Symmetries for Quantum Theory

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Abstract: Five conceptually distinct notions of symmetry in quantum theory are studied in the algebraic setting where a quantum system is characterized by a von Neumann algebra of observables and the set of normal states on the algebra. It is shown that all five symmetry notions are closely related and that the glue binding them together is the concept of a Jordan $*$-automorphism. For factor algebras a Jordan $*$-automorphism reduces either to an $*$-automorphism or a $*$-anti-automorphism. If the algebra is put in standard form then a $*$-automorphism is always unitarily implementable, whereas a $*$-anti-automorphism is always anti-unitarily implementable. However, there is no guarantee that a general von Neumann algebra admits $*$-anti-automorphisms or, if it does, that it admits order two (or involutory) $*$-anti-automorphisms). For non-factor algebras there can be genuine Jordan $*$-automorphisms that are neither $*$-automorphisms nor $*$-anti-automorphisms, and implementation is possible only through partial isometries. These developments enable generalized versions of Wigner's theorem on the implementation of transition probability preserving symmetries for von Neumann algebras. This review is largely an exercise in connecting the dots in existing mathematics and physics literature. But in the service of the philosophy of physics it is an exercise worth doing since the practitioners in this field seem largely unaware of or unappreciative of this literature and how it fits together to yield a multifaceted but unified picture of quantum symmetries. Along the way various interpretations issues worthy of further discussion are flagged.

1 Introduction

Various notions of symmetry in quantum theory have been studied in the algebraic setting, both in terms of C^*-algebras and von Neumann algebras. The latter seems the preferable setting because, arguably, C^*-algebras contain too many states and not enough observables.[1] Some care is needed in carrying over results from C^*-algebras to von Neumann algebras since results

[1] See Clifton and Halvorson (2001), Ruetsche (2011, Ch. 6), and Wald (1994, Ch. 4.6).

for C^*-algebras often appeal to all states on such algebras whereas the results one wants for von Neumann algebras should refer to normal states; and again in the C^*-algebra setting some results make heavy use of pure states whereas some von Neumann algebras do not admit any normal pure states. However, the developments discussed below will suggest a partial retreat to C^*-algebras (see Section 7).

Five conceptually distinct notions of symmetry are formulated for general von Neumann algebras. It is shown that all five symmetry notions are closely related and that the glue binding them together is the concept of a Jordan automorphism.

For factor algebras a Jordan automorphism reduces either to a $*$-automorphism or a $*$-anti-automorphism. If the algebra is put in standard form then a $*$-automorphism is always unitarily implementable, whereas a $*$-anti-automorphism is always anti-unitarily implementable. However, there is no guarantee that a general von Neumann algebra admits $*$-anti-automorphisms or, if it does, that it admits involutory $*$-anti-automorphisms. For non-factor algebras there can be genuine Jordan $*$-automorphisms that are neither $*$-automorphisms nor $*$-anti-automorphisms, and implementation is possible only through partial isometries. These developments enable generalized versions of Wigner's theorem on the implementation of transition probability preserving symmetries to be proven for most if not all Neumann algebras.

This review is largely an exercise in connecting the dots in existing mathematics and physics literature. But in the service of the philosophy of physics it is an exercise worth doing since the practitioners in this field seem largely unaware of or unappreciative of this literature and how it fits together to yield a multifaceted but unified picture of quantum symmetries. Along the way various interpretations issues worthy of further discussion are flagged.

2 Notions of symmetry for QM in the setting of von Neumann algebras

Throughout $\mathfrak{N}$ stands for a von Neumann algebra[2] and $\mathcal{S}(\mathfrak{N})$ for the normal states on $\mathfrak{N}$. A state on a von Neumann algebra $\mathfrak{N}$ is a positive linear map $\phi : \mathfrak{N} \to \mathbb{C}$ such that $\phi(I) = 1$. A normal state ϕ on $\mathfrak{N}$ has various equivalent characterizations: ϕ is completely additive on any family of mutually orthogonal projections in $\mathfrak{N}$; ϕ is ultraweakly continuous[3]; there is a trace class operator ρ (aka density matrix) with $Tr(\rho) = 1$ such that $\phi(A) = Tr(\rho A)$ for all $A \in \mathfrak{N}$ (Bratelli and Robinson 1987, Theorem 2.4.21). Five notions of symmetry studied here can be formulated in terms of transformations of $\mathfrak{N}$ and $\mathcal{S}(\mathfrak{N})$.

1) *State space symmetry.* The idea here is that a symmetry should preserve the structure of the space of states. In particular, it should respect the facts that the set of normal states is closed under convex linear combinations and that the map $A \mapsto \phi(A)$, $A \in \mathfrak{N}$ and $\phi \in \mathcal{S}(\mathfrak{N})$, is continuous.

> *Def. 1.* A state space symmetry is a bijection $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ that is an affine map (i.e. $\Phi(\lambda\phi_1 + (1-\lambda)\phi_2) = \lambda\Phi(\phi_1) + (1-\lambda)\Phi(\phi_2)$ for all $\phi_1, \phi_2 \in \mathcal{S}(\mathfrak{N})$ and all $\lambda \in [0,1]$) and is w^*-continuous (i.e. for any sequence $\{\phi_n\} \in \mathcal{S}(\mathfrak{N})$ such that $\phi_n(A) \to \phi(A)$ for all $A \in \mathfrak{N}$, $\Phi(\phi_n(A)) \to \Phi(\phi(A))$ for all $A \in \mathfrak{N}$).[4]

2) *Transition probability symmetry.* In ordinary QM where the algebra of observables is $\mathfrak{B}(\mathcal{H})$, the von Neumann algebra of all bounded operators on $\mathcal{H}$, transition probability symmetry is usually discussed in terms of transitions between normal pure states on $\mathfrak{B}(\mathcal{H})$. In this setting normal pure

[2] A concrete von Neumann algebra $\mathfrak{N}$ is a *-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$ such that $\mathfrak{N}$ is closed in the weak operator topology, or equivalently (by von Neumann's double commutant theorem), $\mathfrak{N} = \mathfrak{N}'' := (\mathfrak{N}')'$, where $\mathfrak{N}'$ denotes the commutant of $\mathfrak{N}$. For the mathematical background on von Neumann algebras, see Bratelli and Robinson (1987) and Kadison and Ringrose (1997). W^*-algebras, the abstract version of von Neumann algebras, will be introduced in Section 7.

[3] The ultraweak topology is also called the σ-weak topology. For $\mathfrak{N}$ acting on $\mathcal{H}$ it is the weakest topology on $\mathfrak{N}$ such that the elements of the predual of $\mathfrak{N}$ are continuous. This topology is intrinsic to the algebra in the sense that it is independent of the Hilbert space representation of $\mathfrak{N}$. Ultraweak convergence implies weak convergence.

[4] Convergence $\phi_n \to \phi$ in the w^* topology is pointwise convergence, i.e. for any $A \in \mathfrak{N}$ and any $\epsilon > 0$ there is an $N(A, \epsilon)$ such that $|\phi_n(A) - \phi(A)| \leq \epsilon$ for all $n > N(A, \epsilon)$.

states coincide with vector states[5], and the transition probability between pure states ϕ_1, ϕ_2 with representing (unit) vectors $\xi_{\phi_1}, \xi_{\phi_2} \in \mathcal{H}$, is given by the familiar formula $|\langle \xi_{\phi_1}, \xi_{\phi_2} \rangle|^2$. But some von Neumann algebras do not admit any normal pure states (for example, Type III algebras). Nevertheless, there are well-defined notions of transition probability between any pair of normal states on an arbitrary von Neumann algebra, and some of these notions turn out to be useful in the analysis of symmetries. Here we use that of Raggio (1982), which relies on the standard form of a von Neumann algebra.[6]

