Cahn–Hilliard–Brinkman models for tumour growth: Modelling, analysis and optimal control

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Abstract

Phase field models recently gained a lot of interest in the context of tumour growth models. In this work we study several diffuse interface models for tumour growth in a bounded domain with sufficiently smooth boundary. The basic model consists of a Cahn–Hilliard type equation for the concentration of tumour cells coupled to a convection-reaction-diffusion-type equation for an unknown species acting as a nutrient and a Brinkman-type equation for the velocity. The system is equipped with Neumann boundary conditions for the phase field and the chemical potential, a Robin-type boundary condition for the nutrient and a “no-friction” boundary condition for the velocity which allows us to consider solution dependent source terms.

We derive the model from basic thermodynamic principles, conservation laws for mass and momentum and constitutive assumptions. Using the method of formal matched asymptotics, we relate our diffuse interface model with free boundary problems for tumour growth that have been studied earlier.

For the basic model, we show the existence of weak solutions under suitable assumptions on the source terms and the potential by using a Galerkin method, energy estimates and compactness arguments. If the velocity satisfies a no-slip boundary condition and is divergence free, we can establish the existence of weak solutions for degenerate mobilities and singular potentials.

From a modelling point of view, it seems to be more appropriate to describe the nutrient evolution by a so-called quasi-static equation of reaction-diffusion type. For this model we establish existence of both weak and strong solutions for regular potentials and a continuous dependence result yields the uniqueness of weak solutions and thus the model is well-posed. These results build the basis to study an optimal control problem where the control acts as a cytotoxic drug. Moreover, we rigorously prove the zero viscosity limit in two and three space dimensions which allows us to relate the Cahn–Hilliard–Brinkman model with Cahn–Hilliard–Darcy models which have been studied earlier.

Finally, we also analyse the model with quasi-static nutrients and classical singular potentials like the logarithmic and double-obstacle potential which enforce the phase field to stay in the physical relevant range. Under suitable assumptions on the source terms, we can establish the existence of weak solutions for these kinds of potentials.

Zusammenfassung

Phasenfeldmodelle stießen in jüngster Zeit auf großes Interesse im Zusammenhang mit Tumorwachstumsmodellen. In dieser Arbeit untersuchen wir mehrere diffuse Grenzschichtmodelle für Tumorwachstum in einem beschränkten Gebiet mit ausreichend glattem Rand. Das Ausgangsmodell besteht aus einer Cahn-Hilliard-Gleichung für die Konzentration von Tumorzellen gekoppelt mit einer Konvektions-Reaktions-Diffusions-Gleichung für eine unbekannte Spezies, die als Nährstoff dient, und einer Brinkman-Gleichung für die Geschwindigkeit. Wir vervollständigen das System mit Neumann-Randbedingungen für das Phasenfeld und das chemische Potential, einer Robin-Randbedingung für den Nährstoff und einer reibungsfreien Randbedingung für die Geschwindigkeit, die es uns ermöglicht, lösungsabhängige Quellterme zu berücksichtigen.

Wir leiten das Modell aus thermodynamischen Grundprinzipien, Erhaltungssätzen für Masse und Impuls und konstitutiven Annahmen her. Mithilfe von formaler asymptotischer Analysis setzen wir unser diffuses Grenzschichtmodell mit zuvor untersuchten freien Randwertproblemen für Tumorwachstum in Verbindung.

Für das Ausgangsmodell zeigen wir die Existenz von schwachen Lösungen unter geeigneten Annahmen an die Quellterme und das Potenzial unter Verwendung einer Galerkin-Methode, Energieabschätzungen und Kompaktheitsargumenten. Falls die Geschwindigkeit eine Haftbedingung am Rand erfüllt und divergenzfrei ist, können wir die Existenz schwacher Lösungen für degenerierte Mobilitäten und singuläre Potentiale nachweisen.

Aus Modellierungssicht erscheint es realistischer, die Nährstoffentwicklung durch eine sogenannte quasi-statische Reaktions-Diffusions-Gleichung zu beschreiben. Für dieses Modell zeigen wir, dass sowohl schwache als auch starke Lösungen für reguläre Potenziale existieren und diese Lösungen stetig von den Anfangswerten abhängen. Daraus folgt die Eindeutigkeit schwacher Lösungen, sodass das Modell wohlgestellt ist. Diese Ergebnisse bilden die Grundlage für die Untersuchung eines Optimalsteuerungsproblems, bei dem die Kontrolle als cytotoxisches Medikament wirkt. Darüber hinaus beweisen wir rigoros den Grenzwert der verschwindenden Viskositäten in zwei und drei Raumdimensionen, wodurch wir das Cahn–Hilliard–Brinkman-Modell mit zuvor untersuchten Cahn–Hilliard–Darcy-Modellen in Beziehung setzen können.

Schließlich analysieren wir das Modell auch mit quasi-statischen Nährstoffen und klassischen singulären Potentialen wie dem logarithmischen und dem Doppelhindernispotential, die garantieren, dass das Phasenfeld im physikalisch relevanten Bereich bleibt. Unter geeigneten Annahmen an die Quellterme zeigen wir die Existenz von schwachen Lösungen für diese Art von Potenzialen.

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1

Introduction

“Cancer: Finding Beauty in the Beast” - the title of this art-science collaboration (see [110]) is an excellent metaphor for the long-standing endeavour of mathematicians to describe tumour growth by mathematical models. The author of [28] depicts the important role of mathematics in cancer research as follows: “Through the development and solution of mathematical models that describe different aspects of solid tumour growth, applied mathematics has the potential to prevent excessive experimentation and also to provide biologists with complementary and valuable insight into the mechanisms that may control the development of solid tumours.” Particular examples are the prediction of the patient’s response to specific therapies and the development of new patient-specific treatment strategies which prevent undesirable side effects and resistance of the patient to the therapy, see for example [45, 136].

One of the earliest continuum models for spherical symmetric, avascular solid tumour growth goes back to the seminal work of Greenspan [96]. This model has been formulated as a free boundary problem and important mechanisms like adhesion and necrosis, that is the uncontrolled and unplanned cell death due to a lack of nutrients, have already been incorporated. It has been assumed that the tumour consumes nutrients like for example oxygen or glucose and consists of an outer rim of proliferating or viable cells and a necrotic core which forms due to an undersupply of nutrients. The cell motion is assumed to be proportional to the pressure gradient caused by the birth or death of cells. The model of Greenspan has served as a basis for many other works which used variants and extensions of this model, see, e. g., [25, 46, 65, 66, 73, 114, 116, 122, 143].

As a young tumour does not have its own vascular system and must therefore consume growth factors like nutrients or oxygen from the surrounding host tissue, in the early stage of growth the tumour may undergo morphological instabilities like fingering or folding (see, e. g., [44, 46]) to overcome diffusional limitations. This leads to highly challenging mathematical problems when modelling the tumour in the context of free boundary problems since changes in topology have to be tracked.

To overcome these difficulties, it has turned out that diffuse interface models – where the sharp interface is replaced by a narrow transition layer and the tumour is treated as a collection of cells – are a good strategy to describe the evolution and interactions of different species. In contrast to free boundary problems, there is no need to explicitly track the interface or to enforce complicated boundary conditions across the interface, see, e. g., [137]. Moreover, tissue interfaces may be more realistically represented by the diffuse interface framework since phase boundaries between tissues may not be well delineated, see [67]. These models are typically based on a multiphase approach, on balance laws for the single constituents, like mass and momentum balance, on constitutive laws and on thermodynamic principles. Several additional variables describing the extracellular matrix (ECM), growth factors or inhibitors can be incorporated into these models, and biological mechanisms like chemotaxis, apoptosis or necrosis and effects of stress, plasticity or viscoelasticity can be included, see [45, 86, 119]. Apoptosis is the process of programmed cell death and chemotaxis describes the movement of the tumour towards regions with higher nutrient concentrations, see Chapter 2 for more details.

In this thesis, we will always consider a mixture of two components consisting of tumour and surrounding tissue. Denoting by φ the difference of tumour and healthy volume fractions, and so $\varphi = 1$ in the tumour and $\varphi = -1$ in the healthy region, the species or mass balance law in its general form reads as

$$\partial_t \varphi + \operatorname{div}(\varphi \mathbf{v}) + \operatorname{div}(\mathbf{J}_\varphi) = \Gamma_\varphi,$$

where $\mathbf{J}_\varphi$ is the diffusive flux, $\mathbf{v}$ is the mixture velocity and Γ_φ is a source or sink term. In order to identify the flux $\mathbf{J}_\varphi$, it is essential to define the energy associated with the system. To account for adhesive forces between the tumour and healthy components, it has been suggested in [137] to take the well-known Ginzburg–Landau free energy which is given by

$$f(\varphi, \nabla\varphi) = \frac{\beta}{\varepsilon}\psi(\varphi) + \frac{\beta\varepsilon}{2}|\nabla\varphi|^2,$$

where ε and β are positive parameters related to the interfacial thickness and surface tension, respectively. The non-negative function $\psi \colon \mathbb{R} \to \mathbb{R}_+$ is of double-well structure with two minima in or near the pure phases $\varphi = \pm 1$. Typical examples are the logarithmic potential $\psi_{\log}$ which was originally proposed by Cahn and Hilliard [32], the double obstacle potential ψ_{do}, and the smooth double-well potential ψ which is an approximation of both $\psi_{\log}$ and ψ_{do}. They are defined by

$$\psi_{\log}(r) = \tfrac{\theta}{2}\left(\ln(1+r)(1+r) + \ln(1-r)(1-r)\right) + \tfrac{\theta_c}{2}(1-r^2) \quad \forall\, r \in (-1,1),$$

$$\psi_{\mathrm{do}}(r) = \begin{cases} \tfrac{1}{2}(1-r^2) & \text{for } |r| \le 1, \\ +\infty & \text{else,} \end{cases} \qquad \psi(r) = \tfrac{1}{4}(1-r^2)^2,$$

with positive constants $0 < \theta < \theta_c$. The term $\frac{\beta}{\varepsilon}\psi(\varphi)$ favours the pure phases $\varphi = \pm 1$ while the gradient term penalises too rapid spatial changes of φ. In the absence of velocity and source terms, the mass balance law and the Ginzburg–Landau energy lead to the famous Cahn–Hilliard equation in which the flux is given by

$$\mathbf{J}_\varphi = -m(\varphi)\nabla\mu \quad \text{where} \quad \mu = -\beta\varepsilon\Delta\varphi + \frac{\beta}{\varepsilon}\psi'(\varphi)$$

for a non-negative mobility function $m(\varphi)$. The Cahn–Hilliard equation has been derived in the seminal work [32] by Cahn and Hilliard and has been studied by several authors, see, e. g., [19, 20, 31, 33, 61, 62]. Originally introduced to model phase separation in binary alloys, the Cahn–Hilliard equation has become one of the most popular phase field models with various applications like for example image inpainting (see [15, 36, 88]), two-phase flow (see [3, 21]), topology optimisation (see [17, 18]) and tumour growth.

A common feature that a tumour shares with any other living tissue is the requirement of nutrient supply in order to grow. Consequently, we need to account for an additional species like oxygen or glucose in our model. Following the approach in [119], the conservation law and the energy contribution for the unknown species σ acting as a nutrient read as

$$\partial_t\sigma + \mathrm{div}(\sigma\mathbf{v}) + \mathrm{div}(\mathbf{J}_\sigma) = -\Gamma_\sigma, \quad N(\varphi,\sigma) = \frac{\chi_\sigma}{2}|\sigma|^2 + \chi_\varphi\sigma(1-\varphi),$$

where $\mathbf{J}_\sigma$ is a diffusive flux, Γ_σ is a term related to sources or sinks, and χ_σ, χ_φ are non-negative constants referred to as nutrient diffusion and chemotaxis parameter. By N we denote the nutrient free energy density which consists of one part which increases the energy in the presence of nutrients and another part referred to as chemotaxis energy and accounting for interactions between the nutrient and the tumour. As before, we may identify the flux as $\mathbf{J}_\sigma = -n(\varphi)(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi)$ where $n(\cdot)$ is a non-negative mobility function, and the chemical potential μ has to be extended by adding a chemotaxis term $-\chi_\varphi\sigma$ which drives the tumour towards regions with higher nutrient concentration. We can now write our system as

$$\begin{aligned}
\partial_t\varphi + \mathrm{div}(\varphi\mathbf{v}) &= \mathrm{div}(m(\varphi)\nabla\left(-\beta\varepsilon\Delta\varphi + \beta\varepsilon^{-1}\psi'(\varphi) - \chi_\varphi\sigma\right)) + \Gamma_\varphi,\\
\partial_t\sigma + \mathrm{div}(\sigma\mathbf{v}) &= \mathrm{div}(n(\varphi)\left(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi\right)) - \Gamma_\sigma.
\end{aligned}$$

Models of this form are referred to as Cahn–Hilliard type models and have been studied in the absence of velocity, that is, setting $\mathbf{v} = \mathbf{0}$, in many contributions like for example [39, 74, 82, 85, 101, 103]. For mixtures consisting of more than two components it is more suitable to describe tumour growth dynamics by so-called multiphase Cahn–Hilliard type models, see, e. g., [13, 47, 67, 76, 86, 120, 137]. These models are more realistic if, for example, the tumour undergoes necrosis.

However, neglecting the velocity may be too restrictive since living biological tissues in general exhibit viscoelastic properties, see [78, 79, 125]. As pointed out in [64, 66], it is reasonable to consider Stokes flow as an approximation of certain types of viscoelastic behaviour since relaxation times of elastic materials are rather short (see [79]). Therefore, many authors used Stokes flow to describe the tumour as a viscous fluid, see [30,35,65,68,71]. Classically, as pointed out earlier, velocities in tumour growth models are modelled with the help of Darcy's law. In these models the velocity is assumed to be proportional to the pressure gradient caused by the birth of new cells and by the deformation of the tissue, see [26, 66, 96]. Brinkman's law now interpolates between the viscous fluid and the Darcy-type models, see for example [51, 121, 134, 143], and can be derived from the momentum balance law when neglecting inertial effects. In the context of tumour growth this is a reasonable assumption since the Reynolds number is quite small. Brinkman's law was first proposed in [24] and has been derived rigorously by several authors using a homogenisation argument for the Stokes equation, see [10,124]. In this thesis, the general form of Brinkman's law including adhesion forces is given by

$$-\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}) + \nu(\varphi)\mathbf{v} = -\beta\varepsilon\mathrm{div}(\nabla\varphi\otimes\nabla\varphi), \quad \mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}},$$

where $\mathbf{D}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}})$ is the symmetrised velocity gradient, p is the pressure, and $\eta(\cdot)$, $\lambda(\cdot)$ and $\nu(\cdot)$ are non-negative functions related to shear and bulk viscosity as well as permeability. Moreover, the source term $\Gamma_{\mathbf{v}}$ can be derived from single species laws and is usually closely related to Γ_φ. Brinkman's law can be interpreted as an interpolation between Stokes flow and Darcy's law since the former one is approximated on small length scales whereas the latter one on large length scales, see [53].

Summarising the above equations we obtain a coupled Cahn–Hilliard–Brinkman system for tumour growth which serves as the basis for this thesis and is given by

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= \Gamma_{\mathbf{v}} && \text{in } \Omega \times (0,T) =: \Omega_T, \\
-\operatorname{div}(\mathbf{T}(\varphi,\mathbf{v},p)) + \nu(\varphi)\mathbf{v} &= -\operatorname{div}(\beta\varepsilon\nabla\varphi \otimes \nabla\varphi) && \text{in } \Omega_T, \\
\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) &= \operatorname{div}(m(\varphi)\nabla\mu) + \Gamma_\varphi && \text{in } \Omega_T, \\
\mu &= \beta\varepsilon^{-1}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma && \text{in } \Omega_T, \\
\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) &= \operatorname{div}(n(\varphi)\nabla(\chi_\sigma\sigma - \chi_\varphi\varphi)) - \Gamma_\sigma && \text{in } \Omega_T,
\end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded domain, $T > 0$ is a fixed final time and

$$\mathbf{T}(\varphi,\mathbf{v},p) := 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}$$

is called the viscous stress tensor. In most parts of the thesis, we will supplement the system (1.1) with initial and boundary conditions of the form

$$\begin{aligned}
\nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega \times (0,T) =: \Sigma_T, \\
\nabla\sigma\cdot\mathbf{n} &= K(\sigma_\infty - \sigma) && \text{on } \Sigma_T, \\
\mathbf{T}(\varphi,\mathbf{v},p)\mathbf{n} &= \mathbf{0} && \text{on } \Sigma_T, \\
\varphi(0) = \varphi_0, \quad \sigma(0) &= \sigma_0 && \text{in } \Omega,
\end{aligned} \tag{1.2}$$

where φ_0, σ_0 and σ_∞ are given functions, $\partial\Omega$ is the boundary of Ω and $K \geq 0$ is a constant related to the boundary permeability.
We will refer to (1.1)-(1.2) as the full model. In many parts of this thesis, we will use a so-called quasi-static nutrient equation given by

$$0 = \Delta\sigma - \Gamma_\sigma \qquad \text{in } \Omega_T. \tag{1.3}$$

This seems to be more realistic from the modelling point of view since the timescale of nutrient diffusion is usually quite small compared to the tumour doubling timescale. For contributions in direction of the classical Cahn–Hilliard–Brinkman system, i. e., without source terms and nutrients, we refer to [21,42] for the local model and [49,50] for the non-local model. Moreover, we mention the recent work [77] where they studied a (non-)local Cahn–Hilliard–Darcy–Forchheimer–Brinkman model for tumour growth.

In the following, we will outline the main novelties and difficulties of our model.

- A very important feature of our model is that the source term $\Gamma_{\mathbf{v}}$ may depend on φ and σ. Although this condition is of high practical relevance due to the relation between $\Gamma_{\mathbf{v}}$ and Γ_φ, many authors have worked with prescribed source terms $\Gamma_{\mathbf{v}}$ not depending on variables of the diffuse interface model, see e.g. [81, 105]. This is related to the fact that boundary conditions of the form

  $$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \qquad \text{or} \qquad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega$$

  require a source term $\Gamma_{\mathbf{v}}$ which fulfils the compatibility condition

  $$\int_\Omega \Gamma_{\mathbf{v}} \,\mathrm{d}x = \int_\Omega \operatorname{div}(\mathbf{v}) \,\mathrm{d}x = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n} \,\mathrm{d}\mathcal{H}^{d-1} = 0.$$

  Also in the case of inhomogeneous boundary conditions in the form given above, a compatibility condition has to be satisfied. In the case of a solution dependent source term, it is in general not possible to fulfil such a condition. In the literature, there are only a few contributions in this direction, see, e. g. [47, 83], where they consider a quasi-static nutrient equation. To the author's best knowledge, there is no contribution concerning existence of weak solutions for Cahn–Hilliard type models for tumour growth with solution dependent source terms, velocity effects and with a nutrient equation of the form $(1.1)_5$.

- The term $\mathbf{T}(\varphi, \mathbf{v}, p)\mathbf{n}$ characterises effects due to friction on the boundary. The boundary condition $(1.2)_3$ is one of the main features of our model and can be referred to as a "no-friction" condition. It allows us to consider solution dependent source terms and is quite useful in applications, see [95, App. III, 4.4], and very popular for finite element discretisations of the Navier–Stokes equation since it appears naturally in the variational formulation of $(1.1)_2$. In numerical simulations, it can be used to implement boundary conditions in an unbounded domain, for example a channel of infinite length. In this context, we also want to refer to the so-called classical "do-nothing" boundary condition

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{0},$$

  see, e. g., [102]. To the author's best knowledge, there are no contributions in the literature for Cahn–Hilliard-type models for tumour growth with the no-friction boundary condition

- The presence of source terms causes several new difficulties in the analysis. In particular, the most important properties of the classical Cahn–Hilliard equation are not fulfilled by our model, like the decrease of energy and the mass conservation property for φ, and classical arguments for the analysis do no longer work. Crucial in the analysis is the estimate for the chemical potential which requires a bound on the mean of μ to apply Poincaré's inequality. However, the mean of μ is related to the growth of the potential $\psi(\cdot)$ and therefore classical singular potentials cannot be included in the analysis in general. In many contributions, the potential is required to grow at most quadratically, see, e. g., [81–83]. We will apply a new estimate that allows us to consider potentials with higher order growth in some situations and, in particular, can be applied to the classical double-well potential. Moreover, we remark that the velocity is not divergence free and thus classical arguments for Stokes-like equations do not apply since the pressure cannot be eliminated in the weak formulation.

Structure of the thesis We will now outline the structure of this thesis.

A fundamental biological and mathematical background is provided in Chapter 2. In the first part we will introduce the biological notions and aspects related to tumour growth. In the second part we provide auxiliary results that will be applied in this thesis. Most of them are concerned with Galerkin schemes and the analysis of Brinkman or Stokes subsystems. We will give a detailed proof for weak and strong solutions of the Brinkman subsystem with solution dependent viscosities and permeability supplemented with Neumann-type boundary conditions for the stress tensor. The corresponding results seem not to be available in the literature in this form and may therefore be of independent interest.

In Chapter 3 we first derive the diffuse interface model using thermodynamic principles, constitutive assumptions, balance laws for mass and momentum, and the Lagrange multiplier method. We then discuss further aspects of modelling and we use the method of formally matched asymptotics in order to relate our model with free boundary problems for tumour growth that appeared earlier in the literature. Lastly, we will present numerical simulations which give insights into the qualitative behaviour of the model. The simulations have been made by Robert Nürnberg from Imperial College London (see [56]).

Existence of weak solutions for the full model in a very general setting will be established in Chapter 4. The proof is based on a Galerkin approximation, on energy estimates and compactness arguments. The chapter is based on the work [54].

Partial results of the work [55] will be presented in Chapter 5. More precisely, we will prove well-posedness and existence of strong solutions for the model with quasi-static nutrient equation.

This will serve as the basis for the optimal control problem which we study in Chapter 9. We point out that the analysis includes the classical double-well potential which is the standard smooth potential used in the literature.

In Chapter 6 we analyse several singular limits of the model with quasi-static nutrients. We remark that the results for three dimensions are part of the work [55]. In the limit of large boundary permeability, i. e., sending $K \to \infty$ in $(1.2)_2$, we recover weak solutions with a Dirichlet boundary condition for σ. More interestingly, we investigate the zero viscosity limit in the Cahn–Hilliard–Brinkman system (CHB) which allows us to relate our model to former Cahn–Hilliard–Darcy models (CHD) for tumour growth in the literature, see, e. g., [83]. In three space dimension we can show that weak solutions of the CHB model converge to a weak solution of the CHD model. In two space dimensions, we show that strong solutions of the CHB model converge to strong solutions of the CHD system. In particular, we establish uniqueness of weak solutions for the CHD tumour model and we prove a qualitative estimate between strong solutions of the CHB and CHD models in a similar fashion as in [21]. We remark that the CHD model without source terms and nutrient is referred to as Cahn–Hilliard–Hele–Shaw system and has been investigated in, e. g., [48, 63, 112].

A variant of the CHB model with one-sided degenerate mobilities and singular potentials will be analysed in Chapter 7. The mobility degenerates in $\varphi = -1$ and we allow for a singularity of $\psi(\cdot)$ in $\varphi = -1$. Typical examples are so-called single-well potentials of Lennard–Jones type, see, e. g., [5, 6]. In contrast to the rest of this thesis, we consider a no-slip boundary condition for the velocity and we set $\Gamma_{\mathbf{v}} = 0$, i. e.,

$$\operatorname{div}(\mathbf{v}) = 0 \quad \text{a. e. in } \Omega_T, \qquad \mathbf{v} = \mathbf{0} \quad \text{a. e. on } \Sigma_T.$$

We establish the existence of weak solutions for the full model based on arguments in [62]. However, we cannot apply the ideas directly since solutions for the non-degenerate mobility are not regular enough in order to justify a testing procedure in the style of [62]. Our idea is to add a regularisation term $\delta\partial_t\mathbf{v}$, $\delta > 0$, in the Brinkman equation in order to obtain more regular solutions for the system with non-degenerate mobility. We then regularise potential and mobility with the same parameter δ and establish estimates independent of $\delta > 0$ which allows us to obtain solutions for the degenerate mobility by sending $\delta \to 0$. Due to the no-slip boundary condition, the result remains valid for the Stokes equation since estimates can be obtained independent of the permeability ν. Our result seems to be the first for local Cahn–Hilliard type models for tumour growth with source terms and degenerate mobility. For the non-local version, we refer to [75] where they consider a two-sided degenerate mobility.

Under certain conditions on the source terms we can establish existence of solutions for the model with quasi-static nutrients and singular potentials, see Chapter 8. The results are part of the work [59] and include the logarithmic and double obstacle potential which are the most relevant examples. For our analysis it suffices to prescribe conditions on Γ_φ and $\Gamma_{\mathbf{v}}$ in the pure phases $\varphi = \pm 1$. In order to control the source terms we come up with a new estimate which allows us to bound the convex parts of the regularised potentials on the boundary independent of the regularisation parameter. We use the ideas presented in the work [88] to control the mean of φ since classical arguments as in [19] fail which is due to the fact that mass is not conserved. By sending the viscosities to zero we establish the corresponding results for the CHD model. Finally, we prove existence of solutions for the stationary model without velocity. In the context of Cahn–Hilliard type models with singular potentials and source terms we mention the works [76, 85], where the first one is in the absence of velocity and with Dirichlet boundary conditions and the second one considers a multi-phase model with a different boundary condition for μ. We point out that our methods can be used in a similar fashion for the so-called Cahn–Hilliard–Oono equation, see [93], and for models with applications to image inpainting, see [15, 88].

Finally, in Chapter 9 we study an optimal control problem where the medication by cytotoxic drugs acts as the control. The results are part of the contributions [57, 58]. We modify the nutrient equation by adding a term that models the supply of nutrients from an existing vasculature and we impose a Neumann boundary condition, i. e.,

$$0 = \Delta\sigma + \mathcal{B}(\sigma_B - \sigma) - \Gamma_\sigma \quad \text{in } \Omega_T, \qquad \nabla\sigma \cdot \mathbf{n} = 0 \quad \text{on } \Sigma_T,$$

where $\mathcal{B}$ is a positive constant and σ_B is a given nutrient supply from the vasculature. We establish the fundamental requirements of calculus of variations, the existence of a global optimal control and first order necessary optimality conditions. Similar results have been obtained for classical Cahn–Hilliard models in [16, 38, 41, 91, 104, 141, 142] and Cahn–Hilliard type models for tumour growth in, e. g., [40, 84, 106, 128–130]. In the last part of the chapter we establish second order sufficient conditions for local and global optimality. Finally, we investigate local and global uniqueness of optimal controls. To the author's best knowledge, this is the first contribution regarding second order optimality conditions for Cahn–Hilliard type models for tumour growth.

2

Biological and mathematical background

2.1 Fundamental biological aspects of tumour growth

In this section we present basic notions of biology which play a key role to understand mechanisms and processes related to tumour growth. Since the biology of humans and the inherent mechanisms in the human body are extensive and complicated, giving a complete description of cancer biology is far beyond the scope of this work. However, we aim to describe the key mechanisms and structures needed to understand the models we consider in this work. Once we have sketched the typical tissue structure, we will describe the different stages of tumour growth, from early stages where the tumour mainly grows by consuming nutrients from the surrounding environment, to later stages where the tumour has built its own vascular system. This part is inspired by and collected from the very well written and detailed biological textbooks [9, 109] and the work [45].

2.1.1 The tissue structure

Basically, we can subdivide the tissue into four groups (see [109, Chap. 1.2]).

(i) The supporting tissue called **mesenchyme** consisting of connective tissue like fibroblasts (which make collagen and elastin fibres as well as associated proteins), blood vessels, lymphatics, bone, cartilage and muscles; the **stroma** contains, e. g., the fibroblasts, blood vessels, lymphatics or collagen fibres, and is a part of the mesenchyme. The **extracellular matrix (ECM)** builds the main part of the stroma and consists of fibres like collagen or elastin surrounded by water and proteins.

(ii) The tissue-specific cells called **epithelium** – these are the specific cells of different organs like, e. g., skin, intestine or liver.

(iii) The **haematolymphoid system** consisting of 'defence' cells like lymphocytes, macrophages or lymphoid cells.

(iv) The **nervous system** which is divided into the central and peripheral nervous system, the first one consisting of the brain and spinal cord whereas the latter is comprised of nerves leading from the central system.

Structure and function of individual tissues are maintained by tissue-specific cells that are arranged in a standard pattern (see Figure 2.1 and [109, Figure 1.1]). Tissue-specific cells are grouped in a layer of epithelium, separated by a semipermeable **basement membrane** from the mesenchyme. The connective tissue consists mainly of the stroma which may be supported by a layer of bones or muscles, and is supplied with, for example, nutrients and nervous control by blood and lymphatic vessels or nerves, depending on the tissue-specific needs.

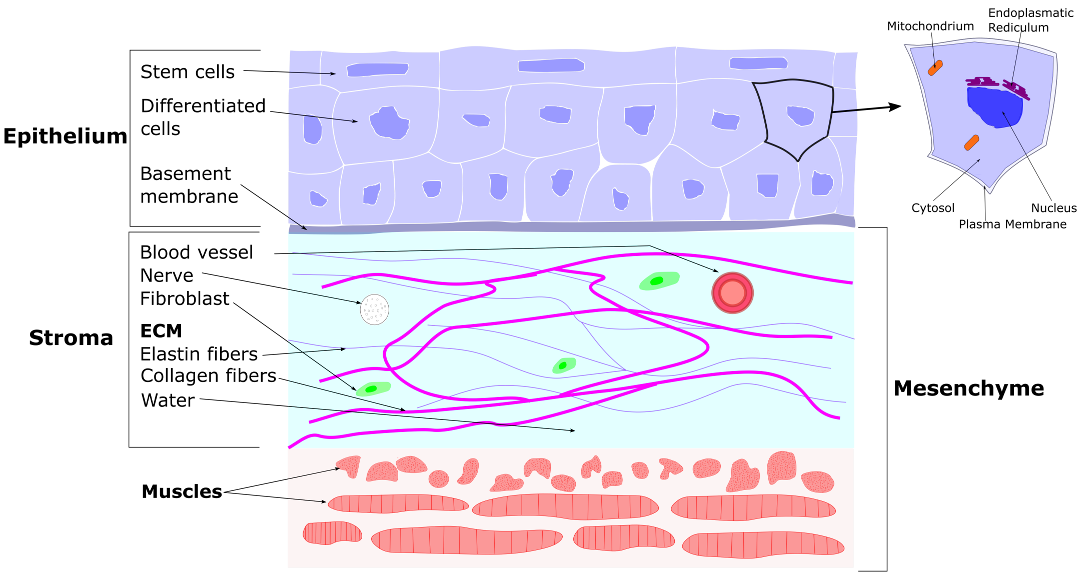

Figure 2.1: Typical tissue structure.

The **epithelium** mainly consists of two cell-types.

(i) **Differentiated cells** usually only differentiate into a specific type of cell which is due to a so-called cell memory phenomenon. The cells remember changes in gene expression and maintain their choice through subsequent cell generations, see [9, Chap. 7, p. 454].

(ii) **Stem cells** are specialized cells and provide an indefinite supply of new differentiated cells if those are, for example, lost or discarded, see [9, Chap. 23, p. 1417].

The structure of both differentiated and stem cells can basically be divided into three parts (see [45, Sec. 2.1.3]). The inner part consists of the **nucleus** that contains the cell's DNA. It is surrounded by the so-called **cytoplasm** consisting mainly of the cell liquid **cytosol** and containing organelles carrying out the functions of the cell, like, for example, **mitochondria** and **endoplasmic reticulum**.
Each cell is enclosed by a semi-impermeable **plasma membrane** separating the cytoplasm from the surrounding extracellular tissue and containing proteins that transfer information across the membrane, possibly triggering a changing behaviour of the cell. Moreover, the plasma membrane keeps the nutrient gathered in the cell and excretes waste products into the environment. For more information, see [9, Chap. 1].

2.1.2 Tumour growth as a multistage process

We first need to explain some crucial terms that are involved during the whole process.

Growth and proliferation (see [109, Chap. 1.3]): Although the term growth is often used for both of these processes, it is important to distinguish these two processes in the context of tumour growth.
We use the term **growth** to describe an increase of, for example, cell, tissue or tumour size. It is worth mentioning that there is a very precise mechanism allowing individual organs to reach a certain size which is usually never exceeded.
The term **proliferation** here means an increase of cell number realised by division. In the case that a part of the tissue is injured, damaged cells are replaced by division (proliferation) of the surviving cells. This process is mostly completed by special reserve or stem cells which can divide in order to substitute organ-specific cells. Proliferation is a multi-stage process and involves in particular the final stage called **mitosis** where two copies of the DNA are separated and two nuclei with this new DNA emerge.

Apoptosis (see [109, Chap. 12.2]) is the process of programmed cell death. Understanding this mechanism is of high importance for cancer researchers to develop new effective strategies and medicines since the effectiveness of, e. g., drug-based cytotoxic cancer therapy relies on the ability to kill cancer cells by inducing apoptosis.

In normal tissue – at least in an adult body – proliferation and apoptosis are rather balanced. As explained above, once a cell is injured or its DNA is damaged, the process of controlled cell death (apoptosis) starts and the damaged cell is replaced by a new one via proliferation.

The early stages - avascular tumour growth (see [45, Chap. 2.2.1 - 2.2.3])

In order to describe the early stages of tumour growth it is important to understand the effects that trigger the initial growth phase. The initial formation of tumour tissue is a multistage process referred to as **carcinogenesis**.
It starts with a genetic mutation of normal cells which triggers the formation of one or a small colony of tumour cells. If these mutations can overcome their natural repair mechanisms, they can further mutate which enables them to ignore neighbouring signals that would inhibit their growth.
In such a case the colony reaches the next stage of carcinogenesis in which low apoptosis favours the formation of a highly proliferative tumour colony. This ensemble of cells is referred to as **in situ** cancer, that means, it is situated in the epithelium and is usually rather small. It is more likely that the colony develops by mutations of stem cells rather than differentiated cells, since the latter ones are restricted in proliferation and cannot divide unlimited. In fact, after a limited number of divisions, differentiated cells either rest in an quiescent state or die by apoptosis.

In conclusion, the evolution of an initial tumour colony is mostly triggered by an imbalance of low apoptosis and high or fast cell proliferation.

At that time the young tumour colony does not have its own vascular system and must therefore consume growth factors and vital nutrients like oxygen or glucose from the surrounding stroma. Nourishment diffuses from the vascularized stroma, enters the epithelium where the tumour is located, and is uptaken by the cancerous cells to proliferate rapidly. As the extent of the tumour ensemble increases, cells in the middle are affected by **hypoxia** which means that they suffer from an undersupply of vital oxygen and, as a consequence, their rate of proliferation and the growth rate of the tumour declines, see [27]. If the concentration of oxygen or glucose in the tumour centre becomes smaller than a critical value, these cells undergo **necrosis**, that is,

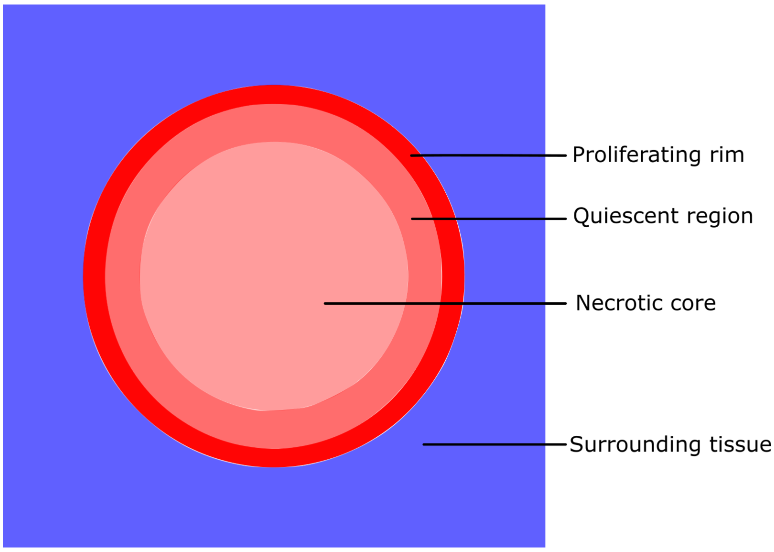

Figure 2.2: Structure of the tumour after necrosis, see also [140, Scheme 1].

uncontrolled and unplanned cell death due to a lack of nutrients. Once the process of necrosis has started, the tumour typically consists of three layers: a necrotic core, a region of quiescent cells which do not proliferate, and an outer viable rim of proliferating tumour cells, see Figure 2.2.

Apart from vascularization which will be described later on, there are other mechanisms that allow the tumour to overcome nutrient limitation. By interacting with its environment, the tumour can mechanically displace or compress its surroundings like the basement membrane. For instance, the tumour can release enzymes to degrade and remodel the ECM which possibly creates new fuel for growth. Degradation of the ECM leads to additional space and reduced pressure in the tumour's micro-environment. As a result the tumour invades and may undergo morphological instabilities like fingering or folding along the directions of low mechanical pressure, see, for example, [44, 46].
A similar effect is observed during **chemotaxis**, describing the movement (of the tumour) towards regions with higher concentrations of a soluble substrate (like oxygen) along the concentration gradient. In this context we also mention a process called **haptotaxis**, describing the movement towards directions with higher concentrations of a substrate-bound (chemo-)attractant.
Moreover, some tumours can mutate in order to build active glucose transporters (e. g., SGLTs) on their cell membrane to be independent of nutrient diffusion (so-called passive transport). These transporters trigger the movement of, for example, glucose towards the tumour colony even against the gradient of nutrient concentration. Such a process is referred to as **active transport**. For more information regarding chemotaxis and active transport, see, for example, [87] and the references therein.

The vascularization stage (see [9, Chap. 23], [45, Chap. 2.2.4])

During the process of **vascular growth** new capillary vessels are developed by sprouting or division from the pre-existing host vasculature. This process is referred to as **angiogenesis** and it is responsible for permanent remodelling and extension of the capillary network. It occurs during the whole life-time of a human being and mostly occurs if a part of the tissue suffers from a lack of blood supply and sends out complex signals, in particular so-called **vascular endothelial growth factors (VEGFs)**, in order to trigger the growth of new vessels.
Every vessel consists of a lumen surrounded by a shin sheet of endothelial cells which play a

crucial role during the process of angiogenesis, and a basal lamina separating the vessel from the surrounding outer layers.
Once the size of a tumour gets big enough, interior cells suffer from hypoxia. Those hypoxic cells produce **hypoxia-inducible factors (HIFs)** stimulating the transcription and secretion of so-called **tumour angiogenic growth factors (TAFs)** like VEGFs. As the VEGF proteins diffuse from the hypoxic region, a VEGF gradient emerges. Once the endothelial cells of the vessel detect this gradient they are stimulated to proliferate and they secrete proteins in order to find a way through the basal lamina. A new capillary forms, directed towards the VEGF source via chemotaxis or haptotaxis, and this new capillary links with another existing vessel or capillary, resulting in a new vasculature providing direct oxygen or nutrient supply to the cancer tissue and allowing for rapid growth of the tumour. For a more detailed description of this process, we refer, for example, to [9, Chap. 23].
In well-oxygenated tissue there are enzymes that switch of the production of, for example, HIFs once the new capillary has formed. However, this may not be the case in tumour tissue where new vessels can evolve even if the cells are well-supplied with oxygen or nutrients. Those new vessels may be less efficient, their neovasculature may be leaky, less stiff or collapsing when faced with tissue stress, and the basal lamina may have defects. As a consequence, drug therapy may be inefficient since drugs may not reach the tumour tissue.

The last stage in the process of tumour growth is referred to as **metastasis** and commonly causes death. It involves many complex phenomena, like, for example, genetic instabilities, increasing HIF production and loss of **cell-cell adhesion**. The tumours invade their surroundings and may develop secondary cancers. In many cases lymphatic vessels build the basis to enable the tumour escaping from its primary organ. Some of the cancer cells are transported to and arrested by lymph nodes. They may be destroyed or they build new tumours. The process of metastasis can also be a result of tumour cells entering blood vessels and moving to other organs. It is worth mentioning that, although being the most harmful stage, metastasis is still poorly understood. We refer to [45] and references therein for more information regarding this process.

2.2 Notation

We first fix some notation. Throughout this thesis we denote by $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, a bounded domain with boundary $\partial\Omega$, and by $T > 0$ a fixed final time. We denote $\Omega_T := \Omega \times (0,T)$, $\Sigma_T := \partial\Omega \times (0,T)$, and for $t \in (0,T)$ we write $\Omega_t := \Omega \times (0,t)$. For a (real) Banach space X we denote by $\|\cdot\|_X$ its norm, by X^* the dual space, and by $\langle \cdot , \cdot \rangle_X$ the duality pairing between X^* and X. For an inner product space X the inner product is denoted by $(\cdot , \cdot)_X$. We define the scalar product of two matrices by

$$\mathbf{A} \colon \mathbf{B} := \sum_{j,k=1}^{d} a_{jk} b_{jk} \quad \text{for } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d},$$

and the divergence of a matrix by

$$\operatorname{div}(\mathbf{A}) := \left(\sum_{k=1}^{d} \partial_{x_k} a_{jk}(x) \right)_{j=1}^{d} \qquad \forall\, \mathbf{A} \in \mathbb{R}^{d\times d}.$$

By $\mathbf{n}$ we will denote the outer unit normal on $\partial\Omega$. For the standard Lebesgue and Sobolev spaces with $1 \le p \le \infty$, $k > 0$, we use the notation $L^p := L^p(\Omega)$ and $W^{k,p} := W^{k,p}(\Omega)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$, respectively. In the case $p = 2$ we use $H^k := W^{k,2}$ and the norm $\|\cdot\|_{H^k}$. For $\beta \in (0,1)$ and $r \in (1,\infty)$ we will denote the Lebesgue and Sobolev spaces on the boundary by $L^p(\partial\Omega)$ and $W^{\beta,r}(\partial\Omega)$ with corresponding norms $\|\cdot\|_{L^p(\partial\Omega)}$ and $\|\cdot\|_{W^{\beta,r}(\partial\Omega)}$ (see, e. g., [133, Chap. I.3] for more details). In the case $r = 2$ we use $H^\beta(\partial\Omega) := W^{\beta,r}(\partial\Omega)$. We denote the space $W_0^{k,p}$ as the completion of $C_0^\infty(\Omega)$ with respect to the $W^{k,p}$-norm and we set $H_0^k := W_0^{k,2}$. By $\mathbf{L}^p$, $\mathbf{W}^{k,p}$, $\mathbf{H}^k$, $\mathbf{L}^p(\partial\Omega)$, $\mathbf{H}^\beta(\partial\Omega)$, $\mathbf{W}^{\beta,r}(\partial\Omega)$, $\mathbf{W}_0^{k,p}$ and $\mathbf{H}_0^k$ we will denote the corresponding spaces of vector valued and matrix valued functions. For Bochner spaces we use the notation $L^p(X) := L^p(0,T;X)$ for a Banach space X with $p \in [1,\infty]$. If $X = L^p$ we will sometimes identify $L^p(0,T;L^p)$ with $L^p(\Omega_T)$. We define

$$\|\cdot\|_{A\cap B} := \|\cdot\|_A + \|\cdot\|_B$$

for two or more Bochner spaces A and B. For the dual space X^* of a Banach space X we introduce the (generalised) mean value by

$$v_\Omega := \frac{1}{|\Omega|}\int_\Omega v \,\mathrm{d}x \quad \text{for } v \in L^1, \quad v_\Omega^* := \frac{1}{|\Omega|}\langle v, 1\rangle_X \quad \text{for } v \in X^*.$$

Moreover, we introduce the function spaces

$$\begin{aligned} L_0^2 &:= \{w \in L^2 \colon w_\Omega = 0\}, \quad (H^1)_0^* := \left\{f \in (H^1)^* \colon f_\Omega^* = 0\right\}, \\ H_N^2 &:= \left\{w \in H^2 \colon \nabla w \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}. \end{aligned}$$

For problems related to the Stokes equation we define the space of smooth and divergence-free vector fields with compact support in Ω by

$$\mathcal{V} := \left\{\mathbf{v} \in C_0^\infty(\Omega;\mathbb{R}^d) \colon \operatorname{div}(\mathbf{v}) = 0\right\},$$

and we define

$$\mathbf{H} := \overline{\mathcal{V}}^{\mathbf{L}^2}, \qquad \mathbf{V} := \overline{\mathcal{V}}^{\mathbf{H}^1}.$$

Then, it is well-known (see, e. g., [22, Lem. IV.3.4, Thm. IV.3.5]) that $\mathbf{V} = \left\{\mathbf{v} \in \mathbf{H}_0^1 \colon \operatorname{div}(\mathbf{v}) = 0\right\}$ and $\mathbf{H} = \left\{\mathbf{v} \in \mathbf{L}^2 \colon \operatorname{div}(\mathbf{v}) = 0,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\right\}$. Finally, for $1 < q < \infty$ we define

$$\mathbf{L}^q_{\operatorname{div}}(\Omega) := \{\mathbf{f} \in \mathbf{L}^q \colon \operatorname{div}(\mathbf{f}) \in L^q\}$$

equipped with the norm

$$\|\mathbf{f}\|_{\mathbf{L}^q_{\operatorname{div}}(\Omega)} := \left(\|\mathbf{f}\|^q_{\mathbf{L}^q} + \|\operatorname{div}(\mathbf{f})\|^q_{L^q}\right)^{\frac{1}{q}},$$

where div is the weak divergence.

2.3 Auxiliary results

We divide this section into several parts which are related to specific topics in this thesis. First, we recall some general results.

2.3.1 General auxiliary results

We start by stating the following generalised version of Hölder's and Young's inequalities:

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a bounded domain. Let $v \in L^p$, $1 \leq p \leq \infty$, and $w \in L^q$, $1 \leq q \leq \infty$. Then the product vw belongs to L^r where*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q},$$

and

$$\|vw\|_{L^r} \leq \|v\|_{L^p}\|w\|_{L^q}. \tag{2.1}$$

Lemma 2.2 *Let $a, b \in \mathbb{R}$ and $p, q \in (1, \infty)$. Then, for every $\delta > 0$ it holds*

$$|ab| \leq \delta |a|^p + \frac{(\delta p)^{1-q}}{q}|b|^q \quad \textit{with} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.2}$$

Proof. We follow the arguments in [11, proof of Lem. 1.14]. Without loss of generality, we assume $a, b > 0$. Concavity of the logarithm yields

$$\ln(ab) = \ln(a) + \ln(b) = \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q) \leq \ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \Longrightarrow ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Therefore, for $\delta > 0$ we have

$$ab = (p\delta)^{\frac{1}{p}} a \frac{1}{(p\delta)^{\frac{1}{p}}} b \leq \delta a^p + \frac{1}{q}\left((p\delta)^{-\frac{1}{p}}\right)^{\frac{p}{p-1}} b^q = \delta a^p + \frac{(\delta p)^{1-q}}{q} b^q,$$

and the proof is complete. □

We recall Poincaré's inequality with mean value for H^1.

Lemma 2.3 *(see [80, Thm. II.5.4]) Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz-boundary. Then, for all $f \in W^{1,q}$, $1 \leq q < \infty$ there exists a constant C_P depending only on Ω, q and d such that*

$$\|f\|_{L^q} \leq C_P \left(\|\nabla f\|_{\mathbf{L}^q} + |f_\Omega|\right) \quad \forall\, f \in W^{1,q}, \tag{2.3a}$$

or equivalently

$$\|f - f_\Omega\|_{L^q} \leq C_P \|\nabla f\|_{\mathbf{L}^q} \quad \forall\, f \in W^{1,q}. \tag{2.3b}$$

Furthermore, we will use the following generalised Gagliardo–Nirenberg inequality:

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz boundary and for $m \in \mathbb{N}$, $1 \leq q, r \leq \infty$, let $f \in W^{m,r} \cap L^q$. Moreover, consider any integer $j \in [0, m)$ and any $\theta \in \left[\frac{j}{m}, 1\right]$ such that there exists $p \in [1, \infty]$ satisfying*

$$j - \frac{d}{p} = \left(m - \frac{d}{r}\right)\theta + (1 - \theta)\left(-\frac{d}{q}\right).$$

If $r \in (1, \infty)$ and $m - j - \frac{d}{r}$ is a non-negative integer, we assume in addition that $\theta < 1$. Then there exists a positive constant C depending only on Ω, d, m, j, p, q, r and θ such that

$$\|D^j f\|_{L^p} \leq C \|f\|_{W^{m,r}}^{\theta} \|f\|_{L^q}^{1-\theta}. \tag{2.4}$$

Proof. See, e. g., [4, Thm. 5.8], [118, Thm. 1] and references therein. □

In the following we introduce the notions of linear and compact operators as well as (bi-)dual spaces.

Definition 2.5 (see [11, Sec. 3.2, Def. 3.5 and Sec. 6.2]) Let X, Y be normed $\mathbb{K}$-spaces where $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$. Then, we define the set of linear operators by

$$\mathcal{L}(X, Y) := \{S \colon X \to Y \,|\, S \text{ is linear and continuous}\},$$

and the set of compact operators by

$$\mathcal{K}(X; Y) := \left\{S \in \mathcal{L}(X, Y) \,|\, \overline{S(B_1(0))} \text{ is compact in } Y\right\}.$$

The dual space of X is defined by $X^* := \mathcal{L}(X; \mathbb{K})$ and the bi-dual space by $X^{**} := (X^*)^*$.

Lemma 2.6 *(see [11, Sec. 6.2.1]) Let X be a normed space. Then the mapping $J_X \in \mathcal{L}(X; X^{**})$ defined by*

$$\langle J_X(x), x^* \rangle_{X^*} := \langle x^*, x \rangle_X \quad \forall\, x^* \in X^*$$

is an isometry. In particular, it holds that

$$\|x\|_X = \sup_{x^* \in X^* \setminus \{0\}} \frac{\langle x^*, x \rangle_X}{\|x^*\|_{X^*}}. \tag{2.5}$$

Lemma 2.7 *(see [11, Sec. 10.1]) Let X, Y be normed spaces. Then the mapping ${}^*\colon \mathcal{L}(X,Y) \to \mathcal{L}(Y^*, X^*)$ which assigns to $S \in \mathcal{L}(X,Y)$ an operator $S^* \in \mathcal{L}(Y^*, X^*)$ defined via*

$$\langle S^* y^*, x \rangle_X := \langle y^*, Sx \rangle_Y \quad \forall\, x \in X,\ y^* \in Y^* \tag{2.6}$$

is an isometric embedding. The operator $S^* \in \mathcal{L}(Y^*, X^*)$ is called dual or adjoint operator of $S \in \mathcal{L}(X,Y)$.

Theorem 2.8 *(Schauder, see [11, Thm. 10.6]) Let X, Y be Banach spaces and $S \in \mathcal{L}(X;Y)$. Then it holds*

$$S \in \mathcal{K}(X,Y) \Longleftrightarrow S^* \in \mathcal{K}(Y^*, X^*).$$

Lemma 2.9 *Let X, Y be separable Banach spaces such that X is densely and continuously embedded into Y, i. e., there exists a continuous embedding $i\colon X \to Y$ such that $\overline{i(X)}^{Y} = Y$. Then, Y^* is continuously embedded into X^*. Moreover, if X is reflexive, the embedding is dense.*

Proof. First, we note that $i^* \in \mathcal{L}(Y^*, X^*)$. Let $f \in Y^*$ such that $i^*(f) = 0$. Then, it follows by definition that $\langle f, i(x) \rangle_Y = 0$ for all $x \in X$ and since $\overline{i(X)}^{Y} = Y$, this implies $f = 0$. Consequently, i^* is injective which yields the first assertion.
Now, let X be reflexive and let $h \in X^{**}$ such that $h(i^*(f)) = 0$ for all $f \in Y^*$. Then, there exists $x \in X$ such that $J_X(x) = h$ and

$$0 = \langle h, i^*(f) \rangle_{X^*} = \langle J_X(x), i^*(f) \rangle_{X^*} = \langle i^*(f), x \rangle_X = \langle f, i(x) \rangle_Y \quad \forall\, f \in Y^*.$$

Since i is injective and by (2.5), this implies $x = 0$, hence $h = J_X(x) = 0$. Therefore, we infer that

$$h(i^*(f)) = 0 \quad \forall\, f \in Y^* \Longrightarrow h(x^*) = 0 \quad \forall\, x^* \in X^*.$$

By the Hahn–Banach theorem we obtain $\overline{i^*(Y^*)}^{X^*} = X^*$ which completes the proof. □

2.3.2 Results related to Galerkin schemes

Galerkin schemes are a common approach to prove existence of solutions for PDE systems. The procedure can roughly be described as follows:

(i) construct a Schauder basis by means of eigenfunctions of a certain differential operator.

(ii) solve the PDE system on finite dimensional subspaces given by the span of eigenfunctions. For time-dependent problems, this mostly can be done by solving a system of (non-linear) ODEs.

(iii) prove that solutions are independent of the dimension of the finite dimensional subspaces.

(iv) use compactness arguments to recover the solution of the original problem when passing to the limit in the approximating system.

In the following we will present auxiliary results related to the individual steps of the Galerkin scheme.

Construction of a Schauder basis – the Neumann–Laplace operator

In this part we assume that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is bounded domain. We begin with the following definition:

Definition 2.10 The Neumann–Laplace operator $-\Delta_N \colon H^1 \to (H^1)_0^*$ is defined through

$$\langle -\Delta_N u, v \rangle_{H^1} := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \forall\, u, v \in H^1.$$

Remark 2.11 For arbitrary $u \in H^1$, the element $-\Delta_N u$ belong to $(H^1)_0^*$ since

$$\langle -\Delta_N u, 1 \rangle_{H^1} = \int_\Omega \nabla u \cdot \nabla 1 \,\mathrm{d}x = 0.$$

Lemma 2.12 *The following statements hold true:*

(i) *for every $f \in (H^1)_0^*$ there exists a unique $u \in H^1 \cap L_0^2$ such that $-\Delta_N u = f$,*

(ii) *for every $g \in L_0^2$ the mapping $f \colon v \mapsto (g, v)_{L^2}$ defines an element $f \in (H^1)_0^*$.*

Proof. We define the space $V := H^1 \cap L_0^2$. Then, applying Poincaré's inequality and the Lax–Milgram theorem, there exists a unique $u \in V$ solving

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \langle f, v \rangle_{H^1} \quad \forall\, v \in V.$$

Since $f \in (H^1)_0^*$, this identity holds also for all $v \in H^1$ which implies (i).
Assertion (ii) follows due to Hölder's inequality and the proof is complete. □

In particular, the inverse operator $(-\Delta_N)^{-1} \colon (H^1)_0^* \to H^1 \cap L_0^2$ is well-defined.

Lemma 2.13 *Let $w \in H^1$. Then, it holds*

$$(-\Delta_N)^{-1}(-\Delta_N w) = w - w_\Omega.$$

Proof. Setting $f = -\Delta_N w \in (H^1)_0^*$, we obtain

$$\langle f, v \rangle_{H^1} = \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega \nabla (w - w_\Omega) \cdot \nabla v \,\mathrm{d}x \quad \forall\, v \in H^1.$$

Since $w - w_\Omega \in H^1 \cap L_0^2$, this implies $(-\Delta_N)^{-1}(-\Delta_N w) = (-\Delta_N)^{-1} f = w - w_\Omega$ which completes the proof. □

Corollary 2.14 *The following statements holds true:*

(i) the embeddings $H_N^2 \subset H^1 \subset L^2 \simeq (L^2)^* \subset (H^1)^* \subset (H_N^2)^*$ are dense and continuous,

(ii) the embeddings $H_N^2 \subset\subset H^1 \subset\subset L^2$ and $(L^2)^* \subset\subset (H^1)^* \subset\subset (H_N^2)^*$ are compact.

Proof. (i): from standard Sobolev embedding theorems (see Lemma 2.32), it follows that $H_N^2 \subset H^1 \subset L^2$ with continuous embeddings. Moreover, it is well known that the embedding $H^1 \subset L^2$ is dense. From [82, Lemma 3.1], we obtain that the embedding $H_N^2 \subset H^1$ is dense. Then, since H_N^2, H^1, L^2 are separable, reflexive Banach spaces, applying Lemma 2.9 yields that the embeddings $(L^2)^* \subset (H^1)^* \subset (H_N^2)^*$ are dense and continuous. Finally, the embedding $L^2 \subset (L^2)^*$ is dense and continuous since L^2 is a separable Hilbert space.

(ii): it is well-known that the embeddings $H^2_N \subset\subset H^1 \subset\subset L^2$ are compact. Therefore, Theorem 2.8 implies the compactness of the embeddings $(L^2)^* \subset\subset (H^1)^* \subset\subset (H^2_N)^*$ which completes the proof. $\square$

Furthermore, the identifications $\langle u, v\rangle_{H^1} = (u, v)_{L^2}$, $\langle u, w\rangle_{H^2_N} = (u, w)_{L^2}$ hold for all $u \in L^2$, $v \in H^1$ and $w \in H^2_N$.

Lemma 2.15 *Let f_1, $f_2 \in (H^1)^*_0$. Then, the expression*

$$(f_1, f_2)_{(H^1)^*_0} = \int_\Omega \nabla\big((-\Delta_N)^{-1} f_1\big) \cdot \nabla\big((-\Delta_N)^{-1} f_1\big)\,\mathrm{d}x$$

defines a scalar product on $(H^1)^*_0$. Moreover, it holds that

$$\langle f_1, (-\Delta_N)^{-1} f_2\rangle_{H^1} = \int_\Omega \nabla\big((-\Delta_N)^{-1} f_1\big) \cdot \nabla\big((-\Delta_N)^{-1} f_2\big)\,\mathrm{d}x = \langle f_2, (-\Delta_N)^{-1} f_1\rangle_{H^1} \qquad (2.7)$$

and

$$(f, v)_{(H^1)^*_0} = \big((-\Delta_N)^{-1} f, v\big)_{L^2} \quad \forall f \in (H^1)^*_0,\ v \in L^2_0. \qquad (2.8)$$

Proof. Symmetry and linearity are obvious. Moreover, for $f \in (H^1)^*_0$ it holds

$$(f, f)_{(H^1)^*_0} = \|\nabla\big((-\Delta_N)^{-1} f\big)\|^2_{\mathbf{L}^2} \geq 0$$

and

$$\begin{aligned}(f, f)_{(H^1)^*_0} = 0 \Longleftrightarrow \|\nabla\big((-\Delta_N)^{-1} f\big)\|_{\mathbf{L}^2} = 0 \Longleftrightarrow (-\Delta_N)^{-1} f = c \Longleftrightarrow (-\Delta_N)^{-1} f = 0\\ \Longleftrightarrow (-\Delta_N)\big((-\Delta_N)^{-1} f\big) = f = 0\end{aligned}$$

where we used that $(-\Delta_N)^{-1} f \in H^1 \cap L^2_0$ and $(-\Delta_N)\big((-\Delta_N)^{-1} f\big) = f$. Identity (2.7) follows by definition of the Neumann–Laplace operator and the scalar product along with $(-\Delta_N)((-\Delta_N)^{-1} f_i) = f_i$, $i = 1, 2$. Finally, (2.8) is a consequence of

$$\begin{aligned}(f, v)_{(H^1)^*_0} &= \int_\Omega \nabla\big((-\Delta_N)^{-1} f\big) \cdot \nabla\big((-\Delta_N)^{-1} v\big)\,\mathrm{d}x = \langle -\Delta_N\big((-\Delta_N)^{-1} v\big), (-\Delta_N)^{-1} f\rangle_{H^1}\\ &= \langle v, (-\Delta_N)^{-1} f\rangle_{H^1} = \big((-\Delta_N)^{-1} f, v\big)_{L^2},\end{aligned}$$

where we used that

$$-\Delta_N\big((-\Delta_N)^{-1} v\big) = v \quad \forall v \in H^1 \cap L^2_0$$

and $\langle h_1, h_2\rangle_{H^1} = (h_1, h_2)_{L^2}$ for all $h_1 \in L^2$, $h_2 \in H^1$. $\square$

Lemma 2.16 *The Neumann–Laplace operator $-\Delta_N \colon \mathcal{D}(-\Delta_N) \subset (H^1)^*_0 \to (H^1)^*_0$ is positive definite and self-adjoint with $\mathcal{D}(-\Delta_N) = H^1 \cap L^2_0$.*

Proof. Applying Poincaré's inequality and using $(-\Delta_N)^{-1}(-\Delta_N u) = u$ for all $u \in H^1 \cap L^2_0$, the first assertion follows due to

$$(-\Delta_N u, u)_{(H^1)^*_0} = \int_\Omega |u|^1\,\mathrm{d}x \geq C\|u\|^2_{L^2} \quad \forall u \in H^1 \cap L^2_0.$$

Now, since $(-\Delta_N)^{-1}(-\Delta_N u) = u$ for all $u \in \mathcal{D}(-\Delta_N)$, for $v \in \mathcal{D}(-\Delta_N)$ it follows from the definition of the scalar product that

$$(v, -\Delta_N u)_{(H^1)^*_0} = (-\Delta_N v, u)_{(H^1)^*_0} \quad \forall u \in \mathcal{D}(-\Delta_n).$$

This implies $\mathcal{D}(-\Delta_N) \subset \mathcal{D}((-\Delta_N)^*)$ and $-\Delta_N v = (-\Delta_N)^* v$ for all $v \in \mathcal{D}(-\Delta_N)$.

Let $v \in \mathcal{D}((-\Delta_N)^*)$, $w = (-\Delta_N)^* v \in (H^1)_0^*$ and define $\tilde{w} = (-\Delta_N)^{-1} w \in \mathcal{D}(-\Delta_N)$. Then, we have

$$(v, (-\Delta_N)u)_{(H^1)_0^*} = (w, u)_{(H^1)_0^*} = ((-\Delta_N)\tilde{w}, u)_{(H^1)_0^*} = (\tilde{w}, (-\Delta_N)u)_{(H^1)_0^*} \quad \forall u \in \mathcal{D}(-\Delta_N).$$

Since $-\Delta_N\big(\mathcal{D}(-\Delta_N)\big) = (H^1)_0^*$, this yields $v = \tilde{w} \in \mathcal{D}(-\Delta_N)$. Consequently, we have

$$((-\Delta_N)v, u)_{(H^1)_0^*} = (v, (-\Delta_N)u)_{(H^1)_0^*} = (w, u)_{(H^1)_0^*} \quad \forall u \in \mathcal{D}(-\Delta_N).$$

The denseness of $\mathcal{D}(-\Delta_N)$ in $(H^1)_0^*$ implies $(-\Delta_N)v = w = (-\Delta_N)^* v$ and therefore the operator $-\Delta_N \colon \mathcal{D}(-\Delta_N) \subset (H^1)_0^* \to (H^1)_0^*$ is self-adjoint. □

Lemma 2.17 *The inverse Neumann–Laplace operator $\mathcal{L} \coloneqq (-\Delta_N)^{-1} \colon L_0^2 \to L_0^2$ is positive definite, symmetric and compact.*

Proof. Let f, $g \in L_0^2$ such that $y = \mathcal{L}f$, $z = \mathcal{L}g$. Then, we have

$$\begin{aligned}
(\mathcal{L}f, f)_{L^2} &= \int_\Omega yf \,\mathrm{d}x = \int_\Omega y(-\Delta_N)y \,\mathrm{d}x = \int_\Omega |\nabla y|^2 \,\mathrm{d}x \geq 0,\\
(\mathcal{L}f, g)_{L^2} &= \int_\Omega yg \,\mathrm{d}x = \int_\Omega y(-\Delta_N)z \,\mathrm{d}x = \int_\Omega \nabla y \cdot \nabla z \,\mathrm{d}x = (f, \mathcal{L}g)_{L^2}.
\end{aligned}$$

Now, let $\{f_n\}_{n\in\mathbb{N}} \subset L_0^2$ be a sequence and denote by $\{z_n = \mathcal{L}f_n\}_{n\in\mathbb{N}} \subset H^1 \cap L_0^2$ the corresponding solution sequence. Using elliptic regularity theory (see Lemma 2.32 below), it follows that $z_n \in H_N^2$ for all $n \in \mathbb{N}$. Then, due to the compact embedding $H_N^2 \hookrightarrow\hookrightarrow H^1$ and reflexive weak compactness, it follows that there is a subsequence (again labelled by n) such that $z_n \to z \in H^1 \cap L_0^2$ as $n \to \infty$. This completes the proof. □

Corollary 2.18 *The eigenfunctions of the Neumann–Laplace operator form an orthonormal Schauder basis in L^2 which is also a Schauder basis of H_N^2.*

Proof. The previous lemma and the spectral theorem (see [11, Thm. 10.12]) yield the existence of a countable set of eigenfunctions $\{v_i\}_{i\in\mathbb{N}}$ of the inverse Neumann–Laplace operator that forms a complete orthonormal system in L_0^2. The corresponding eigenvalues converge to zero as $i \to \infty$. The eigenfunctions of the Neumann–Laplace operator are therefore given by $w_1 = 1/\sqrt{|\Omega|}$ and $w_i = v_{i-1}$ for $i \geq 2$ and $\{w_i\}_{i\in\mathbb{N}}$ is a Schauder basis of L^2. Using elliptic regularity theory (see Lemma 2.32 below) we obtain $w_i \in H_N^2$. In the following we denote by λ_i, $i \in \mathbb{N}$, the corresponding eigenvalues to w_i, $i \in \mathbb{N}$, and we note that $\lambda_1 = 0$. For every $g \in H_N^2$ and $g_n \coloneqq \sum_{i=1}^n (g, w_i)_{L^2} w_i$ we obtain

$$\Delta g_n = \sum_{i=1}^n (g, w_i)_{L^2} \Delta w_i = \sum_{i=1}^n (g, \lambda_i w_i)_{L^2} w_i = \sum_{i=1}^n (g, \Delta w_i)_{L^2} w_i = \sum_{i=1}^n (\Delta g, w_i)_{L^2} w_i.$$

Consequently, we know that Δg_n converges strongly to Δg in L^2 and using elliptic regularity theory we infer that $g_n \to g$ strongly in H_N^2 as $n \to \infty$. This completes the proof. □

Now, we prove a convergence result for the projection onto H^1.

Lemma 2.19 *Let $\{w_i\}_{i\in\mathbb{N}}$ be the eigenfunctions of the Neumann–Laplace operator with corresponding eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}}$, i. e.,*

$$-\Delta w_i = \lambda_i w_i \quad \text{in } \Omega, \tag{2.9a}$$

$$\nabla w_i \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{2.9b}$$

By $P_n^{L^2}$ we denote the L^2-orthogonal projection onto the n-dimensional subspace span$\{w_1, \dots, w_n\}$. Then, $\{w_i\}_{i\in\mathbb{N}}$ is orthogonal in H^1 and for all $v \in H^1$ it holds

$$P_n^{L^2} v \to v \quad \text{in } H^1 \quad \text{as } n \to \infty \qquad \text{and} \qquad \|P_n^{L^2} v\|_{H^1} \leq \|v\|_{H^1}.$$

In particular, $\{w_i\}_{i\in\mathbb{N}}$ is a Schauder basis of H^1.

Proof. By the previous lemma it holds that $\{w_i\}_{i\in\mathbb{N}}$ is an orthonormal Schauder basis of L^2. The H^1 inner product is defined by

$$(v, w)_{H^1} := \int_\Omega \nabla v \cdot \nabla v + \int_\Omega vw \quad \forall\, v, w \in H^1.$$

Testing (2.9a) with w_j in L^2 and integrating by parts we obtain

$$\int_\Omega \nabla w_i \cdot \nabla w_j = \lambda_i \int_\Omega w_i w_j = \lambda_i \delta_{ij}.$$

Hence, we deduce that

$$(w_i, w_j)_{H^1} = (1 + \lambda_i)\delta_{ij}. \tag{2.10}$$

This means that $\{w_i\}_{i\in\mathbb{N}}$ is a orthonormal Schauder basis in L^2 which is orthogonal in H^1. In general, if $\{v_i\}_{i\in\mathbb{N}}$ is an orthogonal set in an inner product space H, the projection of $x \in H$ onto the n-dimensional subspace $\mathcal{V}_n := \operatorname{span}\{v_1, \dots, v_n\}$, $n \in \mathbb{N}$, is defined by

$$P_n^H x = \sum_{i=1}^{n} \left(x, \frac{1}{\alpha_i} v_i \right)_H \frac{1}{\alpha_i} v_i \quad \text{with} \quad \alpha_i = \|v_i\|_H \quad \forall\, 1 \leq i \leq n. \tag{2.11}$$

From (2.10) we deduce that

$$\|w_i\|_{H^1} = \sqrt{1 + \lambda_i} \quad \forall\, i \in \mathbb{N}.$$

Hence, we can define the H^1-orthogonal projection of $v \in H^1$ onto the n-dimensional subspace $\mathcal{W}_n := \operatorname{span}\{w_1, \dots, w_n\}$ by

$$P_n^{H^1} v = \sum_{i=1}^{n} \frac{1}{1 + \lambda_i} (v, w_i)_{H^1} w_i. \tag{2.12}$$

Partial integration yields

$$(v, w_i)_{H^1} = \int_\Omega \nabla v \cdot \nabla w_i + \int_\Omega v w_i = \lambda_i \int_\Omega v w_i + \int_\Omega v w_i = (1 + \lambda_i)(v, w_i)_{L^2} \quad \forall\, i \in \mathbb{N}.$$

This means

$$P_n^{H^1} v = \sum_{i=1}^{n} \frac{1}{1 + \lambda_i} (v, w_i)_{H^1} w_i = \sum_{i=1}^{n} (v, w_i)_{L^2} w_i = P_n^{L^2} v. \tag{2.13}$$

Since $\{w_i\}_{i\in\mathbb{N}}$ is a orthonormal Schauder basis in L^2, we know that

$$\sum_{i=1}^{n} (v, w_i)_{L^2} w_i \to v \quad \text{in } L^2 \quad \text{as } n \to \infty.$$

In order to show that $P_n^{H^1} v \xrightarrow{n\to\infty} v$ in H^1, it is enough to show that $\overline{\operatorname{span}\{w_i \colon i \in \mathbb{N}\}}^{H^1} = H^1$, see, e. g., [11, Section 9.7]. This is equivalent to show the following statement:

$$f \in H^1,\ (f, w_i)_{H^1} = 0 \quad \forall\, i \in \mathbb{N} \Rightarrow f = 0.$$

Taking $f \in H^1$ such that $(f, w_i)_{H^1} = 0$ for all $i \in \mathbb{N}$ and integrating by parts, we have

$$0 = (f, w_i)_{H^1} = \int_\Omega \nabla f \cdot \nabla w_i + \int_\Omega f w_i = (1 + \lambda_i) \int_\Omega f w_i.$$

Since $1 + \lambda_i > 0$ this implies

$$(f, w_i)_{L^2} = 0 \quad \forall\, i \in \mathbb{N}.$$

Using that $\{w_i\}_{i \in \mathbb{N}}$ is a Schauder basis in L^2 yields $f = 0$ and therefore $P_n^{H^1} v \xrightarrow{n \to \infty} v$ in H^1. Since $P_n^{H^1} v = P_n^{L^2} v$ for all $v \in H^1$, this implies that $P_n^{L^2} v \to v$ in H^1 and $\|P_n^{L^2} v\|_{H^1} \leq \|v\|_{H^1}$ for all $v \in H^1$, see [11, Section 9.7], and the proof is complete. □

Results on ODE theory

The results in this part are collected from [100, Chapter I]. By $D \subset \mathbb{R}^{d+1}$ we will always denote an open set and we write an element in D as $(t, \mathbf{x})$, with $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^d$. Furthermore, let $f\colon D \to \mathbb{R}^d$. For $(t_0, \mathbf{x_0}) \in D$ given we consider the following initial value problem:

(IVP) find an interval $I \subset \mathbb{R}$ containing t_0 and a function $\mathbf{x}$ defined on I such that

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = f(t, \mathbf{x}(t)) \quad \text{on } I, \qquad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{2.14}$$

Definition 2.20 We call $\mathbf{x}$ a classical solution of (2.14) on I if

(i) $\mathbf{x} \in C^1(I; \mathbb{R}^d)$, (ii) $(t, \mathbf{x}(t)) \in D \quad \forall\, t \in I$, (iii) $\mathbf{x}$ satisfies (2.14).

The following result is referred to as (Cauchy–)Peano existence theorem and can, e. g., be found in [100, Thm. 1.1].

Lemma 2.21 *Let f be continuous on D. Then, for any $(t_0, \mathbf{x}_0)$ there exists at least one classical solution $\mathbf{x}$ of* (2.14)*.*

Another concern is whether we can specify the existence interval I more precisely. To this end, we give the following definition:

Definition 2.22 Let $\mathbf{x}$ be a solution of (2.14) on some interval I. We call $\hat{\mathbf{x}}$ a continuation of $\mathbf{x}$ if the following conditions hold:

(i) $\hat{\mathbf{x}}$ is defined on an interval $\hat{I}$ with $I \subset\subset \hat{I}$,

(ii) $\hat{\mathbf{x}}$ coincides with $\mathbf{x}$ on I,

(iii) $\hat{\mathbf{x}}$ satisfies (2.14) on $\hat{I}$.

A solution $\mathbf{x}$ is called non-continuable if there exists no continuation, i. e., the interval I is the maximal existence interval of $\mathbf{x}$.

We have the following result:

Lemma 2.23 *Let $f\colon D \to \mathbb{R}^d$ be continuous and bounded and let $\mathbf{x}$ be a solution of* (2.14) *on some interval I. Then, there exists a continuation of $\mathbf{x}$ to a maximal existence interval (a, b). Let $\hat{\mathbf{x}}$ be the extension of $\mathbf{x}$ such that $\hat{\mathbf{x}}$ satisfies* (2.14) *on (a, b). Then, $(t, \hat{\mathbf{x}}(t))$ tends to the boundary of D as $t \to a$ and $t \to b$.*

Proof. See [100, Thm. 2.1]. □

Now, we investigate under which assumptions on f we have uniqueness of solutions. We have the following definition of local Lipschitz continuity.

Definition 2.24 A function $f\colon D \to \mathbb{R}^d$ is said to be locally Lipschitz continuous with respect to the second variable if for any closed bounded set $U \subset D$ there is a $k = k(U)$ such that

$$|f(t, \mathbf{y}_1) - f(t, \mathbf{y}_2)| \leq k|\mathbf{y}_1 - \mathbf{y}_2| \quad \forall\, (t, \mathbf{y}_1),\, (t, \mathbf{y}_2) \in U.$$

The following result is referred to as Picard–Lindelöf existence theorem.

Lemma 2.25 *Let $f\colon D \to \mathbb{R}^d$ be continuous and locally Lipschitz continuous with respect to the second variable. Then, for any $(t_0, \mathbf{x}_0) \in D$ there exists a unique classical solution $\mathbf{x}$ of* (IVP)*.*

Proof. See [100, Thm. 3.1]. □

Let us assume that we have a classical solution $\mathbf{x}$ of (IVP) on some existence interval I. Integrating the first identity in (2.14) with respect to time from t_0 to $t \in I$ and using $\mathbf{x}(t_0) = \mathbf{x}_0$ we obtain

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} f(t, \mathbf{x}(t))\, \mathrm{d}t \quad \forall\, t \in I. \tag{2.15}$$

The formulations (2.14) and (2.15) are equivalent if f is continuous, see, e. g., [100, Lem. 1.1]. Therefore, we can seek for a solution of the integral equation (2.15) instead of solving (IVP). If f is not continuous on D we cannot expect to obtain a classical solution of (IVP).
However, the right hand side of (2.15) does not necessarily require that f is continuous. Indeed, the integral in (2.15) is defined if, for example, f is dominated by and L^1-function on D. This allows us to solve (IVP) at least in an extended sense.
It turns out that it is enough to demand that f satisfies the so-called Carathéodory conditions which are defined as follows:

Definition 2.26 We say that $f\colon D \to \mathbb{R}^d$ satisfies the Carathéodory conditions on D if

(i) f is measurable in t for each fixed $\mathbf{x}$,

(ii) f is continuous in $\mathbf{x}$ for almost every t,

(iii) for each compact set $U \subset\subset D$ there is an integrable function $m_U(t)$ such that

$$|f(t, \mathbf{x})| \leq m_U(t) \quad \forall\, (t, \mathbf{x}) \in U. \tag{2.16}$$

We have the following existence result:

Lemma 2.27 *Let $f\colon D \to \mathbb{R}^d$ satisfy the Carathéodory conditions on D. Then, for any $(t_0, \mathbf{x}_0) \in D$ there exists an absolutely continuous function $\mathbf{x}$ defined on a real interval I such that* (2.15) *is satisfied. Moreover, $\mathbf{x}$ satisfies the first identity in* (2.14) *for almost every $t \in I$.*

Proof. See [100, Thm. 5.1]. We remark that our definition of Carathéodory condition is slightly different than in [100]. Indeed, in [100] the function f is required to be continuous in $\mathbf{x}$ for all t. By a closer inspection of [100, Proof of Thm. 5.1] we see that it is enough to ask for the continuity in $\mathbf{x}$ for almost every t. □

We have the following analogous statement to Lemma 2.23 (see [100, Thm. 5.2] for a proof):

Lemma 2.28 *Let $f\colon D \to \mathbb{R}^d$ satisfy the Carathéodory conditions and let $\mathbf{x}$ be a solution of* (2.15) *on some interval I. Then, there exists a continuation of $\mathbf{x}$ to a maximal existence interval (a, b). Let $\hat{\mathbf{x}}$ be the extension of $\mathbf{x}$ such that $\hat{\mathbf{x}}$ satisfies* (2.15) *on (a, b). Then, $(t, \hat{\mathbf{x}}(t))$ tends to the boundary of D as $t \to a$ and $t \to b$.*

Finally, we give a criterion for uniqueness of solutions for (2.15).

Lemma 2.29 *Let $f\colon D \to \mathbb{R}^d$ satisfy the Carathéodory conditions on D and suppose that for each compact set $U \subset\subset D$ there is an integrable function $k_U(t)$ such that*

$$|f(t, \mathbf{y}_2) - f(t, \mathbf{y}_1)| \leq k_U(t)|\mathbf{y}_2 - \mathbf{y}_1| \quad \forall\, (t, \mathbf{y}_2),\ (t, \mathbf{y}_1) \in U.$$

Then, for any $(t_0, \mathbf{x}_0) \in D$ there exists a unique solution $\mathbf{x}$ of (2.15).

Proof. See the proof of [100, Thm. 5.3]. □

Results related to a priori estimates

We start with a Gronwall inequality in integral form (see [82, Lemma 3.1]).

