# On Magnetic Forces and Work

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We address a long-standing debate over whether classical magnetic forces can do work, ultimately answering the question in the affirmative. In detail, we couple a classical particle with intrinsic spin and elementary dipole moments to the electromagnetic field, derive the appropriate generalization of the Lorentz force law, show that the particle's dipole moments must be collinear with its spin axis, and argue that the magnetic field does mechanical work on the particle's elementary magnetic dipole moment. As consistency checks, we calculate the overall system's energy-momentum and angular momentum, and show that their local conservation equations lead to the same force law and therefore the same conclusions about magnetic forces and work. We also compute the system's Belinfante-Rosenfeld energy-momentum tensor.

## I. INTRODUCTION

Textbook treatments and research articles on classical electromagnetism, such as [1, 2], often suggest that magnetic fields cannot do mechanical work. On the other hand, everyday examples of bar magnets lifting other bar magnets would seem to suggest otherwise. In this paper, we show that there exists a classical way to understand how magnetic fields can indeed do work.[3]

We start in Section II with a review of the kinematics of classical relativistic point particles with intrinsic spin and permanent, elementary dipole moments. In Section III, we couple a particle of this kind to the electromagnetic field and derive its dynamics, showing, in particular, that magnetic forces can classically do work on the particle via its elementary magnetic dipole moment. We also show as a matter of self-consistency that the particle's elementary dipole moments must be collinear with the particle's intrinsic spin. In Section IV, we derive expressions for the overall system's energy-momentum and angular momentum, and show that their associated conservation laws lead to the same equations of motion as before, thereby providing further confirmation that magnetic fields can do work on a particle with elementary dipole moments. We conclude with one more new result by calculating the system's Belinfante-Rosenfeld energy-momentum tensor.

## II. THE KINEMATICS OF A RELATIVISTIC ELEMENTARY DIPOLE

To start, we will need a relativistic description of the kinematics of a classical particle with intrinsic spin.

### A. The Phase Space for a Relativistic Massive Particle with Spin

Following [4–6], we model the particle's kinematics using spacetime coordinates $X^\mu = (cT, \mathbf{X})^\mu$, energy $E$, four-momentum $p^\mu = (E/c, \mathbf{p})^\mu$, positive inertial mass $m > 0$, and antisymmetric spin tensor $S^{\mu\nu}$ by identifying the particle's phase space as a transitive or "irreducible" group action (or homogeneous space) of the orthochronous Poincaré group. The states in this phase space take the form $(X, p, S)$ and are each obtained from the reference state $(0, (mc, \mathbf{0}), S_0)$ by an appropriate Poincaré transformation $(a, \Lambda) \in \mathbb{R}^4 \ltimes O(1,3)$ according to

$$(X, p, S) = (a, \Lambda(mc, \mathbf{0}), \Lambda S_0 \Lambda^{\mathrm{T}}). \tag{1}$$

Here the coordinates $X^\mu = a^\mu$ and the variable Lorentz-transformation matrix $\Lambda^\mu{}_\nu$ are treated as the particle's fundamental phase-space variables, with the condition that $\Lambda^{\mathrm{T}} \eta \Lambda = \eta = \mathrm{diag}(-1, +1, +1, +1)$.

### B. Charge and Elementary Dipole Moments

We can couple the particle to the electromagnetic field by assigning the particle an electric-monopole charge $q$ and an antisymmetric elementary dipole tensor $m^{\mu\nu}$, so that the particle is an elementary dipole.

We note that elementary dipoles of this kind are neither of the Ampère model, which consist of loops of moving electric monopoles, nor of the Gilbert model, which consist of pairs of hypothetical magnetic monopoles. In particular, the elementary dipoles that we examine here represent a classical extension of Maxwell's original theory of electromagnetism, as Maxwell's theory includes dipoles only of the Ampère type.[7]

We let $u^\mu \equiv dX^\mu/d\lambda$ denote the particle's four-velocity and $\gamma \equiv u^0/c$ denote the particle's associated Lorentz factor, where $u^\mu$ is not generically normalized to $u^2 = -c^2$ unless the worldline parameter $\lambda$ is taken to be the particle's proper time $\tau$. The particle's four-velocity then

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takes the form

$$u^\mu = (\gamma c, \gamma \mathbf{v})^\mu. \tag{2}$$

The particle has four-dimensional electric-monopole current density

$$j_{\mathrm{e}}^\nu(\mathbf{x}, t) = (\rho_{\mathrm{e}}(\mathbf{x}, t)c,\, \mathbf{J}_{\mathrm{e}}(\mathbf{x}, t))^\nu = q u^\nu \frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \tag{3}$$

and elementary-dipole density

$$M^{\mu\nu} = m^{\mu\nu}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}), \tag{4}$$

with overall current density

$$j^\nu(\mathbf{x}, t) = j_{\mathrm{e}}^\nu(\mathbf{x}, t) + \partial_\mu M^{\mu\nu}(\mathbf{x}, t), \tag{5}$$

where $(1/\gamma)\delta^3(\mathbf{x}-\mathbf{X})$ is the Lorentz-invariant form of the three-dimensional Dirac delta function.