A standard form for a von Neumann algebra $\mathfrak{N}$ consists of a quadruple $(\mathfrak{N}, \mathcal{H}, J, \mathcal{C})$ where $J : \mathcal{H} \to \mathcal{H}$ is a conjugation[7] and $\mathcal{C} \subset \mathcal{H}$ is a self-dual cone[8] such that (i) $J\mathfrak{N}J = \mathfrak{N}'$, (ii) $JZJ = Z^*$ for all $Z \in \mathcal{Z}(\mathfrak{N})$ (where $\mathcal{Z}(\mathfrak{N}) := \mathfrak{N} \cap \mathfrak{N}'$ is the center of $\mathfrak{N}$), (iii) $J\xi = \xi$ for all $\xi \in \mathcal{C}$, and (iv) $A(JAJ)\mathcal{C} \subseteq \mathcal{C}$ for all $A \in \mathfrak{N}$. Every von Neumann algebra can be put in standard form in the sense that it is $*$-isomorphic to a von Neumann algebra in standard form (Haagerup 1975, Theorem 1.6). If $\mathfrak{N}$ acting on $\mathcal{H}$ is in standard form then for any normal state $\phi \in \mathcal{S}(\mathfrak{N})$ there is a unique vector $\xi_\phi \in \mathcal{C}$ such that $\phi(A) = \langle \xi_\phi, A\xi_\phi \rangle$ for all $A \in \mathfrak{N}$. For $\phi_1, \phi_2 \in \mathcal{S}(\mathfrak{N})$ the Raggio transition probability is defined as $P_R(\phi_1, \phi_2) := \langle \xi_{\phi_1}, \xi_{\phi_2} \rangle$, which is real number lying in $[0, 1]$, equal to 0 if and only if ξ_{ϕ_1} and ξ_{ϕ_2} are orthogonal, and equal to 1 if and only if $\xi_{\phi_1} = \xi_{\phi_2}$. Since the standard form of a von Neumann algebra is unique up to a unitary transformation the Raggio transition probability is independent of the chosen standard form. For $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ Raggio transition probability for normal pure states reproduces the familiar sense of transition probability. The standard form will also play a role later in the discussion of the Hilbert space implementation of symmetries.

These preliminary remarks lead us to posit:

Def. 2. A (Raggio) transition probability symmetry for $\mathfrak{N}$ is

[5] A state ϕ on $\mathfrak{N}$ acting on $\mathcal{H}$ is a vector state iff there is a $\xi \in \mathcal{H}$ such that $\phi(A) = \langle \xi, A\xi \rangle$ for all $A \in \mathfrak{N}$. A state ϕ is mixed iff there are distinct states φ_1 and φ_2 and $\lambda_1, \lambda_2 \in \mathbb{R}$, with $0 < \lambda_1, \lambda_2 < 1$ and $\lambda_1 + \lambda_2 = 1$, such that $\phi = \lambda_1\varphi_1 + \lambda_2\varphi_2$. A pure state is a non-mixed state.

[6] Other generalized concepts of transition probability are discussed in Raggio (1984) and Leung et al. (2016).

[7] A conjugation J is a conjugate linear map, $J(c_1\xi_1 + c_2\xi_2) = \bar{c}_1 J\xi_1 + \bar{c}_2 J\xi_2$, $c_1, c_2 \in \mathbb{C}$ and $\xi_1, \xi_2 \in \mathcal{H}$, that is norm preserving and has the property $J^2 = I$. For example in a $L^2_{\mathbb{C}}(\mathbb{R})$ realization of a separable $\mathcal{H}$, $(J\psi)(x) := \overline{\psi(x)}$ defines a conjugation.

[8] Self-dual means that $\mathcal{C} = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0 \text{ for all } \eta \in \mathcal{C}\}$.

a bijection $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ such that $P_R(\Phi(\phi_1), \Phi(\phi_2)) = P_R(\phi_1, \phi_2)$ for all $\phi_1, \phi_2 \in \mathcal{S}(\mathfrak{N})$.

The justification for using this concept of transition probability lies in its usefulness in extending symmetry notions to arbitrary von Neumann algebras.

3) *Algebra symmetry.* The idea here is that a symmetry is a mapping that preserves the structure of the algebra of observables. The most straightforward—if naive—way to cash this in is in terms of a $*$-automorphism of $\mathfrak{N}$, a bijection $\Theta : \mathfrak{N} \to \mathfrak{N}$ satisfying

$$\begin{aligned} (i)\ \ \Theta(\lambda A + \mu B) &= \lambda\Theta(A) + \mu\Theta(B) \\ (ii)\ \ \Theta(A^*) &= \Theta(A)^* \\ (iii)\ \Theta(AB) &= \Theta(A)\Theta(B). \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}$ and $A, B \in \mathfrak{N}$. However, it seems more in touch with experiments to focus on the self-adjoint elements $\mathfrak{N}_{SA} \subset \mathfrak{N}$. Although $\mathfrak{N}_{SA}$ is not an algebra with respect to the usual notion of multiplication it is an algebra under Jordan multiplication $A \circ B := \frac{1}{2}(AB + BA)$. Thus, mirroring the conditions for a $*$-automorphism, a Jordan $*$-automorphism of $\mathfrak{N}_{SA}$ is defined as a bijection $\theta : \mathfrak{N}_{SA} \to \mathfrak{N}_{SA}$ such that for all $\lambda, \mu \in \mathbb{R}$ and $A, B \in \mathfrak{N}_{SA}$

$$\begin{aligned} \theta(\lambda A + \mu B) &= \lambda\theta(A) + \mu\theta(B) \\ \theta(A^*) &= \theta(A)^* \\ \theta(A \circ B) &= \theta(A) \circ \theta(B). \end{aligned}$$

A Jordan $*$-automorphism θ can be extended to a bijection $\Theta : \mathfrak{N} \to \mathfrak{N}$ using the fact that any $A \in \mathfrak{N}$ can be uniquely decomposed as $A = R + iS$ with $R, S \in \mathfrak{N}_{SA}$, and by setting $\Theta(A) := \theta(R) + i\theta(S)$. So a Jordan $*$-automorphism of $\mathfrak{N}$ is defined a bijection Θ of $\mathfrak{N}$ such that for all $\lambda, \mu \in \mathbb{R}$ and $A, B \in \mathfrak{N}$

$$\begin{aligned} (i)\ \ \Theta(\lambda A + \mu B) &= \lambda\Theta(A) + \mu\Theta(B) \\ (ii)\ \ \Theta(A^*) &= \Theta(A)^* \\ (iii')\ \Theta(AB + BA) &= \Theta(A)\Theta(B) + \Theta(B)\Theta(A). \end{aligned}$$

Thus, we posit

> *Def. 3.* An algebra symmetry is a Jordan $*$-automorphism of $\mathfrak{N}$.

By this definition a $*$-automorphism is an algebra symmetry as is a $*$-anti-automorphism, the latter of which is a bijection of $\mathfrak{N}$ satisfying

$$\begin{aligned} (i)\ \ \Theta(\lambda A + \mu B) &= \lambda\Theta(A) + \mu\Theta(B) \\ (ii)\ \ \Theta(A^*) &= \Theta(A)^* \\ (iii'')\ \ \Theta(AB) &= \Theta(B)\Theta(A). \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}$ and $A, B \in \mathfrak{N}$.

4) Expectation value symmetry. This is the symmetry notion that is closest to experimental results. The version that comes from Roberts and Roepstorff (1969) combines bijections of states and observables. So let $\Theta : \mathfrak{N} \to \mathfrak{N}$ and $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ be bijections.

> *Def. 4.* An expectation value symmetry is a pair (Φ, Θ) that preserves expectation values, i.e. $\Phi(\phi)(\Theta(A)) = \phi(A)$ for all $\phi \in \mathcal{S}(\mathfrak{N})$ and all $A \in \mathfrak{N}$.

5) Quantum event space symmetry (aka quantum logic symmetry).

The projections $\mathcal{P}(\mathfrak{N})$ of $\mathfrak{N}$ can be thought of as representing quantum events or propositions. $\mathcal{P}(\mathfrak{N})$ is a lattice when equipped with the natural partial order $\leq$ where, for $E, F \in \mathcal{P}(\mathfrak{N})$, $E \leq F$ iff $range(E) \subseteq range(F)$. Meet '$\wedge$' and join '$\vee$' are defined respectively as the least upper bound and greatest lower bound. Lattice complementation E^c is taken to be orthocomplementation, i.e. $E^c := E^\perp = I - E$. This orthocomplemented lattice is an example of what is sometimes called a quantum logic.