Lemma 2.30 *Let α, β, u and v be real-valued functions defined on $[0, T]$ for $T > 0$. Assume that α is integrable and bounded on $[0, T]$, β is non-negative and continuous, u is continuous, v is non-negative and integrable. If u and v satisfy the integral inequality*

$$u(s) + \int_0^s v(t)\,\mathrm{d}t \leq \alpha(s) + \int_0^s \beta(t)u(t)\,\mathrm{d}t \quad \text{for all } s \in (0, T],$$

then, it holds for all $s \in (0, T]$ that

$$u(s) + \int_0^s v(t)\,\mathrm{d}t \leq \alpha(s) + \int_0^s \alpha(t)\beta(t) \exp\Bigl(\int_0^t \beta(r)\,\mathrm{d}r\Bigr)\,\mathrm{d}t. \tag{2.17}$$

The following Gronwall-type inequality can be found in [52]:

Lemma 2.31 *Let u be a continuous, real-valued function defined on $[0, T]$ with $T > 0$. Moreover, let α, β, γ be real-valued, non-negative, integrable functions on $[0, T]$ with α bounded on the same interval and assume that*

$$u(s) \leq \alpha(s) + \int_0^s (\beta(t)u(t) + \gamma(t))\,\mathrm{d}t \quad \forall s \in [0, T].$$

Then, it holds

$$u(s) \leq \int_0^s \gamma(t)\,\mathrm{d}t + \sup_{0 \leq s \leq T} a(s) \exp\left(\int_0^s \beta(t)\,\mathrm{d}t\right) \quad \forall\, s \in [0, T].$$

Now, we recall results for elliptic regularity theory.

Lemma 2.32 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with $C^{1,1}$-boundary. Furthermore, let $\beta, \lambda > 0$, $1 < p < \infty$, $f \in L^p$, $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$ and $h \in W^{2-\frac{1}{p},p}(\partial\Omega)$. Let $u, v, w \in H^1$ be the* *solutions of*

$$\begin{aligned} -\Delta u + \lambda u &= f \quad \text{in } \Omega, & \nabla w \cdot \mathbf{n} &= g \quad \text{on } \partial\Omega, \\ -\Delta v &= f \quad \text{in } \Omega, & \nabla v \cdot \mathbf{n} + \beta v &= g \quad \text{on } \partial\Omega, \\ -\Delta w &= f \quad \text{in } \Omega, & w &= h \quad \text{on } \partial\Omega. \end{aligned}$$

Then, it holds $u, v, w \in W^{2,p}$ *and*

$$\begin{aligned} &\|u\|_{W^{2,p}} \le C \left(\|f\|_{L^p} + \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right), \qquad \|v\|_{W^{2,p}} \le C \left(\|f\|_{L^p} + \|g\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right), \\ &\|w\|_{W^{2,p}} \le C \left(\|f\|_{L^p} + \|h\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} \right) \end{aligned} \tag{2.18}$$

for a constant C *independent of* u, v, w.

If, in addition, for $k \ge 1$ *the domain* Ω *has a* $C^{k+1,1}$*-boundary,* $f \in W_p^k$, $g \in W^{1+k-\frac{1}{p},p}(\partial\Omega)$ *and* $h \in W^{2+k-\frac{1}{p},p}(\partial\Omega)$, *then* $u, v, w \in W_p^{k+2}$ *and*

$$\begin{aligned} \|u\|_{W^{k+2,p}} &\le C \left(\|f\|_{W^{k,p}} + \|g\|_{W^{1+k-\frac{1}{p},p}(\partial\Omega)} \right), \\ \|v\|_{W^{k+2,p}} &\le C \left(\|f\|_{W^{k,p}} + \|g\|_{W^{1+k-\frac{1}{p},p}(\partial\Omega)} \right), \\ \|w\|_{W^{k+2,p}} &\le C \left(\|f\|_{W^{k,p}} + \|h\|_{W^{2+k-\frac{1}{p},p}(\partial\Omega)} \right). \end{aligned} \tag{2.19}$$

Proof. This follows from an application of [97, Thm. 2.4.2.5 - 2.4.2.7, Thm. 2.5.1.1]. □

Lemma 2.33 *(trace and extension operator) Let* $\Omega \subset \mathbb{R}^d$, $d \ge 2$, *be a bounded domain with Lipschitz-boundary and let* $1 < q < \infty$. *Then, there exists a bounded linear operator* $T\colon W^{1,q} \to W^{1-\frac{1}{q},q}(\partial\Omega)$ *such that*

$$\|Tu\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \le C(\Omega, q) \|u\|_{W^{1,q}} \quad \forall\, u \in W^{1,q}, \tag{2.20}$$

and $T(u) = u|_{\partial\Omega}$ *for all* $u \in C^\infty(\Omega)$. *Furthermore, there exists a bounded linear operator* $T_e\colon W^{1-\frac{1}{q},q}(\partial\Omega) \to W^{1,q}$ *satisfying* $TT_e(u) = u$ *for all* $u \in W^{1-\frac{1}{q},q}(\partial\Omega)$ *and*

$$\|T_e u\|_{W^{1,q}} \le C(\Omega, q) \|u\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \quad \forall\, u \in W^{1-\frac{1}{q},q}(\partial\Omega). \tag{2.21}$$

Proof. See [117, Chap. 2, Thm. 5.5, Thm. 5.7] □

The following interpolation inequality will also be of importance:

Lemma 2.34 *(see [80, Thm. II.4.1]) Let* $\Omega \subset \mathbb{R}^d$, $d \ge 2$, *be a bounded domain with Lipschitz boundary and let* $u \in W^{1,q}$ *with* $q \in [1, \infty)$. *Assume*

$$\begin{aligned} &r \in [q, q(d-1)/(d-q)] \quad \text{if } q < d, \\ &r \in [q, \infty) \quad\quad\quad\quad\quad\;\; \text{if } q \ge d. \end{aligned}$$

Then, it holds that

$$\|u\|_{L^r(\partial\Omega)} \le C \left(\|u\|_{L^q}^{1-\alpha} \|u\|_{W^{1,q}}^{\alpha} + \|u\|_{L^q}^{(1-\frac{1}{r})(1-\alpha)} \|u\|_{W^{1,q}}^{\frac{1}{r}+\alpha(1-\frac{1}{r})} \right) \tag{2.22}$$

for a positive constant $C = C(d, r, q, \Omega)$ *and* $\alpha = \frac{d(r-q)}{q(r-1)}$.

Results for the limiting process

We first state a generalised version of Lebesgue dominated convergence theorem.

Lemma 2.35 *Let $k \in \mathbb{N}$, $1 \le p < \infty$, and let $f_k, f \colon \Omega \to \mathbb{R}$ be measurable functions. Furthermore, let $\{g_k\}_{k\in\mathbb{N}}$ be a sequence of measurable and non-negative functions such that $g_k \to g$ in L^1 as $k \to \infty$. Then, if*

$$f_k \to f \quad \text{a. e. in } \Omega \quad \text{as } k \to \infty, \quad |f_k|^p \le g_k \quad \text{a. e. and for all } k \in \mathbb{N},$$

it follows that

$$f_k, f \in L^p \quad \forall\, k \in \mathbb{N}, \quad f_k \to f \quad \text{in } L^p \quad \text{as } k \to \infty.$$

Proof. See [11, Thm. 3.25]. □

We also state an Aubin–Lions type lemma.

Lemma 2.36 *Let X, Y, Z be Banach spaces with compact embedding $X \subset\subset Y$ and continuous embedding $Y \subset Z$. Then, the following statements hold:*

(i) for $1 \le p < \infty$ the embedding $W^{1,1}(0,T;Z) \cap L^p(0,T;X) \subset\subset L^p(0,T;Y)$ is compact,

(ii) for $r > 1$ the embedding $W^{1,r}(0,T;Z) \cap L^\infty(0,T;X) \subset\subset C([0,T];Y)$ is compact.

Proof. See [132, Sec. 8, Cor. 4]. □

In order to apply these kinds of embeddings we need to characterise Sobolev embedding properties.

Lemma 2.37 *(see [11, Thm. 10.9, Thm. 10.13], [4, Thm. 4.12, Thm. 6.3]) Let $\Omega \subset \mathbb{R}^d$, $d \ge 1$, be a bounded domain with Lipschitz-boundary and define $W^{0,p} := L^p$ for $1 \le p \le \infty$. Furthermore, let $k_1, k_2 \ge 0$ be integers, $1 \le p_1 \le \infty$, $1 \le p_2 < \infty$, and $0 \le \alpha \le 1$. Then, it holds that:*

(i) if $k_1 \ge k_2$ and

$$k_1 - \frac{d}{p_1} \ge k_2 - \frac{d}{p_2},$$

then the embedding $W^{k_1,p_1} \subset W^{k_2,p_2}$ exists and is continuous. In particular, there exists a positive constant $C = C(\Omega, d, k_1, k_2, p_1, p_2)$ such that

$$\|u\|_{W^{k_2,p_2}} \le C\|u\|_{W^{k_1,p_1}} \quad \forall\, u \in W^{k_1,p_1}.$$

(ii) if $k_1 > k_2$ and

$$k_1 - \frac{d}{p_1} > k_2 - \frac{d}{p_2},$$

then the embedding $W^{k_1,p_1} \subset\subset W^{k_2,p_2}$ is compact.

(iii) if $k_1 > k_2$ and

$$k_1 - \frac{d}{p_1} > k_2,$$

the embedding $W^{k_1,p_1} \subset\subset W^{k_2,\infty}$ is continuous.

(iv) if $k_1 \geq 1$, $\alpha \in (0,1)$ *and*

$$k_1 - \frac{d}{p_1} = k_2 + \alpha,$$

then the embedding $W^{k_1,p_1} \subset C^{k_2,\alpha}(\bar{\Omega})$ *is continuous.*

(v) if $k_1 \geq 1$ *and*

$$k_1 - \frac{d}{p_1} > k_2 + \alpha,$$

then the embedding $W^{k_1,p_1} \subset\subset C^{k_2,\alpha}(\bar{\Omega})$ *is continuous and compact where* $C^{k_2,0}(\bar{\Omega}) := C^{k_2}(\bar{\Omega})$.

2.4 Results for a Stokes resolvent system

In the subsequent chapters we will analyse various tumour growth models involving variants of the Stokes equation. Since we will consider Neumann-type boundary conditions for the velocity in combination with non-constant viscosities and permeability, we need non-standard results that are rather hard to find in the literature. In particular, the solvability of the Stokes resolvent system plays a crucial role in this thesis since it corresponds to a Brinkman equation with positive permeability. For the reader's convenience we provide the proofs required for the analysis by using ideas presented in [1, 22, 80, 126, 133].

We first recall Korn's inequality (see [37, Thm. 6.3-3]).

Lemma 2.38 *Let* $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *be a bounded domain with Lipschitz-boundary and let* $\mathbf{u} \in \mathbf{H}^1$. *Then, there exists a constant* C_K *depending only on* Ω *such that*

$$\|\mathbf{u}\|_{\mathbf{H}^1} \leq C_K \left(\|\mathbf{u}\|_{\mathbf{L}^2}^2 + \int_\Omega \mathbf{D}\mathbf{u} \colon \mathbf{D}\mathbf{u} \, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{2.23}$$

As usual for Stokes-like equations, the properties of the operator div and ∇ and their relation play a crucial role. We therefore recall the most important results in the following.

Lemma 2.39 *(see [80, Sec. III.3]) Let* $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *be a bounded domain with Lipschitz-boundary and let* $1 < q < \infty$. *Then, for every* $f \in L^q$ *and* $\mathbf{a} \in \mathbf{W}^{1-1/q,q}(\partial\Omega)$ *satisfying*

$$\int_\Omega f \, \mathrm{d}x = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{d-1}, \tag{2.24}$$

there exists at least one solution $\mathbf{u} \in \mathbf{W}^{1,q}$ *of the problem*

$$\mathrm{div}(\mathbf{u}) = f \quad \text{in } \Omega, \qquad \mathbf{u} = \mathbf{a} \quad \text{on } \partial\Omega,$$

and the estimate

$$\|\mathbf{u}\|_{\mathbf{W}^{1,q}} \leq C \left(\|f\|_{L^q} + \|\mathbf{a}\|_{\mathbf{W}^{1-1/q,q}(\partial\Omega)} \right) \tag{2.25}$$

holds for a positive constant C *depending only on* Ω *and* q.

Proof. Let T_e be the operator defined in Lemma 2.33. Then, it holds

$$\int_\Omega \mathrm{div}(T_e \mathbf{a}) \, \mathrm{d}x = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}\mathcal{H}^{d-1} = \int_\Omega f \, \mathrm{d}x, \quad \|\mathrm{div}(T_e \mathbf{a})\|_{L^q} \leq C(\Omega, q) \|\mathbf{a}\|_{\mathbf{W}^{1-1/q,q}(\partial\Omega)}.$$

Since $\int_\Omega f - \mathrm{div}(T_e\mathbf{a})\,\mathrm{d}x = 0$ and using the last inequality, we can use [133, Chap. II, Lemma 2.1.1] to conclude that there exists $\mathbf{w} \in \mathbf{W}^{1,q}$ satisfying

$$\mathrm{div}(\mathbf{w}) = f - \mathrm{div}(T_e\mathbf{a}) \quad \text{in } \Omega, \qquad \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega,$$

and

$$\|\mathbf{w}\|_{\mathbf{W}^{1,q}} \leq C(\Omega,q)\left(\|f\|_{L^q} + \|\mathrm{div}(T_e\mathbf{a})\|_{L^q}\right) \leq C(\Omega,q)\left(\|f\|_{L^q} + \|\mathbf{a}\|_{\mathbf{W}^{1-1/q,q}(\partial\Omega)}\right).$$

Consequently, by (2.21) the function $\mathbf{u} = \mathbf{w} + T_e\mathbf{a}$ satisfies

$$\mathrm{div}(\mathbf{u}) = f \quad \text{in } \Omega, \quad \mathbf{w} = \mathbf{a} \quad \text{on } \partial\Omega, \quad \|\mathbf{u}\|_{\mathbf{W}^{1,q}} \leq C(\Omega,q)\left(\|f\|_{L^q} + \|\mathbf{a}\|_{\mathbf{W}^{1-1/q,q}(\partial\Omega)}\right)$$

which completes the proof. □

Lemma 2.40 *(see [133, Lem. II.2.2.2]) Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz-boundary, let $1 < q < \infty$ and $\frac{1}{q'} + \frac{1}{q} = 1$. Then, for any $\mathbf{f} \in (\mathbf{W}_0^{1,q'})^*$ satisfying*

$$\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{W}_0^{1,q'}} = 0 \quad \forall\, \mathbf{v} \in \mathcal{V},$$

there exists a unique pressure $p \in L^q$ such that

$$\int_\Omega p \,\mathrm{d}x = 0, \qquad \mathbf{f} = \nabla p \quad \text{in } (\mathbf{W}_0^{1,q'})^*, \tag{2.26}$$

and

$$\|p\|_{L^q} \leq C\|\mathbf{f}\|_{\left(\mathbf{W}_0^{1,q'}\right)^*} \tag{2.27}$$

for a positive constant C depending only on Ω and q.

Moreover, we will use the following result.

Lemma 2.41 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lipschitz-boundary. For all $q \in (1,\infty)$ the space $\mathbf{L}^q_{\mathrm{div}}(\Omega)$ equipped with the norm*

$$\|\mathbf{u}\|_{\mathbf{L}^q_{\mathrm{div}}(\Omega)} := \left(\|\mathbf{u}\|^q_{\mathbf{L}^q} + \|\mathrm{div}(\mathbf{u})\|^q_{L^q}\right)^{\frac{1}{q}}$$

is a (reflexive) Banach space. Moreover, there exits a continuous trace operator $T_{\mathbf{n}} \colon \mathbf{L}^q_{\mathrm{div}}(\Omega) \to \left(W^{1-\frac{1}{q'},q'}(\partial\Omega)\right)^*$, $\frac{1}{q} + \frac{1}{q'} = 1$, such that $T_{\mathbf{n}}(\mathbf{w}) = \mathbf{w}\cdot\mathbf{n}$ for all $\mathbf{w} \in \left(C^\infty(\bar{\Omega})\right)^d$. It holds that

$$\langle T_{\mathbf{n}}(\mathbf{u}), \Phi \rangle_{W^{1-\frac{1}{q'},q'}(\partial\Omega)} = \int_\Omega \mathbf{u}\cdot\nabla\Phi \,\mathrm{d}x + \int_\Omega \Phi\,\mathrm{div}(\mathbf{u})\,\mathrm{d}x \quad \forall\, \mathbf{u} \in \mathbf{L}^q_{\mathrm{div}}(\Omega),\ \forall\, \Phi \in W^{1,q'}, \tag{2.28}$$

and

$$\|T_{\mathbf{n}}(\mathbf{u})\|_{\left(W^{1-\frac{1}{q'},q'}(\partial\Omega)\right)^*} \leq C_{\mathrm{div}}\|\mathbf{u}\|_{\mathbf{L}^q_{\mathrm{div}}(\Omega)} \quad \forall\, \mathbf{u} \in \mathbf{L}^q_{\mathrm{div}}(\Omega) \tag{2.29}$$

with a constant C_{div} depending only on Ω and q.

Proof. See [80, Thm. III.2.2] and [131, Sec. 5]. □

2.4.1 Weak solutions of the Stokes resolvent system

In this part we will prove existence and uniqueness of weak solutions for a Stokes resolvent system with variable viscosities and a non-constant permeability. Our arguments are based on those presented in [22]. Throughout this part we make the following assumptions:

Assumptions 2.42 *The functions η, λ and ν belong to $C^0(\mathbb{R})$ and fulfil*

$$\eta_0 \le \eta(t) \le \eta_1, \quad 0 \le \lambda(t) \le \lambda_0, \quad \nu_0 \le \nu(t) \le \nu_1 \quad \forall\, t \in \mathbb{R} \tag{2.30}$$

for positive constants η_0, η_1, ν_1 and non-negative constants λ_0, ν_0. Furthermore, we assume that $s \in (1,2]$ if $d = 2$ and $s \in [\frac{6}{5}, 2]$ if $d = 3$.

The main result of this subsection is the following:

Proposition 2.43 *Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$, be a bounded domain with $C^{1,1}$-boundary, let $c \in H^r$ for $r > \frac{1}{2}$ and let $\nu_0 > 0$. Then, for every set of data fulfilling $(\mathbf{f}, g, \mathbf{f}_b) \in \mathbf{L}^s \times L^2 \times (\mathbf{H}^{1/2})^*$, there exists a unique weak solution pair $(\mathbf{v}, p) \in \mathbf{H}^1 \times L^2$ of*

$$-\mathrm{div}(\mathbf{T}_c(\mathbf{v}, p)) + \nu(c)\mathbf{v} = \mathbf{f} \quad \text{in } \Omega, \tag{2.31a}$$

$$\mathrm{div}(\mathbf{v}) = g \quad \text{in } \Omega, \tag{2.31b}$$

$$\mathbf{T}_c(\mathbf{v}, p)\mathbf{n} = \mathbf{f}_b \quad \text{on } \partial\Omega, \tag{2.31c}$$

where

$$\mathbf{T}_c(\mathbf{v}, p) = 2\eta(c)\mathbf{D}\mathbf{v} + \lambda(c)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}.$$

In addition, it holds

$$\|\mathbf{v}\|_{\mathbf{H}^1} + \|p\|_{L^2} \le C\left(\|\mathbf{f}\|_{\mathbf{L}^s} + \|g\|_{L^2} + \|\mathbf{f}_b\|_{(\mathbf{H}^{1/2})^*}\right) \tag{2.32}$$

with a constant C depending only on Ω, s, η_0, η_1, λ_0, ν_0 and ν_1.

Remark 2.44 (i) In contrast to classical Stokes problems it is not enough to assume that $f \in (\mathbf{H}^1)^*$. This is due to the boundary condition (2.31c) which involves first derivatives of the velocity field. In fact, assuming $\mathbf{f} \in (\mathbf{H}^1)^*$ we can at best get that $\mathrm{div}(\mathbf{T}(\mathbf{v}, p))$ belongs to $(\mathbf{H}^1)^*$. In this case the trace of $\mathbf{T}(\mathbf{v}, p)$ is not an element of $(\mathbf{H}^{\frac{1}{2}})^*$.

(ii) The assumption on s will be needed later on to show existence of strong solutions to (2.31). For the proof of Proposition 2.50 it is sufficient to consider the case $s \le 2$. For the case $s > 2$ the arguments are more involved and will not be presented in this work.

The general idea to prove Proposition 2.43 is the reduction to the case $g = 0$. To this end we need the following lemma:

Lemma 2.45 *Let $\Omega \subset \mathbb{R}^d, d = 2,3,$ be a bounded domain with Lipschitz-boundary, let $c \in H^r$ for $r > \frac{1}{2}$ and let $\nu_0 \ge 0$. Then, for every $\mathbf{f}_1 \in (\mathbf{H}_0^1)^*$, $g_1 \in L_0^2$, there exists a unique weak solution pair $(\mathbf{v}, p) \in \mathbf{H}_0^1 \times L_0^2$ of*

$$-\mathrm{div}(2\eta(c)\mathbf{D}\mathbf{v} + \lambda(c)\mathrm{div}(\mathbf{v})\mathbf{I}) + \nabla p + \nu(c)\mathbf{v} = \mathbf{f}_1 \quad \text{in } \Omega, \tag{2.33a}$$

$$\mathrm{div}(\mathbf{v}) = g_1 \quad \text{in } \Omega, \tag{2.33b}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{2.33c}$$

satisfying

$$\|\mathbf{v}\|_{\mathbf{H}^1} + \|p\|_{L^2} \le C\left(\|\mathbf{f}_1\|_{(\mathbf{H}_0^1)^*} + \|g_1\|_{L^2}\right) \tag{2.34}$$

with a positive constant C depending only on Ω, η_0, η_1, λ_0, ν_0 and ν_1.

Proof. Because of Lemma 2.39 there exists $\mathbf{v}_1 \in \mathbf{H}_0^1$ satisfying

$$\mathrm{div}(\mathbf{v}_1) = g_1 \quad \text{a.e. in } \Omega, \quad \|\mathbf{v}_1\|_{\mathbf{H}_0^1} \le C\|g_1\|_{L^2}. \tag{2.35}$$

We then seek for $\mathbf{v}$ of the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_2$ satisfies (2.33) in the weak sense for $g_1 = 0$ and $\mathbf{f}_1$ replaced by

$$\tilde{\mathbf{f}}_1 = \mathbf{f}_1 + \mathrm{div}(2\eta(c)\mathbf{D}\mathbf{v}_1 + \lambda(c)\mathrm{div}(\mathbf{v}_1)\mathbf{I}) - \nu(c)\mathbf{v}_1.$$

Due to the assumptions on $\mathbf{f}_1$ and using (2.35) we have

$$\|\tilde{\mathbf{f}}_1\|_{(\mathbf{H}_0^1)^*} \leq C\left(\|\mathbf{f}_1\|_{(\mathbf{H}_0^1)^*} + \|g_1\|_{L^2}\right) \tag{2.36}$$

for a constant C depending on Ω, η_1, λ_0 and ν_1. Now, we define the function space

$$\mathbf{W} \coloneqq \{\mathbf{w} \in \mathbf{H}_0^1 \colon \mathrm{div}(\mathbf{w}) = 0 \text{ a. e. in } \Omega\}.$$

Furthermore, we define a bilinear form $a\colon \mathbf{W} \times \mathbf{W} \to \mathbb{R}$ and a linear functional $l\colon \mathbf{W} \to \mathbb{R}$ by

$$a(\mathbf{w}_1, \mathbf{w}_2) = \int_\Omega 2\eta(c)\mathbf{D}\mathbf{w}_1 \colon \mathbf{D}\mathbf{w}_2 + \nu(c)\mathbf{w}_1 \cdot \mathbf{w}_2 \, \mathrm{d}x, \quad l(\mathbf{w}) = \langle \tilde{\mathbf{f}}_1, \mathbf{w}\rangle_{\mathbf{H}_0^1}.$$

By Hölder's inequality, the assumptions on $\eta(\cdot)$, $\nu(\cdot)$, and by the definition of the duality product, it is easy to check that a and l are well-defined.
Moreover, it is obvious that a is bilinear. Using Hölder's and Young's inequalities along with Korn's inequality for trace free functions, it is easy to check that a is also continuous and coercive. Therefore, the Lax–Milgram theorem guarantees the existence of a unique $\mathbf{v}_2 \in \mathbf{W}$ solving

$$a(\mathbf{v}_2, \mathbf{w}) = l(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathbf{W}$$

which is equivalent to

$$\int_\Omega 2\eta(c)\mathbf{D}\mathbf{v}_2 \colon \mathbf{D}\mathbf{w} + \nu(c)\mathbf{v}_2 \cdot \mathbf{w} \, \mathrm{d}x = \langle \tilde{\mathbf{f}}_1, \mathbf{w}\rangle_{\mathbf{H}_0^1} \quad \forall\, \mathbf{w} \in \mathbf{W}. \tag{2.37}$$

Choosing $\mathbf{w} = \mathbf{v}_2$ in (2.37) and applying Young's, Hölder's and Korn's inequalities, it follows that

$$\|\mathbf{v}_2\|_{\mathbf{H}^1} \leq \left(\frac{C_K^2}{\min\{2\eta_0, \nu_0\}}\right) \|\tilde{\mathbf{f}}_1\|_{(\mathbf{H}_0^1)^*}. \tag{2.38}$$

Furthermore, by (2.37) we see that

$$\langle -\mathrm{div}(2\eta(c)\mathbf{D}\mathbf{v}_2) + \nu(c)\mathbf{v}_2 - \tilde{\mathbf{f}}_1, \mathbf{w}\rangle_{\mathbf{H}_0^1} = 0 \quad \forall\, \mathbf{w} \in \mathbf{W}. \tag{2.39}$$

By Lemma 2.40 and (2.38) we obtain the existence of a unique pressure $p \in L_0^2(\Omega)$ such that

$$-\mathrm{div}(2\eta(c)\mathbf{D}\mathbf{v}_2) + \nu(c)\mathbf{v}_2 + \nabla p = \tilde{\mathbf{f}}_1 \quad \text{in } (\mathbf{H}_0^1)^*, \qquad \|p\|_{L^2} \leq C\|\tilde{\mathbf{f}}_1\|_{(\mathbf{H}_0^1)^*} \tag{2.40}$$

with a constant C depending only on Ω, η_0, η_1, ν_0 and ν_1. Then, by construction we see that $(\mathbf{v}, p) \in \mathbf{H}_0^1 \times L_0^2$ is a weak solution of (2.33) and satisfies (2.34) which completes the proof. □

We can now prove the main result of this subsection.

Proof of Proposition 2.43. We divide the proof into several steps.

Step 1: First we aim to reduce the problem to the case $g = 0$. For $(x_1, \ldots, x_d)^\intercal \in \Omega$ we define

$$\mathbf{v}_1 = \frac{1}{d}(x_1, \ldots, x_d)^\intercal \quad \mathbf{v}_0 \coloneqq g_\Omega \mathbf{v}_1, \quad p_0 \coloneqq g_\Omega\left(\frac{2\eta(c)}{d} + \lambda(c)\right),$$

where $g_\Omega = \frac{1}{|\Omega|}\int_\Omega g \, \mathrm{d}x$, and we observe that $\mathbf{v}_0 \in \mathbf{H}^1$, $p_0 \in L^2$. Furthermore, it holds that $\mathrm{div}(\mathbf{v}_0) = g_\Omega$ in Ω and

$$\int_\Omega \mathbf{T}_c(\mathbf{v}_0, p_0) \colon \nabla\mathbf{\Phi} \, \mathrm{d}x = 0 \quad \forall\, \mathbf{\Phi} \in \mathbf{H}^1, \quad \mathbf{T}_c(\mathbf{v}_0, p_0)\mathbf{n} = \mathbf{0} \quad \text{a. e. on } \partial\Omega. \tag{2.41}$$

The second identity follows since $c \in H^r(\partial\Omega)$ for $r > \frac{1}{2}$ and thus $T_c(\mathbf{v}_0, p_0) \in \mathbf{L}^2(\partial\Omega)$. Now, let $(\mathbf{w}_0, p_0) \in \mathbf{H}^1_0 \times L^2_0$ be the unique weak solution of

$$-\mathrm{div}(\mathbf{T}_c(\mathbf{w}_0, \pi_0)) + \nu(c)\mathbf{w}_0 = \mathbf{0} \quad \text{in } \Omega, \quad \mathrm{div}(\mathbf{w}_0) = g - g_\Omega \quad \text{in } \Omega, \quad \mathbf{w}_0 = \mathbf{0} \quad \text{on } \partial\Omega$$

which exists according to Lemma 2.45 and satisfies the estimate

$$\|\mathbf{w}_0\|_{\mathbf{H}^1} + \|p_0\|_{L^2} \le C\|g\|_{L^2}. \tag{2.42}$$

This gives $\mathbf{T}_c(\mathbf{w}_0, \pi_0) \in \mathbf{L}^2$ and $\mathrm{div}(\mathbf{T}_c(\mathbf{w}_0, \pi_0)) = \nu(c)\mathbf{w}_0 \in \mathbf{L}^2$. Using Lemma 2.41 and (2.42) yields $\mathbf{T}_c(\mathbf{w}_0, \pi_0)\mathbf{n} \in (\mathbf{H}^{\frac{1}{2}}(\partial\Omega))^*$ and

$$\|\mathbf{T}_c(\mathbf{w}_0, \pi_0)\mathbf{n}\|_{(\mathbf{H}^{\frac{1}{2}}(\partial\Omega))^*} \le C\|g\|_{L^2}. \tag{2.43}$$

It remains to show that there exists a unique weak solution $(\mathbf{w}, \pi)$ of the system

$$\begin{aligned} -\mathrm{div}(\mathbf{T}_c(\mathbf{w}, \pi)) + \nu(c)\mathbf{w} = \tilde{\mathbf{f}} &:= \mathbf{f} - \nu(c)\mathbf{v}_0 && \text{in } \Omega, \\ \mathrm{div}(\mathbf{w}) &= 0 && \text{in } \Omega, \\ \mathbf{T}_c(\mathbf{w}, \pi)\mathbf{n} = \mathbf{F}_b &:= \mathbf{f}_b - \mathbf{T}_c(\mathbf{w}_0, \pi_0)\mathbf{n} && \text{on } \partial\Omega. \end{aligned}$$

Indeed, if such a solution exists one can check that $(\mathbf{v}, p) := (\mathbf{w} + \mathbf{w}_0 + \mathbf{v}_0, \pi + \pi_0 + p_0)$ satisfy (2.31). Moreover, by (2.42)-(2.43) it follows that

$$\|\tilde{\mathbf{f}}\|_{\mathbf{L}^s} \le C\left(\|\mathbf{f}\|_{\mathbf{L}^s} + \|g\|_{L^2}\right) \quad \text{and} \quad \|\mathbf{F}_b\|_{(\mathbf{H}^{1/2})^*} \le C\left(\|\mathbf{f}_b\|_{(\mathbf{H}^{1/2})^*} + \|g\|_{L^2}\right) \tag{2.44}$$

with C depending only on Ω and ν_1. To keep the notation clear, we will write $\mathbf{f}_b$, $\mathbf{f}$, $\mathbf{v}$ and p instead of $\mathbf{F}_b$, $\tilde{\mathbf{f}}$, $\mathbf{w}$ and π in the following.

Step 2: We introduce the function space $\mathbf{W} := \{\mathbf{w} \in \mathbf{H}^1 \colon \mathrm{div}(\mathbf{w}) = 0 \text{ a. e. in } \Omega\}$ and we define a bilinear form and a linear functional on $\mathbf{W}$ by

$$a(\mathbf{w}_1, \mathbf{w}_2) = \int_\Omega 2\eta(c)\mathbf{D}\mathbf{w}_1 \colon \mathbf{D}\mathbf{w}_2 + \nu(c)\mathbf{w}_1 \cdot \mathbf{w}_2 \,\mathrm{d}x, \quad l(\mathbf{w}) = \langle \mathbf{f}_b, \mathbf{w}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)} + \int_\Omega \mathbf{f} \cdot \mathbf{w} \,\mathrm{d}x.$$

Employing the continuous embeddings $\mathbf{H}^1 \hookrightarrow \mathbf{H}^{1/2}(\partial\Omega)$ and $\mathbf{W} \hookrightarrow (\mathbf{L}^r)^*$ one can check that a and l are well-defined. Then, it is straightforward to check that the conditions needed to apply the Lax–Milgram theorem are fulfilled (see proof of Lemma 2.45 for details). This gives the existence of a unique $\mathbf{v} \in \mathbf{W}$ fulfilling

$$a(\mathbf{v}, \mathbf{w}) = l(\mathbf{w}) \quad \forall\, \mathbf{w} \in \mathbf{W},$$

or equivalently

$$\int_\Omega 2\eta(c)\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{w} + \nu(c)\mathbf{v} \cdot \mathbf{w} \,\mathrm{d}x - \langle \mathbf{f}_b, \mathbf{w}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)} = \int_\Omega \mathbf{f} \cdot \mathbf{w} \,\mathrm{d}x \quad \forall\, \mathbf{w} \in \mathbf{W}. \tag{2.45}$$

Furthermore, with similar arguments as in the proof of Lemma 2.45 and by (2.44) we get

$$\|\mathbf{v}\|_{\mathbf{H}^1} \le C\left(\|\mathbf{f}\|_{(\mathbf{H}^1)^*} + \|g\|_{L^2} + \|\mathbf{f}_b\|_{(\mathbf{H}^{1/2})^*}\right). \tag{2.46}$$

We now define the function space

$$\mathcal{V} := \{\mathbf{w} \in (C_0^\infty(\Omega))^d \colon \mathrm{div}(\mathbf{w}) = 0 \text{ a. e. in } \Omega\}.$$

Since clearly $\mathcal{V} \subset \mathbf{W}$, by (2.45) and Lemma 2.40 we obtain the existence of a unique pressure $p \in L^2_0$ such that

$$-\,\mathrm{div}(2\eta(c)\mathbf{D}\mathbf{v}) + \nu(c)\mathbf{v} + \nabla p = \mathbf{f}, \quad \|p\|_{L^2} \le C\left(\|\mathbf{f}\|_{(\mathbf{H}^1)^*} + \|g\|_{L^2} + \|\mathbf{f}_b\|_{(\mathbf{H}^{1/2})^*}\right). \tag{2.47}$$

Writing $(2.47)_1$ in the equivalent way

$$-\operatorname{div}(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I}) = \mathbf{f} - \nu(c)\mathbf{v} \in \mathbf{L}^s, \tag{2.48}$$

we see that (2.46) implies $2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I} \in \mathbf{L}^s_{\text{div}}(\Omega)$. Let $\frac{1}{s'} + \frac{1}{s} = 1$ which implies $s' \geq 2$ due to the assumptions on s. Using (2.48) and applying Lemma 2.41 we obtain that

$$\int_\Omega 2\eta(c)\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{w} + \nu(c)\mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x - \langle(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n}, \mathbf{w}\rangle_{\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)} = \int_\Omega \mathbf{f}\cdot\mathbf{w}\,\mathrm{d}x$$

for all $\mathbf{w} \in \mathbf{W} \cap \mathbf{W}^{1,s'}$. Comparing this with (2.45) we see that

$$\langle(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n} - \mathbf{f}_b, \mathbf{w}\rangle_{\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)} = 0 \quad \forall\, \mathbf{w} \in \mathbf{W} \cap \mathbf{W}^{1,s'}. \tag{2.49}$$

From now on we will use that $\mathbf{W}^{1-\frac{1}{s'},s'}(\partial\Omega) = \mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)$.

Step 3: Let $\boldsymbol{\psi} \in \mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)$ such that $\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = 0$. Then, by Lemma 2.39 there exists a solution $\mathbf{w} \in \mathbf{W}^{1,s'}$ satisfying

$$\operatorname{div}(\mathbf{w}) = 0 \quad \text{in } \Omega, \qquad \mathbf{w} = \boldsymbol{\psi} \quad \text{on } \partial\Omega.$$

Hence, it holds $\mathbf{w} \in \mathbf{W} \cap \mathbf{W}^{1,s'}$ which shows that (2.49) holds for all $\boldsymbol{\psi} \in \mathbf{W}^{\frac{1}{s'},s}(\partial\Omega)$ fulfilling $\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = 0$.

Step 4: Define $\boldsymbol{\psi}_0 = \mathbf{n}$ which belongs to $\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)$. Then, it holds

$$\int_{\partial\Omega} \boldsymbol{\psi}_0\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\partial\Omega} |\mathbf{n}|^2\,\mathrm{d}\mathcal{H}^{d-1} = |\partial\Omega| > 0.$$

For any $\boldsymbol{\psi} \in \mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)$ we set

$$\boldsymbol{\psi}_1 = \boldsymbol{\psi} - \frac{1}{|\partial\Omega|}\left(\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\right)\boldsymbol{\psi}_0,$$

hence

$$\boldsymbol{\psi} = \boldsymbol{\psi}_1 + \frac{1}{|\partial\Omega|}\left(\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\right)\boldsymbol{\psi}_0.$$

Since by definition $\int_{\partial\Omega} \boldsymbol{\psi}_1\cdot\mathbf{n}\mathrm{d}\mathcal{H}^{d-1} = 0$ we know from Step 3 that

$$\langle(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n} - \mathbf{f}_b, \boldsymbol{\psi}_1\rangle_{\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)} = 0.$$

Introducing the number

$$C_0 = \frac{1}{|\partial\Omega|}\langle(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n} - \mathbf{f}_b, \boldsymbol{\psi}_0\rangle_{\mathbf{W}^{\frac{1}{s'},s}(\partial\Omega)},$$

this implies

$$\langle(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n} - \mathbf{f}_b, \boldsymbol{\psi}\rangle_{\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)} = C_0\int_{\partial\Omega} \boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}.$$

Consequently, for any $\boldsymbol{\psi} \in \mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)$ we have

$$\langle(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n} - \mathbf{f}_b, \boldsymbol{\psi}\rangle_{\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)} = \int_{\partial\Omega} (C_0\mathbf{n})\cdot\boldsymbol{\psi}\,\mathrm{d}\mathcal{H}^{d-1}.$$

This proves that

$$(2\eta(c)\mathbf{D}\mathbf{v} - p\mathbf{I})\mathbf{n} - \mathbf{f}_b = C_0\mathbf{n} \quad \text{in } \left(\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)\right)^*.$$

Setting $\tilde{p} = p + C_0$ we see that

$$(2\eta(c)\mathbf{D}\mathbf{v} - \tilde{p}\mathbf{I})\mathbf{n} - \mathbf{f}_b = \mathbf{0} \quad \text{in } \left(\mathbf{W}^{\frac{1}{s},s'}(\partial\Omega)\right)^*.$$

Therefore, we have solved (in a weak sense) the problem

$$\begin{aligned} -\mathrm{div}(\mathbf{T}_c(\mathbf{v},\tilde{p})) + \nu(c)\mathbf{v} &= \mathbf{f} && \text{in } \Omega, \\ \mathrm{div}(\mathbf{v}) &= 0 && \text{in } \Omega, \\ \mathbf{T}_c(\mathbf{v},\tilde{p})\mathbf{n} &= \mathbf{f}_b && \text{on } \partial\Omega. \end{aligned}$$

In particular, the pressure $\tilde{p}$ is defined in a unique way. Finally, using (2.44) and (2.46)-(2.47), we can establish the estimate (2.32) which completes the proof. □

Later on, we will also need a continuous dependence result for the system (2.31). To this end, we make the following additional assumptions:

Assumptions 2.46 *The functions η, λ and ν fulfil Assumptions 2.42 and, in addition, for all r, $s \in \mathbb{R}$ it holds that*

$$|\eta(r) - \eta(s)| \le L_\eta|r-s|, \quad |\lambda(r) - \lambda(s)| \le L_\lambda|r-s|, \quad |\nu(r) - \nu(s)| \le L_\nu|r-s| \tag{2.50}$$

for positive constants L_η, L_λ and L_ν.

The following continuous dependence result will be important in the Galerkin scheme in Chapter 4.

Proposition 2.47 *Let $\Omega \subset \mathbb{R}^d, d = 2,3$, be a bounded domain with $C^{1,1}$-boundary and let Assumptions 2.46 hold. Furthermore, let $(\mathbf{f}_i, g_i, c_i) \in \mathbf{L}^2 \times L^2 \times (L^\infty \cap H^r)$, $i = 1,2$, $r > \frac{1}{2}$, be given and let $(\mathbf{v}_i, p_i) \in \mathbf{H}^1 \times L^2$, $i = 1,2$, be the unique weak solution pairs of*

$$-\mathrm{div}(2\eta(c_i)\mathbf{D}\mathbf{v}_i + \lambda(c_i)\mathrm{div}(\mathbf{v}_i)\mathbf{I} - p_i\mathbf{I}) + \nu(c_i)\mathbf{v}_i = \mathbf{f}_i \quad \text{in } \Omega, \tag{2.51a}$$

$$\mathrm{div}(\mathbf{v}_i) = g_i \quad \text{in } \Omega, \tag{2.51b}$$

$$(2\eta(c_i)\mathbf{D}\mathbf{v}_i + \lambda(c_i)\mathrm{div}(\mathbf{v}_i)\mathbf{I} - p_i\mathbf{I})\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \tag{2.51c}$$

according to Proposition 2.43. Then, the differences $(\mathbf{v}, p) := (\mathbf{v}_2 - \mathbf{v}_1, p_2 - p_1)$ fulfil

$$\|\mathbf{v}\|_{\mathbf{H}^1} + \|p\|_{L^2} \le C\left(\|\mathbf{f}_2 - \mathbf{f}_1\|_{L^2} + \|g_2 - g_1\|_{L^2} + (\|\mathbf{f}_1\|_{L^2} + \|g_1\|_{L^2})\|c_2 - c_1\|_{L^\infty}\right) \tag{2.52}$$

with a constant C depending only on Ω, η_0, η_1, λ_0, ν_0, ν_1, L_η, L_λ and L_ν.

Proof. In the following we denote by C a generic constant depending only on Ω, η_0, η_1, λ_0, ν_0, ν_1, L_η, L_λ and L_ν. We denote $\mathbf{v} := \mathbf{v}_2 - \mathbf{v}_1$, $p := p_2 - p_1$, $\mathbf{f} := \mathbf{f}_2 - \mathbf{f}_1$, $g := g_2 - g_1$ and $c := c_2 - c_1$. Then, a straightforward calculation shows that $(\mathbf{v}, p)$ satisfies

$$\begin{aligned} &\int_\Omega \big(2\eta(c_2)\mathbf{D}\mathbf{v} + \lambda(c_2)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}\big) : \mathbf{D}\mathbf{\Phi} + \nu(c_2)\mathbf{v}\cdot\mathbf{\Phi}\,\mathrm{d}x \\ &= -\int_\Omega 2(\eta(c_2) - \eta(c_1))\mathbf{D}\mathbf{v}_1 : \mathbf{D}\mathbf{\Phi} + (\lambda(c_2) - \lambda(c_1))\mathrm{div}(\mathbf{v}_1)\mathbf{I} : \mathbf{D}\mathbf{\Phi}\,\mathrm{d}x \\ &\quad - \int_\Omega (\nu(c_1) - \nu(c_2))\mathbf{v}_1\cdot\mathbf{\Phi} - \mathbf{f}\cdot\mathbf{\Phi}\,\mathrm{d}x \quad \forall \mathbf{\Phi} \in \mathbf{H}^1, \end{aligned} \tag{2.53}$$

and

$$\mathrm{div}(\mathbf{v}) = g. \tag{2.54}$$

Due to Lemma 2.39 there exists a solution $\mathbf{u} \in \mathbf{H}^1$ satisfying

$$\operatorname{div}(\mathbf{u}) = g \quad \text{a.e. in } \Omega, \quad \mathbf{u} = \left(\frac{1}{|\partial\Omega|}\int_\Omega g \,\mathrm{d}x\right)\mathbf{n} \quad \text{a.e. on } \partial\Omega, \qquad \|\mathbf{u}\|_{\mathbf{H}^1} \le C\|g\|_{L^2}. \quad (2.55)$$

Choosing $\mathbf{\Phi} = \mathbf{v} - \mathbf{u}$ in (2.53) and using the assumptions on $\eta(\cdot)$ and $\nu(\cdot)$ we obtain

$$\begin{aligned} 2\eta_0\|\mathbf{D}\mathbf{v}\|^2_{\mathbf{L}^2} + \nu_0\|\mathbf{v}\|^2_{L^2} \le & \int_\Omega 2\eta(c_2)\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu(c_2)\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x \\ & - \int_\Omega 2(\eta(c_2)-\eta(c_1))\mathbf{D}\mathbf{v}_1 \colon \mathbf{D}(\mathbf{v}-\mathbf{u})\,\mathrm{d}x \\ & - \int_\Omega (\nu(c_1)-\nu(c_2))\mathbf{v}_1\cdot(\mathbf{v}-\mathbf{u}) - \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x. \end{aligned}$$

By (2.1), (2.2), (2.30), (2.32), (2.50) and (2.55) it is straightforward to check that the r. h. s. of this inequality can be controlled by

$$|\mathrm{RHS}| \le \eta_0\|\mathbf{D}\mathbf{v}\|^2_{\mathbf{L}^2} + \frac{\nu_0}{2}\|\mathbf{v}\|^2_{L^2} + C\left(\|\mathbf{f}\|^2_{L^2} + \|g\|^2_{L^2} + (\|\mathbf{f}_1\|^2_{L^2} + \|g_1\|^2_{L^2})\|c\|^2_{L^\infty}\right).$$

Combining the last two inequalities and using (2.23), this implies

$$\|\mathbf{v}\|_{\mathbf{H}^1} \le C\left(\|\mathbf{f}\|_{L^2} + \|g\|_{L^2} + (\|\mathbf{f}_1\|_{L^2} + \|g_1\|_{L^2})\|c\|_{L^\infty}\right). \quad (2.56)$$

Using Lemma 2.39 there exists a solution $\mathbf{q} \in \mathbf{H}^1$ satisfying

$$\operatorname{div}(\mathbf{q}) = p \quad \text{a.e. in } \Omega, \quad \mathbf{q} = \left(\frac{1}{|\partial\Omega|}\int_\Omega p \,\mathrm{d}x\right)\mathbf{n} \quad \text{a.e. on } \partial\Omega, \qquad \|\mathbf{q}\|_{\mathbf{H}^1} \le C\|p\|_{L^2}. \quad (2.57)$$

Choosing $\mathbf{\Phi} = \mathbf{q}$ in (2.53) gives

$$\begin{aligned} \|p\|^2_{L^2} = & \int_\Omega \big(2\eta(c_2)\mathbf{D}\mathbf{v} + \lambda(c_2)\operatorname{div}(\mathbf{v})\mathbf{I}\big) \colon \mathbf{D}\mathbf{q} + \nu(c_2)\mathbf{v}\cdot\mathbf{q}\,\mathrm{d}x \\ & + \int_\Omega 2(\eta(c_2)-\eta(c_1))\mathbf{D}\mathbf{v}_1 \colon \mathbf{D}\mathbf{q} + (\lambda(c_2)-\lambda(c_1))\operatorname{div}(\mathbf{v}_1)\mathbf{I} \colon \mathbf{D}\mathbf{q}\,\mathrm{d}x \\ & + \int_\Omega (\nu(c_1)-\nu(c_2))\mathbf{v}_1\cdot\mathbf{q} - \mathbf{f}\cdot\mathbf{q}\,\mathrm{d}x. \end{aligned}$$

Applying (2.1), (2.2), (2.30), (2.32), (2.50) and (2.55)-(2.57) we can control the r. h. s. of this inequality by

$$|\mathrm{RHS}| \le \frac{1}{2}\|p\|^2_{L^2} + C\left(\|\mathbf{f}\|^2_{L^2} + \|g\|^2_{L^2} + (\|\mathbf{f}_1\|^2_{L^2} + \|g_1\|^2_{L^2})\|c\|^2_{L^\infty}\right).$$

Consequently, the last two inequalities imply that

$$\|p\|_{L^2} \le C\left(\|\mathbf{f}\|_{L^2} + \|g\|_{L^2} + (\|\mathbf{f}_1\|_{L^2} + \|g_1\|_{L^2})\|c\|_{L^\infty}\right).$$

Together with (2.56) this completes the proof. □

2.4.2 Strong solutions of the Stokes resolvent system

Throughout this part we make the following assumptions:

Assumptions 2.48 *The viscosities fulfil η, $\lambda \in C^2(\mathbb{R})$ and*

$$\eta_0 \le \eta(t) \le \eta_1, \qquad 0 \le \lambda(t) \le \lambda_0 \quad \forall\, t \in \mathbb{R}$$

for positive constants η_0, η_1, and a non-negative constant λ_0. The function ν belongs to $C^0(\mathbb{R})$ and fulfils

$$\nu_0 \le \nu(t) \le \nu_1 \quad \forall\, t \in \mathbb{R} \quad (2.58)$$

for positive constants ν_0 and ν_1. Furthermore, we assume that $s > 1$ if $d = 2$ and $s \ge \frac{6}{5}$ if $d = 3$.

For constant viscosities we have the following result:

Lemma 2.49 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^{2,1}$-boundary and let $c \in H^r$, $r > \frac{1}{2}$. Furthermore, we assume $\eta(\cdot) \equiv \eta$ and $\lambda(\cdot) \equiv \lambda$ for constants $\eta > 0$ and $\lambda \geq 0$. Then, for every $g \in W^{1,s}$, $\mathbf{f} \in \mathbf{L}^s$ and $\mathbf{f}_b \in \mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)$ there exists a unique solution $(\mathbf{v}, p) \in \mathbf{W}^{2,s} \times W^{1,s}$ of the system*

$$-\mathrm{div}(\mathbf{T}(\mathbf{v}, p)) + \nu(c)\mathbf{v} = \mathbf{f} \qquad a.\,e.\ in\ \Omega, \tag{2.59a}$$

$$\mathrm{div}(\mathbf{v}) = g \qquad a.\,e.\ in\ \Omega, \tag{2.59b}$$

$$\mathbf{T}(\mathbf{v}, p)\mathbf{n} = \mathbf{f}_b \qquad a.\,e.\ on\ \partial\Omega, \tag{2.59c}$$

where $\mathbf{T}(\mathbf{v}, p) \coloneqq 2\eta\mathbf{D}\mathbf{v} + \lambda\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}$. Furthermore, the estimate

$$\|\mathbf{v}\|_{\mathbf{W}^{2,s}} + \|p\|_{W^{1,s}} \leq C\left(\|\mathbf{f}\|_{\mathbf{L}^s} + \|g\|_{W^{1,s}} + \|\mathbf{f}_b\|_{\mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)}\right) \tag{2.60}$$

holds for a positive constant C depending only on Ω, s, η, λ, ν_0 and ν_1.

Proof. First we observe that $\mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega) \subset (\mathbf{H}^{\frac{1}{2}}(\partial\Omega))^*$ and $W^{1,s} \subset L^2$. Applying Proposition 2.43, there exists a unique weak solution pair $(\mathbf{v}_1, p_1) \in \mathbf{H}^1 \times L^2$ of (2.59). We divide the proof into two cases. This is due to the fact that in three space dimension it holds that $\mathbf{v}_1 \in \mathbf{H}^1 \subset \mathbf{L}^6 \not\subseteq \mathbf{L}^s$ for $s > 6$.

Case 1 ($s \leq 6$): Applying Lemma 2.32 there exists a unique $q \in W^{3,s}$ satisfying

$$-\Delta q = g \quad \text{in } \Omega, \qquad q = 0 \quad \text{on } \partial\Omega, \qquad \|q\|_{W^{3,s}} \leq C\|g\|_{W^{1,s}}. \tag{2.61}$$

We define $\mathbf{v}_2 \coloneqq -\nabla q$ and we consider the system

$$-\mathrm{div}(\mathbf{T}(\tilde{\mathbf{v}}, \tilde{p})) + \nu_0\tilde{\mathbf{v}} = \mathbf{f} - \mathrm{div}(2\eta\mathbf{D}\mathbf{v}_2 + \lambda g\mathbf{I}) + \nu_0\mathbf{v}_2 + (\nu_0 - \nu(c))\mathbf{v}_1 =: \tilde{\mathbf{f}} \quad \text{in } \Omega, \tag{2.62a}$$

$$\mathrm{div}(\tilde{\mathbf{v}}) = 0 \quad \text{in } \Omega, \tag{2.62b}$$

$$\mathbf{T}(\tilde{\mathbf{v}}, \tilde{p})\mathbf{n} = \mathbf{f}_b + (2\eta\mathbf{D}\mathbf{v}_2 + \lambda g\mathbf{I})\mathbf{n} =: \mathbf{F}_b \quad \text{on } \partial\Omega. \tag{2.62c}$$

Let $E\colon \mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega) \to \mathbf{W}^{1,s}$ be a bounded, linear extension operator satisfying $(E\mathbf{h})|_{\partial\Omega} = \mathbf{h}$ for all $\mathbf{h} \in \mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)$. Then, $(E\mathbf{F}_b)|_{\partial\Omega} = \mathbf{F}_b$ and by [127, Thm. 1.1] there exists a unique strong solution $(\tilde{\mathbf{v}}, \tilde{p}) \in \mathbf{W}^{2,s} \times W^{1,s}$ satisfying (2.62) and

$$\|\tilde{\mathbf{v}}\|_{\mathbf{W}^{2,s}} + \|\tilde{p}\|_{W^{1,s}} \leq C\left(\|\tilde{\mathbf{f}}\|_{\mathbf{L}^s} + \|\mathbf{F}_b\|_{\mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)}\right). \tag{2.63}$$

It is straightforward to check that $(\mathbf{v}, p) \coloneqq (\tilde{\mathbf{v}} - \mathbf{v}_1 - \mathbf{v}_2, \tilde{p} - p_1) \in H^1 \times L^2$ is a weak solution of (2.59) with data $(\mathbf{f}, g, \mathbf{f}_b) = (\mathbf{0}, 0, \mathbf{0})$ and by Proposition 2.43, it follows that $\mathbf{v}_1 = \tilde{\mathbf{v}} - \mathbf{v}_2$, $p_1 = \tilde{p}$. By the definition of $\mathbf{v}_2$ and using $(2.61)_3$ along with (2.63), it follows that $(\mathbf{v}_1, p_1) \in \mathbf{W}^{2,s} \times W^{1,s}$ is a strong solution of (2.59) satisfying (2.60) which completes the proof for the case $s \leq 6$.

Case 2 ($s > 6$): Sobolev embedding theory yields $(\mathbf{f}, g, \mathbf{f}_b) \in \mathbf{L}^6 \times W^{1,6} \times \mathbf{W}^{1-\frac{1}{6},6}(\partial\Omega)$ and therefore the case $s \leq 6$ implies $\mathbf{v}_1 \in \mathbf{W}^{2,6} \subset \mathbf{W}^{1,\infty}$. Then, we can define $\tilde{\mathbf{f}}$ as before to obtain $\tilde{\mathbf{f}} \in \mathbf{L}^s$. The remaining arguments are exactly the same as in the case $s \leq 6$ which completes the proof. □

We now prove the following result for the non-constant viscosity case.

Proposition 2.50 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^{2,1}$-boundary. Assume that $\mathbf{f} \in \mathbf{L}^s$, $g \in W^{1,s}$, $\mathbf{f}_b \in \mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)$ and $c \in W^{1,r}$ with $r > d$ and $s \leq r$. Then, there exists* *a unique solution* $(\mathbf{v},p)\in\mathbf{W}^{2,s}\times W^{1,s}$ *of the system*

$$-\mathrm{div}(2\eta(c)\mathbf{D}\mathbf{v}+\lambda(c)\mathrm{div}(\mathbf{v})\mathbf{I})+\nu(c)\mathbf{v}+\nabla p=\mathbf{f}\quad a.\,e.\ in\ \Omega, \tag{2.64a}$$

$$\mathrm{div}(\mathbf{v})=g\quad a.\,e.\ in\ \Omega, \tag{2.64b}$$

$$(2\eta(c)D\mathbf{v}+\lambda(c)\mathrm{div}(\mathbf{v})\mathbf{I}-p\mathbf{I})\mathbf{n}=\mathbf{f}_b\quad a.\,e.\ on\ \partial\Omega, \tag{2.64c}$$

satisfying

$$\|\mathbf{v}\|_{\mathbf{W}^{2,s}}+\|p\|_{W^{1,s}}\le C\left(\|\mathbf{f}\|_{\mathbf{L}^s}+\|g\|_{W^{1,s}}+\|\mathbf{f}_b\|_{\mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)}\right) \tag{2.65}$$

for a positive constant C *depending only on* Ω, s, η_0, η_1, λ_0, ν_0, ν_1 *and* $\|c\|_{W^{1,r}}$.

Proof. We divide the proof into two steps and we use arguments presented in [1].

Step 1: The case $\lambda(\cdot)\equiv 0$:
First we observe that $\mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)\subset(\mathbf{H}^{\frac{1}{2}}(\partial\Omega))^*$ and $W^{1,s}\subset L^2$ and therefore by Proposition 2.43 there exists a unique weak solution pair $(\mathbf{v},p)\in\mathbf{H}^1\times L^2$ of (2.64) with $\lambda(\cdot)\equiv 0$ which means that

$$\mathrm{div}(\mathbf{v})=g\quad \text{a.e. in }\Omega,$$

and

$$\int_\Omega(2\eta(c)\mathbf{D}\mathbf{v}-p\mathbf{I})\colon\nabla\boldsymbol{\phi}+\nu(c)\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x-\langle\mathbf{f}_b\,,\boldsymbol{\phi}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}=\int_\Omega\mathbf{f}\cdot\boldsymbol{\phi}\,\mathrm{d}x\quad\forall\,\boldsymbol{\phi}\in\mathbf{H}^1. \tag{2.66}$$

We define $\boldsymbol{\varphi}:=\eta(c)^{-1}\boldsymbol{\phi}$ and calculate

$$\begin{aligned}
&\int_\Omega(2\mathbf{D}\mathbf{v}-(\eta(c)^{-1}p)\mathbf{I})\colon\nabla\boldsymbol{\phi}+\nu(c)\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x-\langle\eta(c)^{-1}\mathbf{f}_b\,,\boldsymbol{\phi}\rangle_{(\mathbf{H}^{1/2})^*,\mathbf{H}^{1/2}}\\
&=\int_\Omega(2\eta(c)\mathbf{D}\mathbf{v}-p\mathbf{I})\colon\nabla(\eta(c)^{-1}\boldsymbol{\phi})+\nu(c)\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x-\langle\mathbf{f}_b\,,\eta(c)^{-1}\boldsymbol{\phi}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}\\
&\quad-\int_\Omega 2\eta(c)\mathbf{D}\mathbf{v}\colon\big(\nabla(\eta(c)^{-1})\otimes\boldsymbol{\phi}\big)-p\,\nabla(\eta(c)^{-1})\cdot\boldsymbol{\phi}\,\mathrm{d}x\\
&=\int_\Omega(2\eta(c)\mathbf{D}\mathbf{v}-p\mathbf{I})\colon\big(\nabla\boldsymbol{\varphi}-\nabla(\eta(c)^{-1})\otimes\boldsymbol{\phi}\big)+\nu(c)\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{d}x-\langle\mathbf{f}_b\,,\boldsymbol{\varphi}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}\\
&=\int_\Omega(\eta(c)^{-1}\mathbf{f})\cdot\boldsymbol{\phi}\,\mathrm{d}x+\int_\Omega\nu(c)(\mathbf{v}-\eta(c)^{-1}\mathbf{v})\cdot\boldsymbol{\phi}-\int_\Omega(2\eta(c)\mathbf{D}\mathbf{v}-p\mathbf{I})\colon\nabla(\eta(c)^{-1})\otimes\boldsymbol{\phi}\,\mathrm{d}x\\
&=:\langle\mathbf{h}\,,\boldsymbol{\phi}\rangle_\Omega.
\end{aligned}$$

We see that the pair $(\tilde{\mathbf{v}},\tilde{p}):=(\mathbf{v},\eta(c)^{-1}p)\in\mathbf{H}^1\times L^2$ is a weak solution of (2.59) for the data $\mathbf{f}=\mathbf{h}$, $g=g$, and with $\mathbf{f}_b$ replaced by $\eta(c)^{-1}\mathbf{f}_b$. Due to the assumptions on $\mathbf{f}_b$, $\eta(\cdot)$ and c, it follows that $\eta(c)^{-1}\mathbf{f}_b\in\mathbf{W}^{1-\frac{1}{s},s}(\partial\Omega)$.

Our aim is to show that $\mathbf{h}\in\mathbf{L}^s$. To this end we observe that

$$|\langle\mathbf{h}\,,\boldsymbol{\phi}\rangle_\Omega|\le C\|f\|_{\mathbf{L}^s}\|\Phi\|_{L^{\frac{s}{s-1}}}+C\left(\|\mathbf{v}\|_{\mathbf{H}^1}+\|p\|_{L^2}\right)(1+\|\nabla c\|_{\mathbf{L}^r})\|\boldsymbol{\Phi}\|_{\mathbf{L}^{s_0'}}, \tag{2.67}$$

where

$$\frac{1}{s_0}=\frac{1}{r}+\frac{1}{2},\qquad\frac{1}{s_0'}+\frac{1}{s_0}=1.$$

Observe that the assumptions on r guarantee that

$$s_0>1\quad\text{if }d=2,\qquad s_0>\frac{6}{5}\quad\text{if }d=3.$$

In the case $s\le\frac{2r}{2+r}=s_0$ we observe that $h\in\mathbf{L}^s$.
We now consider the case $s>\frac{2r}{2+r}$. Using $\mathbf{h}\in\mathbf{L}^{s_0}$ and applying Lemma 2.49 we obtain

$(\tilde{\mathbf{v}}, \tilde{p}) \in \mathbf{W}^{2,s_0} \times W^{1,s_0}$. Due to the assumptions on $\eta(\cdot)$ and c this implies $p \in W^{1,s_0}$ and by (2.60) we have the estimate

$$\|\mathbf{v}\|_{\mathbf{W}^{2,s_0}} + \|p\|_{W^{1,s_0}} \leq C(1 + \|\nabla c\|_{\mathbf{L}^r}) \left(\|\mathbf{f}\|_{\mathbf{L}^{s_0}} + \|g\|_{W^{1,s_0}} + \|\mathbf{f}_b\|_{\mathbf{W}^{1-\frac{1}{s_0},s_0}(\partial\Omega)} \right). \tag{2.68}$$

Hence, we have $(\mathbf{v}, p) \in \mathbf{W}^{1,p_0} \times L^{p_0}$ with

$$\frac{1}{p_0} = \frac{1}{r} + \frac{1}{2} - \frac{1}{d} = \frac{1}{s_0} - \frac{1}{d}.$$

Since $r > d$ it follows $p_0 > 2$. Using (2.67) we see that $\mathbf{h} \in \mathbf{L}^{\min(s,s_1,d)}$ where

$$\frac{1}{s_1} = \frac{1}{r} + \frac{1}{p_0} = \frac{1}{r} + \frac{1}{2} - \left(\frac{1}{d} - \frac{1}{r} \right).$$

If $s_1 < \min(s, d)$ we can repeat the above step and after k steps we obtain $\mathbf{h} \in \mathbf{L}^{\min(s,s_k,d)}$ where

$$\frac{1}{s_k} = \frac{1}{2} + \frac{1}{r} - k \left(\frac{1}{d} - \frac{1}{r} \right). \tag{2.69}$$

Since $r > d$, we see that the r. h. s. of this equation is strictly monotone decreasing. Now, we consider two cases:

Case 1 ($s \leq d$): In this case we have after k steps that $\mathbf{h} \in \mathbf{L}^{\min(s,s_k)}$. Since the sequence on the r. h. s. of (2.69) is monotonically decreasing, after a finite number of steps we deduce that $s_k \geq s$ and consequently $\mathbf{h} \in \mathbf{L}^s$. With similar arguments as above we then obtain the estimate (2.65).

Case 2 ($s > d$): With exactly the same arguments as in Case 1 we obtain after a finite number of steps that $(\mathbf{v}, p) \in \mathbf{W}^{2,d} \times W^{1,d}$. Now, we take $\tilde{p} = \frac{2dr}{r-d} \in (1, \infty)$ which implies $(\mathbf{v}, p) \in \mathbf{W}^{1,\tilde{p}} \times L^{\tilde{p}}$ due to Sobolev embedding theory. Furthermore, since $r > d$ we observe that

$$\frac{1}{r} + \frac{1}{\tilde{p}} = \frac{2d + r - d}{2dr} = \frac{d + r}{2dr} < \frac{2r}{2dr} = \frac{1}{d}.$$

Since $s \leq r$ this implies $(\mathbf{v}, p) \in \mathbf{W}^{2,s} \times W^{1,s}$ which completes the proof for the case $\lambda(\cdot) \equiv 0$.

Step 2: The case $\lambda(\cdot) \neq 0$: Let $(\tilde{\mathbf{v}}, \tilde{p}) \in \mathbf{W}^{2,s} \times W^{1,s}$ be a solution of (2.64) with $\lambda(c) \equiv 0$. Define $\mathbf{v} := \tilde{\mathbf{v}}$ and $p = \tilde{p} + \lambda(c)g$. Since $\operatorname{div}(v) = g$ it follows that $\nabla\tilde{p} = \nabla p - \operatorname{div}(\lambda(c)\operatorname{div}(\mathbf{v})\mathbf{I})$. Then, it is easy to check that $(\mathbf{v}, p) \in \mathbf{W}^{2,s} \times W^{1,s}$ is a solution of (2.64) with $\lambda(\cdot) \neq 0$ and

$$\begin{aligned} \|p\|_{\mathbf{W}^{1,s}} &\leq C\left(\|\tilde{p}\|_{W^{1,s}} + \|\lambda(c)g\|_{W^{1,s}}\right) \\ &\leq C\|\tilde{p}\|_{W^{1,s}} + C\left(\|c\|_{L^\infty}\|\nabla g\|_{\mathbf{L}^s} + \|\nabla c\|_{\mathbf{L}^r}\|g\|_{L^q}\right) \\ &\leq C\|\tilde{p}\|_{W^{1,s}} + C\|c\|_{W^{1,r}}\|g\|_{W^{1,s}} \quad \text{with } \frac{1}{q} + \frac{1}{r} = \frac{1}{s}. \end{aligned}$$

The last inequality follows from the Sobolev embedding $W^{1,s} \subset L^q$ resulting from

$$1 - \frac{d}{s} > -\frac{d}{q} \iff \frac{1}{d} - \frac{1}{s} > -\frac{1}{q} = \frac{1}{r} - \frac{1}{s} \iff \frac{1}{d} > \frac{1}{r} \iff r > d.$$

Together with (2.69) this shows (2.65). Uniqueness follows from linearity of the system (2.64) and by (2.65). □

In order to show continuous dependence of the system (2.64) with respect to c we require additional assumptions on the nonlinearities.

Assumptions 2.51 *The permeability function satisfies* $\nu \in C^0(\mathbb{R})$ *and*

$$|\nu(t) - \nu(s)| \leq L_\nu |t - s| \quad \forall\, s, t \in \mathbb{R}$$

for a positive constant L_ν

In order to use Proposition 2.50 within a Galerkin scheme we will employ the following continuous dependence result:

Proposition 2.52 *Let* $\Omega \subset \mathbb{R}^d$*,* $d = 2, 3$*, be a bounded domain with* $C^{2,1}$*-boundary and let Assumptions 2.51 hold. Furthermore, let* $(\mathbf{f}_i, g_i, c_i) \in \mathbf{L}^2 \times H^1 \times W^{1,r}$*,* $i = 1, 2$*, with* $r > d$ *be given and let* $(\mathbf{v}_i, p_i) \in \mathbf{H}^2 \times H^1$*,* $i = 1, 2$*, be the unique strong solution pairs of*

$$-\mathrm{div}(2\eta(c_i)\mathbf{D}\mathbf{v}_i + \lambda(c_i)\mathrm{div}(\mathbf{v}_i)\mathbf{I} - p_i\mathbf{I}) + \nu(c_i)\mathbf{v}_i = \mathbf{f}_i \quad \text{in } \Omega, \tag{2.70a}$$

$$\mathrm{div}(\mathbf{v}_i) = g_i \quad \text{in } \Omega, \tag{2.70b}$$

$$(2\eta(c_i)\mathbf{D}\mathbf{v}_i + \lambda(c_i)\mathrm{div}(\mathbf{v}_i)\mathbf{I} - p_i\mathbf{I})\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega \tag{2.70c}$$

according to Proposition 2.50. Then, the differences $(\mathbf{v}, p) \coloneqq (\mathbf{v}_2 - \mathbf{v}_1, p_2 - p_1)$ *fulfil*

$$\|\mathbf{v}\|_{\mathbf{H}^2} + \|p\|_{H^1} \leq C \left(\|\mathbf{f}_2 - \mathbf{f}_1\|_{\mathbf{L}^2} \|g_2 - g_1\|_{H^1} + (\|\mathbf{f}_1\|_{\mathbf{L}^2} + \|g_1\|_{H^1}) \|c_2 - c_1\|_{W^{1,r}} \right) \tag{2.71}$$

with a constant C *depending only on* Ω*,* η_0*,* η_1*,* λ_0*,* ν_0*,* ν_1*,* L_ν*,* $\|c_1\|_{W^{1,r}}$ *and* $\|c_2\|_{W^{1,r}}$*.*

Proof. In the following we denote by C a generic constant depending only on Ω, η_0, η_1, λ_0, ν_0, ν_1, L_η, $\|c_1\|_{W^{1,r}}$ and $\|c_2\|_{W^{1,r}}$. We denote $\mathbf{v} \coloneqq \mathbf{v}_2 - \mathbf{v}_1$, $p \coloneqq p_2 - p_1$, $\mathbf{f} \coloneqq \mathbf{f}_2 - \mathbf{f}_1$, $g \coloneqq g_2 - g_1$ and $c \coloneqq c_2 - c_1$. Then, a straightforward calculation shows that $(\mathbf{v}, p)$ satisfies

$$-\mathrm{div}(2\eta(c_2)\mathbf{D}\mathbf{v} + \lambda(c_2)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}) + \nu(c_2)\mathbf{v} = \tilde{f} \quad \text{in } \Omega, \tag{2.72a}$$

$$\mathrm{div}(\mathbf{v}) = g \quad \text{in } \Omega, \tag{2.72b}$$

$$(2\eta(c_2)\mathbf{D}\mathbf{v} + \lambda(c_2)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I})\mathbf{n} = \mathbf{f}_b \quad \text{on } \partial\Omega, \tag{2.72c}$$

where

$$\begin{aligned} \tilde{\mathbf{f}} &= \mathbf{f} + \mathrm{div}\big(2(\eta(c_2) - \eta(c_1))\mathbf{D}\mathbf{v}_1 + (\lambda(c_2) - \lambda(c_1))g_1\mathbf{I}\big) - (\nu(c_2) - \nu(c_1))\mathbf{v}_1, \\ \mathbf{f}_b &= (2(\eta(c_1) - \eta(c_2))\mathbf{D}\mathbf{v}_1 + (\lambda(c_1) - \lambda(c_2))g_1\mathbf{I})\,\mathbf{n}. \end{aligned}$$

Now, we observe that

$$\begin{aligned} \mathrm{div}\big(2(\eta(c_2) - \eta(c_1))\mathbf{D}\mathbf{v}_1\big) &= 2\mathbf{D}\mathbf{v}_1\big(\eta'(c_2)(\nabla c_2 - \nabla c_1) + (\eta'(c_2) - \eta'(c_1))\nabla c_1\big) \\ &\quad + (\eta(c_2) - \eta(c_1))\mathrm{div}(2\mathbf{D}\mathbf{v}_1). \end{aligned}$$

Setting $\frac{1}{r'} + \frac{1}{r} = \frac{1}{2}$ and using $r > d$ we can check that $H^1 \subset L^{r'}$ and $W^{1,r} \subset L^\infty$. Invoking the assumptions on $\eta(\cdot)$ yields

$$\begin{aligned} \|2\mathbf{D}\mathbf{v}_1\big(\eta'(c_2)(\nabla c_2 - \nabla c_1)\|_{\mathbf{L}^2} &\leq C\|2\mathbf{D}\mathbf{v}_1\|_{\mathbf{L}^{r'}} \|\nabla c\|_{\mathbf{L}^r} \leq C\|\mathbf{v}_1\|_{\mathbf{H}^2}\|c\|_{W^{1,r}}, \\ \|2\mathbf{D}\mathbf{v}_1(\eta'(c_2) - \eta'(c_1))\nabla c_1\|_{\mathbf{L}^2} &\leq C\|2\mathbf{D}\mathbf{v}_1\|_{\mathbf{L}^{r'}} \|\nabla c_1\|_{\mathbf{L}^r}\|c\|_{L^\infty} \leq C\|\mathbf{v}_1\|_{\mathbf{H}^2}\|c\|_{W^{1,r}}, \\ \|(\eta(c_2) - \eta(c_1))\mathrm{div}(2\mathbf{D}\mathbf{v}_1)\|_{\mathbf{L}^2} &\leq C\|c\|_{L^\infty}\|\mathrm{div}(\mathbf{D}\mathbf{v}_1)\|_{\mathbf{L}^2} \leq C\|\mathbf{v}_1\|_{\mathbf{H}^2}\|c\|_{W^{1,r}}, \end{aligned}$$

and therefore

$$\|\mathrm{div}\big(2(\eta(c_2) - \eta(c_1))\mathbf{D}\mathbf{v}_1\big)\|_{\mathbf{L}^2} \leq C\|\mathbf{v}_1\|_{\mathbf{H}^2}\|c\|_{W^{1,r}}.$$

With similar arguments we obtain

$$\|\mathrm{div}\big((\lambda(c_2) - \lambda(c_1))g_1\mathbf{I}\big) - (\nu(c_2) - \nu(c_1))\mathbf{v}_1\|_{\mathbf{L}^2} \leq C\|c\|_{W^{1,r}}\left(\|g_1\|_{H^1} + \|\mathbf{v}_1\|_{\mathbf{H}^2}\right).$$

From the last two inequalities and (2.65) we obtain

$$\|\tilde{\mathbf{f}}\|_{\mathbf{L}^2} \leq C\left(\|\mathbf{f}\|_{\mathbf{L}^2} + (\|\mathbf{f}_1\|_{\mathbf{L}^2} + \|g_1\|_{H^1})\|c\|_{W^{1,r}}\right). \tag{2.73}$$

With similar arguments it follows that

$$\|2(\eta(c_1) - \eta(c_2))\mathbf{D}\mathbf{v}_1 + (\lambda(c_1) - \lambda(c_2))g_1\mathbf{I}\|_{\mathbf{H}^1} \leq C(\|\mathbf{f}_1\|_{\mathbf{L}^2} + \|g_1\|_{H^1})\|c\|_{W^{1,r}}.$$

Using the trace theorem and the assumptions on $\partial\Omega$, this implies

$$\|\mathbf{f}_b\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C(\|\mathbf{f}_1\|_{\mathbf{L}^2} + \|g_1\|_{H^1})\|c\|_{W^{1,r}}. \tag{2.74}$$

Applying (2.65) to (2.72) and using (2.73)-(2.74) we deduce that

$$\|\mathbf{v}\|_{\mathbf{H}^2} + \|p\|_{H^1} \leq C\left(\|\mathbf{f}\|_{\mathbf{L}^2}\|g\|_{H^1} + (\|\mathbf{f}_1\|_{\mathbf{L}^2} + \|g_1\|_{H^1})\|c\|_{W^{1,r}}\right)$$

which completes the proof. □

3

Modelling aspects

Using basic thermodynamic principles and the Lagrange multiplier method of Liu and Müller, we will derive a general Cahn–Hilliard–Brinkman model for tumour growth including effects like, for example, diffusion, chemotaxis, active transport, proliferation and apoptosis. This model will serve as the basis for this thesis and several variants of this model will be analysed. We will consider a partial mixing of a fluid consisting of two components and we follow the ideas presented in [3, 87]. Furthermore, we use basic ideas of continuum mechanics, see, e. g., [60, 99].

In the second part of this chapter, we will discuss several additional modelling aspects like, for example, specific forms of source terms, pressure reformulations, a general energy inequality, boundary conditions and non-dimensionalisation arguments.

Then, we will use the method of formally matched asymptotics to derive some sharp interface models for tumour growth which are related to free boundary problems that have been studied earlier in the literature.

In the last part of this chapter we will show numerical simulations which give further insights into the model and the influence of different parameters.

3.1 Derivation of the model

Let us consider a bounded domain $\Omega \in \mathbb{R}^d$, $d \in \{1,2,3\}$, and a mixture consisting of tumour and healthy cells. We denote the first and second component as the healthy and tumour tissues, respectively. Furthermore, we introduce ρ_i, $i = 1,2$, (actual mass of the component matter per volume in the mixture) and $\bar{\rho}_i$, $i = 1,2$ (mass density of a pure component i). The mass density of the mixture is denoted by $\rho \coloneqq \rho_1 + \rho_2$. We define

$$u_i = \frac{\rho_i}{\bar{\rho}_i}$$

as the volume fraction of component i and

$$c_i = \frac{\rho_i}{\rho}$$

as the mass concentration of the i-th component and we note that $c_1 + c_2 = 1$. Physically we expect $\rho_i \in [0, \bar{\rho}_i]$ and thus $u_i \in [0, 1]$. By $\mathbf{v}_i$, $i = 1, 2$, we denote the velocity of component i and we make the following assumptions on our model.

(i) The excess volume due to mixing of the components is zero, i. e.,

$$u_1 + u_2 = 1. \tag{3.1}$$

(ii) We allow for mass exchange between the two components. Growth of the tumour is represented by mass transfer of healthy to tumour tissue and vice versa.

(iii) We choose a volume-averaged mixture velocity, i. e.,

$$\mathbf{v} := u_1\mathbf{v}_1 + u_2\mathbf{v}_2. \tag{3.2}$$

(iv) We assume the existence of a general chemical species acting as a nutrient for the tumour, like, for example, oxygen or glucose. The concentration of this species is denoted by σ and it is transported by the velocity $\mathbf{v}$ and a diffusive flux $\mathbf{J}_\sigma$.

We remark that the choice of the mixture velocity is in contrast to [113] where they use a barycentric/mass-averaged mixture velocity $\tilde{\mathbf{v}} := c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ leading to a more complicated expression for the continuity equation.

3.1.1 Balance laws

We now study the balance laws for mass and momentum.