It follows immediately from (3) that the particle's electric-monopole density $\rho_{\mathrm{e}} = j_{\mathrm{e}}^t/c$, its electric-monopole current density $\mathbf{J}_{\mathrm{e}} = (j_{\mathrm{e}}^x, j_{\mathrm{e}}^y, j_{\mathrm{e}}^z)$, and its three-velocity $\mathbf{v} \equiv d\mathbf{X}/dt$ satisfy the basic relationship

$$\mathbf{J}_{\mathrm{e}} = \rho_{\mathrm{e}}\mathbf{v}. \tag{6}$$

We emphasize that no such relationship holds for the particle's elementary dipole moments, which, again, are not assumed to arise from any underlying motion of electric monopoles.

As in [2], by introducing suitable four-vectors $\pi^\mu$ and $\mu^\mu$ and antisymmetric tensors

$$\pi^{\mu\nu} \equiv \frac{1}{mc}(p^\mu \pi^\nu - p^\nu \pi^\mu), \tag{7}$$

$$\mu^{\mu\nu} \equiv \frac{1}{mc}\epsilon^{\mu\nu\rho\sigma} p_\rho \mu_\sigma, \tag{8}$$

we can write the particle's elementary dipole tensor in terms of an electric part $\pi^{\mu\nu}$ and a magnetic part $\mu^{\mu\nu}$ as

$$m^{\mu\nu} = \pi^{\mu\nu} + \mu^{\mu\nu}, \tag{9}$$

or, equivalently, as

$$m^{\mu\nu} \equiv \begin{pmatrix} 0 & c\pi_x & c\pi_y & c\pi_z \\ -c\pi_x & 0 & -\mu_z & \mu_y \\ -c\pi_y & \mu_z & 0 & -\mu_x \\ -c\pi_z & -\mu_y & \mu_x & 0 \end{pmatrix}^{\mu\nu}. \tag{10}$$

Here $\epsilon^{\mu\nu\rho\sigma}$ is the four-dimensional Levi-Civita symbol (with $\epsilon_{txyz} \equiv +1$), and $\pi^\nu(\lambda)$ and $\mu^\mu(\lambda)$ are related to their reference values $\pi_0^\mu \equiv (0, \boldsymbol{\pi}_0)^\mu$ and $\mu_0^\mu \equiv (0, \boldsymbol{\mu}_0)^\mu$ and the particle's variable Lorentz-transformation matrix $\Lambda^\mu{}_\nu(\lambda)$ according to

$$\pi^\mu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda)\pi_0^\nu, \tag{11}$$

$$\mu^\mu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda)\mu_0^\nu. \tag{12}$$

## III. THE DYNAMICS OF A RELATIVISTIC ELEMENTARY DIPOLE

Next, we turn to a discussion of the particle's dynamics.

### A. The Action Functional for a Relativistic Massive Particle with Spin

In the absence of external interactions, as shown in [4–6], we can encode the dynamics of a particle with intrinsic spin in terms of the manifestly covariant action functional

$$\begin{aligned} S_{\mathrm{particle}}[X, \Lambda] &= \int d\lambda\, \frac{1}{2} J_{\mu\nu}\dot{\theta}^{\mu\nu} \\ &= \int d\lambda \left( p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right), \end{aligned} \tag{13}$$

where $\lambda$ is a smooth and monotonic but otherwise arbitrary parameter along the particle's worldline, $J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}$ is the particle's total angular-momentum tensor, $L_{\mu\nu} \equiv X_\mu p_\nu - X_\nu p_\mu$ is its orbital angular-momentum tensor, $\theta^{\mu\nu}$ is an antisymmetric tensor of boost and angular degrees of freedom, and we ignore irrelevant boundary terms. Consistency of the particle's dynamics with the required invariance of the quantities $p^2 \equiv -m^2c^2$ and $s^2 \equiv (1/2)S_{\mu\nu}S^{\mu\nu}$ requires the auxiliary phase-space condition

$$p_\mu S^{\mu\nu} = 0. \tag{14}$$

### B. The Particle's Equations of Motion

Our next step will be to couple the particle to the electromagnetic field and obtain the particle's equations of motion, from which we will be able to infer the appropriate generalization of the Lorentz force law.