> *Def. 5.* An event space symmetry (aka quantum logic symmetry) is a bijection $\Pi : \mathcal{P}(\mathfrak{N}) \to \mathcal{P}(\mathfrak{N})$ such that for all $E, F \in \mathcal{P}(\mathfrak{N})$
>
> (α) $\Pi(E) \leq \Pi(F)$ iff $E \leq F$
>
> (β) $\Pi(0) = 0$ and $\Pi(I) = I$
>
> (γ) $\Pi(E^\perp) = \Pi(E)^\perp$

and

(δ) For any family $\{E_\alpha\} \subset \mathcal{P}(\mathfrak{N})$, $\vee_\alpha \Pi(E_\alpha) = \Pi(\vee_\alpha E_\alpha)$ and $\wedge_\alpha \Pi(E_\alpha) = \Pi(\wedge_\alpha E_\alpha)$.

There are no doubt other useful and interesting symmetry notions in the setting of von Neumann algebras, but these five cover much of the territory. It is therefore remarkable that they turn out to be closely bound; and, as seen below, the glue that holds them together is Jordan $*$-automorphisms.

3 Linking symmetry concepts

Relating 1) and 3). Let $\Theta : \mathfrak{N} \to \mathfrak{N}$ be a bijection. Then for any $\phi \in \mathcal{S}(\mathfrak{N})$, $\Theta^*(\phi)(A) := \phi(\Theta(A))$, $A \in \mathfrak{N}$, defines a linear functional Θ^* from $\mathcal{S}(\mathfrak{N})$ to $\mathbb{C}$. It is easy to verify that if Θ is a Jordan $*$-automorphism then Θ^* is a bijective map of $\mathcal{S}(\mathfrak{N})$ onto itself; and furthermore Θ^* is a w^*-continuous affine map (see Emch 1971, pp. 153-154). Thus, if Θ is a Jordan $*$-automorphism then $\Phi = \Theta^*$ is a state space symmetry. The technique of proof used in Kadison (1965) and Emch (1971, Theorem 2, p. 155) serves to establish the converse: If the bijection $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ is a w^*-continuous affine map then there is a Jordan $*$-automorphism $\Theta : \mathfrak{N} \to \mathfrak{N}$ such that $\Phi = \Theta^*$.[9] Following Emch (1971, p. 155) the upshot of this mutual implication relation between state space and algebraic symmetries may be glossed as demonstrating the equivalence of the Heisenberg and Schrödinger pictures as they are drawn in the present algebraic setting.

Relating 2) and 3). Here the heavy lifting has been done by Leung et al. (2016). They prove that if $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ is a bijection then Φ preserves Raggio transition probabilities iff there is a Jordan $*$-automorphism $\Theta : \mathfrak{N} \to \mathfrak{N}$ such that $\Phi = \Theta^*$.

Relating 4) to 1) and 3) (and thus to 2)). The technique of proof used in Roberts and Roepstorff (1969) for C^*-algebras serves to establish that if (Φ, Θ) is an expectation value symmetry then Φ is a w^*-continuous affine map, and Θ is a Jordan $*$-automorphism. This gets us from 4) to 1) and 3)

[9] The Kadison and Emch results are for C^*- algebras, and they use the fullness of the space of all states on such an algebra. The normal states $\mathcal{S}(\mathfrak{N})$ of a von Neumann algebra $\mathfrak{N}$ are also full: if $A \in \mathfrak{N}$ is positive then there is a $\phi \in \mathcal{S}(\mathfrak{N})$ such that $\phi(A) > 0$.

and, thus, to 2). In the other direction, suppose that $\Theta : \mathfrak{N} \to \mathfrak{N}$ is a Jordan $*$-automorphism. Then $\Theta_*(\phi)(A) = \phi(\Theta^{-1}(A))$, $A \in \mathfrak{N}$, defines a bijective map Θ_* of $\mathcal{S}(\mathfrak{N})$ onto itself. Setting $\Phi = \Theta_*$, (Φ, Θ) is an expectation value symmetry. And similarly, if Φ is a state space symmetry then there is a Jordan $*$-automorphism Θ such that (Φ, Θ) is an expectation value symmetry

Relating 5) to 1)-4). In one direction we want to establish that if Θ is a Jordan $*$-automorphism of $\mathfrak{N}$ then its restriction to $\mathcal{P}(\mathfrak{N})$ is a logic symmetry of $\mathcal{P}(\mathfrak{N})$. A Jordan $*$-automorphism is order preserving (Bratelli and Robinson 1979, Theorem 3.2.3) so it satisfies property (α) for a logic symmetry. It then follows directly from the definitions of '$\wedge$' and '$\vee$' that a Jordan $*$-automorphism satisfies (δ) for a finite family of projections, and from this plus the fact that a Jordan $*$-automorphism is continuous in the strong operator topology (see Emch 1972, Result 1, p. 156) it follows that (δ) is satisfied. The proof of the satisfaction of properties (β) and (γ) for a logic symmetry is straightforward. From the fact that a Jordan $*$-automorphism is real linear it follows that $\Theta(O) = \Theta(I - I) = \Theta(I) - \Theta(I) = O$. Furthermore, $\Theta(2I^2) = 2\Theta(I) = 2\Theta(I)^2$, which implies that $\Theta(I)$ equals I or O; but it can't be the latter since then $\Theta(I) = \Theta(O)$ contradicting the injective character of Θ. From this we get for any projection $E \in \mathfrak{N}$, $\Theta(E^\perp) = \Theta(I - E) = \Theta(I) - \Theta(E) = I - \Theta(E) = \Theta(E)^\perp$.

Establishing that a logic symmetry of $\mathcal{P}(\mathfrak{N})$ extends to a Jordan $*$-automorphism of $\mathfrak{N}$ is far from trivial. That this is so with only a mild restriction on $\mathfrak{N}$ can be inferred from a remarkable theorem of Dye (1955) showing that, provided $\mathfrak{N}$ contains no Type I_2 summand, an orthoautomorphism of $\mathcal{P}(\mathfrak{N})$ (i.e. a bijection of $\mathcal{P}(\mathfrak{N})$ that preserves orthogonality in both directions) is implemented by a unique Jordan $*$-automorphism of $\mathfrak{N}$. To apply the theorem to the case at hand note that if Π is a logic symmetry of $\mathcal{P}(\mathfrak{N})$ then $[E, F] = 0$ iff $[\Pi(E), \Pi(F)] = 0$ for all $E, F \in \mathcal{P}(\mathfrak{N})$, and, thus, Π preserves orthogonality in both directions.[10]

Upshot: The five conceptually distinct symmetry concepts sketched above all coalesce around the notion of a Jordan $*$-automorphism. An algebra symmetry (= Jordan $*$-automorphism) is a logic symmetry; and, with the mild restriction on the algebra noted above, every logic symmetry is uniquely extendible to an algebra symmetry. An algebra symmetry Θ generates a state

[10] See also Kruszynski (1976). His proof relies of Gleason's theorem which, in generalized form, implies Dye's theorem (see Section 9 below).

space symmetry $\Phi = \Theta^*$, and every state space symmetry is generated by an algebra symmetry. Similarly, every Raggio transition probability symmetry is generated by an algebra symmetry, and every algebra symmetry generates a Raggio transition probability symmetry. Finally, if (Φ, Θ) is an expectation value symmetry then Φ is a state space symmetry and Θ is an algebra symmetry; if Φ is a state space symmetry then there is an algebra symmetry Θ such that (Φ, Θ) is an expectation value symmetry; and if Θ is an algebra symmetry there is a state space symmetry such that (Φ, Θ) is an expectation value symmetry.

Stepping back from the technicalia, the root notion behind all five symmetry concepts is that a symmetry transformation in quantum theory is a transformation that preserves some physically important structure identified by the theory. For each of the five concepts the structure preserving transformation is either a Jordan $*$-automorphism or is generated by a Jordan $*$-automorphism. An apparently stronger notion of symmetry transformation would require that the transformation preserves structure that is so central to the description of a quantum system that the transformation produces physically equivalent descriptions of the system, at least so far as the descriptive apparatus of the quantum theory can discern. Arguably, if physical equivalence is limited to what is experimentally testable then preservation of expectation values is sufficient as well as necessary for generating physically equivalent descriptions. If so, Jordan $*$-automorphisms are the key to generating equivalent descriptions in the algebraic approach to quantum theory.