Balance of mass

The mass balance law in its local form for the two components is given by

$$\partial_t \rho_i + \operatorname{div}(\rho_i \mathbf{v}_i) = \Gamma_i, \quad i = 1, 2, \tag{3.3}$$

with source or sink terms Γ_i, $i = 1, 2$. Dividing (3.3) by $\bar{\rho}_i$, $i = 1, 2$, we obtain the identities

$$\partial_t u_i + \operatorname{div}(u_i \mathbf{v}_i) = \frac{\Gamma_i}{\bar{\rho}_i}, \quad i = 1, 2. \tag{3.4}$$

Using (3.1)-(3.2) and (3.4) yields

$$\operatorname{div}(\mathbf{v}) = \operatorname{div}(u_1\mathbf{v}_1) + \operatorname{div}(u_2\mathbf{v}_2) = \sum_{i=1}^{2}\left(\frac{\Gamma_i}{\bar{\rho}_i} - \partial_t u_i\right) = \frac{\Gamma_1}{\bar{\rho}_1} + \frac{\Gamma_2}{\bar{\rho}_2} =: \Gamma_{\mathbf{v}}. \tag{3.5}$$

We introduce the fluxes

$$\mathbf{J}_i := \rho_i(\mathbf{v}_i - \mathbf{v}), \quad \mathcal{J} := \mathbf{J}_1 + \mathbf{J}_2, \quad \mathbf{J} := -\frac{1}{\bar{\rho}_1}\mathbf{J}_1 + \frac{1}{\bar{\rho}_2}\mathbf{J}_2,$$

where $\mathbf{J}_i$ describes the remaining diffusive flux after subtracting the flux resulting from mathematical transport along the mixture velocity. Using the identity

$$\mathcal{J} + \rho\mathbf{v} = \mathbf{J}_1 + \mathbf{J}_2 + \rho\mathbf{v} = \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2$$

in conjunction with (3.3), the equation for the mixture density reads as

$$\partial_t \rho + \operatorname{div}(\rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2) = \partial_t\rho + \operatorname{div}(\rho\mathbf{v} + \mathcal{J}) = \Gamma_1 + \Gamma_2. \tag{3.6}$$

In particular, we see that the flux of the mixture is decomposed into one part representing mathematical transport along the mixture velocity and another part describing additional fluxes. In some models it is assumed that there is no gain or loss of mass locally which is the case if $\Gamma_1 = -\Gamma_2$ in (3.6). From now on we denote by $\varphi := u_2 - u_1$ the difference in volume fractions of the two components. Recalling $\rho_i = \bar{\rho}_i u_i$ and using the identity

$$\operatorname{div}(u_i \mathbf{v}_i) = \operatorname{div}\left(\frac{\rho_i}{\bar{\rho}_i}\mathbf{v}_i\right) = \operatorname{div}\left(\frac{\rho_i}{\bar{\rho}_i}(\mathbf{v}_i - \mathbf{v} + \mathbf{v})\right) = \frac{1}{\bar{\rho}_i}\operatorname{div}(\mathbf{J}_i) + \operatorname{div}(u_i\mathbf{v}),$$

from (3.4) we obtain

$$\partial_t u_i + \frac{1}{\bar{\rho}_i}\operatorname{div}(\mathbf{J}_i) + \operatorname{div}(u_i\mathbf{v}) = \frac{\Gamma_i}{\bar{\rho}_i}.$$

Subtracting the equation for u_1 from the equation for u_2 yields

$$\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) + \operatorname{div}(\mathbf{J}) = \frac{\Gamma_2}{\bar{\rho}_2} - \frac{\Gamma_1}{\bar{\rho}_1} =: \Gamma_\varphi. \tag{3.7}$$

In particular, using $u_1 + u_2 = 1$ gives

$$u_2 = \frac{1+\varphi}{2}, \quad u_1 = \frac{1-\varphi}{2}$$

which means that $\{x \in \Omega : \varphi(x) = 1\}$ represents the region of pure tumour tissue whereas $\{x \in \Omega : \varphi(x) = -1\}$ is the region of pure healthy tissue. From the definition of ρ and u_i, $i = 1, 2$, it follows that

$$\rho = \rho(\varphi) = \frac{\bar{\rho}_1 + \bar{\rho}_2}{2} + \frac{\bar{\rho}_2 - \bar{\rho}_1}{2}\varphi,$$

and therefore ρ depends linearly on φ. Moreover, we see that

$$\rho = \bar{\rho}_1 \quad \text{if } \varphi = -1, \qquad \rho = \bar{\rho}_2 \quad \text{if } \varphi = 1.$$

For the nutrient we postulate the balance law

$$\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) + \operatorname{div}\mathbf{J}_\sigma = -\Gamma_\sigma, \tag{3.8}$$

where Γ_σ is a source or sink term, $\sigma\mathbf{v}$ models transport by the volume-averaged velocity and $\mathbf{J}_\sigma$ represents other transport mechanisms.

Balance of linear momentum

We make the following assumptions for our model.

(i) As in [3], we consider the mixture as a single fluid with volume-averaged velocity $\mathbf{v}$ which satisfies the balance law of linear momentum of continuum mechanics.

(ii) We assume that inertial forces are negligible which can be justified as the Reynolds number for biological processes like tumour growth is usually very small. Since gravity plays no role in our model of interest and other body forces are difficult to imagine, we neglect body forces.

(iii) Contact forces are represented by a stress tensor $\mathbf{T}$, and we assume an additional source $\mathbf{m}$ in the momentum balance equation which could for example represent momentum exchange.

(iv) We assume that the stress tensor is symmetric, isotropic and depends on $\nabla\mathbf{v}$, φ, μ, σ and $\nabla\varphi$.

With all these assumptions, the balance of linear momentum takes the form

$$\operatorname{div}(\mathbf{T}) + \mathbf{m} = 0, \tag{3.9}$$

where $\mathbf{T}$ and $\mathbf{m}$ have to be specified by constitutive assumptions.

3.1.2 Consequences of frame indifference

In the following we apply the same arguments as in [60, 99]. The constitutive law for the stress tensor is assumed to be of the form

$$\mathbf{T} = \hat{\mathbf{T}}(\varphi, \mu, \sigma, \nabla\varphi, \nabla\mathbf{v}).$$

Dependence of the stress tensor on $\nabla\mathbf{v}$

In the following, we suppress the dependence on $(\varphi, \mu, \sigma, \nabla\varphi)$ and denote $\mathbf{L} := \nabla\mathbf{v}$. We allow for observer changes of the form

$$(t, x) \mapsto (t^*, x^*) = (t, \mathbf{a}(t) + \mathbf{Q}(t)x)$$

with smooth functions $\mathbf{a}\colon \mathbb{R}_+ \to \mathbb{R}^n$ and $\mathbf{Q}\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ satisfying $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$ and $\det(\mathbf{Q}) = 1$. The vector $\mathbf{a}$ realises a translation while $\mathbf{Q}$ is an orthogonal matrix. Under a change of observer, the quantities $\mathbf{T}$ and $\mathbf{L}$ transform to $\mathbf{Q}^\intercal\mathbf{T}^*\mathbf{Q}$ and $\mathbf{Q}\mathbf{L}\mathbf{Q}^\intercal + \mathbf{Q}'\mathbf{Q}^\intercal$ (see, e. g., [60]). Moreover, the assumption of isotropy for the stress tensor requires that the constitutive law does not change. Hence, the relation

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{L})\mathbf{Q}^\intercal = \hat{\mathbf{T}}\left(\mathbf{Q}\mathbf{L}\mathbf{Q}^\intercal + \mathbf{Q}'\mathbf{Q}^\intercal\right) \tag{3.10}$$

has to hold for all $\mathbf{Q}$. Let $\mathbf{\Omega}_0$ be an arbitrary skew-symmetric matrix and define $\mathbf{Q}$ as the unique solution of the initial value problem

$$\mathbf{Q}'(t) = \mathbf{\Omega}_0\mathbf{Q}(t), \quad \mathbf{Q}(0) = \mathbf{I} \quad \forall\, t \geq 0.$$

It can be checked that $\mathbf{Q}$ satisfies $\mathbf{Q}^\intercal\mathbf{Q} = \mathbf{I}$ and $\det(\mathbf{Q}) = 1$ and therefore (3.10) has to hold for all such $\mathbf{Q}$. At time $t = 0$ we find that

$$\hat{\mathbf{T}}(\mathbf{L}) = \hat{\mathbf{T}}(\mathbf{L} + \mathbf{\Omega}_0).$$

Using the identities

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} := \frac{1}{2}(\mathbf{L} + \mathbf{L}^\intercal), \quad \mathbf{W} := \frac{1}{2}(\mathbf{L} - \mathbf{L}^\intercal)$$

we obtain

$$\hat{\mathbf{T}}(\mathbf{L}) = \hat{\mathbf{T}}(\mathbf{D} + \mathbf{W} + \mathbf{\Omega}_0). \tag{3.11}$$

These relations must hold for all skew-symmetric matrices $\mathbf{\Omega}_0$ and all $\mathbf{L}$. We now fix $\mathbf{L}$ and choose

$$\mathbf{\Omega}_0 = -\mathbf{W}$$

in (3.11) to obtain that

$$\hat{\mathbf{T}}(\mathbf{L}) = \hat{\mathbf{T}}(\mathbf{D}).$$

The r. h. s. of this identity depends only on the symmetric part of $\mathbf{L}$ and therefore $\hat{\mathbf{T}}(\mathbf{L}) = \hat{\mathbf{T}}(\mathbf{D})$. Finally, we take $\mathbf{Q}$ constant in time in (3.10) to get the additional restriction

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{D})\mathbf{Q}^\intercal = \hat{\mathbf{T}}(\mathbf{Q}\mathbf{D}\mathbf{Q}^\intercal). \tag{3.12}$$

Dependence of the stress tensor on $\nabla\varphi$

Let $\mathbf{Q}$ be an arbitrary orthogonal matrix satisfying $\det(\mathbf{Q}) = 1$. In the following, we suppress the dependence on $(\varphi, \mu, \sigma, \mathbf{Dv})$ and we restrict to the three-dimensional case. Similar as before, we require that

$$\mathbf{Q}\hat{\mathbf{T}}(\nabla\varphi) = \hat{\mathbf{T}}(\mathbf{Q}\nabla\varphi)\mathbf{Q} \quad \forall\, \mathbf{Q}. \tag{3.13}$$

Let $\bar{\mathbf{Q}}$ be the matrix that realises a rotation around $\nabla\varphi$ with the angle of rotation given by $\frac{\pi}{2}$. Then, it holds that $\bar{\mathbf{Q}}\nabla\varphi = \nabla\varphi$ and thus $\nabla\varphi$ is an eigenvector of $\bar{\mathbf{Q}}$ with corresponding eigenvalue 1, and the eigenspace has dimension 1. From (3.13) we obtain

$$\bar{\mathbf{Q}}\hat{\mathbf{T}}(\nabla\varphi) = \hat{\mathbf{T}}(\nabla\varphi)\bar{\mathbf{Q}},$$

and applying both sides to $\nabla\varphi$ gives

$$\bar{\mathbf{Q}}\hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi = \hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi.$$

Hence, $\hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi$ is an eigenvector of $\bar{\mathbf{Q}}$ with corresponding eigenvalue 1. Since the eigenspace of $\nabla\varphi$ has dimension 1, this implies

$$\hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi = \alpha(\nabla\varphi)\nabla\varphi \tag{3.14}$$

for a function $\alpha\colon \mathbb{R}^3 \to \mathbb{R}$. For the vector $\mathbf{q} = \hat{\mathbf{q}}(\nabla\varphi) = \hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi$ we obtain by using (3.13) that

$$\hat{\mathbf{q}}(\mathbf{Q}\nabla\varphi) = \hat{\mathbf{T}}(\mathbf{Q}\nabla\varphi)\mathbf{Q}\nabla\varphi = \mathbf{Q}\hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi = \mathbf{Q}\hat{\mathbf{q}}(\nabla\varphi),$$

and thus the expression on the left hand side of (3.14) is isotropic. Therefore, the r. h. s. of (3.14) has to isotropic, and we conclude from (3.14) that

$$\alpha(\mathbf{Q}\nabla\varphi)\mathbf{Q}\nabla\varphi = \alpha(\nabla\varphi)\mathbf{Q}\nabla\varphi,$$

and so, if $\alpha \neq 0$, it follows

$$\alpha(\nabla\varphi) = \alpha(\mathbf{Q}\nabla\varphi).$$

Now, for any vector $\mathbf{a}$ with $|\mathbf{a}| = |\nabla\varphi|$, there exists an orthogonal matrix $\mathbf{Q}$ with $\det(\mathbf{Q}) = 1$ such that $\mathbf{a} = \mathbf{Q}\nabla\varphi$, and therefore

$$\hat{\mathbf{T}}(\nabla\varphi)\nabla\varphi = \alpha(|\nabla\varphi|)\nabla\varphi. \tag{3.15}$$

Due to the symmetry of $\mathbf{T}$, there exist eigenvectors $\mathbf{v}_i$, $i = 2, 3$, of $\mathbf{T}$ with corresponding eigenvalues $\alpha_i(\nabla\varphi)$, $i = 2, 3$, such that $\{\nabla\varphi/|\nabla\varphi|, \mathbf{v}_2, \mathbf{v}_3\}$ forms an orthonormal basis of $\mathbb{R}^3$. Moreover, it holds that

$$\hat{\mathbf{T}}(\nabla\varphi) = \alpha(|\nabla\varphi|)\frac{\nabla\varphi}{|\nabla\varphi|} \otimes \frac{\nabla\varphi}{|\nabla\varphi|} + \alpha_2(\nabla\varphi)\mathbf{v}_2 \otimes \mathbf{v}_2 + \alpha_3(\nabla\varphi)\mathbf{v}_3 \otimes \mathbf{v}_3. \tag{3.16}$$

Let $\mathbf{Q}_1$ be the matrix that realises a rotation around $\nabla\varphi$ such that $\mathbf{Q}_1\mathbf{v}_2 = \mathbf{v}_3$ and $\mathbf{Q}_1\mathbf{v}_3 = -\mathbf{v}_2$. Then, using (3.13) with $\mathbf{Q} = \mathbf{Q}_1$, we obtain by using (3.16) that

$$\begin{aligned}
&\alpha(|\nabla\varphi|)\frac{\nabla\varphi}{|\nabla\varphi|} \otimes \frac{\nabla\varphi}{|\nabla\varphi|} + \alpha_2(\nabla\varphi)\mathbf{v}_3 \otimes \mathbf{v}_2 - \alpha_3(\nabla\varphi)\mathbf{v}_2 \otimes \mathbf{v}_3 \\
&\quad = \left(\alpha(|\nabla\varphi|)\frac{\nabla\varphi}{|\nabla\varphi|} \otimes \frac{\nabla\varphi}{|\nabla\varphi|} + \alpha_2(\nabla\varphi)\mathbf{v}_2 \otimes \mathbf{v}_2 + \alpha_3(\nabla\varphi)\mathbf{v}_3 \otimes \mathbf{v}_3\right)\mathbf{Q}_1.
\end{aligned}$$

Applying both sides to $\mathbf{v}_2$ or $\mathbf{v}_3$ we obtain

$$\alpha_2(\nabla\varphi)\mathbf{v}_i = \alpha_3(\nabla\varphi)\mathbf{v}_i \quad \text{for } i = 2, 3 \implies \alpha_2(\nabla\varphi) = \alpha_3(\nabla\varphi).$$

Together with (3.16), this gives

$$\hat{\mathbf{T}}(\nabla\varphi) = \alpha(|\nabla\varphi|)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|} + \alpha_2(\nabla\varphi)\left(\mathbf{I} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right), \tag{3.17}$$

and therefore

$$\hat{\mathbf{T}}(\mathbf{Q}\nabla\varphi) = \mathbf{Q}\hat{\mathbf{T}}(\nabla\varphi)\mathbf{Q}^{\mathsf{T}} + (\alpha_2(\mathbf{Q}\nabla\varphi) - \alpha_2(\nabla\varphi))\mathbf{Q}\left(\mathbf{I} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)\mathbf{Q}^{\mathsf{T}}.$$

In order to fulfil (3.13) we require that

$$(\alpha_2(\mathbf{Q}\nabla\varphi) - \alpha_2(\nabla\varphi))\mathbf{Q}\left(\mathbf{I} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)\mathbf{Q}^{\mathsf{T}}\mathbf{u} = \mathbf{0} \quad \forall\, \mathbf{u}\in\mathbb{R}^3\setminus\{(0,0,0)^{\mathsf{T}}\}. \tag{3.18}$$

We now argue by contradiction in order to show that α_2 depends only on $|\nabla\varphi|$. Assume that there exists $\mathbf{Q}_2$ such that $\alpha_2(\mathbf{Q}_2\nabla\varphi) \neq \alpha_2(\nabla\varphi)$. This implies $\mathbf{Q}_2\nabla\varphi \neq \nabla\varphi$ and there exists a vector $\mathbf{x}\in\mathbb{R}^3\setminus\{(0,0,0)^{\mathsf{T}}\}$ such that $\mathbf{Q}_2^{\mathsf{T}}\mathbf{x} = \mathbf{v}_1$, which implies

$$(\alpha_2(\mathbf{Q}_2\nabla\varphi) - \alpha_2(\nabla\varphi))\mathbf{Q}_2\left(\mathbf{I} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right)\mathbf{Q}_2^{\mathsf{T}}\mathbf{x} = (\alpha_2(\mathbf{Q}_2\nabla\varphi) - \alpha_2(\nabla\varphi))\mathbf{x} \neq 0.$$

This is a contradiction to (3.18) and therefore $\alpha_2(\mathbf{Q}\nabla\varphi) = \alpha_2(\nabla\varphi)$ for all orthogonal matrices $\mathbf{Q}$. Then, arguing as above we obtain $\alpha_2(\nabla\varphi) = \alpha_2(|\nabla\varphi|)$, and (3.17) implies

$$\hat{\mathbf{T}}(\nabla\varphi) = \alpha(|\nabla\varphi|)\frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|} + \alpha_2(|\nabla\varphi|)\left(\mathbf{I} - \frac{\nabla\varphi}{|\nabla\varphi|}\otimes\frac{\nabla\varphi}{|\nabla\varphi|}\right). \tag{3.19}$$

3.1.3 Energy inequality and the Lagrange multiplier method:

In an isothermal situation, i. e., the system's temperature remains constant, the second law of thermodynamics is formulated as an energy inequality, see, e. g., [98]. Thus, the specific form of the stress tensor and the fluxes for φ and σ depends on the choice of a suitable system energy. Since we have neglected inertia effects in the momentum balance law, we assume that there is no contribution of kinetic energy. For a model including inertia effects we refer to [2] where the authors deduce a Navier–Stokes–Cahn–Hilliard system. We postulate a free energy of the form

$$e = \hat{e}(\varphi, \nabla\varphi, \sigma). \tag{3.20}$$

A discussion of the situation when source terms are present can be found in, e. g., [99, Chap. 62]. We denote by $V(t)\subset\Omega$ an arbitrary volume which is transported with the fluid velocity. Using the second law of thermodynamics in an isothermal situation, the following energy inequality has to hold

$$\begin{aligned}\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} e(\varphi,\nabla\varphi,\sigma)\,\mathrm{d}x}_{\substack{\text{Change of}\\ \text{energy}}} &\leq \underbrace{-\int_{\partial V(t)} \mathbf{J}_e\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}}_{\substack{\text{Energy flux across}\\ \text{the boundary}}} + \underbrace{\int_{\partial V(t)} (\mathbf{T}\mathbf{n})\cdot\mathbf{v}\,\mathrm{d}\mathcal{H}^{d-1}}_{\substack{\text{Working due to}\\ \text{macroscopic stresses}}}\\ &\quad + \underbrace{\int_{V(t)} c_{\mathbf{v}}\Gamma_{\mathbf{v}} + c_\varphi\Gamma_\varphi + c_\sigma(-\Gamma_\sigma)\,\mathrm{d}x}_{\text{Supply of energy}},\end{aligned} \tag{3.21}$$

where $\mathbf{n}$ is the outer unit normal to $\partial V(t)$, $\mathbf{J}_e$ is an energy flux yet to be determined and $c_{\mathbf{v}}$, c_φ and c_σ are unknown multipliers which have to be specified. Furthermore, the second boundary

term describes working due to the macroscopic stresses, see, e. g., [3].
We introduce the material derivative of a function f by

$$\partial_t^\bullet f := \partial_t f + \nabla f \cdot \mathbf{v}.$$

Following the arguments in, e. g., [3, 87], we now apply the Lagrange multiplier method of Liu and Müller which has been developed in [111]. More precisely, we introduce Lagrange multipliers $\lambda_{\mathbf{v}}$, λ_φ and λ_σ for (3.5) and (3.7)-(3.8). The following identity can be easily verified upon using the momentum balance equation:

$$-\int_{\partial V(t)} (\mathbf{Tn}) \cdot \mathbf{v} \, d\mathcal{H}^{d-1} = -\int_{V(t)} \operatorname{div}(\mathbf{T}) \cdot \mathbf{v} + \mathbf{T} \colon \nabla \mathbf{v} \, dx = \int_{V(t)} \mathbf{m} \cdot \mathbf{v} - \mathbf{T} \colon \nabla \mathbf{v} \, dx.$$

Therefore, using Reynold's transport theorem and (3.21), the following local dissipation inequality has to be fulfilled for arbitrary values of $(\varphi, \sigma, \nabla\varphi, \nabla\sigma, \mathbf{v}, \Gamma_{\mathbf{v}}, \Gamma_\varphi, \Gamma_\sigma, \partial_t^\bullet\varphi, \partial_t^\bullet\sigma)$

$$\begin{aligned}
-\mathcal{D} := {} & \partial_t^\bullet e + e\operatorname{div}(\mathbf{v}) + \operatorname{div}(\mathbf{J}_e) - \mathbf{T} \colon \nabla \mathbf{v} + \mathbf{m} \cdot \mathbf{v} - c_{\mathbf{v}}\Gamma_{\mathbf{v}} - c_\varphi \Gamma_\varphi + c_\sigma \Gamma_\sigma \\
& - \lambda_{\mathbf{v}}(\operatorname{div}(\mathbf{v}) - \Gamma_{\mathbf{v}}) \\
& - \lambda_\varphi(\partial_t^\bullet\varphi + \varphi\operatorname{div}(\mathbf{v}) + \operatorname{div}(\mathbf{J}_\varphi) - \Gamma_\varphi) \\
& - \lambda_\sigma(\partial_t^\bullet\sigma + \sigma\operatorname{div}(\mathbf{v}) + \operatorname{div}(\mathbf{J}_\sigma) + \Gamma_\sigma) \leq 0.
\end{aligned}$$

For the first term in the definition of $-\mathcal{D}$ we calculate

$$\partial_t^\bullet e = \frac{\partial e}{\partial \varphi}\partial_t^\bullet\varphi + \frac{\partial e}{\partial \nabla\varphi}\partial_t^\bullet(\nabla\varphi) + \frac{\partial e}{\partial\sigma}\partial_t^\bullet\sigma.$$

We therefore arrive at the inequality

$$\begin{aligned}
-\mathcal{D} = {} & \operatorname{div}(\mathbf{J}_e - \lambda_\varphi \mathbf{J}_\varphi - \lambda_\sigma \mathbf{J}_\sigma) + \nabla\lambda_\varphi \cdot \mathbf{J}_\varphi + \nabla\lambda_\sigma \cdot \mathbf{J}_\sigma \\
& + \partial_t^\bullet\varphi\left(\frac{\partial e}{\partial\varphi} - \lambda_\varphi\right) + \partial_t^\bullet\sigma\left(\frac{\partial e}{\partial\sigma} - \lambda_\sigma\right) \\
& - \mathbf{T} \colon \nabla\mathbf{v} + \mathbf{m}\cdot\mathbf{v} + \frac{\partial e}{\partial\nabla\varphi} \cdot \partial_t^\bullet(\nabla\varphi) \\
& + (c_\sigma - \lambda_\sigma)\Gamma_\sigma + (\lambda_{\mathbf{v}} - c_{\mathbf{v}})\Gamma_{\mathbf{v}} + (\lambda_\varphi - c_\varphi)\Gamma_\varphi \\
& + (e - \lambda_\varphi\varphi - \lambda_\sigma\sigma - \lambda_{\mathbf{v}})\operatorname{div}(\mathbf{v}) \leq 0.
\end{aligned}$$

Using the identity

$$\partial_{x_j}(\partial_t^\bullet\varphi) = \partial_t\partial_{x_j}\varphi + \mathbf{v}\cdot\nabla(\partial_{x_j}\varphi) + \partial_{x_j}\mathbf{v}\cdot\nabla\varphi = \partial_t^\bullet(\partial_{x_j}\varphi) + \partial_{x_j}\mathbf{v}\cdot\nabla\varphi$$

we calculate

$$\operatorname{div}\left(\partial_t^\bullet\varphi\frac{\partial e}{\partial\nabla\varphi}\right) = \partial_t^\bullet\varphi\operatorname{div}\left(\frac{\partial e}{\partial\nabla\varphi}\right) + \partial_t^\bullet(\nabla\varphi)\cdot\frac{\partial e}{\partial\nabla\varphi} + \nabla\mathbf{v}\colon\left(\nabla\varphi\otimes\frac{\partial e}{\partial\nabla\varphi}\right).$$

Therefore, we can rewrite $-\mathcal{D}$ as

$$\begin{aligned}
-\mathcal{D} = {} & \operatorname{div}\left(\mathbf{J}_e - \lambda_\varphi\mathbf{J}_\varphi - \lambda_\sigma\mathbf{J}_\sigma + \partial_t^\bullet\varphi\frac{\partial e}{\partial\nabla\varphi}\right) + \nabla\lambda_\varphi\cdot\mathbf{J}_\varphi + \nabla\lambda_\sigma\cdot\mathbf{J}_\sigma \\
& + \partial_t^\bullet\varphi\left(\frac{\partial e}{\partial\varphi} - \operatorname{div}\left(\frac{\partial e}{\partial\nabla\varphi}\right) - \lambda_\varphi\right) + \partial_t^\bullet\sigma\left(\frac{\partial e}{\partial\sigma} - \lambda_\sigma\right) \\
& - \left(\mathbf{T} + \left(\nabla\varphi\otimes\frac{\partial e}{\partial\nabla\varphi}\right)\right) \colon \nabla\mathbf{v} + \mathbf{m}\cdot\mathbf{v} \\
& + (c_\sigma - \lambda_\sigma)\Gamma_\sigma + (\lambda_{\mathbf{v}} - c_{\mathbf{v}})\Gamma_{\mathbf{v}} + (\lambda_\varphi - c_\varphi)\Gamma_\varphi \\
& + (e - \lambda_\varphi\varphi - \lambda_\sigma\sigma - \lambda_{\mathbf{v}})\operatorname{div}(\mathbf{v}) \leq 0.
\end{aligned} \tag{3.22}$$

Finally, we define the chemical potential as

$$\mu := \frac{\partial e}{\partial\varphi} - \operatorname{div}\left(\frac{\partial e}{\partial\nabla\varphi}\right).$$

3.1.4 Constitutive assumptions:

To fulfil (3.22) we can argue as in, e. g., [3,87] and we make the following constitutive assumptions

$$\mathbf{J}_e = \lambda_\sigma \mathbf{J}_\sigma + \lambda_\varphi \mathbf{J}_\varphi - \partial_t^\bullet \varphi \frac{\partial e}{\partial \nabla \varphi}, \quad c_{\mathbf{v}} = \lambda_{\mathbf{v}}, \tag{3.23a}$$

$$c_\varphi = \lambda_\varphi = \frac{\partial e}{\partial \varphi} - \operatorname{div}\left(\frac{\partial e}{\partial \nabla \varphi}\right) = \mu, \quad c_\sigma = \lambda_\sigma = \frac{\partial e}{\partial \sigma}, \tag{3.23b}$$

$$\mathbf{J}_\varphi = -m(\varphi)\nabla\mu, \quad \mathbf{J}_\sigma = -n(\varphi)\nabla\left(\frac{\partial e}{\partial \sigma}\right), \tag{3.23c}$$

where $m(\varphi)$ and $n(\varphi)$ are non-negative mobilities corresponding to a generalised Fick's law (see [3]). In principle, $m(\cdot)$ and $n(\cdot)$ could also depend on additional variables like μ and σ. With these choices (3.22) simplifies to

$$-\left(\mathbf{T} + \left(\nabla\varphi \otimes \frac{\partial e}{\partial \nabla\varphi}\right)\right) : \nabla\mathbf{v} + \mathbf{m}\cdot\mathbf{v} + (e - \lambda_\varphi\varphi - \lambda_\sigma\sigma - \lambda_{\mathbf{v}})\operatorname{div}(\mathbf{v}) \le 0. \tag{3.24}$$

We now introduce the unknown pressure p and we rewrite the stress tensor as

$$\mathbf{T} = \mathbf{S} - p\mathbf{I} \quad \text{where} \quad \mathbf{S} = \mathbf{T} + p\mathbf{I}. \tag{3.25}$$

An easy calculation yields the identity

$$\left(\nabla\varphi \otimes \frac{\partial e}{\partial \nabla\varphi}\right) : \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^\intercal) = \frac{1}{2}\left(\nabla\varphi \otimes \frac{\partial e}{\partial \nabla\varphi} - \frac{\partial e}{\partial \nabla\varphi} \otimes \nabla\varphi\right) : \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^\intercal).$$

Since the skew symmetric part of $\nabla\mathbf{v}$ can attain arbitrary values (see, e. g., [3]) and by the symmetry of $\mathbf{T}$ we conclude from (3.24) that

$$\nabla\varphi \otimes \frac{\partial e}{\partial \nabla\varphi} = \frac{\partial e}{\partial \nabla\varphi} \otimes \nabla\varphi$$

which implies

$$\left|\frac{\partial e}{\partial \nabla\varphi}\right|^2 |\nabla\varphi|^2 = \left(\nabla\varphi \cdot \frac{\partial e}{\partial \nabla\varphi}\right)^2.$$

The last identity yields

$$\frac{\partial e}{\partial \nabla\varphi}(\varphi, \nabla\varphi, \sigma) = a(\varphi, \nabla\varphi, \sigma)\nabla\varphi \tag{3.26}$$

for some real valued function $a(\varphi, \nabla\varphi, \sigma)$. Since $\mathbf{S}$ is symmetric we have

$$\mathbf{S} \colon \nabla\mathbf{v} = \mathbf{S} \colon \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^\intercal) + \mathbf{S} \colon \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^\intercal) = \mathbf{S} \colon \mathbf{D}\mathbf{v}.$$

For the pressure we use $\mathbf{I} \colon \mathbf{D}\mathbf{v} = \operatorname{tr}(\mathbf{D}\mathbf{v})$ to obtain

$$-p\mathbf{I} \colon \nabla\mathbf{v} = -p\mathbf{I} \colon \mathbf{D}\mathbf{v} - p\mathbf{I} \colon \frac{1}{2}(\nabla\mathbf{v} - (\nabla\mathbf{v})^\intercal) = -p\operatorname{tr}(\mathbf{D}\mathbf{v}) = -p\operatorname{div}(\mathbf{v}).$$

Invoking the last two identities and using (3.25) we get

$$\mathbf{T} \colon \nabla\mathbf{v} = \mathbf{S} \colon \mathbf{D}\mathbf{v} - p\operatorname{div}(\mathbf{v}).$$

This identity allows us to rewrite (3.24) as

$$-\left(\mathbf{S} + (\nabla\varphi \otimes a(\varphi, \nabla\varphi, \sigma)\nabla\varphi)\right) : \mathbf{D}\mathbf{v} + \mathbf{m}\cdot\mathbf{v} + (e - \lambda_\varphi\varphi - \lambda_\sigma\sigma + p - \lambda_{\mathbf{v}})\operatorname{div}(\mathbf{v}) \le 0.$$

In order to control the mass exchange term we set

$$\lambda_{\mathbf{v}} := e - \lambda_\varphi \varphi - \lambda_\sigma \sigma + p,$$

and therefore it remains to fulfil the inequality

$$\big(\mathbf{S} + (\nabla\varphi \otimes a(\varphi, \nabla\varphi, \sigma)\nabla\varphi)\big) : \mathbf{D}\mathbf{v} - \mathbf{m}\cdot\mathbf{v} \geq 0.$$

Similar as in, e. g., [3] and motivated by Newton's linear rheological law we make the constitutive assumption

$$\mathbf{S} + \nabla\varphi \otimes a(\varphi, \nabla\varphi, \sigma)\nabla\varphi = 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I},$$

where $\eta(\cdot)$ and $\lambda(\cdot)$ are non-negative functions referred to as shear and bulk viscosities. This means that, on account of the last identity, the dissipation inequality (3.22) holds provided

$$-\mathbf{m}\cdot\mathbf{v} \geq 0.$$

Using similar arguments as in, e. g., [120] we choose

$$\mathbf{m} := -\nu(\varphi)\mathbf{v},$$

where $\nu(\cdot)$ represents the permeability and is also referred to as "drag" coefficient function. Summarising, the constitutive assumptions are given by

$$\mathbf{J}_e = \frac{\partial e}{\partial \sigma}\mathbf{J}_\sigma + \mu \mathbf{J}_\varphi - \partial_t^\bullet \varphi \frac{\partial e}{\partial \nabla\varphi}, \tag{3.27a}$$

$$c_\varphi = \lambda_\varphi = \frac{\partial e(\varphi, \nabla\varphi, \sigma)}{\partial \varphi} - \mathrm{div}\left(\frac{\partial e(\varphi, \nabla\varphi, \sigma)}{\partial \nabla\varphi}\right) = \mu, \quad c_\sigma = \lambda_\sigma = \frac{\partial e(\varphi, \nabla\varphi, \sigma)}{\partial \sigma}, \tag{3.27b}$$

$$c_{\mathbf{v}} = \lambda_{\mathbf{v}} = e - \lambda_\varphi \varphi - \lambda_\sigma \sigma + p, \tag{3.27c}$$

$$\mathbf{J}_\varphi = -m(\varphi)\nabla\mu, \quad \mathbf{J}_\sigma = -n(\varphi)\nabla\left(\frac{\partial e(\varphi, \nabla\varphi, \sigma)}{\partial \sigma}\right), \quad \mathbf{m} = -\nu(\varphi)\mathbf{v}, \tag{3.27d}$$

$$\mathbf{S} + (\nabla\varphi \otimes a(\varphi, \nabla\varphi, \sigma)\nabla\varphi) = 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I}. \tag{3.27e}$$

Furthermore, we showed that

$$\frac{\partial e(\varphi, \nabla\varphi, \sigma)}{\partial \nabla\varphi} = a(\varphi, \nabla\varphi, \sigma)\nabla\varphi.$$

We remark that by (3.19) we require $a(\varphi, \nabla\varphi, \sigma) = a(\varphi, |\nabla\varphi|, \sigma)$. The energy flux $\mathbf{J}_e$ in (3.27a) is chosen such that the divergence term in (3.22) vanishes. It contains classical terms like $\mu\mathbf{J}_\varphi$ and $\frac{\partial e}{\partial \sigma}\mathbf{J}_\sigma$ which describe energy flux due to mass diffusion and the non-classical term $\partial_t^\bullet \varphi \frac{\partial e}{\partial \nabla\varphi}$ describing working due to microscopic stresses. For more details see, e. g., [3, 87]. Collecting the results above, we arrive at the following dissipation inequality

$$\mathcal{D} = 2\eta(\varphi)|\mathbf{D}\mathbf{v}|^2 + \lambda(\varphi)(\mathrm{div}(\mathbf{v}))^2 + \nu(\varphi)|\mathbf{v}|^2 + m(\varphi)|\nabla\mu|^2 + n(\varphi)\left|\nabla\frac{\partial e}{\partial \sigma}\right|^2 \geq 0.$$

Hence, dissipation is produced by the following processes: viscosity effects on the velocity, changes in volume, dissipation at the pores of the mixture due to the flow, and transport along $\nabla\mu$ and $\nabla\frac{\partial e}{\partial \sigma}$.

3.1.5 The model equations:

From now on we assume a general energy of the form

$$e(\varphi, \nabla\varphi, \sigma) = f(\varphi, \nabla\varphi) + N(\varphi, \sigma).$$

The first term accounts for interfacial energy of the diffuse interface, whereas the second term represents the energy contribution due to the presence of the nutrient and the interaction between the tumour tissue and the nutrients. For more details regarding the second energy term, we refer to [101]. Furthermore, we assume that f is of Ginzburg–Landau type, that is,

$$f(\varphi, \nabla\varphi) = \frac{\beta}{\varepsilon}\psi(\varphi) + \frac{\beta\varepsilon}{2}|\nabla\varphi|^2,$$

where ψ is a potential with minima at $s = \pm 1$, typically the classical double-well potential, and the parameters $\beta > 0$ and $\varepsilon > 0$ are related to the surface tension and the interfacial thickness, respectively.

With this choice we calculate

$$\frac{\partial e}{\partial \varphi} = \frac{\beta}{\varepsilon}\psi'(\varphi) + N_\varphi, \quad \frac{\partial e}{\partial \nabla\varphi} = \beta\varepsilon\nabla\varphi, \quad a(\varphi, \nabla\varphi, \sigma) = \beta\varepsilon, \quad \frac{\partial e}{\partial \sigma} = N_\sigma,$$

where N_φ and N_σ denote the derivatives of $N(\varphi, \sigma)$ with respect to φ and σ, respectively. In the following we use the relation (3.25).

Recalling (3.5), (3.7)-(3.9) and using the constitutive assumptions (3.27) we obtain the following general Cahn–Hilliard–Brinkman model for tumour growth

$$\operatorname{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}, \tag{3.28a}$$

$$-\operatorname{div}(2\eta(\varphi)D\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I}) + \nu(\varphi)\mathbf{v} + \nabla p = -\operatorname{div}(\beta\varepsilon\nabla\varphi \otimes \nabla\varphi), \tag{3.28b}$$

$$\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = \operatorname{div}(m(\varphi)\nabla\mu) + \Gamma_\varphi, \tag{3.28c}$$

$$\mu = \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi + N_\varphi, \tag{3.28d}$$

$$\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) = \operatorname{div}(n(\varphi)\nabla N_\sigma) - \Gamma_\sigma, \tag{3.28e}$$

where

$$\Gamma_{\mathbf{v}} = \frac{\Gamma_2}{\bar\rho_2} + \frac{\Gamma_1}{\bar\rho_1}, \quad \Gamma_\varphi = \frac{\Gamma_2}{\bar\rho_2} - \frac{\Gamma_1}{\bar\rho_1}.$$

3.2 Further aspects of modelling

3.2.1 Specific source terms

We now outline specific choices of source terms which are commonly used in the literature.

(i) Assuming no gain or loss of mass locally (see (3.6)), we demand that

$$\Gamma_2 = -\Gamma_1 =: \Gamma.$$

Then, there is a close relation between the source terms $\Gamma_{\mathbf{v}}$ and Γ_φ given by

$$\Gamma_\varphi = \frac{\Gamma_2}{\bar\rho_2} - \frac{\Gamma_1}{\bar\rho_1} = \left(\frac{1}{\bar\rho_1} + \frac{1}{\bar\rho_2}\right)\Gamma, \qquad \Gamma_{\mathbf{v}} = \frac{\Gamma_2}{\bar\rho_2} + \frac{\Gamma_1}{\bar\rho_1} = \left(\frac{1}{\bar\rho_2} - \frac{1}{\bar\rho_1}\right)\Gamma. \tag{3.29}$$

In the following we set

$$\alpha := \frac{1}{\bar\rho_2} - \frac{1}{\bar\rho_1}, \qquad \beta := \frac{1}{\bar\rho_1} + \frac{1}{\bar\rho_2}. \tag{3.30}$$

(ii) Using linear kinetics (see, e. g., [81, 87]) we choose

$$\Gamma := (\mathcal{P}\sigma - \mathcal{A})h(\varphi), \qquad \Gamma_\sigma = \mathcal{C}\sigma h(\varphi), \tag{3.31}$$

where $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{C}$ are non-negative constants related to proliferation, apoptosis and consumption. The function $h(\cdot)$ interpolates linearly between $h(-1) = 0$ and $h(1) = 1$ and can be extended constant outside of the interval $[-1, 1]$. We refer to [87] for the motivation of these specific source terms.

(iii) Using linear phenomenological laws for chemical reactions, in [101] it was suggested to take

$$\Gamma_\varphi = \Gamma_\sigma = P(\varphi)(N_\sigma - \mu) \tag{3.32}$$

for a non-negative proliferation function $P(\cdot)$. These kind of source terms have, e. g., been studied in [39, 74]. In [101] it has been proposed to take

$$P(\varphi) = \begin{cases} \delta P_0(1+\varphi) & \varphi \geq -1, \\ 0 & \text{else} \end{cases}$$

for positive constants δ and P_0, where δ is usually very small. In contrast, the authors in [103] considered a proliferation function given by

$$P(\varphi) = \begin{cases} 2\varepsilon^{-1} P_0 \sqrt{\psi(\varphi)} & \varphi \in [-1,1], \\ 0 & \text{else.} \end{cases}$$

(iv) Taking $\Gamma_1 = 0$ and $\Gamma = \Gamma_2$ one obtains

$$\Gamma_\varphi = \Gamma_{\mathbf{v}} = \frac{1}{\bar\rho_2}\Gamma.$$

This choice will be of importance when deriving the formal asymptotic sharp interface limit for a mobility of the form $m(\varphi) = m_0\varepsilon$ with a positive constant m_0, where source terms of the form (3.29) with Γ as in (3.31) do not fulfil a corresponding compatibility condition.

3.2.2 Pressure reformulations

We consider different reformulations of the pressure leading to some variants of equation (3.28b).

(i) We first redefine the pressure as $q := p + \frac{\beta}{\varepsilon}\psi(\varphi) + \frac{\beta\varepsilon}{2}|\nabla\varphi|^2$ and use (3.28d) to obtain

$$\nabla q = \nabla p + (\mu - N_\varphi)\nabla\varphi + \beta\varepsilon \mathrm{div}(\nabla\varphi \otimes \nabla\varphi).$$

Hence, (3.28b) can be rewritten as

$$-\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I}) + \nu(\varphi)\mathbf{v} + \nabla q = (\mu - N_\varphi)\nabla\varphi. \tag{3.33a}$$

(ii) Defining $r := p + e = p + \frac{\beta}{\varepsilon}\psi(\varphi) + \frac{\beta\varepsilon}{2}|\nabla\varphi|^2 + N(\varphi,\sigma)$ yields

$$\nabla r = \nabla p + \mu\nabla\varphi + N_\sigma\nabla\sigma + \beta\varepsilon\mathrm{div}(\nabla\varphi\otimes\nabla\varphi).$$

Hence, we can reformulate (3.28b) by

$$-\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I}) + \nu(\varphi)\mathbf{v} + \nabla r = \mu\nabla\varphi + N_\sigma\nabla\sigma. \tag{3.33b}$$

(iii) If we choose $\tilde p := p + e - \mu\varphi - N_\sigma\sigma$ we get (see (ii))

$$\nabla\tilde p = \nabla p + \beta\varepsilon\mathrm{div}(\nabla\varphi\otimes\nabla\varphi) - \varphi\nabla\mu - \sigma\nabla N_\sigma,$$

and (3.28b) transforms to

$$-\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I}) + \nu(\varphi)\mathbf{v} + \nabla\tilde p = -\varphi\nabla\mu - \sigma\nabla N_\sigma. \tag{3.33c}$$

(iv) Choosing $\tilde q := p + e - \mu\varphi$, we obtain

$$\nabla\tilde q = \nabla p + N_\sigma\nabla\sigma - \varphi\nabla\mu + \beta\varepsilon\mathrm{div}(\nabla\varphi\otimes\nabla\varphi),$$

and consequently

$$-\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I}) + \nu(\varphi)\mathbf{v} + \nabla\tilde q = -\varphi\nabla\mu + N_\sigma\nabla\sigma. \tag{3.33d}$$

3.2.3 A general energy inequality:

We now deduce an energy inequality for (3.28) with the pressure as defined in (ii). Furthermore, we define the viscous stress tensor by

$$\mathbf{T}(\mathbf{v},p) := 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}.$$

Then, the system under consideration is given by

$$\begin{aligned} \mathrm{div}(\mathbf{v}) &= \Gamma_{\mathbf{v}} && \text{in } \Omega_T, && (3.34a)\\ -\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I}) + \nu(\varphi)\mathbf{v} + \nabla p &= \mu\nabla\varphi + N_\sigma\nabla\sigma && \text{in } \Omega_T, && (3.34b)\\ \partial_t\varphi + \mathrm{div}(\varphi\mathbf{v}) &= \mathrm{div}(m(\varphi)\nabla\mu) + \Gamma_\varphi && \text{in } \Omega_T, && (3.34c)\\ \mu &= \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi + N_\varphi && \text{in } \Omega_T, && (3.34d)\\ \partial_t\sigma + \mathrm{div}(\sigma\mathbf{v}) &= \mathrm{div}(n(\varphi)\nabla N_\sigma) - \Gamma_\sigma && \text{in } \Omega_T. && (3.34e) \end{aligned}$$

From now on we assume that there exists a solution to this system which is regular enough to carry out all the calculations. Multiplying (3.34c) with μ, (3.34d) with $-\partial_t\varphi$ and (3.34e) with N_σ, integrating over Ω and by by parts, we obtain

$$\begin{aligned} &\int_\Omega \partial_t\varphi\mu + \mu\nabla\varphi\cdot\mathbf{v} + \varphi\Gamma_{\mathbf{v}}\mu + m(\varphi)|\nabla\mu|^2 - \Gamma_\varphi\mu \,\mathrm{d}x - \int_{\partial\Omega} m(\varphi)\mu\nabla\mu\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = 0,\\ &-\int_\Omega \partial_t\varphi(\mu - N_\varphi)\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{\beta}{\varepsilon}\psi(\varphi) + \tfrac{\beta\varepsilon}{2}|\nabla\varphi|^2\,\mathrm{d}x - \int_{\partial\Omega}\beta\varepsilon\partial_t\varphi\nabla\varphi\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = 0,\\ &\int_\Omega \partial_t\sigma N_\sigma + N_\sigma\nabla\sigma\cdot\mathbf{v} + N_\sigma\sigma\Gamma_{\mathbf{v}} + n(\varphi)|\nabla N_\sigma|^2 + \Gamma_\sigma N_\sigma\,\mathrm{d}x - \int_{\partial\Omega} n(\varphi)N_\sigma\nabla N_\sigma\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = 0. \end{aligned}$$

Moreover, we multiply (3.34b) with $\mathbf{v}$ and integrate over Ω and by parts to get

$$\begin{aligned} &\int_\Omega 2\eta(\varphi)|\mathbf{D}\mathbf{v}|^2 + \lambda(\varphi)(\mathrm{div}(\mathbf{v}))^2 + \nu(\varphi)|\mathbf{v}|^2 - p\Gamma_{\mathbf{v}} - (\mu\nabla\varphi + N_\sigma\nabla\sigma)\cdot\mathbf{v}\,\mathrm{d}x\\ &\quad - \int_{\partial\Omega}\mathbf{T}(\mathbf{v},p)\mathbf{n}\cdot\mathbf{v}\,\mathrm{d}\mathcal{H}^{d-1} = 0, \end{aligned}$$

where we applied the identities $\mathbf{D}\mathbf{v} : \nabla\mathbf{v} = \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{v}$ and $\mathrm{div}(\mathbf{v})\mathbf{I} : \nabla\mathbf{v} = (\mathrm{div}(\mathbf{v}))^2$. Summing up the last four equations we obtain the energy identity

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{\beta}{\varepsilon}\psi(\varphi) + \tfrac{\beta\varepsilon}{2}|\nabla\varphi|^2 + N(\varphi,\sigma)\,\mathrm{d}x + \int_\Omega m(\varphi)|\nabla\mu|^2 + n(\varphi)|\nabla N_\sigma|^2\,\mathrm{d}x\\ &\quad + \int_\Omega 2\eta(\varphi)|\mathbf{D}\mathbf{v}|^2 + \lambda(\varphi)(\mathrm{div}(\mathbf{v}))^2 + \nu(\varphi)|\mathbf{v}|^2\,\mathrm{d}x - \int_\Omega \Gamma_\varphi\mu - \Gamma_\sigma N_\sigma\,\mathrm{d}x\\ &\quad + \int_\Omega (\mu\varphi + N_\sigma\sigma - p)\,\Gamma_{\mathbf{v}}\,\mathrm{d}x - \int_{\partial\Omega} m(\varphi)\mu\nabla\mu\cdot\mathbf{n} + n(\varphi)N_\sigma\nabla N_\sigma\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad - \int_{\partial\Omega}\beta\varepsilon\partial_t\varphi\nabla\varphi\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} - \int_{\partial\Omega}\mathbf{T}(\mathbf{v},p)\mathbf{n}\cdot\mathbf{v}\,\mathrm{d}\mathcal{H}^{d-1} = 0. \qquad (3.35) \end{aligned}$$

3.2.4 Boundary and initial conditions

We prescribe homogeneous Neumann boundary conditions for the phase field variable, the chemical potential and the stress tensor, i. e.,

$$\begin{aligned} \nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} &= 0 && \text{a. e. on } \Sigma_T, && (3.36a)\\ \mathbf{T}(\mathbf{v},p)\mathbf{n} &= \mathbf{0} && \text{a. e. on } \Sigma_T. && (3.36b) \end{aligned}$$

For the nutrient we may prescribe Robin-type boundary conditions of the form

$$n(\varphi)\nabla N_\sigma \cdot \mathbf{n} = K(\sigma_\infty - \sigma) \qquad \text{a. e. on } \Sigma_T \tag{3.36c}$$

for a constant $K \geq 0$ referred to as the boundary permeability and σ_∞ denoting a given nutrient supply at the boundary. We may see σ_∞ as a far-field nutrient level outside of Ω, and recalling (3.27d) we can rewrite (3.36c) as

$$\mathbf{J}_\sigma \cdot \mathbf{n} = K(\sigma - \sigma_\infty).$$

Thus, we see that there is nutrient outflow if $\sigma > \sigma_\infty$, i. e., the nutrient concentration on the boundary is higher than the far-field nutrient level, and inflow if $\sigma_\infty > \sigma$. The rate of inflow or outflow depends on the boundary permeability K. Finally, we impose the initial conditions

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{a. e. in } \Omega \tag{3.36d}$$

with prescribed functions φ_0, σ_0. The Robin boundary condition (3.36c) can be interpreted as an interpolation between Neumann and Dirichlet boundary conditions. Indeed, the case $K = 0$, that means no boundary permeability, corresponds to the Neumann type boundary condition

$$n(\varphi)\nabla N_\sigma \cdot \mathbf{n} = 0 \quad \text{a. e. on } \Sigma_T,$$

whereas formally sending $K \to \infty$ gives a Dirichlet boundary condition of the form

$$\sigma = \sigma_\infty \quad \text{a. e. on } \Sigma_T.$$

Finally, we remark that (3.36a)-(3.36c) are chosen in such a way that the boundary terms in (3.35) simplify to

$$K \int_{\partial\Omega} N_\sigma(\sigma_\infty - \sigma) \, \mathrm{d}\mathcal{H}^{d-1}.$$

This can also be realised with no-slip boundary conditions for the velocity $\mathbf{v}$.

3.2.5 Specific form of the nutrient energy

For the rest of the thesis we consider a nutrient energy density of the form

$$N(\varphi, \sigma) := \frac{\chi_\sigma}{2}|\sigma|^2 + \chi_\varphi \sigma(1 - \varphi) \tag{3.37}$$

for positive constants χ_σ and χ_φ referred to as the nutrient diffusion and chemotaxis parameter, respectively.

The first term characterises energy effects due to the presence of the nutrient, i. e., a high concentration of nutrients leads to a high energy of the system. The second term accounts for chemotaxis effects, i. e., tumour cells move towards regions of high nutrient concentration. We refer to [101] for more details regarding this form of the nutrient energy. Using (3.37) we compute

$$N_\sigma = \chi_\sigma \sigma + \chi_\varphi(1 - \varphi), \qquad N_\varphi = -\chi_\varphi \sigma.$$

Therefore, the fluxes $\mathbf{J}_\varphi$ and $\mathbf{J}_\sigma$ are given by

$$\mathbf{J}_\varphi = -m(\varphi)\nabla\left(\tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma\right), \qquad \mathbf{J}_\sigma = -n(\varphi)\nabla\left(\chi_\sigma\sigma - \chi_\varphi\varphi\right).$$

There are two non-standard contributions in the definition of $\mathbf{J}_\varphi$ and $\mathbf{J}_\sigma$. The term $m(\varphi)\nabla(\chi_\varphi\sigma)$ drives the tumour cells towards regions of high nutrient concentrations and is referred to as chemotaxis.

Moreover, we encounter a term of the form $n(\varphi)\nabla(\chi_\varphi \varphi)$ driving the nutrients towards regions with higher tumour concentrations. This effect is called active transport and seems to be counter-intuitive at first glance. However, it can be observed for malign tumours in, e. g., the avascular growth phase. Indeed, to overcome nutrient limitations, some tumours express more glucose transporters to provide an increasing glucose transport through the cell membrane. We remark that this term is only active on the interface and we refer to [87] for more details.

In general we can decouple chemotaxis and active transport mechanisms by introducing for $\lambda > 0$ a new mobility

$$\mathcal{D}(\varphi) \coloneqq \lambda^{-1} n(\varphi)\chi_\varphi, \qquad \chi_\sigma \coloneqq \lambda^{-1}\chi_\varphi. \tag{3.38}$$

Then, the fluxes can be rewritten as

$$\mathbf{J}_\varphi = -m(\varphi)\nabla\left(\tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma\right), \qquad \mathbf{J}_\sigma = -\mathcal{D}(\varphi)\nabla\left(\sigma - \lambda\varphi\right). \tag{3.39}$$

By formally sending $\lambda \to 0$ we can switch of active transport while preserving the chemotaxis mechanism.

3.2.6 Non-dimensionalisation arguments

The nutrient equation

In Chapter 5 we will consider a model variant of (3.28) where the nutrient is assumed to evolve quasi-statically meaning that the time derivative of σ does not appear in (3.34e) and the nutrient evolution is driven by the tumour evolution. This can be motivated using a non-dimensionalisation argument. Recalling the decoupling of chemotaxis and active transport mechanisms and assuming for simplicity that $\mathcal{D}(\varphi) = \mathcal{D}_\sigma$ where $\mathcal{D}_\sigma$ is a positive nutrient diffusion constant, we have the following equation describing the evolution of nutrient

$$\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) = \mathcal{D}_\sigma \operatorname{div}(\nabla\sigma - \lambda\nabla\varphi) - \mathcal{C}h(\varphi)\sigma, \tag{3.40}$$

where λ and $\mathcal{C}$ denote the active transport and nutrient consumption rate, respectively. Furthermore, $h(\cdot)$ is an interpolation function satisfying $h(-1) = 0$ and $h(1) = 1$.

We now introduce the rescaled quantities

$$\sigma' = \frac{\sigma}{\sigma_\infty}, \quad x' = \frac{x}{L}, \quad t' = \frac{t}{T}, \quad \mathbf{v}' = \frac{\mathbf{v}}{\bar{\mathbf{v}}},$$

where σ_∞ is the characteristic nutrient concentration. The quantities L and T are the characteristic length and time scales determining the characteristic velocity given by

$$\bar{\mathbf{v}} = \frac{L}{T}.$$

Computing all the quantities in (3.40) in the new variables x' and t' yields

$$\frac{\sigma_\infty}{T}\left(\partial_{t'}\sigma' + \operatorname{div}_{x'}(\sigma'\mathbf{v}')\right) = \frac{\sigma_\infty\mathcal{D}_\sigma}{L^2}\Delta_{x'}\sigma' - \frac{\mathcal{D}_\sigma\lambda}{L^2}\Delta_{x'}\varphi' - \mathcal{C}\sigma_\infty h(\varphi')\sigma'. \tag{3.41}$$

Then, we can define the time scales for diffusion, active transport and consumption by

$$T_{\mathcal{D}} \coloneqq \frac{L^2}{\mathcal{D}_\sigma}, \qquad T_\lambda \coloneqq \frac{L^2\sigma_\infty}{\mathcal{D}_\sigma\lambda}, \qquad T_{\mathcal{C}} \coloneqq \frac{1}{\mathcal{C}}.$$

Since we are interested in the evolution of the tumour, the time scale of interest is the tumour doubling time scale denoted by T_{TD}. Choosing $T = T_{TD}$ and dropping the primes in (3.41) we obtain

$$\frac{T_{\mathcal{D}}}{T_{TD}}\left(\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v})\right) = \Delta\sigma - \frac{T_{\mathcal{D}}}{T_\lambda}\Delta\varphi - \frac{T_{\mathcal{D}}}{T_{\mathcal{C}}}h(\varphi)\sigma.$$

Since experimental values indicate that the tumour doubling timescale is much longer than nutrient diffusion timescale (days compared to minutes or seconds), see, e. g., [29], we have that $\frac{T_{\mathcal{D}}}{T_{TD}} \ll 1$. Therefore, it is reasonable to replace the last equation by

$$0 = \Delta\sigma - \theta\Delta\varphi - \alpha h(\varphi)\sigma, \tag{3.42}$$

where θ denotes the ratio between nutrient diffusion and active transport timescale, whereas α is the ratio between nutrient diffusion and consumption timescale. In some situations it might be reasonable to assume that $T_{\mathcal{D}}$ and T_λ are of the same order, see, e. g., [82]. However, there might be situations where $T_{\mathcal{D}} \ll T_\lambda$, and thus the active transport term in (3.42) can be neglected. Moreover, formally sending $\lambda \to 0$ we see that $\theta = \frac{T_{\mathcal{D}}}{T_\lambda} = \frac{\lambda}{\sigma_\infty} \to 0$ and therefore (3.42) reads as

$$0 = \Delta\sigma - \alpha h(\varphi)\sigma.$$

Brinkman's equation

In the following we analyse the Brinkman equation (3.34b) via a non-dimensionalisation argument. For simplicity we set $\Gamma_{\mathbf{v}} = 0$ in (3.34a) and we assume that the viscosities and porosity are constant. Then, (3.34b) reduces to

$$-\eta\Delta\mathbf{v} + \nu\mathbf{v} + \nabla p = \mu\nabla\varphi + N_\sigma\nabla\sigma.$$

We now introduce a new scaling $x' = \frac{x}{L}$ where L is the characteristic length. Calculating all the quantities with respect to x', dropping the primes and multiplying the resulting equations by L, we obtain

$$-\frac{\eta}{L}\Delta\mathbf{v} + L\nu\mathbf{v} + \nabla p = \mu\nabla\varphi + N_\sigma\nabla\sigma.$$

This allows us to make the following observations: On small length scales, i. e., $L \ll 1$, Brinkman's equation approximates Stokes flow, whereas on larger length scales, i. e., $L \gg 1$, it is an approximation of Darcy's law, see also [53].

3.3 Formally matched asymptotics

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain. In the following we formally derive the sharp interface limit of the system

$$\operatorname{div}(\mathbf{v}) = \bar{\rho}_2^{-1}\Gamma_2(\varphi,\sigma,\mu) + \bar{\rho}_1^{-1}\Gamma_1(\varphi,\sigma,\mu), \tag{3.43a}$$

$$-\operatorname{div}(\mathbf{T}(\mathbf{v},p,\varphi)) + \nu(\varphi)\mathbf{v} = (\mu + \chi_\varphi\sigma)\nabla\varphi, \tag{3.43b}$$

$$\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = \operatorname{div}(m(\varphi)\nabla\mu) + \bar{\rho}_2^{-1}\Gamma_2(\varphi,\sigma,\mu) - \bar{\rho}_1^{-1}\Gamma_1(\varphi,\sigma,\mu), \tag{3.43c}$$

$$\mu = \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma, \tag{3.43d}$$

$$\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) = \operatorname{div}(n(\varphi)(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi)) - \Gamma_\sigma(\varphi,\sigma,\mu), \tag{3.43e}$$

where

$$\mathbf{T}(\mathbf{v},p,\varphi) := 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}.$$

We will focus on the double-well potential given by

$$\psi(\varphi) = \frac{1}{4}(1-\varphi^2)^2, \tag{3.44}$$

and satisfying

$$\psi'(\varphi) = \varphi^3 - \varphi, \quad \psi''(\varphi) = 3\varphi^2 - 1.$$

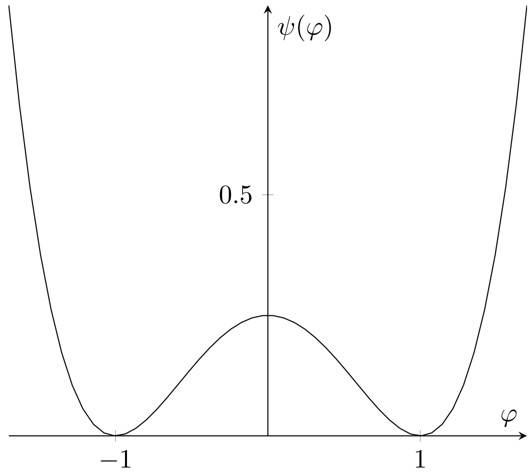

Figure 3.1: Plot of the double-well potential.

We plot the double-well potential in Figure 3.1.

Moreover, we assume that $\eta(\cdot)$, $\lambda(\cdot)$, $\nu(\cdot)$ are smooth with $\eta(\cdot)$, $\nu(\cdot)$ positive such that $\eta(-1) = \eta_1$, $\eta(1) = \eta_2$, $\nu(-1) = \nu_1$, $\nu(1) = \nu_2$, and $\lambda(\cdot)$ non-negative such that $\lambda(-1) = \lambda_1$, $\lambda(1) = \lambda_2$. For the mobility $m(\cdot)$ we consider the following three cases:

$$m(\varphi) = \begin{cases} m_0 & \text{Case (i)}, \\ \varepsilon m_0 & \text{Case (ii)}, \\ \frac{m_1}{2}(1+\varphi)^2 & \text{Case (iii)}. \end{cases} \tag{3.45}$$

3.3.1 Outer Expansion

Assumptions

We make the following assumptions (compare [87]).

(i) For any $\varepsilon > 0$ small enough there exists a family $(\varphi_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)_{\varepsilon>0}$ of solutions to (3.43a)-(3.43e) which are sufficiently smooth.

(ii) We assume that
$$\Sigma(\varepsilon) := \{(t,x) \in [0,T] \times \Omega \colon \varphi_\varepsilon(t,x) = 0\}$$
are evolving hypersurfaces (see, e. g., [14, Def. 23]) that do not intersect with $\partial\Omega$ and we define
$$\Sigma(\varepsilon, t) := \{x \in \Omega \colon \varphi_\varepsilon(t,x) = 0\}.$$
We assume that for every $\varepsilon > 0$ small enough and for each time $t \in [0,T]$ the domain Ω can be divided into two open subdomains
$$\Omega_+(\varepsilon,t) := \{x \in \Omega \colon \varphi_\varepsilon(t,x) > 0\}, \quad \Omega_-(\varepsilon,t) := \{x \in \Omega \colon \varphi_\varepsilon(t,x) < 0\}$$
separated by $\Sigma(\varepsilon,t)$ such that $\Omega_+(t,\varepsilon)$ is enclosed by $\Sigma(\varepsilon,t)$. Thus, for all $\varepsilon > 0$ small enough and all $t \in [0,T]$ it holds that
$$\Omega = \Omega_+(\varepsilon,t) \cup \Sigma(\varepsilon,t) \cup \Omega_-(\varepsilon,t), \quad \Sigma(\varepsilon,t) = \partial\Omega_+(\varepsilon,t), \quad \Omega_+(\varepsilon,t) = \Omega \setminus \overline{\Omega_-(\varepsilon,t)}.$$
We show a sketch of the typical situation in Figure 3.2.

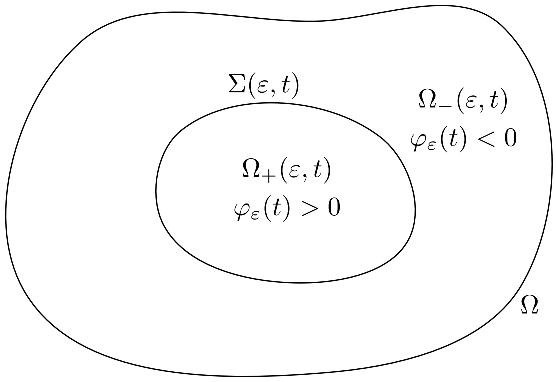

Figure 3.2: Typical situation for the formal asymptotic analysis.

(iii) We assume that $(\varphi_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon)_{\varepsilon>0}$ have an asymptotic expansion in ε in the bulk regions away from $\Sigma(\varepsilon)$ (**outer expansion**), and another expansion in the interfacial region close to $\Sigma(\varepsilon)$ (**inner expansion**).

(iv) The zero level sets of φ_ε depend smoothly on t and ε and converge as $\varepsilon \to 0$ to a limiting evolving hypersurface $\Sigma(0)$ which evolves with normal velocity $\mathcal{V}$.

We use the notation $(3.43\text{d})_O^a$ and $(3.43\text{d})_I^a$ for the terms resulting from the order a outer and inner expansions of (3.43d), respectively.

Expansion to leading order

We assume that $f_\varepsilon \in \{\varphi_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon, \mu_\varepsilon, \sigma_\varepsilon\}$ can be expanded by

$$f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots .$$

Then, to leading order $(3.43\text{d})_O^{-1}$ yields

$$-\beta\psi'(\varphi_0) = 0. \tag{3.46}$$

Stable solutions of (3.46) are the minima of $\psi(\cdot)$ and they are given by $\varphi_0 = \pm 1$. Consequently, we define

$$\Omega_T := \{x \in \Omega \colon \varphi_0(x) = 1\}, \quad \Omega_H := \{x \in \Omega \colon \varphi_0(x) = -1\}. \tag{3.47}$$

The typical situation for Ω_T and Ω_H is shown in Figure 3.3.

Since $\nabla\varphi_0 = \mathbf{0}$, $\partial_t\varphi_0 = 0$ in Ω_T and Ω_H, we obtain for the equations to zeroth order that

$$\operatorname{div}(\mathbf{v}_0) = \tfrac{1}{\bar\rho_2}\Gamma_2(\varphi_0,\sigma_0,\mu_0) + \tfrac{1}{\bar\rho_1}\Gamma_1(\varphi_0,\sigma_0,\mu_0), \tag{3.48a}$$

$$-\operatorname{div}(\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0)) + \nu(\varphi_0)\mathbf{v}_0 = 0, \tag{3.48b}$$

$$-\operatorname{div}(m(\varphi_0)\nabla\mu_0) = \tfrac{1}{\bar\rho_2}\Gamma_2(\varphi_0,\sigma_0,\mu_0)(1-\varphi_0) - \tfrac{1}{\bar\rho_1}\Gamma_1(\varphi_0,\sigma_0,\mu_0)(1+\varphi_0), \tag{3.48c}$$

$$\partial_t\sigma_0 + \operatorname{div}(\sigma_0\mathbf{v}_0) = \operatorname{div}(n(\varphi_0)\chi_\sigma\nabla\sigma_0) + \Gamma_\sigma(\varphi_0,\sigma_0,\mu_0), \tag{3.48d}$$

where

$$\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0) = 2\eta(\varphi_0)\mathbf{D}\mathbf{v}_0 + \lambda(\varphi_0)\operatorname{div}(\mathbf{v}_0)\mathbf{I} - p\mathbf{I}.$$

We now analyse the three different cases for (3.43c) according to the mobilities introduced in (3.45).

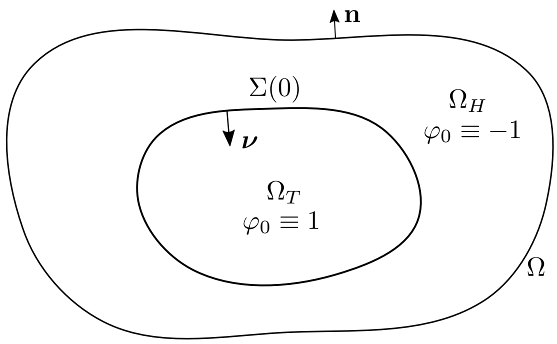

Figure 3.3: The tumour and healthy regions Ω_T and Ω_H.

Case (i) ($m(\varphi) = m_0$): In this case we obtain

$$-m_0\Delta\mu_0 = \bar{\rho}_2^{-1}\Gamma_2(\varphi_0,\sigma_0,\mu_0)(1-\varphi_0) - \bar{\rho}_1^{-1}\Gamma_1(\varphi_0,\sigma_0,\mu_0)(1+\varphi_0). \tag{3.49a}$$

Case (ii) ($m(\varphi) = \varepsilon m_0$): The mobility is rescaled and the chemical potential does not contribute to the equations at zeroth order. Indeed, we have

$$\bar{\rho}_2^{-1}\Gamma_2(\varphi_0,\sigma_0,\mu_0)(1-\varphi_0) = \bar{\rho}_1^{-1}\Gamma_1(\varphi_0,\sigma_0,\mu_0)(1+\varphi_0). \tag{3.49b}$$

Case (iii) ($m(\varphi) = \frac{m_1}{2}(1+\varphi)^2$): The degenerate mobility case leads to

$$-\operatorname{div}(\tfrac{m_1}{2}(1+\varphi_0)^2\,\nabla\mu_0) = \bar{\rho}_2^{-1}\Gamma_2(\varphi_0,\sigma_0,\mu_0)(1-\varphi_0) - \bar{\rho}_1^{-1}\Gamma_1(\varphi_0,\sigma_0,\mu_0)(1+\varphi_0). \tag{3.49c}$$

Remark 3.1 (i) In order to fulfil (3.49b) we have to assume that

$$\Gamma_1(1,\sigma_0,\mu_0) = 0 \quad \text{and} \quad \Gamma_2(-1,\sigma_0,\mu_0) = 0. \tag{3.50}$$

Furthermore, we observe that for general source terms the chemical potential μ_0 appears on the r. h. s. of (3.48a) although the bulk equations for μ_0 remain undetermined. Therefore, it is reasonable to assume that the source terms are either independent of μ, i. e.,

$$\Gamma_1 = \Gamma_1(\varphi,\sigma), \quad \Gamma_2 = \Gamma_2(\varphi,\sigma), \tag{3.51}$$

or we may ask for

$$\Gamma_1(\pm 1,\sigma,\mu) = 0, \quad \Gamma_2(\pm 1,\sigma,\mu) = 0. \tag{3.52}$$

To fulfil (3.50) and (3.51) we could choose

$$\Gamma_1 \equiv 0, \quad \Gamma_2(\varphi,\sigma) := \frac{\bar{\rho}_2}{2}\left(\frac{1}{\bar{\rho}_2} - \frac{1}{\bar{\rho}_1}\right)(\mathcal{P}\sigma - \mathcal{A})(1+\varphi),$$

where $\mathcal{P}$ and $\mathcal{A}$ are non-negative constants related to proliferation and apoptosis, respectively. In this case the source terms in (3.43a), (3.43c) coincide and are of the form

$$\Gamma_\varphi(\varphi,\sigma) = \Gamma_{\mathbf{v}}(\varphi,\sigma) = \frac{\alpha}{2}(\mathcal{P}\sigma - \mathcal{A})(1+\varphi),$$

where

$$\alpha := \frac{1}{\bar{\rho}_2} - \frac{1}{\bar{\rho}_2}.$$

Equation (3.50) can be interpreted as follows:

- in the pure tumour phases, there can be no growth of healthy cells,
- in regions of unmixed healthy tissue, there is no spontaneous growth of tumour cells.

In a situation where we assume no gain or loss of mass locally, i. e., $\Gamma_2 = -\Gamma_1$, condition (3.50) implies that

$$\Gamma_1(\pm 1, \sigma_0, \mu_0) = \Gamma_2(\pm 1, \sigma_0, \mu_0) = 0$$

which coincides with (3.52). Hence, death and growth are restricted to the interfacial region and we may choose, for example,

$$\Gamma_1(\varphi, \sigma, \mu) = \gamma_1(\varphi, \sigma, \mu)(1 - \varphi^2)_+$$

for a function γ_1 to be specified. Alternatively we could use phenomenological laws to describe growth and death by choosing

$$\Gamma_2 = -\Gamma_1 = P_1(\varphi)(\chi_\sigma \sigma + \chi_\varphi(1 - \varphi) - \mu),$$

where $P_1(\cdot)$ is a proliferation function satisfying $P_1(\pm 1) = 0$. For instance, we could take $P_1(\varphi) = \frac{1}{4}(1 - \varphi^2)^2$.

(ii) In the healthy region (3.49c) simplifies to

$$0 = 2\bar{\rho}_2^{-1}\Gamma_2(-1, \sigma_0, \mu_0).$$

This is a compatibility for the source term Γ_2. For similar reasons as before we can assume that either the source terms are independent of μ or

$$\Gamma_1(-1, \sigma, \mu) = \Gamma_2(-1, \sigma, \mu) = 0.$$

Reasonable choices are

$$\Gamma_2(\varphi, \sigma) = \gamma_2(\varphi, \sigma)(1 + \varphi)_+$$

for some function γ_2, or

$$\frac{\Gamma_2}{\bar{\rho}_2} = -\frac{\Gamma_1}{\bar{\rho}_1} = P_2(\varphi)(\chi_\sigma \sigma + \chi_\varphi(1 - \varphi) - \mu),$$

where $P_2(\varphi) = p_0(1 + \varphi)_+$. This can be interpreted as a scaled zero excess of total mass and we have

$$\Gamma_\varphi = 2P_2(\varphi)(\chi_\sigma \sigma + \chi_\varphi(1 - \varphi) - \mu), \quad \Gamma_{\mathbf{v}} = 0.$$

If the mobility was degenerate in both phases we would obtain the same condition as in (3.50).

(iii) Similar conditions have to hold for the source term Γ_σ. From now on we assume that the source terms are independent of μ.

3.3.2 Inner Expansion

New Coordinates and matching conditions

This subsection uses ideas presented in [3] and [89]. We denote by $\Sigma(0)$ the smooth evolving interface which is assumed to be the limit of the zero level sets of φ_ε as $\varepsilon \to 0$ (see, e. g., [89] for details). We now introduce new coordinates in a neighbourhood of $\Sigma(0)$. To this end, we choose a time interval $I \subset \mathbb{R}$ and a spatial parameter domain $U \subset \mathbb{R}^{d-1}$ and we define a local parametrisation of $\Sigma(0)$ by

$$\gamma \colon I \times U \to \mathbb{R}^d.$$

By $\boldsymbol{\nu}$ we denote the unit normal to $\Sigma(0)$ pointing into the tumour region. Close to $\gamma(I \times U)$ we consider the signed distance function $d(t,x)$ of a point x to $\Sigma(0,t)$ with $d(t,x) > 0$ if $x \in \Omega_T$ and $d(t,x) < 0$ if $x \in \Omega_H$. We introduce a local parametrisation of $I \times \mathbb{R}^d$ near $\gamma(I \times U)$ using the rescaled distance $z = \frac{d}{\varepsilon}$ by

$$G^\varepsilon(t,s,z) \coloneqq (t, \gamma(t,s) + \varepsilon z \boldsymbol{\nu}(t,s))$$

with $s \in U \subset \mathbb{R}^{d-1}$. We show a sketch of the situation in Figure 3.4.

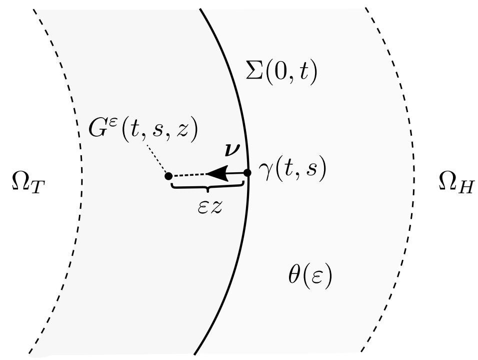

Figure 3.4: Schematic sketch of the inner region close to $\Sigma(0)$.

The (scalar) normal velocity is given by

$$\mathcal{V} = \partial_t \gamma \cdot \boldsymbol{\nu},$$

and we observe that $(G^\varepsilon)^{-1}(t,x) =: (t,s,z)(t,x)$ fulfils

$$\partial_t z = \frac{1}{\varepsilon}\partial_t d = -\frac{1}{\varepsilon}\mathcal{V}.$$

In particular, it holds that $\boldsymbol{\nu}(t,x) = \nabla d(t,x)$ on $\Sigma(0,t)$.

