry of the dynamics under constant translations in spacetime together with Noether's theorem (256) to determine the canonical energy-momentum tensor for the overall system.

To begin, we consider infinitesimal translations for which the particle's degrees of freedom $X^\mu(\lambda)$ and $\Lambda^\mu{}_\nu(\lambda)$ transform according to

$$\left. \begin{aligned} X^\mu(\lambda) &\mapsto X'^\mu(\lambda) \equiv X^\mu(\lambda) + \epsilon^\mu, \\ \Lambda^\mu{}_\nu(\lambda) &\mapsto \Lambda'^\mu{}_\nu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda), \end{aligned} \right\} \tag{324}$$

where ϵ^μ is a four-vector consisting of small, constant components. In order for this transformation to be a symmetry of the action functional, we will need the gauge field $A_\mu(x)$ to transform in such a way that its new value $A'_\mu(x')$ at the new spacetime point $x'^\mu = x^\mu + \epsilon^\mu$ is equal to its original value $A_\mu(x)$ at the original spacetime point x^μ:

$$A'_\mu(x') = A_\mu(x). \tag{325}$$

Replacing $x'^\mu \equiv x^\mu + \epsilon^\mu$ with x^μ and replacing x^μ with $x^\mu - \epsilon^\mu$, we obtain the following infinitesimal transformation rule for the gauge field:

$$\begin{aligned} A_\mu(x) \mapsto A'_\mu(x) &\equiv A_\mu(x-\epsilon) \\ &= A_\mu(x) - \partial_\nu A_\mu(x)\epsilon^\nu. \end{aligned} \tag{326}$$

We therefore identify

$$\delta X^\mu = \epsilon^\mu = \delta^\mu_\nu \epsilon^\nu \qquad \Longrightarrow \; g_{X^\mu,\nu} = \delta^\mu_\nu, \tag{327}$$

$$\delta A_\mu = -\partial_\nu A_\mu \epsilon^\nu \qquad \Longrightarrow \; g_{A_\mu,\nu} = -\partial_\nu A_\mu. \tag{328}$$

We can write the system's action functional (323) as

$$S[X,\Lambda,A] = \int dt\, L,$$

with a standard, non-covariant Lagrangian

$$\begin{aligned} L = p_\nu \frac{dX^\nu}{dt} &+ \frac{1}{2}\mathrm{Tr}\left[S\frac{d\Lambda}{dt}\Lambda^{-1}\right] \\ &+ \int d^3x \left(-\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}\right) \\ &+ q\frac{dX^\nu}{dt}A_\nu \\ &- \frac{1}{2c}\sqrt{-(dX/dt)^2}m^{\mu\nu}F_{\mu\nu}. \end{aligned} \tag{329}$$

Before we can employ Noether's theorem, it will be crucial to determine the correct functions f_ν that appear on the right-hand side of (255),

$$\delta_\epsilon L = \frac{df_\nu}{dt}\epsilon^\nu.$$

Only the second line in (329) gives a nonzero contribution, and we find

$$\begin{aligned} f_\nu &= \int d^3x\, \frac{1}{c}\delta^t_\nu\left(\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}\right) \\ &= \int d^3x\, \frac{1}{c}(-n_\mu)\delta^\mu_\nu\left(\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}\right), \end{aligned} \tag{330}$$

where, as before, $n_\mu \equiv (-1,\mathbf{0})_\mu$ is a unit timelike four-vector.