Given the charge and elementary dipole moments outlined above, the overall action functional for the elementary dipole and the electromagnetic field is given by

$$\begin{aligned} S[X, \Lambda, A] &\equiv S_{\mathrm{particle}}[X, \Lambda] + S_{\mathrm{field}}[A] + S_{\mathrm{int}}[X, \Lambda, A] \\ &= \int d\lambda \left( p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) \qquad (S_{\mathrm{particle}}) \\ &\quad + \int dt \int d^3x \left( -\frac{1}{4\mu_0} F^{\mu\nu}F_{\mu\nu} \right) \qquad (S_{\mathrm{field}}) \\ &\quad + \int dt \int d^3x\, j^\nu A_\nu \qquad (S_{\mathrm{int}}), \end{aligned} \tag{15}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual Faraday tensor, $j^\nu = j_{\mathrm{e}}^\nu + \partial_\mu M^{\mu\nu}$ is the particle's overall current density, and the interaction term in the final line ensures that extremizing the action functional with respect to the electromagnetic gauge field $A_\mu$ yields the Maxwell equations in their usual form. The first line in this action functional ($S_{\mathrm{particle}}$) is fixed by group theory, the second line ($S_{\mathrm{field}}$) defines the vacuum in the pure Maxwell theory, and the third line ($S_{\mathrm{int}}$) provides the canonical coupling between the particle and the electromagnetic field in a manner consistent with the Maxwell equations and the particle's features as laid out in the previous section.

After an integration by parts, we can write the interaction term in the final line as

$$S_{\mathrm{int}}[X,\Lambda,A] = \int dt \int d^3x \left( j_{\mathrm{e}}^{\nu} A_\nu - \frac{1}{2} M^{\mu\nu} F_{\mu\nu} \right). \quad (16)$$

Collecting together all the terms that involve the particle's degrees of freedom, we obtain

$$S_{\mathrm{particle+int}}[X,\Lambda,A] = \int d\lambda \left( p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] \right) + \int dt \int d^3x\, j_{\mathrm{e}}^{\nu} A_\nu + \int dt \int d^3x \left( -\frac{1}{2} \right) M^{\mu\nu} F_{\mu\nu}, \quad (17)$$

which we can further reduce to the form

$$S_{\mathrm{particle+int}}[X,\Lambda,A] = \int d\lambda\, \mathscr{L}_{\mathrm{particle+int}}, \quad (18)$$

for a manifestly covariant Lagrangian defined by

$$\mathscr{L}_{\mathrm{particle+int}} \equiv p_\mu \dot{X}^\mu + \frac{1}{2}\mathrm{Tr}[S\dot{\Lambda}\Lambda^{-1}] + q\dot{X}^\nu A_\nu - \frac{1}{2c}\sqrt{-\dot{X}^2}\, m^{\mu\nu} F_{\mu\nu}. \quad (19)$$

It follows from a straightforward calculation that the particle's equations of motion, expressed in terms of the particle's proper time $\tau$, are then

$$\begin{aligned} \frac{dp^\mu}{d\tau} &= -qu_\nu F^{\nu\mu} - \frac{1}{2} m^{\rho\sigma}\partial^\mu F_{\rho\sigma} - \frac{1}{2c^2}\frac{d}{d\tau}(u^\mu m^{\rho\sigma}F_{\rho\sigma}) \\ &= -qu_\nu F^{\nu\mu} - \frac{1}{2} m^{\rho\sigma}(\eta^{\mu\nu} + u^\mu u^\nu)\partial_\nu F_{\rho\sigma} \\ &\quad - \frac{1}{2c^2}\frac{d}{d\tau}(u^\mu m^{\rho\sigma})F_{\rho\sigma}, \end{aligned} \quad (20)$$

as obtained in [6, 8, 9], and

$$\frac{dS^{\mu\nu}}{d\tau} = -(u^\mu p^\nu - u^\nu p^\mu) - (m^{\mu\rho}F^{\nu}{}_{\rho} - m^{\nu\rho}F^{\mu}{}_{\rho}), \quad (21)$$

which generalizes the results of [6, 8, 10].