Let $b(t,x)$ be a scalar function and define $B(t, s(t,x), z(t,x)) = b(t,x)$. Then, in the new coordinate system we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t,x) = \partial_t B + \partial_z B \partial_t z + \nabla B \cdot \partial_t s = -\frac{1}{\varepsilon}\mathcal{V}\partial_z B + \text{h.o.t.}.$$

In the following, we will often suppress the dependence on t. For the gradient of b we have

$$\nabla_x b = \nabla_{\Sigma_{\varepsilon z}} B + \frac{1}{\varepsilon}\partial_z B \boldsymbol{\nu},$$

where $\nabla_{\Sigma_{\varepsilon z}}$ is the surface gradient on $\Sigma_{\varepsilon z} \coloneqq \{\gamma(s) + \varepsilon z \boldsymbol{\nu} \colon s \in U\}$.
For a vector quantity $\mathbf{j}(t,x) = \mathbf{J}(t, s(t,x), z(t,x))$ we obtain

$$\nabla_x \cdot \mathbf{j} = \frac{1}{\varepsilon}\partial_z \mathbf{J} \cdot \boldsymbol{\nu} + \operatorname{div}_{\Sigma_{\varepsilon z}} \mathbf{J}$$

with $\operatorname{div}_{\Sigma_{\varepsilon z}}$ being the surface divergence. Furthermore, it holds

$$\Delta_x b(t,x) = \frac{1}{\varepsilon^2}\partial_{zz} B - \frac{1}{\varepsilon}\kappa \partial_z B + \text{h.o.t.},$$

where κ is the mean curvature. In addition, we have

$$\begin{aligned}
\nabla_{\Sigma_{\varepsilon z}} B(s,z) &= \nabla_{\Sigma(0)} B(s,z) + \text{h.o.t.}, \\
\operatorname{div}_{\Sigma_{\varepsilon z}} \mathbf{J}(s,z) &= \operatorname{div}_{\Sigma(0)} \mathbf{J}(s,z) + \text{h.o.t.}, \\
\Delta_{\Sigma_{\varepsilon z}} B(s,z) &= \Delta_{\Sigma(0)} B(s,z) + \text{h.o.t.}.
\end{aligned}$$

Summarising all the identities deduced so far yields

$$\frac{\mathrm{d}}{\mathrm{d}t}b(s,z) = -\frac{1}{\varepsilon}\mathcal{V}\partial_z B + \text{h. o. t.}, \tag{3.53a}$$

$$\nabla_x b(s,z) = \frac{1}{\varepsilon}\partial_z B\boldsymbol{\nu} + \nabla_{\Sigma(0)}B + \text{h. o. t.}, \tag{3.53b}$$

$$\Delta_x b(s,z) = \frac{1}{\varepsilon^2}\partial_{zz}B - \frac{1}{\varepsilon}\kappa\partial_z B + \text{h. o. t.}, \tag{3.53c}$$

$$\mathrm{div}_x\mathbf{j} = \frac{1}{\varepsilon}\partial_z\mathbf{J}\cdot\boldsymbol{\nu} + \mathrm{div}_{\Sigma(0)}\mathbf{J} + \text{h. o. t.}. \tag{3.53d}$$

Using (3.53b)-(3.53c) component-wise we obtain

$$\nabla_x\mathbf{j} = \frac{1}{\varepsilon}\partial_z\mathbf{J}\otimes\boldsymbol{\nu} + \nabla_{\Sigma(0)}\mathbf{J} + \text{h. o. t.}, \tag{3.53e}$$

$$\Delta_x\mathbf{j} = \frac{1}{\varepsilon^2}\partial_{zz}\mathbf{J} - \frac{1}{\varepsilon}\kappa\partial_z\mathbf{J} + \text{h. o. t.}. \tag{3.53f}$$

We denote the variables φ_ε, μ_ε, σ_ε, $\mathbf{v}_\varepsilon$, p_ε, in the new coordinate system by Φ_ε, Ξ_ε, C_ε, $\mathbf{V}_\varepsilon$, P_ε, and we assume the following inner expansion

$$F_\varepsilon(s,z) = F_0(s,z) + \varepsilon F_1(s,z) + \varepsilon^2 F_2(s,z) + \dots$$

for $F_\varepsilon \in \{\Phi_\varepsilon, \Xi_\varepsilon, C_\varepsilon, \mathbf{V}_\varepsilon, P_\varepsilon\}$. The assumption that the zero level sets of φ_ε converge to $\Sigma(0)$ implies

$$\Phi_0(t,s,z=0) = 0.$$

We will employ the matching conditions (see [87])

$$\lim_{z\to\pm\infty} F_0(t,s,z) = f_0^{\pm}(t,x), \tag{3.54a}$$

$$\lim_{z\to\pm\infty} \partial_z F_0(t,s,z) = 0, \tag{3.54b}$$

$$\lim_{z\to\pm\infty} \partial_z F_1(t,s,z) = \nabla f_0^{\pm}(t,x)\cdot\boldsymbol{\nu}, \tag{3.54c}$$

where

$$f_0^{\pm}(t,x) := \lim_{\delta\searrow 0} f_0(t, x\pm\delta\boldsymbol{\nu}) \quad \text{for } x\in\Sigma(0,t). \tag{3.55}$$

Moreover, we introduce the notation

$$[f]_H^T := \lim_{\delta\searrow 0} f(t,x+\delta\boldsymbol{\nu}) - \lim_{\delta\searrow 0} f(t,x-\delta\boldsymbol{\nu}) \quad \text{for } x\in\Sigma(0,t) \tag{3.56}$$

to denote the jump of a quantity f across the interface.

Inner Expansion to leading order

Step 1: From $(3.43\text{d})_I^{-1}$ we obtain

$$\partial_{zz}\Phi_0 - \psi'(\Phi_0) = 0. \tag{3.57}$$

Since $\Phi_0(t,s,z=0) = 0$ we can choose Φ_0 independent of s and t, hence, Φ_0 solves

$$\Phi_0''(z) - \psi'(\Phi_0(z)) = 0, \quad \Phi_0(0) = 0, \quad \Phi_0(\pm\infty) = \pm 1, \tag{3.58}$$

where we used (3.54a). The unique solution of (3.58) is given by

$$\Phi_0(z) = \tanh\left(\frac{z}{\sqrt{2}}\right).$$

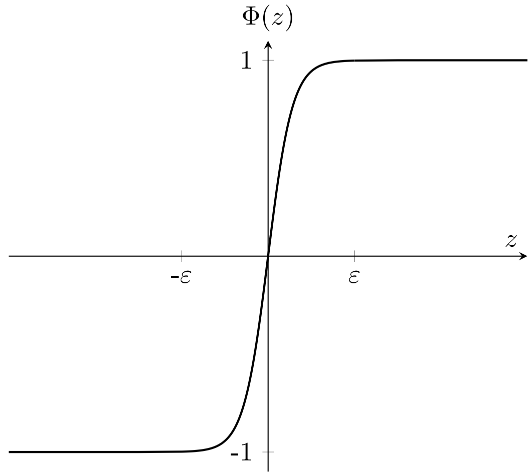

Figure 3.5: Plot of the optimal profile.

The optimal profile Φ_0 is shown in Figure 3.5.
Multiplying $(3.58)_1$ with $\Phi_0'(z)$ yields

$$\frac{1}{2}((\Phi_0'(z))^2)' = (\psi(\Phi_0(z)))' \quad \forall\,|z| < \infty.$$

Integrating from $-\infty$ to $\tilde{z}$ with $|\tilde{z}| < \infty$, using (3.54a)-(3.54b) and $\psi(-1) = 0$, we obtain

$$\frac{1}{2}|\Phi_0'(z)|^2 = \psi(\Phi_0(z)) \quad \forall\,|z| < \infty \tag{3.59}$$

which is referred to as the so-called equipartition of energy.
Step 2: From $(3.43\text{a})_I^{-1}$ we obtain (using (3.53d))

$$\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu} = 0. \tag{3.60}$$

Due to $\partial_z \boldsymbol{\nu} = 0$ this implies

$$\partial_z(\mathbf{V}_0 \cdot \boldsymbol{\nu}) = 0. \tag{3.61}$$

Integrating by parts, this gives

$$0 = \int_{-\infty}^{\infty} \partial_z(\mathbf{V}_0 \cdot \boldsymbol{\nu})\,\mathrm{d}z = [\mathbf{V}_0 \cdot \boldsymbol{\nu}]_{-\infty}^{\infty}.$$

Hence, the matching condition (3.54a) yields

$$[\mathbf{v}_0]_H^T \cdot \boldsymbol{\nu} := \mathbf{v}_0^+ \cdot \boldsymbol{\nu} - \mathbf{v}_0^- \cdot \boldsymbol{\nu} = 0. \tag{3.62}$$

Step 3: We now analyse (3.43c) by considering each term individually. First of all, on the l. h. s. of (3.43c) the leading order terms are of magnitude $\theta(\varepsilon^{-1})$ and given by

$$\begin{aligned}
\partial_t \varphi &= -\frac{1}{\varepsilon}\mathcal{V}\Phi_0' + \text{h. o. t.}, \\
\nabla_x \varphi \cdot \mathbf{v} &= \left(\frac{1}{\varepsilon}\partial_z \Phi \boldsymbol{\nu} + \nabla_{\Sigma(0)}\Phi + \text{h. o. t.}\right)(\mathbf{V}_0 + \text{h. o. t.}) = \frac{1}{\varepsilon}\partial_z \Phi \mathbf{V}_0 \cdot \boldsymbol{\nu} + \text{h. o. t.}, \\
\varphi\, \mathrm{div}_x(\mathbf{v}) &= (\Phi_0 + \text{h. o. t.})\left(\frac{1}{\varepsilon}\partial_z \mathbf{V} \cdot \boldsymbol{\nu} + \mathrm{div}_{\Sigma(0)}\mathbf{V} + \text{h. o. t.}\right) = \frac{1}{\varepsilon}\Phi_0 \partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu} + \text{h. o. t.}.
\end{aligned} \tag{3.63}$$

The terms $\bar{\rho}_2^{-1}\Gamma_2$ and $\bar{\rho}_1^{-1}\Gamma_1$ are of magnitude $\theta(1)$ at leading order and they therefore do not contribute. We distinguish again the three cases for the mobilities:

Case (i) $(m(\varphi) = m_0)$: Using (3.53c), at order $\theta(\varepsilon^{-2})$ we obtain

$$m_0\Delta_x\mu = m_0\partial_{zz}\Xi_0.$$

Recalling (3.63), from $(3.43\text{c})_I^{-2}$ we get

$$m_0\partial_{zz}\Xi_0 = 0.$$

Upon integrating and using the matching condition (3.54b) we obtain

$$\partial_z\Xi_0 = 0 \quad \forall\,|z| < \infty.$$

Integrating again from $-\infty$ to ∞ and using (3.54a), this yields

$$[\mu_0]_H^T = 0. \tag{3.64}$$

Case (ii) $(m(\varphi) = \varepsilon m_0)$: Using $\operatorname{div}(\varepsilon m_0\nabla\mu) = \varepsilon m_0\Delta\mu$ and (3.53c) we have

$$\operatorname{div}_x(\varepsilon m_0\nabla_x\mu) = \frac{1}{\varepsilon}\partial_z(m_0\partial_z\Xi_0) + \text{h.o.t.}.$$

In conjunction with (3.63) we therefore obtain from $(3.43\text{c})_I^{-1}$ that

$$-\mathcal{V}\Phi_0' + \partial_z(\Phi_0\mathbf{V}_0)\cdot\boldsymbol{\nu} = \partial_z(m_0\partial_z\Xi_0). \tag{3.65}$$

The identities $\mathcal{V} = \partial_t\gamma\cdot\boldsymbol{\nu}$, $\partial_z(\partial_t\gamma) = 0$ and $\partial_z\boldsymbol{\nu} = \mathbf{0}$ imply

$$\partial_z\mathcal{V} = \partial_z(\partial_t\gamma)\cdot\boldsymbol{\nu} + \partial_t\gamma\cdot\partial_z\boldsymbol{\nu} = 0.$$

Using $\partial_z\boldsymbol{\nu} = 0$ and (3.61), integrating by parts yields

$$\int_{-\infty}^{+\infty} -\mathcal{V}\Phi_0'(z) + \partial_z(\Phi_0\mathbf{V}_0)\cdot\boldsymbol{\nu}\,\mathrm{d}z = [(-\mathcal{V} + \mathbf{V}_0\cdot\boldsymbol{\nu})\Phi_0]_{-\infty}^{+\infty} = 2(-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu}).$$

Employing the matching condition (3.54b) gives

$$\int_{-\infty}^{+\infty} \partial_z(m_0\partial_z\Xi_0)\mathrm{d}z = \lim_{z\to+\infty}(m_0\partial_z\Xi_0(z)) - \lim_{z\to-\infty}(m_0\partial_z\Xi_0(z)) = 0.$$

Combining the last two identities with (3.65), we end up at

$$2(-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu}) = 0. \tag{3.66}$$

In particular, we obtain from (3.61)-(3.62) and (3.65) that

$$\partial_{zz}\Xi_0 = (-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu})\Phi_0' = 0,$$

which together with the matching condition (3.54b) implies that $\partial_z\Xi_0 = 0$ for all $|z| < \infty$. Hence, we obtain that Ξ_0 is independent of z.

Case (iii) $\left(m(\varphi) = \frac{m_1}{2}(1+\varphi)^2\right)$: With similar arguments as above we obtain from $(3.43\text{c})_I^{-2}$ that

$$\tfrac{m_1}{2}\partial_z((1+\Phi_0)^2\,\partial_z\Xi_0) = 0.$$

Integrating this inequality in time from $-\infty$ to z with $|z| < \infty$ and using the matching condition (3.54b) gives

$$\tfrac{m_1}{2}(1+\Phi_0)^2\partial_z\Xi_0(t,s,z) = 0 \quad \forall\,|z| < \infty.$$

Since $|\Phi_0(z)| < 1$ for $|z| < \infty$, this implies that

$$\partial_z \Xi_0(t,s,z) = 0 \quad \forall\,|z| < \infty, \tag{3.67}$$

and therefore Ξ_0 is independent of z.

Step 4: Using $\partial_z \boldsymbol{\nu} = \mathbf{0}$ and applying similar calculations as for (3.43c), from $(3.43e)_I^{-2}$ we obtain

$$\partial_z(n(\Phi_0)\chi_\sigma \partial_z C_0) - \partial_z(n(\Phi_0)\chi_\varphi \partial_z \Phi_0) = 0.$$

Integrating this identity from $-\infty$ to z with $|z| < \infty$ and using (3.54b) yields

$$n(\Phi_0)(\chi_\sigma \partial_z C_0 - \chi_\varphi \Phi_0'(z)) = 0 \quad \forall\,|z| < \infty.$$

Since $n(\Phi_0) > 0$, this means

$$\chi_\sigma \partial_z C_0(t,s,z) = \chi_\varphi \Phi_0'(z) \quad \forall\,|z| < \infty. \tag{3.68}$$

Upon integrating and using (3.54a) we see that

$$[\sigma_0]_H^T = [C_0(t,s,z)]_{-\infty}^{+\infty} = \int_{-\infty}^{\infty} \partial_z C_0(t,s,z)\,\mathrm{d}z = \frac{\chi_\varphi}{\chi_\sigma}\int_{-\infty}^{\infty} \Phi_0'(z)\,\mathrm{d}z = 2\frac{\chi_\varphi}{\chi_\sigma}. \tag{3.69}$$

Step 5: Finally, we analyse $(3.43b)_I^{-2}$. Using (3.53e) we obtain

$$\begin{aligned}
\nabla_x \mathbf{v} &= \nabla_{\Sigma(0)} \mathbf{V} + \frac{1}{\varepsilon}\partial_z \mathbf{V} \otimes \boldsymbol{\nu} + \text{h.o.t.}\,, \\
D_x \mathbf{v} &= \frac{1}{2}(\nabla_{\Sigma(0)} \mathbf{V} + (\nabla_{\Sigma(0)} \mathbf{V})^{\mathsf{T}}) + \frac{1}{2\varepsilon}(\partial_z \mathbf{V} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \partial_z \mathbf{V}) + \text{h.o.t.}\,.
\end{aligned}$$

We define $\mathcal{E}(A) = \frac{1}{2}(A + A^{\mathsf{T}})$ for a quadratic matrix and use the last two identities together with $\partial_z \boldsymbol{\nu} = 0$ to obtain

$$\begin{aligned}
\operatorname{div}_x(\eta(\varphi) D_x \mathbf{v}) &= \frac{1}{\varepsilon^2}\partial_z(\eta(\Phi)\mathcal{E}(\partial_z \mathbf{V} \otimes \boldsymbol{\nu}))\boldsymbol{\nu} + \frac{1}{\varepsilon}\partial_z(\eta(\Phi)\mathcal{E}(\nabla_{\Sigma(0)}\mathbf{V}))\boldsymbol{\nu} \\
&\quad + \frac{1}{\varepsilon}\operatorname{div}_{\Sigma(0)}(\eta(\Phi)\mathcal{E}(\partial_z \mathbf{V} \otimes \boldsymbol{\nu})) + \operatorname{div}_{\Sigma(0)}(\eta(\Phi)\mathcal{E}(\nabla_{\Sigma(0)}\mathbf{V})) \\
&= \frac{1}{\varepsilon^2}\partial_z(\eta(\Phi)\mathcal{E}\left(\partial_z \mathbf{V} \otimes \boldsymbol{\nu}\right)\boldsymbol{\nu}) + \frac{1}{\varepsilon}\partial_z(\eta(\Phi)\mathcal{E}\left(\nabla_{\Sigma(0)}\mathbf{V}\right)\boldsymbol{\nu}) \\
&\quad + \frac{1}{\varepsilon}\operatorname{div}_{\Sigma(0)}(\eta(\Phi)\mathcal{E}(\partial_z \mathbf{V} \otimes \boldsymbol{\nu})) + \operatorname{div}_{\Sigma(0)}(\eta(\Phi)\mathcal{E}(\nabla_{\Sigma(0)}\mathbf{V})) + \text{h.o.t.}\,.
\end{aligned} \tag{3.70}$$

Furthermore, using (3.53b) gives

$$\nabla_x p = \nabla_{\Sigma(0)} P + \frac{1}{\varepsilon}\partial_z P \boldsymbol{\nu} + \text{h.o.t.}\,. \tag{3.71}$$

For the forcing term we obtain

$$(\mu + \chi_\varphi \sigma)\nabla_x \varphi = (\Xi_0 + \chi_\varphi C_0 + \text{h.o.t.})\left(\nabla_{\Sigma(0)}\Phi + \frac{1}{\varepsilon}\partial_z \Phi + \text{h.o.t.}\right). \tag{3.72}$$

Now, with similar arguments as above and recalling (3.60) we obtain at order $\theta(\varepsilon^{-2})$

$$\begin{aligned}
\operatorname{div}_x(\lambda(\varphi)\operatorname{div}_x(\mathbf{v})\mathbf{I}) &= \lambda'(\varphi)\operatorname{div}_x(\mathbf{v})\mathbf{I}\,\nabla_x\varphi + \lambda(\varphi)\nabla_x(\operatorname{div}_x(\mathbf{v})) \\
&= \lambda'(\Phi)(\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu})\mathbf{I}\,\partial_z\Phi_0\boldsymbol{\nu} + \lambda(\Phi)\partial_z(\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} \\
&= 0.
\end{aligned} \tag{3.73}$$

Using (3.70)-(3.73), from $(3.43\text{b})_I^{-2}$ we obtain

$$\partial_z(2\eta(\Phi_0)\mathcal{E}(\partial_z \mathbf{V}_0 \otimes \boldsymbol{\nu})\boldsymbol{\nu}) = 0. \tag{3.74}$$

Due to (3.61) we have

$$(\boldsymbol{\nu} \otimes \partial_z \mathbf{V}_0)\boldsymbol{\nu} = (\partial_z \mathbf{V}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} = 0.$$

Together with (3.74) and the identity $(\partial_z \mathbf{V}_0 \otimes \boldsymbol{\nu})\boldsymbol{\nu} = \partial_z \mathbf{V}_0$, this implies

$$\partial_z(\eta(\Phi_0)\partial_z \mathbf{V}_0) = 0.$$

Integrating from $-\infty$ to z with $|z| < \infty$, using the matching condition (3.54b) and the positivity of $\eta(\cdot)$, this gives

$$\partial_z \mathbf{V}_0 = 0 \quad \forall\, |z| < \infty. \tag{3.75}$$

Once more integrating and using the matching condition (3.54a) yields

$$[v_0]_H^T = 0. \tag{3.76}$$

Inner Expansion to higher order

We will now expand the equations in the inner regions to the next highest order.

Step 1: From $(3.43\text{d})_I^0$, we obtain

$$\beta\Phi_1\psi''(\Phi_0) + \beta\kappa\Phi_0' - \beta\partial_{zz}\Phi_1 - \chi_\varphi C_0 = \Xi_0.$$

Multiplying by Φ_0' and integrating from $-\infty$ to $+\infty$ yields

$$\int_{-\infty}^{\infty} \Xi_0(t,s)\Phi_0'(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} \beta(\psi'(\Phi_0))'\Phi_1 - \beta\partial_{zz}\Phi_1\Phi_0' + \beta\kappa|\Phi_0'|^2 - \chi_\varphi C_0\Phi_0'\,\mathrm{d}z. \tag{3.77}$$

Using (3.54a)-(3.54b), (3.57) and $\psi'(\pm 1) = 0$, integration by parts gives

$$\begin{aligned}\int_{-\infty}^{\infty} (\psi'(\Phi_0))'\Phi_1 - \partial_{zz}\Phi_1\Phi_0'\mathrm{d}z &= [\psi'(\Phi_0)\Phi_1 - \partial_z\Phi_1\Phi_0']_{-\infty}^{+\infty} \\ &\quad - \int_{-\infty}^{\infty} \partial_z\Phi_1(\psi'(\Phi_0) - \Phi_0'')\,\mathrm{d}z = 0.\end{aligned} \tag{3.78}$$

Recalling that Ξ_0 is independent of z and applying the matching condition (3.54a) we have

$$\int_{-\infty}^{+\infty} \Xi_0(t,s)\Phi_0'(z)\,\mathrm{d}z = 2\mu_0. \tag{3.79}$$

By the equi-partition of energy (3.59) we compute

$$\begin{aligned}\int_{-\infty}^{\infty} |\Phi_0'(z)|^2\,\mathrm{d}z &= \int_{-\infty}^{\infty} |\Phi_0'(z)|\sqrt{2\psi(\Phi_0(z))}\,\mathrm{d}z = \int_{-1}^{1} \sqrt{2\psi(y)}\,\mathrm{d}y \\ &= \frac{1}{\sqrt{2}}\int_{-1}^{1}(1-y^2)\,\mathrm{d}y = \frac{2\sqrt{2}}{3} =: \tau,\end{aligned}$$

and obtain

$$\int_{-\infty}^{+\infty} \beta\kappa|\Phi_0'(z)|^2\,\mathrm{d}z = \beta\kappa\int_{-\infty}^{+\infty} 2\psi(\Phi_0(z))\,\mathrm{d}z = \beta\kappa\tau. \tag{3.80}$$

Finally, by (3.68), we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} \chi_\varphi C_0\Phi_0'(z)\,\mathrm{d}z = \chi_\sigma\int_{-\infty}^{+\infty} \partial_z C_0(t,s,z)C_0(t,s,z)\,\mathrm{d}z &= \frac{\chi_\sigma}{2}\int_{-\infty}^{+\infty} \partial_z(|C_0|^2)\,\mathrm{d}z \\ &= \frac{\chi_\sigma}{2}[|\sigma_0|^2]_H^T.\end{aligned} \tag{3.81}$$

Collecting (3.77)-(3.81) gives

$$2\mu_0 = \beta\kappa\tau - \frac{\chi_\sigma}{2}[|\sigma_0|^2]_H^T. \tag{3.82}$$

This is a solvability condition for Φ_1, the so-called **Gibbs–Thomas equation**.

Step 2: With similar arguments as above and using (3.68), equation $(3.43\text{e})_I^{-1}$ gives

$$(-\mathcal{V} + \mathbf{V}_0 \cdot \boldsymbol{\nu})\partial_z C_0 = \partial_z(n(\Phi_0)(\chi_\sigma \partial_z C_1 - \chi_\varphi \partial_z \Phi_1)).$$

Employing the matching condition (3.54c) and $\nabla\varphi_0 = \mathbf{0}$ together with $\partial_z \mathcal{V} = 0$ and (3.61), this yields

$$\begin{aligned}(-\mathcal{V} + \mathbf{v}_0 \cdot \boldsymbol{\nu})[\sigma_0]_H^T &= \int_{-\infty}^{\infty} (-\mathcal{V} + \mathbf{V}_0 \cdot \boldsymbol{\nu})\partial_z C_0 \,\mathrm{d}z \\ &= \int_{-\infty}^{+\infty} \partial_z(n(\Phi_0)(\chi_\sigma \partial_z C_1 - \chi_\varphi \partial_z \Phi_1)) \,\mathrm{d}z = \chi_\sigma [n(\varphi_0)\nabla\sigma_0]_H^T \cdot \boldsymbol{\nu}. \end{aligned} \tag{3.83}$$

Step 3: Similar as in [3] we analyse (3.43c) only for the mobilities (3.45), (i) and (iii) since the case (3.45), (ii) is rescaled and therefore does not contribute to the sharp interface limit.

Case (i) ($m(\varphi) = m_0$): Using $\partial_z \Xi_0 = 0$ and (3.60), from $(3.43\text{c})_I^{-1}$ we obtain

$$(-\mathcal{V} + \mathbf{V}_0 \cdot \boldsymbol{\nu})\Phi_0' = m_0 \partial_{zz}\Xi_1.$$

Integrating with respect to z from $-\infty$ to ∞, using (3.61)-(3.62) and the matching condition (3.54c), this yields

$$2(-\mathcal{V} + \mathbf{v}_0 \cdot \boldsymbol{\nu}) = m_0[\nabla\mu_0]_H^T \cdot \boldsymbol{\nu}. \tag{3.84}$$

Case (iii) $\big(m(\varphi) = m_1(1+\varphi)^2\big)$: With similar arguments as above we obtain

$$(-\mathcal{V} + \mathbf{V}_0 \cdot \boldsymbol{\nu})\Phi_0' = \tfrac{m_1}{2}\partial_z\left((1+\Phi_0)^2 \partial_z \Xi_1\right).$$

Using the matching conditions (3.54a), (3.54c) and the same arguments as for (3.84), this entails

$$(-\mathcal{V} + \mathbf{v}_0 \cdot \boldsymbol{\nu}) = m_1 \nabla\mu_0^T \cdot \boldsymbol{\nu}. \tag{3.85}$$

Step 4: Finally, we consider the momentum balance equation (3.43b) at order $\theta(\varepsilon^{-1})$. Recalling (3.70) and (3.75), the term $\mathrm{div}_x(2\eta(\varphi)D_x\mathbf{v})$ at order $\theta(\varepsilon^{-1})$ gives

$$\partial_z\big(2\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_1 \otimes \boldsymbol{\nu})\boldsymbol{\nu} + 2\eta(\Phi_0)\mathcal{E}(\nabla_{\Sigma(0)}\mathbf{V}_0)\boldsymbol{\nu}\big). \tag{3.86}$$

Using that $\partial_z\mathbf{V}_0 = 0$, at order $\theta(\varepsilon^{-1})$ the term $\mathrm{div}_x(\eta(\varphi)\mathrm{div}_x(\mathbf{v})\mathbf{I})$ yields

$$\partial_z\big(\lambda(\Phi_0)(\partial_z\mathbf{V}_1 \cdot \boldsymbol{\nu} + \mathrm{div}_{\Sigma(0)}\mathbf{V}_0)\big)\boldsymbol{\nu}.$$

Using (3.71), the term $\nabla_x p$ contributes with

$$\partial_z P_0 \boldsymbol{\nu}. \tag{3.87}$$

Furthermore, we obtain from (3.72) that

$$(\mu + \chi_\varphi\sigma)\nabla\varphi = (\Xi_0 + \chi_\varphi C_0)\Phi_0'\boldsymbol{\nu} \tag{3.88}$$

at order $\theta(\varepsilon^{-1})$. Combining (3.86)-(3.88), at order $\theta(\varepsilon^{-1})$ we get

$$\begin{aligned}&-\partial_z\big(2\eta(\Phi_0)\mathcal{E}(\partial_z\mathbf{V}_1 \otimes \boldsymbol{\nu})\boldsymbol{\nu} + 2\eta(\Phi_0)\mathcal{E}(\nabla_{\Sigma(0)}\mathbf{V}_0)\boldsymbol{\nu} + \lambda(\Phi_0)(\partial_z\mathbf{V}_1 \cdot \boldsymbol{\nu} + \mathrm{div}_{\Sigma(0)}\mathbf{V}_0)\boldsymbol{\nu} - P_0\boldsymbol{\nu}\big) \\ &\quad = (\Xi_0 + \chi_\varphi C_0)\Phi_0'\boldsymbol{\nu}. \end{aligned} \tag{3.89}$$

Since matching requires $\lim_{z\to\pm\infty}\partial_z\mathbf{V}_1(z) = \nabla\mathbf{v}_0^{\pm}\boldsymbol{\nu}$, we conclude

$$\begin{aligned}
&\partial_z\mathbf{V}_1\otimes\boldsymbol{\nu} + \nabla_{\Sigma(0)}\mathbf{V}_0 \to \nabla_x\mathbf{v}_0 \quad \text{for} \quad z\to\pm\infty,\\
&\partial_z\mathbf{V}_1\cdot\boldsymbol{\nu} + \mathrm{div}_{\Sigma(0)}\mathbf{V}_0 \to \mathrm{div}_x\mathbf{v}_0 \quad \text{for} \quad z\to\pm\infty.
\end{aligned}$$

Integrating (3.89) with respect to z from $-\infty$ to $+\infty$ and using (3.54a), this implies

$$-[2\eta(\varphi_0)\mathcal{E}(\nabla_x\mathbf{v}_0) + \lambda(\varphi_0)\mathrm{div}(\mathbf{v}_0)\mathbf{I} - p_0\mathbf{I}]_H^T\boldsymbol{\nu} = \int_{-\infty}^{+\infty}(\Xi_0(t,s) + \chi_\varphi C_0(t,s,z))\Phi_0'(z)\boldsymbol{\nu}\,\mathrm{d}z.$$

Together with (3.79) and (3.81)-(3.82), we end up at

$$[\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0)]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}. \tag{3.91}$$

3.3.3 Equations of the formal sharp interface limit

For the readers convenience we summarise the sharp interface models for the different mobilities:

Case (i) ($m(\varphi) = m_0$) The equations in the bulk are given by

$$\begin{aligned}
-\mathrm{div}(\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0)) + \nu(\varphi_0)\mathbf{v}_0 &= 0 && \text{in } \Omega_T\cup\Omega_H,\\
\mathrm{div}(\mathbf{v}_0^T) &= \bar\rho_2^{-1}\Gamma_2(1,\sigma_0^T) + \bar\rho_1^{-1}\Gamma_1(1,\sigma_0^T) && \text{in } \Omega_T,\\
\mathrm{div}(\mathbf{v}_0^H) &= \bar\rho_2^{-1}\Gamma_2(-1,\sigma_0^H) + \bar\rho_1^{-1}\Gamma_1(-1,\sigma_0^H) && \text{in } \Omega_H,\\
-m_0\Delta\mu_0^T &= -2\bar\rho_1^{-1}\Gamma_1(1,\sigma_0^T) && \text{in } \Omega_T,\\
-m_0\Delta\mu_0^H &= 2\bar\rho_2^{-1}\Gamma_2(-1,\sigma_0^H) && \text{in } \Omega_H,\\
\partial_t\sigma_0^T + \mathrm{div}(\sigma_0^T\mathbf{v}_0^T) &= \mathrm{div}(n(1)\chi_\sigma\nabla\sigma_0^T) - \Gamma_\sigma(1,\sigma_0^T) && \text{in } \Omega_T,\\
\partial_t\sigma_0^H + \mathrm{div}(\sigma_0^H\mathbf{v}_0^H) &= \mathrm{div}(n(-1)\chi_\sigma\nabla\sigma_0^H) - \Gamma_\sigma(-1,\sigma_0^H) && \text{in } \Omega_H.
\end{aligned}$$

Furthermore, on $\Sigma(0)$ we have the free boundary conditions

$$\begin{aligned}
&[\mathbf{v}_0]_H^T = \mathbf{0}, \qquad [\mu_0]_H^T = 0, \qquad [\sigma_0]_H^T = 2\tfrac{\chi_\varphi}{\chi_\sigma},\\
&2\mu_0 = \beta\kappa\tau - \tfrac{\chi_\sigma}{2}[|\sigma_0|^2]_H^T, \qquad (-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu})[\sigma_0]_H^T = [n(\varphi_0)\nabla\sigma_0]_H^T\cdot\boldsymbol{\nu},\\
&2(-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu}) = m_0[\nabla\mu_0]_H^T\cdot\boldsymbol{\nu}, \qquad [\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0)]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}.
\end{aligned}$$

Case (ii) ($m(\varphi) = \varepsilon m_0$) The equations in the bulk are given by

$$\begin{aligned}
-\mathrm{div}(\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0)) + \nu(\varphi_0)\mathbf{v}_0 &= 0 && \text{in } \Omega_T\cup\Omega_H,\\
\mathrm{div}(\mathbf{v}_0^T) &= \bar\rho_2^{-1}\Gamma_2(1,\sigma_0^T) && \text{in } \Omega_T,\\
\mathrm{div}(\mathbf{v}_0^H) &= \bar\rho_1^{-1}\Gamma_1(-1,\sigma_0^H) && \text{in } \Omega_H,\\
\partial_t\sigma_0^T + \mathrm{div}(\sigma_0^T\mathbf{v}_0^T) &= \mathrm{div}(n(1)\chi_\sigma\nabla\sigma_0^T) - \Gamma_\sigma(1,\sigma_0^T) && \text{in } \Omega_T,\\
\partial_t\sigma_0^H + \mathrm{div}(\sigma_0^H\mathbf{v}_0^H) &= \mathrm{div}(n(-1)\chi_\sigma\nabla\sigma_0^H) - \Gamma_\sigma(-1,\sigma_0^H) && \text{in } \Omega_H.
\end{aligned}$$

Furthermore, on $\Sigma(0)$ we have the free boundary conditions

$$\begin{aligned}
&[\mathbf{v}_0]_H^T = \mathbf{0}, \qquad [\sigma_0]_H^T = 2\tfrac{\chi_\varphi}{\chi_\sigma}, \qquad 0 = [n(\varphi_0)\nabla\sigma_0]_H^T\cdot\boldsymbol{\nu},\\
&\mathcal{V} = \mathbf{v}_0\cdot\boldsymbol{\nu}, \qquad [\mathbf{T}(\mathbf{v}_0,p_0,\varphi_0)]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}.
\end{aligned}$$

Case (iii) $\big(m(\varphi) = m_1(1+\varphi)^2\big)$ The equations in the bulk are given by

$$\begin{aligned}
-\mathrm{div}(\mathbf{T}(\mathbf{v}_0, p_0, \varphi_0)) + \nu(\varphi_0)\mathbf{v}_0 &= 0 && \text{in } \Omega_T \cup \Omega_H,\\
\mathrm{div}(\mathbf{v}_0^T) &= \bar{\rho}_2^{-1}\Gamma_2(1,\sigma_0^T) + \bar{\rho}_1^{-1}\Gamma_1(1,\sigma_0^T) && \text{in } \Omega_T,\\
\mathrm{div}(\mathbf{v}_0^H) &= \bar{\rho}_1^{-1}\Gamma_1(-1,\sigma_0^H) && \text{in } \Omega_H,\\
-m_1\Delta\mu_0^T &= -\bar{\rho}_1^{-1}\Gamma_1(1,\sigma_0^T) && \text{in } \Omega_T,\\
\partial_t\sigma_0^T + \mathrm{div}(\sigma_0^T\mathbf{v}_0^T) &= \mathrm{div}(n(1)\chi_\sigma\nabla\sigma_0^T) - \Gamma_\sigma(1,\sigma_0^T) && \text{in } \Omega_T,\\
\partial_t\sigma_0^H + \mathrm{div}(\sigma_0^H\mathbf{v}_0^H) &= \mathrm{div}(n(-1)\chi_\sigma\nabla\sigma_0^H) - \Gamma_\sigma(-1,\sigma_0^H) && \text{in } \Omega_H.
\end{aligned}$$

Furthermore, on $\Sigma(0)$ we have the free boundary conditions

$$\begin{aligned}
&[\mathbf{v}_0]_H^T = \mathbf{0}, \qquad [\sigma_0]_H^T = 2\tfrac{\chi_\varphi}{\chi_\sigma}, \qquad 2\mu_0 = \beta\kappa\tau - \tfrac{\chi_\sigma}{2}[|\sigma_0|^2]_H^T,\\
&(-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu})[\sigma_0]_H^T = [n(\varphi_0)\nabla\sigma_0]_H^T\cdot\boldsymbol{\nu}, \qquad (-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu}) = m_1\nabla\mu_0^T\cdot\boldsymbol{\nu},\\
&[\mathbf{T}(\mathbf{v}_0, p_0, \varphi_0)]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}.
\end{aligned}$$

3.3.4 Specific sharp interface models

The limit of vanishing active transport, Darcy's law and Stokes' flow

We consider (3.43a)-(3.43e) with quasi-static nutrients and the mobility (3.45), (ii) along with constant viscosities and permeability. Moreover, we decouple chemotaxis and active transport according to (3.38), we set

$$\mathcal{D}(\varphi) = \frac{1+\varphi}{2} + \mathcal{D}\frac{1-\varphi}{2}$$

for a constant $\mathcal{D} > 0$, and we choose

$$\Gamma_1 \equiv 0, \qquad \Gamma_2(\varphi,\sigma) = \frac{\bar{\rho}_2}{2}\left(\frac{1}{\bar{\rho}_2} - \frac{1}{\bar{\rho}_1}\right)(\mathcal{P}\sigma - \mathcal{A})(1+\varphi), \qquad \Gamma_\sigma(\varphi,\sigma) = \frac{\mathcal{C}}{2}\sigma(1+\varphi).$$

This gives the following system of equations

$$\begin{aligned}
\mathrm{div}(\mathbf{v}) &= \frac{\alpha}{2}(\mathcal{P}\sigma - \mathcal{A})(1+\varphi),\\
-\mathrm{div}(\mathbf{T}(\mathbf{v},p)) + \nu\mathbf{v} &= (\mu + \chi_\varphi\sigma)\nabla\varphi,\\
\partial_t\varphi + \nabla\varphi\cdot\mathbf{v} &= \mathrm{div}(\varepsilon m_0\nabla\mu) + \frac{\alpha}{2}(\mathcal{P}\sigma - \mathcal{A})(1-\varphi^2),\\
\mu &= \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma,\\
0 &= \mathrm{div}(\mathcal{D}(\varphi)\nabla\sigma) - \lambda\mathrm{div}(\mathcal{D}(\varphi)\nabla\varphi) - \mathcal{C}\sigma(1+\varphi),
\end{aligned}$$

where $\mathbf{T}(\mathbf{v},p) = 2\eta\mathbf{D}\mathbf{v} + \lambda\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}$. With slightly different argument as above (see also [87]) and sending $\lambda \to 0$, we obtain

$$-\mathrm{div}(\mathbf{T}(\mathbf{v}_0, p_0)) + \nu\mathbf{v}_0 = 0 \qquad \text{in } \Omega_T \cup \Omega_H, \tag{3.92a}$$

$$\mathrm{div}(\mathbf{v}_0) = \begin{cases} \alpha(\mathcal{P}\sigma_0^T - \mathcal{A}) & \text{in } \Omega_T,\\ 0 & \text{in } \Omega_H, \end{cases} \tag{3.92b}$$

$$\Delta\sigma_0 = \begin{cases} \mathcal{C}\sigma_0 & \text{in } \Omega_T,\\ 0 & \text{in } \Omega_H, \end{cases} \tag{3.92c}$$

and the free boundary conditions on $\Sigma(0)$ are given by

$$\begin{aligned}
&[\mathbf{v}_0]_H^T = \mathbf{0}, \qquad [\sigma_0]_H^T = 0, \qquad \nabla\sigma_0^T\cdot\boldsymbol{\nu} = \mathcal{D}\nabla\sigma_0^H\cdot\boldsymbol{\nu},\\
&\mathcal{V} = \mathbf{v}_0\cdot\boldsymbol{\nu}, \qquad [\mathbf{T}(\mathbf{v}_0, p_0)]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}.
\end{aligned} \tag{3.92d}$$

This model is a special case of the two-phase free boundary problem in [143] where they present numerical simulations for (3.92). Similar models have been studied in [43]. For a one-phase model with Brinkman's law for the velocity we refer to [122].
Sending the viscosities to 0 in (3.92), we can express the velocity in terms of the pressure and we obtain the following Darcy-type model

$$-\Delta p_0 = \begin{cases} \nu\, \alpha(\mathcal{P}\sigma_0^T - \mathcal{A}) & \text{in } \Omega_T, \\ 0 & \text{in } \Omega_H, \end{cases}$$
$$\Delta \sigma_0 = \begin{cases} \mathcal{C}\sigma_0 & \text{in } \Omega_T, \\ 0 & \text{in } \Omega_H, \end{cases}$$

where free boundary conditions on $\Sigma(0)$ are given by

$$[\sigma_0]_H^T = 0, \quad \nabla\sigma_0^T \cdot \boldsymbol{\nu} = \mathcal{D}\nabla\sigma_0^H \cdot \boldsymbol{\nu}, \quad \frac{1}{\nu}[\nabla p_0]_H^T \cdot \boldsymbol{\nu} = 0, \quad \mathcal{V} = -\frac{1}{\nu}\nabla p_0 \cdot \boldsymbol{\nu}, \quad [p_0]_H^T = -\beta\kappa\tau.$$

Similar models have been studied in, e. g., [46, 96, 114, 115]. We remark that the continuity condition for $\mathbf{v}_0$ across the interface (see (3.76)) is based on the positivity of the shear viscosity.
Sending the permeability to zero in (3.92), i. e., $\nu \to 0$, we obtain a Stokes model given by

$$-\mathrm{div}(2\eta\mathbf{D}\mathbf{v}_0 + \lambda\mathrm{div}(\mathbf{v}_0)\mathbf{I} - p_0\mathbf{I}) = 0 \qquad \text{in } \Omega_T \cup \Omega_H,$$
$$\mathrm{div}(\mathbf{v}_0) = \begin{cases} \alpha(\mathcal{P}\sigma_0^T - \mathcal{A}) & \text{in } \Omega_T, \\ 0 & \text{in } \Omega_H, \end{cases}$$
$$\Delta \sigma_0 = \begin{cases} \mathcal{C}\sigma_0 & \text{in } \Omega_T, \\ 0 & \text{in } \Omega_H, \end{cases}$$

and the free boundary conditions on $\Sigma(0)$ are given by

$$[\mathbf{v}_0]_H^T = \mathbf{0}, \qquad [\sigma_0]_H^T = 0, \qquad \nabla\sigma_0^T \cdot \boldsymbol{\nu} = \mathcal{D}\nabla\sigma_0^H \cdot \boldsymbol{\nu},$$
$$\mathcal{V} = \mathbf{v}_0 \cdot \boldsymbol{\nu}, \qquad [2\eta\mathbf{D}\mathbf{v}_0 + \lambda\mathrm{div}(\mathbf{v}_0)\mathbf{I} - p_0\mathbf{I}]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}.$$

For similar models, we refer to [65, 66, 68–70, 72, 138].

The tumour as a viscous fluid surrounded by extracellular fluid

We now consider the same model as in the last part but with non-constant viscosities and permeability and with $\alpha = 1$. Moreover, we model the tumour as a viscous fluid and the surroundings (e. g., the extracellular fluid) as an inviscid fluid (see, e. g., [27, 30]) by setting

$$\eta(\varphi) = \frac{1+\varphi}{2}\eta_0, \quad \lambda(\varphi) = \frac{1+\varphi}{2}\lambda_0, \quad \nu(\varphi) = \frac{1-\varphi}{2}\nu_0$$

for positive constants η_0, λ_0 and ν_0. Hence, we consider the model

$$\begin{aligned} \mathrm{div}(\mathbf{v}) &= \tfrac{1}{2}(\mathcal{P}\sigma - \mathcal{A})(1+\varphi), \\ -\mathrm{div}(2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}) + \nu(\varphi)\mathbf{v} &= (\mu + \chi_\varphi\sigma)\nabla\varphi, \\ \partial_t\varphi + \nabla\varphi\cdot\mathbf{v} &= \mathrm{div}(\varepsilon m_0\nabla\mu) + \tfrac{1}{2}(\mathcal{P}\sigma - \mathcal{A})(1-\varphi^2), \\ \mu &= \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma, \\ 0 &= \mathrm{div}(\mathcal{D}(\varphi)\nabla\sigma) - \mathcal{C}\sigma(1+\varphi). \end{aligned}$$

Then, the equations in the bulk are given by

$$-\mathrm{div}(2\eta_0\,\mathbf{D}\mathbf{v}_0^T + \lambda_0\,\mathrm{div}(\mathbf{v}_0^T)\mathbf{I} - p_0^T\,\mathbf{I}) = 0 \quad \text{in } \Omega_T, \tag{3.93a}$$

$$\mathbf{v}_0^H = -\nu_0^{-1}\nabla p_0^H \quad \text{in } \Omega_H, \tag{3.93b}$$

$$\mathrm{div}(\mathbf{v}_0) = \begin{cases} \mathcal{P}\sigma_0^T - \mathcal{A} & \text{in } \Omega_T, \\ 0 & \text{in } \Omega_H, \end{cases} \tag{3.93c}$$

$$\Delta\sigma_0 = \begin{cases} \mathcal{C}\sigma_0 & \text{in } \Omega_T, \\ 0 & \text{in } \Omega_H, \end{cases} \tag{3.93d}$$

and the free boundary conditions on $\Sigma(0)$ by

$$\begin{aligned} &[\mathbf{v}_0]_H^T \cdot \boldsymbol{\nu} = 0, \qquad [\sigma_0]_H^T = 0, \qquad \nabla\sigma_0^T\cdot\boldsymbol{\nu} = \mathcal{D}\nabla\sigma_0^H\cdot\boldsymbol{\nu}, \\ &\mathcal{V} = \mathbf{v}_0\cdot\boldsymbol{\nu}, \qquad (2\eta_0\,\mathbf{D}\mathbf{v}_0^T + \lambda_0\,\mathrm{div}(\mathbf{v}_0^T)\mathbf{I} - p_0^T\,\mathbf{I})\boldsymbol{\nu} = -(\beta\kappa\tau + p_0^H)\boldsymbol{\nu}. \end{aligned} \tag{3.93e}$$

By using the Darcy law and the continuity equation in Ω_H, we can rewrite (3.93a)-(3.93c) as

$$\begin{aligned} -\mathrm{div}(2\eta_0\mathbf{D}\mathbf{v}_0^T + \lambda_0\mathrm{div}(\mathbf{v}_0^T)\mathbf{I} - p_0^T\mathbf{I}) &= 0 && \text{in } \Omega_T, \\ \mathrm{div}(\mathbf{v}_0^T) &= \mathcal{P}\sigma_0^T - \mathcal{A} && \text{in } \Omega_T, \\ -\Delta p_0^H &= 0 && \text{in } \Omega_H, \end{aligned}$$

and $(3.93\mathrm{e})_1$ as

$$\mathbf{v}_0^T\cdot\boldsymbol{\nu} = -\nu_0^{-1}\nabla p_0^H\cdot\boldsymbol{\nu} \qquad \text{on } \Sigma(0).$$

Linear phenomenological laws for chemical reactions

We consider the quasi-static model (3.43a)-(3.43e), i. e., we neglect the l. h. s. of (3.43e), and we take a mobility of the form mobility $m(\varphi) = \varepsilon m_0$. We assume that $\bar{\rho}_1 = \bar{\rho}_2 = 1$ and

$$\Gamma_2 = -\Gamma_1 = P(\varphi)(\chi_\sigma\sigma + \chi_\varphi(1-\varphi) - \mu), \qquad \Gamma_\sigma = 2P(\varphi)(\chi_\sigma\sigma + \chi_\varphi(1-\varphi) - \mu),$$

where

$$P(\varphi) = P_0\,\tfrac{1}{4}(1-\varphi^2)^2$$

for a positive constant P_0. Moreover, we assume $n(\varphi) = n_0$, we rescale

$$n_0 = \varepsilon\chi_\varphi^{-1}, \quad \chi_\sigma = \varepsilon^{-1}\chi_\varphi,$$

and we take constant viscosities and permeability. With these choices the system (3.43a)-(3.43e) reads

$$\mathrm{div}(\mathbf{v}) = 0, \tag{3.94a}$$

$$-\mathrm{div}(2\eta\mathbf{D}\mathbf{v} + \lambda\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}) + \nu\mathbf{v} = (\mu + \chi_\varphi\sigma)\nabla\varphi, \tag{3.94b}$$

$$\partial_t\varphi + \mathrm{div}(\varphi\mathbf{v}) = \varepsilon m_0\Delta\mu + 2P_0\psi(\varphi)\left(\tfrac{\chi_\varphi}{\varepsilon}\sigma + \chi_\varphi(1-\varphi) - \mu\right), \tag{3.94c}$$

$$\mu = \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma, \tag{3.94d}$$

$$0 = \Delta\sigma - \varepsilon\Delta\varphi - 2P_0\psi(\varphi)\left(\tfrac{\chi_\varphi}{\varepsilon}\sigma + \chi_\varphi(1-\varphi) - \mu\right). \tag{3.94e}$$

The equations in the outer regions are given by

$$\begin{aligned} -\mathrm{div}(2\eta\mathbf{D}\mathbf{v}_0 + \lambda\mathrm{div}(\mathbf{v}_0)\mathbf{I} - p_0\mathbf{I}) + \nu\mathbf{v}_0 &= 0 && \text{in } \Omega_T\cup\Omega_H, \\ \mathrm{div}(\mathbf{v}_0) &= 0 && \text{in } \Omega_T\cup\Omega_H, \\ -\Delta\sigma_0 &= 0 && \text{in } \Omega_T\cup\Omega_H. \end{aligned}$$

Inner expansion to leading order With similar arguments as above we obtain from $(3.94\text{b})_I^{-2}$ and $(3.94\text{e})_I^{-2}$ that

$$[\mathbf{v}_0]_H^T = [\sigma_0]_H^T = 0, \quad \partial_z \mathbf{V}_0 = \partial_z C_0 = 0 \quad \forall\, |z| < \infty. \tag{3.95}$$

From $(3.94\text{c})_I^{-1}$ we get

$$-\mathcal{V}\Phi_0' + \partial_z(\Phi_0 \mathbf{V}_0)\cdot\boldsymbol{\nu} = \partial_z(m_0\partial_z \Xi_0) + 2\chi_\varphi P_0 \psi(\Phi_0) C_0. \tag{3.96}$$

The equipartition of energy is given by

$$\int_{-\infty}^{\infty} 2\psi(\Phi_0(z))\mathrm{d}z = \tau.$$

Therefore, integrating (3.96) with respect to z from $-\infty$ to ∞ and using (3.95) yields

$$2(-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu}) = P_0\chi_\varphi \tau\sigma_0. \tag{3.97}$$

From $(3.94\text{e})_I^{-1}$ we obtain that

$$[\nabla\sigma_0]_H^T\cdot\boldsymbol{\nu} = P_0\chi_\varphi\tau\sigma_0. \tag{3.98}$$

Then, with similar arguments as above we arrive at the following limit problem

$$\begin{aligned} -\mathrm{div}(2\eta\mathbf{D}\mathbf{v}_0 + \lambda\mathrm{div}(\mathbf{v}_0)\mathbf{I} - p_0\mathbf{I}) + \nu\mathbf{v}_0 &= 0 \quad \text{in } \Omega_T\cup\Omega_H, \\ \mathrm{div}\mathbf{v}_0 &= 0 \quad \text{in } \Omega_T\cup\Omega_H, \\ -\Delta\sigma_0 &= 0 \quad \text{in } \Omega_T\cup\Omega_H, \end{aligned}$$

with the free boundary conditions on $\Sigma(0)$ given by

$$\begin{aligned} &[\mathbf{v}_0]_H^T = \mathbf{0}, \qquad [\sigma_0]_H^T = 0, \qquad [\nabla\sigma_0]_H^T\cdot\boldsymbol{\nu} = P_0\chi_\varphi\tau\sigma_0, \\ &2(-\mathcal{V} + \mathbf{v}_0\cdot\boldsymbol{\nu}) = P_0\chi_\varphi\tau\sigma_0, \qquad [2\eta\mathbf{D}\mathbf{v}_0 + \lambda\mathrm{div}(\mathbf{v}_0)\mathbf{I} - p_0\mathbf{I}]_H^T\boldsymbol{\nu} = -\beta\kappa\tau\boldsymbol{\nu}. \end{aligned}$$

The double obstacle potential

We will now present the main differences for the double obstacle potential given by

$$\psi(\varphi) := \frac{1}{2}(1-\varphi^2) + I_{[-1,1]}(\varphi), \quad I_{[-1,1]}(\varphi) = \begin{cases} 0 & \text{if } |\varphi| \le 1, \\ +\infty & \text{else.} \end{cases} \tag{3.99}$$

We plot the double obstacle potential in Figure 3.6 and we refer the reader to [87] for more details. The derivative has to be understood in the sense of subdifferentials, i. e.,

$$\psi'(\varphi) = -\varphi + \partial I_{[-1,1]}(\varphi), \quad \partial I_{[-1,1]}(\varphi) = \begin{cases} (-\infty, 0] & \text{if } \varphi = -1, \\ 0 & \text{if } |\varphi| < 1, \\ [0,\infty) & \text{if } \varphi = +1. \end{cases} \tag{3.100}$$

Hence, (3.43d) has to be replaced by

$$\int_\Omega -\mu(\xi-\varphi) - \tfrac{\beta}{\varepsilon}\varphi(\xi-\varphi) + \beta\varepsilon\nabla\varphi\cdot\nabla(\xi-\varphi) - \chi_\varphi\sigma(\xi-\varphi)\,\mathrm{d}x \ge 0 \tag{3.101}$$

for all $\xi \in \mathcal{K} := \{\xi \in H^1 : |\xi| \le 1 \text{ a. e. in } \Omega\}$.

Expanding (3.101) in the outer region, we require $\varphi_0 = \pm 1$ and Ω_T and Ω_H can be defined as before.

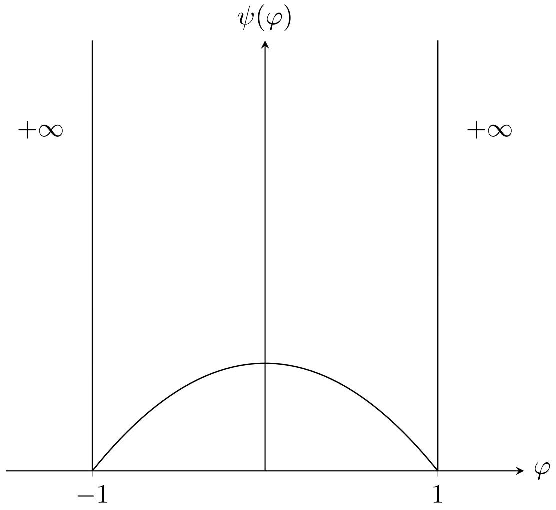

Figure 3.6: Plot of the double obstacle potential.

In the following we assume that the inner variable Φ_ε is monotone increasing with z and we assume that the interfacial layer has finite thickness $2l$ with $l = \frac{\pi}{2}$. Furthermore, we assume that

$$\Phi_\varepsilon(t, s, z = \tfrac{\pi}{2}) = +1, \quad \Phi_\varepsilon(t, s, z = -\tfrac{\pi}{2}) = -1. \tag{3.102}$$

The constant τ is now defined as

$$\tau = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(z)\,\mathrm{d}z = \frac{\pi}{2}.$$

From $(3.101)_I^{-1}$ and (3.102) we obtain

$$\Phi_0(z) = \begin{cases} +1 & \text{if } z \geq \frac{\pi}{2}, \\ \sin(z) & \text{if } |z| \leq \frac{\pi}{2}, \\ -1 & \text{if } z \leq -\frac{\pi}{2}, \end{cases} \qquad \Phi_1(t, s, \pm\tfrac{\pi}{2}) = 0.$$

Moreover, the equipartition of energy (see (3.59)) holds and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\Phi_0'(z)|^2\,\mathrm{d}z = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(\Phi_0(z))\,\mathrm{d}z = \tau.$$

Finally, from $(3.101)_I^0$, we obtain (compare (3.82))

$$2\mu_0 = \beta\kappa\tau - \frac{\chi_\sigma}{2}[|\sigma_0|^2]_H^T. \tag{3.103}$$

Hence, except from the definition of τ we obtain the same equations for the sharp interface limit as for the double-well potential.

3.4 Numerical results

In this part we aim to show several simulations for the tumour growth model derived in this chapter. The simulations are provided by Dr. Robert Nürnberg from Imperial College London (see [56]). We consider the system

$$\operatorname{div}(\mathbf{v}) = \alpha\tfrac{1}{2}(\mathcal{P}\sigma - \mathcal{A})(\varphi+1) \quad \text{in } \Omega_T, \tag{3.104a}$$

$$-\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu\mathbf{v} = (\mu + \chi_\varphi\sigma)\nabla\varphi \quad \text{in } \Omega_T, \tag{3.104b}$$

$$\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = \operatorname{div}(m(\varphi)\nabla\mu) + \rho_S\tfrac{1}{2}(\mathcal{P}\sigma - \mathcal{A})(1+\varphi) \quad \text{in } \Omega_T, \tag{3.104c}$$

$$\mu = \tfrac{\beta}{\varepsilon}\psi'(\varphi) - \beta\varepsilon\Delta\varphi - \chi_\varphi\sigma \quad \text{in } \Omega_T, \tag{3.104d}$$

$$0 = \operatorname{div}(\mathcal{D}(\nabla\sigma - \chi\nabla\varphi)) - \tfrac{1}{2}\mathcal{C}\sigma(\varphi+1) \quad \text{in } \Omega_T, \tag{3.104e}$$

where

$$\mathbf{T}(\mathbf{v},p) = 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I},$$

and with mobilities of the form (3.45), that means

$$\text{(i) } m(\varphi) = m_0, \qquad \text{(ii) } m(\varphi) = \varepsilon m_0, \qquad \text{(iii) } m(\varphi) = m_0\tfrac{1}{2}(1+\varphi)^2. \tag{3.105}$$

We supplement the system with initial and boundary conditions of the form

$$\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = 0, \quad \sigma = \sigma_B \quad \text{on } \Sigma_T, \tag{3.106a}$$

$$\mathbf{T}(\mathbf{v},p)\mathbf{n} = 0 \quad \text{on } \partial_1\Omega\times(0,T), \quad \mathbf{v} = \mathbf{0} \quad \text{on } \partial_2\Omega\times(0,T), \tag{3.106b}$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega, \tag{3.106c}$$

where σ_B is a given function and $\partial_1\Omega$, $\partial_2\Omega$, are measurable such that

$$\partial_1\Omega\cup\partial_2\Omega = \partial\Omega \quad \text{and} \quad \partial_1\Omega\cap\partial_2\Omega = \emptyset.$$

In (3.104) we denote by $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{C}$ the proliferation, apoptosis and consumption rate. Moreover, the parameters $\mathcal{D}$, χ_φ, χ and β are related to nutrient diffusion, chemotaxis, active transport and surface tension. The remaining variables and parameters are defined as before. In the case (3.105), (ii) we always set $\rho_S = \alpha$ in order to fulfil (3.49b). We remark that setting $\eta(\cdot) = \lambda(\cdot) \equiv 0$ leads to a Cahn–Hilliard–Darcy model.
In the following, we choose $\psi(\cdot)$ as the double obstacle potential (see (3.99)), and we define the function spaces

$$\mathcal{K} := \{f\in H^1 : |f|\le 1 \text{ a.e. in } \Omega\}, \quad \mathbf{H}^1_{\partial_1\Omega} := \{\mathbf{u}\in\mathbf{H}^1 : \mathbf{u} = \mathbf{0} \text{ a.e. on } \partial_1\Omega\}.$$

We call $(\varphi,\sigma,\mu,\mathbf{v},p)$ weak solution of (3.104) and (3.106) if

$$\varphi\in H^1((H^1)^*)\cap L^2(H^1), \quad \mu\in L^2(H^1), \quad \sigma\in\left(\sigma_B + L^2(H^1_0)\right), \quad \mathbf{v}\in L^2(\mathbf{H}^1_{\partial_1\Omega}), \quad p\in L^2(L^2),$$

such that $\varphi(0) = \varphi_0$ a.e. in Ω, $\varphi(t)\in\mathcal{K}$ for a.e. $t\in(0,T)$, and

$$\int_\Omega \mathbf{T}(\mathbf{v},p) : \nabla\mathbf{\Phi} + \nu\mathbf{v}\cdot\mathbf{\Phi}\,\mathrm{d}x = \int_\Omega(\mu+\chi_\varphi\sigma)\nabla\varphi\cdot\mathbf{\Phi}\,\mathrm{d}x,$$

$$\int_\Omega \operatorname{div}(\mathbf{v})\Phi\,\mathrm{d}x = \alpha\tfrac{1}{2}\int_\Omega(\mathcal{P}\sigma-\mathcal{A})(\varphi+1)\,\Phi\,\mathrm{d}x,$$

$$\langle\partial_t\varphi,\xi\rangle_{H^1} + \int_\Omega\nabla\varphi\cdot\mathbf{v}\,\xi\,\mathrm{d}x = -\int_\Omega m(\varphi)\nabla\mu\cdot\nabla\xi - \tfrac{1}{2}(\rho_S-\alpha\varphi)(\mathcal{P}\sigma-\mathcal{A})(\varphi+1)\,\xi\,\mathrm{d}x,$$

$$\int_\Omega\left(\mu + \tfrac{\beta}{\varepsilon}\varphi + \chi_\varphi\sigma\right)(\zeta-\varphi)\,\mathrm{d}x \le \int_\Omega\beta\varepsilon\nabla\varphi\cdot\nabla(\zeta-\varphi)\,\mathrm{d}x,$$

$$\int_\Omega\mathcal{D}\,(\nabla\sigma-\chi\nabla\varphi)\cdot\nabla\phi\,\mathrm{d}x = -\int_\Omega\tfrac{1}{2}\mathcal{C}\sigma\,(\varphi+1)\,\phi\,\mathrm{d}x$$

for a.e. $t\in(0,T)$ and all $\mathbf{\Phi}\in\mathbf{H}_{\partial_1\Omega}$, $\Phi\in L^2$, $\xi\in H^1$, $\zeta\in\mathcal{K}$, $\phi\in H^1_0$.

The weak formulation is used for a finite element method based on a semi-implicit Euler discretization, see [56]. Unless otherwise stated, we will always use the following set of parameters

$$\begin{aligned} &\varepsilon = 0.02, \quad \alpha = 0.5, \quad \rho_S = 2, \quad \mathcal{P} = 0.1, \quad \mathcal{A} = 0, \quad \mathcal{C} = 2, \quad \chi_\varphi = 5, \\ &\mathcal{D} = 1, \quad \sigma_B = 1, \quad \chi = 0.02, \quad \lambda = 0, \quad \nu = 100, \quad \partial_1\Omega = \Omega, \quad \Omega = (-3,3)^2, \end{aligned} \tag{3.107}$$

and the initial profile shown in Figure 3.7.

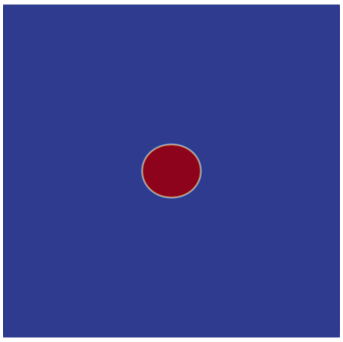

Figure 3.7: Initial tumour size.

We will now systematically interpret the influence of different parameters in our model.

3.4.1 Brinkman's and Darcy's law

In the following we investigate the relation of the Cahn–Hilliard–Brinkman (CHB) and Cahn–Hilliard–Darcy (CHD) models. Indeed, in Theorem 6.7 we will prove a qualitative estimate for the solutions of the CHB and CHD model in two space dimensions. Thus, for small viscosities we expect a similar qualitative behaviour of solutions to the corresponding systems. For the mobility we take $m(s) = \frac{1}{2}(1+s)^2$ which corresponds to (3.105), (iii), with $m_0 = 1$. In Figure 3.8 we show the tumour for both the CHD and CHB model for $\eta = 10^{-5}$ at time $t = 12$. We see that the the qualitative behaviour for both models is similar for low viscosities which validates the qualitative estimate.

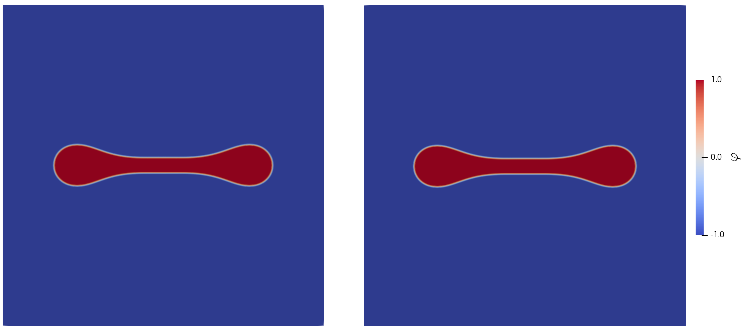

Figure 3.8: Tumour at time $t = 12$ for $\beta = 0.1$, left side for the CHD model, right side for the CHB model with $\eta = 10^{-5}$.

3.4.2 Influence of mobility and surface tension

We now investigate the influence of the mobility and the surface tension. In Figure 3.9 we show the evolutions with $\eta = 10^{-5}$ and for different mobilities. The formal asymptotic analysis in the previous section indicates that the mobility (3.105), (ii), corresponds to a free boundary problem where the interface is transported solely by the fluid velocity.

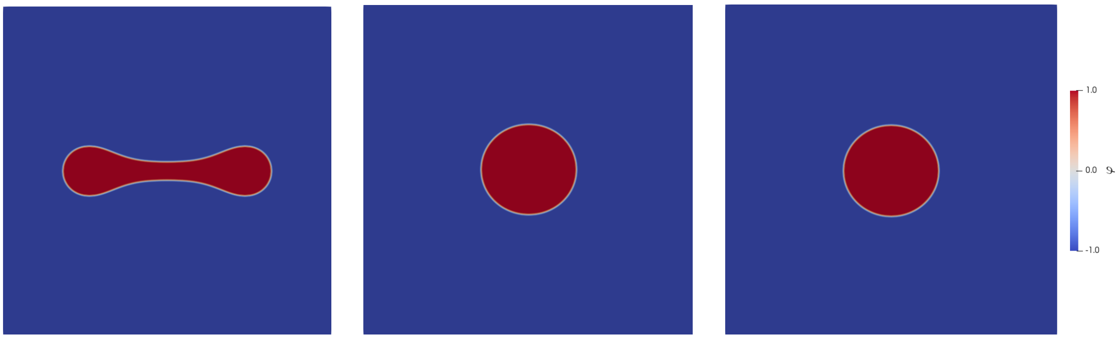

Figure 3.9: Tumour at time $t = 9$ for $\eta = 10^{-5}$, $\beta = 0.1$ and $\alpha = \rho_S = 2$, but with different mobilities, left $m(\varphi) = \frac{1}{2}(1+\varphi)^2$, middle $m(\varphi) = \varepsilon$, right $m(\varphi) = 10^{-3}\,\varepsilon$.

Thus, we see that a one-sided degenerate mobility causes instabilities while pure transport stabilises the interface. Moreover, having a closer look we see that the thickness of the interface is smaller for the mobility $m(\varphi) = 10^{-3}\,\varepsilon$.
As the Ginzburg–Landau energy models adhesion forces, it can be expected that a reduction of the parameter $\beta > 0$ reduces surface tension forces and leads to instabilities. In Figure 3.10, we compare the tumour evolutions for $\beta \in \{0.1, 0.01\}$ with $\eta = 0.1$ and for the mobility $m(\varphi) = \frac{1}{2}(1+\varphi)^2$. We see that the instabilities are more pronounced for $\beta = 0.01$ and the fingers are longer and thinner.

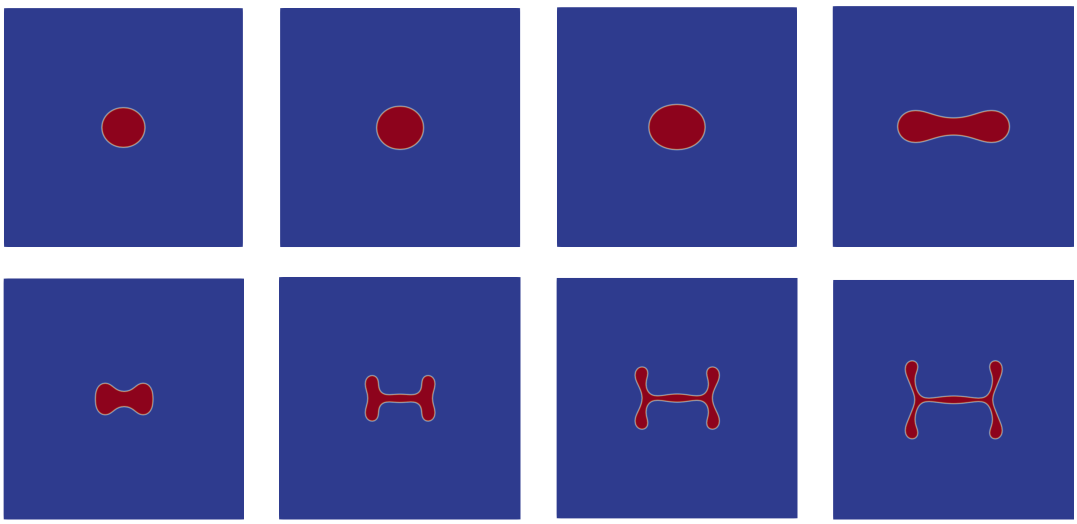

Figure 3.10: Evolution of the tumour with $m(\varphi) = \frac{1}{2}(1+\varphi)^2$ and $\eta = 0.1$, above for $\beta = 0.1$ at time $t = 1, 3, 6, 10$, below for $\beta = 0.01$ at time $t = 1, 1.5, 2, 2.5$.

However, in Figure 3.11 we see that a reduced surface tension does not cause instabilities if the mobility is of the form $m(\varphi) = \varepsilon$. Hence, the stabilising effect of pure transport seems to be stronger than the destabilising effect of a reduced surface tension.

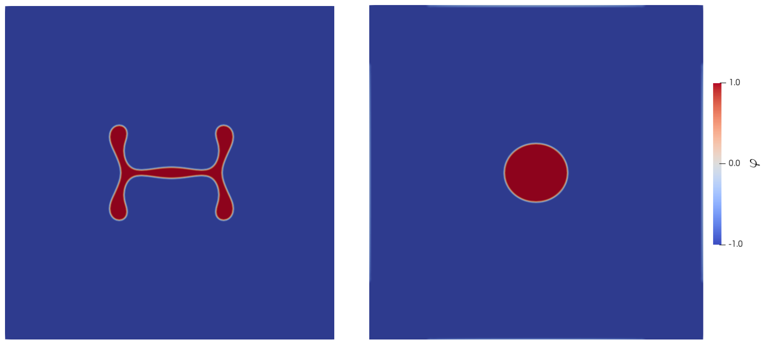

Figure 3.11: Evolution of the tumour with $\eta = 10^{-5}$ and $\beta = 0.01$, left for $m(\varphi) = \frac{1}{2}(1+\varphi)^2$ at time $t = 2.2$, right for $m(\varphi) = \varepsilon$ at time $t = 5$.

3.4.3 Influence of the viscosity

Next we investigate the influence of the viscosity and we always take the one-sided degenerate mobility $m(\varphi) = \frac{1}{2}(1+\varphi)^2$.

In Figure 3.12, we compare the tumour at time $t = 2.5$ for constant viscosities $\eta \in \{0.1, 100\}$ and the Neumann boundary condition for the stress tensor. We see that the results look nearly identical. We also plot the velocity magnitude which is slightly bigger for $\eta = 0.1$. Thus, it seems that the influence of viscosity in the case of stress free boundary conditions is rather low.

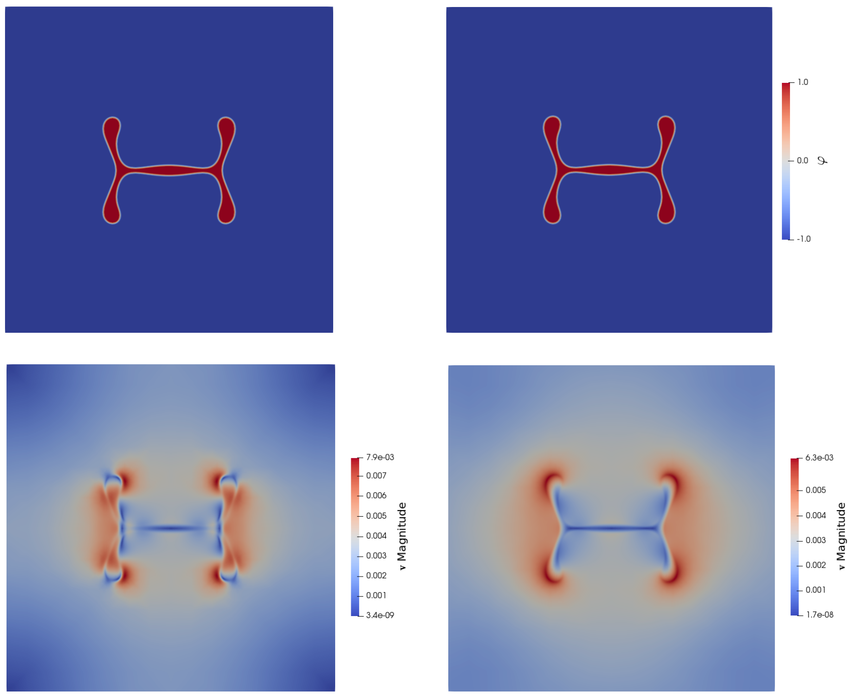

Figure 3.12: Tumour and velocity for $\beta = 0.01$ at time $t = 2.5$, left for $\eta = 0.1$, right for $\eta = 100$, on top the tumour and below the velocity magnitude.

In the case of no-slip conditions on one part of the boundary we observe a different situation. In Figure 3.13, we plot the evolution for $\eta \in \{0.1, 10\}$ with $\nu = 0$, $\beta = 0.1$ and a no-slip boundary condition on the left boundary, i. e., $\partial_2\Omega = \{-3\} \times (-3, 3)$. We see that for low viscosity the tumour evolves radially symmetric whereas instabilities appear if the viscosity is higher.

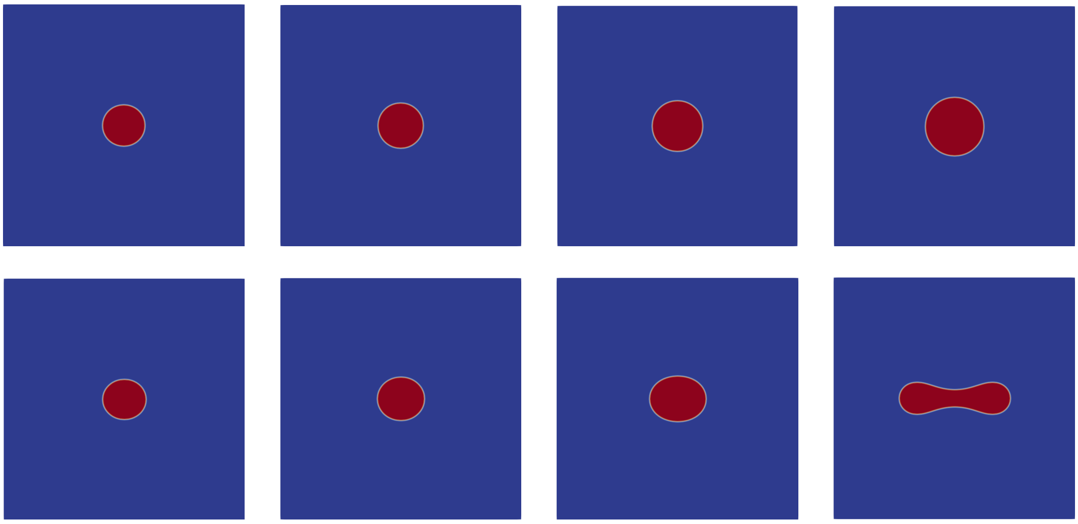

Figure 3.13: Evolution of the tumour at time $t = 1, 3, 6, 10$ with $\beta = 0.1$, $\nu = 0$ and a no-slip boundary condition on the left boundary, on top for $\eta = 0.1$ and below for $\eta = 10$.

We also show the velocity magnitudes at $t = 10$ in Figure 3.14. Although the maximal magnitudes are almost the same, we see more regions with high velocity if the viscosity is bigger, that means for $\eta = 10$. It is also worth noticing that the velocity field is no longer symmetric as observed in Figure 3.12 which is due to the no-slip boundary condition.

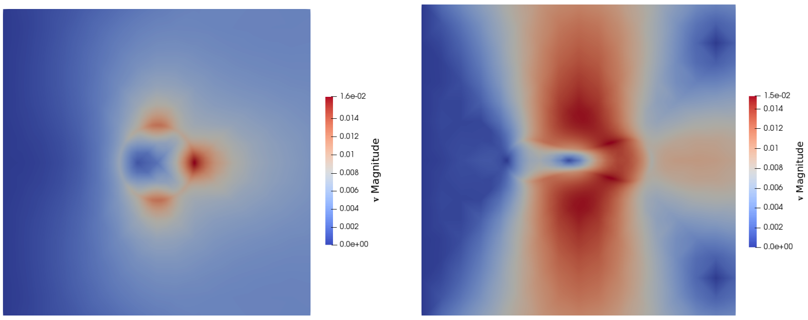

Figure 3.14: The velocity magnitude at time $t = 10$ with $\beta = 0.1$, $\nu = 0$ and a no-slip boundary condition on the left boundary, left for $\eta = 0.1$, right for $\eta = 10$.

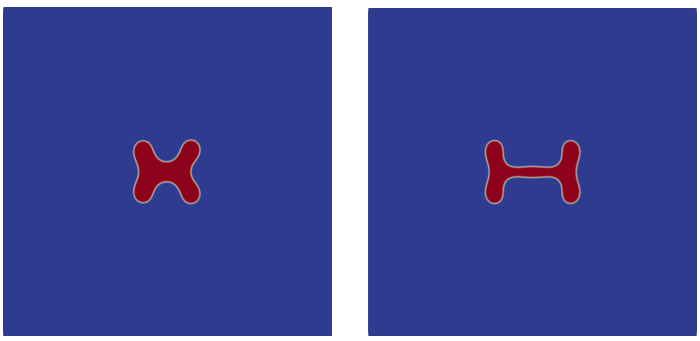

Figure 3.15: The tumour at time $t = 1.5$ with $\beta = 0.01$, $\nu = 0$ and a no-slip boundary condition on the left boundary, left for $\eta = 0.1$, right for $\eta = 10$.

If we decrease the surface tension to $\beta = 0.01$, we can observe the development of fingers for both viscosities and we see that the tumour evolves asymmetric and elongates towards the right boundary, see Figure 3.15

Finally, we investigate the influence of different viscosities for the no-slip boundary condition. We denote by η_+ and η_- the viscosities in the tumour and healthy phase, respectively. In Figure 3.16, we show the tumour at time $t = 10$ for different cases.

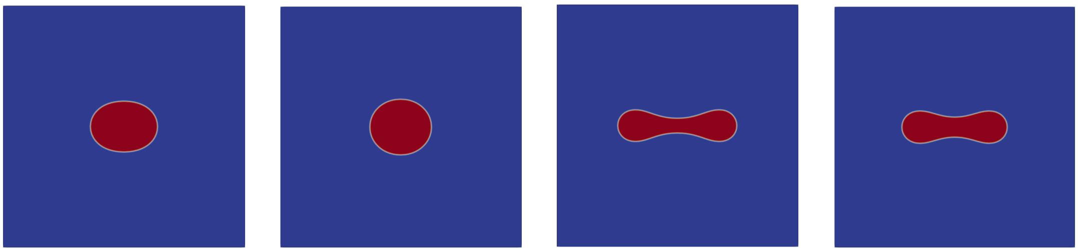

Figure 3.16: Tumour at time $t = 10$ with $\beta = 0.1$, $\nu = 0$ and a no-slip b. c. on the left boundary, with $\eta_- = 0.01$, $\eta_+ = 1$; $\eta_- = 1$, $\eta_+ = 0.01$; $\eta_- = 0.01$, $\eta_+ = 10$; $\eta_- = 10$, $\eta_+ = 0.01$.

It can be seen that a large difference between the viscosities leads to a more interesting evolution. Moreover, instabilities are more pronounced if the viscosity in the surroundings is lower than in the tumour tissue. Thus, the tumour tends to grow towards directions with least resistance. This effect has also been observed in a theoretical analysis in [65].