In any case, enough has been said to make it evident that to understand quantum symmetry we need to understand more about Jordan $*$-automorphisms and their implementations.

4 Jordan $*$-automorphisms in ordinary QM

For ordinary non-relativistic QM (sans superselection rules) the algebra is $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$. For $\mathfrak{B}(\mathcal{H})$ the normal pure states on $\mathfrak{B}(\mathcal{H})$ are in one-to-one correspondence with the rays of $\mathcal{H}$. A Wigner symmetry $\mathbf{T}$ is a bijection of the unit rays of $\mathcal{H}$ that preserves the standard notion of transition probability between the states corresponding to any pair of rays. Wigner (1931) announced the fundamental theorem that any such symmetry $\mathbf{T}$ is implemented by a unique (up to phase factor) vector map $T : \mathcal{H} \to \mathcal{H}$ that is either unitary U or anti-unitary V.[11] In the former case $\Theta_U(A) := UAU^*$, $A \in \mathfrak{B}(\mathcal{H})$, is a $*$-automorphism. In the latter case $\Theta_V(A) := VA^*V^*$ is also a Jordan $*$-automorphism, but it is an *-anti–automorphism rather than a $*$-automorphism.[12]

Apart from its negative function of showing concretely that the implication that a Jordan $*$-automorphism is a $*$-automorphism is not valid in physically interesting cases, this example has the positive virtue of leading one to wonder which features of ordinary QM generalize to arbitrary von Neumann algebras. In particular,

Q1: Are Jordan $*$-automorphisms of a von Neumann algebra exhausted by $*$-automorphisms and *-anti–automorphisms?

Q2: $\mathfrak{B}(\mathcal{H})$ admits both non-trivial $*$-automorphisms and *-anti–automorphisms. Is the same true of arbitrary $\mathfrak{N}$?

Q3: That a $*$-automorphism (respectively, a $*$-anti-automorphism) Θ of $\mathfrak{N}$ acting on $\mathcal{H}$ is unitarily implementable (respectively, is anti-unitarily implementable) means that there is a unitary $U : \mathcal{H} \to \mathcal{H}$ (respectively, anti-unitary $V : \mathcal{H} \to \mathcal{H}$) such that $\Theta(A) = UAU^*$ (respectively, $\Theta(A) = VA^*V^*$) for all $A \in \mathfrak{N}$. A unitarily implementable $*$-automorphism is sometimes called a spatial automorphism (e.g. Dixmier 1984 and Emch 1972), and that terminology will be adopted here.[13] By extension of language an anti-unitarily implementable $*$-anti-automorphism will also be called spatial. Similarly, for $\mathfrak{N}_i \subseteq \mathfrak{B}(\mathcal{H}_i)$, $i = 1, 2$, a *-isomorphism (respectively, *-anti–isomorphism) $\Theta : \mathfrak{N}_1 \to \mathfrak{N}_2$ is called spatial a if there is a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\Theta(A) = UAU^*$ for all $A \in \mathfrak{N}_1$ (respectively, an anti-unitary $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\Theta(A) = VA^*V^*$ for all $A \in \mathfrak{N}_1$). All $*$-automorphisms and all $*$-anti-automorphisms of $\mathfrak{B}(\mathcal{H})$ are spatial. Is this also true for arbitrary $\mathfrak{N}$? Are all *-isomorphisms and all *-anti-isomorphisms between arbitrary von Neumann algebras spatial?

Q4: Do the answers to the first three questions lead to interesting gener-

[11] A bijection $V : \mathcal{H} \to \mathcal{H}$ is anti-unitary iff it is conjugate linear, $V(c_1\xi_1 + c_2\xi_2) = \overline{c}_2 V\xi_1 + \overline{c}_2^* V\xi_2)$ with $c_1, c_2 \in \mathbb{C}$ and $\xi_1, \xi_2 \in \mathcal{H}$, has an inverse V^{-1}, and is norm preserving, i.e. $||V\xi|| = ||\xi||$ for all $\xi \in \mathcal{H}$.

[12] There were gaps in Wigner's proof that were not filled in until three decades later; see Uhlhorn (1963) and Bargmann (1964). Uhlhorn's version of Wigner's theorem opens the way for a generalization to arbitrary von Neumann algebras; see Section 9 below.

[13] If a *-automorphism of $\mathfrak{N}$ is spatial and the implementing unitary belongs to $\mathfrak{N}$ then the automorphism is called inner. In what follows issues of spatiality are pursued without regard to whether or not the automorphism is inner. But the importance of inner automorphisms will be noted in Section 10.

alizations of Wigner's theorem for general von Neumann algebras?

The answer to the first question is given in the next section. The answers to the remainder unfold in successive sections.

5 Jordan $*$-automorphisms are sums of partial $*$-automorphisms and partial $*$-anti–automorphisms

A fundamental result due to Kadison (1951, Theorem 10 and Cor. 11)[14] (see also Bratelli and Robinson 1979, Theorem 3.2.3) shows that if Θ is a Jordan $*$-automorphism of $\mathfrak{N}$ then there is a projection E in the center $\mathcal{Z}(\mathfrak{N})$ of $\mathfrak{N}$ such that $\Theta(AB) = \Theta(A)\Theta(B)E + \Theta(B)\Theta(A)(I - E)$ for all $A, B \in \mathfrak{N}$.

Until further notice I will concentrate on factor algebras, i.e. algebras $\mathfrak{N}$ such that $\mathcal{Z}(\mathfrak{N})$ =$\{\lambda I\}$, $\lambda \in \mathbb{R}$. For a factor algebra $\mathfrak{N}$ a central projection E is either I or O so that any Jordan $*$-automorphism of $\mathfrak{N}$ is either a $*$-automorphism or an *-anti–automorphism, and the study of quantum symmetries cum Jordan $*$-automorphisms is then reduced to the study of these objects. However, for non-factors genuine Jordan $*$-automorphisms are possible, and for such automorphisms neither $*$-automorphisms nor *-anti–automorphisms by themselves, but only in combination, are realizations of the various notions of symmetry studied above.

6 The existence of $*$-automorphisms and $*$-anti–automorphisms

Any von Neumann algebra admits $*$-automorphisms galore. If $\mathfrak{N}$ acts on $\mathcal{H}$ choose a unitary $U : \mathcal{H} \to \mathcal{H}$ with $U \in \mathfrak{N}$.[15] Then $\Theta(A) := UAU^*$, $A \in \mathfrak{N}$, defines $*$-automorphism of $\mathfrak{N}$. Not every $*$-automorphism is of this form, i.e. not every $*$-automorphism is inner. But is it the case that every $*$-automorphism is spatial, i.e. implemented by a unitary that is not necessarily in the algebra? This question will be treated in the following section. Before turning to this matter some comments on *-anti–automorphisms are in order.

[14] Note that by a C^*-isomorphism Kadison (1951) means a Jordan *-isomorphism and not a *-isomorphism of the algebra.

[15] Recall that any $\mathfrak{N}$ is generated by its unitary elements in the sense that if $\{U\}$ is the set of unitaries in $\mathfrak{N}$ then $\{U\}'' = \mathfrak{N}$.

Which von Neumann algebras admit *-anti–automorphisms? Type I factors for sure. But apparently the general answer is not known. Why is there a worry here? Let $J : \mathcal{H} \to \mathcal{H}$ be a conjugation. For $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ and any conjugation J of $\mathcal{H}$, the map $\Theta_J(A) := JA^*J$, $A \in \mathfrak{B}(\mathcal{H})$, is a bijection of $\mathfrak{B}(\mathcal{H})$ having all of the properties required for a $*$-anti-automorphism. Moreover Θ_J is of order 2 (aka involutory), i.e. $\Theta_J^2 = I$. But before concluding that for some conjugation J of the Hilbert space on which $\mathfrak{N} \neq \mathfrak{B}(\mathcal{H})$ acts $\Theta_J(A)$ defines a $*$-anti-automorphism of $\mathfrak{N}$ it needs to be established that $JA^*J \in \mathfrak{N}$ for all $A \in \mathfrak{N}$. Størmer (1967, Theorem 4.5) shows that for a factor $\mathfrak{N}$ there exists a conjugation J such that $J\mathfrak{N}J = \mathfrak{N}$ iff $\mathfrak{N}$ admits an $*$-anti-automorphism order 2.