### C. The Non-Relativistic Limit with Time-Independent External Fields

In the non-relativistic limit and ignoring self-field effects—so that we can replace the overall electric and magnetic field with the external fields $\mathbf{E}_{\mathrm{ext}}$ and $\mathbf{B}_{\mathrm{ext}}$—the equations of motion (20)–(21) reduce to

$$\frac{dE}{dt} \approx \mathbf{v}\cdot(q\mathbf{E}_{\mathrm{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\mathrm{ext}} + \boldsymbol{\mu}\cdot\mathbf{B}_{\mathrm{ext}})), \quad (22)$$

$$\frac{d\mathbf{p}}{dt} \approx q(\mathbf{E}_{\mathrm{ext}} + \mathbf{v}\times\mathbf{B}_{\mathrm{ext}}) + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\mathrm{ext}} + \boldsymbol{\mu}\cdot\mathbf{B}_{\mathrm{ext}}), \quad (23)$$

$$\frac{d\mathbf{J}}{dt} \approx \mathbf{X}\times\frac{d\mathbf{p}}{dt} + \boldsymbol{\pi}\times\mathbf{E}_{\mathrm{ext}} + \boldsymbol{\mu}\times\mathbf{B}_{\mathrm{ext}}, \quad (24)$$

where the particle's four-momentum in this limit is

$$p^\mu = (E/c, \mathbf{p})^\mu \approx (mc^2 + (1/2)m\mathbf{v}^2, \mathbf{p})^\mu, \quad (25)$$

and the particle's overall angular-momentum pseudovector $\mathbf{J}$ is made up of orbital and spin contributions according to

$$\mathbf{J} \equiv \mathbf{L} + \mathbf{S} = (L^{yz}, L^{zx}, L^{xy}) + (S^{yz}, S^{zx}, S^{xy}). \quad (26)$$

The dynamical equation (23) tells us that the electromagnetic force on the particle is

$$\mathbf{F} = q\mathbf{E}_{\mathrm{ext}} + q\mathbf{v}\times\mathbf{B}_{\mathrm{ext}} + \nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\mathrm{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\mathrm{ext}}). \quad (27)$$

We observe that the usual Lorentz force law, $q\mathbf{E}_{\mathrm{ext}} + q\mathbf{v}\times\mathbf{B}_{\mathrm{ext}}$, is enhanced in the presence of the particle's elementary dipole moments by the appearance of two additional dipole terms, $\nabla(\boldsymbol{\pi}\cdot\mathbf{E}_{\mathrm{ext}}) + \nabla(\boldsymbol{\mu}\cdot\mathbf{B}_{\mathrm{ext}})$, in which the magnetic field appears on an equal footing with the electric field. Accordingly, the magnetic field contributes to the work done by the external electromagnetic field, $W \equiv \int d\mathbf{X}\cdot\mathbf{F}$:

$$W = \int_A^B dt\,(q\mathbf{v}\cdot\mathbf{E}_{\mathrm{ext}}) + \Delta(\boldsymbol{\pi}\cdot\mathbf{E}_{\mathrm{ext}}) + \Delta(\boldsymbol{\mu}\cdot\mathbf{B}_{\mathrm{ext}}). \quad (28)$$

Moreover the rate at which work is done is in agreement with the dynamical equation (22).

We have reached the key conclusion of this paper—namely, that magnetic forces can do work on classical particles with elementary dipole moments.[11] We next turn to a detailed treatment of self-consistency conditions on the particle's dynamics, as well as obtain the necessary formulas for determining the particle's four-velocity $u^\mu$ in the presence of a nonzero electromagnetic field. Later on, we will analyze electromagnetic forces and work done on the particle from the standpoint of local conservation laws.

### D. Implications of Self-Consistency

Taking a derivative of the phase-space condition $p_\mu S^{\mu\nu}$ from (14) yields the self-consistency requirement

$$\frac{dp_\mu}{d\tau}S^{\mu\nu} + p_\mu\frac{dS^{\mu\nu}}{d\tau} = 0,$$

which entails that the particle's four-momentum $p^\mu$ and its four-velocity $u^\mu = dX^\mu/d\tau$ (now normalized to $u^2 = -c^2$) are related by

$$p^\mu = m_{\text{eff}} u^\mu + b^\mu. \tag{29}$$

Here $m_{\text{eff}}$, which plays the role of an effective mass, is defined by

$$m_{\text{eff}} \equiv -\frac{m^2c^2}{p \cdot u}, \tag{30}$$

and the four-vector $b^\mu$, which is orthogonal to the particle's four-momentum, $b \cdot p = 0$, is given by

$$b^\mu \equiv \frac{1}{p \cdot u}\left(\frac{dp_\nu}{d\tau} S^{\nu\mu} - p_\nu (m^{\nu\rho} F^\mu{}_\rho - m^{\mu\rho} F^\nu{}_\rho)\right). \tag{31}$$

As in [6], we regard (29) as an implicit formula for the particle's four-velocity $u^\mu$. This formula ensures, in particular, that the particle's four-momentum $p^\mu$ has constant norm-squared $p^2 = -m^2c^2$.