Putting everything together, and recalling our expression (214) for the particle's manifestly covariant Lagrangian $\mathscr{L} \equiv \mathscr{L}_{\text{particle+int}}$ together with our formula (323) for the overall system's Lagrangian density $\mathcal{L}$, Noether's theorem (256) then tells us that the conserved canonical four-momentum of the overall system is

$$\begin{aligned} P_\nu &= \frac{\partial\mathscr{L}}{\partial\dot{X}^\rho}g_{X^\rho,\nu} + \int d^3x\,(-n_\mu)\frac{\partial\mathcal{L}}{\partial(c\partial_\mu A_\rho)}g_{A_\rho,\nu} - f_\nu \\ &= p_\nu + qA_\nu + \frac{1}{2c^2}u_\nu m^{\sigma\tau}F_{\sigma\tau} \\ &\quad + \frac{1}{c}\int d^3x\,(-n_\mu)\left(H^{\mu\rho}\partial_\nu A_\rho - \delta^\mu_\nu\left(\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}\right)\right) \\ &= \frac{1}{c}\int d^3x\,(-n_\mu)T^\mu_{\text{can},\nu}, \end{aligned} \tag{331}$$

where we have identified the overall system's canonical energy-momentum tensor as

$$\begin{aligned} T^{\mu\nu}_{\text{can}} = u^\mu p^\nu &\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\ &+ H^{\mu\rho}\partial^\nu A_\rho + j^\mu_{\text{e}}A^\nu - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma} \\ &+ \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}). \end{aligned} \tag{332}$$

Here we have invoked the definition (151) of the auxiliary Faraday tensor $H^{\mu\nu}$, specialized to the case $Q^{\mu\nu} = M^{\mu\nu}$ in which quadrupole moments and higher multipole moments are absent,

$$\begin{aligned} H^{\mu\nu} &= \frac{1}{\mu_0}F^{\mu\nu} + M^{\mu\nu} \\ &= \frac{1}{\mu_0}F^{\mu\nu} + m^{\mu\nu}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \end{aligned} \tag{333}$$

and j^μ_{e} is the particle's electric-monopole current density (169),

$$j^\mu_{\text{e}} = qu^\mu\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}).$$

The terms $H^{\mu\rho}\partial^\nu A_\rho + j^\mu_{\text{e}}A^\nu$ in the canonical energy-momentum tensor (332) do not look gauge invariant. However, we can use the auxiliary inhomogeneous Maxwell equation (152) to write the interaction term $j^\mu_{\text{e}}A^\nu$ as

$$j^\mu_{\text{e}}A^\nu = -H^{\mu\rho}\partial_\rho A^\nu + \partial_\rho(H^{\mu\rho}A^\nu), \tag{334}$$

so when the equations of motion hold, the canonical energy-momentum tensor (332) is equivalent to

$$\begin{aligned} T^{\mu\nu}_{\text{can}} = u^\mu p^\nu &\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\ &+ H^{\mu\rho}F^\nu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2 \\ &+ \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\ &+ \partial_\rho(H^{\mu\rho}A^\nu). \end{aligned} \tag{335}$$

The last term in (335) is a total spacetime divergence, and taking its spacetime divergence on its μ index yields zero:

$$\partial_\mu\partial_\rho(H^{\mu\rho}A^\nu) = 0.$$

Moreover, the integral of its $\nu = t$ component over three-dimensional space gives a boundary term that vanishes if we assume that our fields go to zero sufficiently rapidly at spatial infinity:

$$\int d^3x\,\partial_\rho(H^{\mu\rho}A^\nu) = \int d^3x\,\nabla\cdot(\cdots) = 0.$$

We can therefore ignore this term in our calculations ahead.

Notice the crucial role played here by the interaction term $j_{\mathrm{e}}^{\mu}A^{\nu}$, which gave us the correction $-H^{\mu\rho}\partial_\rho A^\nu$ that we needed to yield a gauge-invariant combination $H^{\mu\rho}F^{\nu}{}_{\rho}$ in the canonical energy-momentum tensor (335). Despite the fact that it arises from the interaction term $j_{\mathrm{e}}^{\mu}A^{\nu}$, it is natural to regard the correction $-H^{\mu\rho}\partial_\rho A^\nu$ as part of the electromagnetic field's internal energy, even when dipoles are absent and $H^{\mu\rho}$ reduces to $(1/\mu_0)F^{\mu\rho}$.