4

Cahn–Hilliard–Brinkman model for tumour growth

In this chapter we aim to analyse the model (3.34) supplemented with (3.36). Since it has no bearing on the analysis, we set $\beta = 1$ for the rest of this thesis. We will consider the nutrient energy density given by (3.37), that is

$$N(\varphi, \sigma) = \frac{\chi_\sigma}{2}|\sigma|^2 + \chi_\varphi \sigma(1-\varphi),$$

where χ_σ and χ_φ are referred to as nutrient diffusion and chemotaxis parameter, and we denote

$$N_\sigma := \frac{\partial}{\partial\sigma}N(\varphi,\sigma) = \chi_\sigma\sigma + \chi_\varphi(1-\varphi) \quad N_\varphi := \frac{\partial}{\partial\varphi}N(\varphi,\sigma) = -\chi_\varphi\sigma.$$

To get a first impression of the difficulties arising in the analysis, we recall the energy inequality (3.35) given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{-1}\psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2 + N(\varphi,\sigma)\,\mathrm{d}x + \int_\Omega m(\varphi)|\nabla\mu|^2 + n(\varphi)|\nabla N_\sigma|^2\,\mathrm{d}x \\
&\quad + \int_\Omega 2\eta(\varphi)|\mathbf{D}\mathbf{v}|^2 + \lambda(\varphi)(\mathrm{div}(\mathbf{v}))^2 + \nu(\varphi)|\mathbf{v}|^2\,\mathrm{d}x + \int_{\partial\Omega} KN_\sigma(\sigma-\sigma_\infty)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad = \int_\Omega \Gamma_\varphi\mu - \Gamma_\sigma N_\sigma + (p - \mu\varphi - N_\sigma\sigma)\Gamma_{\mathbf{v}}\,\mathrm{d}x.
\end{aligned}$$

In order to bound the right hand side of this inequality we need to make suitable assumptions on the source terms $\Gamma_{\mathbf{v}}$, Γ_φ and Γ_σ. In particular, we have to assume that $\Gamma_{\mathbf{v}}$ is uniformly bounded in order to control the triple products $\mu\varphi\Gamma_{\mathbf{v}}$ and $N_\sigma\sigma\Gamma_{\mathbf{v}}$. As an immediate consequence we have to assume that $\Gamma_{\mathbf{v}}$ is independent of μ. Indeed, to pass to the limit within the Galerkin scheme, all the occurring terms have to be linear in μ. If $\Gamma_{\mathbf{v}}$ would depend linearly on μ, this contradicts the uniform boundedness of $\Gamma_{\mathbf{v}}$. Moreover, we observe that the pressure appears without control. This problem can be circumvented by using the method of subtracting the divergence and estimating the pressure a posteriori. In order to get an estimate on the velocity, the positivity of $\eta(\cdot)$ and $\nu(\cdot)$ is crucial. Finally, we notice that $N(\varphi,\sigma)$ is not positive in general as we cannot guarantee that φ and σ stay in the physical relevant intervals $[-1,1]$ and $[0,1]$, respectively. Therefore, we have to impose a smallness assumption on ε which is not a problem in applications since the parameter ε is usually very small.

The results in this chapter are based on the work [54]. We consider the model (3.34), i. e.,

$$\begin{aligned} \operatorname{div}(\mathbf{v}) &= \Gamma_{\mathbf{v}} && \text{a. e. in } \Omega_T, && (4.1a)\\ -\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu(\varphi)\mathbf{v} &= \mu\nabla\varphi + N_\sigma\nabla\sigma && \text{a. e. in } \Omega_T, && (4.1b)\\ \partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) &= \operatorname{div}(m(\varphi)\nabla\mu) + \Gamma_\varphi && \text{a. e. in } \Omega_T, && (4.1c)\\ \mu &= \varepsilon^{-1}\psi'(\varphi) - \varepsilon\Delta\varphi + N_\varphi && \text{a. e. in } \Omega_T, && (4.1d)\\ \partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) &= \operatorname{div}(n(\varphi)\nabla N_\sigma) - \Gamma_\sigma && \text{a. e. in } \Omega_T, && (4.1e) \end{aligned}$$

where the viscous stress tensor is given by

$$\mathbf{T}(\mathbf{v},p) = 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}.$$

Moreover, we equip the system with the boundary and initial conditions introduced in (3.36), i. e.,

$$\begin{aligned} &\nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} = 0 && \text{a. e. on } \Sigma_T, && (4.2a)\\ &\mathbf{T}(\mathbf{v},p)\mathbf{n} = \mathbf{0} && \text{a. e. on } \Sigma_T, && (4.2b)\\ &n(\varphi)\nabla N_\sigma\cdot\mathbf{n} = K(\sigma_\infty - \sigma) && \text{a. e. on } \Sigma_T, && (4.2c)\\ &\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 && \text{a. e. in } \Omega && (4.2d) \end{aligned}$$

for a boundary permeability constant $K \geq 0$, a given nutrient supply σ_∞ at the boundary and for given functions φ_0 and σ_0.

4.1 Assumptions and main result

We make the following assumptions.

Assumptions 4.1

(A1) *The constants ε and χ_σ are positive and fixed and χ_φ, K are fixed, non-negative constants.*

(A2) *The mobilities $m(\cdot)$, $n(\cdot)$ are continuous on $\mathbb{R}$ and satisfy*

$$m_0 \leq m(t) \leq m_1, \quad n_0 \leq n(t) \leq n_1 \quad \forall\, t \in \mathbb{R}$$

for positive constants m_0, m_1, n_0 and n_1.

(A3) *The viscosities fulfil $\eta, \lambda \in C^1(\mathbb{R})$ with bounded first derivatives and*

$$\eta_0 \leq \eta(t) \leq \eta_1, \quad 0 \leq \lambda(t) \leq \lambda_0 \quad \forall\, t \in \mathbb{R}$$

for positive constants η_0, η_1 and a non-negative constant λ_0. The permeability fulfils $\nu \in C^0(\mathbb{R})$ and

$$\nu_0 \leq \nu(s) \leq \nu_1, \quad |\nu(r) - \nu(s)| \leq L_\nu|r - s| \quad \forall\, r, s \in \mathbb{R}$$

for positive constants ν_0, ν_1 and L_ν.

(A4) *The functions Γ_φ and Γ_σ are of the form*

$$\begin{aligned} \Gamma_\varphi(\varphi,\sigma,\mu) &= \Lambda_\varphi(\varphi,\sigma) - \theta_\varphi(\varphi,\sigma)\mu,\\ \Gamma_\sigma(\varphi,\sigma,\mu) &= \Lambda_\sigma(\varphi,\sigma) - \theta_\sigma(\varphi,\sigma)\mu, \end{aligned}$$

where $\theta_\varphi, \theta_\sigma \colon \mathbb{R}^2 \to \mathbb{R}$ *are continuous bounded functions with* θ_φ *non-negative, and* $\Lambda_\varphi, \Lambda_\sigma \colon \mathbb{R}^2 \to \mathbb{R}$ *are continuous with linear growth, i. e.,*

$$|\theta_i(\varphi,\sigma)| \le R_0, \quad |\Lambda_i(\varphi,\sigma)| \le R_0(1+|\varphi|+|\sigma|) \quad \textit{for } i \in \{\varphi,\sigma\}$$

such that

$$|\Gamma_\varphi| + |\Gamma_\sigma| \le R_0(1+|\varphi|+|\sigma|+|\mu|)$$

for some positive constant R_0.

(A5) *The function* $\Gamma_{\mathbf{v}} \in C^0(\mathbb{R}^2,\mathbb{R})$ *is assumed to be bounded and Lipschitz-continuous, i. e.,*

$$|\Gamma_{\mathbf{v}}(\varphi,\sigma)| \le \gamma_0, \quad |\Gamma_{\mathbf{v}}(\varphi_2,\sigma_2) - \Gamma_{\mathbf{v}}(\varphi_1,\sigma_1)| \le L\left(|\varphi_2-\varphi_1| + |\sigma_2-\sigma_1|\right)$$

for positive constants γ_0 *and* L.

(A6) *The function* $\psi \in C^2(\mathbb{R})$ *is non-negative and satisfies*

$$\psi(t) \ge R_1|t|^2 - R_2 \quad \forall\, t \in \mathbb{R}$$

for some constants R_1, R_2 *with* R_1 *positive, and either one of the following holds:*

1.) if θ_φ *is non-negative and bounded, then there exist positive constants* R_3, R_4 *such that*

$$|\psi(t)| \le R_3(1+|t|^2), \quad |\psi'(t)| \le R_4(1+|t|), \quad |\psi''(t)| \le R_4 \quad \forall\, t \in \mathbb{R}.$$

2.) if θ_φ *is positive and bounded, that is*

$$R_0 \ge \theta_\varphi(t,s) \ge R_5 > 0 \quad \forall\, t,s \in \mathbb{R}$$

for a positive constant R_5, *then*

$$|\psi''(t)| \le R_6(1+|t|^q) \quad \forall\, t \in \mathbb{R}$$

for $q \in [0,4)$ *and for a positive constants* R_6.

Furthermore, we assume that

$$\frac{1}{\varepsilon} > \frac{2\chi_\varphi^2}{\chi_\sigma R_1}.$$

(A7) *The initial and boundary data satisfy*

$$\sigma_\infty \in L^2(0,T;L^2(\partial\Omega)), \quad \varphi_0 \in H^1, \quad \sigma_0 \in L^2.$$

Remark 4.2 Due to the relation of ε to the thickness of the diffuse interface which is typically very small, the assumption on ε in (A6) in practice means no restriction. Furthermore, the Lipschitz-continuity of $\Gamma_{\mathbf{v}}$ is needed within the Galerkin ansatz to guarantee continuity of velocity and pressure under perturbations of φ and σ, see Proposition 2.47. We further remark that our analysis includes source terms of the form (3.31) and (3.32) if we choose, for example, $P(\varphi) = \max\left(0, \min\left(2\delta P_0, \delta P_0(1+\varphi)\right)\right)$ in (3.32) for positive constants δ and P_0, or

$$\Gamma = (\mathcal{P}\max(0,\min(1,\sigma)) - \mathcal{A})h(\varphi), \quad h(\varphi) = \max\left(0, \min\left(1, \tfrac{1}{2}(1+\varphi)\right)\right), \quad \Gamma_\sigma = \mathcal{C}\sigma h(\varphi)$$

in (3.31), where $\mathcal{P}$, $\mathcal{A}$ and $\mathcal{C}$ are the same constants as in (3.31). We cannot use exactly the same form as in (3.31) as $\Gamma_{\mathbf{v}}$ needs to be bounded uniformly.

We now introduce the weak formulation of (4.1)-(4.2).

Definition 4.3 (Weak solution for (4.1)-(4.2)) We call a quintuple $(\varphi, \sigma, \mu, \mathbf{v}, p)$ a weak solution of (4.1)-(4.2) if

$$
\begin{aligned}
&\varphi \in H^1(0,T;(H^1)^*) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2), \\
&\sigma \in W^{1,\frac{4}{3}}(0,T;(H^1)^*) \cap L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \\
&\mu \in L^2(0,T;H^1), \quad \mathbf{v} \in L^2(0,T;\mathbf{H}^1), \quad p \in L^{\frac{4}{3}}(0,T;L^2)
\end{aligned}
$$

such that

$$
\begin{aligned}
&\operatorname{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma) \quad \text{a.e. in } \Omega_T, \quad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega, \\
&\langle \sigma(0), \zeta \rangle_{H^1} = \langle \sigma_0, \zeta \rangle_{H^1} \quad \forall \zeta \in H^1,
\end{aligned}
$$

and

$$
\int_\Omega T(\mathbf{v},p) \colon \nabla \mathbf{\Phi} + \nu(\varphi)\mathbf{v} \cdot \mathbf{\Phi} \,\mathrm{d}x = \int_\Omega (\mu \nabla \varphi + N_\sigma \nabla \sigma) \cdot \mathbf{\Phi} \,\mathrm{d}x, \tag{4.3a}
$$

$$
\langle \partial_t \varphi, \Phi \rangle_{H^1} = \int_\Omega -m(\varphi)\nabla\mu \cdot \nabla\Phi + \Gamma_\varphi \Phi - (\nabla\varphi \cdot \mathbf{v} + \varphi\Gamma_{\mathbf{v}})\,\Phi \,\mathrm{d}x \tag{4.3b}
$$

$$
\int_\Omega \mu\Phi \,\mathrm{d}x = \int_\Omega \varepsilon^{-1}\Psi'(\varphi)\Phi + \varepsilon\nabla\varphi \cdot \nabla\Phi - \chi_\varphi \sigma \Phi \,\mathrm{d}x, \tag{4.3c}
$$

$$
\begin{aligned}
\langle \partial_t \sigma, \Phi \rangle_{H^1} = &\int_\Omega -n(\varphi)\nabla N_\sigma \cdot \nabla\Phi - \Gamma_\sigma \Phi - (\nabla\sigma \cdot \mathbf{v} + \sigma\Gamma_{\mathbf{v}})\Phi \,\mathrm{d}x \\
&+ \int_{\partial\Omega} K(\sigma_\infty - \sigma)\Phi \,\mathrm{d}\mathcal{H}^{d-1}
\end{aligned} \tag{4.3d}
$$

for a. e. $t \in (0,T)$ and for all $\mathbf{\Phi} \in \mathbf{H}^1$, $\Phi \in H^1$.

The main goal of this chapter is to prove the following existence result:

Theorem 4.4 *(Weak solutions for* (4.1)-(4.2)*) Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^{1,1}$-boundary $\partial\Omega$. Suppose Assumptions 4.1 are satisfied. Then, there exists a weak solution quintuple $(\varphi, \sigma, \mu, \mathbf{v}, p)$ for* (4.1)-(4.2) *in the sense of Definition 4.3. Moreover, the estimate*

$$
\begin{aligned}
&\|\varphi\|_{H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^2)} + \|\sigma\|_{W^{1,\frac{4}{3}}((H^1)^*) \cap L^\infty(L^2) \cap L^2(H^1)} \\
&\quad + \|\mu\|_{L^2(H^1)} + K^{\frac{1}{2}} \|\sigma\|_{L^2(L^2(\partial\Omega))} + \|p\|_{L^{\frac{4}{3}}(L^2)} \\
&\quad + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)} + \|\operatorname{div}(\varphi\mathbf{v})\|_{L^2(L^{\frac{3}{2}})} + \|\operatorname{div}(\sigma\mathbf{v})\|_{L^{\frac{4}{3}}((H^1)^*)} \leq C
\end{aligned} \tag{4.4}
$$

holds for a constant C independent of $(\varphi, \mu, \sigma, \mathbf{v}, p)$. The constant C is also uniformly bounded for $K \in (0, 1]$.

Remark 4.5 An additional term $\mathcal{B}(\sigma_B - \sigma)$ could be included in the nutrient equation (4.1e), where $\mathcal{B} \geq 0$ is a constant and σ_B is a given function. Provided that σ_B is regular enough we can still establish the result of Theorem 4.4, and actually all the other results in this thesis remain true. The term $\mathcal{B}(\sigma_B - \sigma)$ models the nutrient supply from an existing vasculature and will be explained in more detail in Chapter 9.

4.2 Existence of weak solutions (Proof of Theorem 4.4)

4.2.1 Galerkin approximation

We will construct approximate solutions by applying a Galerkin approximation with respect to φ, μ and σ and at the same time solve for $\mathbf{v}$ and p in the corresponding whole function spaces. As Galerkin basis for φ, μ and σ, we use the eigenfunctions of the Neumann–Laplace operator $\{w_i\}_{i\in\mathbb{N}}$ that form an orthonormal Schauder basis in L^2 which is also a basis of H^2_N (see Chapter 2).
We fix $k \in \mathbb{N}$ and define

$$\mathcal{W}_k \coloneqq \operatorname{span}\{w_1, \dots, w_k\}.$$

Our aim is to find functions of the form

$$\varphi_k(t,x) = \sum_{i=1}^k a_i^k(t) w_i(x), \quad \mu_k(t,x) = \sum_{i=1}^k b_i^k(t) w_i(x), \quad \sigma_k(t,x) = \sum_{i=1}^k c_i^k(t) w_i(x)$$

satisfying the approximation problem

$$\int_\Omega \partial_t \varphi_k v \,\mathrm{d}x = \int_\Omega -m(\varphi_k)\nabla\mu_k \cdot \nabla v + \Gamma_{\varphi,k} v - (\nabla\varphi_k \cdot \mathbf{v}_k + \varphi_k \Gamma_{\mathbf{v},k}) v \,\mathrm{d}x, \tag{4.5a}$$

$$\int_\Omega \mu_k v \,\mathrm{d}x = \int_\Omega \varepsilon \nabla\varphi_k \cdot \nabla v + \varepsilon^{-1}\psi'(\varphi_k) v - \chi_\varphi \sigma_k v \,\mathrm{d}x, \tag{4.5b}$$

$$\begin{aligned}\int_\Omega \partial_t \sigma_k v \,\mathrm{d}x = &\int_\Omega -n(\varphi_k)(\chi_\sigma \nabla\sigma_k - \chi_\varphi \nabla\varphi_k) \cdot \nabla v - \Gamma_{\sigma,k} v - (\nabla\sigma_k \cdot \mathbf{v}_k + \sigma_k \Gamma_{\mathbf{v},k}) v \,\mathrm{d}x \\ &+ \int_{\partial\Omega} K(\sigma_\infty - \sigma_k) v \,\mathrm{d}\mathcal{H}^{d-1},\end{aligned} \tag{4.5c}$$

which has to hold for all $v \in \mathcal{W}_k$ where $\Gamma_{\varphi,k} \coloneqq \Gamma_\varphi(\varphi_k, \sigma_k, \mu_k)$, $\Gamma_{\sigma,k} \coloneqq \Gamma_\sigma(\varphi_k, \sigma_k, \mu_k)$ and $\Gamma_{\mathbf{v},k} \coloneqq \Gamma_{\mathbf{v}}(\varphi_k, \sigma_k)$. Furthermore, we define the velocity $\mathbf{v}_k$ and the pressure p_k as the weak solutions of (2.31) with

$$\mathbf{f} = \mu_k \nabla\varphi_k + N_{\sigma,k}\nabla\sigma_k, \quad g = \Gamma_{\mathbf{v},k}, \quad c = \varphi_k, \quad \mathbf{f}_b = \mathbf{0},$$

where

$$N_{\sigma,k} \coloneqq \frac{\partial}{\partial\sigma} N(\varphi_k, \sigma_k) = \chi_\sigma \sigma_k + \chi_\varphi (1 - \varphi_k).$$

Using the continuous embedding $H^2_N \hookrightarrow L^\infty$ and (A5), straightforward arguments yield that

$$\mu_k \nabla\varphi_k + N_{\sigma,k}\nabla\sigma_k \in L^2, \quad \Gamma_{\mathbf{v},k} \in L^2.$$

Therefore, by Proposition 2.43, we obtain that $(\mathbf{v}_k, p_k) \in \mathbf{H}^1 \times L^2$ and the following equations are satisfied

$$\int_\Omega \mathbf{T}(\mathbf{v}_k, p_k) \colon \nabla\mathbf{\Phi} + \nu(\varphi_k)\mathbf{v}_k \cdot \mathbf{\Phi} \,\mathrm{d}x = \int_\Omega (\mu_k \nabla\varphi_k + N_{\sigma,k}\nabla\sigma_k) \cdot \mathbf{\Phi} \,\mathrm{d}x \quad \forall\, \mathbf{\Phi} \in \mathbf{H}^1, \tag{4.5d}$$

$$\operatorname{div}(\mathbf{v}_k) = \Gamma_{\mathbf{v},k} \quad \text{a.e. in } \Omega. \tag{4.5e}$$

We define the following matrices with components

$$(\mathbf{S}^k_m)_{ji} \coloneqq \int_\Omega m(\varphi_k)\nabla w_i \cdot \nabla w_j \,\mathrm{d}x, \qquad (\mathbf{S}^k_n)_{ji} \coloneqq \int_\Omega n(\varphi_k)\nabla w_i \cdot \nabla w_j \,\mathrm{d}x \quad \forall\, 1 \le i,j \le n,$$

and introduce for all $1 \leq i, j \leq n$ the notation

$$\begin{aligned}
\psi_j^k &:= \int_\Omega \psi'(\varphi_k) w_j \,\mathrm{d}x, & \boldsymbol{\psi}^k &:= (\psi_1^k, \dots, \psi_k^k)^\intercal, \\
(\mathbf{M}_{\partial\Omega})_{ji} &:= \int_{\partial\Omega} w_i w_j \,\mathrm{d}\mathcal{H}^{d-1}, & \mathbf{S}_{ij} &:= \int_\Omega \nabla w_i \cdot \nabla w_j \,\mathrm{d}x, \\
G_j^k &:= \int_\Omega \Gamma_\varphi(\varphi_k, \sigma_k, \mu_k) w_j \,\mathrm{d}x, & \mathbf{G}^k &:= (G_1^k, \dots, G_k^k)^\intercal, \\
F_j^k &:= \int_\Omega \Gamma_\sigma(\varphi_k, \sigma_k, \mu_k) w_j \,\mathrm{d}x, & \mathbf{F}^k &:= (F_1^k, \dots, F_k^k)^\intercal, \\
\Sigma_j^k &:= \int_{\partial\Omega} \sigma_\infty w_j \,\mathrm{d}\mathcal{H}^{d-1}, & \boldsymbol{\Sigma}^k &:= (\Sigma_1^k, \dots, \Sigma_k^k)^\intercal, \\
(\mathbf{C}^k)_{ji} &:= \int_\Omega \nabla w_i \cdot \mathbf{v}_k w_j \,\mathrm{d}x, & (\mathbf{D}^k)_{ij} &:= \int_\Omega w_i w_j \Gamma_{\mathbf{v}}(\varphi_k, \sigma_k) \,\mathrm{d}x,
\end{aligned}$$

and we denote by δ_{ij} the Kronecker-delta. Furthermore, we define the vectors $\mathbf{a}^k := (a_1^k, \dots, a_k^k)^\intercal$, $\mathbf{b}^k := (b_1^k, \dots, b_k^k)^\intercal$ and $\mathbf{c}^k := (c_1^k, \dots, c_k^k)^\intercal$. Inserting $v = w_j$, $1 \leq j \leq k$, in (4.5a)-(4.5c) and using the above introduced notation, we get a system of ODEs equivalent to (4.5a)-(4.5c), given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}^k = -\mathbf{S}_m^k \mathbf{b}^k + \mathbf{G}^k - (\mathbf{C}^k + \mathbf{D}^k)\mathbf{a}^k, \tag{4.6a}$$

$$\mathbf{b}^k = \varepsilon \mathbf{S}\mathbf{a}^k + \varepsilon^{-1}\boldsymbol{\psi}^k - \chi_\varphi \mathbf{c}^k, \tag{4.6b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}^k = \mathbf{S}_n^k(\chi_\varphi \mathbf{a}^k - \chi_\sigma \mathbf{c}^k) - \mathbf{F}^k - (\mathbf{C}^k + \mathbf{D}^k)\mathbf{c}^k - K\mathbf{M}_{\partial\Omega}\mathbf{c}^k + K\boldsymbol{\Sigma}^k, \tag{4.6c}$$

where $\mathbf{v}_k$, p_k are defined as above. We complete the system with the initial conditions

$$(\mathbf{a}^k)_i(0) = \int_\Omega \varphi_0 w_i \,\mathrm{d}x \quad \forall\, 1 \leq i \leq k, \tag{4.7a}$$

$$(\mathbf{c}^k)_i(0) = \int_\Omega \sigma_0 w_i \,\mathrm{d}x \quad \forall\, 1 \leq i \leq k, \tag{4.7b}$$

where we have

$$\left\| \sum_{i=1}^k (\mathbf{a}^k)_i(0) w_i \right\|_{H^1} \leq \|\varphi_0\|_{H^1}, \qquad \left\| \sum_{i=1}^k (\mathbf{c}^k)_i(0) w_i \right\|_{L^2} \leq \|\sigma_0\|_{L^2}.$$

Substituting (4.6b) and $\mathbf{v}_k$ into (4.6a), (4.6c), we obtain a coupled system of ODEs for $\mathbf{a}^k$ and $\mathbf{c}^k$ where $\boldsymbol{\psi}^k$, $\mathbf{F}^k$, $\mathbf{G}^k$, $\mathbf{S}_m^k$, $\mathbf{S}_n^k$, $\mathbf{C}^k$ and $\mathbf{D}^k$ depend non-linearly on the solutions $\mathbf{a}^k$ and $\mathbf{c}^k$. Owing to the continuity of $m(\cdot)$, $n(\cdot)$, $\psi'(\cdot)$, $\Gamma_{\mathbf{v}}(\cdot,\cdot)$ and the source terms and due to (A3) and the stability of the system (2.31) under perturbations (cf. Proposition 2.47), we obtain that the right hand side of (4.6) depends continuously on $(\mathbf{a}^k, \mathbf{c}^k)$.
Therefore, Lemma 2.27 ensures that there exists $T_k^* \in (0, \infty]$ such that (4.6)-(4.7) has at least one solution triple $\mathbf{a}^k, \mathbf{b}^k, \mathbf{c}^k$ with $\mathbf{a}^k, \mathbf{b}^k, \mathbf{c}^k \in H^1([0, T_k^*), \mathbb{R}^k)$ (where we used the relation (4.6b) for $\mathbf{b}^k$). Hence, (4.5a)-(4.5c) admits at least one solution triplet $(\varphi_k, \mu_k, \sigma_k) \in (H^1([0, T_k^*); \mathcal{W}_k))^3$. Furthermore, we can define $\mathbf{v}_k$ and p_k as the solutions of (4.5d)-(4.5e). With similar arguments as above, we obtain that $(\mathbf{v}_k(t), p_k(t)) \in \mathbf{H}^1 \times L^2$ for all $t \in [0, T_k^*)$. We remark that the Cauchy–Peano theorem cannot be applied since the coefficients $\boldsymbol{\Sigma}^k$ are not continuous in time.

4.2.2 A priori estimates

Let δ_{ij} denote the Kronecker-delta. We choose $v = b_j^k w_j$ in (4.5a), $v = \frac{\mathrm{d}}{\mathrm{d}t} a_j^k w_j$ in (4.5b) and $v = \chi_\sigma c_j^k w_j + \chi_\varphi(\sqrt{|\Omega|}\delta_{1j} - a_j^k) w_j$ in (4.5c) and sum the resulting identities over $j = 1, \dots, k$, to obtain

$$
\begin{aligned}
\int_\Omega \partial_t\varphi_k\mu_k \,\mathrm{d}x &= \int_\Omega -m(\varphi_k)|\nabla\mu_k|^2 + \Gamma_{\varphi,k}\mu_k - (\nabla\varphi_k\cdot\mathbf{v}_k + \varphi_k\Gamma_{\mathbf{v},k})\mu_k \,\mathrm{d}x,\\
\int_\Omega \mu_k\partial_t\varphi_k \,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{-1}\psi(\varphi_k) + \frac{\varepsilon}{2}|\nabla\varphi_k|^2 \,\mathrm{d}x + \int_\Omega N_{\varphi,k}\partial_t\varphi_k \,\mathrm{d}x,\\
\int_\Omega \partial_t\sigma_k N_{\sigma,k} \,\mathrm{d}x &= \int_\Omega -n(\varphi_k)|\nabla N_{\sigma,k}|^2 - \Gamma_{\sigma,k}N_{\sigma,k} - (\nabla\sigma_k\cdot\mathbf{v}_k + \sigma_k\Gamma_{\mathbf{v},k})N_{\sigma,k} \,\mathrm{d}x\\
&\quad + \int_{\partial\Omega} K(\sigma_\infty - \sigma_k)N_{\sigma,k} \,\mathrm{d}\mathcal{H}^{d-1},
\end{aligned}
$$

where we used that

$$
N_{\varphi,k} := \frac{\partial}{\partial\varphi}N(\varphi_k,\sigma_k) = -\chi_\varphi\sigma_k.
$$

Summing up the three identities yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{-1}\psi(\varphi_k) + \frac{\varepsilon}{2}|\nabla\varphi_k|^2 + N(\varphi_k,\sigma_k) \,\mathrm{d}x\\
&\qquad + \int_\Omega m(\varphi_k)|\nabla\mu_k|^2 + n(\varphi_k)|\nabla N_{\sigma,k}|^2 \,\mathrm{d}x + \int_{\partial\Omega} K\chi_\sigma|\sigma_k|^2 \,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad = \int_\Omega \Gamma_{\varphi,k}\mu_k - \Gamma_{\sigma,k}N_{\sigma,k} \,\mathrm{d}x + \int_{\partial\Omega} K(\sigma_\infty N_{\sigma,k} - \sigma_k\chi_\varphi(1-\varphi_k)) \,\mathrm{d}\mathcal{H}^{d-1}\\
&\qquad - \int_\Omega (\nabla\varphi_k\cdot\mathbf{v}_k + \varphi_k\Gamma_{\mathbf{v},k})\mu_k + (\nabla\sigma_k\cdot\mathbf{v}_k + \sigma_k\Gamma_{\mathbf{v},k})N_{\sigma,k} \,\mathrm{d}x.
\end{aligned}
\tag{4.8}
$$

For the Stokes subsystem, we would like to take $\mathbf{v}_k$ as a test function in (4.5d). Then, we would have to get an estimate for p_k without having any a priori estimates on the solutions. Therefore, we use the so called method of subtracting the divergence.
Due to the assumptions on Ω and $\Gamma_\mathbf{v}$ (in particular $\Gamma_{\mathbf{v},k} \in L^\infty$ for all $k \in \mathbb{N}$) and using Lemma 2.39, for every $q \in (1,\infty)$ there exists a solution $\mathbf{u}_k \in \mathbf{W}^{1,q}$ (not necessarily unique) of the problem

$$
\begin{aligned}
\operatorname{div}(\mathbf{u}_k) &= \Gamma_{\mathbf{v},k} && \text{in } \Omega,\\
\mathbf{u}_k &= \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v},k} \,\mathrm{d}x\right)\mathbf{n} =: \mathbf{a}_k && \text{on } \partial\Omega,
\end{aligned}
$$

satisfying the estimate

$$
\|\mathbf{u}_k\|_{\mathbf{W}^{1,q}} \le c\|\Gamma_{\mathbf{v},k}\|_{L^q} \tag{4.9}
$$

with a constant c depending only on q and Ω. We remark that the compatibility condition (2.24) is fulfilled since

$$
\int_{\partial\Omega} \mathbf{a}_k\cdot\mathbf{n} \,\mathrm{d}\mathcal{H}^{d-1} = \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v},k} \,\mathrm{d}x\right)\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{n} \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \Gamma_{\mathbf{v},k} \,\mathrm{d}x.
$$

Choosing $\mathbf{\Phi} = \mathbf{v}_k - \mathbf{u}_k$ in (4.5d) and using (4.5e), we end up at

$$
\begin{aligned}
\int_\Omega 2\eta(\varphi_k)|\mathbf{D}\mathbf{v}_k|^2 + \nu(\varphi_k)|\mathbf{v}_k|^2 \,\mathrm{d}x &= \int_\Omega 2\eta(\varphi_k)\mathbf{D}\mathbf{v}_k : \nabla\mathbf{u}_k + \nu(\varphi_k)\mathbf{v}_k\cdot\mathbf{u}_k \,\mathrm{d}x\\
&\quad + \int_\Omega (\mu_k\nabla\varphi_k + N_{\sigma,k}\nabla\sigma_k)\cdot(\mathbf{v}_k - \mathbf{u}_k) \,\mathrm{d}x.
\end{aligned}
$$

Adding this identity to (4.8) gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{-1}\psi(\varphi_k) + \frac{\varepsilon}{2}|\nabla\varphi_k|^2 + N(\varphi_k,\sigma_k)\,\mathrm{d}x + \int_\Omega 2\eta(\varphi_k)|\mathbf{D}\mathbf{v}_k|^2 + \nu(\varphi_k)|\mathbf{v}_k|^2\,\mathrm{d}x \\
&\quad + \int_\Omega m(\varphi_k)|\nabla\mu_k|^2 + n(\varphi_k)|\nabla N_{\sigma,k}|^2\,\mathrm{d}x + \int_{\partial\Omega} K\chi_\sigma|\sigma_k|^2\,\mathrm{d}\mathcal{H}^{d-1} \\
&= \int_\Omega \Gamma_{\varphi,k}\mu_k - \Gamma_{\sigma,k}N_{\sigma,k}\,\mathrm{d}x + \int_{\partial\Omega} K(\sigma_\infty N_{\sigma,k} - \sigma_k\chi_\varphi(1-\varphi_k))\,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad - \int_\Omega (\nabla\varphi_k\cdot\mathbf{u}_k + \varphi_k\Gamma_{\mathbf{v},k})\mu_k + (\nabla\sigma_k\cdot\mathbf{u}_k + \sigma_k\Gamma_{\mathbf{v},k})N_{\sigma,k}\,\mathrm{d}x \\
&\quad + \int_\Omega 2\eta(\varphi_k)\mathbf{D}\mathbf{v}_k : \nabla\mathbf{u}_k + \nu(\varphi_k)\mathbf{v}_k\cdot\mathbf{u}_k\,\mathrm{d}x. \qquad (4.10)
\end{aligned}$$

We now estimate the terms on the right hand side of this identity individually.

Estimates for the Stokes terms

Using Hölder's and Young's inequalities and inequality (4.9) with $q = 2$, we see that

$$\begin{aligned}
&\left|\int_\Omega 2\eta(\varphi_k)\mathbf{D}\mathbf{v}_k : \nabla\mathbf{u}_k + \nu(\varphi_k)\mathbf{v}_k\cdot\mathbf{u}_k\,\mathrm{d}x\right| \\
&\le 2\eta_1\|\mathbf{D}\mathbf{v}_k\|_{\mathbf{L}^2}\|\nabla\mathbf{u}_k\|_{\mathbf{L}^2} + \nu_1\|\mathbf{v}_k\|_{\mathbf{L}^2}\|\mathbf{u}_k\|_{\mathbf{L}^2} \\
&\le \eta_0\|\mathbf{D}\mathbf{v}_k\|_{\mathbf{L}^2}^2 + \frac{\nu_0}{2}\|\mathbf{v}_k\|_{\mathbf{L}^2}^2 + \left(\frac{\eta_1^2}{\eta_0} + \frac{\nu_1^2}{2\nu_0}\right)\|\mathbf{u}_k\|_{\mathbf{H}^1}^2 \\
&\le \eta_0\|\mathbf{D}\mathbf{v}_k\|_{\mathbf{L}^2}^2 + \frac{\nu_0}{2}\|\mathbf{v}_k\|_{\mathbf{L}^2}^2 + C(q,|\Omega|)\left(\frac{\eta_1^2}{\eta_0} + \frac{\nu_1^2}{2\nu_0}\right)\|\Gamma_{\mathbf{v},k}\|_{L^2}^2 \\
&\le \eta_0\|\mathbf{D}\mathbf{v}_k\|_{\mathbf{L}^2}^2 + \frac{\nu_0}{2}\|\mathbf{v}_k\|_{\mathbf{L}^2}^2 + C(q,|\Omega|)\left(\frac{\eta_1^2}{\eta_0} + \frac{\nu_1^2}{2\nu_0}\right)\gamma_0^2, \qquad (4.11)
\end{aligned}$$

where we used (A3), (A5), and where $C(q,|\Omega|)$ depends on the constant arising in (4.9).

Estimates for the boundary term

Using again Hölder's and Young's inequalities together with the trace theorem, we see that

$$\begin{aligned}
&\left|\int_{\partial\Omega} K(\sigma_\infty N_{\sigma,k} - \sigma_k\chi_\varphi(1-\varphi_k))\,\mathrm{d}\mathcal{H}^{d-1}\right| \\
&\le \frac{K\chi_\sigma}{2}\|\sigma_k\|_{L^2(\partial\Omega)}^2 + \left(\frac{2K\chi_\varphi^2}{\chi_\sigma} + \frac{K\chi_\varphi}{2}\right)\left(|\Omega| + \|\varphi_k\|_{L^2(\partial\Omega)}^2\right) + K(\chi_\varphi + \chi_\sigma)\|\sigma_\infty\|_{L^2(\partial\Omega)}^2 \\
&\le \frac{K\chi_\sigma}{2}\|\sigma_k\|_{L^2(\partial\Omega)}^2 + C_1\left(1 + \|\varphi_k\|_{H^1}^2\right) + C_2\|\sigma_\infty\|_{L^2(\partial\Omega)}^2, \qquad (4.12)
\end{aligned}$$

where

$$C_1 := \left(\frac{2K\chi_\varphi^2}{\chi_\sigma} + \frac{K\chi_\varphi}{2}\right)(|\Omega| + C_{\mathrm{tr}}^2), \quad C_2 := K(\chi_\varphi + \chi_\sigma),$$

and C_{tr} is the constant resulting from the trace theorem.

Energy inequality for non-negative θ_φ

First we deduce an estimate for the L^2-norm of μ_k. Inserting $v = b_j^k w_j$ into (4.5b) and summing over $j = 1,\dots,k$, yields

$$\int_\Omega |\mu_k|^2\,\mathrm{d}x = \int_\Omega \varepsilon^{-1}\psi'(\varphi_k)\mu_k + \varepsilon\nabla\varphi_k\cdot\nabla\mu_k - \chi_\varphi\sigma_k\mu_k\,\mathrm{d}x.$$

Using Hölder's and Young's inequalities together with the assumptions on ψ (see (A6)), we obtain

$$\begin{aligned}\|\mu_k\|_{L^2}^2 &\le \int_\Omega \varepsilon^{-1}R_4(1+|\varphi_k|)|\mu_k| + \varepsilon|\nabla\varphi_k||\nabla\mu_k| + \chi_\varphi|\sigma_k||\mu_k| \,\mathrm{d}x \\ &\le \frac{1}{2}\|\mu_k\|_{L^2}^2 + \frac{2R_4^2}{\varepsilon^2}\left(|\Omega| + \|\varphi_k\|_{L^2}^2\right) + \frac{\varepsilon}{2}\left(\|\nabla\varphi_k\|_{\mathbf{L}^2}^2 + \|\nabla\mu_k\|_{\mathbf{L}^2}^2\right) + \chi_\varphi^2\|\sigma_k\|_{L^2}^2,\end{aligned}$$

and consequently

$$\|\mu_k\|_{L^2}^2 \le \frac{4R_4^2}{\varepsilon^2}\left(|\Omega| + \|\varphi_k\|_{L^2}^2\right) + \varepsilon\left(\|\nabla\varphi_k\|_{\mathbf{L}^2}^2 + \|\nabla\mu_k\|_{\mathbf{L}^2}^2\right) + 2\chi_\varphi^2\|\sigma_k\|_{L^2}^2. \tag{4.13}$$

By (A4) we observe that

$$\Gamma_\varphi(\varphi_k,\sigma_k,\mu_k)\mu_k = \Lambda_\varphi(\varphi_k,\sigma_k)\mu_k - \theta_\varphi(\varphi_k,\sigma_k)|\mu_k|^2.$$

Therefore, we can neglect the non-positive term $-\theta_\varphi(\varphi_k,\sigma_k)|\mu_k|^2$ on the r. h. s. of (4.10). Using (A4) and Hölder's inequality (in the following, we will write $\Lambda_{i,k} \coloneqq \Lambda_i(\varphi_k,\sigma_k)$ for $i=\varphi,\sigma$), we can estimate the first term on the r. h. s. of (4.10) by

$$\begin{aligned}&\left|\int_\Omega \Lambda_{\varphi,k}\mu_k - \Gamma_{\sigma,k}(\chi_\sigma\sigma_k + \chi_\varphi(1-\varphi_k)) \,\mathrm{d}x\right| \\ &\le \|\Lambda_{\varphi,k}\|_{L^2}\|\mu_k\|_{L^2} + (\|\Lambda_{\sigma,n}\|_{L^2} + R_0\|\mu_k\|_{L^2})\left(\|\chi_\sigma\sigma_k + \chi_\varphi(1-\varphi_k)\|_{L^2}\right) \\ &\le R_0\left((1+\chi_\varphi)(|\Omega|^{\frac{1}{2}} + \|\varphi_k\|_{L^2}) + (1+\chi_\sigma)\|\sigma_k\|_{L^2}\right)\|\mu_k\|_{L^2} \\ &\quad + R_0(|\Omega|^{\frac{1}{2}} + \|\varphi_k\|_{L^2} + \|\sigma_k\|_{L^2})\left(\chi_\varphi|\Omega|^{\frac{1}{2}} + \chi_\sigma\|\sigma_k\|_{L^2} + \chi_\varphi\|\varphi_k\|_{L^2}\right).\end{aligned}$$

Using Young's inequality, we obtain

$$\left|\int_\Omega \Lambda_{\varphi,k}\mu_k - \Gamma_{\sigma,k}(\chi_\sigma\sigma_k + \chi_\varphi(1-\varphi_k)) \,\mathrm{d}x\right| \le \delta\|\mu_k\|_{L^2}^2 + C_{3,\delta}\left(1+\|\varphi_k\|_{L^2}^2\right) + C_{4,\delta}\|\sigma_k\|_{L^2}^2 \tag{4.14}$$

with constants

$$\begin{aligned}C_{3,\delta} &\coloneqq \left(\frac{3R_0^2}{4\delta}(1+\chi_\varphi)^2 + R_0\left(1+\chi_\varphi+\chi_\varphi^2\right)\right)(1+|\Omega|), \\ C_{4,\delta} &\coloneqq \frac{3R_0^2}{4\delta}(1+\chi_\sigma)^2 + R_0(1+\chi_\sigma+\chi_\sigma^2),\end{aligned}$$

and $\delta > 0$ to be chosen later. It remains to estimate the third and fourth integral on the r. h. s. of (4.10). Using (A5), (4.9) and the continuous embedding $L^\infty \hookrightarrow L^q$ for all $q \in (1,\infty)$, we observe that

$$\|\mathbf{u}_k\|_{\mathbf{W}^{1,q}} \le c(q,\Omega)\|\Gamma_{\mathbf{v},k}\|_{L^q} \le c(q,\Omega)\|\Gamma_{\mathbf{v},k}\|_{L^\infty} \le c(q,\Omega,\gamma_0)$$

for all $q \in (1,\infty)$. Using (2.1)-(2.2) and the Sobolev embedding $W^{1,q} \subset L^\infty$, $q \in (3,\infty)$, we obtain

$$\begin{aligned}&\left|\int_\Omega (\nabla\varphi_k\cdot\mathbf{u}_k + \varphi_k\Gamma_{\mathbf{v},k})\mu_k \,\mathrm{d}x\right| \\ &\le \left(\|\nabla\varphi_k\|_{\mathbf{L}^2}\|\mathbf{u}_k\|_{\mathbf{L}^\infty} + \|\varphi_k\|_{L^2}\|\Gamma_{\mathbf{v},k}\|_{L^\infty}\right)\|\mu_k\|_{L^2} \\ &\le C(q,|\Omega|)\|\Gamma_{\mathbf{v},k}\|_{L^\infty}\left(\|\nabla\varphi_k\|_{\mathbf{L}^2} + \|\varphi_k\|_{L^2}\right)\|\mu_k\|_{L^2} \\ &\le \frac{\gamma_0^2 C(q,|\Omega|)}{2\delta}\left(\|\varphi_k\|_{L^2}^2 + \|\nabla\varphi_k\|_{\mathbf{L}^2}^2\right) + \delta\|\mu_k\|_{L^2}^2 \quad \forall\, q\in(3,\infty)\end{aligned} \tag{4.15}$$

with $\delta > 0$ to be chosen later. With similar arguments we deduce for $q \in (3, \infty)$ that

$$
\begin{aligned}
&\left| \int_\Omega (\nabla \sigma_k \cdot \mathbf{u}_k + \sigma_k \Gamma_{\mathbf{v},k}) N_{\sigma,k} \, \mathrm{d}x \right| \\
&\quad \leq \left(\|\nabla \sigma_k\|_{\mathbf{L}^2} \|\mathbf{u}_k\|_{\mathbf{L}^\infty} + \|\sigma_k\|_{L^2} \|\Gamma_{\mathbf{v},k}\|_{\mathbf{L}^\infty} \right) \|N_{\sigma,k}\|_{L^2} \\
&\quad \leq \gamma_0 C(q, |\Omega|) \|\nabla \sigma_k\| \|N_{\sigma,k}\|_{L^2} + \gamma_0 \|\sigma_k\|_{L^2} \|N_{\sigma,k}\|_{L^2} \\
&\quad \leq \left(C(q, |\Omega|) C_{5,\tilde{\delta}} + C_6 \right) \left(1 + \|\varphi_k\|_{L^2}^2 + \|\sigma_k\|_{L^2}^2 \right) + \tilde{\delta} \|\nabla \sigma_k\|_{\mathbf{L}^2}^2
\end{aligned}
\tag{4.16}
$$

with

$$
C_{5,\tilde{\delta}} \coloneqq \frac{\gamma_0^2 (3\chi_\varphi^2 (1 + |\Omega|) + 3\chi_\sigma^2)}{2\tilde{\delta}}, \qquad C_6 \coloneqq \gamma_0 \left(1 + \chi_\sigma + \frac{\chi_\varphi}{2} (1 + |\Omega|) \right),
$$

and $\tilde{\delta} > 0$ to be chosen later. Furthermore, using Hölder's and Young's inequalities we deduce that

$$
\|\chi_\sigma \nabla \sigma_k\|_{\mathbf{L}^2}^2 = \|\nabla N_{\sigma,k} + \chi_\varphi \nabla \varphi_k\|_{\mathbf{L}^2}^2 \leq 2 \left(\|\nabla N_{\sigma,k}\|_{\mathbf{L}^2}^2 + \|\chi_\varphi \nabla \varphi_k\|_{\mathbf{L}^2}^2 \right). \tag{4.17}
$$

In the following we fix $q \in (3, \infty)$ and we denote by C_K the constant arising in Korn's inequality. Choosing δ, $\tilde{\delta}$ small enough and applying (A3), (A6) along with (4.11)-(4.17) in (4.10) yields the energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varepsilon^{-1} \psi(\varphi_k) + \frac{\varepsilon}{2} |\nabla \varphi_k|^2 + \frac{\chi_\sigma}{2} |\sigma_k|^2 + \chi_\varphi \sigma_k (1 - \varphi_k) \, \mathrm{d}x \\
&\quad + \frac{\min(\eta_0, \nu_0/2)}{C_K^2} \|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + \frac{m_0}{2} \|\nabla \mu_k\|_{\mathbf{L}^2}^2 + \frac{n_0 \chi_\sigma^2}{2} \|\nabla \sigma_k\|_{\mathbf{L}^2}^2 + \frac{K \chi_\sigma}{2} \|\sigma_k\|_{L^2(\partial\Omega)}^2 \\
&\leq \bar{C}_b \left(1 + \|\nabla \varphi_k\|_{\mathbf{L}^2}^2 + \|\sigma_k\|_{L^2}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2 \right) + \frac{\bar{C}_b}{R_1} \left(\|\psi(\varphi_k)\|_{L^1} + R_2 |\Omega| \right)
\end{aligned}
$$

with a constant $\bar{C}_b$ depending on the system parameters, but not on $k \in \mathbb{N}$. Integrating with respect to time from 0 to $s \in (0, T]$ gives

$$
\begin{aligned}
&\varepsilon^{-1} \|\psi(\varphi_k(s))\|_{L^1} + \frac{\varepsilon}{2} \|\nabla \varphi_k(s)\|_{\mathbf{L}^2}^2 + \frac{\chi_\sigma}{2} \|\sigma_k(s)\|_{L^2}^2 + \int_\Omega \chi_\varphi \sigma_k(s) (1 - \varphi_k(s)) \, \mathrm{d}x \\
&\quad + \int_0^s \frac{\min(\eta_0, \nu_0/2)}{C_K^2} \|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + \frac{m_0}{2} \|\nabla \mu_k\|_{\mathbf{L}^2}^2 + \frac{n_0 \chi_\sigma^2}{2} \|\nabla \sigma_k\|_{\mathbf{L}^2}^2 + \frac{K \chi_\sigma}{2} \|\sigma_k\|_{L^2(\partial\Omega)}^2 \, \mathrm{d}t \\
&\leq \bar{C}_b \left(1 + \frac{R_2 |\Omega|}{R_1} \right) T + \bar{C}_b \int_0^s \|\nabla \varphi_k\|_{\mathbf{L}^2}^2 + \|\sigma_k\|_{L^2}^2 + \frac{1}{R_1} \|\psi(\varphi_k)\|_{L^1} \, \mathrm{d}t \\
&\quad + \bar{C}_b \|\sigma_\infty\|_{L^2(0,T;L^2(\partial\Omega))}^2 + \varepsilon^{-1} \|\psi(\varphi_0)\|_{L^1} + \frac{\varepsilon}{2} \|\varphi_0\|_{H^1}^2 + \frac{\chi_\sigma}{2} \|\sigma_0\|_{L^2}^2.
\end{aligned}
\tag{4.18}
$$

Since $\varphi_0 \in H^1$, $\sigma_0 \in L^2$ and $\psi(\varphi_0) \in L^1$ by (A6), we observe that

$$
C_I \coloneqq \varepsilon^{-1} \|\psi(\varphi_0)\|_{L^1} + \frac{\varepsilon}{2} \|\varphi_0\|_{H^1}^2 + \frac{\chi_\sigma}{2} \|\sigma_0\|_{L^2}^2 < \infty.
$$

Using Hölder's and Young's inequalities together with (A6), we obtain

$$
\begin{aligned}
&\left| \int_\Omega \chi_\varphi \sigma_k(s) (1 - \varphi_k(s)) \, \mathrm{d}x \right| \\
&\quad \leq \frac{3\chi_\sigma}{8} \|\sigma_k(s)\|_{L^2}^2 + \frac{\chi_\varphi^2}{\chi_\sigma R_1} \|\psi(\varphi_k(s))\|_{L^1} + \left(\frac{\chi_\varphi^2 R_2}{\chi_\sigma R_1} + \frac{2\chi_\varphi^2}{\chi_\sigma} \right) |\Omega| \\
&\quad \leq \frac{3\chi_\sigma}{8} \|\sigma_k(s)\|_{L^2}^2 + \frac{1}{2\varepsilon} \|\psi(\varphi_k(s))\|_{L^1} + \left(\frac{\chi_\varphi^2 R_2}{\chi_\sigma R_1} + \frac{2\chi_\varphi^2}{\chi_\sigma} \right) |\Omega|.
\end{aligned}
$$

Substituting this inequality into (4.18) yields

$$\begin{aligned}
&\min\left(\frac{1}{2\varepsilon},\frac{\varepsilon}{2},\frac{\chi_\sigma}{8}\right)\big(\|\psi(\varphi_k(s))\|_{L^1}+\|\nabla\varphi_k(s)\|^2_{\mathbf{L}^2}+\|\sigma_k(s)\|^2_{L^2}\big)\\
&\quad+\int_0^s\frac{\min(\eta_0,\nu_0/2)}{C_K^2}\|\mathbf{v}_k\|^2_{\mathbf{H}^1}+\frac{m_0}{2}\|\nabla\mu_k\|^2_{\mathbf{L}^2}+\frac{n_0\chi_\sigma^2}{2}\|\nabla\sigma_k\|^2_{\mathbf{L}^2}+\frac{K\chi_\sigma}{2}\|\sigma_k\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\\
&\le\tilde{C}_b\big(1+T+\|\sigma_\infty\|^2_{L^2(L^2(\partial\Omega))}\big)+C_I+\tilde{C}_b\int_0^s\|\nabla\varphi_k\|^2_{\mathbf{L}^2}+\|\sigma_k\|^2_{L^2}+\|\psi(\varphi_k)\|_{L^1}\,\mathrm{d}t,
\end{aligned}\tag{4.19}$$

where

$$\tilde{C}_b:=\max\left(\bar{C}_b\left(1+\frac{R_2|\Omega|}{R_1}\right),\left(\frac{\chi_\varphi^2R_2}{\chi_\sigma R_1}+\frac{2\chi_\varphi^2}{\chi_\sigma}\right)|\Omega|,\frac{\bar{C}_b}{R_1}\right).$$

Setting

$$\alpha:=\tilde{C}_b\left(1+T+\|\sigma_\infty\|^2_{L^2(0,T;L^2(\partial\Omega))}\right)+C_I,\quad \beta:=\tilde{C}_b,$$

and noting that

$$\alpha\left(1+\int_0^s\beta\exp\left(\int_0^t\beta\mathrm{d}r\right)\,\mathrm{d}t\right)=\alpha(1+\exp(\beta s)-1)\le\alpha\exp(\beta T),$$

an application of Lemma 2.30 to (4.19) gives

$$\begin{aligned}
&\sup_{s\in(0,T]}\big(\|\psi(\varphi_k(s))\|_{L^1}+\|\nabla\varphi_k(s)\|^2_{\mathbf{L}^2}+\|\sigma_k(s)\|^2_{L^2}\big)\\
&\quad+\int_0^T\|\mathbf{v}_k\|^2_{\mathbf{H}^1}+\|\nabla\mu_k\|^2_{\mathbf{L}^2}+\|\nabla\sigma_k\|^2_{\mathbf{L}^2}+K\|\sigma_k\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\le C
\end{aligned}\tag{4.20}$$

with a constant C independent of $k\in\mathbb{N}$. In particular, the constant C is bounded uniformly for $K\in(0,1]$. In the following we will use the constant C as a generic constant which may change its value even within one line. Using (A6) and (4.13), as an immediate consequence of (4.20) we obtain

$$\sup_{s\in(0,T]}\|\varphi_k(s)\|_{H^1}+\int_0^T\|\mu_k\|^2_{H^1}\,\mathrm{d}t\le C.\tag{4.21}$$

Energy inequality for positive θ_φ

We assume that (A6), 2.) is valid. Then, arguing as above, the specific form of Γ_φ yields

$$\Gamma_{\varphi,k}\mu_k=\Lambda_{\varphi,k}\mu_k-\theta_\varphi(\varphi_k,\sigma_k)|\mu_k|^2.$$

We move the second term on the r. h. s. of this equation to the l. h. s. of (4.10). Then, we can perform exactly the same estimates as before, but we do not need (4.13). We remark that estimate (4.13) was the only reason why we needed assumption (A6), 1.). Again choosing δ and $\tilde{\delta}$ small enough, we arrive at the inequality (compare (4.19))

$$\begin{aligned}
&\min\left(\frac{1}{2\varepsilon},\frac{\varepsilon}{2},\frac{\chi_\sigma}{8}\right)\big(\|\psi(\varphi_k(s))\|_{L^1}+\|\nabla\varphi_k(s)\|^2_{\mathbf{L}^2}+\|\sigma_k(s)\|^2_{L^2}\big)\\
&\quad+\int_0^s C_{11}\|\mathbf{v}_k\|^2_{\mathbf{H}^1}+m_0\|\nabla\mu_k\|^2_{\mathbf{L}^2}+\frac{R_5}{2}\|\mu_k\|^2_{L^2}+\frac{n_0\chi_\sigma^2}{2}\|\nabla\sigma_k\|^2_{\mathbf{L}^2}+\frac{K\chi_\sigma}{2}\|\sigma_k\|^2_{L^2(\partial\Omega)}\,\mathrm{d}t\\
&\le C\big(1+T+\|\sigma_\infty\|^2_{L^2(0,T;L^2(\partial\Omega))}\big)+C_I+C\int_0^s\|\nabla\varphi_k\|^2_{\mathbf{L}^2}+\|\sigma_k\|^2_{L^2}+\|\psi(\varphi_k)\|_{L^1}\,\mathrm{d}t
\end{aligned}\tag{4.22}$$

with C_I as defined above. Here, we have the term $m_0\|\nabla\mu_k\|^2_{\mathbf{L}^2}$ instead of $\frac{m_0}{2}\|\nabla\mu_k\|^2_{\mathbf{L}^2}$ because we do not use (4.13). We still have

$$\|\psi(\varphi_0)\|_{L^1}<\infty$$

since

$$\|\psi(\varphi_0)\|_{L^1} \le C\left(1+\|\varphi_0\|_{L^6}^6\right) \le C\left(1+\|\varphi_0\|_{H^1}^6\right) < \infty$$

due to (A6) and the Sobolev embedding $H^1 \subset L^6$.
Applying Lemma 2.30 and using similar arguments as above along with (A6), from (4.22) we obtain (4.20)-(4.21).

Estimates for the pressure

Using (4.5d) and (4.5e) we deduce that

$$\begin{aligned}\int_\Omega p_k \mathrm{div}(\mathbf{\Phi})\,\mathrm{d}x &= \int_\Omega (2\eta(\varphi_k)\mathbf{D}\mathbf{v}_k + \lambda(\varphi_k)\Gamma_{\mathbf{v},k}\mathbf{I})\colon \nabla\mathbf{\Phi}\,\mathrm{d}x \\ &\quad + \int_\Omega (\nu(\varphi_k)\mathbf{v}_k - \mu_k\nabla\varphi_k - N_{\sigma,k}\nabla\sigma_k)\cdot\mathbf{\Phi}\,\mathrm{d}x \end{aligned} \tag{4.23}$$

for all $\mathbf{\Phi} \in \mathbf{H}^1$. Now, we define a family of functionals on $\mathbf{H}^1$ by

$$\mathcal{F}_k(\mathbf{\Phi}) \coloneqq \int_\Omega (2\eta(\varphi_k)\mathbf{D}\mathbf{v}_k + \lambda(\varphi_k)\Gamma_{\mathbf{v},k}\mathbf{I})\colon \nabla\mathbf{\Phi} + \nu(\varphi_k)\mathbf{v}_k\cdot\mathbf{\Phi} - (\mu_k\nabla\varphi_k + N_{\sigma,k}\nabla\sigma_k)\cdot\mathbf{\Phi}\,\mathrm{d}x$$

for all $\mathbf{\Phi} \in \mathbf{H}^1$. Using Hölder's inequality, (A3) and the Sobolev embedding $H^1 \subset L^6$, we obtain

$$\begin{aligned}|\mathcal{F}_k(\mathbf{\Phi})| &\le C\left(\|\mathbf{v}_k\|_{\mathbf{H}^1} + \|\Gamma_{\mathbf{v},k}\|_{L^2} + \|\mathbf{v}_k\|_{\mathbf{L}^2} + \|\mu_k\nabla\varphi_k\|_{\mathbf{L}^{\frac{6}{5}}} + \|N_{\sigma,k}\nabla\sigma_k\|_{\mathbf{L}^{\frac{6}{5}}}\right)\|\mathbf{\Phi}\|_{\mathbf{H}^1} \\ &\le C\left(1 + \|\mathbf{v}_k\|_{\mathbf{H}^1} + \|\mu_k\|_{L^3}\|\nabla\varphi_k\|_{\mathbf{L}^2} + \|N_{\sigma,k}\|_{L^3}\|\nabla\sigma_k\|_{\mathbf{L}^2}\right)\|\mathbf{\Phi}\|_{\mathbf{H}^1}\end{aligned}$$

with $C = C(\Omega, \gamma_0, \eta_1, \lambda_0, \nu_0, \nu_1)$. Taking the supremum over all $\mathbf{\Phi} \in \mathbf{H}^1$ with $\|\mathbf{\Phi}\|_{\mathbf{H}^1} \le 1$, we deduce that

$$\|\mathcal{F}_k\|_{(\mathbf{H}^1)^*} \le C\left(1 + \|\mathbf{v}_k\|_{\mathbf{H}^1} + \|\mu_k\|_{L^3}\|\nabla\varphi_k\|_{\mathbf{L}^2} + \|N_{\sigma,k}\|_{L^3}\|\nabla\sigma_k\|_{\mathbf{L}^2}\right). \tag{4.24}$$

Now, (4.23) implies

$$\mathcal{F}_k(\mathbf{\Phi}) = \int_\Omega p_k \mathrm{div}(\mathbf{\Phi})\,\mathrm{d}x \quad \forall\,\mathbf{\Phi} \in \mathbf{H}^1. \tag{4.25}$$

Invoking Lemma 2.39 we deduce that there is at least one solution $\mathbf{q}_k \in \mathbf{H}^1$ of the system

$$\mathrm{div}(\mathbf{q}_k) = p_k \quad \text{a.e. in } \Omega, \qquad \mathbf{q}_k = \frac{1}{|\partial\Omega|}\left(\int_\Omega p_k\,\mathrm{d}x\right)\mathbf{n} \quad \text{a.e. on } \partial\Omega$$

satisfying

$$\|\mathbf{q}_k\|_{\mathbf{H}^1} \le C_d\|p_k\|_{L^2} \tag{4.26}$$

with C_d depending only on Ω. Notice that the compatibility condition (2.24) is satisfied since

$$\int_{\partial\Omega} \mathbf{q}_k\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = \frac{1}{|\partial\Omega|}\left(\int_\Omega p_k\,\mathrm{d}x\right)\int_{\partial\Omega}\mathbf{n}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega p_k\,\mathrm{d}x.$$

Choosing $\mathbf{\Phi} = \mathbf{q}_k$ in (4.25) and using Young's inequality along with (4.26), we obtain

$$\|p_k\|_{L^2}^2 = \mathcal{F}_k(\mathbf{q}_k) \le \|\mathcal{F}_k\|_{(\mathbf{H}^1)^*}\|\mathbf{q}_k\|_{\mathbf{H}^1} \le C_d\|\mathcal{F}_k\|_{(\mathbf{H}^1)^*}\|p_k\|_{L^2} \le \frac{C_d^2}{2}\|\mathcal{F}_k\|_{(\mathbf{H}^1)^*}^2 + \frac{1}{2}\|p_k\|_{L^2}^2$$

which gives

$$\|p_k\|_{L^2} \le C_d\|\mathcal{F}_k\|_{(\mathbf{H}^1)^*}.$$

Using Young's and Hölder's inequalities together with (4.24), the last inequality implies

$$\int_0^T \|p_k\|_{L^2}^{\frac{4}{3}} \,\mathrm{d}t \le C \int_0^T 1 + \|\mathbf{v}_k\|_{\mathbf{H}^1}^{\frac{4}{3}} + \|\mu_k\|_{L^3}^{\frac{4}{3}} \|\nabla\varphi_k\|_{\mathbf{L}^2}^{\frac{4}{3}} + \|N_{\sigma,k}\|_{L^3}^{\frac{4}{3}} \|\nabla\sigma_k\|_{\mathbf{L}^2}^{\frac{4}{3}} \,\mathrm{d}t$$
$$\le C\left(1 + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)}^{\frac{4}{3}} + \|\mu_k\|_{L^2(L^3)}^{\frac{4}{3}} \|\nabla\varphi_k\|_{L^4(\mathbf{L}^2)}^{\frac{4}{3}} + \|N_{\sigma,k}\|_{L^4(L^3)}^{\frac{4}{3}} \|\nabla\sigma_k\|_{L^2(\mathbf{L}^2)}^{\frac{4}{3}}\right).$$

Due to (4.20)-(4.21), the first three terms on the r. h. s. of this inequality and the term $\|\nabla\sigma_k\|_{L^2(\mathbf{L}^2)}^{\frac{4}{3}}$ are bounded. Using the continuous embedding $L^\infty(L^2) \cap L^2(H^1) \hookrightarrow L^4(L^3)$ resulting from Gagliardo–Nirenberg's inequality and the bounds (4.20)-(4.21), by the specific form of $N_{\sigma,k}$ we obtain $N_{\sigma,k} \in L^4(L^3)$ with bounded norm (independent of $k \in \mathbb{N}$). Consequently, the r. h. s. of the last inequality is bounded independent of $k \in \mathbb{N}$ which implies

$$\|p_k\|_{L^{\frac{4}{3}}(L^2)} \le C. \tag{4.27}$$

Remark 4.6 Alternatively we may also use a Poincaré-type inequality which is a consequence of Nečas' inequality (see [22, Thm IV.1.1 and Prop. IV.1.7]). Indeed, for all $q \in L^2$ there exists a constant C_N such that

$$\|q\|_{L^2} \le C_N \left(\frac{1}{|\Omega|}\left|\int_\Omega q \,\mathrm{d}x\right| + \|\nabla q\|_{(\mathbf{H}_0^1)^*}\right). \tag{4.28}$$

Taking $\mathbf{\Phi} \in \mathbf{H}_0^1$ arbitrary in (4.23) and using the same estimates as above yields

$$\|\nabla p_k\|_{L^{\frac{4}{3}}((\mathbf{H}_0^1)^*)} \le C.$$

With similar arguments and taking, for example, $\mathbf{\Phi} = (x_1, 0, 0)^\intercal$ in (4.23), we obtain

$$\|(p_k)_\Omega\|_{L^{\frac{4}{3}}(0,T)} \le C.$$

Combining the last two estimates and using (4.28) gives (4.27).

Higher order estimates for φ_k

We aim to deduce estimates for φ_k in $L^2(H^2)$. Using Gagliardo–Nirenberg's inequality and the Sobolev embedding $H^1 \subset L^6$, we have

$$\|\varphi_k\|_{L^\infty} \le C\|\varphi_k\|_{H^1}^{\frac{1}{2}} \|\varphi_k\|_{H^2}^{\frac{1}{2}}.$$

Applying elliptic regularity theory, this implies

$$\|\varphi_k\|_{L^\infty} \le C\|\varphi_k\|_{H^1}^{\frac{1}{2}} \left(\|\varphi_k\|_{L^2}^{\frac{1}{2}} + \|\Delta\varphi_k\|^{\frac{1}{2}}\right). \tag{4.29}$$

Choosing $v = \lambda_j a_j^k w_j$ in (4.5b), integrating by parts and summing the resulting equations over $j = 1, \ldots, k$, yields

$$\varepsilon\|\Delta\varphi_k\|_{L^2}^2 = \int_\Omega \nabla\mu_k \cdot \nabla\varphi_k - \varepsilon^{-1}\psi''(\varphi_k)|\nabla\varphi_k|^2 + \chi_\varphi \nabla\sigma_k \cdot \nabla\varphi_k \,\mathrm{d}x.$$

Due to Hölder's inequality and the assumption on $\psi(\cdot)$, we therefore get

$$\varepsilon\|\Delta\varphi_k\|_{L^2}^2 \le \|\nabla\mu_k\|_{\mathbf{L}^2}\|\nabla\varphi_k\|_{\mathbf{L}^2} + \chi_\varphi\|\nabla\sigma_k\|_{\mathbf{L}^2}\|\nabla\varphi_k\|_{\mathbf{L}^2} + \int_\Omega C\left(1 + |\varphi_k|^q\right)|\nabla\varphi_k|^2 \,\mathrm{d}x.$$

Integrating in time from 0 to T and applying Hölder's inequality along with (4.20)-(4.21) gives

$$\begin{aligned}\varepsilon\|\Delta\varphi_k\|^2_{L^2(L^2)} &\le \|\nabla\mu_k\|_{L^2(\mathbf{L}^2)}\|\nabla\varphi_k\|_{L^2(\mathbf{L}^2)} + \chi_\varphi\|\nabla\sigma_k\|_{L^2(\mathbf{L}^2)}\|\nabla\varphi_k\|_{L^2(\mathbf{L}^2)} \\ &\quad + C\int_0^T\int_\Omega (1+|\varphi_k|^q)\,|\nabla\varphi_k|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le C\left(1+\int_0^T\int_\Omega |\varphi_k|^q|\nabla\varphi_k|^2\,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{4.30}$$

In the case $q=0$, applying (4.20) yields

$$\int_0^T\int_\Omega |\varphi_k|^q|\nabla\varphi_k|^2\,\mathrm{d}x\,\mathrm{d}t = \|\nabla\varphi_k\|^2_{L^2(\mathbf{L}^2)} \le C.$$

In the case $q\in(0,4)$, we use Hölder's inequality and (4.29) to calculate

$$\begin{aligned}\int_0^T\int_\Omega |\varphi_k|^q|\nabla\varphi_k|^2\,\mathrm{d}x\,\mathrm{d}t &\le C\int_0^T \|\varphi_k\|^q_{L^\infty}\|\nabla\varphi_k\|^2_{\mathbf{L}^2}\,\mathrm{d}t \\ &\le C\int_0^T \|\varphi_k\|^2_{H^1}\|\varphi_k\|^{\frac{q}{2}}_{H^1}\left(\|\varphi_k\|^{\frac{q}{2}}_{L^2}+\|\Delta\varphi_k\|^{\frac{q}{2}}_{L^2}\right)\,\mathrm{d}t \\ &\le C\left(\|\varphi_k\|^{q+2}_{L^\infty(H^1)} + \int_0^T \|\varphi_k\|^{\frac{q+4}{2}}_{H^1}\|\Delta\varphi_k\|^{\frac{q}{2}}_{L^2}\,\mathrm{d}t\right).\end{aligned}$$

Observing $\frac{4}{q}>1$, we can use (2.2) to estimate the last integral on the r. h. s. of this inequality by

$$\int_0^T \|\varphi_k\|^{\frac{q+4}{2}}_{H^1}\|\Delta\varphi_k\|^{\frac{q}{2}}_{L^2}\,\mathrm{d}t \le C\|\varphi_k\|^{\frac{2(q+4)}{4-q}}_{L^\infty(H^1)} + \frac{\varepsilon}{2}\|\Delta\varphi_k\|^2_{L^2(L^2)}.$$

Invoking the last three inequalities together with (4.30) we obtain

$$\frac{\varepsilon}{2}\|\Delta\varphi_k\|^2_{L^2(L^2)} \le C.$$

Using elliptic regularity theory in conjunction with (4.21), this implies

$$\|\varphi_k\|_{L^2(H^2)} \le C.$$

Together with (4.20)-(4.21) and (4.27), we therefore deduce that

$$\|\varphi_k\|_{L^\infty(H^1)\cap L^2(H^2)} + \|\sigma_k\|_{L^\infty(L^2)\cap L^2(H^1)} + \|\mu_k\|_{L^2(H^1)} + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)} + \|p_k\|_{L^{\frac{4}{3}}(L^2)} \le C. \tag{4.31}$$

Regularity for the convection terms and the time derivatives

By Hölder's inequality and the Sobolev embedding $\mathbf{H}^1\subset\mathbf{L}^6$ we obtain that

$$\begin{aligned}\|\nabla\varphi_k\cdot\mathbf{v}_k\|^2_{L^2(0,T;L^{\frac{3}{2}})} = \int_0^T \|\nabla\varphi_k\cdot\mathbf{v}_k\|^2_{L^{\frac{3}{2}}}\,\mathrm{d}t &\le \int_0^T \|\mathbf{v}_k\|^2_{\mathbf{L}^6}\|\nabla\varphi_k\|^2_{\mathbf{L}^2}\,\mathrm{d}t \\ &\le C\int_0^T \|\mathbf{v}_k\|^2_{\mathbf{H}^1}\|\varphi_k\|^2_{H^1}\,\mathrm{d}t \\ &\le C\|\varphi_k\|^2_{L^\infty(0,T;H^1)}\|\mathbf{v}_k\|^2_{L^2(0,T;\mathbf{H}^1)}.\end{aligned}$$

Using the boundedness of $\Gamma_{\mathbf{v}}$ and (4.21) we infer

$$\|\varphi_k\Gamma_{\mathbf{v},k}\|^2_{L^2(0,T;L^{\frac{3}{2}})} \le C\|\varphi_k\|^2_{L^2(0,T;L^{\frac{3}{2}})} \le C$$

with a constant C depending on γ_0. From the last two inequalities and (4.20)-(4.21) we deduce that

$$\|\mathrm{div}(\varphi_k \mathbf{v}_k)\|_{L^2(0,T;L^{\frac{3}{2}})} \le C. \tag{4.32}$$

Taking an arbitrary $\zeta \in L^4(0,T;H^1)$ and integrating by parts we obtain

$$\begin{aligned}\int_0^T \int_\Omega \mathrm{div}(\sigma_k \mathbf{v}_k)\zeta \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\partial\Omega} \zeta\sigma_k \mathbf{v}_k \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &\quad - \int_0^T \int_\Omega \sigma_k \mathbf{v}_k \cdot \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}\tag{4.33}$$

Due to (2.4) with $j = 0$, $p = 3$, $m = 1$, $r = q = 2$, we have

$$\|\sigma_k\|_{L^3} \le C\|\sigma_k\|_{L^2}^{\frac{1}{2}}\|\sigma_k\|_{H^1}^{\frac{1}{2}}.$$

Then, by Hölder's inequality and the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ along with (4.31), we can estimate the second term on the r. h. s. of (4.33) by

$$\begin{aligned}\left|\int_0^T \int_\Omega \sigma_k \mathbf{v}_k \cdot \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t\right| &\le \int_0^T \|\sigma_k\|_{L^3}\|\mathbf{v}_k\|_{\mathbf{L}^6}\|\nabla\zeta\|_{\mathbf{L}^2}\,\mathrm{d}t \\ &\le C\int_0^T \|\sigma_k\|_{L^2}^{\frac{1}{2}}\|\sigma_k\|_{H^1}^{\frac{1}{2}}\|\mathbf{v}_k\|_{\mathbf{H}^1}\|\zeta\|_{H^1}\,\mathrm{d}t \\ &\le C\|\sigma_k\|_{L^\infty(L^2)}^{\frac{1}{2}}\|\sigma_k\|_{L^2(H^1)}^{\frac{1}{2}}\|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)}\|\zeta\|_{L^4(H^1)} \\ &\le C\|\zeta\|_{L^4(H^1)}. \end{aligned}$$

Furthermore, using (2.22) with $r = q = 2$ (hence $\alpha = 0$) gives

$$\|\sigma_k\|_{L^2(\partial\Omega)} \le C\left(\|\sigma_k\|_{L^2} + \|\sigma_k\|_{L^2}^{\frac{1}{2}}\|\sigma_k\|_{H^1}^{\frac{1}{2}}\right).$$

By (4.20) this implies

$$\|\sigma_k\|_{L^4(L^2(\partial\Omega))} \le C.$$

Together with Hölder's inequality, the trace theorem and (4.31), we obtain

$$\begin{aligned}\left|\int_0^T \int_{\partial\Omega} \zeta\sigma_k \mathbf{v}_k \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\right| &\le C\int_0^T \|\sigma_k\|_{L^2(\partial\Omega)}\|\mathbf{v}_k\|_{\mathbf{L}^4(\partial\Omega)}\|\zeta\|_{L^4(\partial\Omega)}\,\mathrm{d}t \\ &\le C\int_0^T \|\sigma_k\|_{L^2(\partial\Omega)}\|\mathbf{v}_k\|_{\mathbf{H}^1}\|\zeta\|_{H^1}\,\mathrm{d}t \\ &\le C\|\sigma_k\|_{L^4(L^2(\partial\Omega))}\|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)}\|\zeta\|_{L^4(H^1)} \\ &\le C\|\zeta\|_{L^4(H^1)}. \end{aligned}$$

From the above estimates and (4.33) we get

$$\left|\int_0^T \int_\Omega \mathrm{div}(\sigma_k \mathbf{v}_k)\zeta \,\mathrm{d}x\,\mathrm{d}t\right| \le C\|\zeta\|_{L^4(H^1)},$$

and taking the supremum over all $\zeta \in L^4(H^1)$ yields

$$\|\mathrm{div}(\sigma_k \mathbf{v}_k)\|_{L^{\frac{4}{3}}(0,T;(H^1)^*)} \le C. \tag{4.34}$$

Now, we denote by $\{\zeta_{kj}\}_{1\le j\le k}$ the coefficients of $\zeta \in L^4(0,T;H^1)$ such that $\mathbb{P}_k\zeta = \sum_{j=1}^k \zeta_{kj} w_j$. Then, taking $v = \zeta_{kj} w_j$ in (4.5c), summing over $j = 1,\dots,k$, and integrating the resulting identity in time from 0 to T yields

$$\left|\int_0^T \int_\Omega \partial_t \sigma_k \zeta \,\mathrm{d}x\,\mathrm{d}t\right| \le \int_0^T n_1(\chi_\sigma \|\nabla\sigma_k\|_{\mathbf{L}^2} + \chi_\varphi \|\nabla\varphi_k\|_{\mathbf{L}^2})\|\nabla\mathbb{P}_k\zeta\|_{\mathbf{L}^2} + \|\Gamma_{\sigma,k}\|_{L^2}\|\mathbb{P}_k\zeta\|_{L^2}\,\mathrm{d}t$$
$$+ \int_0^T \|\mathrm{div}(\sigma_k \mathbf{v}_k)\|_{(H^1)^*}\|\mathbb{P}_k\zeta\|_{H^1}\,\mathrm{d}t$$
$$+ \int_0^T K(\|\sigma_\infty\|_{L^2(\partial\Omega)} + \|\sigma_k\|_{L^2(\partial\Omega)})\|\mathbb{P}_k\zeta\|_{L^2(\partial\Omega)}\,\mathrm{d}t.$$

Using (4.20)-(4.21) we obtain

$$\|\Gamma_{\sigma,k}\|_{L^2(L^2)} \le C(R_0,|\Omega|,T)\left(1 + \|\varphi_k\|_{L^2(L^2)} + \|\sigma_k\|_{L^2(L^2)} + \|\mu_k\|_{L^2(L^2)}\right) \le C.$$

Then, Hölder's inequality, the trace theorem and the estimate $\|\mathbb{P}_k\zeta\|_{H^1} \le C\|\zeta\|_{H^1}$ yield

$$\left|\int_0^T \int_\Omega \partial_t \sigma_k \zeta \,\mathrm{d}x\,\mathrm{d}t\right| \le C\left(1 + \|\mathrm{div}(\sigma_k\mathbf{v}_k)\|_{L^{\frac{4}{3}}((H^1)^*)}\right)\|\zeta\|_{L^4(H^1)}.$$

By taking the supremum over all $\zeta \in L^4(H^1)$ and using (4.20)-(4.21) along with (4.34), we end up with

$$\|\partial_t\sigma_k\|_{L^{\frac{4}{3}}((H^1)^*)} \le C. \tag{4.35}$$

With similar arguments we can show that

$$\|\partial_t\varphi_k\|_{L^2((H^1)^*)} \le C. \tag{4.36}$$

Notice that we have lower time regularity for the time derivative of φ_k compared to the convection term since the regularity of the time derivative depends on the term $\nabla\mu_k$.