For vanishing field, $F_{\mu\nu} = 0$, the relationship (29) reduces to the familiar equation $p^\mu = mu^\mu$, as expected. On the other hand, when the electromagnetic field is nonzero, $F_{\mu\nu} \neq 0$, (29) has the form

$$p^\mu = mu^\mu + (\text{terms of order } 1/c^2). \tag{32}$$

This relation ensures that there is no ambiguity over whether we should identify the particle's energy $E$ as $p^t c$ or $u^t mc^2$ for the purposes of quantifying the work done by the field on the particle in the non-relativistic regime.

Invoking the spin tensor's equation of motion (21), together with the phase-space condition (14), $p_\mu S^{\mu\nu} = 0$, and the constancy of the particle's spin-squared scalar $s^2 \equiv (1/2) S_{\mu\nu} S^{\mu\nu}$, we find

$$\begin{aligned} \frac{d}{d\tau}(s^2) &= \frac{d}{d\tau}\left(\frac{1}{2} S_{\mu\nu} S^{\mu\nu}\right) \\ &= (S^\rho{}_\mu m^{\mu\sigma} - S^\sigma{}_\mu m^{\mu\rho}) F_{\rho\sigma} = 0, \end{aligned} \tag{33}$$

which yields the condition

$$S^\rho{}_\mu m^{\mu\sigma} = S^\sigma{}_\mu m^{\mu\rho}. \tag{34}$$

In the particle's reference state, this equality produces the relations

$$\left.\begin{aligned} \boldsymbol{\pi}_0 \times \mathbf{S}_0 = 0, \\ \boldsymbol{\mu}_0 \times \mathbf{S}_0 = 0, \end{aligned}\right\} \tag{35}$$

which dictate that the particle's elementary electric and magnetic dipole moments must be collinear with the particle's spin pseudovector $\mathbf{S}_0$:

$$\left.\begin{aligned} \boldsymbol{\pi}_0 &= \frac{1}{c}\,\Xi\,\mathbf{S}_0, \\ \boldsymbol{\mu}_0 &= \Gamma\,\mathbf{S}_0. \end{aligned}\right\} \tag{36}$$

Here $\Xi$ is a pseudoscalar and $\Gamma$ is the particle's scalar gyromagnetic ratio. We can understand these relationships physically as telling us that if the particle's elementary-dipole vectors were not collinear with the particle's spin axis, then torques exerted on the particle by the electromagnetic field would cause the particle's overall spin to speed up or slow down, in violation of the constancy of $s^2$.

## IV. CONSERVATION LAWS

For completeness, we verify that the equations of motion (20)–(21) also follow from local conservation of energy-momentum and angular momentum. To begin, we recall the relevant version of Noether's theorem, which states that if a system's dynamics has a continuous symmetry,

$$\begin{aligned} q_\alpha \mapsto q'_\alpha &= q_\alpha + \delta_\epsilon q_\alpha, \\ \delta_\epsilon q_\alpha &= \sum_b g_{q_\alpha, b} \epsilon_b, \end{aligned} \tag{37}$$

where the quantities $\epsilon_b$ parameterize the symmetry and the quantities $g_{q_\alpha,b}$ characterize its precise form, then we have the following conservation law:

$$Q_b \equiv \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} g_{q_\alpha, b} - f_b, \quad \frac{dQ}{dt} = 0. \tag{38}$$

Here $Q_b$ are a set of conserved quantities, $L$ is the system's Lagrangian, $q_\alpha$ are its degrees of freedom, and the functions $f_b$ are related to the change in the Lagrangian according to

$$\begin{aligned} L &\mapsto L + \delta_\epsilon L, \\ \delta_\epsilon L &= \frac{d}{dt}\left(\sum_b f_b \epsilon_b\right) = \sum_b \frac{df_b}{dt} \epsilon_b. \end{aligned} \tag{39}$$

### A. Local Conservation of Energy-Momentum

In order to employ Noether's theorem to obtain the overall system's energy-momentum tensor, we examine the behavior of the system under a translation in spacetime by an infinitesimal four-vector $\epsilon^\mu$. The particle's phase-space variables transform as

$$\left.\begin{aligned} X^\mu(\lambda) &\mapsto X'^\mu(\lambda) \equiv X^\mu(\lambda) + \epsilon^\mu, \\ \Lambda^\mu{}_\nu(\lambda) &\mapsto \Lambda'^\mu{}_\nu(\lambda) \equiv \Lambda^\mu{}_\nu(\lambda), \end{aligned}\right\} \tag{40}$$

and the electromagnetic gauge potential transforms as

$$\begin{aligned} A_\mu(x) \mapsto A'_\mu(x) &\equiv A_\mu(x - \epsilon) \\ &= A_\mu(x) - \partial_\nu A_\mu(x) \epsilon^\nu. \end{aligned} \tag{41}$$