We can divide up $T^{\mu\nu}_{\mathrm{can}}$ into the canonical energy-momentum tensor for the particle alone,

$$T^{\mu\nu}_{\mathrm{can,particle}} \equiv u^\mu p^\nu \frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \tag{336}$$

and the canonical energy-momentum tensor for the field,

$$\begin{aligned} T^{\mu\nu}_{\mathrm{can,field}} \equiv{}& H^{\mu\rho}F^{\nu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2 \\ &+ \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\ &+ \partial_\rho(H^{\mu\rho}A^\nu), \end{aligned} \tag{337}$$

which we can equivalently write as

$$\begin{aligned} T^{\mu\nu}_{\mathrm{can,field}} ={}& H^{\mu\rho}F^{\nu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4}(H^{\rho\sigma}+M^{\rho\sigma})F_{\rho\sigma} \\ &+ \frac{1}{2}\left(\eta^{\mu\nu}+\frac{u^\mu u^\nu}{c^2}\right)M^{\rho\sigma}F_{\rho\sigma} \\ &+ \partial_\rho(H^{\mu\rho}A^\nu). \end{aligned} \tag{338}$$

We have

$$\int d^3x\, T^{t\nu}_{\mathrm{can,particle}} = p^\nu c, \tag{339}$$

as expected, and [18]

$$\begin{aligned} \int d^3x\, T^{t\nu}_{\mathrm{can,field}} ={}& \int d^3x \left(H^{t\rho}F^{\nu}{}_{\rho} - \eta^{t\nu}\frac{1}{4\mu_0}F^2\right) \\ &+ \frac{1}{2c}u^\nu m^{\rho\sigma}F_{\rho\sigma}. \end{aligned} \tag{340}$$

In close analogy with the construction (82) from the example of our xy system, we can integrate the local conservation law (274), $\partial_\mu T^{\mu\nu}_{\mathrm{can}} = 0$, over three-dimensional space to compute the time derivative of the particle's four-momentum p^ν:

$$\begin{aligned} \frac{dp^\nu}{dt} &= \frac{1}{c}\frac{d}{dt}\int d^3x\, T^{t\nu}_{\mathrm{can,particle}} \\ &= -\frac{1}{c}\frac{d}{dt}\int d^3x\, T^{t\nu}_{\mathrm{can,field}} \\ &= \int d^3x \left(-\partial_\mu\left(H^{\mu\rho}F^{\nu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2\right)\right) \\ &\quad - \frac{1}{2c^2}\frac{d}{dt}(u^\nu m^{\rho\sigma}F_{\rho\sigma}). \end{aligned}$$

By a straightforward calculation, we have

$$\begin{aligned} &-\partial_\mu\left(H^{\mu\rho}F^{\nu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2\right) \\ &\qquad = -j_{\mathrm{e},\rho}F^{\rho\nu} - M^{\mu\rho}\partial_\mu F^{\nu}{}_{\rho}, \end{aligned}$$

and so, using $dt/d\tau = \gamma$ from (122), we obtain

$$\frac{dp^\nu}{d\tau} = -qu_\mu F^{\mu\nu} + m_{\rho\mu}\partial^\mu F^{\nu\rho} - \frac{1}{2c^2}\frac{d}{d\tau}(u^\nu m^{\rho\sigma}F_{\rho\sigma}).$$

Invoking the electromagnetic Bianchi identity (42),

$$\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} = 0,$$

we can write the second term as

$$m_{\rho\mu}\partial^\mu F^{\nu\rho} = -\frac{1}{2}m_{\rho\sigma}\partial^\nu F^{\rho\sigma}.$$

Relabeling indices, we find

$$\frac{dp^\mu}{d\tau} = -qu_\nu F^{\nu\mu} - \frac{1}{2}m_{\rho\sigma}\partial^\mu F^{\rho\sigma} - \frac{1}{2c^2}\frac{d}{d\tau}(u^\mu m^{\rho\sigma}F_{\rho\sigma}),$$

which precisely replicates the particle's equation of motion (219).