4.2.3 Passing to the limit

At this point, we summarise the estimates (4.27), (4.31)-(4.32) and (4.34)-(4.36) to deduce

$$\|\varphi_k\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^2)} + \|\sigma_k\|_{W^{1,\frac{4}{3}}((H^1)^*)\cap L^\infty(L^2)\cap L^2(H^1)} + \|\mu_k\|_{L^2(H^1)}$$
$$+ \|\mathrm{div}(\varphi_k\mathbf{v}_k)\|_{L^2(L^{\frac{3}{2}})} + \|\mathrm{div}(\sigma_k\mathbf{v}_k)\|_{L^{\frac{4}{3}}((H^1)^*)} + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)} + \|p_k\|_{L^{\frac{4}{3}}(L^2)} \le C. \tag{4.37}$$

Using standard compactness arguments (Lemma 2.36 and reflexive weak compactness), the compact embeddings

$$H^{j+1}(\Omega) = W^{j+1,2}(\Omega) \hookrightarrow\hookrightarrow W^{j,r} \quad \forall\, j\in\mathbb{Z},\ j\ge 0,\ 1\le r<6,$$

and $L^2 \hookrightarrow\hookrightarrow (H^1)^*$, we obtain, at least for a subsequence which will again be labelled by k, the convergence properties

$$\begin{aligned}
\varphi_k &\to \varphi && \text{weakly-star} && \text{in } H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^2),\\
\sigma_k &\to \sigma && \text{weakly-star} && \text{in } W^{1,\frac{4}{3}}((H^1)^*)\cap L^\infty(L^2)\cap L^2(H^1),\\
\mu_k &\to \mu && \text{weakly} && \text{in } L^2(H^1),\\
p_k &\to p && \text{weakly} && \text{in } L^{\frac{4}{3}}(L^2),\\
\mathbf{v}_k &\to \mathbf{v} && \text{weakly} && \text{in } L^2(\mathbf{H}^1),\\
\mathrm{div}(\varphi_k\mathbf{v}_k) &\to \tau && \text{weakly} && \text{in } L^2(L^{\frac{3}{2}}),\\
\mathrm{div}(\sigma_k\mathbf{v}_k) &\to \theta && \text{weakly} && \text{in } L^{\frac{4}{3}}((H^1)^*),\\
\mathrm{div}(\mathbf{v}_k) &\to \mathrm{div}(\mathbf{v}) && \text{weakly} && \text{in } L^2(L^2)
\end{aligned}$$

for some limit functions $\tau \in L^2(L^{\frac{3}{2}})$ and $\theta \in L^{\frac{4}{3}}((H^1)^*)$. Furthermore, we have the strong convergences

$$\begin{aligned} \varphi_k &\to \varphi \quad \text{strongly} \quad \text{in } C^0(L^r) \cap L^2(W^{1,r}) \quad \text{and a.e. in } \Omega_T, \\ \sigma_k &\to \sigma \quad \text{strongly} \quad \text{in } C^0((H^1)^*) \cap L^2(L^r) \quad \text{and a.e. in } \Omega_T \end{aligned}$$

for all $r \in [1,6)$. From now on, we fix $1 \le j \le k$ and $\xi \in L^2$, $\mathbf{\Phi} \in \mathbf{H}^1$, $\delta \in C_0^\infty(0,T)$. Then, since the eigenfunctions $\{w_j\}_{j\in\mathbb{N}}$ belong to H^2_N we observe that $\delta w_j \in C^\infty(H^2_N)$ for all $j \in \mathbb{N}$. Furthermore, we have $\delta\xi \in C^\infty(L^2)$ and $\delta\mathbf{\Phi} \in C^\infty(\mathbf{H}^1)$. Inserting $v = w_j$ in (4.5a)-(4.5c), multiplying the resulting equations with δ and integrating over $(0,T)$ yields

$$\int_0^T \delta(t) \left(\int_\Omega (\partial_t \varphi_k - \Gamma_{\varphi,k} + \nabla\varphi_k \cdot \mathbf{v}_k + \varphi_k \Gamma_{\mathbf{v},k}) w_j + m(\varphi_k)\nabla\mu_k \cdot \nabla w_j \,\mathrm{d}x \right) \mathrm{d}t = 0, \tag{4.38a}$$

$$\int_0^T \delta(t) \left(\int_\Omega (\mu_k - \varepsilon^{-1}\psi'(\varphi_k) + \chi_\varphi \sigma_k) w_j - \varepsilon \nabla\varphi_k \cdot \nabla w_j \,\mathrm{d}x \right) \mathrm{d}t = 0, \tag{4.38b}$$

$$\begin{aligned} &\int_0^T \delta(t) \left(\int_\Omega (\partial_t \sigma_k + \Gamma_{\sigma,k} + \nabla\sigma_k \cdot \mathbf{v}_k + \sigma_k \Gamma_{\mathbf{v},k}) w_j + n(\varphi_k)\nabla N_{\sigma,k} \nabla w_j \,\mathrm{d}x \right) \mathrm{d}t \\ &\quad - \int_0^T \delta(t) \left(\int_{\partial\Omega} K(\sigma_\infty - \sigma_k) w_j \,\mathrm{d}\mathcal{H}^{d-1} \right) \mathrm{d}t = 0. \end{aligned} \tag{4.38c}$$

Furthermore, multiplying (4.5d) with δ and integrating in time from 0 to T gives

$$\int_0^T \delta(t) \left(\int_\Omega \mathbf{T}(\mathbf{v}_k, p_k) \colon \nabla\mathbf{\Phi} + \nu(\varphi_k)\mathbf{v}_k \cdot \mathbf{\Phi} - (\mu_k \nabla\varphi_k + N_{\sigma,k}\nabla\sigma_k) \cdot \mathbf{\Phi} \,\mathrm{d}x \right) \mathrm{d}t = 0. \tag{4.38d}$$

With similar arguments, (4.5e) gives

$$\int_0^T \delta(t) \left(\int_\Omega \mathrm{div}(\mathbf{v}_k)\xi \,\mathrm{d}x \right) \mathrm{d}t = \int_0^T \delta(t) \left(\int_\Omega \Gamma_{\mathbf{v},k}\xi \,\mathrm{d}x \right) \mathrm{d}t. \tag{4.38e}$$

Now, we pass to the limit in (4.38).

Step 1: (4.38a) Since $\delta w_j \in C^\infty(H^2) \hookrightarrow L^2((H^1)^*)$ we obtain

$$\int_0^T \int_\Omega \partial_t \varphi_k \delta w_j \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \delta(t) \langle \partial_t \varphi, w_j \rangle_{H^1} \,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.39}$$

By continuity of $m(\cdot)$ and since $\varphi_k \to \varphi$ a.e. in Ω_T as $k \to \infty$, we observe that $m(\varphi_k) \to m(\varphi)$ a.e. in Ω_T. Using the boundedness of $m(\cdot)$ and applying Lebesgue dominated convergence theorem to $(m(\varphi_k) - m(\varphi))^2 |\delta|^2 |\nabla w_j|^2$, we obtain

$$\|(m(\varphi_k) - m(\varphi))\delta \nabla w_j\|_{L^2(\mathbf{L}^2)} \to 0 \quad \text{as } k \to \infty.$$

Then, the weak convergence $\nabla\mu_k \rightharpoonup \nabla\mu$ in $L^2(\mathbf{L}^2)$ as $k \to \infty$ implies

$$\int_0^T \int_\Omega \delta m(\varphi_k) \nabla w_j \cdot \nabla\mu_k \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta m(\varphi) \nabla w_j \cdot \nabla\mu \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.40}$$

Using the Sobolev embedding $H^2 \subset L^\infty$, we have

$$\begin{aligned} \int_0^T \int_\Omega |\delta|^2 |w_j|^2 |\nabla\varphi_k - \nabla\varphi|^2 \,\mathrm{d}x \,\mathrm{d}t &\le \int_0^T |\delta|^2 \|\nabla\varphi_k - \nabla\varphi\|_{\mathbf{L}^2}^2 \|w_j\|_{L^\infty}^2 \,\mathrm{d}t \\ &\le C \|\delta\|_{L^\infty(0,T)}^2 \|w_j\|_{H^2}^2 \|\varphi_k - \varphi\|_{L^2(H^1)}^2 \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Therefore, $\delta w_j \nabla\varphi_k \to \delta w_j \nabla\varphi$ strongly in $L^2(\mathbf{L}^2)$ as $k \to \infty$. Then, by the product of weak-strong convergence we obtain

$$\int_0^T \int_\Omega \delta w_j \nabla\varphi_k \cdot \mathbf{v}_k \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta w_j \nabla\varphi \cdot \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.41}$$

Using the boundedness and continuity of $\Gamma_{\mathbf{v}}(\cdot,\cdot)$, with similar arguments as for (4.40) we obtain

$$\|(\Gamma_{\mathbf{v}}(\varphi_k,\sigma_k) - \Gamma_{\mathbf{v}}(\varphi,\sigma))\delta w_j\|_{L^2(\Omega_T)} \to 0 \quad \text{as } k \to \infty$$

which implies

$$\int_0^T \int_\Omega \delta w_j \varphi_k \Gamma_{\mathbf{v}}(\varphi_k,\sigma_k) \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta w_j \varphi \Gamma_{\mathbf{v}}(\varphi,\sigma) \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.42}$$

Now, using the estimate

$$\|\varphi_k\|_{H^1}^4 \le C\|\varphi_k\|_{L^2}^2 \|\varphi_k\|_{H^2}^2$$

together with (4.37) and the strong convergence $\varphi_k \to \varphi$ in $C^0(L^2)$ as $k \to \infty$, we obtain that $\varphi_k \to \varphi$ strongly in $L^4(H^1)$ as $k \to \infty$. Using the weak convergence $\mathbf{v}_k \rightharpoonup \mathbf{v}$ in $L^2(\mathbf{H}^1)$, by the product of weak-strong convergence we obtain $\operatorname{div}(\varphi_k \mathbf{v}) \rightharpoonup \operatorname{div}(\varphi\mathbf{v})$ in $L^{\frac{4}{3}}(L^{\frac{3}{2}})$ as $k \to \infty$. By uniqueness of weak limits, this implies $\operatorname{div}(\varphi\mathbf{v}) = \tau$.

Recalling the specific form of $\Gamma_{\varphi,k}$ given by $\Gamma_{\varphi,k} = \Lambda_\varphi(\varphi_k,\sigma_k) - \theta_\varphi(\varphi_k,\sigma_k)\mu_k$ and using that $\varphi_k \to \varphi$ and $\sigma_k \to \sigma$ a. e. in Ω_T together with the continuity and boundedness of $\theta_\varphi(\cdot,\cdot)$, Lebesgue dominated convergence theorem implies

$$\int_0^T \int_\Omega |\delta w_j(\theta_\varphi(\varphi_k,\sigma_k) - \theta_\varphi(\varphi,\sigma))|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } k \to \infty.$$

Therefore, $\delta w_j \theta_\varphi(\varphi_k,\sigma_k) \to \delta w_j \theta_\varphi(\varphi,\sigma)$ strongly in $L^2(\Omega_T)$ as $k \to \infty$. Together with the weak convergence $\mu_k \rightharpoonup \mu$ in $L^2(\Omega_T)$ we conclude that

$$\int_0^T \int_\Omega \delta w_j \theta_\varphi(\varphi_k,\sigma_k)\mu_k \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta w_j \theta_\varphi(\varphi,\sigma)\mu \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.43}$$

We now analyse the other term in the definition of $\Gamma_{\varphi,k}$. Applying the inequality $||a|-|b|| \le |a-b|$, we obtain

$$\int_0^T \int_\Omega |(\delta w_j)(|\varphi_k| - |\varphi|)| \,\mathrm{d}x\,\mathrm{d}t \le \|\delta w_j\|_{L^2(\Omega_T)} \|\varphi_k - \varphi\|_{L^2(\Omega_T)} \to 0 \quad \text{as } k \to \infty,$$

and

$$\int_0^T \int_\Omega |(\delta w_j)(|\sigma_k| - |\sigma|)| \,\mathrm{d}x\,\mathrm{d}t \le \|\delta w_j\|_{L^2(\Omega_T)} \|\sigma_k - \sigma\|_{L^2(\Omega_T)} \to 0 \quad \text{as } k \to \infty.$$

This implies

$$R_0(1 + |\varphi_k| + |\sigma_k|)|\delta w_j| \to R_0(1 + |\varphi| + |\sigma|)|\delta w_j| \quad \text{strongly in } L^1(\Omega_T) \quad \text{as } k \to \infty.$$

Since $\varphi_k \to \varphi$ and $\sigma_k \to \sigma$ a. e. in Ω_T as $k \to \infty$, the continuity of $\Lambda_\varphi(\cdot,\cdot)$ yields

$$\delta w_j \Lambda_\varphi(\varphi_k,\sigma_k) \to \delta w_j \Lambda_\varphi(\varphi,\sigma) \quad \text{a. e. in } \Omega_T \quad \text{as } k \to \infty.$$

Using

$$|\delta w_j \Lambda_\varphi(\varphi_k,\sigma_k)| \le |\delta w_j| R_0(1 + |\varphi_k| + |\varphi_k|) \in L^1(\Omega_T) \quad \forall\, k \ge 1,$$

by the generalised Lebesgue dominated convergence theorem (see Lemma 2.35) we obtain

$$\int_0^T \int_\Omega \delta w_j \Lambda_\varphi(\varphi_k, \sigma_k) \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta w_j \Lambda_\varphi(\varphi, \sigma) \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty.$$

Together with (4.43), this implies

$$\int_0^T \int_\Omega \delta w_j \Gamma_\varphi(\varphi_k, \sigma_k, \mu_k) \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta w_j \Gamma_\varphi(\varphi, \sigma, \mu) \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.44}$$

Step 2: We now analyse (4.38b). Since $\mu_k \rightharpoonup \mu$, $\sigma_k \rightharpoonup \sigma$ and $\nabla\varphi_k \rightharpoonup \nabla\varphi$ in $L^2(\Omega_T)$ and $L^2(\mathbf{L}^2)$, respectively, we easily deduce

$$\begin{aligned}
&\int_0^T \delta(t) \left(\int_\Omega (\mu_k + \chi_\varphi \sigma_k) w_j - \varepsilon \nabla\varphi_k \cdot \nabla w_j \,\mathrm{d}x \right) \mathrm{d}t \\
&\quad \to \int_0^T \delta(t) \left(\int_\Omega (\mu + \chi_\varphi \sigma) w_j - \varepsilon \nabla\varphi \cdot \nabla w_j \,\mathrm{d}x \right) \mathrm{d}t \quad \text{as } k \to \infty.
\end{aligned} \tag{4.45}$$

Recalling that $\varphi_k \to \varphi$ strongly in $C^0(L^r)$ for $r \in [1,6)$, for $s \in [1,5)$ and $q = \frac{6}{5}s < 6$ we obtain

$$\begin{aligned}
\int_0^T \int_\Omega ||\varphi_k - \varphi|^s \delta w_j| \,\mathrm{d}x \,\mathrm{d}t &\le \int_0^T |\delta| \|w_j\|_{L^6} \|\varphi_k - \varphi\|_{L^{\frac{6s}{5}}}^s \,\mathrm{d}t \\
&\le C\|\delta\|_{L^6(0,T)} \|w_j\|_{H^1} \|\varphi_k - \varphi\|_{L^q(L^q)}^s \\
&\to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

This implies

$$|\varphi_k - \varphi|^s \delta w_j \to 0 \quad \text{strongly in } L^1(\Omega_T) \quad \text{as } k \to \infty.$$

Furthermore, we have

$$\begin{aligned}
&|\varphi_k|^s |\delta w_j| \le C(s)(|\varphi_k - \varphi|^s + |\varphi|^s)|\delta w_j| \in L^1(\Omega_T) \quad \text{for all } k \ge 1, \\
&(|\varphi_k - \varphi|^s + |\varphi|^s)|\delta w_j| \to |\varphi|^s |\delta w_j| \quad \text{a.e. in } \Omega_T \quad \text{as } k \to \infty, \\
&\int_0^T \int_\Omega (|\varphi_k - \varphi|^s + |\varphi|^s)|\delta w_j| \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega |\varphi|^s |\delta w_j| \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty.
\end{aligned}$$

Again using the generalised Lebesgue dominated convergence theorem, we obtain

$$\int_0^T \int_\Omega |\varphi_k|^s \delta w_j \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega |\varphi|^s \delta w_j \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.46}$$

Now, by continuity of $\psi'(\cdot)$ we have $\psi'(\varphi_k) \to \psi'(\varphi)$ a.e. in Ω_T as $k \to \infty$. Furthermore, by the growth assumption on $\psi'(\cdot)$ we observe that

$$|\psi'(\varphi_k)\delta w_j| \le C(1 + |\varphi_k|^s)|\delta w_j| \in L^1(\Omega_T) \quad \text{for all } k \in \mathbb{N},\ s \in [1,5).$$

Using once more the generalised Lebesgue dominated convergence theorem and (4.46), we get

$$\int_0^T \int_\Omega \varepsilon^{-1} \psi'(\varphi_k) \delta w_j \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \varepsilon^{-1} \psi'(\varphi) \delta w_j \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.47}$$

Step 3: We now pass to the limit in (4.38e). Since $\varphi_k \to \varphi$, $\sigma_k \to \sigma$ a.e. in Ω_T as $k \to \infty$, the continuity and boundedness of $\Gamma_{\mathbf{v}}$ and similar arguments as for (4.40) imply

$$\int_0^T \int_\Omega \delta(t) \Gamma_{\mathbf{v},k} \xi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta(t) \Gamma_{\mathbf{v}}(\varphi, \sigma) \xi \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } k \to \infty.$$

Recalling the weak convergence $\mathrm{div}(\mathbf{v}_k) \rightharpoonup \mathrm{div}(\mathbf{v})$ in $L^2(L^2)$ as $k \to \infty$, we deduce

$$\int_0^T \int_\Omega \delta(t)\mathrm{div}(\mathbf{v}_k)\xi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta(t)\mathrm{div}(\mathbf{v})\xi \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k \to \infty.$$

This allows us to pass to the limit $k \to \infty$ in (4.38e) to obtain

$$\int_0^T \delta(t) \int_\Omega \mathrm{div}(\mathbf{v})\xi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \delta(t) \int_\Omega \Gamma_{\mathbf{v}}(\varphi,\sigma)\xi \,\mathrm{d}x\,\mathrm{d}t. \tag{4.48}$$

In particular, since this holds for all $\delta \in C_0^\infty(0,T)$ and all $\xi \in L^2$, we have

$$\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma) \quad \text{a.e. in } \Omega_T. \tag{4.49}$$

Step 4: With similar arguments as for (4.39)-(4.40) and (4.44), we obtain

$$\begin{aligned}\int_0^T \int_\Omega \partial_t\sigma_k \delta w_j \,\mathrm{d}x\,\mathrm{d}t &\to \int_0^T \delta(t)\langle \partial_t\sigma, w_j\rangle_{H^1} \,\mathrm{d}t,\\ \int_0^T \int_\Omega \delta n(\varphi_k)\nabla N_{\sigma,k}\cdot\nabla w_j \,\mathrm{d}x\,\mathrm{d}t &\to \int_0^T \int_\Omega \delta n(\varphi)(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi)\cdot\nabla w_j \,\mathrm{d}x\,\mathrm{d}t,\\ \int_0^T \int_\Omega \delta w_j \Gamma_\sigma(\varphi_k,\sigma_k,\mu_k) \,\mathrm{d}x\,\mathrm{d}t &\to \int_0^T \int_\Omega \delta w_j \Gamma_\sigma(\varphi,\sigma,\mu) \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.50}$$

as $k \to \infty$. For the boundary term in (4.38c) we first recall the continuous embedding $H^1 \hookrightarrow L^4(\partial\Omega)$. Then, by the weak convergence of $\sigma_k \rightharpoonup \sigma$ in $L^2(\Sigma_T)$ we conclude

$$\int_0^T \delta(t)\left(\int_{\partial\Omega} \sigma_k w_j \,\mathrm{d}\mathcal{H}^{d-1}\right)\mathrm{d}t \to \int_0^T \delta(t)\left(\int_{\partial\Omega} \sigma w_j \,\mathrm{d}\mathcal{H}^{d-1}\right)\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.51}$$

To pass to the limit in the convection term of (4.5c), we first show that

$$\int_0^T \int_\Omega \sigma\Gamma_{\mathbf{v}}(\varphi,\sigma)\delta w_j \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \sigma\mathrm{div}(\mathbf{v})\delta w_j \,\mathrm{d}x\,\mathrm{d}t. \tag{4.52}$$

Indeed, a short calculation yields

$$\begin{aligned}\int_0^T \int_\Omega |\delta|^2|w_j|^2|\sigma_k-\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T |\delta|^2\|w_j\|_{L^6}^2\|\sigma_k-\sigma\|_{L^3}^2 \,\mathrm{d}t\\ &\le C\|\delta\|_{L^\infty(0,T)}^2\|w_j\|_{H^1}^2\|\sigma_k-\sigma\|_{L^2(L^3)}^2\\ &\to 0 \quad \text{as } k\to\infty,\end{aligned}$$

where we used that $\sigma_k \to \sigma$ strongly in $L^2(L^3)$. Therefore, we obtain that $\sigma_k\delta w_j \to \sigma\delta w_j$ strongly in $L^2(L^2)$. With similar arguments as for (4.48), this implies (4.52). Now, as $\delta w_j \in C^\infty(H^2) \hookrightarrow L^4(H^1)$, the weak convergence $\mathrm{div}(\sigma_k\mathbf{v}_k) \rightharpoonup \theta$ in $L^{\frac{4}{3}}((H^1)^*)$ implies

$$\int_0^T \int_\Omega \mathrm{div}(\sigma_k\mathbf{v}_k)\delta w_j \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \delta(t)\langle\theta, w_j\rangle_{H^1} \,\mathrm{d}t \quad \text{as } k\to\infty. \tag{4.53}$$

Integrating by parts, we see that

$$\begin{aligned}\int_0^T \int_\Omega \mathrm{div}(\sigma_k\mathbf{v}_k)\delta w_j \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\partial\Omega} \delta w_j\sigma_k\mathbf{v}_k\cdot\mathbf{n} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\ &\quad - \int_0^T \int_\Omega \delta\sigma_k\mathbf{v}_k\cdot\nabla w_j \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{4.54}$$

To proceed further, we now prove that $\sigma_k \to \sigma$ strongly in $L^2(L^3(\partial\Omega))$ as $k \to \infty$. Using Lemma 2.34 with $r = 3$, $q = 2$, it follows that

$$\|\sigma_k - \sigma\|^2_{L^3(\partial\Omega)} \leq C\left(\|\sigma_k - \sigma\|^{\frac{1}{2}}_{L^2}\|\sigma_k - \sigma\|^{\frac{3}{2}}_{H^1} + \|\sigma_k - \sigma\|^{\frac{1}{3}}_{L^2}\|\sigma_k - \sigma\|^{\frac{5}{3}}_{H^1}\right).$$

Integrating this inequality in time from 0 to T and using Hölder's inequality we obtain

$$\|\sigma_k - \sigma\|^2_{L^2(L^3(\partial\Omega))} \leq C\left(\|\sigma_k - \sigma\|^{\frac{1}{2}}_{L^2(L^2)}\|\sigma_k - \sigma\|^{\frac{3}{2}}_{L^2(H^1)} + \|\sigma_k - \sigma\|^{\frac{1}{3}}_{L^2(L^2)}\|\sigma_k - \sigma\|^{\frac{5}{3}}_{L^2(H^1)}\right).$$

Due to the boundedness of $\sigma_k - \sigma \in L^2(H^1)$ and invoking the strong convergence $\sigma_k \to \sigma$ in $L^2(L^2)$ as $k \to \infty$, this implies

$$\|\sigma_k - \sigma\|_{L^2(L^3(\partial\Omega))} \to 0 \quad \text{as } k \to \infty,$$

hence $\sigma_k \to \sigma$ strongly in $L^2(L^3(\partial\Omega))$ as $k \to \infty$. Using the continuous embedding $\mathbf{H}^1 \hookrightarrow \mathbf{L}^4(\partial\Omega)$ resulting from the trace theorem, we therefore have (after another extraction of subsequences)

$$\sigma_k \to \sigma \quad \text{strongly in } L^2(L^3(\partial\Omega)), \qquad \mathbf{v}_k \rightharpoonup \mathbf{v} \quad \text{weakly in } L^2(\mathbf{L}^4(\partial\Omega)) \quad \text{as } k \to \infty.$$

Again by the trace theorem and the continuous embeddings $H^2 \hookrightarrow W^{1,6}$, $W^{\frac{5}{6},6}(\partial\Omega) \hookrightarrow L^6(\partial\Omega)$, we observe that $w_j \in H^2(\Omega) \hookrightarrow L^6(\partial\Omega)$. Since the outer unit normal is continuous, we calculate

$$\begin{aligned}\int_0^T \int_{\partial\Omega} |\delta|^2|\mathbf{n}|^2|\sigma_k - \sigma|^2|w_j|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t &\leq \int_0^T |\delta|^2\|w_j\|^2_{L^6(\partial\Omega)}\|\sigma_k - \sigma\|^2_{L^3(\partial\Omega)}\,\mathrm{d}t \\ &\leq C\|\delta\|^2_{L^\infty(0,T)}\|w_j\|^2_{H^2}\|\sigma_k - \sigma\|^2_{L^2(L^3(\partial\Omega))} \\ &\to 0 \quad \text{as } k \to \infty,\end{aligned}$$

meaning $\delta w_j\sigma_k\mathbf{n} \to \delta w_j\sigma\mathbf{n}$ strongly in $L^2(\mathbf{L}^2(\partial\Omega))$ as $k \to \infty$. Then, by the product of weak-strong convergence we obtain

$$\int_0^T \int_{\partial\Omega} \delta w_j\sigma_k\mathbf{v}_k \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \to \int_0^T \int_{\partial\Omega} \delta w_j\sigma\mathbf{v} \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \quad \text{as } k \to \infty. \tag{4.55}$$

Furthermore, since $\sigma_k \to \sigma$ strongly in $L^2(L^r)$ for all $r \in [1, 6)$ as $k \to \infty$, we get

$$\begin{aligned}\int_0^T \int_\Omega |\delta|^2|\nabla w_j|^2|\sigma_k - \sigma|^2\,\mathrm{d}x\,\mathrm{d}t &\leq \int_0^T |\delta|^2\|\nabla w_j\|^2_{L^6}\|\sigma_k - \sigma\|^2_{L^3}\,\mathrm{d}t \\ &\leq C\|\delta\|^2_{L^\infty(0,T)}\|w_j\|^2_{H^2}\|\sigma_k - \sigma\|^2_{L^2(L^3)} \\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Then, since $\mathbf{v}_k \rightharpoonup \mathbf{v}$ weakly in $L^2(\mathbf{H}^1)$ as $k \to \infty$, by the product of weak-strong convergence we have

$$\int_0^T \int_\Omega \delta\sigma_k\mathbf{v}_k \cdot \nabla w_j\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta\sigma\mathbf{v} \cdot \nabla w_j\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k \to \infty.$$

Consequently, recalling (4.53) and (4.55), we can pass to the limit in (4.54) to obtain

$$\int_0^T \delta(t)\langle\theta, w_j\rangle_{H^1}\,\mathrm{d}t = \int_0^T \int_{\partial\Omega} \delta w_j\sigma\mathbf{v} \cdot \mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t - \int_0^T \int_\Omega \delta\sigma\mathbf{v} \cdot \nabla w_j\,\mathrm{d}x\,\mathrm{d}t. \tag{4.56}$$

Again integrating by parts yields

$$\int_0^T \delta(t)\langle\theta, w_j\rangle_{H^1}\,\mathrm{d}t = \int_0^T \int_\Omega \mathrm{div}(\sigma\mathbf{v})\delta w_j\,\mathrm{d}x\,\mathrm{d}t,$$

hence $\mathrm{div}(\sigma\mathbf{v}) = \theta$ in the sense of distributions. In particular, by (4.52) we have

$$\int_0^T \delta(t)\langle\theta, w_j\rangle_{H^1}\,\mathrm{d}t = \int_0^T\int_\Omega \nabla\sigma\cdot\mathbf{v}\delta w_j + \sigma\Gamma_{\mathbf{v}}(\varphi,\sigma)\delta w_j\,\mathrm{d}x\,\mathrm{d}t.$$

Step 5: Finally, we pass to the limit in (4.38d). Recalling that $\delta\mathbf{\Phi}\in C^\infty(\mathbf{H}^1)$, by continuity of $\eta(\cdot)$, $\lambda(\cdot)$, $\nu(\cdot)$, and since $\varphi_k\to\varphi$ a.e. in Ω_T as $k\to\infty$, we observe that $\eta(\varphi_k)\to\eta(\varphi)$, $\lambda(\varphi_k)\to\lambda(\varphi)$ and $\nu(\varphi_k)\to\nu(\varphi)$ a.e. in Ω_T as $k\to\infty$. Invoking the boundedness of $\eta(\cdot)$, $\lambda(\cdot)$ and $\nu(\cdot)$, applying Lebesgue dominated convergence theorem to $(\eta(\varphi_k)-\eta(\varphi))^2|\delta|^2|\nabla\mathbf{\Phi}|^2$, $(\lambda(\varphi_k)-\lambda(\varphi))^2|\delta|^2|\nabla\mathbf{\Phi}|^2$ and $(\nu(\varphi_k)-\nu(\varphi))^2|\delta|^2|\nabla\mathbf{\Phi}|^2$ gives

$$\begin{aligned}
&\|(\eta(\varphi_k)-\eta(\varphi))\delta\nabla\mathbf{\Phi}\|_{L^2(\mathbf{L}^2)}\to 0 \quad \text{as } k\to\infty,\\
&\|(\lambda(\varphi_k)-\lambda(\varphi))\delta\nabla\mathbf{\Phi}\|_{L^2(\mathbf{L}^2)}\to 0 \quad \text{as } k\to\infty,\\
&\|(\nu(\varphi_k)-\nu(\varphi))\delta\nabla\mathbf{\Phi}\|_{L^2(\mathbf{L}^2)}\to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Therefore, by the weak convergence $\mathbf{v}_k\rightharpoonup\mathbf{v}$ in $L^2(\mathbf{H}^1)$, $\mathrm{div}(\mathbf{v}_k)\rightharpoonup\mathrm{div}(\mathbf{v})$ in $L^2(L^2)$ and $p_k\rightharpoonup p$ in $L^{\frac{4}{3}}(L^2)$ as $k\to\infty$, we easily deduce that

$$\begin{aligned}
\int_0^T\int_\Omega \delta\mathbf{T}(\mathbf{v}_k,p_k)\colon\nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t &\to \int_0^T\int_\Omega \delta\mathbf{T}(\mathbf{v},p)\colon\nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t,\\
\int_0^T\int_\Omega \delta\nu(\varphi_k)\mathbf{v}_k\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t &\to \int_0^T\int_\Omega \delta\nu(\varphi)\mathbf{v}\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}\tag{4.57}$$

as $k\to\infty$ where we used that $\delta\mathbf{\Phi}\in L^4(\mathbf{H}^1)$. Using $\varphi_k\to\varphi$ strongly in $L^2(W^{1,3})$ as $k\to\infty$ and the Sobolev embedding $\mathbf{H}^1\subset\mathbf{L}^6$, we have

$$\begin{aligned}
\int_0^T\int_\Omega |\delta|^2|\mathbf{\Phi}|^2|\nabla\varphi_k-\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t &\le \int_0^T |\delta|^2\|\mathbf{\Phi}\|_{\mathbf{L}^6}^2\|\nabla\varphi_k-\nabla\varphi\|_{\mathbf{L}^3}^2\,\mathrm{d}t\\
&\le C\|\delta\|_{L^\infty(0,T)}^2\|\mathbf{\Phi}\|_{\mathbf{H}^1}^2\|\varphi_k-\varphi\|_{L^2(W^{1,3})}^2\\
&\to 0 \quad \text{as } k\to\infty,
\end{aligned}$$

meaning $\delta\mathbf{\Phi}\cdot\nabla\varphi_k\to\delta\mathbf{\Phi}\cdot\nabla\varphi$ strongly in $L^2(L^2)$ as $k\to\infty$. By the product of weak-strong convergence it follows

$$\int_0^T\int_\Omega \delta\mu_k\nabla\varphi_k\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \delta\mu\nabla\varphi\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k\to\infty. \tag{4.58}$$

By the specific form of $N_{\sigma,k}$ and since $\varphi_k\to\varphi$, $\sigma_k\to\sigma$ strongly in $L^2(L^3)$ as $k\to\infty$, using a similar argument as for (4.58) yields

$$N_{\sigma,k}\delta\mathbf{\Phi}\to N_\sigma\delta\mathbf{\Phi} \quad \text{strongly in } L^2(\mathbf{L}^2) \quad \text{as } k\to\infty.$$

Consequently, by the product of weak-strong convergence we obtain

$$\int_0^T\int_\Omega \delta N_{\sigma,k}\nabla\sigma_k\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \delta N_\sigma(\varphi,\sigma)\nabla\sigma\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } k\to\infty. \tag{4.59}$$

Now, we can pass to the limit in (4.38a)-(4.38e) to obtain

$$\begin{aligned}
\int_0^T \delta(t)\langle \partial_t\varphi , w_j\rangle_{H^1} \,\mathrm{d}t &= \int_0^T \delta(t)\left(\int_\Omega -m(\varphi)\nabla\mu\cdot\nabla w_j + \Gamma_\varphi w_j \,\mathrm{d}x\right)\,\mathrm{d}t \\
&\quad - \int_0^T \delta(t)\left(\int_\Omega \nabla\varphi\cdot\mathbf{v}w_j + \varphi\Gamma_{\mathbf{v}} w_j\,\mathrm{d}x\right)\,\mathrm{d}t, \\
\int_0^T\int_\Omega \delta(t)\mu w_j\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \delta(t)(\varepsilon^{-1}\psi'(\varphi)w_j + \varepsilon\nabla\varphi\cdot\nabla w_j - \chi_\varphi\sigma w_j)\,\mathrm{d}x\,\mathrm{d}t, \\
\int_0^T \delta(t)\langle \partial_t\sigma , w_j\rangle_{H^1} \,\mathrm{d}t &= \int_0^T\int_\Omega \delta(t)(-n(\varphi)\nabla N_\sigma\cdot\nabla w_j - \Gamma_\sigma w_j)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T\int_\Omega \delta(t)(\nabla\sigma\cdot\mathbf{v}w_j + \sigma\Gamma_{\mathbf{v}}w_j)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \delta(t)\left(\int_{\partial\Omega} K(\sigma_\infty-\sigma)w_j\,\mathrm{d}\mathcal{H}^{d-1}\right)\,\mathrm{d}t, \\
\int_0^T\int_\Omega \delta(t)\mathbf{T}(\mathbf{v},p)\colon\nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega \delta(t)(-\nu(\varphi)\mathbf{v} + \mu\nabla\varphi + N_\sigma\nabla\sigma)\cdot\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t, \\
\int_0^T \delta(t)\left(\int_\Omega \mathrm{div}(\mathbf{v})\Phi\,\mathrm{d}x\right)\mathrm{d}t &= \int_0^T \delta(t)\left(\int_\Omega \Gamma_{\mathbf{v}}\Phi\,\mathrm{d}x\right)\,\mathrm{d}t.
\end{aligned} \tag{4.60}$$

Since these equations hold for every $\delta \in C_0^\infty(0,T)$, we obtain that $(\varphi,\sigma,\mu,\mathbf{v},p)$ satisfies (4.3) with $\Phi = w_j$ for almost all $t\in(0,T)$ and all $j \geq 1$. Furthermore, the last equation in (4.60) implies $\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma)$ a. e. in Ω_T. As $\{w_j\}_{j\in\mathbb{N}}$ is a Schauder basis of H^2_N and H^2_N is dense in H^1, we obtain that $(\varphi,\sigma,\mu,\mathbf{v},p)$ satisfies (4.3b)-(4.3d) for all $\Phi\in H^1$ and (4.3a) for all $\mathbf{\Phi}\in\mathbf{H}^1$.

Step 6: It remains to show that the initial conditions and the energy estimate are satisfied. To this end, we notice that (4.7a)-(4.7b) imply $\varphi_k(0) = \mathbb{P}_k\varphi_0$ and $\sigma_k(0) = \mathbb{P}_k\sigma_0$ where $\mathbb{P}_k$ denotes the L^2-orthogonal projection onto the finite-dimensional subspaces $\mathcal{W}_k$. Since $\{w_j\}_{j\in\mathbb{N}}$ is a Schauder basis in L^2, it holds that $\mathbb{P}_k\varphi_0\to\varphi_0$ in L^2 as $k\to\infty$. Furthermore, we know that $\varphi_k\to\varphi$ strongly in $C^0([0,T];L^2)$, meaning $\varphi_k(0)\to\varphi(0)$ strongly in L^2. But this already implies $\varphi(0)=\varphi_0$. Furthermore, since σ belongs to $C^0([0,T];(H^1)^*)$ we see that $\sigma(0)$ is well-defined as an element of $(H^1)^*$. Furthermore, using the strong convergence $\sigma_k\to\sigma$ in $C^0([0,T];(H^1)^*)$ we obtain for arbitrary $\zeta\in H^1$ that

$$\lim_{n\to\infty}\langle \sigma_k(0), \zeta\rangle_{H^1} = \langle\sigma(0),\zeta\rangle_{H^1}.$$

By the strong convergence $\mathbb{P}_k\sigma_0\to\sigma_0$ in L^2, this implies

$$\langle\sigma_0,\zeta\rangle_{H^1} = \lim_{n\to\infty}\langle\mathbb{P}_k\sigma_0,\zeta\rangle_{H^1} = \lim_{n\to\infty}\langle\sigma_k(0),\zeta\rangle_{H^1} = \langle\sigma(0),\zeta\rangle_{H^1}$$

which yields $\sigma(0)=\sigma_0$ in $(H^1)^*$. Finally, the estimate (4.4) follows from (4.37) by weak (weak-star) lower-semicontinuity of norms and dual norms.

Remark 4.7 If the boundary of Ω satisfies $\partial\Omega\in C^{2,1}$, one can show that $\varphi\in L^2(H^3)$ with bounded norm. Indeed, inserting $v=\lambda_j^2 a_j^k w_j$ in (4.5b), integrating by parts and over $(0,T)$ and summing the resulting equations over $j=1,\dots,k$, we obtain

$$\varepsilon\|\nabla\Delta\varphi_k\|^2_{L^2(\mathbf{L}^2)} = -\int_0^T\int_\Omega \nabla\left(\mu_k+\chi_\varphi\sigma_k\right)\cdot\nabla\Delta\varphi_k - \varepsilon^{-1}\psi''(\varphi_k)\nabla\varphi_k\cdot\nabla\Delta\varphi_k\,\mathrm{d}x\,\mathrm{d}t. \tag{4.61}$$

For the first term on the r. h. s. of (4.61), applying Hölder's and Young's inequality gives

$$\left|\int_0^T\int_\Omega \nabla\left(\mu_k+\chi_\varphi\sigma_k\right)\cdot\nabla\Delta\varphi_k\,\mathrm{d}x\,\mathrm{d}t\right| \leq C\left(\|\mu_k\|^2_{L^2(H^1)} + \|\sigma_k\|^2_{L^2(H^1)}\right) + \frac{\varepsilon}{4}\|\nabla\Delta\varphi_k\|^2_{L^2(\mathbf{L}^2)}.$$

If $\psi(\cdot)$ satisfies assumptions (A6), 2.), the second term on the r. h. s. of (4.61) can be estimated by

$$\int_0^T \int_\Omega \varepsilon^{-1}\psi''(\varphi_k)\nabla\varphi_k\cdot\nabla\Delta\varphi_k \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_\Omega \varepsilon^{-1}R_6(1+|\varphi_k|^q)|\nabla\varphi_k||\nabla\Delta\varphi_k|\,\mathrm{d}x\,\mathrm{d}t.$$

Applying Young's and Hölder's inequality, we obtain

$$\int_0^T \int_\Omega \varepsilon^{-1}R_6|\nabla\varphi_k||\nabla\Delta\varphi_k|\,\mathrm{d}x\,\mathrm{d}t \le C\|\nabla\varphi_k\|^2_{L^2(\mathbf{L}^2)} + \frac{\varepsilon}{8}\|\nabla\Delta\varphi_k\|^2_{L^2(\mathbf{L}^2)}.$$

Furthermore, using the inequality $\|\varphi_k\|_{L^\infty} \le C\big(\|\varphi_k\|_{H^1} + \|\varphi_k\|^{\frac{3}{4}}_{H^1}\|\nabla\Delta\varphi_k\|^{\frac{1}{4}}_{\mathbf{L}^2}\big)$ gives

$$\begin{aligned}
&\int_0^T \int_\Omega \varepsilon^{-1}R_6|\varphi_k|^q|\nabla\varphi_k||\nabla\Delta\varphi_k|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \int_0^T \varepsilon^{-1}R_6\|\varphi_k\|^q_{L^\infty}\|\nabla\varphi_k\|_{\mathbf{L}^2}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}\,\mathrm{d}t \\
&\quad\le C\int_0^T \|\varphi_k\|^{\frac{3q+4}{4}}_{H^1}\|\nabla\Delta\varphi_k\|^{\frac{q+4}{4}}_{\mathbf{L}^2} + \|\varphi_k\|^{q+1}_{H^1}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}\,\mathrm{d}t \\
&\quad\le C\int_0^T \|\varphi_k\|^{\frac{3q+4}{4}}_{H^1}\|\nabla\Delta\varphi_k\|^{\frac{q+4}{4}}_{\mathbf{L}^2}\,\mathrm{d}t + C\|\varphi_k\|^{2(q+1)}_{L^\infty(H^1)} + \frac{\varepsilon}{8}\|\nabla\Delta\varphi_k\|^2_{L^2(\mathbf{L}^2)},
\end{aligned}$$

where we used Young's inequality. Observing that $8/(q+4) > 1$ since $q \in [0,4)$, we can use Young's generalised inequality for the first term on the r. h. s. of this inequality to obtain

$$C\int_0^T \|\varphi_k\|^{\frac{3q+4}{4}}_{H^1}\|\nabla\Delta\varphi_k\|^{\frac{q+4}{4}}_{\mathbf{L}^2}\,\mathrm{d}t \le C\int_0^T \|\varphi_k\|^{\frac{2(3q+4)}{4-q}}_{H^1}\,\mathrm{d}t + \frac{\varepsilon}{4}\|\nabla\Delta\varphi_k\|^2_{L^2(\mathbf{L}^2)}.$$

Invoking the last five inequalities in (4.61) and using (4.37), we see that

$$\frac{\varepsilon}{4}\|\nabla\Delta\varphi_k\|^2_{L^2(\mathbf{L}^2)} \le C.$$

Together with (4.37), using elliptic regularity theory and $\partial\Omega \in C^{2,1}$ implies

$$\|\varphi_k\|_{L^2(H^3)} \le C.$$

The case where $\psi(\cdot)$ satisfies assumptions (A6), 1.) corresponds to the case (A6), 2.) with $q = 0$.

5

Cahn–Hilliard–Brinkman model for tumour growth with quasi-static nutrients

In this chapter we will consider a variant of the model analysed in Chapter 4. Instead of imposing (4.1e), the nutrients will evolve quasi-statically which has a twofold meaning. On the one hand, we will neglect the time derivative on the left hand side of (4.1e). On the other hand, the tumour's evolution affects the nutrients by consumption and therefore the nutrient concentration varies in time.
We shortly recap the motivation for the modified nutrient equation. Denoting by T_{TD}, $T_{\mathcal{D}}$, T_λ and $T_{\mathcal{C}}$ the timescales for tumour doubling, nutrient diffusion, active transport and consumption, respectively, and following the arguments in Chapter 3, by non-dimensionalising (4.1e) and using a source term of the form (3.31) we obtain

$$\frac{T_{\mathcal{D}}}{T_{TD}}(\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v})) = \Delta\sigma - \frac{T_{\mathcal{D}}}{T_\lambda}\Delta\varphi - \frac{T_{\mathcal{D}}}{T_{\mathcal{C}}}h(\varphi)\sigma.$$

As outlined in Chapter 3, experimental values indicate that $\frac{T_{\mathcal{D}}}{T_{TD}} \ll 1$. Furthermore, neglecting active transport mechanisms and assuming $\frac{T_{\mathcal{D}}}{T_{\mathcal{C}}} \sim \mathcal{O}(1)$, the nutrient equation reads as

$$0 = \Delta\sigma - h(\varphi)\sigma.$$

We point out that it is possible to neglect active transport mechanisms while keeping chemotaxis via the decoupling (3.38)-(3.39). Furthermore, we remark that the energy of the new model is given by

$$E(\varphi,\sigma) = \int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2 + \varepsilon^{-1}\psi(\varphi)\,\mathrm{d}x,$$

thus, in contrast to (4.1), there is no contribution from the nutrient free energy. Finally, we will use the pressure reformulation according to (3.33a), thus (4.1b) can be replaced by

$$-\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu(\varphi)\mathbf{v} = (\mu + \chi\sigma)\nabla\varphi,$$

where χ denotes the chemotaxis parameter. In particular, the modified form of the forcing term is more suitable to deduce a priori estimates for the new model.

The results are part of the work [55]. We consider the system

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= \Gamma_{\mathbf{v}}(\varphi,\sigma) && \text{a.e. in } \Omega_T, && (5.1a)\\
-\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu(\varphi)\mathbf{v} &= (\mu+\chi\sigma)\nabla\varphi && \text{a.e. in } \Omega_T, && (5.1b)\\
\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) &= \operatorname{div}(m(\varphi)\nabla\mu) + \Gamma_{\varphi}(\varphi,\sigma) && \text{a.e. in } \Omega_T, && (5.1c)\\
\mu &= \varepsilon^{-1}\psi'(\varphi) - \varepsilon\Delta\varphi - \chi\sigma && \text{a.e. in } \Omega_T, && (5.1d)\\
0 &= \Delta\sigma - h(\varphi)\sigma && \text{a.e. in } \Omega_T, && (5.1e)
\end{aligned}$$

where the viscous stress tensor is defined by

$$\mathbf{T}(\mathbf{v},p) \coloneqq 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I},$$

and χ is a non-negative chemotaxis parameter. We equip the system with initial and boundary conditions of the form

$$\begin{aligned}
\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} &= 0 && \text{a.e. on } \Sigma_T, && (5.2a)\\
\nabla\sigma\cdot\mathbf{n} &= K(\sigma_\infty - \sigma) && \text{a.e. on } \Sigma_T, && (5.2b)\\
\mathbf{T}(\mathbf{v},p)\mathbf{n} &= \mathbf{0} && \text{a.e. on } \Sigma_T, && (5.2c)\\
\varphi(0) &= \varphi_0 && \text{a.e. in } \Omega, && (5.2d)
\end{aligned}$$

where φ_0, σ_∞ are given functions and K is a positive permeability constant.

In the following we outline the main challenges arising in the analysis. When testing the Brinkman equation with $\mathbf{v}$, we have to estimate the term $\int_\Omega p\operatorname{div}(\mathbf{v})\,\mathrm{d}x$. Hence, we need to get an estimate on $\|p\|_{L^2}$ in the absence of any a priori estimates. To overcome this difficulty we will use the so-called method of "Subtracting the divergence". More precisely, we choose $\mathbf{v}-\mathbf{u}$ as a test function in (5.1b) where $\mathbf{u}$ satisfies

$$\operatorname{div}(\mathbf{u}) = \Gamma_{\mathbf{v}} \quad \text{in } \Omega, \quad \mathbf{u} = \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v}}\,\mathrm{d}x\right)\mathbf{n} \quad \text{on } \partial\Omega.$$

As a result we avoid to control the pressure a priori, but we now have to bound the term

$$\begin{aligned}
\int_\Omega \mu\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x &= \int_\Omega (\mu-\mu_\Omega)\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x + \mu_\Omega\int_\Omega \nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x\\
&= \int_\Omega (\mu-\mu_\Omega)\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x + \mu_\Omega\left(\frac{1}{|\partial\Omega|}\int_\Omega \Gamma_{\mathbf{v}}\,\mathrm{d}x\int_{\partial\Omega}\varphi\,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega \varphi\Gamma_{\mathbf{v}}\,\mathrm{d}x\right),
\end{aligned}$$

where $\mu_\Omega = \frac{1}{|\Omega|}\int_\Omega \mu\,\mathrm{d}x$. To control the boundary integral, we will derive an estimate for the $L^\rho(\partial\Omega)-$norm for φ where $\rho\in[2,6]$ is an exponent related to the growth rate of the potential $\psi(\cdot)$.

Furthermore, we comment on the assumption $\sigma_\infty \in L^4(L^2(\partial\Omega))$ which is not needed to prove existence of weak solutions, but crucial to establish well-posedness of the system. Indeed, this allows us to estimate the velocity in $L^{\frac{8}{3}}(0,T;\mathbf{H}^1)$ (see proof of Theorem 5.5) and, as a consequence, we can handle the term

$$\int_\Omega 2(\eta(\varphi_1)-\eta(\varphi_2))\mathbf{D}\mathbf{v}_2 \colon \nabla\mathbf{v}\,\mathrm{d}x$$

in the proof of Theorem 5.7. We remark that this term does not arise in the case of constant viscosity.

Finally, in the proof for existence of strong solutions we will derive an estimate for the time derivative of the nutrient concentration by using a difference quotient method. This argument is needed due to the fact that the L^2-orthogonal projection $\mathbb{P}_n$ onto the n-dimensional Galerkin solution spaces is not continuous on the whole space H^2. Indeed, when testing (5.1d) with $\Delta\partial_t\varphi$ in the Galerkin scheme and integrating by parts twice, we encounter the term

$$\int_0^T \int_\Omega \chi\Delta(\mathbb{P}_n\sigma)\partial_t\varphi \, \mathrm{d}x \, \mathrm{d}t.$$

Although we can control σ in $L^2(H^2)$, an estimate of $\Delta\mathbb{P}_n\sigma \in L^2(L^2)$ can not be deduced due to (5.2b). If the time derivative of σ fulfils $\partial_t\sigma \in L^2(H^1)$, a control of $\Delta\mathbb{P}_n\sigma \in L^2(L^2)$ is not needed, see proof of Theorem 5.11.

5.1 Main results

We make the following assumptions:

Assumptions 5.1

(A1) *The positive constants ε, K, T are fixed and χ is a fixed, non-negative constant. Furthermore, the function $\sigma_\infty \in L^2(L^2(\partial\Omega))$ and the initial datum $\varphi_0 \in H^1$ are prescribed.*

(A2) *The mobility $m(\cdot)$ is continuous on $\mathbb{R}$ and satisfies*

$$m_0 \le m(t) \le m_1 \quad \forall\, t \in \mathbb{R}$$

for positive constants m_0 and m_1.

(A3) *The viscosities fulfil $\eta, \lambda \in C^2(\mathbb{R})$ with bounded first derivatives and*

$$\eta_0 \le \eta(t) \le \eta_1, \quad 0 \le \lambda(t) \le \lambda_0 \quad \forall\, t \in \mathbb{R}$$

for positive constants η_0, η_1 and a non-negative constant λ_0. The permeability function fulfils $\nu \in C^0(\mathbb{R})$ and

$$\nu_0 \le \nu(r) \le \nu_1, \qquad |\nu(r) - \nu(s)| \le L_\nu |r - s| \quad \forall\, r, s \in \mathbb{R}$$

for positive constants ν_0, ν_1 and L_ν.

(A4) *The source terms are of the form*

$$\Gamma_{\mathbf{v}}(\varphi, \sigma) = b_{\mathbf{v}}(\varphi)\sigma + f_{\mathbf{v}}(\varphi), \quad \Gamma_\varphi(\varphi, \sigma) = b_\varphi(\varphi)\sigma + f_\varphi(\varphi)$$

where $b_{\mathbf{v}}, f_{\mathbf{v}} \in C^1(\mathbb{R})$ are bounded with bounded first derivatives and $b_\varphi, f_\varphi \in C^0(\mathbb{R})$ are bounded functions. The function $h \in C^0(\mathbb{R})$ is bounded, non-negative and Lipschitz-continuous with Lipschitz constant L_h.

(A5) *The function $\psi \in C^2(\mathbb{R})$ is non-negative and satisfies one of the following conditions:*

(i) it can be written as

$$\psi(s) = \psi_1(s) + \psi_2(s) \quad \forall\, s \in \mathbb{R}$$

with $\psi_1, \psi_2 \in C^2(\mathbb{R})$ and

$$\begin{aligned} R_1(1 + |s|^{\rho-2}) \le \psi_1''(s) &\le R_2(1 + |s|^{\rho-2}) \quad &&\forall\, s \in \mathbb{R}, \\ |\psi_2''(s)| &\le R_3 &&\forall\, s \in \mathbb{R}, \end{aligned}$$

where R_1, R_2 and R_3 are positive constants with $R_1 < R_2$ and $\rho \in (2, 6]$.

(ii) it fulfils

$$\psi(s) \geq R_0|s|^2 - R_1, \quad |\psi'(s)| \leq R_2\left(1+|s|\right), \quad |\psi''(s)| \leq R_3 \quad \forall\, s \in \mathbb{R},$$

where R_i, $i = 0, \ldots, 3$, are positive constants.

Remark 5.2 Using (A5), it is straightforward to check that there exist positive constants R_i, $i = 4, 5, 6$, such that

$$\psi(s) \geq R_4|s|^\rho - R_5 \quad \forall\, s \in \mathbb{R} \tag{5.3}$$

and

$$|\psi'(s)| \leq R_6(1 + |s|^{\rho-1}) \quad \forall\, s \in \mathbb{R} \tag{5.4}$$

for $\rho \in [2, 6]$.

Remark 5.3 It is easy to check that our assumptions are fulfilled by the classical double-well potential $\psi(s) = \frac{1}{4}(1 - s^2)^2$ which approximates singular potentials of logarithmic type. Note that the double-well potential does not ensure that the order parameter φ lies in the physical relevant range $[-1, 1]$. However, also with the smooth double-well potential, convergence to a sharp interface model holds true, see, e. g., [87] for the Darcy case or Chapter 3.
In some situations it might be more appropriate to use so-called single-well Lennard–Jones type potentials, see e.g. [5, 6, 12, 27], since cell-cell interactions are expected to be attractive at a moderate cell volume fraction and repulsive at higher densities. However, these potentials are not included in our analysis.
Finally, we point out that singular potentials (logarithmic type, double obstacle type) are quite delicate to handle if source terms are present. The analysis of those problems requires more restrictive assumptions on the source terms Γ_φ and $\Gamma_{\mathbf{v}}$ and different techniques and will be investigated later on, see Chapter 8.

We now introduce the weak formulation of (5.1)-(5.2).

Definition 5.4 (Weak solution for (5.1)-(5.2)) We call a quintuple $(\varphi, \sigma, \mu, \mathbf{v}, p)$ a weak solution of (5.1)-(5.2) if

$$\begin{aligned}
&\varphi \in H^1(0,T;(H^1)^*) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^3), \quad \mu \in L^2(0,T;H^1), \\
&\sigma \in L^2(0,T;H^1), \quad \mathbf{v} \in L^2(0,T;\mathbf{H}^1), \quad p \in L^2(0,T;L^2),
\end{aligned}$$

such that

$$\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi, \sigma) \quad \text{a.e. in } \Omega_T, \qquad \varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega,$$

and

$$0 = \int_\Omega \mathbf{T}(\mathbf{v}, p) \colon \nabla\mathbf{\Phi} + \nu(\varphi)\mathbf{v} \cdot \mathbf{\Phi} - (\mu + \chi\sigma)\nabla\varphi \cdot \mathbf{\Phi} \, \mathrm{d}x, \tag{5.5a}$$

$$0 = \langle \partial_t\varphi, \Phi \rangle_{H^1} + \int_\Omega m(\varphi)\nabla\mu \cdot \nabla\Phi + (\nabla\varphi \cdot \mathbf{v} + \varphi\Gamma_{\mathbf{v}}(\varphi, \sigma) - \Gamma_\varphi(\varphi, \sigma))\Phi \, \mathrm{d}x, \tag{5.5b}$$

$$0 = \int_\Omega (\mu + \chi\sigma)\Phi - \varepsilon^{-1}\Psi'(\varphi)\Phi - \varepsilon\nabla\varphi \cdot \nabla\Phi \, \mathrm{d}x, \tag{5.5c}$$

$$0 = \int_\Omega \nabla\sigma \cdot \nabla\Phi + h(\varphi)\sigma\Phi \, \mathrm{d}x + \int_{\partial\Omega} K(\sigma - \sigma_\infty)\Phi \, \mathrm{d}\mathcal{H}^{d-1} \tag{5.5d}$$

for a. e. $t \in (0, T)$ and for all $\mathbf{\Phi} \in \mathbf{H}^1$, $\Phi \in H^1$.

We have the following theorem concerning weak solutions of (5.1)- (5.2):

Theorem 5.5 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^3-boundary and assume that Assumptions 5.1 is fulfilled. Then, there exists a solution quintuple $(\varphi, \mu, \sigma, \mathbf{v}, p)$ of* (5.1)-(5.2) *in the sense of Definition 5.4 and the estimate*

$$\begin{aligned} &\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)} + \|\mu\|_{L^2(H^1)} + \|\sigma\|_{L^2(H^1)} \\ &\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^{\frac{3}{2}})} + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \leq C \end{aligned} \tag{5.6}$$

holds with a constant C independent of $(\varphi, \mu, \sigma, \mathbf{v}, p)$.
If in addition $\sigma_\infty \in L^4(L^2(\partial\Omega))$, we have

$$\sigma \in L^4(0,T;H^1), \quad \mu \in L^4(0,T;L^2), \quad \mathbf{v} \in L^{\frac{8}{3}}(0,T;\mathbf{H}^1),$$

and

$$\|\sigma\|_{L^4(H^1)} + \|\mu\|_{L^4(L^2)} + \|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)} + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} \leq C. \tag{5.7}$$

To prove continuous dependence on the initial and boundary data we need to make the following additional assumptions.

Assumptions 5.6

(B1) *The mobility $m(\cdot)$ is constant, without loss of generality we assume $m(\cdot) \equiv 1$.*

(B2) *The functions $b_\varphi(\cdot)$ and $f_\varphi(\cdot)$ are Lipschitz continuous with Lipschitz constants L_b and L_f, respectively.*

(B3) *For ψ' and ψ'', we assume that*

$$\begin{aligned} |\psi'(s_1) - \psi'(s_2)| &\leq k_1(1 + |s_1|^4 + |s_2|^4)|s_1 - s_2| \quad \forall\, s_1, s_2 \in \mathbb{R}, \\ |\psi''(s_1) - \psi''(s_2)| &\leq k_2(1 + |s_1|^3 + |s_2|^3)|s_1 - s_2| \quad \forall\, s_1, s_2 \in \mathbb{R} \end{aligned}$$

for some positive constants k_1 and k_2.

Under these assumptions we can establish the following continuous dependence result:

Theorem 5.7 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with C^3-boundary and assume that Assumptions 5.1 and 5.6 hold. Then, for any two weak solution quintuples $(\varphi_i, \mu_i, \sigma_i, \mathbf{v}_i, p_i)$, $i = 1, 2$, of* (5.1)-(5.2) *in the sense of definition* (5.4) *satisfying*

$$\begin{aligned} &\varphi \in H^1(0,T;(H^1)^*) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^3), \quad \mu \in L^4(0,T;L^2) \cap L^2(0,T;H^1), \\ &\sigma \in L^4(0,T;H^1), \quad \mathbf{v} \in L^{\frac{8}{3}}(0,T;\mathbf{H}^1), \quad p \in L^2(0,T;L^2) \end{aligned}$$

with $\sigma_{i,\infty} \in L^4(L^2(\partial\Omega))$ and $\varphi_i(0) = \varphi_{i,0} \in H^1$ for $i = 1, 2$, it holds that

$$\begin{aligned} &\sup_{t\in(0,T]} \left(\|\varphi_1(t) - \varphi_2(t)\|_{H^1}^2\right) + \|\varphi_1 - \varphi_2\|_{H^1((H^1)^*)\cap L^2(H^3)}^2 + \|\mu_1 - \mu_2\|_{L^2(H^1)}^2 \\ &\quad + \|\sigma_1 - \sigma_2\|_{L^2(H^1)}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\mathbf{H}^1)}^2 + \|p_1 - p_2\|_{L^2(L^2)}^2 \\ &\quad \leq C\left(\|\varphi_{1,0} - \varphi_{2,0}\|_{H^1}^2 + \|\sigma_{1,\infty} - \sigma_{2,\infty}\|_{L^4(L^2(\partial\Omega))}^2\right) \end{aligned} \tag{5.8}$$

for a positive constant C depending on Ω, T, ε, χ, L_h, L_b, L_f, L_ν, K, k_1, k_2, R_1, R_2, R_3, ρ, η_0, η_1, λ_0, ν_0, ν_1, $\|\varphi_i\|_{L^\infty(H^1)\cap L^2(H^3)}$, $\|\mu_i\|_{L^2(H^1)}$, $\|\sigma_i\|_{L^4(H^1)}$, $\|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)}$, $\|b_{\mathbf{v}}(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|f_{\mathbf{v}}(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|b_\varphi(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|f_\varphi(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|h(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|\eta(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|\lambda(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$.

Remark 5.8 For $i = 1, 2$, it holds that $\varphi_i \in H^1((H^1)^*) \cap L^2(H^3)$ and $\nabla\varphi_i \cdot \mathbf{n} = 0$ almost everywhere on Σ_T. Therefore, [123, Lemma 4.1] implies $\varphi_i \in C^0([0, T]; H^1)$ for $i = 1, 2$, and the supremum in (5.8) is well defined.

In the following we introduce the notion of strong solutions.

Definition 5.9 (Strong solution for (5.1)-(5.2)) We call a quintuple $(\varphi, \sigma, \mu, \mathbf{v}, p)$ a strong solution of (5.1)-(5.2) if

$$\begin{aligned}&\varphi \in H^1(0,T;L^2) \cap L^\infty(0,T;H^2) \cap L^2(0,T;H^4), \quad \mu \in L^2(0,T;H^2),\\&\sigma \in L^2(0,T;H^2), \quad \mathbf{v} \in L^2(0,T;\mathbf{H}^2), \quad p \in L^2(0,T;H^1),\end{aligned}$$

and (5.1)-(5.2) are fulfilled almost everywhere in the respective sets.

For the existence of strong solutions, we make the following additional assumptions:

Assumptions 5.10

(C1) *The mobility $m(\cdot)$ is constant, without loss of generality we assume $m(\cdot) \equiv 1$.*

(C2) *The boundary datum $\sigma_\infty \in H^1(0,T;H^{\frac{1}{2}}(\partial\Omega))$ and the initial datum $\varphi_0 \in H^2_N$ are prescribed.*

(C3) *The function $\psi \in C^3(\mathbb{R})$ fulfils*

$$|\psi'''(s)| \le k_3(1 + |s|^3) \quad \forall\, s \in \mathbb{R}$$

for a positive constant k_3.

We have the following result concerning strong solutions:

Theorem 5.11 *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $C^{3,1}$-boundary and assume that Assumptions 5.1 and 5.10 hold. Then, there exists a solution quintuple $(\varphi, \mu, \sigma, \mathbf{v}, p)$ of* (5.1)-(5.2) *in the sense of Definition 5.9. Furthermore, we have*

$$\begin{aligned}&\varphi \in C^0(\overline{\Omega_T}), \quad \mu \in L^\infty(0,T;L^2), \quad \sigma \in H^1(0,T;H^1) \cap L^\infty(0,T;H^2),\\&\mathbf{v} \in L^8(0,T;\mathbf{H}^2), \quad p \in L^8(0,T;H^1),\end{aligned}$$

and the estimate

$$\begin{aligned}&\|\varphi\|_{H^1(L^2)\cap C^0(\overline{\Omega_T})\cap L^\infty(H^2)\cap L^2(H^4)} + \|\sigma\|_{H^1(H^1)\cap L^\infty(H^2)} + \|\mu\|_{L^\infty(L^2)\cap L^2(H^2)}\\&\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathbf{v}\|_{L^8(\mathbf{H}^2)} + \|p\|_{L^8(H^1)} \le C \qquad (5.9)\end{aligned}$$

holds for a positive constant C independent of $(\varphi, \mu, \sigma, \mathbf{v}, p)$.

5.2 Well-posedness of the model

We will first prove existence of weak solutions.

5.2.1 Proof of Theorem 5.5

The idea of the proof is to apply a Galerkin approximation, to derive uniform estimates and then pass to the limit in the Galerkin scheme.

Galerkin approximation

We briefly present the Galerkin scheme. We construct approximating solutions by applying a Galerkin approximation with respect to φ and μ and at the same time solving for σ, $\mathbf{v}$ and p in the corresponding whole function spaces. We use the eigenfunctions $\{w_i\}_{i\in\mathbb{N}}$ of the Neumann–Laplace operator that form an orthonormal Schauder basis in L^2 which is also a Schauder basis of H^2_N, see Chapter 2.
We fix $k \in \mathbb{N}$ and define

$$\mathcal{W}_k := \operatorname{span}\{w_1, \dots, w_k\},$$

and we denote by $\mathbb{P}_k$ the L^2-orthogonal projection onto the k-dimensional subspaces $\mathcal{W}_k$. Our aim is to find functions of the form

$$\varphi_k(t,x) = \sum_{i=1}^{k} a_i^k(t) w_i(x), \qquad \mu_k(t,x) = \sum_{i=1}^{k} b_i^k(t) w_i(x),$$

satisfying the approximation problem

$$\int_\Omega \partial_t \varphi_k v \,\mathrm{d}x = \int_\Omega -\mathrm{m}(\varphi_k)\nabla\mu_k \cdot \nabla v + \Gamma_{\varphi,k} v - (\nabla\varphi_k \cdot \mathbf{v}_k + \varphi_k \Gamma_{\mathbf{v},k}) v \,\mathrm{d}x, \tag{5.10a}$$

$$\int_\Omega \mu_k v \,\mathrm{d}x = \int_\Omega \varepsilon \nabla\varphi_k \cdot \nabla v + \varepsilon^{-1}\psi'(\varphi_k) v - \chi\sigma_k v \,\mathrm{d}x \tag{5.10b}$$

for all $v \in \mathcal{W}_k$, where $\Gamma_{\varphi,k} := \Gamma_\varphi(\varphi_k, \sigma_k)$ and $\Gamma_{\mathbf{v},k} := \Gamma_{\mathbf{v}}(\varphi_k, \sigma_k)$. Furthermore, we define σ_k as the unique weak solution of

$$0 = \Delta\sigma_k - h(\varphi_k)\sigma_k \quad \text{in } \Omega, \qquad \nabla\sigma_k \cdot \mathbf{n} = K(\sigma_\infty - \sigma_k) \quad \text{on } \partial\Omega, \tag{5.10c}$$

and the velocity $\mathbf{v}_k$ and the pressure p_k as the solutions of (2.64) with

$$\mathbf{f} = (\mu_k + \chi\,\mathbb{P}_k\sigma_k)\nabla\varphi_k, \quad g = \Gamma_{\mathbf{v},k}, \quad c = \varphi_k, \quad \mathbf{f}_b = \mathbf{0}.$$

We complete the system with the initial condition $\varphi_k(0) = \mathbb{P}_k\,\varphi_0$. Due to the assumptions on σ_∞ and $h(\cdot)$, it follows that $\sigma_k \in H^1$. Furthermore, using the continuous embedding $H^2_N \hookrightarrow L^\infty$ and the assumptions on $\Gamma_{\mathbf{v}}$, an easy calculation yields that

$$(\mu_k + \chi\mathbb{P}_k\sigma_k)\nabla\varphi_k \in \mathbf{L}^2, \quad \Gamma_{\mathbf{v},k} \in H^1 \cap L^2.$$

Therefore, by Proposition 2.50, we obtain that $(\mathbf{v}_k, p_k) \in \mathbf{H}^2 \times H^1$ and

$$-\mathrm{div}(T(\mathbf{v}_k, p_k)) + \nu(\varphi_k)\mathbf{v}_k = (\mu_k + \chi\mathbb{P}_k\sigma_k)\nabla\varphi_k \qquad \text{a. e. in } \Omega, \tag{5.10d}$$

$$\mathrm{div}(\mathbf{v}_k) = \Gamma_{\mathbf{v},k} \qquad \text{a. e. in } \Omega, \tag{5.10e}$$

$$T(\mathbf{v}_k, p_k)\mathbf{n} = \mathbf{0} \qquad \text{a. e. on } \partial\Omega. \tag{5.10f}$$

Then, it is straightforward to check that (5.10a)-(5.10f) together with the initial condition for φ_k is equivalent to a coupled system of ODEs in the k unknowns a_i^k, $1 \le i \le k$. Owing to the continuity of $\Gamma_{\mathbf{v}}$, Γ_φ, h, m and ψ' and due to the stability of (5.10d)-(5.10f) under perturbations of $\mathbf{f}$, g and φ_k (see Proposition 2.52) and the stability of (5.10c) under perturbations of φ_k, Lemma 2.27 ensures that there exists some $T_k^* \in (0,\infty]$ such that (5.10a)-(5.10c) admits at least one solution triplet $(\varphi_k, \mu_k, \sigma_k) \in (H^1([0,T_k^*); \mathcal{W}_k))^2 \times L^2([0,T_k^*); H^1)$. Finally, we can define velocity and pressure via (5.10d)-(5.10f) and by Proposition 2.50 we have $(\mathbf{v}_k, p_k) \in \mathbf{H}^2 \times H^1$ for almost all $t \in [0, T_k^*)$.
We remark that a similar scheme as in the proof of Theorem 4.4 for velocity and pressure could be applied. However, the ansatz we make here can also be used to prove strong solutions (with a slight modification for the nutrient concentration), see proof of Theorem 5.11.

A priori estimates

We now derive a priori estimates for $(\varphi_k, \mu_k, \sigma_k, \mathbf{v}_k, p_k)$. By C we denote a positive constant not depending on $k \in \mathbb{N}$ which may vary from line to line. Furthermore, we will omit the subscript k and we write Γ_φ, $\Gamma_{\mathbf{v}}$ instead of $\Gamma_\varphi(\varphi,\sigma)$ and $\Gamma_{\mathbf{v}}(\varphi,\sigma)$.

Estimating the nutrient concentration Testing the weak formulation of (5.10c) with σ and using the non-negativity of $h(\cdot)$ we obtain

$$\int_\Omega |\nabla\sigma|^2 \,\mathrm{d}x + K\int_{\partial\Omega} |\sigma|^2 \,\mathrm{d}\mathcal{H}^{d-1} \le K\int_{\partial\Omega} \sigma\sigma_\infty \,\mathrm{d}\mathcal{H}^{d-1}.$$

Using Hölder's and Young's inequalities we have

$$\left| K\int_{\partial\Omega} \sigma\sigma_\infty \,\mathrm{d}\mathcal{H}^{d-1}\right| \le \frac{K}{2}\|\sigma\|^2_{L^2(\partial\Omega)} + \frac{K}{2}\|\sigma_\infty\|^2_{L^2(\partial\Omega)}.$$

Invoking the last two inequalities and Poincaré's inequality yields

$$\|\sigma\|_{H^1} \le C\|\sigma_\infty\|_{L^2(\partial\Omega)}. \tag{5.11}$$

Moreover, by the Sobolev embedding $H^1 \subset L^p$, $p \in [1,6]$ and (A4), we have

$$\|\Gamma_\varphi\|_{L^p} + \|\Gamma_{\mathbf{v}}\|_{L^p} \le C\left(1 + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right) \quad \forall\, p \in [2,6]. \tag{5.12}$$

An energy identity In the following we will omit the projection $\mathbb{P}_k$ for better readability. However, we point out that the projection $\mathbb{P}_k$ is continuous on H^1, see Chapter 2. Invoking (A4) and (5.11)-(5.12), by Lemma 2.39 there exists a solution $\mathbf{u} \in \mathbf{H}^1$ of the problem

$$\operatorname{div}(\mathbf{u}) = \Gamma_{\mathbf{v}} \quad \text{in } \Omega, \qquad \mathbf{u} = \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v}} \,\mathrm{d}x\right)\mathbf{n} =: \mathbf{a} \quad \text{on } \partial\Omega$$

satisfying the estimate

$$\|\mathbf{u}\|_{\mathbf{H}^1} \le C\|\Gamma_{\mathbf{v}}\|_{L^2} \le C\left(1 + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right). \tag{5.13}$$

Multiplying (5.10d) with $\mathbf{v} - \mathbf{u}$ and using (5.10e)-(5.10f), testing (5.10a) with $\mu + \chi\sigma$, (5.10b) with $\partial_t\varphi$ and summing the resulting identities, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{-1}\psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2 \,\mathrm{d}x + \int_\Omega m(\varphi)|\nabla\mu|^2 \,\mathrm{d}x + \int_\Omega 2\eta(\varphi)|\mathbf{D}\mathbf{v}|^2 + \nu(\varphi)|\mathbf{v}|^2 \,\mathrm{d}x \\
&\quad = \int_\Omega -m(\varphi)\chi\nabla\mu\cdot\nabla\sigma + (\Gamma_\varphi - \varphi\Gamma_{\mathbf{v}})(\mu + \chi\sigma) \,\mathrm{d}x \\
&\qquad + \int_\Omega 2\eta(\varphi)\mathbf{D}\mathbf{v} \colon \nabla\mathbf{u} + \nu(\varphi)\mathbf{v}\cdot\mathbf{u} \,\mathrm{d}x - \int_\Omega (\mu + \chi\sigma)\nabla\varphi\cdot\mathbf{u} \,\mathrm{d}x.
\end{aligned} \tag{5.14}$$

Estimating the right hand side of the energy identity Using Hölder's and Young's inequalities together with (A3) and (5.13) gives

$$\begin{aligned}
\left|\int_\Omega 2\eta(\varphi)\mathbf{D}\mathbf{v} \colon \nabla\mathbf{u} + \nu(\varphi)\mathbf{v}\cdot\mathbf{u} \,\mathrm{d}x\right| &\le \|\sqrt{\eta(\varphi)}\mathbf{D}\mathbf{v}\|^2_{\mathbf{L}^2} + \frac{\nu_0}{2}\|\mathbf{v}\|^2_{\mathbf{L}^2} \\
&\quad + C\left(1 + \|\eta(\varphi)\|_{L^\infty}\right)\|\Gamma_{\mathbf{v}}\|^2_{L^2}.
\end{aligned} \tag{5.15}$$

To estimate the terms involving $\Gamma_{\mathbf{v}}$ and Γ_φ, we first derive estimates for the mean $(\mu+\chi\sigma)_\Omega$. Choosing $v=1$ in (5.10b) and using (5.4) leads to

$$\left|\int_\Omega \mu+\chi\sigma \,\mathrm{d}x\right| = \left|\int_\Omega \varepsilon^{-1}\psi'(\varphi)\,\mathrm{d}x\right| \le \varepsilon^{-1}R_6\int_\Omega 1+|\varphi|^{\rho-1}\,\mathrm{d}x \le C\left(1+\|\varphi\|_{L^{\rho-1}}^{\rho-1}\right),$$

hence

$$|(\mu+\chi\sigma)_\Omega| \le C\left(1+\|\varphi\|_{L^{\rho-1}}^{\rho-1}\right) \le C\left(1+\|\varphi\|_{L^{\rho}}^{\rho-1}\right).$$

In particular, using Young's inequality, the Sobolev embedding $L^\rho\subset L^2$, $\rho\in[2,6]$, and (5.3), this implies

$$|(\mu+\chi\sigma)_\Omega| \le C\left(1+\|\varphi\|_{L^\rho}^{\rho-1}\right) \le C\left(1+\|\varphi\|_{L^\rho}^{\rho}\right) \le C\left(1+\|\psi(\varphi)\|_{L^1}\right), \tag{5.16}$$

$$\|\varphi\|_{L^2}|(\mu+\chi\sigma)_\Omega| \le C\left(\|\varphi\|_{L^2}+\|\varphi\|_{L^\rho}^{\rho}\right) \le C\left(1+\|\varphi\|_{L^\rho}^{\rho}\right) \le C\left(1+\|\psi(\varphi)\|_{L^1}\right). \tag{5.17}$$

Using Hölder's, Poincaré's and Young's inequalities along with (5.11)-(5.12) and (5.16)-(5.17), we obtain

$$\begin{aligned}\left|\int_\Omega \Gamma_\varphi(\mu+\chi\sigma)\,\mathrm{d}x\right| &\le C_P\|\Gamma_\varphi\|_{L^2}\left(|(\mu+\chi\sigma)_\Omega|+\|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}\right)\\ &\le C\left(1+\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)\left(1+|(\mu+\chi\sigma)_\Omega|\right)+\frac{m_0}{8}\|\nabla\mu\|_{\mathbf{L}^2}^2\\ &\le C\left(1+\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)\left(1+\|\psi(\varphi)\|_{L^1}\right)+\frac{m_0}{8}\|\nabla\mu\|_{\mathbf{L}^2}^2.\end{aligned}$$

With similar arguments and using the Sobolev embedding $H^1\subset L^6$, it holds that

$$\begin{aligned}\left|\int_\Omega \Gamma_{\mathbf{v}}\varphi(\mu+\chi\sigma)\,\mathrm{d}x\right| &\le C\|\Gamma_{\mathbf{v}}\|_{L^3}\|\varphi\|_{L^2}\|\mu+\chi\sigma\|_{L^6}\\ &\le C\|\Gamma_{\mathbf{v}}\|_{L^3}\|\varphi\|_{L^2}\left(|(\mu+\chi\sigma)_\Omega|+\|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}\right)\\ &\le C\left(1+\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)\left(1+\|\varphi\|_{L^2}^2+\|\varphi\|_{L^2}|(\mu+\chi\sigma)_\Omega|\right)+\frac{m_0}{8}\|\nabla\mu\|_{\mathbf{L}^2}^2\\ &\le C\left(1+\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)\left(1+\|\psi(\varphi)\|_{L^1}\right)+\frac{m_0}{8}\|\nabla\mu\|_{\mathbf{L}^2}^2.\end{aligned}$$

Combining the last two inequalities yields

$$\left|\int_\Omega(\Gamma_\varphi-\varphi\Gamma_{\mathbf{v}})(\mu+\chi\sigma)\,\mathrm{d}x\right| \le C\left(1+\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)\left(1+\|\psi(\varphi)\|_{L^1}\right)+\frac{m_0}{4}\|\nabla\mu\|_{\mathbf{L}^2}^2. \tag{5.18}$$

For the first term on the r. h. s. of (5.14), applying Hölder's and Young's inequalities, (A2) and (5.11), we obtain

$$\left|\int_\Omega m(\varphi)\chi\nabla\mu\cdot\nabla\sigma\,\mathrm{d}x\right| \le m_1\chi\|\nabla\mu\|_{\mathbf{L}^2}\|\nabla\sigma\|_{\mathbf{L}^2} \le \frac{m_0}{8}\|\nabla\mu\|_{\mathbf{L}^2}^2+C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2. \tag{5.19}$$

Estimating the remaining term

For the remaining term on the r. h. s. of (5.14), we claim that the following bound holds:

$$\left|\int_\Omega(\mu+\chi\sigma)\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x\right| \le C\left(1+\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)\left(1+\|\psi(\varphi)\|_{L^1}+\|\nabla\varphi\|_{\mathbf{L}^2}^2\right)+\frac{m_0}{8}\|\nabla\mu\|_{\mathbf{L}^2}^2 \tag{5.20}$$

The proof of this inequality is divided into two cases.