A $*$-anti-automorphism order 2 need not be of this form. For $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$ and $\dim(\mathcal{H})$ either finite and even or else infinite there is an anti-conjugation[16] $J' : \mathcal{H} \to \mathcal{H}$. Then $\Theta_{J'}(A) := -J'A^*J'$ is a $*$-anti-automorphism of order 2 of $\mathfrak{B}(\mathcal{H})$.[17] Further, when $\mathcal{H}$ admits an anti-conjugation J' then $\Theta_{J'} \neq \Theta_J$ for any conjugation J (see Strømer 1967, Lemma 3.9). And Θ_J and $\Theta_{J'}$ are, up to conjugacy, the only *-anti–automorphisms of $\mathfrak{B}(\mathcal{H})$ (see Ayupov 1995).[18] But again, for $\mathfrak{N} \neq \mathfrak{B}(\mathcal{H})$ it needs to be established that $-J'A^*J' \in \mathfrak{N}$ for all $A \in \mathfrak{N}$ before concluding that $\Theta_{J'}$ is a $*$-anti-automorphism of a $\mathfrak{N}$.

[Digression. When $\dim(\mathcal{H}) = 2$ the Hilbert space is isomorphic to $\mathbb{C}^2$, and linear operators are elements of $M(2, \mathbb{C})$, $2x2$ complex matrices. $\Theta\begin{pmatrix} x & y \\ w & z \end{pmatrix} := \begin{pmatrix} z & -y \\ -w & x \end{pmatrix}$ defines a $*$-anti-automorphism of order 2. It is implemented as $\Theta_{J'}(A) = -J'A^*J'$, $A \in M(2, \mathbb{C})$, where J' is an anti-conjugation of $\mathbb{C}^2$ given by $J'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \overline{y} \\ -\overline{x} \end{pmatrix}$ (see Strømer 1967, p. 363).[19] There is an interesting connection here with the quaternions. A quaternion $a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ (where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternion units satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$) is represented on $M(2, \mathbb{C})$ as $\begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$.

[16] A map $J' : \mathcal{H} \to \mathcal{H}$ that is conjugate linear, norm preserving, and has the property $J'^2 = -I$.

[17] $\Theta_{J'}(A)$ can be written as $J'A^*J'^*$ since $J'^* = J'^{-1} = -J'$.

[18] Two *-anti–automorphisms Θ_1 and Θ_2 of $\mathfrak{N}$ are conjugate if and only if there is a *-automorphism Ψ of $\mathfrak{N}$ such that $\Theta_2 = \Psi^{-1} \circ \Theta_1 \circ \Psi$.

[19] J' does not have a representation in $M(2, \mathbb{C})$ since it is not a linear operator.

Quaternions are precisely the elements of $M(2,\mathbb{C})$ satisfying $\Theta_{J'}(A^*) = A$.]

These remarks indicate the character of $*$-anti-automorphisms but do little towards establishing which algebras, beyond $\mathfrak{B}(\mathcal{H})$, admit them. Some sobering negative results are known. Connes (1975a, 1975b) gave examples of a Type III factor and a Type II_1 factor where there are no $*$-anti-automorphisms at all. Subsequently Jones (1980) gave an example of Type II_1 factor which admits $*$-anti-automorphisms but none are of order 2.

For a factor algebra $\mathfrak{N}$ that does not admit $*$-anti-automorphisms the study of Jordan $*$-automorphisms boils down to the study of $*$-automorphisms. But one might adopt the attitude that such an $\mathfrak{N}$ is not an appropriate arena for quantum physics since it cannot accommodate symmetries that correspond to $*$-anti-automorphisms; indeed, if $\mathfrak{N}$ that does not admit $*$-anti-automorphisms of order 2 it would seem that time reversal and CPT reversal symmetries cannot be accommodated since two successive applications of each of the symmetries should not result in any change in the physical situation.

7 Hilbert space implementations of Jordan $*$-automorphisms for factor algebras

7.1 Hilbert space implementations of $*$-isomorphisms and $*$-anti-automorphisms

A von Neumann factor algebra is either Type I, II_1, II_∞, or III (Takesaki 1979, Cor. 1.20, p. 297). It is known that factors of Type I, Type II_1, and Type III all have the property that any $*$-automorphism is spatial (see Emch 1971, pp. 156-158). However, it is also known that Type II_∞ factors with a II_1 commutant admit non-spatial $*$-automorphisms (Kadison 1955).[20] Spatiality of *-isomorphisms holds for some kinds of non-factor algebras, e.g. those with abelian commutants (Dixmier 1984, III.3.2, Cor. 1 of Prop. 3), but spatiality of *-isomorphisms of non-factors is not to be expected as the norm.

Nevertheless, there is a more liberal sense in which $*$-automorphisms of

[20] Type II algebras contain finite dimensional projections but no minimal projections. In Type II_1 algebras the identity operator is finite dimensional whereas in Type II_∞ it is infinite dimensional.

any von Neumann algebras can be counted as spatial. The motivation for this liberal construal starts with the attitude that algebras that are $*$-isomorphic may be counted as the "same" even though they act on different Hilbert spaces. This attitude can be underpinned by a more abstract form of von Neumann algebras that are freed from a Hilbert space representation. One way to achieve this goal is by means of a partial retreat into abstract C^*-algebras which are characterized independently of a Hilbert space representation. A W^*-algebra is a special type of C^*-algebra, namely a C^*-algebra with a pre-dual, i.e. an algebra that is the dual of a Banach space. A C^*-algebra is $*$-isomorphic to a von Neumann algebra if and only if it is a dual of Banach space, which is to say that it is a W^*-algebra (Sakai 1988, Theorem 1.16.7). Different but $*$-isomorphic concrete von Neumann algebras acting on different Hilbert spaces may be regarded as different representations of the same abstract von Neumann/W^*-algebra. The analysis of quantum symmetries given Sections 2-4 above can be carried out on W^*-algebras rather than on concrete von Neumann algebras since the analysis depends only on the structure of the algebra (in particular, on the Jordan $*$-automorphisms of the algebra) and the space of normal states on the algebra, things that are invariant over the isomorphism class of concrete von Neumann algebras.[21] From this more abstract perspective one is free to choose whichever Hilbert space representation of the W^*-algebra at issue is convenient in discussing Hilbert space implementation of quantum symmetries. And since Raggio transition probability, which was used to bring transition probability symmetry into our family of symmetry notions for general von Neumann algebras, requires the use of the standard form of algebras, it seems natural to use this form in assessing the Hilbert space implementation of symmetries. I will refer this as the standard implementation.

Recall that if $\mathfrak{N}_i \subseteq \mathfrak{B}(\mathcal{H}_i)$, $i = 1, 2$, are are Neumann algebras (factors or non-factors) in standard form then there are conjugations J_i of $\mathcal{H}_i$ such

[21] As with concrete von Neumann algebras, the normal states of a W^*-algebra can be characterized equivalently as the completely additive states or the ultraweakly continuous states; but obviously the existence of a density operator representation makes no sense for abstract W^*-algebra. Since a *-isomorphism $\alpha: \mathfrak{N}_1 \to \mathfrak{N}_2$ is automatically ultraweakly continuous, if ϕ is a normal state on $\mathfrak{N}_1$ then $\phi \circ \alpha^{-1}$ is a normal state of $\mathfrak{N}_2$. Whether or not a normal state of $\mathfrak{N}$ is a vector state does, of course, depend on the Hilbert space on which $\mathfrak{N}$ acts. Not all normal states are vector states; but if ϕ is a normal state on $\mathfrak{N}$ acting on $\mathcal{H}$ then ϕ is a vector state on $\mathfrak{N}$ acting on $\mathcal{H} \otimes \ell^2(\mathbb{N})$ as $\mathfrak{N} \otimes I_{\ell^2(\mathbb{N})}$ (see Jones 2009, Theorem 7.1.3).

that (a) $J_i \mathfrak{N}_i J_i = \mathfrak{N}_i'$ and (b) $J_i Z_i J_i = Z_i^*$ for all $Z_i \in \mathcal{Z}(\mathfrak{N}_i)$. And note that if conditions (a) and (b) are satisfied then any $*$-isomorphism of $\mathfrak{N}_1$ onto $\mathfrak{N}_2$ is spatial (Dixmier 1957, III.1.5, Theorem 6) and, a fortiori, any $*$-automorphism of an algebra satisfying (a) and (b) is spatial. Hence, all *-isomorphisms and all $*$-automorphisms of von Neumann algebras, factors and non-factors, count as spatial in the standard sense.