F. Local Conservation of Angular Momentum

Observe that the overall system's canonical energy-momentum tensor (335) is not symmetric on its two indices, $T^{\mu\nu}_{\mathrm{can}} \neq T^{\nu\mu}_{\mathrm{can}}$, reflecting the fact that it does not encode the system's intrinsic spin. To analyze local conservation of angular momentum for the overall system comprehensively, we return once again to the full action functional (323):

$$\begin{aligned} S[X,\Lambda,A] ={}& \int d\lambda \left(p_\nu \dot{X}^\nu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}]\right) \\ &+ \int dt \int d^3x \left(-\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}\right) \\ &+ \int d\lambda\, q\dot{X}^\nu A_\nu \\ &- \frac{1}{2c}\int d\lambda\, \sqrt{-\dot{X}^2}m^{\mu\nu}F_{\mu\nu}. \end{aligned}$$

Our next goal will be to invoke the symmetry of this action functional under Lorentz transformations along with Noether's theorem to compute the system's canonical angular-momentum tensor.

We start by noting that under an active ($-d\theta^{\rho\sigma} \mapsto +\epsilon^{\rho\sigma}$) infinitesimal Lorentz transformation (33),

$$\Lambda_{\mathrm{inf}} = 1 + \frac{i}{2}\epsilon^{\rho\sigma}\sigma_{\rho\sigma}, \tag{341}$$

the particle's degrees of freedom $X^\mu(\lambda)$ and $\Lambda^{\mu}{}_{\nu}(\lambda)$ transform according to

$$\left.\begin{aligned} X^\mu(\lambda) \mapsto X'^\mu(\lambda) &\equiv (\Lambda_{\mathrm{inf}}X(\lambda))^\mu \\ &= X^\mu(\lambda) + \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^{\mu}{}_{\nu}X^\nu(\lambda), \\ \Lambda^{\mu}{}_{\nu}(\lambda) \mapsto \Lambda'^{\mu}{}_{\nu}(\lambda) &\equiv (\Lambda_{\mathrm{inf}}\Lambda(\lambda))^{\mu}{}_{\nu} \\ &= \Lambda^{\mu}{}_{\nu}(\lambda) + \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^{\mu}{}_{\lambda}\Lambda^{\lambda}{}_{\nu}(\lambda), \end{aligned}\right\} \tag{342}$$

where $\epsilon^{\rho\sigma}$ is an antisymmetric tensor consisting of small constants. Note that the lower Lorentz index on $\Lambda^\mu{}_\nu(\lambda)$ does not participate in the second transformation rule, which fundamentally arises from the composition property $\Lambda' \equiv \Lambda_{\mathrm{inf}}\Lambda(\lambda)$. Observe also that we can rephrase this second transformation rule as the statement that the underlying antisymmetric array $\theta^{\mu\nu}(\lambda)$ of boost and angular parameters transforms as

$$\theta^{\mu\nu}(\lambda) \mapsto \theta'^{\mu\nu}(\lambda) \equiv \theta^{\mu\nu}(\lambda) + \epsilon^{\mu\nu}. \tag{343}$$

Meanwhile, the gauge field $A_\mu(x)$ transforms as

$$\begin{aligned} A_\mu(x) \mapsto A'_\mu(x) &\equiv (A(\Lambda_{\mathrm{inf}}^{-1}x)\Lambda_{\mathrm{inf}}^{-1})_\mu \\ &\equiv A_\lambda((1-(i/2)\epsilon^{\rho\sigma}\sigma_{\rho\sigma})x)(\delta^\lambda_\mu - (i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\lambda{}_\mu) \\ &= A_\mu(x) - \partial_\nu A_\mu(x)(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda \\ &\quad - A_\lambda(x)(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\lambda{}_\mu. \end{aligned} \tag{344}$$