By an application of Noether's theorem to the particle's manifestly covariant Lagrangian $\mathscr{L} \equiv \mathscr{L}_{\text{particle+int}}$ defined by (18) and the Lagrangian density $\mathcal{L}$ for the overall system defined in terms of the action functional $S[X, \Lambda, A] \equiv \int dt \int d^3x\, \mathcal{L}$ from (15), one finds that the overall system's conserved four-momentum is expressible as

$$\begin{aligned} P_\nu &= \frac{\partial \mathscr{L}}{\partial \dot{X}^\rho} g_{X^\rho,\nu} + \int d^3x\,(-n_\mu)\frac{\partial \mathcal{L}}{\partial(c\partial_\mu A_\rho)} g_{A_\rho,\nu} - f_\nu \\ &= p_\nu + qA_\nu + \frac{1}{2c^2} u_\nu m^{\sigma\tau} F_{\sigma\tau} \\ &\quad + \frac{1}{c}\int d^3x\,(-n_\mu)\bigg(H^{\mu\rho}\partial_\nu A_\rho - \delta^\mu_\nu\Big(\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}\Big)\bigg) \\ &= \frac{1}{c}\int d^3x\,(-n_\mu) T^\mu_{\text{can},\nu}, \end{aligned} \tag{42}$$

where $n_\mu \equiv (-1, \mathbf{0})_\mu$ is a unit timelike vector orthogonal to the three-dimensional spatial hypersurface of integration. In this expression, the overall system's canonical energy-momentum tensor is given by

$$T^{\mu\nu}_{\text{can}} = T^{\mu\nu}_{\text{can,particle}} + T^{\mu\nu}_{\text{can,field}}, \tag{43}$$

with the contributions from the particle and the field given respectively by[12]

$$T^{\mu\nu}_{\text{can,particle}} \equiv u^\mu p^\nu \frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \tag{44}$$

and

$$\begin{aligned} T^{\mu\nu}_{\text{can,field}} \equiv\; & H^{\mu\rho}F^\nu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2 \\ & + \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}) \\ & + \partial_\rho(H^{\mu\rho}A^\nu). \end{aligned} \tag{45}$$

Here $H^{\mu\nu}$ is the auxiliary Faraday tensor:

$$\begin{aligned} H^{\mu\nu} &\equiv \frac{1}{\mu_0}F^{\mu\nu} + M^{\mu\nu} \\ &= \frac{1}{\mu_0}F^{\mu\nu} + m^{\mu\nu}\frac{1}{\gamma}\delta^3(\mathbf{x} - \mathbf{X}). \end{aligned} \tag{46}$$

The last term in (45) is a total spacetime divergence with vanishing divergence $\partial_\mu\partial_\rho(H^{\mu\rho}A^\nu) = 0$ on its $\mu$ index, and its temporal component $\partial_\rho(H^{t\rho}A^\nu)$ has vanishing integral over three-dimensional space under the assumption that the fields go to zero sufficiently rapidly at spatial infinity. We emphasize that in our approach, all the terms in the overall system's canonical energy-momentum tensor follow from the systematic application of Noether's theorem to the relevant action functionals.

We can integrate the local conservation law $\partial_\mu T^{\mu\nu}_{\text{can}} = 0$ over three-dimensional space to compute the time derivative of the particle's four-momentum $p^\nu$:

$$\begin{aligned} \frac{dp^\nu}{dt} &= \frac{1}{c}\frac{d}{dt}\int d^3x\, T^{t\nu}_{\text{can,particle}} \\ &= -\frac{1}{c}\frac{d}{dt}\int d^3x\, T^{t\nu}_{\text{can,field}} \\ &= \int d^3x\bigg(-\partial_\mu\Big(H^{\mu\rho}F^\nu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^2\Big)\bigg) \\ &\quad - \frac{1}{2c^2}\frac{d}{dt}(u^\nu m^{\rho\sigma}F_{\rho\sigma}) \\ &= -qu_\mu F^{\mu\nu} + m_{\rho\mu}\partial^\mu F^{\nu\rho} - \frac{1}{2c^2}\frac{d}{d\tau}(u^\nu m^{\rho\sigma}F_{\rho\sigma}). \end{aligned}$$

After invoking the electromagnetic Bianchi identity $\partial^\mu F^{\nu\rho} + \partial^\rho F^{\mu\nu} + \partial^\nu F^{\rho\mu} = 0$, we obtain the equation of motion (20).