Using this result only a simple argument is needed to show that any $*$-anti-isomorphism between algebras in standard form is anti-unitarily implementable. Let $\mathfrak{N}_i \subseteq \mathfrak{B}(\mathcal{H}_i)$, $i = 1, 2$, be in standard form so that there are conjugations J_i of $\mathcal{H}_i$ as in (a) and (b), and let α be a $*$-anti-automorphism from $\mathfrak{N}_1$ onto $\mathfrak{N}_2$. We wish to show that α is implementable by an anti-unitary V. Towards this end note that the map $\beta : \mathfrak{N}_1 \to \mathfrak{N}_1'$ given by $A \to J_1 A^* J_1$ is a $*$-anti-isomorphism and, consequently, as the composition of two $*$-anti-automorphisms the map $\gamma := \alpha \circ \beta^{-1} : \mathfrak{N}_1' \to \mathfrak{N}_2$ is a $*$-isomorphism. But since $\mathfrak{N}_1'$ also satisfies (a) and (b) with the same J_1 as $\mathfrak{N}_1$ we know that γ is implementable by a unitary U. So $\alpha(A) = \gamma(\beta(A)) = U(JA^*J)U^* = (UJ)A^*(UJ)^*$, which is to say that α is implemented by the anti-unitary $V := UJ$.[22]

7.2 Implementing Jordan $*$-isomorphisms for factor algebras

If the standard sense of spatiality of *-automorphisms and $*$-anti–automorphisms is employed then the problem of Hilbert space implementations of Jordan $*$-automorphisms is solved for factor algebras. To repeat, a Jordan $*$-automorphism of a factor algebra is either a $*$-automorphism or a $*$-anti–automorphism, and, as we have just seen, *-automorphisms and $*$-anti–automorphisms are respectively unitarily and anti-unitarily implementable in the standard sense. The Hilbert space implementation of Jordan $*$-automorphisms for non-factors is a different matter and requires a more subtle treatment.

[22] This is essentially a rearrangement of Theorem 1 of Kadison (1998), which supplies a different proof of the unitary implementability of γ.

8 Hilbert space implementations of Jordan $*$-automorphisms for non-factors

The result on the spatiality in the standard sense for $*$-automorphisms and $*$-anti-automorphisms of von Neumann algebras applies to non-factors as well as factors. But non-factors may admit genuine Jordan $*$-automorphisms that are neither fish nor fowl, neither $*$-automorphisms nor $*$-anti-automorphisms, and in such cases the above account of Hilbert space implementations of Jordan $*$-automorphisms for factors is not applicable. Non-factor algebras cannot be swept under the rug because the application of quantum theory cannot be confined to factor algebras—non-factor algebras are encountered in a variety of contexts, including cases where superselection rules[23] arise and the thermodynamic limit in quantum statistical mechanics. The obvious need for an account of the Hilbert space implementation of Jordan $*$-automorphisms for non-factors went unmet for many years until it was satisfied by an elegant treatment by Riekers and Roos (1989).[24]

Recall that if Θ is a Jordan $*$-automorphism of $\mathfrak{N}$ acting on $\mathcal{H}$ then there is a decomposing projection, i.e. a central projection $E \in \mathcal{Z}(\mathfrak{N})$ such that $\Theta(A) = \Theta_1(A)E + \Theta_2(A)(I - E)$, $A \in \mathfrak{N}$, where $\Theta_1(AB) = \Theta(A)\Theta(B)$ and $\Theta_2(AB) = \Theta(B)\Theta(A)$, $A, B \in \mathfrak{N}$, act as partial $*$-automorphisms and $*$-anti-automorphisms respectively. In the Riekers-Roos scheme these partial $*$-automorphisms and $*$-anti-automorphisms are implemented as partial isometries of $\mathcal{H}$. A partial isometry is a linear or anti-linear map $W : \mathcal{H} \to \mathcal{H}$ that is an isometry on $(\ker W)^{\perp}$. $(\ker W)^{\perp}$ is called the initial space, and $rangeW$ is called the final space. W is a partial isometry iff W^*W is a projection from $\mathcal{H}$ onto $(\ker W)^{\perp}$. Further, $\ker W^* = (rangeW)^{\perp}$, and WW^* is a projection onto $\ker W^*$. Unitaries/anti-unitaries are partial isometries with initial and final spaces consisting of the entire $\mathcal{H}$. A pair of partial isometries (X, Y) of $\mathcal{H}$, with X linear and Y conjugate linear, are said to implement (Θ, E) if

[23] A non-trivial center for the algebra of observables is a characteristic feature of one conception of superselection rules; see Earman (2018).

[24] Their results are framed in terms of Jordan $*$-isomorphisms. Here I restate them in terms of $*$-automorphisms in order to maintain parallism with statements for factor algebras given above.

$$\begin{aligned} XX^* &= E \qquad X^*X = \Theta^{-1}(E) \\ YY^* &= E^{\perp} \quad\; Y^*Y = \Theta^{-1}(E^{\perp}) \\ \Theta(A) &= XAX^* + YA^*Y^*, \;\; A \in \mathfrak{N}. \end{aligned} \tag{RR}$$

Riekers and Roos (1989) establish the existence of such implementations of Jordan $*$-automorphisms of the standard form of a von Neumann algebra as well as the uniqueness once the decomposing projection E is specified. In general the decomposing projection is not unique; but a maximal decomposing projection is unique. If $\mathfrak{N}$ is a factor then either $E = I$ or $E = 0$. In the former case Θ is a $*$-automorphism, and the linear X is a total isometry and, therefore, a unitary; in the latter case Θ is a $*$-anti-automorphism, and the anti-linear Y is a total isometry and, therefore, an anti-unitary. Pure $*$-automorphisms and pure $*$-anti-automorphism are, of course, Jordan $*$-automorphisms, and the (RR) representation implies the spatiality $*$-automorphisms and $*$-anti-automorphisms of algebras, factors or non-factors, in standard form.

9 Generalized Wigner theorems for von Neumann algebras

Many different results have been labeled a "generalized Wigner theorem." Here I give my own (admittedly idiosyncratic) take on what deserves to wear that mantle.[25] The first goal is to generalize Wigner's result from the special case of $\mathfrak{B}(\mathcal{H})$ to as general a von Neumann algebra as possible and, more specifically, to reach the conclusion that a putative symmetry transformation generates a Jordan $*$-automorphism of the algebra. And for the generalization to count as a generalized Wigner theorem the putative symmetry transformation must be recognizably related to Wigner's original notion of symmetry. The second goal is to spell out the Hilbert space implementation of the Jordan $*$-automorphism for the von Neumann algebras covered by the first generalization, and to show that the implementation conforms to Wigner's form, or else to explain why this is not possible.

[25] For an overview of the history of Wigner's theorem and proposed generalizations see Chavalier (2007). For a sampling of other opinions on this matter, see Baker and Halvorson (2013), Kharatyan (1974), Shultz (1982), and Valente (2019).

The first candidate for a generalized Wigner theorem takes off from a result of Uhlhorn (1963) showing that, provided $\dim(\mathcal{H}) > 2$, the conclusion of Wigner's theorem for $\mathfrak{B}(\mathcal{H})$ can be derived from the apparently weaker condition that the ray mapping of $\mathcal{H}$ is a bijection preserving orthogonality of rays in both directions. The rays of $\mathcal{H}$ are in one-one correspondence with the minimal projections of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ so a bijection of unit rays preserving orthogonality can be lifted to an orthoautomorphism of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$. If h is a bijection of minimal projections of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ define the associated orthoautomorphism $\widetilde{h}$ of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ as follows: for $E \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$, $\widetilde{h}(E)$ is the largest projection orthogonal to $h(E_0)$ for every minimal $E_0 \in \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ orthogonal to E (Araki 1990, p. 125). If $\widetilde{h}_1$ and $\widetilde{h}_2$ are orthoautomorphisms of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ that agree on the minimal projections then they are the same. Dye's theorem can then be invoked to conclude that the orthoautomorphism of $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ is a Jordan $*$-automorphism of $\mathfrak{B}(\mathcal{H})$, which is always implemented by either a unitary or an anti-unitary transformation of $\mathcal{H}$, as Wigner's theorem showed. Thus, for $\dim(\mathcal{H}) > 2$ Wigner's theorem may be regarded as a special case of Dye's theorem. From this perspective, the generalization of Wigner's theorem to any von Neumann algebra containing no Type I_2 summand is already given by Dye's theorem. So our first generalization of Wigner's theorem is simply Dye's theorem:

> (W1) If $\mathfrak{N}$ is a von Neumann algebra containing no Type I_2 summand then any orthoautomorphism of $\mathcal{P}(\mathfrak{N})$ extends to a unique Jordan $*$-automorphism of $\mathfrak{N}$.