We can therefore identify

$$\begin{aligned} \delta X^\mu =& \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\nu X^\nu \\ &\implies g_{X^\mu,\rho\sigma} = i[\sigma_{\rho\sigma}]^\mu{}_\nu X^\nu, \end{aligned} \tag{345}$$

$$\begin{aligned} \delta\theta^{\mu\nu} &= \epsilon^{\mu\nu} = \frac{1}{2}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)\epsilon^{\rho\sigma} \\ &\implies g_{\theta^{\mu\nu},\rho\sigma} = \delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho, \end{aligned} \tag{346}$$

$$\begin{aligned} \delta A_\mu &= -\partial_\nu A_\mu(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda - A_\nu(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\nu{}_\mu \\ &\implies g_{A_\mu,\rho\sigma} = -\partial_\nu A_\mu i[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda - A_\nu i[\sigma_{\rho\sigma}]^\nu{}_\mu. \end{aligned} \tag{347}$$

Finally, the functions $f_{\rho\sigma}$ that appear on the right-hand side of (255),

$$\delta_\epsilon L = \frac{1}{2}\frac{df_{\rho\sigma}}{dt}\epsilon^{\rho\sigma},$$

are given by

$$\begin{aligned} f_{\rho\sigma} &= \int d^3x\,\frac{1}{c}\delta^t_\nu\left(\frac{1}{4\mu_0}F^2\right)i[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda \\ &= \int d^3x\,\frac{1}{c}(-n_\mu)\delta^\mu_\nu\left(\frac{1}{4\mu_0}F^2\right)i[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda, \end{aligned} \tag{348}$$

where, as usual, $n_\mu \equiv (-1,\mathbf{0})_\mu$.

We then have from Noether's theorem (256) that the conserved angular-momentum tensor of the overall system is, up to an overall minus sign, given by

$$\begin{aligned} -J_{\nu\rho} &= \frac{\partial\mathscr{L}}{\partial\dot{X}^\alpha}g_{X^\alpha,\nu\rho} + \frac{1}{2}\frac{\partial\mathscr{L}}{\partial\dot{\theta}^{\alpha\beta}}g_{\theta^{\alpha\beta},\nu\rho} \\ &\quad + \int d^3x\,(-n_\mu)\frac{\partial\mathcal{L}}{\partial(c\partial_\mu A_\alpha)}g_{A_\alpha,\nu\rho} - f_{\nu\rho} \\ &= -(p_\alpha + qA_\alpha - \frac{1}{2}(-u_\alpha/c^2)m^{\sigma\lambda}F_{\sigma\lambda})(X_\nu\delta^\alpha_\rho - X_\rho\delta^\alpha_\nu) \\ &\quad - S_{\nu\rho} \\ &\quad - \frac{1}{c}\int d^3x\,(-n_\mu)\left(H^{\mu\alpha} - \delta^\mu_\sigma\left(\frac{1}{4\mu_0}F^2\right)\right) \\ &\qquad\qquad \times \partial_\sigma A_\alpha(x_\nu\delta^\sigma_\rho - x_\rho\delta^\sigma_\nu) \\ &\quad - \frac{1}{c}\int d^3x\,(-n_\mu)(H^\mu{}_\nu A_\rho - H^\mu{}_\rho A_\nu) \\ &= -\int d^3x\,(-n_\mu)\mathcal{J}^\mu_{\mathrm{can},\nu\rho}, \end{aligned} \tag{349}$$

where the overall system's canonical angular-momentum flux tensor is

$$\begin{aligned} \mathcal{J}^{\mu\nu\rho}_{\mathrm{can}} &= \frac{1}{c}(x^\nu T^{\mu\rho}_{\mathrm{can}} - x^\rho T^{\mu\nu}_{\mathrm{can}}) \\ &\quad + \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) + \frac{1}{c}(H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu). \end{aligned} \tag{350}$$

Here $T^{\mu\nu}_{\mathrm{can}}$ is the canonical energy-momentum tensor (332):

$$\begin{aligned} T^{\mu\nu}_{\mathrm{can}} &= u^\mu p^\nu\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\ &\quad + H^{\mu\rho}\partial^\nu A_\rho + j^\mu_{\mathrm{e}}A^\nu - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma} \\ &\quad + \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}). \end{aligned}$$