The case $\rho=2$: Using Hölder's, Young's and Poincaré's inequalities, the Sobolev embeddings

$\mathbf{H}^1 \subset \mathbf{L}^6$, $H^1 \subset L^3$, and (5.3), (5.13), (5.16), we obtain

$$\begin{aligned}
\left| \int_\Omega (\mu + \chi\sigma)\nabla\varphi \cdot \mathbf{u} \,\mathrm{d}x \right| &\leq \|\mu + \chi\sigma\|_{L^3} \|\nabla\varphi\|_{\mathbf{L}^2} \|\mathbf{u}\|_{\mathbf{L}^6} \\
&\leq C\big(|(\mu+\chi\sigma)_\Omega| + \|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}\big)\|\nabla\varphi\|_{\mathbf{L}^2}\|\mathbf{u}\|_{\mathbf{H}^1} \\
&\leq C\big(1 + \|\varphi\|_{L^2} + \|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}\big)\|\nabla\varphi\|_{\mathbf{L}^2}\left(1 + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right) \\
&\leq C\left(1 + \|\sigma_\infty\|^2_{L^2(\partial\Omega)}\right)\left(1 + \|\varphi\|^2_{L^2} + \|\nabla\varphi\|^2_{\mathbf{L}^2}\right) + \frac{m_0}{8}\|\nabla\mu\|^2_{\mathbf{L}^2} \\
&\leq C\left(1 + \|\sigma_\infty\|^2_{L^2(\partial\Omega)}\right)\left(1 + \|\psi(\varphi)\|_{L^1} + \|\nabla\varphi\|^2_{\mathbf{L}^2}\right) + \frac{m_0}{8}\|\nabla\mu\|^2_{\mathbf{L}^2}
\end{aligned}$$

which shows (5.20).

The case $\rho \in (2,6]$: In this case, we need a more subtle argument. Testing (5.10b) with $-\Delta\varphi$, integrating by parts and using (A5), it holds

$$\int_\Omega \varepsilon|\Delta\varphi|^2 + \varepsilon^{-1}\psi_1''(\varphi)|\nabla\varphi|^2 \,\mathrm{d}x = -\varepsilon^{-1}\int_\Omega \psi_2''(\varphi)|\nabla\varphi|^2 \,\mathrm{d}x + \int_\Omega \nabla(\mu+\chi\sigma)\cdot\nabla\varphi \,\mathrm{d}x.$$

Neglecting the non-negative term $\varepsilon \int_\Omega |\Delta\varphi|^2 \,\mathrm{d}x$ on the l. h. s. of this equation and using (A5), (i) along with Young's inequality leads to

$$\int_\Omega |\varphi|^{\rho-2}|\nabla\varphi|^2 \,\mathrm{d}x \leq \left(1 + \frac{R_3}{R_1} + \frac{\varepsilon^2}{4\delta R_1^2}\right)\|\nabla\varphi\|^2_{\mathbf{L}^2} + \delta\|\nabla(\mu+\chi\sigma)\|^2_{\mathbf{L}^2}$$

with $\delta > 0$ to be chosen later. Together with the identity

$$\left|\nabla\left(\frac{2|\varphi|^{\frac{\rho}{2}}}{\rho}\right)\right| = |\varphi|^{\frac{\rho-2}{2}}|\nabla\varphi|$$

we therefore obtain

$$\left\|\nabla(|\varphi|^{\frac{\rho}{2}})\right\|^2_{\mathbf{L}^2} \leq \frac{\rho^2}{4}\left(1 + \frac{R_3}{R_1} + \frac{\varepsilon^2}{4\delta R_1^2}\right)\|\nabla\varphi\|^2_{\mathbf{L}^2} + \frac{\delta\rho^2}{4}\|\nabla(\mu+\chi\sigma)\|^2_{\mathbf{L}^2}.$$

Applying the trace theorem yields

$$\begin{aligned}
\left\||\varphi|^{\frac{\rho}{2}}\right\|^2_{L^2(\partial\Omega)} &\leq C_{tr}^2\left(\left\||\varphi|^{\frac{\rho}{2}}\right\|^2_{L^2} + \left\|\nabla\left(|\varphi|^{\frac{\rho}{2}}\right)\right\|^2_{\mathbf{L}^2}\right) \\
&\leq C_{tr}^2\left(\|\varphi\|^\rho_{L^\rho} + \frac{\rho^2}{4}\left(1 + \frac{R_3}{R_1} + \frac{\varepsilon^2}{4\delta R_1^2}\right)\|\nabla\varphi\|^2_{\mathbf{L}^2} + \frac{\delta\rho^2}{4}\|\nabla(\mu+\chi\sigma)\|^2_{\mathbf{L}^2}\right),
\end{aligned}$$

hence

$$\|\varphi\|^\rho_{L^\rho(\partial\Omega)} \leq C_{tr}^2\left(\|\varphi\|^\rho_{L^\rho} + \frac{\rho^2}{4}\left(1 + \frac{R_3}{R_1} + \frac{\varepsilon^2}{4\delta R_1^2}\right)\|\nabla\varphi\|^2_{\mathbf{L}^2} + \frac{\delta\rho^2}{4}\|\nabla(\mu+\chi\sigma)\|^2_{\mathbf{L}^2}\right). \tag{5.21}$$

Now, upon integrating by parts and recalling that $\operatorname{div}(\mathbf{u}) = \Gamma_{\mathbf{v}}$ in Ω and $\mathbf{u} = |\partial\Omega|^{-1}(\int_\Omega \Gamma_{\mathbf{v}} \,\mathrm{d}x)\mathbf{n}$ on $\partial\Omega$, we calculate

$$\begin{aligned}
\int_\Omega (\mu+\chi\sigma)\nabla\varphi\cdot\mathbf{u} \,\mathrm{d}x &= \int_\Omega \big(\mu + \chi\sigma - (\mu+\chi\sigma)_\Omega\big)\nabla\varphi\cdot\mathbf{u} \,\mathrm{d}x + (\mu+\chi\sigma)_\Omega \int_\Omega \nabla\varphi\cdot\mathbf{u} \,\mathrm{d}x \\
&= \int_\Omega \big(\mu + \chi\sigma - (\mu+\chi\sigma)_\Omega\big)\nabla\varphi\cdot\mathbf{u} \,\mathrm{d}x \\
&\quad + (\mu+\chi\sigma)_\Omega\left(\frac{1}{|\partial\Omega|}\int_\Omega \Gamma_{\mathbf{v}} \,\mathrm{d}x \int_{\partial\Omega} \varphi \,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega \varphi\Gamma_{\mathbf{v}} \,\mathrm{d}x\right).
\end{aligned} \tag{5.22}$$

Using Hölder's, Young's and Poincaré's inequalities, the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ and (5.13), it is straightforward to check that

$$\left|\int_\Omega (\mu + \chi\sigma - (\mu+\chi\sigma)_\Omega)\nabla\varphi \cdot \mathbf{u}\,\mathrm{d}x\right| \le C_{\delta_1}(1 + \|\sigma_\infty\|^2_{L^2(\partial\Omega)})(1+\|\nabla\varphi\|^2_{\mathbf{L}^2}) + \delta_1\|\nabla\mu\|^2_{\mathbf{L}^2}$$

with $\delta_1 > 0$ to be chosen. Using (5.13), (5.17) and Hölder's inequality, we obtain

$$\begin{aligned}\left|(\mu+\chi\sigma)_\Omega \int_\Omega \varphi\Gamma_{\mathbf{v}}\,\mathrm{d}x\right| &\le |(\mu+\chi\sigma)_\Omega| \|\Gamma_{\mathbf{v}}\|_{L^2}\|\varphi\|_{L^2} \\ &\le C\left(1 + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right)(1 + \|\psi(\varphi)\|_{L^1}).\end{aligned}$$

Now, using Hölder's and Young's inequalities, (5.3), (5.13), (5.16), (5.21) and recalling $\rho > 2$ gives

$$\begin{aligned}\left|(\mu+\chi\sigma)_\Omega \frac{1}{|\partial\Omega|}\int_\Omega \Gamma_{\mathbf{v}}\,\mathrm{d}x \int_{\partial\Omega}\varphi\,\mathrm{d}\mathcal{H}^{d-1}\right| &\le C\left(1 + \|\varphi\|^{\rho-1}_{L^\rho}\right)\|\Gamma_{\mathbf{v}}\|_{L^2}\|\varphi\|_{L^\rho(\partial\Omega)} \\ &\le C_{\delta_2}(1+\|\varphi\|^\rho_{L^\rho})\|\Gamma_{\mathbf{v}}\|^{\frac{\rho}{\rho-1}}_{L^2} + \delta_2\|\varphi\|^\rho_{L^\rho(\partial\Omega)} \\ &\le C_{\delta_2}\left(1 + \|\sigma_\infty\|^2_{L^2(\partial\Omega)}\right)(1 + \|\psi(\varphi)\|_{L^1}) + \delta_2\|\varphi\|^\rho_{L^\rho(\partial\Omega)},\end{aligned}$$

where we used that $\frac{\rho-1}{\rho} + \frac{1}{\rho} = 1$. Employing the last three inequalities in (5.22), using (5.3), (5.21) and choosing δ, δ_1, δ_2 small enough, we finally obtain

$$\left|\int_\Omega (\mu+\chi\sigma)\nabla\varphi\cdot\mathbf{u}\,\mathrm{d}x\right| \le \left(1 + \|\sigma_\infty\|^2_{L^2(\partial\Omega)}\right)\left(1 + \|\psi(\varphi)\|_{L^1} + \|\nabla\varphi\|^2_{\mathbf{L}^2}\right) + \frac{m_0}{8}\|\nabla\mu\|^2_{\mathbf{L}^2},$$

which implies (5.20).

An application of (5.15) and (5.18)-(5.20) in (5.14) along with (A3), (5.3) and (5.11) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varepsilon^{-1}\psi(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2\,\mathrm{d}x + \frac{m_0}{4}\|\nabla\mu\|^2_{\mathbf{L}^2} + \frac{\nu_0}{2}\|\mathbf{v}\|^2_{\mathbf{L}^2} + \|\sqrt{\eta(\varphi)}\mathbf{D}\mathbf{v}\|^2_{\mathbf{L}^2} + \|\sigma\|^2_{H^1} \\ &\quad \le \alpha(t)\left(1 + \|\nabla\varphi(t)\|^2_{\mathbf{L}^2} + \|\psi(\varphi(t))\|_{L^1}\right),\end{aligned}$$

and, recalling (A1) and (A3),

$$\alpha(t) := C\left(1 + \|\eta(\varphi(t))\|_{L^\infty}\right)\left(1 + \|\sigma_\infty(t)\|^2_{L^2(\partial\Omega)}\right) \in L^1(0,T).$$

Integrating the last inequality in time from 0 to $s \in (0,T]$ and applying Gronwall's lemma (see Lemma 2.31) leads to

$$\begin{aligned}&\varepsilon^{-1}\|\psi(\varphi(s))\|_{L^1} + \frac{\varepsilon}{2}\|\nabla\varphi(s)\|^2_{\mathbf{L}^2} + \int_0^s \frac{m_0}{4}\|\nabla\mu\|^2_{\mathbf{L}^2} + \frac{\nu_0}{2}\|\mathbf{v}\|^2_{\mathbf{L}^2} + \|\sqrt{\eta(\varphi)}\mathbf{D}\mathbf{v}\|^2_{\mathbf{L}^2} + \|\sigma\|^2_{H^1}\,\mathrm{d}t \\ &\quad \le \left(\varepsilon^{-1}\|\psi(\varphi_0)\|_{L^1} + \frac{\varepsilon}{2}\|\nabla\varphi_0\|^2_{\mathbf{L}^2} + \int_0^s \alpha(t)\,\mathrm{d}t\right)\exp\left(\int_0^s \alpha(t)\,\mathrm{d}t\right) \quad \forall\, s\in(0,T]. \qquad (5.23)\end{aligned}$$

Due to (A1), (A5) and the Sobolev embedding $H^1 \subset L^6$, we have $\psi(\varphi_0) \in L^1$, $\nabla\varphi_0 \in \mathbf{L}^2$. Then, due to Korn's inequality (see (2.23)) and (A3), taking the supremum over all $s \in (0,T]$ in (5.23) implies

$$\operatorname*{ess\,sup}_{s\in(0,T]}\left(\|\psi(\varphi(s))\|_{L^1} + \|\nabla\varphi(s)\|^2_{\mathbf{L}^2}\right) + \int_0^T \|\mathbf{v}\|^2_{\mathbf{H}^1} + \|\nabla\mu\|^2_{\mathbf{L}^2} + \|\sigma\|^2_{H^1}\,\mathrm{d}t \le C.$$

Recalling (5.3), using Poincaré's inequality, (5.16) and the fact that $\rho \ge 2$, this in particular gives

$$\operatorname*{ess\,sup}_{s\in(0,T]}\|\varphi(s)\|^2_{H^1} + \int_0^T \|\mu\|^2_{H^1}\,\mathrm{d}t \le C.$$

Combining the last two inequalities yields

$$\operatorname*{ess\,sup}_{s\in(0,T]} \left(\|\psi(\varphi(s))\|_{L^1} + \|\varphi(s)\|_{H^1}^2\right) + \int_0^T \|\mathbf{v}\|_{\mathbf{H}^1}^2 + \|\mu\|_{H^1}^2 + \|\sigma\|_{H^1}^2 \,\mathrm{d}t \le C. \tag{5.24}$$

Due to (A4) and the Sobolev embedding $H^1 \subset L^6$, this implies

$$\|\Gamma_{\mathbf{v}}\|_{L^2(L^6)} + \|\Gamma_\varphi\|_{L^2(L^6)} \le C. \tag{5.25}$$

Estimating the pressure By Lemma 2.39 there is at least one solution $\mathbf{q} \in \mathbf{H}^1$ of

$$\operatorname{div}(\mathbf{q}) = p \quad \text{in } \Omega, \qquad \mathbf{q} = \frac{1}{|\partial\Omega|}\left(\int_\Omega p \,\mathrm{d}x\right)\mathbf{n} \quad \text{on } \partial\Omega$$

such that

$$\|\mathbf{q}\|_{\mathbf{H}^1} \le C_d \|p\|_{L^2} \tag{5.26}$$

with C_d depending only on Ω. Notice that the compatibility condition (2.24) is satisfied since

$$\int_{\partial\Omega} \mathbf{q}\cdot\mathbf{n} \,\mathrm{d}\mathcal{H}^{d-1} = \frac{1}{|\partial\Omega|}\left(\int_\Omega p \,\mathrm{d}x\right)\int_{\partial\Omega} \mathbf{n}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega p \,\mathrm{d}x.$$

Multiplying (5.10d) with $\mathbf{q}$ and using (5.10e)-(5.10f) we obtain

$$\int_\Omega |p|^2 \,\mathrm{d}x = \int_\Omega (2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\Gamma_{\mathbf{v}}\mathbf{I})\colon \nabla\mathbf{q}\,\mathrm{d}x + \int_\Omega (\nu(\varphi)\mathbf{v} - (\mu+\chi\sigma)\nabla\varphi)\cdot\mathbf{q}\,\mathrm{d}x.$$

Using (5.26), (A3) and Hölder's and Young's inequalities, an easy calculation shows that

$$\|p\|_{L^2}^2 \le C\big(\|\eta(\varphi)\|_{L^\infty}\|\sqrt{\eta(\varphi)}\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\lambda(\varphi)\|_{L^\infty}^2\|\Gamma_{\mathbf{v}}\|_{L^2}^2 + \|\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\mu+\chi\sigma\|_{L^3}^2\|\nabla\varphi\|_{\mathbf{L}^2}^2\big). \tag{5.27}$$

Integrating this inequality in time from 0 to T and employing Hölder's inequality yields

$$\begin{aligned}\|p\|_{L^2(L^2)}^2 &\le C\left(\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\|\sqrt{\eta(\varphi)}\mathbf{D}\mathbf{v}\|_{L^2(\mathbf{L}^2)}^2 + \|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}^2\|\Gamma_{\mathbf{v}}\|_{L^2(L^2)}^2\right) \\ &\quad + C\left(\|\mathbf{v}\|_{L^2(\mathbf{L}^2)}^2 + \|\mu+\chi\sigma\|_{L^2(L^3)}^2\|\nabla\varphi\|_{L^\infty(\mathbf{L}^2)}^2\right).\end{aligned}$$

By (5.23), (5.24)-(5.25) and (A3), this implies

$$\|p\|_{L^2(L^2)} \le C. \tag{5.28}$$

Higher order estimates for $\boldsymbol{\varphi}$ Our aim is to show that

$$\|\varphi\|_{L^4(H^2)\cap L^2(H^3)} \le C. \tag{5.29}$$

In the case where ψ satisfies (A5), (ii), we observe that $|\nabla\psi(\varphi)| = |\psi''(\varphi)\nabla\varphi| \le C|\nabla\varphi|$ and the bounds for φ along with (5.10b) yield (5.29). Thus, it remains to consider the case where $\psi(\cdot)$ satisfies (A5), (i) and thus $\rho \in (2,6]$. Testing (5.10b) with $-\Delta\varphi$, integrating by parts and neglecting the non-negative term resulting from ψ_1 (see (A5)) leads to

$$\varepsilon\|\Delta\varphi\|_{L^2}^2 + \int_\Omega \varepsilon^{-1}\psi_2''(\varphi)|\nabla\varphi|^2\,\mathrm{d}x \le \int_\Omega \nabla(\mu+\chi\sigma)\cdot\nabla\varphi\,\mathrm{d}x.$$

Using Hölder's inequality and the assumptions on ψ_2, we conclude that

$$\varepsilon\|\Delta\varphi\|_{L^2}^2 \le \varepsilon^{-1}R_3\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}\|\nabla\varphi\|_{\mathbf{L}^2}.$$

Taking the square of this inequality and integrating in time from 0 to T gives

$$\varepsilon^2 \int_0^T \|\Delta\varphi\|_{L^2}^4 \, \mathrm{d}t \le C \int_0^T \|\nabla\varphi\|_{\mathbf{L}^2}^4 + \|\nabla\varphi\|_{\mathbf{L}^2}^2 \|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}^2 \, \mathrm{d}t$$
$$\le C\left(\|\nabla\varphi\|_{L^\infty(\mathbf{L}^2)}^4 + \|\nabla(\mu+\chi\sigma)\|_{L^2(\mathbf{L}^2)}^2 \|\nabla\varphi\|_{L^\infty(\mathbf{L}^2)}^2 \right).$$

Applying elliptic regularity theory and (5.24) we obtain

$$\|\varphi\|_{L^4(H^2)} \le C. \tag{5.30}$$

Next, we test (5.10b) with $\Delta^2\varphi$, integrate by parts and in time over $(0,T)$ to infer that

$$\varepsilon\|\nabla\Delta\varphi\|_{L^2(\mathbf{L}^2)}^2 = -\int_0^T\int_\Omega \nabla(\mu+\chi\sigma)\cdot\nabla\Delta\varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T\int_\Omega \varepsilon^{-1}\psi''(\varphi)\nabla\varphi\cdot\nabla\Delta\varphi \, \mathrm{d}x \, \mathrm{d}t.$$

For the first term on the r. h. s., applying Hölder's and Young's inequality gives

$$\left|\int_0^T\int_\Omega \nabla(\mu+\chi\sigma)\cdot\nabla\Delta\varphi \, \mathrm{d}x \, \mathrm{d}t\right| \le C\left(\|\mu\|_{L^2(H^1)}^2 + \|\sigma\|_{L^2(H^1)}^2\right) + \frac{\varepsilon}{4}\|\nabla\Delta\varphi\|_{L^2(\mathbf{L}^2)}^2.$$

Due to (2.4), (A5), (i) and (5.24), using Hölder's and Young's inequalities yields

$$\left|\int_0^T\int_\Omega \varepsilon^{-1}\psi''(\varphi)\nabla\varphi\cdot\nabla\Delta\varphi \, \mathrm{d}x \, \mathrm{d}t\right| \le C\int_0^T\int_\Omega \left(1+|\varphi|^{\rho-2}\right)|\nabla\varphi||\nabla\Delta\varphi| \, \mathrm{d}x \, \mathrm{d}t$$
$$\le C\int_\Omega \left(1+\|\varphi\|_{L^\infty}^{\rho-2}\right)\|\nabla\varphi\|_{\mathbf{L}^2}\|\nabla\Delta\varphi\|_{\mathbf{L}^2} \, \mathrm{d}t$$
$$\le C\int_0^T \left(1+\|\varphi\|_{H^2}^{\frac{\rho-2}{2}}\right)\|\nabla\Delta\varphi\|_{\mathbf{L}^2} \, \mathrm{d}t$$
$$\le C\left(1+\|\varphi\|_{L^{\rho-2}(H^2)}^{\rho-2}\right) + \frac{\varepsilon}{4}\|\nabla\Delta\varphi\|_{L^2(\mathbf{L}^2)}^2.$$

Recalling that $\rho-2\le 4$, the last three inequalities imply that

$$\frac{\varepsilon}{2}\|\nabla\Delta\varphi\|_{L^2(\mathbf{L}^2)}^2 \le C\left(1+\|\mu\|_{L^2(H^1)}^2 + \|\sigma\|_{L^2(H^1)}^2 + \|\varphi\|_{L^4(H^2)}^4\right) \le C,$$

where we used (5.24) and (5.30). By elliptic regularity theory, (5.24) and (5.30), this gives (5.29).

Regularity for the convection terms and the time derivatives Employing Hölder's and Young's inequalities along with the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ leads to

$$\|\nabla\varphi\cdot\mathbf{v}\|_{L^2(0,T;L^{\frac{3}{2}})}^2 \le C\int_0^T \|\mathbf{v}\|_{\mathbf{H}^1}^2\|\nabla\varphi\|_{\mathbf{L}^2}^2 \, \mathrm{d}t \le C\|\nabla\varphi\|_{L^\infty(\mathbf{L}^2)}^2\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^1)}^2.$$

Using (5.24)-(5.25) we see that

$$\|\varphi\Gamma_{\mathbf{v}}\|_{L^2(L^2)}^2 \le C\|\varphi\|_{L^\infty(H^1)}^2\|\Gamma_{\mathbf{v}}\|_{L^2(L^3)}^2 \le C.$$

From the last two inequalities we deduce that

$$\|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(0,T;L^{\frac{3}{2}})} \le C.$$

In conjunction with (5.24)-(5.25) and using (5.10a), this shows that

$$\|\partial_t\varphi\|_{L^2((H^1)^*)} \le C.$$

Notice that we have lower time regularity for the time derivative of φ compared to the convection term since the regularity of the time derivative depends on the term $\nabla\mu$. Invoking the last two inequalities together with (5.24) and (5.28)-(5.29), we end up with

$$\begin{aligned}&\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)} + \|\sigma\|_{L^2(H^1)} + \|\mu\|_{L^2(H^1)}\\ &\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^{\frac{3}{2}})} + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \leq C. \end{aligned} \tag{5.31}$$

Passing to the limit

We briefly sketch the ideas needed to pass to the limit in the Galerkin scheme. Recalling (5.31), using standard compactness arguments (Lemma 2.36 and reflexive weak compactness) and the compact embeddings

$$H^{j+1}(\Omega) = W^{j+1,2}(\Omega) \hookrightarrow\hookrightarrow W^{j,r} \quad \forall\, j \in \mathbb{Z},\ j \geq 0,\ 1 \leq r < 6,$$

we obtain, at least for a subsequence which will again be labelled by k, the convergence properties

$$\begin{aligned}
\varphi_k &\to \varphi && \text{weakly-star} && \text{in } H^1((H^1)^*) \cap L^\infty(H^1) \cap L^4(H^2) \cap L^2(H^3),\\
\sigma_k &\to \sigma && \text{weakly} && \text{in } L^2(H^1),\\
\mu_k &\to \mu && \text{weakly} && \text{in } L^2(H^1),\\
p_k &\to p && \text{weakly} && \text{in } L^{\frac{4}{3}}(L^2).\\
\mathbf{v}_k &\to \mathbf{v} && \text{weakly} && \text{in } L^2(\mathbf{H}^1),\\
\mathrm{div}(\varphi_k\mathbf{v}_k) &\to \tau && \text{weakly} && \text{in } L^2(L^{\frac{3}{2}})
\end{aligned}$$

for some limit function $\tau \in L^2(L^{\frac{3}{2}})$. Furthermore, we have the strong convergence

$$\varphi_k \to \varphi \quad \text{strongly} \quad \text{in } C^0(L^r) \cap L^4(W^{1,r}) \cap L^2(W^{2,r}) \quad \text{and a. e. in } \Omega_T$$

for all $r \in [1, 6)$.

From now on we fix $1 \leq j \leq k$ and $\xi \in L^2$, $\phi \in H^1$, $\mathbf{\Phi} \in \mathbf{H}^1$, $\delta \in C_0^\infty(0,T)$. Then, since the eigenfunctions $\{w_j\}_{j\in\mathbb{N}}$ belong to H^2_N, we observe that $\delta w_j \in C^\infty(H^2_N)$ for all $j \in \mathbb{N}$. Furthermore, we have $\delta\xi \in C^\infty(L^2)$, $\delta\phi \in C^\infty(H^1)$, $\delta\mathbf{\Phi} \in C^\infty(\mathbf{H}^1)$. Inserting $v = w_j$ in (5.10a)-(5.10b), using the weak formulation of (5.10c), multiplying the resulting equations with δ and integrating over $(0,T)$, we obtain

$$\int_0^T \delta(t)\left(\int_\Omega (\partial_t\varphi_k + \nabla\varphi_k\cdot\mathbf{v}_k + \varphi_k\Gamma_{\mathbf{v},k} - \Gamma_{\varphi,k})\, w_j + m(\varphi_k)\nabla\mu_k\cdot\nabla w_j \,\mathrm{d}x\right) \mathrm{d}t = 0, \tag{5.32a}$$

$$\int_0^T \delta(t)\left(\int_\Omega (\mu_k - \varepsilon^{-1}\psi'(\varphi_k) + \chi\sigma_k)\, w_j - \varepsilon\nabla\varphi_k\cdot\nabla w_j \,\mathrm{d}x\right) \mathrm{d}t = 0, \tag{5.32b}$$

$$\int_0^T \delta(t)\left(\int_\Omega \nabla\sigma_k\cdot\nabla\phi + h(\varphi_k)\sigma_k\phi \,\mathrm{d}x + \int_{\partial\Omega} K(\sigma_k - \sigma_\infty)\phi \,\mathrm{d}\mathcal{H}^{d-1}\right) \mathrm{d}t = 0. \tag{5.32c}$$

Furthermore, we take the $\mathbf{L}^2$-scalar product of (5.10d) with $\mathbf{\Phi}$, multiply with δ and integrate over $(0,T)$ to deduce

$$\int_0^T \delta(t)\left(\int_\Omega \mathbf{T}(\mathbf{v}_k, p_k) \colon \nabla\mathbf{\Phi} + \nu(\varphi_k)\mathbf{v}_k\cdot\mathbf{\Phi} - (\mu_k + \chi\mathbb{P}_k\sigma_k)\nabla\varphi_k\cdot\mathbf{\Phi} \,\mathrm{d}x\right) \mathrm{d}t = 0, \tag{5.32d}$$

where we used (5.10f). With similar arguments, (5.10e) gives

$$\int_0^T \delta(t)\left(\int_\Omega \mathrm{div}(\mathbf{v}_k)\,\xi \,\mathrm{d}x\right) \mathrm{d}t = \int_0^T \delta(t)\left(\int_\Omega \Gamma_{\mathbf{v},k}\,\xi \,\mathrm{d}x\right) \mathrm{d}t. \tag{5.32e}$$

Then, it is easy to check that we can pass to the limit in the linear terms. For the non-linear terms we use the continuity of $\mathbb{P}_k$ on L^2, the strong convergence for φ_k, the assumptions on the source terms, the viscosities, the permeability and the potential $\psi(\cdot)$ together with the product of weak-strong convergence, see Chapter 4 for details. As $\{w_j\}_{j\in\mathbb{N}}$ is a Schauder basis of H^1 and due to the fact that (5.32a)-(5.32e) hold for all $\delta \in C_0^\infty(0,T)$, we know that (5.5) holds along with $\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma)$ a. e. in Ω_T. Attainment of the initial condition follows from the continuity of $\mathbb{P}_k$ on L^2 and due to the strong convergence $\varphi_k \to \varphi$ in $C^0(L^2)$.

Further results on regularity

In the case that $\sigma_\infty \in L^4(L^2(\partial\Omega))$, by (5.11) we obtain

$$\|\sigma\|_{L^4(H^1)} \le C. \tag{5.33}$$

In particular, by (A4) this gives

$$\|\Gamma_{\mathbf{v}}\|_{L^4(L^2)} + \|\Gamma_\varphi\|_{L^4(L^2)} \le C.$$

Thanks to Lemma 2.4, we have the continuous embedding

$$L^\infty(H^1) \cap L^2(H^3) \hookrightarrow L^{20}(L^{10}).$$

Hence, the assumptions on $\psi(\cdot)$ and (5.31) imply

$$\|\psi'(\varphi)\|_{L^4(L^2)} \le C.$$

Taking $\Phi = \mu + \chi\sigma$ in (5.5c) and squaring the resulting identity, an application of Hölder's and Young's inequalities gives

$$\|\mu + \chi\sigma\|_{L^2}^4 \le C\left(\|\psi'(\varphi)\|_{L^2}^4 + \|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}^2 \|\nabla\varphi\|_{\mathbf{L}^2}^2\right).$$

Integrating this inequality in time from 0 to T and using (5.31) together with the bound for $\psi'(\varphi)$, we conclude that

$$\|\mu + \chi\sigma\|_{L^4(L^2)} \le C. \tag{5.34}$$

We now choose $\mathbf{\Phi} = \mathbf{v}$ in (5.5a) and use Young's, Hölder's and Korn's inequality (see (2.23)) along with the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ to obtain

$$\|\mathbf{v}\|_{\mathbf{H}^1}^{\frac{8}{3}} \le C\left(\|p\|_{L^2}^{\frac{4}{3}} \|\Gamma_{\mathbf{v}}\|_{L^2}^{\frac{4}{3}} + \|(\mu+\chi\sigma)\nabla\varphi\|_{\mathbf{L}^{\frac{6}{5}}}^{\frac{8}{3}}\right).$$

Integrating this inequality in time from 0 to T, an application of Hölder's and Young's inequalities leads to

$$\|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)}^{\frac{8}{3}} \le C\left(\|p\|_{L^2(L^2)}^{\frac{4}{3}} \|\Gamma_{\mathbf{v}}\|_{L^4(L^2)}^{\frac{4}{3}} + \|\mu+\chi\sigma\|_{L^4(L^2)}^{\frac{8}{3}} \|\nabla\varphi\|_{L^8(\mathbf{L}^3)}^{\frac{8}{3}}\right).$$

Using the continuous embedding $L^\infty(H^1) \cap L^4(H^2) \hookrightarrow L^8(W^{1,3})$ resulting from Lemma 2.4 and invoking (5.31), (5.34) along with the boundedness of $\Gamma_{\mathbf{v}}$ in $L^4(L^2)$, we conclude that

$$\|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)} \le C. \tag{5.35}$$

Furthermore, using Hölder's and Young's inequalities, (5.31), (5.35) and the continuous embeddings $\mathbf{H}^1 \hookrightarrow \mathbf{L}^6$, $L^\infty(H^1) \cap L^4(H^2) \hookrightarrow L^8(W^{1,3})$ yields

$$\|\nabla\varphi \cdot \mathbf{v}\|_{L^2(0,T;L^2)}^2 \le C\int_0^T \|\mathbf{v}\|_{\mathbf{H}^1}^2 \|\nabla\varphi\|_{\mathbf{L}^3}^2 \,\mathrm{d}t \le C\|\nabla\varphi\|_{L^8(\mathbf{L}^3)}^2 \|\mathbf{v}\|_{L^{\frac{8}{3}}(0,T;\mathbf{H}^1)}^2 \le C.$$

Together with the estimate

$$\|\varphi\Gamma_{\mathbf{v}}\|_{L^2(L^2)}^2 \le C\|\varphi\|_{L^\infty(H^1)}^2 \|\Gamma_{\mathbf{v}}\|_{L^2(L^3)}^2 \le C,$$

this implies

$$\|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} \le C. \tag{5.36}$$

Using (5.33), (5.34)-(5.36) and recalling (5.31), we obtain (5.7) which completes the proof.

5.2.2 Continuous dependence (Proof of Theorem 5.7)

In the following we set $\varepsilon = 1$ since it has no bearing on the analysis. Let $(\varphi_i, \mu_i, \sigma_i, \mathbf{v}_i, p_i)_{i=1,2}$ be two solutions of (5.1)-(5.2) in the sense of Definition 5.4. We denote $\Gamma_{\mathbf{v}}(\varphi_i, \sigma_i) \coloneqq \Gamma_{\mathbf{v},i}$, $\Gamma_{\varphi}(\varphi_i, \sigma_i) \coloneqq \Gamma_{\varphi,i}$, $i = 1, 2$, and $\sigma_\infty \coloneqq \sigma_{1,\infty} - \sigma_{2,\infty}$. Then, the differences $f \coloneqq f_1 - f_2$, $f_i \in \{\varphi_i, \mu_i, \sigma_i, \mathbf{v}_i, p_i\}$, $i = 1, 2$, satisfy

$$\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2} \quad \text{a.e. in } \Omega_T, \quad \varphi(0) = \varphi_{1,0} - \varphi_{2,0} =: \varphi_0 \quad \text{a.e. in } \Omega,$$

and

$$\begin{aligned} 0 &= \int_\Omega (2\eta(\varphi_1)\mathbf{D}\mathbf{v} + \lambda(\varphi_1)\mathrm{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}) \colon \nabla\mathbf{\Phi} + \nu(\varphi_1)\mathbf{v} \cdot \mathbf{\Phi} \,\mathrm{d}x \\ &\quad - \int_\Omega (\mu + \chi\sigma)\nabla\varphi_1 \cdot \mathbf{\Phi} + (\mu_2 + \chi\sigma_2)\nabla\varphi \cdot \mathbf{\Phi} - (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2 \cdot \mathbf{\Phi} \,\mathrm{d}x \\ &\quad + \int_\Omega (2(\eta(\varphi_1) - \eta(\varphi_2))\mathbf{D}\mathbf{v}_2 + (\lambda(\varphi_1) - \lambda(\varphi_2))\mathrm{div}(\mathbf{v}_2)\mathbf{I}) \colon \nabla\mathbf{\Phi} \,\mathrm{d}x, \end{aligned} \tag{5.37a}$$

$$\begin{aligned} 0 &= \langle \partial_t \varphi, \Phi \rangle_{H^1} + \int_\Omega \nabla\mu \cdot \nabla\Phi + (\varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}) - (\Gamma_{\varphi,1} - \Gamma_{\varphi,2}))\Phi \,\mathrm{d}x \\ &\quad + \int_\Omega (\nabla\varphi_1 \cdot \mathbf{v} + \nabla\varphi \cdot \mathbf{v}_2)\Phi + \varphi\Gamma_{\mathbf{v},1}\Phi \,\mathrm{d}x, \end{aligned} \tag{5.37b}$$

$$\begin{aligned} 0 &= \int_\Omega \nabla\sigma \cdot \nabla\Phi \,\mathrm{d}x + \int_\Omega (h(\varphi_1)\sigma + \sigma_2(h(\varphi_1) - h(\varphi_2))\Phi \,\mathrm{d}x \\ &\quad + \int_{\partial\Omega} K(\sigma - \sigma_\infty)\Phi \,\mathrm{d}\mathcal{H}^{d-1} \end{aligned} \tag{5.37c}$$

for a.e. $t \in (0, T)$ and for all $\mathbf{\Phi} \in \mathbf{H}^1$, $\Phi \in H^1$, where μ is given by

$$\mu = \psi'(\varphi_1) - \psi'(\varphi_2) - \Delta\varphi - \chi\sigma \quad \text{a.e. in } \Omega_T. \tag{5.37d}$$

In the following we will frequently use the Sobolev embeddings $H^1 \subset L^6$ and $\mathbf{H}^1 \subset \mathbf{L}^6$. We divide the analysis into several steps.

Step 1: Taking $\Phi = \sigma$ in (5.37c), using the non-negativity of $h(\cdot)$ and applying Hölder's and Young's inequalities, we obtain

$$\int_\Omega |\nabla\sigma|^2 \,\mathrm{d}x + K \int_{\partial\Omega} |\sigma|^2 \,\mathrm{d}\mathcal{H}^{d-1} \leq \frac{L_h^2}{4\delta} \|\sigma_2\|_{L^3}^2 \|\varphi\|_{L^2}^2 + \frac{K}{2}\left(\|\sigma\|_{L^2(\partial\Omega)}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right) + \delta\|\sigma\|_{L^6}^2$$

with $\delta > 0$ to be chosen and where we used (B2). Choosing $\delta > 0$ small enough and using Poincaré's inequality, this implies

$$\|\sigma\|_{H^1} \leq C(K, L_h, \Omega)\left(\|\sigma_2\|_{L^6}\|\varphi\|_{L^2} + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right). \tag{5.38}$$

Step 2: By Lemma 2.39 there exists a solution $\mathbf{u} \in \mathbf{H}^1$ of the problem

$$\mathrm{div}(\mathbf{u}) = \Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2} \quad \text{a.e. in } \Omega, \quad \mathbf{u} = \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2} \,\mathrm{d}x\right)\mathbf{n} \quad \text{a.e. on } \partial\Omega$$

satisfying the estimate

$$\|\mathbf{u}\|_{\mathbf{H}^1} \leq c\|\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}\|_{L^2} \tag{5.39}$$

with a constant c depending only on Ω. Choosing $\mathbf{\Phi} = \mathbf{v} - \mathbf{u}$ in (5.37a) and $\Phi = \varphi - \Delta\varphi$ in (5.37b), integrating by parts, using (5.37d) and summing the resulting identities, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}(\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\varphi\|_{L^2}^2) + \int_\Omega 2\eta(\varphi_1)|\mathbf{D}\mathbf{v}|^2 + \nu(\varphi_1)|\mathbf{v}|^2 + |\Delta\varphi|^2 + |\nabla\Delta\varphi|^2 \,\mathrm{d}x \\
&\quad = \int_\Omega \nabla(\psi'(\varphi_1) - \psi'(\varphi_2)) \cdot (\nabla\Delta\varphi - \nabla\varphi) - ((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}))\Delta\varphi \,\mathrm{d}x \\
&\qquad + \int_\Omega (\nabla\varphi \cdot \mathbf{v}_2 + \varphi\Gamma_{\mathbf{v},1})\Delta\varphi + (\psi'(\varphi_1) - \psi'(\varphi_2))\nabla\varphi_1 \cdot (\mathbf{v} - \mathbf{u}) \,\mathrm{d}x, \\
&\qquad + \int_\Omega (\mu_2 + \chi\sigma_2)\nabla\varphi \cdot (\mathbf{v} - \mathbf{u}) + \Delta\varphi\nabla\varphi_1 \cdot \mathbf{u} \,\mathrm{d}x + \int_\Omega \chi\nabla\sigma \cdot (\nabla\varphi - \nabla\Delta\varphi) \,\mathrm{d}x \\
&\qquad + \int_\Omega ((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}))\varphi - (\nabla\varphi_1 \cdot \mathbf{v} + \nabla\varphi \cdot \mathbf{v}_2)\varphi - \Gamma_{\mathbf{v},1}|\varphi|^2 \,\mathrm{d}x \\
&\qquad + \int_\Omega 2\eta(\varphi_1)\mathbf{D}\mathbf{v} \colon \nabla\mathbf{u} + \nu(\varphi_1)\mathbf{v} \cdot \mathbf{u} - 2(\eta(\varphi_1) - \eta(\varphi_2))\mathbf{D}\mathbf{v}_2 \colon \nabla(\mathbf{v} - \mathbf{u}) \,\mathrm{d}x \\
&\qquad - \int_\Omega (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2 \cdot (\mathbf{v} - \mathbf{u}) \,\mathrm{d}x.
\end{aligned} \tag{5.40}$$

Step 3: We now estimate the terms on the r. h. s. of (5.40) individually and we frequently use Young's, Hölder's and Gagliardo–Nirenberg's inequalities. By (A4) and (B2) it holds

$$|\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}| \le C(|\sigma| + |\sigma_2||\varphi| + |\varphi|), \quad |\Gamma_{\varphi,1} - \Gamma_{\varphi,2}| \le C(|\sigma| + |\sigma_2||\varphi| + |\varphi|).$$

Hence, applying (5.38) leads to

$$\|\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}\|_{L^2} + \|\Gamma_{\varphi,1} - \Gamma_{\varphi,2}\|_{L^2} \le C\left(1 + \|\sigma_2\|_{L^6}\right)\|\varphi\|_{H^1} + C\|\sigma_\infty\|_{L^2(\partial\Omega)}. \tag{5.41}$$

Invoking (5.6) and (5.41) we infer that

$$\begin{aligned}
&\left|\int_\Omega ((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}))\Delta\varphi \,\mathrm{d}x\right| \\
&\quad \le C\big(1 + \|\varphi_2\|_{L^\infty}^2\big)\big((1 + \|\sigma_2\|_{L^6}^2)\|\varphi\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\big) + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2 \\
&\quad \le C\left(1 + \|\varphi_2\|_{H^2} + \|\sigma_2\|_{H^1}^4\right)\|\varphi\|_{H^1}^2 + C\big(1 + \|\varphi_2\|_{H^2}\big)\|\sigma_\infty\|_{L^2(\partial\Omega)}^2 + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2.
\end{aligned} \tag{5.42}$$

By the specific form of $\Gamma_{\mathbf{v}}$, applying (A4) gives

$$\left|\int_\Omega \varphi\Gamma_{\mathbf{v},1}\Delta\varphi \,\mathrm{d}x\right| \le C\left(1 + \|\sigma_1\|_{L^3}^2\right)\|\varphi\|_{H^1}^2 + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2. \tag{5.43}$$

Using (2.18) and the estimate $\|\Delta f\|_{L^2} \le \|\nabla f\|_{\mathbf{L}^2}^{\frac{1}{2}}\|\nabla\Delta f\|_{\mathbf{L}^2}^{\frac{1}{2}}$ holding for all $f \in H_N^2 \cap H^3$, we calculate

$$\begin{aligned}
\left|\int_\Omega \nabla\varphi \cdot \mathbf{v}_2\Delta\varphi \,\mathrm{d}x\right| &\le \|\nabla\varphi\|_{\mathbf{L}^3}\|\mathbf{v}_2\|_{\mathbf{L}^6}\|\Delta\varphi\|_{L^2} \\
&\le C\|\mathbf{v}_2\|_{\mathbf{H}^1}\|\nabla\varphi\|_{\mathbf{L}^2}\|\varphi\|_{H^2}^{\frac{1}{2}}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^{\frac{1}{2}} \\
&\le C\|\mathbf{v}_2\|_{\mathbf{H}^1}\|\varphi\|_{H^1}\left(\|\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2} + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}\right) \\
&\le C\left(1 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2 + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{5.44}$$

Due to (B3), (2.4) and (5.6), we obtain

$$\|\psi'(\varphi_1) - \psi'(\varphi_2)\|_{L^2}^2 \le C\left(1 + \|\varphi_1\|_{H^2}^2 + \|\varphi_2\|_{H^2}^2\right)\|\varphi\|_{H^1}^2. \tag{5.45}$$

Using the elliptic estimate $\|f\|_{H^2} \le C\|f\|_{H^1}^{\frac12}\|f\|_{H^3}^{\frac12}$ holding for all $f \in H^2_N \cap H^3$, by (5.6) we get

$$\begin{aligned}\left|\int_\Omega (\psi'(\varphi_2)-\psi'(\varphi_1))\nabla\varphi_1\cdot\mathbf{v}\,\mathrm{d}x\right| &\le C\|\psi'(\varphi_1)-\psi'(\varphi_2)\|_{L^2}^2\|\nabla\varphi_1\|_{\mathbf{L}^3}^2+\delta_1\|\mathbf{v}\|_{\mathbf{H}^1}^2\\ &\le C\left(1+\|\varphi_1\|_{H^3}^2+\|\varphi_2\|_{H^3}^2\right)\|\varphi\|_{H^1}^2+\delta_1\|\mathbf{v}\|_{\mathbf{H}^1}^2 \qquad (5.46)\end{aligned}$$

with $\delta_1 > 0$ to be chosen later. Recalling (5.39) and (5.41), with similar arguments we deduce that

$$\begin{aligned}&\left|\int_\Omega (\psi'(\varphi_2)-\psi'(\varphi_1))\nabla\varphi_1\cdot\mathbf{u}\,\mathrm{d}x\right|\\ &\quad\le C\|\psi'(\varphi_1)-\psi'(\varphi_2)\|_{L^2}\|\nabla\varphi_1\|_{\mathbf{L}^3}\|\mathbf{u}\|_{\mathbf{L}^6}\\ &\quad\le C\left(1+\|\varphi_1\|_{L^{12}}^4+\|\varphi_2\|_{L^{12}}^4\right)\|\varphi\|_{H^1}\|\nabla\varphi_1\|_{\mathbf{L}^3}\|\mathbf{u}\|_{\mathbf{H}^1}\\ &\quad\le C\left(1+\|\varphi_1\|_{H^3}^2+\|\varphi_2\|_{H^3}^2+\|\sigma_2\|_{L^6}^4\right)\|\varphi\|_{H^2}^2+C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2. \qquad (5.47)\end{aligned}$$

Applying (5.39) and (5.41) gives

$$\begin{aligned}\left|\int_\Omega (\mu_2+\chi\sigma_2)\nabla\varphi\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\right| &\le C\left(1+\|\mu_2\|_{H^1}^2+\|\sigma_2\|_{H^1}^2\right)\|\varphi\|_{H^1}^2\\ &\quad + C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2+\delta_2\|\mathbf{v}\|_{\mathbf{H}^1}^2 \qquad (5.48)\end{aligned}$$

with $\delta_2 > 0$ to be chosen later. By (5.6), (5.39) and (5.41), it holds that

$$\begin{aligned}\left|\int_\Omega \Delta\varphi\nabla\varphi_1\cdot\mathbf{u}\,\mathrm{d}x\right| &\le \|\Delta\varphi\|_{L^2}\|\nabla\varphi_1\|_{\mathbf{L}^3}\|\mathbf{u}\|_{\mathbf{L}^6}\\ &\le C\|\nabla\varphi_1\|_{\mathbf{L}^3}^2\|\mathbf{u}\|_{\mathbf{H}^1}^2+\frac18\|\Delta\varphi\|_{L^2}^2\\ &\le C\left(1+\|\varphi_1\|_{H^2}^2+\|\sigma_2\|_{H^1}^4\right)\|\varphi\|_{H^1}^2\\ &\quad + C\|\varphi_1\|_{H^2}\|\sigma_\infty\|_{L^2(\partial\Omega)}^2+\frac18\|\Delta\varphi\|_{L^2}^2. \qquad (5.49)\end{aligned}$$

Moreover, the assumptions on $\eta(\cdot)$ and $\nu(\cdot)$ guarantee that

$$\begin{aligned}\left|\int_\Omega 2\eta(\varphi_1)\mathbf{D}\mathbf{v}\colon\nabla\mathbf{u}+\nu(\varphi_1)\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x\right| &\le \frac{\eta_0}{2}\|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2}^2+\frac{\nu_0}{4}\|\mathbf{v}\|_{\mathbf{L}^2}^2+C\|\mathbf{u}\|_{\mathbf{H}^1}^2\\ &\le \frac{\eta_0}{2}\|\mathbf{D}\mathbf{v}\|_{\mathbf{L}^2}^2+\frac{\nu_0}{4}\|\mathbf{v}\|_{\mathbf{L}^2}^2+C\left(1+\|\sigma_2\|_{H^1}^2\right)\|\varphi\|_{H^1}^2\\ &\quad + C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2. \qquad (5.50)\end{aligned}$$

Furthermore, using the boundedness of $\eta'(\cdot)$, elliptic regularity and (2.4), (5.39), (5.41), we have

$$\begin{aligned}&\left|\int_\Omega 2(\eta(\varphi_1)-\eta(\varphi_2)\mathbf{D}\mathbf{v}_2\colon\nabla(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\right|\\ &\quad\le C\|\varphi\|_{L^\infty}\|\mathbf{D}\mathbf{v}_2\|_{\mathbf{L}^2}\left(\|\nabla\mathbf{v}\|_{\mathbf{L}^2}+\|\nabla\mathbf{u}\|_{\mathbf{L}^2}\right)\\ &\quad\le C\|\varphi\|_{H^1}^{\frac34}\|\varphi\|_{H^3}^{\frac14}\|\mathbf{D}\mathbf{v}_2\|_{\mathbf{L}^2}\left(\|\nabla\mathbf{v}\|_{\mathbf{L}^2}+\|\nabla\mathbf{u}\|_{\mathbf{L}^2}\right)\\ &\quad\le \delta_3\|\nabla\mathbf{v}\|_{\mathbf{L}^2}^2+C\|\nabla\mathbf{u}\|_{\mathbf{L}^2}^2+C\|\varphi\|_{H^1}^{\frac32}\|\varphi\|_{H^3}^{\frac12}\|\mathbf{D}\mathbf{v}_2\|_{\mathbf{L}^2}^2\\ &\quad\le \delta_3\|\nabla\mathbf{v}\|_{\mathbf{L}^2}^2+C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2+\frac14\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2+C\left(1+\|\mathbf{D}\mathbf{v}_2\|_{\mathbf{L}^2}^{\frac83}+\|\sigma_2\|_{H^1}^2\right)\|\varphi\|_{H^1}^2 \qquad (5.51)\end{aligned}$$

with $\delta_3 > 0$ to be chosen later. Similar arguments lead to

$$\begin{aligned}\left|\int_\Omega (\nu(\varphi_1)-\nu(\varphi_2))\mathbf{v}_2\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\right| &\le \delta_4\|\mathbf{v}\|_{\mathbf{L}^2}^2+C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2\\ &\quad + C\left(1+\|\mathbf{v}_2\|_{\mathbf{H}^1}^2+\|\sigma_2\|_{H^1}^2\right)\|\varphi\|_{H^1}^2 \qquad (5.52)\end{aligned}$$

with $\delta_4 > 0$ to be chosen. Due to (5.41) and since $\varphi_2 \in L^\infty(H^1)$ with bounded norm, we get

$$\left| \int_\Omega (\Gamma_{\varphi,1} - \Gamma_{\varphi,2} - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}))\, \varphi \, \mathrm{d}x \right| \leq C\,(1 + \|\sigma_2\|_{L^6})\, \|\varphi\|_{H^1}^2 + C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2, \quad (5.53)$$

and, applying (5.6) gives

$$\begin{aligned}\left| \int_\Omega (\nabla\varphi_1 \cdot \mathbf{v} + \nabla\varphi \cdot \mathbf{v}_2)\varphi \, \mathrm{d}x \right| &\leq \|\nabla\varphi_1\|_{\mathbf{L}^3}\|\mathbf{v}\|_{\mathbf{L}^6}\|\varphi\|_{L^2} + \|\nabla\varphi\|_{\mathbf{L}^2}\|\mathbf{v}_2\|_{\mathbf{L}^6}\|\varphi\|_{L^3} \\ &\leq C\left(\|\nabla\varphi_1\|_{\mathbf{L}^3}\|\mathbf{v}\|_{\mathbf{H}^1}\|\varphi\|_{H^1} + \|\varphi\|_{H^1}^2\|\mathbf{v}_2\|_{\mathbf{H}^1}\right) \\ &\leq C\,(\|\varphi_1\|_{H^2} + \|\mathbf{v}_2\|_{\mathbf{H}^1})\,\|\varphi\|_{H^1}^2 + \delta_5\|\mathbf{v}\|_{\mathbf{H}^1}^2 \quad (5.54)\end{aligned}$$

with $\delta_5 > 0$ to be chosen later. For the last term on the r. h. s. of (5.40), we employ (A4) to derive the bound

$$\left| \int_\Omega \Gamma_{\mathbf{v},1}|\varphi|^2 \, \mathrm{d}x \right| \leq C\,(1 + \|\sigma_2\|_{L^6})\,\|\varphi\|_{H^1}^2. \quad (5.55)$$

We now estimate the first term on the r. h. s. of (5.40). First, we observe that

$$\nabla(\psi'(\varphi_1) - \psi'(\varphi_2)) = \psi''(\varphi_1)\nabla\varphi_1 - \psi''(\varphi_2)\nabla\varphi_2 = \psi''(\varphi_1)\nabla\varphi + \nabla\varphi_2(\psi''(\varphi_1) - \psi''(\varphi_2)).$$

Due to (A5), (i) and (5.6) along with the estimate $\|\varphi_1\|_{L^\infty}^8 \leq C\|\varphi_1\|_{H^1}^6\|\varphi_1\|_{H^3}^2$, we obtain

$$\int_\Omega |\psi''(\varphi_1)\nabla\varphi|^2 \, \mathrm{d}x \leq C\int_\Omega \left(1 + |\varphi_1|^8\right)|\nabla\varphi|^2 \, \mathrm{d}x \leq C\left(1 + \|\varphi_1\|_{H^3}^2\right)\|\varphi\|_{H^1}^2.$$

Applying (5.6), (B3) and the elliptic estimate $\|f\|_{H^2} \leq C\|f\|_{H^1}^{\frac{1}{2}}\|f\|_{H^3}^{\frac{1}{2}}$ holding for all $f \in H_N^2 \cap H^3$, we conclude that

$$\begin{aligned}\int_\Omega |\nabla\varphi_2(\psi''(\varphi_1) - \psi''(\varphi_2))|^2 \, \mathrm{d}x &\leq C\int_\Omega (1 + |\varphi_1|^6 + |\varphi_2|^6)|\nabla\varphi_2|^2|\varphi|^2 \, \mathrm{d}x \\ &\leq C\left(1 + \|\varphi_1\|_{L^{18}}^6 + \|\varphi_2\|_{L^{18}}^6\right)\|\nabla\varphi_2\|_{\mathbf{L}^6}^2\|\varphi\|_{L^6}^2 \\ &\leq C\left(1 + \|\varphi_1\|_{H^3}^2 + \|\varphi_2\|_{H^3}^2\right)\|\varphi\|_{H^1}^2.\end{aligned}$$

The last two inequalities imply

$$\|\nabla(\psi'(\varphi_1) - \psi'(\varphi_2))\|_{\mathbf{L}^2}^2 \leq C\left(1 + \|\varphi_1\|_{H^3}^2 + \|\varphi_2\|_{H^3}^2\right)\|\varphi\|_{H^1}^2. \quad (5.56)$$

From this, we infer that

$$\begin{aligned}\left| \int_\Omega \nabla(\psi'(\varphi_1) - \psi'(\varphi_2)) \cdot (\nabla\Delta\varphi - \nabla\varphi) \, \mathrm{d}x \right| &\leq C\left(1 + \|\varphi_1\|_{H^3}^2 + \|\varphi_2\|_{H^3}^2\right)\|\varphi\|_{H^1}^2 \\ &\quad + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2. \quad (5.57)\end{aligned}$$

Finally, by (5.38) it follows

$$\left| \int_\Omega \chi\nabla\sigma \cdot (\nabla\varphi - \nabla\Delta\varphi) \, \mathrm{d}x \right| \leq C\left(1 + \|\sigma_2\|_{L^6}^2\right)\|\varphi\|_{H^1}^2 + C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2 + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2. \quad (5.58)$$

Using (5.42)-(5.58) in (5.40) and choosing

$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \frac{\min\{\frac{\nu_0}{4}, \frac{\eta_0}{2}\}}{10C_K^2} =: C_1,$$

where C_K is the constant in Korn's inequality, we end up with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\varphi\|_{L^2}^2\right) + C_1\|\mathbf{v}\|_{\mathbf{H}^1}^2 + \frac{1}{2}\int_\Omega |\Delta\varphi|^2 + |\nabla\Delta\varphi|^2 \, \mathrm{d}x \\ &\quad \leq \alpha_1(t)\|\varphi\|_{H^1}^2 + \alpha_2(t)\|\sigma_\infty\|_{L^2(\partial\Omega)}^2,\end{aligned}$$

where

$$\alpha_1(t) \coloneqq C\left(1+\|\varphi_1\|_{H^3}^2+\|\varphi_2\|_{H^3}^2+\|\mathbf{v}_2\|_{\mathbf{H}^1}^{\frac{8}{3}}+\|\mu_2\|_{H^1}^2+\|\sigma_1\|_{H^1}^2+\|\sigma_2\|_{H^1}^4\right),$$
$$\alpha_2(t) \coloneqq C\left(1+\|\varphi_1\|_{H^2}+\|\varphi_2\|_{H^2}\right).$$

Due to (5.6)-(5.7) it follows that $\alpha_1 \in L^1(0,T)$ and $\alpha_2 \in L^4(0,T)$, where we employed that $\sigma_2 \in L^4(H^1)$ with bounded norm. Then, using a Gronwall argument (see Lemma 2.31) in the last inequality yields

$$\sup_{s\in(0,T]}\|\varphi(s)\|_{H^1}^2+\int_0^T\|\mathbf{v}\|_{\mathbf{H}^1}^2\,\mathrm{d}s+\int_0^T\int_\Omega|\Delta\varphi|^2+|\nabla\Delta\varphi|^2\,\mathrm{d}x\mathrm{d}s \le C\left(\|\varphi_0\|_{H^1}^2+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}^2\right).$$

Together with elliptic regularity theory, this leads to

$$\|\varphi\|_{L^\infty(H^1)\cap L^2(H^3)}+\|\mathbf{v}\|_{L^2(\mathbf{H}^1)} \le C\left(\|\varphi_0\|_{H^1}+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right), \tag{5.59}$$

and from (5.38) and (5.59) we immediately obtain

$$\|\sigma\|_{L^2(H^1)} \le C\left(\|\varphi_0\|_{H^1}+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right). \tag{5.60}$$

Using (2.4), (5.45), (5.56), (5.59) and the boundedness of $\varphi_1,\varphi_2 \in L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)$, it is straightforward to check that

$$\|\psi'(\varphi_1)-\psi'(\varphi_2)\|_{L^2(H^1)} \le C\left(\|\varphi_0\|_{H^1}+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right).$$

Invoking (5.59)-(5.60) and using the relation (5.37d) for μ yields

$$\|\mu\|_{L^2(H^1)} \le C\left(\|\varphi_0\|_{H^1}+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right),$$

and in conjunction with (5.59)-(5.60) this gives

$$\|\varphi\|_{H^1((H^1)^*)} \le C\left(\|\varphi_0\|_{H^1}+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right),$$

where we used (5.37b) for $\partial_t\varphi$. The last two estimates together with (5.59)-(5.60) entail that

$$\begin{aligned}&\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)}+\|\mu\|_{L^2(H^1)}+\|\sigma\|_{L^2(H^1)}+\|\mathbf{v}\|_{L^2(\mathbf{H}^1)}\\&\quad\le C\left(\|\varphi_0\|_{H^1}+\|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right).\end{aligned} \tag{5.61}$$

Step 4: It remains to control the pressure. Let $\mathbf{q} \in \mathbf{H}^1$ be a solution of

$$\operatorname{div}(\mathbf{q}) = p \quad \text{in } \Omega, \quad \mathbf{q} = \frac{1}{|\partial\Omega|}\left(\int_\Omega p\,\mathrm{d}x\right)\mathbf{n} \quad \text{on } \partial\Omega$$

such that

$$\|\mathbf{q}\|_{\mathbf{H}^1} \le c\|p\|_{L^2} \tag{5.62}$$

with c depending only on Ω. Then, choosing $\mathbf{\Phi} = \mathbf{q}$ in (5.37a) we obtain

$$\begin{aligned}\|p\|_{L^2}^2 &= \int_\Omega(2\eta(\varphi_1)\mathbf{D}\mathbf{v}+\lambda(\varphi_1)\operatorname{div}(\mathbf{v})\mathbf{I})\colon\nabla\mathbf{q}+\nu(\varphi_1)\mathbf{v}\cdot\mathbf{q}\,\mathrm{d}x\\&\quad-\int_\Omega(\mu+\chi\sigma)\nabla\varphi_1\cdot\mathbf{q}+(\mu_2+\chi\sigma_2)\nabla\varphi\cdot\mathbf{q}-(\nu(\varphi_1)-\nu(\varphi_2))\mathbf{v}_2\cdot\mathbf{q}\,\mathrm{d}x\\&\quad+\int_\Omega(2(\eta(\varphi_1)-\eta(\varphi_2))\mathbf{D}\mathbf{v}_2+(\lambda(\varphi_1)-\lambda(\varphi_2))\operatorname{div}(\mathbf{v}_2)\mathbf{I})\colon\nabla\mathbf{q}\,\mathrm{d}x.\end{aligned} \tag{5.63}$$

Using (5.6)-(5.7) and (A3), a straightforward calculation shows that

$$\begin{aligned}&\left|\int_\Omega (2\eta(\varphi_1)\mathbf{D}\mathbf{v} + \lambda(\varphi_1)\mathrm{div}(\mathbf{v})\mathbf{I})\colon \nabla\mathbf{q} + \big(\nu(\varphi_1)\mathbf{v} - (\mu+\chi\sigma)\nabla\varphi_1 - (\mu_2+\chi\sigma_2)\nabla\varphi\big)\cdot\mathbf{q}\,\mathrm{d}x\right|\\ &\quad\leq C\left(\|\mathbf{v}\|_{\mathbf{H}^1}^2 + \|\mu+\chi\sigma\|_{H^1}^2 + \|\mu_2+\chi\sigma_2\|_{H^1}^2\|\varphi\|_{H^1}^2\right) + \frac{1}{4}\|p\|_{L^2}^2.\end{aligned}$$

For the remaining terms, we apply (5.6)-(5.7) and (A3) to obtain

$$\begin{aligned}&\left|\int_\Omega (2(\eta(\varphi_1)-\eta(\varphi_2))\mathbf{D}\mathbf{v}_2 + (\lambda(\varphi_1)-\lambda(\varphi_2))\mathrm{div}(\mathbf{v}_2)\mathbf{I})\colon\nabla\mathbf{q} + (\nu(\varphi_1)-\nu(\varphi_2))\mathbf{v}_2\cdot\mathbf{q}\,\mathrm{d}x\right|\\ &\quad\leq C\|\mathbf{v}_2\|_{\mathbf{H}^1}^2\|\varphi\|_{L^\infty}^2 + \frac{1}{4}\|p\|_{L^2}^2.\end{aligned}$$

Invoking the last two inequalities in (5.63), integrating the resulting estimate in time from 0 to T and using Young's generalised inequality, we deduce that

$$\begin{aligned}\|p\|_{L^2(L^2)}^2 &\leq C\Big(\|\mathbf{v}\|_{L^2(\mathbf{H}^1)}^2 + \|\mu+\chi\sigma\|_{L^2(H^1)}^2 + \|\mu_2+\chi\sigma_2\|_{L^2(H^1)}^2\|\varphi\|_{L^\infty(H^1)}^2\Big)\\ &\quad + C\|\mathbf{v}_2\|_{L^{\frac{8}{3}}(\mathbf{H}^1)}^2\|\varphi\|_{L^8(L^\infty)}^2.\end{aligned}$$

Therefore, invoking the continuous embedding $L^\infty(H^1)\cap L^2(H^3)\hookrightarrow L^8(L^\infty)$ along with (5.6)-(5.7) and (5.61), the last inequality implies

$$\|p\|_{L^2(L^2)} \leq C\left(\|\varphi_0\|_{H^1} + \|\sigma_\infty\|_{L^4(L^2(\partial\Omega))}\right). \tag{5.64}$$

In conjunction with (5.61) this leads to (5.8), hence the proof is complete.

5.3 Existence of strong solutions (Proof of Theorem 5.11)

We will now prove Theorem 5.11. The testing procedure can again be justified by a Galerkin scheme. In the following we assume for simplicity and as it has no further consequence for the analysis that $\varepsilon = 1$. Then, with similar arguments as before, we obtain

$$\begin{aligned}&\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)} + \|\sigma\|_{L^4(H^1)} + \|\mu\|_{L^2(H^1)\cap L^4(L^2)}\\ &\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \leq C.\end{aligned} \tag{5.65}$$

The result will now be established in a series of higher order estimates.

Step 1: Observing that (5.5d) is for a. e. $t\in(0,T)$ the weak formulation of

$$\begin{aligned}-\Delta\sigma + h(\varphi)\sigma &= 0 && \text{a. e. in } \Omega,\\ \nabla\sigma\cdot\mathbf{n} + K\sigma &= K\sigma_\infty && \text{a. e. on } \partial\Omega,\end{aligned}$$

by the assumptions on $h(\cdot)$ and by Lemma 2.32 we deduce that

$$\|\sigma\|_{H^2} \leq C\|K\sigma_\infty\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Therefore, invoking the Sobolev embedding $H^2\subset L^\infty$ and the fact that $\sigma_\infty\in H^1(H^{\frac{1}{2}}(\partial\Omega))\hookrightarrow C^0(H^{\frac{1}{2}}(\partial\Omega))$, we have

$$\|\sigma\|_{L^\infty(H^2)\cap L^\infty(\Omega_T)} \leq C. \tag{5.66}$$

By (A3) this yields

$$\|\mathrm{div}(\mathbf{v})\|_{L^\infty(L^\infty)} + \|\Gamma_\varphi\|_{L^\infty(L^\infty)} \leq C. \tag{5.67}$$

Now, for $h > 0$ we introduce the incremental ratio

$$\partial_t^h u(t) = \frac{1}{h}(u(t+h) - u(t)).$$

Then, using (5.5d) we see that

$$\begin{aligned} 0 = &\int_\Omega \nabla \partial_t^h \sigma(t) \cdot \nabla \Phi + \left(\partial_t^h (h(\varphi(t)))\sigma(t+h) + \partial_t^h \sigma(t) h(\varphi(t))\right)\Phi \, \mathrm{d}x \\ &+ \int_{\partial\Omega} K(\partial_t^h \sigma(t) - \partial_t^h \sigma_\infty(t))\Phi \, \mathrm{d}\mathcal{H}^{d-1} \end{aligned}$$

holding for almost every $t \in (0, T-h]$. Choosing $\Phi = \partial_t^h \sigma(t)$, integrating in time from 0 to $T - h$ and using the non-negativity of $h(\cdot)$, we conclude that

$$\begin{aligned} &\int_0^{T-h} \|\nabla \partial_t^h \sigma(t)\|_{\mathbf{L}^2}^2 \, \mathrm{d}t + K \int_0^{T-h} \|\partial_t^h \sigma(t)\|_{L^2(\partial\Omega)}^2 \, \mathrm{d}t \\ &\quad \le \int_0^{T-h} \left(\int_\Omega \partial_t^h (h(\varphi(t)))\sigma(t+h)\partial_t^h \sigma(t) \, \mathrm{d}x + \int_{\partial\Omega} K \partial_t^h \sigma(t) \partial_t^h \sigma_\infty(t) \, \mathrm{d}\mathcal{H}^{d-1} \right) \mathrm{d}t. \end{aligned}$$

To estimate the r. h. s. of this equation, we use (5.66) along with the Lipschitz-continuity of $h(\cdot)$ to get

$$\begin{aligned} \left| \int_0^{T-h} \int_\Omega \partial_t^h (h(\varphi(t)))\sigma(t+h)\partial_t^h \sigma(t) \, \mathrm{d}x \, \mathrm{d}t \right| &\le C \int_0^{T-h} \int_\Omega |\partial_t^h \varphi(t) \partial_t^h \sigma(t)| \, \mathrm{d}x \, \mathrm{d}t \\ &\le C \|\partial_t^h \varphi\|_{L^2(0,T-h;(H^1)^*)} \|\partial_t^h \sigma\|_{L^2(0,T-h;H^1)} \\ &\le C \|\partial_t \varphi\|_{L^2(0,T;(H^1)^*)} \|\partial_t^h \sigma\|_{L^2(0,T-h;H^1)}. \end{aligned}$$

With similar arguments and using the trace theorem, the remaining term can be controlled by

$$\left| \int_0^{T-h} \int_{\partial\Omega} K \partial_t^h \sigma(t) \partial_t^h \sigma_\infty(t) \, \mathrm{d}\mathcal{H}^{d-1} \, \mathrm{d}t \right| \le C \|\partial_t \sigma_\infty\|_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} \|\partial_t^h \sigma\|_{L^2(0,T-h;H^1)}.$$

Invoking the last three inequalities together with (5.65) and (C2), an application of Poincaré's inequality leads to

$$\|\partial_t^h \sigma\|_{L^2(0,T-h;H^1)} \le C \left(\|\partial_t \varphi\|_{L^2(0,T;(H^1)^*)} + \|\partial_t \sigma_\infty\|_{L^2(0,T;H^{\frac{1}{2}})} \right) \le C.$$

Since the constant C is independent of $h > 0$, we infer

$$\|\partial_t \sigma\|_{L^2(H^1)} \le C.$$

Together with (5.66) and the continuous embedding $H^1(H^1) \hookrightarrow C^0(H^1)$, this entails that

$$\|\sigma\|_{H^1(H^1) \cap C^0(H^1) \cap L^\infty(H^2)} \le C. \tag{5.68}$$

Step 2: Choosing $\Phi = \partial_t \varphi$ in (5.5b) and $\Phi = \Delta \partial_t \varphi$ in (5.5c), integrating by parts and summing the resulting identities, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega |\Delta\varphi|^2 \, \mathrm{d}x + \int_\Omega |\partial_t \varphi|^2 = &- \int_\Omega (\mathrm{div}(\varphi \mathbf{v}) - \Gamma_\varphi) \, \partial_t \varphi \, \mathrm{d}x + \chi \int_\Omega \nabla\sigma \cdot \nabla \partial_t \varphi \, \mathrm{d}x \\ &+ \int_\Omega \psi'''(\varphi)|\nabla\varphi|^2 \, \partial_t \varphi + \psi''(\varphi)\Delta\varphi \, \partial_t \varphi \, \mathrm{d}x. \end{aligned} \tag{5.69}$$

We recall that $\Gamma_\varphi,\ \Gamma_{\mathbf{v}} \in L^2(L^2)$ with bounded norm. Then, using Hölder's and Young's inequalities, we can bound the first integral on the r. h. s. of (5.69) by

$$\left|\int_\Omega (\operatorname{div}(\varphi\mathbf{v}) - \Gamma_\varphi)\,\partial_t\varphi \,\mathrm{d}x\right| \le C\left(\|\operatorname{div}(\varphi\mathbf{v})\|_{L^2}^2 + \|\Gamma_\varphi\|_{L^2}^2\right) + \frac{1}{4}\|\partial_t\varphi\|_{L^2}^2.$$

For the last term on the r. h. s. of (5.69), we use Hölder's and Young's inequalities along with (2.4), (A5), (i) and (5.65) to obtain

$$\begin{aligned}\left|\int_\Omega \psi''(\varphi)\Delta\varphi\,\partial_t\varphi\,\mathrm{d}x\right| &\le C\left(1+\|\varphi\|_{L^\infty}^4\right)\|\Delta\varphi\|_{L^2}\|\partial_t\varphi\|_{L^2}\\ &\le \frac{1}{4}\|\partial_t\varphi\|_{L^2}^2 + C\left(1+\|\varphi\|_{H^3}^2\right)\|\Delta\varphi\|_{L^2}^2.\end{aligned}$$

Now, using (C3), Hölder's and Young's inequalities, (2.4), (2.18) and (5.65), we infer

$$\begin{aligned}\left|\int_\Omega \psi'''(\varphi)|\nabla\varphi|^2\,\partial_t\varphi\,\mathrm{d}x\right| &\le C\left(1+\|\varphi\|_{L^\infty}^3\right)\|\nabla\varphi\|_{\mathbf{L}^4}^2\|\partial_t\varphi\|_{L^2}\\ &\le C\left(1+\|\varphi\|_{H^3}^{\frac{3}{4}}\right)\|\varphi\|_{H^2}^{\frac{3}{2}}\|\partial_t\varphi\|_{L^2}\\ &\le C\left(1+\|\varphi\|_{H^3}\right)\left(\|\varphi\|_{L^2}+\|\Delta\varphi\|_{L^2}\right)\|\partial_t\varphi\|_{L^2}\\ &\le C\left(1+\|\varphi\|_{H^3}^2\right)\left(1+\|\Delta\varphi\|_{L^2}^2\right)+\frac{1}{4}\|\partial_t\varphi\|_{L^2}^2.\end{aligned}$$

The remaining term on the r. h. s. of (5.69) can be rewritten by

$$\chi\int_\Omega \nabla\sigma\cdot\nabla\partial_t\varphi\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\chi\int_\Omega \nabla\sigma\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega \nabla\partial_t\sigma\cdot\nabla\varphi\,\mathrm{d}x \quad \text{for a. e. } t\in(0,T).$$

Invoking the last four (in)equalities in (5.69) leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi|^2\,\mathrm{d}x + \frac{1}{4}\int_\Omega |\partial_t\varphi|^2\,\mathrm{d}x &\le C\left(1+\|\operatorname{div}(\varphi\mathbf{v})\|_{L^2}^2+\|\Gamma_\varphi\|_{L^2}^2+\|\varphi\|_{H^3}^2\right)\\ &\quad + C\left(1+\|\varphi\|_{H^3}^2\right)\|\Delta\varphi\|_{L^2}^2\\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}\chi\int_\Omega \nabla\sigma\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega \nabla\partial_t\sigma\cdot\nabla\varphi\,\mathrm{d}x.\end{aligned}$$

Integrating this inequality in time from 0 to $s\in(0,T]$ implies

$$\begin{aligned}\frac{1}{2}\|\Delta\varphi(s)\|_{L^2}^2 + \frac{1}{4}\|\partial_t\varphi\|_{L^2(0,s;L^2)}^2 &\le \|\Delta\varphi_0\|_{L^2}^2 + \int_0^s \alpha_1(t)+\alpha_2(t)\|\Delta\varphi(t)\|_{L^2}^2\,\mathrm{d}t\\ &\quad + \chi\int_\Omega \nabla\sigma(s)\cdot\nabla\varphi(s)\,\mathrm{d}x - \chi\int_\Omega \nabla\sigma(0)\cdot\nabla\varphi_0\,\mathrm{d}x\\ &\quad - \chi\int_0^s\int_\Omega \nabla\partial_t\sigma\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t, \qquad (5.70)\end{aligned}$$

where

$$\alpha_1(t) \coloneqq C\left(1+\|\operatorname{div}(\varphi\mathbf{v})\|_{L^2}^2+\|\Gamma_\varphi\|_{L^2}^2+\|\varphi\|_{H^3}^2\right), \qquad \alpha_2(t)\coloneqq C\left(1+\|\varphi\|_{H^3}^2\right).$$

Now, using (5.65), (5.68) and $\varphi_0\in H_N^2$ we obtain

$$\begin{aligned}\left|\chi\int_\Omega \nabla\sigma(0)\cdot\nabla\varphi_0\,\mathrm{d}x\right| = \left|\chi\int_\Omega \sigma(0)\Delta\varphi_0\,\mathrm{d}x\right| &\le C\left(1+\|\Delta\varphi_0\|_{L^2}^2\right),\\ \left|\chi\int_\Omega \nabla\sigma(s)\cdot\nabla\varphi(s)\,\mathrm{d}x\right| &\le \|\sigma\|_{C^0(H^1)}\sup_{s\in(0,T]}\|\nabla\varphi(s)\| \le C,\\ \left|\chi\int_0^s\int_\Omega \nabla\partial_t\sigma\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t\right| &\le C\int_0^s \|\partial_t\sigma(t)\|_{H^1}\,\mathrm{d}t.\end{aligned}$$

Together with (5.70), this implies

$$\frac{1}{2}\|\Delta\varphi(s)\|_{L^2}^2 + \frac{1}{4}\|\partial_t\varphi\|_{L^2(0,s;L^2)}^2 \le C\left(1+\|\Delta\varphi_0\|_{L^2}^2\right) + \int_0^s \beta_1(t) + \beta_2(t)\|\Delta\varphi(t)\|_{L^2}^2 \,\mathrm{d}t, \quad (5.71)$$

where

$$\beta_1(t) := C\left(1+\|\partial_t\sigma\|_{H^1} + \|\operatorname{div}(\varphi\mathbf{v})\|_{L^2}^2 + \|\Gamma_\varphi\|_{L^2}^2 + \|\varphi\|_{H^3}^2\right), \qquad \beta_2(t) := C\left(1+\|\varphi\|_{H^3}^2\right).$$

Due to (5.65), (5.67) and (5.68), it holds that $\beta_1, \beta_2 \in L^1(0,T)$. Together with the assumption $\varphi_0 \in H^2_N$, an application of Gronwall's lemma in (5.71) yields

$$\|\Delta\varphi\|_{L^\infty(L^2)} + \|\partial_t\varphi\|_{L^2(L^2)} \le C.$$

In combination with (5.65) and elliptic regularity theory, this entails that

$$\begin{aligned}&\|\varphi\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^3)} + \|\sigma\|_{H^1(H^1)\cap C^0(H^1)\cap L^\infty(H^2)} + \|\mu\|_{L^2(H^1)\cap L^4(L^2)} \\ &\quad + \|\operatorname{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \le C. \qquad (5.72)\end{aligned}$$

Step 3: Applying (5.72) along with elliptic regularity theory in (5.5b), and using the relation (5.5c) for μ in conjunction with (5.72), we obtain

$$\|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} \le C. \qquad (5.73)$$

Step 4: We now aim to apply Proposition 2.50. Employing (5.72) and the assumptions on $\Gamma_{\mathbf{v}}$, it is straightforward to check that

$$\|\Gamma_{\mathbf{v}}\|_{L^\infty(H^1)} \le C. \qquad (5.74)$$

Furthermore, since $\nabla\varphi \in L^4(\mathbf{L}^\infty)$, $\mu + \chi\sigma \in L^\infty(L^2)$ with bounded norm, it holds

$$\|(\mu+\chi\sigma)\nabla\varphi\|_{L^4(\mathbf{L}^2)} \le C. \qquad (5.75)$$

Hence, using the assumptions on $\eta(\cdot)$, $\lambda(\cdot)$ and $\nu(\cdot)$, an application of (2.65) yields

$$\|\mathbf{v}\|_{\mathbf{H}^2} + \|p\|_{H^1} \le C\big(\eta_0, \eta_1, \lambda_0, \|\varphi\|_{W^{1,4}}\big)\big(\|(\mu+\chi\sigma)\nabla\varphi\|_{\mathbf{L}^2} + \|\Gamma_{\mathbf{v}}\|_{H^1}\big).$$

Integrating this inequality in time from 0 to T, using (5.74)-(5.75) and recalling $\varphi \in L^\infty(W^{1,4})$ due to the Sobolev embedding $H^2 \subset W^{1,4}$, we conclude

$$\|\mathbf{v}\|_{L^4(\mathbf{H}^2)} + \|p\|_{L^4(H^1)} \le C. \qquad (5.76)$$

Step 5: Finally, due to the compact embedding $H^2 \hookrightarrow C^0(\overline{\Omega})$ and because of (5.72), we obtain

$$\|\varphi\|_{C^0(\overline{\Omega_T})} \le C.$$

Summarising the above estimates we get

$$\begin{aligned}&\|\varphi\|_{H^1(L^2)\cap C^0(\overline{\Omega_T})\cap L^\infty(H^2)\cap L^2(H^3)} + \|\sigma\|_{H^1(H^1)\cap C^0(H^1)\cap L^\infty(H^2)} + \|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} \\ &\quad + \|\operatorname{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathbf{v}\|_{L^4(\mathbf{H}^2)} + \|p\|_{L^4(H^1)} \le C. \qquad (5.77)\end{aligned}$$

These a priori estimates are enough to pass to the limit to show existence of strong solutions. We remark that $\varphi_0 \in H^2_N$ is needed since the projection onto the finite dimensional subspaces is continuous on H^2_N, but not on H^2. For the details we again refer to Chapter 4 and [81].