Our formulas above for the overall system's canonical energy-momentum tensor are new results. By replicating the particle's equation of motion (20), they provide further support for the key claim of this paper—that magnetic forces can classically do work on particles with elementary dipole moments.

### B. Local Conservation of Angular Momentum

Next, we use Noether's theorem to examine the overall system's angular momentum and its local conservation. Under an infinitesimal Lorentz transformation

$$\Lambda_{\text{inf}} = 1 + \frac{i}{2}\epsilon^{\rho\sigma}\sigma_{\rho\sigma}, \tag{47}$$

the particle's phase-space variables transform as

$$\left.\begin{aligned} X^\mu(\lambda) \mapsto X'^\mu(\lambda) &\equiv (\Lambda_{\text{inf}}X(\lambda))^\mu \\ &= X^\mu(\lambda) + \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\nu X^\nu(\lambda), \\ \Lambda^\mu{}_\nu(\lambda) \mapsto \Lambda'^\mu{}_\nu(\lambda) &\equiv (\Lambda_{\text{inf}}\Lambda(\lambda))^\mu{}_\nu \\ &= \Lambda^\mu{}_\nu(\lambda) + \frac{i}{2}\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\mu{}_\lambda \Lambda^\lambda{}_\nu(\lambda). \end{aligned}\right\} \tag{48}$$

The second of these two transformation laws is equivalent to the following transformation rule for the particle's Lorentz parameters $\theta^{\mu\nu}(\lambda)$:

$$\theta^{\mu\nu}(\lambda) \mapsto \theta'^{\mu\nu}(\lambda) \equiv \theta^{\mu\nu}(\lambda) + \epsilon^{\mu\nu}. \tag{49}$$

Meanwhile, the gauge field $A_\mu(x)$ transforms as

$$\begin{aligned} A_\mu(x) \mapsto A'_\mu(x) &\equiv (A(\Lambda^{-1}_{\text{inf}}x)\Lambda^{-1}_{\text{inf}})_\mu \\ &\equiv A_\lambda((1 - (i/2)\epsilon^{\rho\sigma}\sigma_{\rho\sigma})x)(\delta^\lambda_\mu - (i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\lambda{}_\mu) \\ &= A_\mu(x) - \partial_\nu A_\mu(x)(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\nu{}_\lambda x^\lambda \\ &\quad - A_\lambda(x)(i/2)\epsilon^{\rho\sigma}[\sigma_{\rho\sigma}]^\lambda{}_\mu. \end{aligned} \tag{50}$$

Noether's theorem (38) then yields the system's overall angular-momentum tensor, up to an overall minus sign:

$$
\begin{aligned}
-J_{\nu\rho} &= \frac{\partial \mathscr{L}}{\partial \dot{X}^\alpha} g_{X^\alpha,\nu\rho} + \frac{1}{2}\frac{\partial \mathscr{L}}{\partial \dot{\theta}^{\alpha\beta}} g_{\theta^{\alpha\beta},\nu\rho} \\
&\quad + \int d^3x\,(-n_\mu)\frac{\partial \mathcal{L}}{\partial(c\partial_\mu A_\alpha)} g_{A_\alpha,\nu\rho} - f_{\nu\rho} \\
&= -\left(p_\alpha + qA_\alpha - \frac{1}{2}(-u_\alpha/c^2)m^{\sigma\lambda}F_{\sigma\lambda}\right)(X_\nu\delta^\alpha_\rho - X_\rho\delta^\alpha_\nu) \\
&\quad - S_{\nu\rho} \\
&\quad - \frac{1}{c}\int d^3x\,(-n_\mu)\left(H^{\mu\alpha} - \delta^\mu_\sigma\left(\frac{1}{4\mu_0}F^2\right)\right) \\
&\qquad\quad \times \partial_\sigma A_\alpha(x_\nu\delta^\sigma_\rho - x_\rho\delta^\sigma_\nu) \\
&\quad - \frac{1}{c}\int d^3x\,(-n_\mu)(H^\mu{}_\nu A_\rho - H^\mu{}_\rho A_\nu) \\
&= -\int d^3x\,(-n_\mu)\mathcal{J}^\mu_{\mathrm{can},\nu\rho}.
\end{aligned} \tag{51}
$$

Here we have identified the system's canonical angular-momentum flux tensor as

$$\mathcal{J}^{\mu\nu\rho}_{\mathrm{can}} = \mathcal{L}^{\mu\nu\rho} + \mathcal{S}^{\mu\nu\rho}, \tag{52}$$

with orbital contribution

$$\mathcal{L}^{\mu\nu\rho} \equiv x^\nu \frac{1}{c}T^{\mu\rho}_{\mathrm{can}} - x^\rho\frac{1}{c}T^{\mu\nu}_{\mathrm{can}} \tag{53}$$

and spin contribution

$$\mathcal{S}^{\mu\nu\rho} = \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) + \frac{1}{c}(H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu). \tag{54}$$