Observe that the canonical angular-momentum flux tensor (350) has precisely the form (294),

$$\mathcal{J}^{\mu\nu\rho}_{\mathrm{can}} = \mathcal{L}^{\mu\nu\rho} + \mathcal{S}^{\mu\nu\rho},$$

with $\mathcal{L}^{\mu\nu\rho}$ representing the contribution (295) from orbital angular momentum,

$$\mathcal{L}^{\mu\nu\rho} \equiv x^\nu\frac{1}{c}T^{\mu\rho}_{\mathrm{can}} - x^\rho\frac{1}{c}T^{\mu\nu}_{\mathrm{can}},$$

and with $\mathcal{S}^{\mu\nu\rho}$ representing the contribution (296) from the intrinsic spin of both the particle and the electromagnetic field,

$$\mathcal{S}^{\mu\nu\rho} = \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) + \frac{1}{c}(H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu). \tag{351}$$

Specifically, the first term in (351) describes the particle's intrinsic spin,

$$\mathcal{S}^{\mu\nu\rho}_{\mathrm{particle}} = \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \tag{352}$$

and the second term arises from the field's spin,

$$\mathcal{S}^{\mu\nu\rho}_{\text{field}} = \frac{1}{c}(H^{\mu\nu}A^{\rho} - H^{\mu\rho}A^{\nu}). \tag{353}$$

Integrating the local conservation law (297), $\partial_\mu \mathcal{J}^{\mu\nu\rho}_{\text{can}} = 0$, over three-dimensional space and taking advantage of the local conservation (274) of the overall canonical energy-momentum tensor $T^{\mu\rho}_{\text{can}}$, we can compute the time derivative of the particle's spin tensor as follows:

$$\begin{aligned}
\frac{dS^{\nu\rho}}{dt} &= \frac{d}{dt}\int d^3x\, \mathcal{S}^{t\nu\rho}_{\text{particle}} \\
&= -\frac{d}{dt}\int d^3x\, \frac{1}{c}(x^{\nu}T^{t\rho}_{\text{can}} - x^{\rho}T^{t\nu}_{\text{can}} + H^{t\nu}A^{\rho} - H^{t\rho}A^{\nu}) \\
&= -\int d^3x\, \partial_\mu(x^{\nu}T^{\mu\rho}_{\text{can}} - x^{\rho}T^{\mu\nu}_{\text{can}} + H^{\mu\nu}A^{\rho} - H^{\mu\rho}A^{\nu}) \\
&= -\frac{1}{\gamma}(u^{\nu}p^{\rho} - u^{\rho}p^{\nu}) - \frac{1}{\gamma}(m^{\nu\sigma}F^{\rho}{}_{\sigma} - m^{\rho\sigma}F^{\nu}{}_{\sigma}).
\end{aligned}$$

Using $dt/d\tau = \gamma$ from (122) and relabeling indices, we therefore find

$$\frac{dS^{\mu\nu}}{d\tau} = -(u^{\mu}p^{\nu} - u^{\nu}p^{\mu}) - (m^{\mu\rho}F^{\nu}{}_{\rho} - m^{\nu\rho}F^{\mu}{}_{\rho}),$$

which precisely agrees with the particle's equation of motion (234) for $S^{\mu\nu}$.

Now that we have calculated the system's canonical angular-momentum flux tensor $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ and identified the spin flux tensor $\mathcal{S}^{\mu\nu\rho}$, as given by (351), we can construct a symmetric, gauge-invariant energy-momentum tensor (303), $T^{\mu\nu} = T^{\mu\nu}_{\text{can}} + \partial_\rho \mathcal{B}^{\mu\rho\nu}$, from the system's Belinfante-Rosenfeld tensor (299),

$$\begin{aligned}
\mathcal{B}^{\mu\rho\nu} &\equiv \frac{c}{2}(\mathcal{S}^{\mu\nu\rho} + \mathcal{S}^{\nu\mu\rho} + \mathcal{S}^{\rho\mu\nu}) \\
&= -H^{\mu\rho}A^{\nu} + \frac{1}{2}(u^{\mu}S^{\nu\rho} + u^{\nu}S^{\mu\rho} + u^{\rho}S^{\mu\nu})\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}).
\end{aligned} \tag{354}$$