Another generalization which is closer to the language of Wigner's original result and which overcomes the lack of complete generality in (W1) due to its exclusion of algebras with Type I_2 summands can be fashioned by combining Raggio's (1982) notion of transition probability for normal states of von Neumann algebras with the results of Leung et al. (2016).

> (W2) If $\mathfrak{N}$ is a von Neumann algebra and $\Phi : \mathcal{S}(\mathfrak{N}) \rightarrow \mathcal{S}(\mathfrak{N})$ is a bijection of normal states of $\mathfrak{N}$ preserving Raggio transition probabilities then there is a Jordan $*$-automorphism Θ of $\mathfrak{N}$ such that $\Phi = \Theta^*$.

Since the (W2) generalization uses Raggio transition probability which assumes the use of a standard form of the algebra it seems appropriate to use this form in expanding (W2) include Hilbert space implementations of the symmetry. Then using the results of Section 7 on the standard implementation of $*$-automorphisms and $*$-anti-automorphisms we have the following for factor algebras:

> (W2$'$) If $\mathfrak{N}$ is a von Neumann algebra and $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ is a bijection of normal states of $\mathfrak{N}$ preserving Raggio transition probabilities then there is a Jordan $*$-automorphism Θ of $\mathfrak{N}$ such that $\Phi = \Theta^*$. If $\mathfrak{N}$ is a factor then Θ is either a $*$-automorphism or a $*$-anti-automorphism. And if $\mathfrak{N}$ acting on $\mathcal{H}$ is in standard form then accordingly Θ is either implementable by a unitary $U : \mathcal{H} \to \mathcal{H}$ or by anti-unitary $V : \mathcal{H} \to \mathcal{H}$. And accordingly if $\phi \in \mathcal{S}(\mathfrak{N})$ then either $\Phi(\phi)(A) = \Theta^*(\phi)(A) = \phi(\Theta(A)) = \phi(U^*AU)$ or $\Phi(\phi)(A) = \Theta^*(\phi)(A) = \phi(\Theta(A)) = \phi(V^*A^*V)$ for all $A \in \mathfrak{N}$. If ϕ is a vector state with representative vector $\xi_\phi \in \mathcal{H}$ then $\Phi(\phi)$ is also a vector state with representative vector $U\xi_\phi$ or $V\xi_\phi$ according as Θ is implementable by U or by V.

For non-factor algebras the implementation cannot in general duplicate the Wigner form since a Jordan $*$-automorphism of a non-factor may not reduce to a either a $*$-automorphism or a $*$-anti-automorphism. The closest such implementation proceeds via the Riekers-Roos scheme using partial isometries, leading to:

> (W2$''$) If $\mathfrak{N}$ is a von Neumann algebra and $\Phi : \mathcal{S}(\mathfrak{N}) \to \mathcal{S}(\mathfrak{N})$ is a bijection of normal states of $\mathfrak{N}$ preserving Raggio transition probabilities then there is a Jordan $*$-automorphism Θ of $\mathfrak{N}$ such that $\Phi = \Theta^*$. And if $\mathfrak{N}$ acting on $\mathcal{H}$ is standard form then there are partial linear and anti-linear isometries X and Y of $\mathcal{H}$ per the scheme (RR) such that for any $\phi \in \mathcal{S}(\mathfrak{N})$ and any $A \in \mathfrak{N}$

$$\begin{aligned}
\Phi(\phi)(A) &= \Theta^*(\phi)(A) = \phi(\Theta(A)) \\
&= \phi(XAX^* + YA^*Y^*) \\
&= \phi(XAX^*) + \phi(YA^*Y^*).
\end{aligned}$$

This seems to be as general a form of Wigner's theorem as can be achieved with the four corners of the present approach to quantum symmetries.

10 Concluding remarks

The five symmetry concepts studied above form a neat package, except for the fact that the generalized concept of transition probability lacks the clear operational significance that the familiar notion of transition probability possesses for $\mathfrak{N} = \mathfrak{B}(\mathcal{H})$. If ϕ_1 and ϕ_2 are normal pure states on $\mathfrak{B}(\mathcal{H})$ the transition probability from ϕ_1 and ϕ_2 means the probability that a system in state ϕ_1 will give a Yes answer to the measurement of the support projection of state ϕ_2, in which case the post-measurement state of the system is state ϕ_2. This inference follows from a fact and a postulate. The fact is that the support projection for a normal pure state on $\mathfrak{B}(\mathcal{H})$ is a filter for that state. (The support projection for a normal state on $\mathfrak{N}$ is the smallest element of $\mathcal{P}(\mathfrak{N})$ to which the state assigns probability 1. For a normal pure state on $\mathfrak{B}(\mathcal{H})$ the support projection is the projection onto the ray corresponding to said state. A filter for a normal state ϕ on a von Neumann algebra $\mathfrak{N}$ is an element $E_\phi \in \mathcal{P}(\mathfrak{N})$ such that for any normal state ω, $\dfrac{\omega(E_\phi A E_\phi)}{\omega(E_\phi)} = \phi(A)$ for all $A \in \mathfrak{N}$, provided that $\omega(E_\phi) \neq 0$.) The postulate is the Lüders projection postulate, asserting that if a system with algebra $\mathfrak{N}$ initially in state ω returns a Yes answer to a measurement of $E \in \mathcal{P}(\mathfrak{N})$, $\omega(E) \neq 0$, then the post measurement state is $\omega' = \dfrac{\omega(EAE)}{\omega(E)}$. The problem with extending this operational significance to a concept of transition probability for mixed states is that mixed states do not have filters. Of course, one could say that the concept of Raggio transition probability for mixed states acquires physical significance from its demonstrated web of relations to the other symmetry concepts studied above. Nevertheless, it would be reassuring to have a more direct operational indicator of physical significance.

There are also several pieces of unfinished business. To begin, the analysis of quantum symmetries on offer here needs to be extended to include dynamical symmetries. The obvious idea is to express, say, time translation invariance in terms of a one-parameter group Θ_t, $t \in \mathbb{R}$, of Jordan $*$-automorphisms of the von Neumann algebra characterizing the system of interest. The development of this idea cannot parallel the above analysis, e.g. it cannot be the case that for factor algebras Θ_t is either a group of $*$-automorphisms or a group of $*$-anti-automorphisms for the obvious reason that $*$-anti–automorphisms do not form a group, the composition of two *-anti–automorphisms being a $*$-automorphism. If Θ_t is a group of $*$-

automorphisms for an algebra in standard form then the $*$-automorphism group is implemented by a unitary group U_t. If further U_t is strongly continuous then by Stone's theorem it has a self-adjoint generator.[26] And if there is some means to identify *which* one parameter group of $*$-automorphisms expresses time translation invariance then this generator may be identified with the Hamiltonian of the system. It is more usual in physics to proceed the other way round; that is, to start with a Hamiltonian and then exponentiate to get a one-parameter unitary group which supplies both the Schrödinger dynamics (states but not observables evolve) and the Heisenberg dynamics (observables but not the states evolve). From this point of view it is not surprising that the concept of a Jordan $*$-automorphism, as distinct from a $*$-automorphism, does not enter the analysis of time translation symmetry.