We naturally read off the spin flux tensors for the particle and the field respectively as

$$\mathcal{S}^{\mu\nu\rho}_{\mathrm{particle}} = \frac{1}{c}u^\mu S^{\nu\rho}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}), \tag{55}$$

$$\mathcal{S}^{\mu\nu\rho}_{\mathrm{field}} = \frac{1}{c}(H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu). \tag{56}$$

Integrating the local conservation law $\partial_\mu \mathcal{J}^{\mu\nu\rho}_{\mathrm{can}} = 0$ over three-dimensional space and taking advantage of the local conservation $\partial_\mu T^{\mu\rho}_{\mathrm{can}} = 0$ of the overall canonical energy-momentum tensor $T^{\mu\rho}_{\mathrm{can}}$, we can compute the time derivative of the particle's spin tensor as follows:

$$
\begin{aligned}
\frac{dS^{\nu\rho}}{dt} &= \frac{d}{dt}\int d^3x\,\mathcal{S}^{t\nu\rho}_{\mathrm{particle}} \\
&= -\frac{d}{dt}\int d^3x\,\frac{1}{c}(x^\nu T^{t\rho}_{\mathrm{can}} - x^\rho T^{t\nu}_{\mathrm{can}} + H^{t\nu}A^\rho - H^{t\rho}A^\nu) \\
&= -\int d^3x\,\partial_\mu(x^\nu T^{\mu\rho}_{\mathrm{can}} - x^\rho T^{\mu\nu}_{\mathrm{can}} + H^{\mu\nu}A^\rho - H^{\mu\rho}A^\nu) \\
&= -\frac{1}{\gamma}(u^\nu p^\rho - u^\rho p^\nu) - \frac{1}{\gamma}(m^{\nu\sigma}F^\rho{}_\sigma - m^{\rho\sigma}F^\nu{}_\sigma).
\end{aligned}
$$

We therefore see that local conservation of angular momentum yields the equation of motion (21).

### C. The Belinfante-Rosenfeld Energy-Momentum Tensor

The overall system's canonical energy-momentum tensor (43) is not symmetric on its two indices, a feature that is required of the energy-momentum tensor that locally sources the gravitational field in general relativity. To conclude this paper, we follow the standard Belinfante-Rosenfeld construction[13] to construct a properly symmetric energy-momentum tensor, which will likewise represent a new result.

We start by introducing a new tensor

$$
\begin{aligned}
\mathcal{B}^{\mu\rho\nu} &\equiv \frac{c}{2}(\mathcal{S}^{\mu\nu\rho} + \mathcal{S}^{\nu\mu\rho} + \mathcal{S}^{\rho\mu\nu}) \\
&= -H^{\mu\rho}A^\nu + \frac{1}{2}(u^\mu S^{\nu\rho} + u^\nu S^{\mu\rho} + u^\rho S^{\mu\nu})\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}).
\end{aligned} \tag{57}
$$

We then obtain a symmetric, locally conserved energy-momentum tensor $T^{\mu\nu}$ for the overall system from the relation $T^{\mu\nu} = T^{\mu\nu}_{\mathrm{can}} + \partial_\rho \mathcal{B}^{\mu\rho\nu}$:[14]

$$
\begin{aligned}
T^{\mu\nu} &= \frac{1}{2}(u^\mu p^\nu + u^\nu p^\mu)\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\
&\quad + \frac{1}{2}H^{\mu\rho}F^\nu{}_\rho + \frac{1}{2}H^{\nu\rho}F^\mu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma} \\
&\quad + \frac{1}{2c^2}u^\mu u^\nu m^{\rho\sigma}F_{\rho\sigma}\frac{1}{\gamma}\delta^3(\mathbf{x}-\mathbf{X}) \\
&\quad + \frac{1}{2}\partial_\rho(\mathcal{S}^{\mu\nu\rho}_{\mathrm{particle}} + \mathcal{S}^{\nu\mu\rho}_{\mathrm{particle}}).
\end{aligned} \tag{58}
$$

In the free-field limit, this energy-momentum tensor reduces to the standard gauge-invariant Maxwell energy-momentum tensor, as expected:

$$T^{\mu\nu} = \frac{1}{\mu_0}F^{\mu\rho}F^\nu{}_\rho - \eta^{\mu\nu}\frac{1}{4\mu_0}F^{\rho\sigma}F_{\rho\sigma}. \tag{59}$$

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