We obtain [19]

$$\begin{aligned}
T^{\mu\nu} &= \frac{1}{2}(u^{\mu}p^{\nu} + u^{\nu}p^{\mu})\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\
&\quad + \frac{1}{2}H^{\mu\rho}F^{\nu}{}_{\rho} + \frac{1}{2}H^{\nu\rho}F^{\mu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma} \\
&\quad + \frac{1}{2c^2}u^{\mu}u^{\nu}m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\
&\quad + \frac{1}{2}\partial_\rho(\mathcal{S}^{\mu\nu\rho}_{\text{particle}} + \mathcal{S}^{\nu\mu\rho}_{\text{particle}}).
\end{aligned} \tag{355}$$

In the free-field limit, this energy-momentum tensor reduces to (317), as expected:

$$T^{\mu\nu} = \frac{1}{\mu_0}F^{\mu\rho}F^{\nu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}.$$

VI. CONCLUSION

In this paper, we have employed the Lagrangian formulation of classical physics to show that a massive particle with four-momentum p^{μ}, spin tensor $S^{\mu\nu}$, electric charge q, and elementary dipole tensor $m^{\mu\nu}$ in an external electromagnetic field $F_{\mu\nu}$ obeys the relativistic equations of motion (219) and (234):

$$\begin{aligned}
\frac{dp^{\mu}}{d\tau} &= -qu_{\nu}F^{\nu\mu} - \frac{1}{2}m^{\rho\sigma}\partial^{\mu}F_{\rho\sigma} - \frac{1}{2c^2}\frac{d}{d\tau}(u^{\mu}m^{\rho\sigma}F_{\rho\sigma}), \\
\frac{dS^{\mu\nu}}{d\tau} &= -(u^{\mu}p^{\nu} - u^{\nu}p^{\mu}) - (m^{\mu\rho}F^{\nu}{}_{\rho} - m^{\nu\rho}F^{\mu}{}_{\rho}).
\end{aligned}$$

To verify that these equations of motion are compatible with local conservation of energy, momentum, and angular momentum, we have effectively divided up the locally conserved, canonical energy-momentum tensor $T^{\mu\nu}_{\text{can}} = T^{\mu\nu}_{\text{can,particle}} + T^{\mu\nu}_{\text{can,field}}$ of the overall system by defining the canonical energy-momentum tensor for the particle to be (336),

$$T^{\mu\nu}_{\text{can,particle}} \equiv u^{\mu}p^{\nu}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}),$$

and the canonical energy-momentum tensor for the electromagnetic field to be (337),

$$\begin{aligned}
T^{\mu\nu}_{\text{can,field}} &\equiv H^{\mu\rho}F^{\nu}{}_{\rho} - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2 \\
&\quad + \frac{1}{2c^2}u^{\mu}u^{\nu}m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\
&\quad + \partial_\rho(H^{\mu\rho}A^{\nu}).
\end{aligned}$$

The local conservation equation $\partial_\mu T^{\mu\nu}_{\text{can}} = 0$ then translates into the relativistic equation of motion (219) for the particle's four-momentum, and the local conservation law $\partial_\mu \mathcal{J}^{\mu\nu\rho}_{\text{can}} = 0$ satisfied by the overall system's canonical angular-momentum flux tensor $\mathcal{J}^{\mu\nu\rho}_{\text{can}}$ as defined in (350) yields the equation of motion (234) for the particle's spin tensor.