There is also a bit of a puzzle about implementability of symmetries: Since neither the definitions of nor the analysis of the relationships among the five symmetry concepts discussed above rely on Hilbert space implementations, why then is so much attention in the literature devoted to this matter? Riekers and Roos (1989) opine:

> [I]t is a general experience in physics, that in order to construct a symmetry transformation and to calculate with it, one needs an explicit [Hilbert space] implementation. (p. 98)

Does this general experience point to a general truth about symmetry transformation? And, if so, is the truth about a merely pragmatic virtue of explicit Hilbert space implementations, or is it about what is essential to our understanding and deployment of symmetries? The safe answer is something of both. But this requires further thought.

Finally, it is worth mentioning a topic that, at first blush, seems irrelevant to present concerns. The seminal result for quantum probability theory is Gleason's theorem (Gleason 1957). In its original form it applied only to $\mathfrak{B}(\mathcal{H})$ with separable $\mathcal{H}$: Any countably additive probability measure on $\mathcal{P}(\mathfrak{B}(\mathcal{H}))$ with $\dim(\mathcal{H}) > 2$ extends uniquely to a normal state on $\mathfrak{B}(\mathcal{H})$. Several decades of work generalized the theorem to include non-separable $\mathcal{H}$ and most von Neumann algebras: If $\mathfrak{N}$ is a von Neumann algebra acting on $\mathcal{H}$

[26] The issue of whether the unitary implementer of a symmetry is inner has not played any role in the forgoing. But at this juncture it comes to the fore. In order for the self-adjoint generator H of U_t to count as an observable its spectral projections must belong to the algebra, which implies that U_t be inner.

(separable or non-separable) and containing no Type I_2 summand then any completely additive probability measure on $\mathcal{P}(\mathfrak{N})$ extends to a unique normal state on $\mathfrak{N}$. This form of Gleason's theorem implies Dye's theorem (see Bunch and Wright 1993), which as we have seen can be regarded as a generalized version of Wigner's theorem. It is remarkable that key aspects of quantum symmetries are already fixed by the basic structural features of quantum probability, but some connection is hardly surprising if one appreciates how thoroughly a probabilistic theory quantum mechanics is.

References

[1] Araki, H. 1990. "Some of the Legacy of John von Neumann in Physics: Theory of Measurement, Quantum Logic, and von Neumann Algebras in Physics," in J. Glim, J. Impagliazzo, and I. Singers (eds), *The Legacy of John von Neumann*, pp. 119-136. Providence: American Mathematical Society.

[2] Ayupov, S. A. 1995. "Anti-Automorphisms of Factors and Lie Operator Algebras," *Quarterly Journal of Mathematics* (Oxford), 46: 129-140.

[3] Baker, D. J. and Halvorson, H. 2011. "How is Spontaneous Symmetry Breaking Possible? Understanding Wigner's Theorem in Light of Unitary Inequivalence," *Studies in History and Philosophy of Modern Physics* 44: 464-469.

[4] Bargmann, V. 1964. "Note on Wigner's Theorem on Symmetry," *Journal of Mathematical Physics 5*: 862-868.

[5] Bratelli, O. and Robinson, D. W. 1987. Operator algebras and quantum statistical mechanics I, 2nd ed. Springer-Verlag: Berlin.

[6] Bunch, L.J. and Wright, J.D.M. 1993. "On Dye's theorem for operator algebras," *Expositiones Mathematicae* 11: 91-95.

[7] Chavalier, G. 2007. "Wigner's Theorem and Its Generalizations," in K. Engesser, D. M. Gabbay and D. Lehmann (eds.), *Handbook of Quantum Logic and Quantum Structures: Quantum Structures*, pp. 429-475. Amsterdam: Elsevier.

[8] Clifton, R., and Halvorson, H. 2001. "Are Rindler quanta real? Inequivalent particle concepts in quantum field theory," *British Journal for the Philosophy of Science* 52: 417-470.

[9] Connes, A. 1975a. "A factor not anti-isomorphic to itself," *Annals of Mathematics* 101: 536-554.

[10] Connes, A. 1975b. "Sur la classification des facteurs type II, *Compte Rendus de l'Académie des Sciences* (Paris) Sér. A-B 281: 13-15.

[11] Dye, H. A. 1955. "On the Geometry of Projections in Certain Operator Algebras," *Annals of Mathematics* 61: 73-89.

[12] Dixmier, J. 1984. *Von Neumann Algebras*. Paris: Amsterdam: North Holland.

[13] Earman, J. 2018. "Superselection for Philosophers," *Erkenntnis* 69: 377-414.

[14] Emch, G. 1972. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*. New York: John Wiley & Sons.

[15] Gleason, A. W. 1957. "Measures on the Closed Subspaces of a Hilbert Space," *Journal of Mathematics and Mechanics* 6: 885–893.

[16] Haagerup, U. 1975. "The Standard Form of von Neumann Algebras," *Mathematica Scandinavica* 37: 271-283.

[17] Jones, V. F. R. 1980. "A Type II_1 Factor Anti-Isomorphic to Itself but without Involutory Antiautomorphisms," *Mathematica Scandinavica* 46: 103-116.

[18] ___________ 2015. *Von Neumann Algebras*, lecture notes, https://math.vanderbilt.edu/jonesvf/VONNEUMANNALGEBRAS2015/VonNeumann2015.pdf

[19] Kadison, R. V. 1951. "Isometries of Operator Algebras," *Annals of Mathematics* 54: 325-338.

[20] ____________ 1955. "Isomorphisms of factors of infinite type," *Canadian Journal of Mathematics* 7: 322-327.

[21] ____________ 1965. "Transformations of States in Operator Theory and Dynamics," *Topology* 3, Suppl. 2: 177-198.

[22] Kadison, R. V. 1998. "Dual Cones and Tomita-Takesaki Theory," *Contemporary Mathematics* 228: 151-178.

[23] Kadison, R. V. and Ringrose, J. R. 1997. *Fundamentals of the Theory of Operator Algebras*, 2 Vols. Mathematical Society, Providence, R.I.

[24] Kharatyan, S. G. 1974. "Generalization of Wigner's Theorem on Symmetries in the C^*-Algebraic Approach," *Theoretical and Mathematical Physics* 20: 751-753.

[25] Kruszynski, P. 1976. "Automorphisms of Quantum Logics," *Reports on Mathematical Physics* 10: 213-217.

[26] Leung, C.-W., Ng, C.-K., and Wong, N.-C. 2016. "Transition probabilities of normal states determine the Jordan structure of quantum mechanics," *Journal of Mathematical Physics* 57: 015212-1-13

[27] Raggio, G. 1982. "Comparison of Uhlmann's Transition Probability with the One Induced by the Natural Positive Cone of von Neumann Algebras in Standard Form," *Letters in Mathematical Physics* 6: 233-236.

[28] _________ 1984. "Generalized Transition Probabilities and Applications," in L. Accardi, A. Figerio, and V. Gorini (eds), *Quantum Probability and Applications to Irreversible Processes*, pp. 327-335. Berlin: Springer.

[29] Riekers, A. and Roos, H. 1989. "Implementation of Jordan-Isomorphisms for General von Neumann Algebras," *Annales de l'Institut Henri Poincaré - Physique Théorique* 50: 95-113.

[30] Roberts, J. E., and Roepstorff, G. 1969. "Some Basic Concepts of Algebraic Quantum Theory," *Communications in Mathematical Physics* 11: 321-338.

[31] Ruetsche, L. 2011. *Interpreting Quantum Theories*. Oxford: Oxford University Press.

[32] Sakai, S. 1998. *C^*-Algebras and W^*-Algebras*. Springer: Berlin.

[33] Shultz, F. W. 1982. "Pure States as a Dual Object for C^*-Algebras," *Communications in Mathematical Physics* 82: 497-509.

[34] Størmer, E. 1967. "On Anti-Automorphisms of von Neumann Algebras," *Pacific Journal of Mathematics* 21: 349-370.

[35] Takesaki, M. 1979. *Theory of Operator Algebra I*. Berlin: Springer.

[36] Uhlhorn, U. 1962. "Representation of symmetry transformations in quantum mechanics," *Arkiv för Fysik 23*: 307-340.

[37] Valente, G. 2019. "Quantum Symmetry Breaking and Physical Inequivalence," pre-print.

[38] Wald, R. M. 1994. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. Chicago: University of Chicago Press.

[39] Wigner, E. 1931. *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*. Braunchsweig: Vieweg.