In the non-relativistic limit, the equation of motion (219) generalizes the Lorentz force law to (225),

$$\mathbf{F} = q\mathbf{E}_{\text{ext}} + q\mathbf{v}\times\mathbf{B}_{\text{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}}),$$

and gives the power law (227),

$$\begin{aligned}
\frac{dW}{dt} &= \mathbf{v}\cdot(q\mathbf{E}_{\text{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\text{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\text{ext}})) \\
&= \mathbf{v}\cdot\mathbf{F}.
\end{aligned}$$

These formulas are consistent with the fact that magnetic forces cannot do work on *electric monopoles*, but also make clear that magnetic forces are fully capable of doing work on *elementary magnetic dipoles*, in accordance with the basic definition (6) of what it means for a force to do mechanical work on an object in moving the object from a location A to another location B, as we showed explicitly in (226):

$$W \equiv \int_A^B d\mathbf{X} \cdot \mathbf{F}$$
$$= \int_A^B d\mathbf{X} \cdot q\mathbf{E}_{\text{ext}} + \Delta(\boldsymbol{\pi} \cdot \mathbf{E}_{\text{ext}}) + \Delta(\boldsymbol{\mu} \cdot \mathbf{B}_{\text{ext}}).$$

As an interesting aside, these results provide a loophole in the Bohr-van Leeuwen theorem [20], which Niels Bohr first proved in his 1911 doctoral thesis [21] and which was later independently proved by Hendrika Johanna van Leeuwen in her own doctoral thesis in 1919 [22]. The Bohr-van Leeuwen theorem asserts on the basis of the original Lorentz force law (that is, without contributions from elementary dipoles) that a non-rotating system of particles, when treated *classically*, always has a vanishing average magnetization in thermal equilibrium. A key implication of the Bohr-van Leeuwen theorem is that phenomena like diamagnetism cannot arise without quantum mechanics. Our results in this paper provide a theoretical exception to this corollary.

Returning to our equations describing forces and work done on a classical particle with elementary dipole moments, it is important to note that we do not require any external, ad hoc sources of energy and momentum to ensure the validity of these equations. The energy and momentum that flow into the particle are fully accounted for in the energy and momentum that arise from the overall classical action functional describing the coupling of the particle to the electromagnetic field, regardless of whether, at the level of interpretation, we attribute all that energy and momentum to the electromagnetic field alone or to the interactions between the electromagnetic field and the particle.

Magnetic forces can do work. In this paper, we have shown how.

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[5] For more comprehensive pedagogical treatments, see [23–25].
[6] For more detailed examples, see Section 8.3 of [23].
[7] See Chapter 12 of [24] and Chapter 11 of [25] for more extensive treatments.
[8] We present a much more detailed survey in [1].
[9] Like the analogous Lorenz equation $\partial_\mu A^\mu = 0$ in the Proca field theory and in Lorenz gauge of electromagnetism, the condition (101) will end up eliminating unphysical spin states.
[10] Although it will take some work, we will eventually show that these fixed reference values correspond to the particle's rest frame. Keep in mind that up to this point in our discussion, we haven't yet provided a precise relationship between the particle's four-momentum p^μ and its four-velocity $dX^\mu/d\lambda$.
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[14] For a rigorous treatment of self-forces and self-energies, see [12].
[15] Keep in mind the suppressed indices on the elementary dipole tensor and the Lorentz generators in the first line of this calculation.
[16] See [26] for another classical derivation of these conditions. Their quantum-mechanical analogues follow from the Wigner-Eckart theorem.
[17] P. Di Francesco, P. Mathieu, and Sénéchal, *Conformal Field Theory*, 1st ed. (Springer, 1997).
[18] Note that the authors of [12] break up the total energy-momentum tensor differently by including the interaction terms with the energy-momentum tensor for the *particle*. This approach obscures the work being done by the electromagnetic field on the particle, and, indeed, the authors end up concluding that magnetic forces are incapable of doing work on elementary magnetic dipole moments.
[19] This formula differs from the corresponding result in [3], whose energy-momentum tensor yields the correct equations of motion for the particle only after an unjustified four-dimensional integration by parts.
[20] We thank Sebastiano Covone for suggesting the consideration of the Bohr-van Leeuwen theorem in the context of our results.
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