Step 6: Since (5.1d) holds a. e. in Ω_T, we see that φ is a solution of

$$\begin{aligned} \Delta\varphi &= \psi'(\varphi) - \mu - \chi\sigma && \text{a. e. in } \Omega_T, \\ \nabla\varphi \cdot \mathbf{n} &= 0 && \text{a. e. on } \Sigma_T. \end{aligned}$$

Due to the boundedness of $\psi'(\varphi) - \mu - \chi\sigma \in L^2(H^2)$, elliptic regularity theory entails

$$\|\varphi\|_{L^2(H^4)} \leq C.$$

Invoking the continuous embedding $L^\infty(H^1) \cap L^2(H^3) \hookrightarrow L^8(L^\infty)$ and (5.77), this implies $(\mu + \chi\sigma)\nabla\varphi \in L^8(\mathbf{L}^2)$ with bounded norm. Consequently, with the same arguments as used for (5.76) we deduce that

$$\|\mathbf{v}\|_{L^8(\mathbf{H}^2)} + \|p\|_{L^8(H^1)} \leq C$$

which completes the proof.

6

Asymptotic limits

In this chapter we aim to analyse several singular limits for the model considered in Chapter 5.

The first limit concerns the boundary condition (5.2b) which, for a positive permeability constant K, is given by

$$\nabla\sigma \cdot \mathbf{n} = K(\sigma_\infty - \sigma) \qquad \text{on } \Sigma_T. \tag{6.1}$$

Formally, the case $K = 0$ corresponds to a Neumann boundary condition, whereas letting $K \to \infty$, we expect that σ satisfies a Dirichlet boundary condition with datum σ_∞ (see Chapter 3). If σ satisfies a homogeneous Neumann boundary condition, equation (5.1e) gives a control for the gradient of σ but not for σ itself which already indicates that the limit $K \to 0$ cannot be established rigorously. To the contrary, by virtue of Poincaré's inequality, the Dirichlet boundary condition together with the gradient estimate deduced from (5.1e) allows us to prove the limit of large boundary permeability rigorously, see Proposition 6.1.

A second concern is to analyse the relation of (5.1b) with Stokes flow and Darcy's law. In the limit $\nu(\cdot) \to 0$ which corresponds to Stokes flow, we encounter that Korn's inequality does no longer hold and the operator on the left hand side of (5.1b) has a non-trivial kernel consisting of, e. g., rigid motions, hence we cannot establish the zero permeability limit rigorously.
Again, the situation is different if we consider the zero viscosity limit where we recover Darcy's law in the limit in 3D (see Theorem 6.3), although we loose regularity for the velocity field. In two space dimensions, the situation is even better due to improved Sobolev embeddings, and we can show that every strong solution of the Darcy model can be approximated by taking the zero viscosity limit in Brinkman's law (see Theorems 6.5 and 6.7). For the zero viscosity limit, we use similar ideas as presented in [21].
We will first analyse the limit $K \to \infty$ and then the zero viscosity limit.

6.1 The singular limit of large boundary permeability

We aim to prove the following result:

Proposition 6.1 *(The limit $K \to \infty$) Let the assumptions of Theorem 5.5 be fulfilled and assume in addition that $\sigma_\infty \in L^2(H^{\frac{1}{2}}(\partial\Omega))$. Let $K > 0$ and denote by $(\varphi_K, \mu_K, \sigma_K, \mathbf{v}_K, p_K)$ a* *weak solution of* (5.1)-(5.2) *corresponding to* φ_0 *and* K *in the sense of Definition 5.4. Then, as* $K \to \infty$*, we have (at least for a non-relabelled subsequence)*

$$
\begin{aligned}
\varphi_K &\to \varphi && \textit{weakly-star} && \textit{in } H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^3),\\
\sigma_K &\to \sigma && \textit{weakly} && \textit{in } L^2(H^1),\\
\mu_K &\to \mu && \textit{weakly} && \textit{in } L^2(H^1),\\
p_K &\to p && \textit{weakly} && \textit{in } L^2(L^2),\\
\mathbf{v}_K &\to \mathbf{v} && \textit{weakly} && \textit{in } L^2(\mathbf{H}^1),\\
\mathrm{div}(\varphi_K \mathbf{v}_K) &\to \mathrm{div}(\varphi \mathbf{v}) && \textit{weakly} && \textit{in } L^2(L^{\frac{3}{2}}),\\
\sigma_K &\to \sigma_\infty && \textit{strongly} && \textit{in } L^2(L^2(\partial\Omega)),
\end{aligned}
$$

where $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *satisfies*

$$
\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi, \sigma) \quad \textit{a. e. in } \Omega_T, \quad \varphi(0) = \varphi_0 \quad \textit{a. e. in } \Omega, \quad \sigma \in (\sigma_\infty + L^2(0,T;H_0^1)),
$$

and (5.5) *with* (5.5d) *replaced by*

$$
0 = \int_\Omega \nabla\sigma \cdot \nabla\xi + h(\varphi)\sigma\xi \,\mathrm{d}x \tag{6.2}
$$

for a. e. $t \in (0,T)$ *and for all* $\xi \in H_0^1$.

Proof. Due to Theorem 5.5, for every $K > 0$ there exists a solution quintuple $(\varphi_K, \mu_K, \sigma_K, \mathbf{v}_K, p_K)$ solving (5.1)-(5.2) in the sense of Definition 5.4 and enjoying the regularity properties stated in Theorem 5.5. In the following we assume without loss of generality that $K > 1$. Let $E\colon H^{\frac{1}{2}}(\partial\Omega) \to H^1$ be a bounded, linear extension operator satisfying $(Ef)|_{\partial\Omega} = f$ for all $f \in H^{\frac{1}{2}}(\partial\Omega)$ (see Lemma 2.33). Then, choosing $\Phi = \sigma_K - E\sigma_\infty$ in (5.5d) (in the following we omit the operator E), we obtain

$$
\int_\Omega |\nabla\sigma_K|^2 + h(\varphi)|\sigma_K|^2 \,\mathrm{d}x + K\int_{\partial\Omega} |\sigma_K - \sigma_\infty|^2 \,\mathrm{d}\mathcal{H}^{d-1} = \int_\Omega \nabla\sigma_K \cdot \nabla\sigma_\infty + h(\varphi)\sigma_K\sigma_\infty \,\mathrm{d}x. \tag{6.3}
$$

For the first term on the r. h. s. of this equation, we use Hölder's and Young's inequalities and the boundedness of the extension operator to obtain

$$
\left|\int_\Omega \nabla\sigma_K \cdot \nabla\sigma_\infty \,\mathrm{d}x\right| \le \frac{1}{4}\|\nabla\sigma_K\|_{\mathbf{L}^2}^2 + \|\nabla\sigma_\infty\|_{\mathbf{L}^2}^2 \le \frac{1}{4}\|\nabla\sigma_K\|_{\mathbf{L}^2}^2 + C\|\sigma_\infty\|_{H^{\frac{1}{2}}(\partial\Omega)}^2.
$$

With the same arguments and using the boundedness of $h(\cdot)$, we can estimate the second term on the r. h. s. of (6.3) by

$$
\left|\int_\Omega h(\varphi)\sigma_K\sigma_\infty \,\mathrm{d}x\right| \le \delta\|\sigma_K\|_{L^2}^2 + C_\delta\|\sigma_\infty\|_{H^{\frac{1}{2}}(\partial\Omega)}^2.
$$

Using the last two inequalities in (6.3) and neglecting the non-negative term $\int_\Omega h(\varphi)|\sigma_K|^2 \,\mathrm{d}x$ on the l. h. s. of (6.3), we obtain

$$
\frac{3}{4}\int_\Omega |\nabla\sigma_K|^2 + K\int_{\partial\Omega} |\sigma_K - \sigma_\infty|^2 \,\mathrm{d}\mathcal{H}^{d-1} \le \delta\|\sigma_K\|_{L^2}^2 + C_\delta\|\sigma_\infty\|_{H^{\frac{1}{2}}(\partial\Omega)}^2.
$$

From Poincaré's inequality and the boundedness of the operator E, it follows that

$$
\|\sigma_K\|_{L^2}^2 \le \tilde{C}\left(\|\nabla\sigma_K\|_{\mathbf{L}^2}^2 + \|\sigma_K - \sigma_\infty\|_{L^2(\partial\Omega)}^2 + \|\sigma_\infty\|_{H^{\frac{1}{2}}(\partial\Omega)}^2\right)
$$

for a positive constant $\tilde{C}$ independent of K. Employing the last two inequalities and choosing $\delta > 0$ small enough we obtain

$$\int_\Omega |\nabla \sigma_K|^2 \,\mathrm{d}x + K \int_{\partial\Omega} |\sigma_K - \sigma_\infty|^2 \,\mathrm{d}\mathcal{H}^{d-1} \le C \|\sigma_\infty\|^2_{H^{\frac{1}{2}}(\partial\Omega)}.$$

In conjunction with the estimate for $\|\sigma_K\|^2_{L^2}$, this implies

$$\|\sigma_K\|^2_{H^1} + K \int_{\partial\Omega} |\sigma_K - \sigma_\infty|^2 \,\mathrm{d}\mathcal{H}^{d-1} \le C \|\sigma_\infty\|^2_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Integrating this inequality in time from 0 to T and using $\sigma_\infty \in L^2(H^{\frac{1}{2}}(\partial\Omega))$, we conclude that

$$\|\sigma_K\|_{L^2(H^1)} + \sqrt{K} \|\sigma_K - \sigma_\infty\|_{L^2(L^2(\partial\Omega))} \le C,$$

where C is independent of K. Then, with exactly the same arguments as above it follows that

$$\begin{aligned} &\|\varphi_K\|_{H^1((H^1)^*) \cap L^\infty(H^1) \cap L^4(H^2) \cap L^2(H^3)} + \|\sigma_K\|_{L^2(H^1)} + \sqrt{K} \|\sigma_K - \sigma_\infty\|_{L^2(L^2(\partial\Omega))} \\ &\quad + \|\mu_K\|_{L^2(H^1)} + \|\mathrm{div}(\varphi_K \mathbf{v}_K)\|_{L^2(L^{\frac{3}{2}})} + \|\mathbf{v}_K\|_{L^2(\mathbf{H}^1)} + \|p_K\|_{L^2(L^2)} \le C. \end{aligned} \tag{6.4}$$

Using standard compactness arguments (see Lemma 2.36 and reflexive weak compactness), we obtain the convergence properties as stated in Proposition 6.1. Passing to the limit can be carried out with exactly the same arguments as in Chapters 4 and 5. Here, we only present the arguments needed to obtain (6.2). In the following let $\xi \in H^1_0$ be arbitrary. Multiplying (5.5d) with $\delta \in C^\infty_0(0,T)$, integrating in time from 0 to T and noting that $H^1_0 \subset H^1$, we observe that

$$0 = \int_0^T \int_\Omega \delta(\nabla \sigma_K \cdot \nabla \xi + h(\varphi_K)\sigma_K \xi) \,\mathrm{d}x \,\mathrm{d}t \quad \forall\, \xi \in H^1_0. \tag{6.5}$$

Since $h(\cdot)$ is a bounded, continuous function, $\delta\xi \in C^\infty(H^1_0)$ and $\varphi_K \to \varphi$ a.e. in Ω_T, the Lebesgue theorem gives that

$$\|h(\varphi_K)\delta\xi - h(\varphi)\delta\xi\|_{L^2(\Omega_T)} \to 0 \quad \text{as } K \to \infty.$$

Since $\sigma_K \to \sigma$ weakly in $L^2(\Omega_T)$ as $K \to \infty$, by the product of weak-strong convergence we obtain

$$\int_0^T \int_\Omega h(\varphi_K)\sigma_K \xi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega h(\varphi)\sigma \xi \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } K \to \infty.$$

Furthermore, since $\sigma_K \to \sigma$ weakly in $L^2(H^1)$ and as $\delta\xi \in L^2(H^1)$, it follows that

$$\int_0^T \int_\Omega \delta \nabla \sigma_K \cdot \nabla \xi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta \nabla \sigma \cdot \nabla \xi \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } K \to \infty.$$

Therefore, we can pass to the limit in (6.5) to deduce that

$$0 = \int_0^T \int_\Omega \delta(\nabla \sigma \cdot \nabla \xi + h(\varphi)\sigma \xi) \,\mathrm{d}x \,\mathrm{d}t \quad \forall\, \xi \in H^1_0.$$

Since this holds for all $\delta \in C^\infty_0(0,T)$, we can recover (6.2). Finally, from (6.4) we infer that

$$\|\sigma_K - \sigma_\infty\|_{L^2(L^2(\partial\Omega))} \le \frac{C}{\sqrt{K}},$$

where C is independent of K. Sending $K \to \infty$ and recalling that $\sigma_K \to \sigma$ weakly in $L^2(L^2(\partial\Omega))$ as $K \to \infty$, it follows that

$$\sigma = \sigma_\infty \quad \text{a.e. on } \Sigma_T$$

which completes the proof. □

6.2 The singular limit of vanishing viscosities in 3D

As already pointed out above, Brinkman's equation can be interpreted as an interpolation between Darcy's law and Stokes flow. In the singular limit of vanishing viscosities, one can recover a so-called Cahn–Hilliard–Darcy model given by

$$\operatorname{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma) \quad \text{in } \Omega_T, \tag{6.6a}$$

$$\nu(\varphi)\mathbf{v} = -\nabla p + (\mu+\chi\sigma)\nabla\varphi \quad \text{in } \Omega_T, \tag{6.6b}$$

$$\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = \operatorname{div}(m(\varphi)\nabla\mu) + \Gamma_{\varphi}(\varphi,\sigma) \quad \text{in } \Omega_T, \tag{6.6c}$$

$$\mu = \varepsilon^{-1}\psi'(\varphi) - \varepsilon\Delta\varphi - \chi\sigma \quad \text{in } \Omega_T, \tag{6.6d}$$

$$0 = \Delta\sigma - h(\varphi)\sigma \quad \text{in } \Omega_T, \tag{6.6e}$$

and supplemented with the boundary and initial conditions

$$\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_T, \tag{6.7a}$$

$$\nabla\sigma\cdot\mathbf{n} = K(\sigma_\infty - \sigma) \quad \text{on } \Sigma_T, \tag{6.7b}$$

$$p = 0 \quad \text{on } \Sigma_T, \tag{6.7c}$$

$$\varphi(0) = \varphi_0 \quad \text{in } \Omega. \tag{6.7d}$$

We first have to introduce the definition of weak solutions of the Cahn–Hilliard–Darcy system (6.6)-(6.7).

Definition 6.2 (weak solutions of (6.6)-(6.7)) We call a quintuple $(\varphi,\mu,\sigma,\mathbf{v},p)$ weak solution of the Cahn–Hilliard–Darcy system (6.6)-(6.7) if

$$\varphi \in W^{1,\frac{8}{5}}(0,T;(H^1)^*)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^3), \quad \mu\in L^2(0,T;H^1),$$
$$\sigma\in L^2(0,T;H^1), \quad \mathbf{v}\in L^2(0,T;\mathbf{L}^2_{\mathrm{div}}), \quad p\in L^{\frac{8}{5}}(0,T;H^1_0),$$

such that

$$\operatorname{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma) \quad \text{a.e. in } \Omega_T, \qquad \varphi(0)=\varphi_0 \quad \text{a.e. in } \Omega, \qquad p=0 \quad \text{a.e. on } \Sigma_T,$$

and

$$0 = \langle\partial_t\varphi,\phi\rangle_{H^1} + \int_\Omega m(\varphi)\nabla\mu\cdot\nabla\phi \,\mathrm{d}x + \int_\Omega (\nabla\varphi\cdot\mathbf{v} + \varphi\Gamma_{\mathbf{v}}(\varphi,\sigma) - \Gamma_\varphi(\varphi,\sigma))\phi\,\mathrm{d}x, \tag{6.8a}$$

$$0 = \int_\Omega (\mu+\chi\sigma)\phi - \varepsilon^{-1}\psi'(\varphi)\phi - \varepsilon\nabla\varphi\cdot\nabla\phi\,\mathrm{d}x, \tag{6.8b}$$

$$0 = \int_\Omega \nabla\sigma\cdot\nabla\phi + h(\varphi)\sigma\phi\,\mathrm{d}x + \int_{\partial\Omega} K(\sigma-\sigma_\infty)\phi\,\mathrm{d}\mathcal{H}^{d-1}, \tag{6.8c}$$

$$0 = \int_\Omega (\nu(\varphi)\mathbf{v} + \nabla p - (\mu+\chi\sigma)\nabla\varphi)\cdot\mathbf{\Phi}\,\mathrm{d}x \tag{6.8d}$$

for a. e. $t\in(0,T)$ and all $\phi\in H^1$, $\mathbf{\Phi}\in\mathbf{L}^2$.

The following theorem states that solutions of the Cahn–Hilliard–Darcy system can be found as the limit of the Cahn–Hilliard–Brinkman system when the viscosities tend to zero.

Theorem 6.3 *Let $\Omega\subset\mathbb{R}^d$, $d=2,3$, be a bounded domain with C^3-boundary and assume that Assumptions 5.1,* (A1)-(A2), (A4)-(A5) *hold. Furthermore, let $\{\eta_n,\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of function pairs fulfilling Assumptions 5.1,* (A3) *such that*

$$\|\eta_n(\cdot)\|_{C^0(\mathbb{R})}\to 0, \quad \|\lambda_n(\cdot)\|_{C^0(\mathbb{R})}\to 0 \quad \text{as } n\to\infty,$$

and assume in addition that $\nu \in C^1(\mathbb{R})$ fulfils Assumptions 5.1, (A3)*. Let $(\varphi_n, \mu_n, \sigma_n, \mathbf{v}_n, p_n)$ be a sequence of weak solutions of the Cahn–Hilliard–Brinkman system in the sense of Definition 5.4 for $\eta(\cdot) = \eta_n(\cdot)$, $\lambda(\cdot) = \lambda_n(\cdot)$ and originating from $\varphi_0 \in H^1$. Then, at least for a subsequence, $(\varphi_n, \mu_n, \sigma_n, \mathbf{v}_n, p_n)$ converges to a weak solution $(\varphi, \mu, \sigma, \mathbf{v}, p)$ of the Cahn–Hilliard–Darcy system in the sense of Definition 6.2 such that*

$$\begin{aligned}
\varphi_n &\to \varphi && \text{weakly-star} && \text{in } W^{1,\frac{8}{5}}((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^3),\\
\sigma_n &\to \sigma && \text{weakly} && \text{in } L^2(H^1),\\
\mu_n &\to \mu && \text{weakly} && \text{in } L^2(H^1),\\
p_n &\to p && \text{weakly} && \text{in } L^2(L^2),\\
\mathbf{v}_n &\to \mathbf{v} && \text{weakly} && \text{in } L^2(\mathbf{L}^2) \cap L^2\left(\mathbf{L}^2_{\mathrm{div}}(\Omega)\right),\\
2\eta_n(\varphi_n)\mathbf{D}\mathbf{v}_n &\to \mathbf{0} && \text{weakly} && \text{in } L^2(\mathbf{L}^2),\\
\lambda_n(\varphi_n)\mathrm{div}(\mathbf{v}_n)\mathbf{I} &\to \mathbf{0} && \text{weakly} && \text{in } L^2(\mathbf{L}^2),
\end{aligned}$$

and

$$\varphi_n \to \varphi \quad \text{strongly} \quad \text{in } C^0(L^r) \cap L^2(W^{2,r}) \quad \text{and a. e. in } \Omega_T$$

for all $r \in [1, 6)$. Moreover, it holds that

$$\begin{aligned}
\nabla\varphi \cdot \mathbf{n} &= 0 && \text{a. e. on } \Sigma_T,\\
\nu(\varphi)\mathbf{v} &= -\nabla p + (\mu + \chi\sigma)\nabla\varphi && \text{a. e. in } \Omega_T,\\
\mu &= \varepsilon^{-1}\psi'(\varphi) - \varepsilon\Delta\varphi - \chi\sigma && \text{a. e. in } \Omega_T,
\end{aligned}$$

and

$$\begin{aligned}
&\|\varphi\|_{W^{1,\frac{8}{5}}((H^1)^*) \cap L^\infty(H^1) \cap L^4(H^2) \cap L^2(H^3)} + \|\mu\|_{L^2(H^1)} + \|\sigma\|_{L^2(H^1)}\\
&\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^{\frac{8}{5}}(L^{\frac{6}{5}})} + \|\mathbf{v}\|_{L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega))} + \|p\|_{L^{\frac{8}{5}}(H^1_0)} \le C
\end{aligned} \tag{6.9}$$

with a constant C independent of $(\varphi, \mu, \sigma, \mathbf{v}, p)$.

Proof. Let $\{\eta_n, \lambda_n\}_{n\in\mathbb{N}}$ be a sequence of function pairs fulfilling Assumptions 5.1, (A3) such that

$$\|\eta_n(\cdot)\|_{C^0(\mathbb{R})} \to 0, \quad \|\lambda_n(\cdot)\|_{C^0(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality, we may assume that

$$\|\eta_n(\cdot)\|_{L^\infty(\mathbb{R})} \le 1, \quad \|\lambda_n(\cdot)\|_{L^\infty(\mathbb{R})} \le 1.$$

Then, by Theorem 5.5, for every $n \in \mathbb{N}$ there exists a solution quintuple $(\varphi_n, \mu_n, \sigma_n, \mathbf{v}_n, p_n)$ of (5.1)-(5.2) in the sense of Definition 5.4 fulfilling

$$\begin{aligned}
&\varphi_n \in H^1(0,T;(H^1)^*) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^3), \quad \mu_n \in L^2(0,T;H^1),\\
&\sigma_n \in L^2(0,T;H^1), \quad \mathbf{v}_n \in L^2(0,T;\mathbf{H}^1), \quad p_n \in L^2(0,T;L^2),
\end{aligned}$$

such that

$$\mathrm{div}(\mathbf{v}_n) = \Gamma_{\mathbf{v}}(\varphi_n, \sigma_n) \quad \text{a. e. in } \Omega_T, \quad \nabla\varphi_n \cdot \mathbf{n} = 0 \quad \text{a. e. on } \Sigma_T, \quad \varphi_n(0) = \varphi_0 \quad \text{a. e. in } \Omega, \tag{6.10a}$$

and

$$0 = \int_\Omega \mathbf{T}_n(\mathbf{v}_n, p_n) \colon \nabla \mathbf{\Phi} + \nu(\varphi_n)\mathbf{v}_n \cdot \mathbf{\Phi} - (\mu_n + \chi\sigma_n)\nabla\varphi_n \cdot \mathbf{\Phi} \, \mathrm{d}x, \tag{6.10b}$$

$$\begin{aligned} 0 = \langle \partial_t \varphi_n , \Phi \rangle_{H^1} &+ \int_\Omega m(\varphi_n)\nabla\mu_n \cdot \nabla\Phi \, \mathrm{d}x \\ &+ \int_\Omega (\nabla\varphi_n \cdot \mathbf{v}_n + \varphi_n \Gamma_{\mathbf{v}}(\varphi_n, \sigma_n) - \Gamma_\varphi(\varphi_n, \sigma_n))\Phi \, \mathrm{d}x, \end{aligned} \tag{6.10c}$$

$$0 = \int_\Omega \nabla\sigma_n \cdot \nabla\Phi + h(\varphi_n)\sigma_n \Phi \, \mathrm{d}x + \int_{\partial\Omega} K(\sigma_n - \sigma_\infty)\Phi \, \mathrm{d}\mathcal{H}^{d-1} \tag{6.10d}$$

for a. e. $t \in (0, T)$ and for all $\mathbf{\Phi} \in \mathbf{H}^1$, $\Phi \in H^1$, where μ_n is given by

$$\mu_n = \varepsilon^{-1}\Psi'(\varphi_n) - \varepsilon\Delta\varphi_n - \chi\sigma_n \quad \text{a. e. in } \Omega_T, \tag{6.10e}$$

and the viscous stress tensor is defined by

$$\mathbf{T}_n(\mathbf{v}_n, p_n) \coloneqq 2\eta_n(\varphi_n)\mathbf{D}\mathbf{v}_n + \lambda_n(\varphi_n)\mathrm{div}(\mathbf{v}_n)\mathbf{I} - p_n\mathbf{I}.$$

We will denote $\Gamma_{\mathbf{v},n} = \Gamma_{\mathbf{v}}(\varphi_n, \sigma_n)$ and $\Gamma_{\varphi,n} = \Gamma_\varphi(\varphi_n, \sigma_n)$.

A priori estimates

In the following we derive bounds which are independent of $n \in \mathbb{N}$. By C, we denote a generic constant independent of $n \in \mathbb{N}$. Furthermore, we frequently use Hölder's and Young's inequalities.

First, we recall that (A1), (A5) and the Sobolev embedding $H^1 \subset L^6$ imply that $\psi(\varphi_0) \in L^1$, $\nabla\varphi_0 \in \mathbf{L}^2$. Then, using (A1), (5.23) and the bound $\|\eta_n(\cdot)\|_{L^\infty(\mathbb{R})} \leq 1$, taking the supremum over all $s \in (0, T]$ in (5.23) yields

$$\begin{aligned} &\operatorname*{ess\,sup}_{s \in (0,T]} \left(\|\psi(\varphi_n(s))\|_{L^1} + \|\nabla\varphi_n(s)\|_{\mathbf{L}^2}^2 \right) \\ &+ \int_0^T \frac{m_0}{4}\|\nabla\mu_n\|_{\mathbf{L}^2}^2 + \frac{\nu_0}{2}\|\mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\sqrt{\eta_n(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\sigma_n\|_{H^1}^2 \, \mathrm{d}t \leq C. \end{aligned}$$

Recalling (A5) and using Poincaré's inequality along with (5.16), this in particular gives

$$\operatorname*{ess\,sup}_{s \in (0,T]} \|\varphi_n(s)\|_{H^1}^2 + \int_0^T \|\mu_n\|_{H^1}^2 \, \mathrm{d}t \leq C.$$

Now, using exactly the same arguments as in Chapter 5, we obtain

$$\|\varphi_n\|_{L^4(H^2) \cap L^2(H^3)} \leq C.$$

Invoking the last three bounds along with (A4), (6.10a) and the Sobolev embedding $H^1 \subset L^6$, we deduce that

$$\begin{aligned} &\|\varphi_n\|_{L^\infty(H^1) \cap L^4(H^2) \cap L^2(H^3)} + \|\mu_n\|_{L^2(H^1)} + \|\sigma_n\|_{L^2(H^1)} + \|\Gamma_{\mathbf{v},n}\|_{L^2(L^6)} + \|\Gamma_{\varphi,n}\|_{L^2(L^6)} \\ &+ \|\mathrm{div}(\mathbf{v}_n)\|_{L^2(L^6)} + \|\sqrt{\eta_n(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{L^2(\mathbf{L}^2)} + \|\mathbf{v}_n\|_{L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega))} \leq C. \end{aligned} \tag{6.11}$$

Due to (5.27) and using the bounds $\|\eta_n(\cdot)\|_{L^\infty(\mathbb{R})} \leq 1$, $\|\lambda_n(\cdot)\|_{L^\infty(\mathbb{R})} \leq 1$ for every $n \in \mathbb{N}$, we obtain

$$\|p_n\|_{L^2}^2 \leq C \left(\|\sqrt{\eta_n(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\Gamma_{\mathbf{v},n}\|_{L^2}^2 + \|\mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\mu_n + \chi\sigma_n\|_{L^3}^2 \|\nabla\varphi_n\|_{\mathbf{L}^2}^2 \right).$$

Integrating this inequality in time from 0 to T and using (6.11) yields

$$\|p_n\|_{L^2(L^2)} \le C. \tag{6.12}$$

Using (2.4) and (6.11) we obtain

$$\|\nabla\varphi_n \cdot \mathbf{v}_n\|_{L^{\frac{8}{5}}(L^{\frac{6}{5}})} \le C\|\mathbf{v}_n\|_{L^2(\mathbf{L}^2)}\|\nabla\varphi_n\|_{L^8(\mathbf{L}^3)} \le C\|\mathbf{v}_n\|_{L^2(\mathbf{L}^2)}\|\varphi_n\|^{\frac{1}{2}}_{L^\infty(H^1)}\|\varphi_n\|^{\frac{1}{2}}_{L^4(H^2)} \le C,$$

and thus

$$\|\nabla\varphi_n \cdot \mathbf{v}_n\|_{L^{\frac{8}{5}}(L^{\frac{6}{5}})} \le C.$$

By (6.11) and the Sobolev embedding $H^1 \subset L^3$, we calculate

$$\|\varphi_n\Gamma_{\mathbf{v},n}\|_{L^2(L^2)} \le \|\varphi_n\|_{L^\infty(L^3)}\|\Gamma_{\mathbf{v},n}\|_{L^2(L^6)} \le C\|\varphi_n\|_{L^\infty(H^1)}\|\Gamma_{\mathbf{v},n}\|_{L^2(L^6)} \le C.$$

In conjunction with the continuous embedding $L^{\frac{6}{5}} \hookrightarrow (H^1)^*$ and (6.10a), we conclude that

$$\|\mathrm{div}(\varphi_n\mathbf{v}_n)\|_{L^{\frac{8}{5}}(L^{\frac{6}{5}})} \le C.$$

Using the relation (6.10c) for $\partial_t\varphi_n$, (6.11) and the continuous embedding $L^{\frac{6}{5}} \hookrightarrow (H^1)^*$, this yields

$$\|\partial_t\varphi_n\|_{L^{\frac{8}{5}}((H^1)^*)} \le C.$$

Invoking the last two estimates and recalling (6.11)-(6.12) leads to

$$\begin{aligned}\|\varphi_n\|_{W^{1,\frac{8}{5}}((H^1)^*)\cap L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)} + \|\mu_n\|_{L^2(H^1)} + \|\sigma_n\|_{L^2(H^1)} + \|\mathrm{div}(\varphi_n\mathbf{v}_n)\|_{L^{\frac{8}{5}}(L^{\frac{6}{5}})} \\ + \|\mathrm{div}(\mathbf{v}_n)\|_{L^2(L^6)} + \|\sqrt{\eta_n(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{L^2(\mathbf{L}^2)} + \|\mathbf{v}_n\|_{L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega))} + \|p_n\|_{L^2(L^2)} \le C.\end{aligned} \tag{6.13}$$

Passing to the limit

Recalling (6.13), using standard compactness arguments (see Lemma 2.36), reflexive weak compactness and the compact embeddings

$$H^{j+1}(\Omega) = W^{j+1,2}(\Omega) \hookrightarrow\hookrightarrow W^{j,r} \quad \forall\, j \in \mathbb{Z},\ j \ge 0,\ 1 \le r < 6,$$

for a non-relabelled subsequence we obtain

$$\begin{aligned}
\varphi_n &\to \varphi && \text{weakly-star} && \text{in } W^{1,\frac{8}{5}}((H^1)^*) \cap L^\infty(H^1) \cap L^4(H^2) \cap L^2(H^3),\\
\sigma_n &\to \sigma && \text{weakly} && \text{in } L^2(H^1),\\
\mu_n &\to \mu && \text{weakly} && \text{in } L^2(H^1),\\
p_n &\to p && \text{weakly} && \text{in } L^2(L^2),\\
\mathbf{v}_n &\to \mathbf{v} && \text{weakly} && \text{in } L^2(\mathbf{L}^2) \cap L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega)),\\
\mathrm{div}(\varphi_n\mathbf{v}_n) &\to \tau && \text{weakly} && \text{in } L^{\frac{8}{5}}(L^{\frac{6}{5}})
\end{aligned}$$

for some limit function $\tau \in L^{\frac{8}{5}}(L^{\frac{6}{5}})$. Furthermore, using the fact that $\|\eta_n(\cdot)\|_{C^0(\mathbb{R})} \to 0$ and $\|\lambda_n(\cdot)\|_{C^0(\mathbb{R})} \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
&\varphi_n \to \varphi \quad \text{strongly} && \text{in } C^0(L^r) \cap L^4(W^{1,r}) \cap L^2(W^{2,r}) \text{ and a.e. in } \Omega_T,\\
&\eta_n(\varphi_n) \to 0 && \text{a.e. in } \Omega_T,\\
&\lambda_n(\varphi_n) \to 0 && \text{a.e. in } \Omega_T
\end{aligned}$$

as $n \to \infty$ for $r \in [1, 6)$. In the following we fix $\delta \in C_0^\infty(0,T)$, $\Phi \in H^1$, $\mathbf{\Phi} \in \mathbf{H}^1$, $\phi \in L^2$ and we note that $\delta\Phi \in C^\infty(H^1)$, $\delta\mathbf{\Phi} \in C^\infty(\mathbf{H}^1)$, $\delta\phi \in C^\infty(L^2)$. Multiplying (6.10b)-(6.10d) with δ, (6.10e) with $\delta\phi$ and integrating from 0 to T and over Ω_T, respectively, we obtain

$$0 = \int_0^T \delta(t) \left(\int_\Omega \mathbf{T}_n(\mathbf{v}_n, p_n) \colon \nabla \mathbf{\Phi} + \big(\nu(\varphi_n)\mathbf{v}_n - (\mu_n + \chi\sigma_n)\nabla\varphi_n\big) \cdot \mathbf{\Phi} \,\mathrm{d}x \right) \mathrm{d}t, \tag{6.14a}$$

$$0 = \int_0^T \delta(t) \left(\langle \partial_t \varphi_n \,, \Phi \rangle_{H^1} + \int_\Omega m(\varphi_n)\nabla\mu_n \cdot \nabla\Phi + \big(\mathrm{div}(\varphi_n \mathbf{v}_n) - \Gamma_{\varphi,n}\big)\Phi \,\mathrm{d}x \right) \mathrm{d}t, \tag{6.14b}$$

$$0 = \int_0^T \delta(t) \left(\int_\Omega (\mu_n - \varepsilon^{-1}\Psi'(\varphi_n) + \varepsilon\Delta\varphi_n + \chi\sigma_n)\Phi \,\mathrm{d}x \right) \mathrm{d}t, \tag{6.14c}$$

$$0 = \int_0^T \delta(t) \left(\int_\Omega \nabla\sigma_n \cdot \nabla\Phi + h(\varphi_n)\sigma_n\Phi \,\mathrm{d}x + K \int_{\partial\Omega} (\sigma_n - \sigma_\infty)\Phi \ \mathrm{d}\mathcal{H}^{d-1} \right) \mathrm{d}t. \tag{6.14d}$$

Furthermore, we multiply $(6.10\mathrm{a})_1$ with $\delta\phi$ and integrate over Ω_T to obtain

$$\int_0^T \int_\Omega \delta(t)\,\mathrm{div}(\mathbf{v}_n)\phi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \delta(t)\Gamma_{\mathbf{v},n}\phi \,\mathrm{d}x \,\mathrm{d}t. \tag{6.14e}$$

We now analyse each term individually. For (6.14c)-(6.14d) we omit the details and refer to the arguments used in Chapter 4 and [81, Sec. 5].

Step 1 ((6.14e)): Since $\varphi_n \to \varphi$ a. e. in Ω_T as $n \to \infty$ and due to the boundedness of $b_{\mathbf{v}}(\cdot)$ and $f_{\mathbf{v}}(\cdot)$, Lebesgue dominated convergence theorem implies

$$\|\delta\phi\,(b_{\mathbf{v}}(\varphi_n) - b_{\mathbf{v}}(\varphi))\|_{L^2(\Omega_T)} \to 0, \quad \|\delta\phi\,(f_{\mathbf{v}}(\varphi_n) - f_{\mathbf{v}}(\varphi))\|_{L^2(\Omega_T)} \to 0 \quad \text{as } n \to \infty.$$

Together with the weak convergence $\sigma_n \to \sigma$ in $L^2(\Omega_T)$ as $n \to \infty$, by the product of weak-strong convergence we obtain

$$\int_0^T \int_\Omega \delta\,\Gamma_{\mathbf{v},n}\,\phi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta\,\Gamma_{\mathbf{v}}(\varphi,\sigma)\phi \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } n \to \infty. \tag{6.15}$$

Moreover, since $\mathbf{v}_n \rightharpoonup \mathbf{v}$ weakly in $L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega))$ as $n \to \infty$, it follows that $\mathbf{v} \in L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega))$ and $\mathrm{div}(\mathbf{v}_n) \rightharpoonup \mathrm{div}(\mathbf{v})$ weakly in $L^2(L^2)$ as $n \to \infty$. From this considerations and by (6.15), we can pass to the limit $n \to \infty$ in (6.14e) to infer

$$\int_0^T \int_\Omega \delta\,\mathrm{div}(\mathbf{v})\phi \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega \delta\,\Gamma_{\mathbf{v}}(\varphi,\sigma)\phi \,\mathrm{d}x \,\mathrm{d}t,$$

and therefore

$$\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma) \quad \text{a. e. in } \Omega_T. \tag{6.16}$$

Step 2 ((6.14b)): Since $\delta\Phi \in C^\infty(H^1)$ and $\mathrm{div}(\varphi_n\mathbf{v}_n) \rightharpoonup \tau$ weakly in $L^{\frac{8}{5}}(L^{\frac{6}{5}})$ as $n \to \infty$, we have

$$\int_0^T \int_\Omega \delta\,\mathrm{div}(\varphi_n\mathbf{v}_n)\Phi \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta\,\tau\Phi \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } n \to \infty. \tag{6.17}$$

Moreover, by the strong convergence $\nabla\varphi_n \to \nabla\varphi$ in $L^2(\mathbf{L}^3)$ and the Sobolev embedding $H^1 \subset L^6$ it holds

$$\begin{aligned} \int_0^T \int_\Omega |\delta|^2|\Phi|^2|\nabla\varphi_n - \nabla\varphi|^2 \,\mathrm{d}x \,\mathrm{d}t &\le \int_0^T |\delta|^2 \|\Phi\|_{L^6}^2 \|\nabla\varphi_n - \nabla\varphi\|_{\mathbf{L}^3}^2 \,\mathrm{d}t \\ &\le C\|\delta\|_{L^\infty(0,T)}^2 \|\Phi\|_{H^1}^2 \|\nabla\varphi_n - \nabla\varphi\|_{L^2(\mathbf{L}^3)}^2 \to 0 \end{aligned}$$

as $n \to \infty$. This implies $\delta\Phi\nabla\varphi_n \to \delta\Phi\nabla\varphi$ strongly in $L^2(\mathbf{L}^2)$. Together with the weak convergence $\mathbf{v}_n \rightharpoonup \mathbf{v}$ in $L^2(\mathbf{L}^2)$ as $n \to \infty$, by the product of weak-strong convergence we get

$$\int_0^T \int_\Omega \delta\Phi\nabla\varphi_n \cdot \mathbf{v}_n \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \delta\Phi\nabla\varphi \cdot \mathbf{v} \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } n \to \infty. \tag{6.18}$$

Since $\varphi_n \to \varphi$ strongly in $L^2(L^3)$ and a. e. in Ω_T as $n \to \infty$, the boundedness of $b_{\mathbf{v}}(\cdot)$, $f_{\mathbf{v}}(\cdot)$ and Lebesgue dominated convergence theorem guarantee that

$$\|(b_{\mathbf{v}}(\varphi_n)\varphi_n - b_{\mathbf{v}}(\varphi)\varphi)\delta\Phi\|_{L^2(\Omega_T)} \to 0, \qquad \|(f_{\mathbf{v}}(\varphi_n)\varphi_n - f_{\mathbf{v}}(\varphi)\varphi)\delta\Phi\|_{L^2(\Omega_T)} \to 0$$

as $n \to \infty$ where we used that $\Phi \in H^1 \subset L^6$. Together with the weak convergence $\sigma_n \rightharpoonup \sigma$ in $L^2(\Omega_T)$ as $n \to \infty$, this implies

$$\int_0^T \int_\Omega \delta\Phi\Gamma_{\mathbf{v},n}\varphi_n \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta\Phi\Gamma_{\mathbf{v}}(\varphi,\sigma)\varphi \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } n \to \infty. \tag{6.19}$$

Using $(6.10\text{a})_1$, we see that

$$\int_0^T \int_\Omega \delta\mathrm{div}(\varphi_n\mathbf{v}_n)\Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \delta\Phi\nabla\varphi_n \cdot \mathbf{v}_n \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \delta\Phi\Gamma_{\mathbf{v},n}\varphi_n \,\mathrm{d}x\,\mathrm{d}t.$$

Passing to the limit $n \to \infty$ on both sides of this equation and using (6.18)-(6.19), we obtain

$$\int_0^T \int_\Omega \delta\tau\Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \delta\Phi\nabla\varphi \cdot \mathbf{v} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \delta\Phi\Gamma_{\mathbf{v}}(\varphi,\sigma)\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Together with (6.16), this entails

$$\int_0^T \int_\Omega \delta\tau\Phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \delta\mathrm{div}(\varphi\mathbf{v})\Phi \,\mathrm{d}x\,\mathrm{d}t,$$

hence $\mathrm{div}(\varphi\mathbf{v}) = \tau$ a. e. in Ω_T. For the remaining terms in (6.14b), we again refer to Chapter 4 and [81, Sec. 5].

Step 3 ((6.14a)): With exactly the same arguments as used for (6.18), in the limit $n \to \infty$ we have $\delta\mathbf{\Phi} \cdot \nabla\varphi_n \to \delta\mathbf{\Phi} \cdot \nabla\varphi$ strongly in $L^2(L^2)$. Then, recalling that $\mu_n + \chi\sigma_n \rightharpoonup \mu + \chi\sigma$ weakly in $L^2(L^2)$ as $n \to \infty$, by the product of weak-strong convergence we obtain

$$\int_0^T \int_\Omega \delta(\mu_n + \chi\sigma_n)\nabla\varphi_n \cdot \mathbf{\Phi} \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta(\mu + \chi\sigma)\nabla\varphi \cdot \mathbf{\Phi} \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } n \to \infty. \tag{6.20}$$

Recalling that $p_n \rightharpoonup p$, $\mathbf{v}_n \rightharpoonup \mathbf{v}$ weakly in $L^2(L^2)$ and $L^2(\mathbf{L}^2)$ as $n \to \infty$, respectively, and using the identity

$$\int_0^T \int_\Omega \delta p_n\mathbf{I} \colon \nabla\mathbf{\Phi} \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \delta p_n\mathrm{div}(\mathbf{\Phi}) \,\mathrm{d}x\,\mathrm{d}t$$

along with the assumptions on $\nu(\cdot)$ and the convergence $\varphi_n \to \varphi$ a. e. in Ω_T as $n \to \infty$, by the product of weak-strong convergence we obtain

$$\int_0^T \int_\Omega \delta(-p_n\mathrm{div}(\mathbf{\Phi}) + \nu(\varphi_n)\mathbf{v}_n \cdot \mathbf{\Phi}) \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \delta(-p\mathrm{div}(\mathbf{\Phi}) + \nu\mathbf{v}(\varphi) \cdot \mathbf{\Phi}) \,\mathrm{d}x\,\mathrm{d}t \tag{6.21}$$

as $n \to \infty$. Finally, we recall that $\eta_n(\varphi_n) \to 0$ a. e. in Ω_T as $n \to \infty$. Consequently, applying (6.13) yields

$$\begin{aligned} \left|\int_0^T \int_\Omega \delta 2\eta_n(\varphi_n)\mathbf{D}\mathbf{v}_n \colon \nabla\mathbf{\Phi} \,\mathrm{d}x\,\mathrm{d}t\right| &\le C\|\sqrt{\eta_n(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{L^2(\mathbf{L}^2)}\|\delta\sqrt{\eta_n(\varphi_n)}\|_{L^\infty(\Omega_T)}\|\mathbf{\Phi}\|_{\mathbf{H}^1} \\ &\le C\|\sqrt{\eta_n(\varphi_n)}\|_{L^\infty(\Omega_T)}\|\delta\|_{L^\infty(0,T)}\|\mathbf{\Phi}\|_{\mathbf{H}^1} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{6.22}$$

Using that $\lambda_n(\varphi_n) \to 0$ a.e. in Ω_T as $n \to \infty$ and applying (6.13), it follows that

$$\begin{aligned}\left|\int_0^T \int_\Omega \delta\lambda_n(\varphi_n)\mathrm{div}(\mathbf{v}_n)\mathbf{I} \colon \nabla\mathbf{\Phi} \, \mathrm{d}x \, \mathrm{d}t\right| &\leq C\|\mathrm{div}(\mathbf{v}_n)\|_{L^2(L^2)}\|\lambda_n(\varphi_n)\|_{L^3(\Omega_T)}\|\delta\|_{L^\infty(0,T)}\|\mathbf{\Phi}\|_{\mathbf{H}^1} \\ &\leq C\|\lambda_n(\varphi_n)\|_{L^3(\Omega_T)}\|\delta\|_{L^\infty(0,T)}\|\mathbf{\Phi}\|_{\mathbf{H}^1} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{6.23}$$

Step 4: Due to (6.15)-(6.23), we have enough to pass to the limit $n \to \infty$ in (6.14) to obtain that

$$0 = \int_0^T \delta(t) \left(\int_\Omega -p\mathrm{div}(\mathbf{\Phi}) + (\nu(\varphi)\mathbf{v} - (\mu + \chi\sigma)\nabla\varphi) \cdot \mathbf{\Phi} \, \mathrm{d}x \right) \mathrm{d}t, \tag{6.24a}$$

$$0 = \int_0^T \delta(t) \left(\langle \partial_t\varphi, \Phi\rangle_{H^1} + \int_\Omega m(\varphi)\nabla\mu \cdot \nabla\Phi + (\mathrm{div}(\varphi\mathbf{v}) - \Gamma_\varphi(\varphi,\sigma))\Phi \, \mathrm{d}x \right) \mathrm{d}t, \tag{6.24b}$$

$$0 = \int_0^T \delta(t) \left(\int_\Omega (\mu - \varepsilon^{-1}\Psi'(\varphi) + \varepsilon\Delta\varphi + \chi\sigma)\Phi \, \mathrm{d}x \right) \mathrm{d}t, \tag{6.24c}$$

$$0 = \int_0^T \delta(t) \left(\int_\Omega \nabla\sigma \cdot \nabla\Phi + h(\varphi)\sigma\Phi \, \mathrm{d}x + \int_{\partial\Omega} K(\sigma - \sigma_\infty)\Phi \, \mathrm{d}\mathcal{H}^{d-1} \right) \mathrm{d}t \tag{6.24d}$$

for all $\delta \in C_0^\infty(0,T)$ and

$$\mathrm{div}(\mathbf{v}) = \Gamma_{\mathbf{v}}(\varphi,\sigma) \quad \text{a.e. in } \Omega_T. \tag{6.25a}$$

Since (6.24) holds for all $\delta \in C_0^\infty(0,T)$, we deduce that

$$0 = \int_\Omega -p\mathrm{div}(\mathbf{\Phi}) + (\nu(\varphi)\mathbf{v} - (\mu + \chi\sigma)\nabla\varphi) \cdot \mathbf{\Phi} \, \mathrm{d}x, \tag{6.25b}$$

$$0 = \langle \partial_t\varphi, \Phi\rangle_{H^1} + \int_\Omega m(\varphi)\nabla\mu \cdot \nabla\Phi + (\nabla\varphi \cdot \mathbf{v} + \varphi\Gamma_{\mathbf{v}}(\varphi,\sigma) - \Gamma_\varphi(\varphi,\sigma))\Phi \, \mathrm{d}x, \tag{6.25c}$$

$$0 = \int_\Omega (\mu - \varepsilon^{-1}\Psi'(\varphi) + \varepsilon\Delta\varphi + \chi\sigma)\Phi \, \mathrm{d}x, \tag{6.25d}$$

$$0 = \int_\Omega \nabla\sigma \cdot \nabla\Phi + h(\varphi)\sigma\Phi \, \mathrm{d}x + \int_{\partial\Omega} K(\sigma - \sigma_\infty)\Phi \, \mathrm{d}\mathcal{H}^{d-1} \tag{6.25e}$$

holds for a.e. $t \in (0,T)$ and all $\Phi \in H^1$, $\mathbf{\Phi} \in \mathbf{H}^1$. The initial condition is satisfied since $\varphi_n(0) = \varphi_0$ a.e. in Ω and by the strong convergence $\varphi_n \to \varphi$ in $C^0(L^2)$ as $n \to \infty$. By the weak (weak-star) lower semi-continuity of norms and (6.13), we obtain that $(\varphi, \mu, \sigma, \mathbf{v}, p)$ satisfies

$$\begin{aligned} &\|\varphi\|_{W^{1,\frac{8}{5}}((H^1)^*)\cap L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)} + \|\mu\|_{L^2(H^1)} + \|\sigma\|_{L^2(H^1)} + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^{\frac{8}{5}}(L^{\frac{6}{5}})} \\ &\quad + \|\mathrm{div}(\mathbf{v})\|_{L^2(L^6)} + \|\mathbf{v}\|_{L^2(\mathbf{L}^2_{\mathrm{div}}(\Omega))} + \|p\|_{L^2(L^2)} \leq C. \end{aligned} \tag{6.26}$$

Step 5: Using (6.25b) and (6.26), we obtain that p has a weak derivative in $L^{\frac{8}{5}}(L^2)$ and it holds

$$\nabla p = -\nu(\varphi)\mathbf{v} + (\mu + \chi\sigma)\nabla\varphi \quad \text{a.e. in } \Omega_T. \tag{6.27}$$

By (2.4) we have

$$\|\nabla\varphi\|_{\mathbf{L}^3} \leq C\|\nabla\varphi\|_{\mathbf{L}^2}^{\frac{3}{4}}\|\nabla\varphi\|_{\mathbf{H}^2}^{\frac{1}{4}} \leq C\|\varphi\|_{H^1}^{\frac{3}{4}}\|\varphi\|_{H^3}^{\frac{1}{4}}$$

which in turn implies

$$\|\nabla\varphi\|_{L^8(\mathbf{L}^3)} \leq C$$

due to (6.26). Then, using the Sobolev embedding $H^1 \subset L^6$ and (6.26) again, we obtain

$$\|(\mu + \chi\sigma)\nabla\varphi\|_{L^{\frac{8}{5}}(\mathbf{L}^2)} \leq C.$$

Since $\mathbf{v} \in L^2(\mathbf{L}^2)$ and $p \in L^2(L^2)$ with bounded norm, (6.27) yields

$$\|p\|_{L^2(L^2)\cap L^{\frac{8}{5}}(H^1)} \leq C. \tag{6.28}$$

Integrating (6.24a) by parts, we obtain

$$-\int_0^T \delta(t)\left(\int_{\partial\Omega} p\mathbf{\Phi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\right)\,\mathrm{d}t = \int_0^T \delta(t)\left(\int_\Omega (-\nabla p + (\mu+\chi\sigma)\nabla\varphi - \nu(\varphi)\mathbf{v})\cdot\mathbf{\Phi}\,\mathrm{d}x\right)\,dt$$

for all $\mathbf{\Phi} \in \mathbf{H}^1$ and all $\delta \in C_0^\infty(0,T)$. Because of (6.27), this leads to

$$\int_0^T \delta(t)\left(\int_{\partial\Omega} p\mathbf{\Phi}\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1}\right)\,\mathrm{d}t = 0$$

for all $\mathbf{\Phi} \in \mathbf{H}^1$ and all $\delta \in C_0^\infty(0,T)$. Therefore, we obtain

$$p = 0 \quad \text{a.e. on } \Sigma_T.$$

With similar arguments, it is straightforward to show that

$$\mu = \varepsilon^{-1}\psi'(\varphi) - \varepsilon\Delta\varphi - \chi\sigma \quad \text{a.e. in } \Omega_T, \qquad \nabla\varphi\cdot\mathbf{n} = 0 \quad \text{a.e. on } \Sigma_T$$

which completes the proof. □

6.3 The singular limit of vanishing viscosities in 2D

In this part we will analyse the relation between solutions of the Cahn–Hilliard–Brinkman and Cahn–Hilliard–Darcy models in two space dimension. In particular, we will show that there exists a unique strong solution of the Cahn–Hilliard–Darcy model which is the limit of unique strong solutions of the Cahn–Hilliard–Brinkman model as the viscosities tend to zero.

6.3.1 Convergence of strong solutions of the Cahn–Hilliard–Brinkman model

We start with the definition of strong solutions for (6.6)-(6.7)

Definition 6.4 We call a quintuple $(\varphi,\mu,\sigma,\mathbf{v},p)$ strong solution of the Cahn–Hilliard–Darcy system (6.6)-(6.7) if

$$\begin{aligned}&\varphi \in H^1(0,T;L^2)\cap L^2(0,T;H^4), \quad \mu \in L^2(0,T;H^2),\\ &\sigma \in L^2(0,T;H^2), \quad \mathbf{v} \in L^2(0,T;\mathbf{H}^1), \quad p \in L^2(0,T;H_0^1\cap H^2)\end{aligned}$$

such that

$$\varphi(0) = \varphi_0 \quad \text{a.e. in } \Omega$$

and equations (6.6)-(6.7) are fulfilled a.e. in the respective sets.

The following theorem shows that strong solutions of (6.6)-(6.7) can be established via the zero viscosity limit of (5.1)-(5.2).

Theorem 6.5 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^4-boundary and assume that Assumptions 5.1,* (A1)*,* (A4)-(A5) *and Assumptions 5.10 hold. Furthermore, let $\{\eta_n,\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of function pairs fulfilling Assumptions 5.1,* (A3) *such that*

$$\|\eta_n(\cdot)\|_{C^0(\mathbb{R})} \to 0, \quad \|\lambda_n(\cdot)\|_{C^0(\mathbb{R})} \to 0 \quad \textit{as } n\to\infty,$$

and assume in addition that $\nu \in C^1(\mathbb{R})$ *fulfils* (A3)*. Let* $(\varphi_n, \mu_n, \sigma_n, \mathbf{v}_n, p_n)$ *be a sequence of strong solutions of the Cahn–Hilliard–Brinkman system in the sense of Definition 5.9 for* $\eta(\cdot) = \eta_n(\cdot)$*,* $\lambda(\cdot) = \lambda_n(\cdot)$*, and originating from* $\varphi_0 \in H^2_N$*. Then, at least for a subsequence,* $(\varphi_n, \mu_n, \sigma_n, \mathbf{v}_n, p_n)$ *converges to a strong solution* $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *of the Cahn–Hilliard–Darcy system in the sense of Definition 6.2 such that*

$$
\begin{aligned}
\varphi_n &\to \varphi && \textit{weakly-star} && \textit{in } H^1(L^2) \cap L^\infty(H^2) \cap L^2(H^3),\\
\sigma_n &\to \sigma && \textit{weakly-star} && \textit{in } H^1(H^1) \cap L^\infty(H^2),\\
\mu_n &\to \mu && \textit{weakly-star} && \textit{in } L^\infty(L^2) \cap L^2(H^2),\\
p_n &\to p && \textit{weakly} && \textit{in } L^2(L^2),\\
\mathbf{v}_n &\to \mathbf{v} && \textit{weakly} && \textit{in } L^4\left(\mathbf{L}^2_{\mathrm{div}}(\Omega)\right),\\
2\eta_n(\varphi_n)\mathbf{D}\mathbf{v}_n &\to \mathbf{0} && \textit{weakly} && \textit{in } L^2(\mathbf{L}^2),\\
\lambda_n(\varphi_n)\mathrm{div}(\mathbf{v}_n)\mathbf{I} &\to \mathbf{0} && \textit{weakly} && \textit{in } L^2(\mathbf{L}^2),
\end{aligned}
$$

and

$$\varphi_n \to \varphi \quad \textit{strongly} \quad \textit{in } C^0(W^{1,r}) \cap L^2(W^{3,r}) \textit{ and a. e. in } \Omega_T$$

for all $r \in [1, 6)$*. Moreover, the quintuple* $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *fulfils* (6.6)-(6.7) *a. e. in the respective sets and*

$$
\begin{aligned}
&\|\varphi\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^4)} + \|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\sigma\|_{H^1(H^1)\cap L^\infty(H^2)}\\
&\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathbf{v}\|_{L^4(\mathbf{L}^2_{\mathrm{div}}(\Omega))\cap L^2(\mathbf{H}^1)} + \|p\|_{L^2(H^2)} \leq C
\end{aligned}
\tag{6.29}
$$

with a constant C *independent of* $(\varphi, \mu, \sigma, \mathbf{v}, p)$*.*

Proof. In what follows, we will derive estimates independent of $n \in \mathbb{N}$ that can be justified rigorously within the Galerkin scheme presented in Chapter 5. We notice that the the testing procedure cannot be carried out on the continuous level due to a lack of regularity and due to the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi|^2 \,\mathrm{d}x = \int_\Omega \partial_t\varphi\,\Delta^2\varphi \,\mathrm{d}x \quad \text{for a. e. } t \in (0,T)$$

does not hold since $\nabla\Delta\varphi \cdot \mathbf{n} = -\chi\nabla\sigma \cdot \mathbf{n}$ almost everywhere on Σ_T. We will frequently use Hölder's and Young's inequalities and we denote by C a generic constant independent of $n \in \mathbb{N}$. We divide the proof into several steps.

Step 1: Hereinafter, we denote $\Gamma_{\mathbf{v},n} = \Gamma_{\mathbf{v}}(\varphi_n, \sigma_n)$ and $\Gamma_{\varphi,n} = \Gamma_\varphi(\varphi_n, \sigma_n)$. With exactly the same arguments as in the proof of Theorem 6.3, it follows that

$$
\begin{aligned}
&\|\varphi_n\|_{L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)} + \|\sigma_n\|_{L^\infty(H^2)} + \|\mu_n\|_{L^2(H^1)\cap L^4(L^2)} + \|\Gamma_{\mathbf{v},n}\|_{L^\infty(\Omega_T)} + \|\Gamma_{\varphi,n}\|_{L^\infty(\Omega_T)}\\
&\quad + \|\mathbf{v}_n\|_{L^2(\mathbf{L}^2_{\mathrm{div}})} + \|\sqrt{\eta(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{L^2(\mathbf{L}^2)} + \|\mathrm{div}(\mathbf{v}_n)\|_{L^\infty(\Omega_T)} + \|p_n\|_{L^2(L^2)} \leq C.
\end{aligned}
\tag{6.30}
$$

Taking $\mathbf{\Phi} = \mathbf{v}_n$ in (5.5a), we obtain

$$\int_\Omega 2\eta_n(\varphi_n)|\mathbf{D}\mathbf{v}_n|^2 + \lambda_n(\varphi_n)|\mathrm{div}(\mathbf{v}_n)|^2 + \nu(\varphi_n)|\mathbf{v}_n|^2 - p_n\mathrm{div}(\mathbf{v}_n)\,\mathrm{d}x = \int_\Omega (\mu_n + \chi\sigma_n)\nabla\varphi_n \cdot \mathbf{v}_n \,\mathrm{d}x.$$

Due to the non-negativity of $\eta_n(\cdot)$ and $\lambda_n(\cdot)$, we can neglect the first two terms on the l. h. s. of this equation to obtain

$$\int_\Omega \nu(\varphi_n)|\mathbf{v}_n|^2 \,\mathrm{d}x \leq \left|\int_\Omega p_n\mathrm{div}(\mathbf{v}_n) + (\mu_n + \chi\sigma_n)\nabla\varphi_n \cdot \mathbf{v}_n \,\mathrm{d}x\right|.$$

Using the assumptions on $\nu(\cdot)$ and recalling (6.30), this yields

$$\|\mathbf{v}_n\|_{\mathbf{L}^2} \leq C\left(1 + \|p_n\|^{\frac{1}{2}}_{L^2} + \|(\mu_n + \chi\sigma_n)\nabla\varphi_n\|_{\mathbf{L}^2}\right). \tag{6.31}$$

Using (6.30), it follows that $\nabla\varphi_n \in L^{\frac{4q_1}{q_1-2}}(\mathbf{L}^{q_1})$ for all $q_1 \in (2,\infty)$. Then, by the Sobolev embedding $H^1 \subset L^{q_2}$, $q_2 \in (1,\infty)$, and arguing similar as in [81], we infer that

$$\|\partial_t\varphi_n\|_{L^r((H^1)^*)} \le C \quad \forall\, r \in (1,2)\,.$$

With similar arguments as used for (5.68), it follows that

$$\|\sigma_n\|_{W^{1,r}(H^1)\cap C^0([0,T];H^1)} \le C \quad \forall\, r \in (1,2)\,. \tag{6.32}$$

Choosing $\Phi = \partial_t\varphi_n$ in (5.5b), $\Phi = \Delta\partial_t\varphi_n$ in (5.5c), and integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi_n|^2\,\mathrm{d}x + \int_\Omega |\partial_t\varphi_n|^2\,\mathrm{d}x &= -\int_\Omega (\operatorname{div}(\varphi_n\mathbf{v}_n) - \Gamma_{\varphi,n})\partial_t\varphi_n\,\mathrm{d}x \\ &\quad + \chi\int_\Omega \nabla\sigma_n\cdot\nabla\partial_t\varphi_n\,\mathrm{d}x + \int_\Omega \Delta\psi'(\varphi_n)\partial_t\varphi_n\,\mathrm{d}x. \end{aligned}\tag{6.33}$$

Due to (6.30), it is straightforward to show that

$$\left|\int_\Omega (-\Gamma_{\mathbf{v},n}\varphi_n + \Gamma_{\varphi,n})\partial_t\varphi_n\,\mathrm{d}x\right| \le C + \frac{1}{8}\|\partial_t\varphi_n\|_{L^2}^2. \tag{6.34}$$

Gagliardo–Nirenberg's inequality in 2D and elliptic regularity theory guarantee that

$$\begin{aligned}\|\nabla\varphi_n\|_{\mathbf{L}^4} &\le C\|\nabla\varphi_n\|_{\mathbf{L}^2}^{\frac12}\|\nabla\varphi_n\|_{\mathbf{H}^1}^{\frac12} \le C\left(\|\varphi_n\|_{H^1} + \|\varphi_n\|_{H^1}^{\frac12}\|\Delta\varphi_n\|_{L^2}^{\frac12}\right),\\ \|\varphi_n\|_{L^\infty} &\le C\|\varphi_n\|_{H^3}^{\frac14}\|\varphi_n\|_{L^3}^{\frac34}.\end{aligned}$$

Therefore, the assumptions on $\psi(\cdot)$, the Sobolev embedding $H^1 \subset L^3$ and (6.30) imply

$$\begin{aligned}\|\Delta(\psi'(\varphi_n))\|_{L^2}^2 &= \int_\Omega |\psi'''(\varphi_n)|^2|\nabla\varphi_n|^4 + |\psi''(\varphi_n)|^2|\Delta\varphi_n|^2\,\mathrm{d}x\\ &\le C\int_\Omega (1+|\varphi_n|^6)|\nabla\varphi_n|^4 + (1+|\varphi_n|^8)|\Delta\varphi_n|^2\,\mathrm{d}x\\ &\le C\left(1+\|\varphi_n\|_{L^\infty}^6\right)\|\nabla\varphi_n\|_{\mathbf{L}^4}^4 + C\left(1+\|\varphi_n\|_{L^\infty}^8\right)\|\Delta\varphi_n\|_{L^2}^2\\ &\le C\left(1+\|\varphi_n\|_{H^3}^2\right)\left(1+\|\Delta\varphi_n\|_{L^2}^2\right).\end{aligned}$$

Consequently, we deduce

$$\left|\int_\Omega \Delta(\psi'(\varphi_n))\partial_t\varphi_n\,\mathrm{d}x\right| \le C\left(1+\|\varphi_n\|_{H^3}^2\right)\left(1+\|\Delta\varphi_n\|_{L^2}^2\right) + \frac{1}{8}\|\partial_t\varphi_n\|^2. \tag{6.35}$$

Moreover, we observe that

$$\chi\int_\Omega \nabla\sigma_n\cdot\nabla\partial_t\varphi_n\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\chi\int_\Omega \nabla\sigma_n\cdot\nabla\varphi_n\,\mathrm{d}x - \int_\Omega \nabla\partial_t\sigma_n\cdot\nabla\varphi\,\mathrm{d}x \quad \text{for a.e. } t\in(0,T). \tag{6.36}$$

We now analyse the remaining term on the r. h. s. of (6.33). Using Gagliardo–Nirenberg's inequality in 2D, we have

$$\|\nabla\varphi_n\|_{\mathbf{L}^\infty} \le C\|\nabla\varphi_n\|_{\mathbf{L}^2}^{\frac12}\|\nabla\varphi_n\|_{\mathbf{H}^2}^{\frac12} \le C\|\varphi_n\|_{H^1}^{\frac12}\|\varphi_n\|_{H^3}^{\frac12}.$$

Together with (6.30)-(6.31), this entails that

$$\begin{aligned}\|\nabla\varphi_n\cdot\mathbf{v}_n\|_{L^2}^2 &\le \|\mathbf{v}_n\|_{\mathbf{L}^2}^2\|\nabla\varphi_n\|_{\mathbf{L}^\infty}^2\\ &\le C\|\mathbf{v}_n\|_{\mathbf{L}^2}^2\|\varphi_n\|_{H^1}\|\varphi_n\|_{H^3}\\ &\le C\left(1+\|p_n\|_{L^2} + \|(\mu_n+\chi\sigma_n)\nabla\varphi_n\|_{\mathbf{L}^2}^2\right)\|\varphi_n\|_{H^3}.\end{aligned}\tag{6.37}$$

Using (6.30), elliptic regularity theory and Gagliardo–Nirenberg's inequality, we obtain

$$\begin{aligned}
&\|(\mu_n + \chi\sigma_n)\nabla\varphi_n\|_{\mathbf{L}^2}^2 \\
&\quad \le \|\mu_n + \chi\sigma_n\|_{L^4}^2 \|\nabla\varphi_n\|_{\mathbf{L}^4}^2 \\
&\quad \le C\left(\|\mu_n + \chi\sigma_n\|_{L^2}^2 + \|\mu_n + \chi\sigma_n\|_{L^2}\|\nabla(\mu_n + \chi\sigma_n)\|_{\mathbf{L}^2}\right)\|\varphi_n\|_{H^1}(\|\varphi_n\|_{L^2} + \|\Delta\varphi_n\|_{L^2}) \\
&\quad \le C\left(\|\mu_n + \chi\sigma_n\|_{L^2}^2 + \|\mu_n + \chi\sigma_n\|_{L^2}\|\nabla(\mu_n + \chi\sigma_n)\|_{\mathbf{L}^2}\right)(1 + \|\Delta\varphi_n\|_{L^2}).
\end{aligned}$$

Furthermore, using (5.5c), the assumptions on $\psi(\cdot)$ and (6.30) gives

$$\begin{aligned}
\|\mu_n + \chi\sigma_n\|_{L^2} &\le C\left(\|\Delta\varphi_n\|_{L^2} + \|\psi'(\varphi_n)\|_{L^2}\right) \le C\left(1 + \|\Delta\varphi_n\|_{L^2}\right), \\
\|\nabla(\mu_n + \chi\sigma_n)\|_{\mathbf{L}^2} &\le C\left(1 + \|\varphi_n\|_{H^3}\right).
\end{aligned}$$

The last three inequalities imply that

$$\|(\mu_n + \chi\sigma_n)\nabla\varphi_n\|_{\mathbf{L}^2}^2 \le C\left(1 + \|\varphi_n\|_{H^3}\right)\left(1 + \|\Delta\varphi_n\|_{L^2}^2\right).$$

Employing this inequality in (6.37) yields

$$\|\nabla\varphi_n \cdot \mathbf{v}_n\|_{L^2}^2 \le C\left(1 + \|\varphi_n\|_{H^3}^2 + \|p_n\|_{L^2}^2\right)\left(1 + \|\Delta\varphi_n\|_{L^2}^2\right).$$

Consequently, we have

$$\begin{aligned}
\left|\int_\Omega \nabla\varphi_n \cdot \mathbf{v}_n \partial_t\varphi_n \,\mathrm{d}x\right| &\le C\|\nabla\varphi_n \cdot \mathbf{v}_n\|_{L^2}^2 \\
&\le C\left(1 + \|\varphi_n\|_{H^3}^2 + \|p_n\|_{L^2}^2\right)\left(1 + \|\Delta\varphi_n\|_{L^2}^2\right) + \frac{1}{8}\|\partial_t\varphi_n\|_{L^2}^2. \qquad (6.38)
\end{aligned}$$

Invoking (6.34)-(6.36), (6.38) in (6.33) and using (6.30) gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Delta\varphi_n\|_{L^2}^2 + \frac{5}{8}\|\partial_t\varphi_n\|_{L^2}^2 \le{}& \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \chi\nabla\sigma_n \cdot \nabla\varphi_n \,\mathrm{d}x - \int_\Omega \chi\nabla\partial_t\sigma_n \cdot \nabla\varphi \,\mathrm{d}x \\
&+ \alpha_n(t)\left(1 + \|\Delta\varphi_n(t)\|_{L^2}^2\right),
\end{aligned}$$

where

$$\alpha_n(t) = C\left(1 + \|\varphi_n\|_{H^3}^2 + \|p_n\|_{L^2}^2\right).$$

Integrating this inequality in time from 0 to $s \in (0,T]$, we obtain

$$\begin{aligned}
\frac{1}{2}\|\Delta\varphi_n(s)\|_{L^2}^2 + \frac{5}{8}\|\partial_t\varphi_n\|_{L^2(0,s;L^2)}^2 \le{}& \frac{1}{2}\|\Delta\varphi_0\|_{L^2}^2 + \int_0^s \alpha_n(t)\left(1 + \|\Delta\varphi_n(t)\|_{L^2}^2\right)\,\mathrm{d}t \\
&+ \chi\int_\Omega \nabla\sigma_n(s)\cdot\nabla\varphi_n(s)\,\mathrm{d}x - \chi\int_\Omega \nabla\sigma_n(0)\cdot\nabla\varphi_0\,\mathrm{d}x \\
&- \chi\int_0^s\int_\Omega \nabla\partial_t\sigma_n \cdot \nabla\varphi_n \,\mathrm{d}x\,\mathrm{d}t. \qquad (6.39)
\end{aligned}$$

Now, using (6.30), (6.32) and $\varphi_0 \in H_N^2$ yields

$$\begin{aligned}
\left|\chi\int_\Omega \nabla\sigma_n(0)\cdot\nabla\varphi_0\,\mathrm{d}x\right| &\le \|\sigma_n\|_{C^0([0,s];H^1)}\|\nabla\varphi_0\|_{\mathbf{L}^2} \le C, \\
\left|\chi\int_\Omega \nabla\sigma_n(s)\cdot\nabla\varphi_n(s)\,\mathrm{d}x\right| &\le \|\sigma_n\|_{C^0([0,s];H^1)}\|\nabla\varphi_n(s)\|_{\mathbf{L}^2} \le C + \frac{1}{4}\|\Delta\varphi_n(s)\|_{L^2}^2, \\
\left|\chi\int_0^s\int_\Omega \nabla\partial_t\sigma_n\cdot\nabla\varphi_n\,\mathrm{d}x\,\mathrm{d}t\right| &\le C\int_0^s \|\nabla\partial_t\sigma_n\|_{\mathbf{L}^2}\|\nabla\varphi_n\|_{\mathbf{L}^2}\,\mathrm{d}t \le C.
\end{aligned}$$

Recalling (6.30), $\varphi_0 \in H^2_N$, and using the last three inequalities in (6.39), we deduce that

$$\frac{1}{4}\|\Delta\varphi_n(s)\|^2_{L^2} + \frac{1}{2}\|\partial_t\varphi_n\|^2_{L^2(0,s;L^2)} \leq C + \int_0^s \alpha_n(t)\left(1 + \|\Delta\varphi_n(t)\|^2_{L^2}\right)\,\mathrm{d}t,$$

where $\alpha_n \in L^1(0,T)$ due to (6.30). Therefore, using elliptic regularity theory and (6.30), an application of Gronwall's lemma leads to

$$\|\varphi_n\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^3)} \leq C$$

with a constant C independent of $n \in \mathbb{N}$. Then, recalling (6.30)-(6.32), applying the continuous embedding $L^\infty(H^2) \cap L^2(H^3) \hookrightarrow L^4(W^{1,\infty})$ and using the relation (5.5c) for μ_n, we obtain

$$\|\mu_n\|_{L^\infty(L^2)} + \|\sigma_n\|_{H^1(H^1)\cap C^0(H^1)} + \|\mathbf{v}_n\|_{L^4(\mathbf{L}^2)} \leq C.$$

Invoking the last two inequalities along with (6.30), this in particular yields

$$\|\mathrm{div}(\varphi_n\mathbf{v}_n)\|_{L^2(L^2)} \leq C.$$

Hence, from the equation (5.5b) for $\Delta\mu_n$ and using elliptic regularity theory again, we obtain

$$\|\mu_n\|_{L^2(H^2)} \leq C.$$

From the last four inequalities and (6.30), we obtain that

$$\begin{aligned}&\|\varphi_n\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^3)} + \|\sigma_n\|_{H^1(H^1)\cap L^\infty(H^2)} + \|\mu_n\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\mathrm{div}(\varphi_n\mathbf{v}_n)\|_{L^2(L^2)}\\ &\quad + \|\mathbf{v}_n\|_{L^4(\mathbf{L}^2_{\mathrm{div}}(\Omega))} + \|\sqrt{\eta_n(\varphi_n)}\mathbf{D}\mathbf{v}_n\|_{L^2(\mathbf{L}^2)} + \|\mathrm{div}(\mathbf{v}_n)\|_{L^\infty(\Omega_T)} + \|p_n\|_{L^2(L^2)} \leq C. \qquad (6.40)\end{aligned}$$

Step 2: With similar arguments as in the three-dimensional case, we can pass to the limit $n \to \infty$ in (5.5) to deduce the existence of a solution quintuple $(\varphi, \mu, \sigma, \mathbf{v}, p)$ solving

$$\begin{aligned}\mathrm{div}(\mathbf{v}) &= \Gamma_{\mathbf{v}}(\varphi,\sigma) && \text{a.e. in } \Omega_T, && (6.41a)\\ \nu(\varphi)\mathbf{v} &= -\nabla p + (\mu + \chi\sigma)\nabla\varphi && \text{a.e. in } \Omega_T, && (6.41b)\\ \partial_t\varphi + \mathrm{div}(\varphi\mathbf{v}) &= \Delta\mu + \Gamma_\varphi(\varphi,\sigma) && \text{a.e. in } \Omega_T, && (6.41c)\\ \mu &= \psi'(\varphi) - \Delta\varphi - \chi\sigma && \text{a.e. in } \Omega_T, && (6.41d)\\ 0 &= \Delta\sigma - h(\varphi)\sigma && \text{a.e. in } \Omega_T, && (6.41e)\end{aligned}$$

and

$$p = \nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} = 0 \quad \text{a.e. on } \Sigma_T, \qquad \nabla\sigma\cdot\mathbf{n} = K(\sigma_\infty - \sigma) \quad \text{a.e. on } \Sigma_T. \qquad (6.41f)$$

Following the arguments in the proof of Theorem 5.11, we deduce that

$$\|\varphi\|_{L^2(H^4)} \leq C.$$

Furthermore, the following estimates hold (compare (6.28))

$$\begin{aligned}&\|\varphi\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^4)} + \|\sigma\|_{H^1(H^1)\cap L^\infty(H^2)} + \|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)}\\ &\quad + \|\mathbf{v}\|_{L^4(\mathbf{L}^2_{\mathrm{div}}(\Omega))} + \|\mathrm{div}(\mathbf{v})\|_{L^\infty(\Omega_T)} + \|p\|_{L^2(H^1)} \leq C. \qquad (6.42)\end{aligned}$$

Step 3: Due to (6.41b) and (6.41f), we observe that p is for almost every $t \in (0,T)$ a weak solution of

$$\begin{aligned}-\Delta p &= \nu'(\varphi)\nabla\varphi\cdot\mathbf{v} + \nu(\varphi)\mathrm{div}(\mathbf{v}) - \mathrm{div}((\mu+\chi\sigma)\nabla\varphi) && \text{a.e. in } \Omega,\\ p &= 0 && \text{a.e. on } \partial\Omega.\end{aligned}$$

Using elliptic regularity theory and the assumptions on $\nu(\cdot)$, this implies

$$\|p\|_{H^2}^2 \leq C\left(\|\nabla\varphi\|_{\mathbf{L}^\infty}^2\|\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\mathrm{div}(\mathbf{v})\|_{L^2}^2 + \|\mathrm{div}((\mu+\chi\sigma)\nabla\varphi)\|_{L^2}^2 + \|p\|_{L^2}^2\right).$$

Integrating this inequality in time from 0 to T, using (6.42), Gagliardo–Nirenberg's inequality and the continuous embeddings $H^2 \hookrightarrow L^\infty$, $H^1 \hookrightarrow L^4$, $L^\infty(H^2)\cap L^2(H^3) \hookrightarrow L^4(W^{1,\infty})$, we obtain

$$\begin{aligned}\int_0^T \|p\|_{H^2}^2 \,\mathrm{d}t &\leq C\int_0^T \|\nabla\varphi\|_{\mathbf{L}^\infty}^2\|\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\mathrm{div}(\mathbf{v})\|_{L^2}^2 + \|\mathrm{div}((\mu+\chi\sigma)\nabla\varphi)\|_{L^2}^2 + \|p\|_{L^2}^2 \,\mathrm{d}t\\ &\leq C\int_0^T \|\nabla\varphi\|_{\mathbf{L}^\infty}^2\|\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\mathrm{div}(\mathbf{v})\|_{L^2}^2 + \|\mu+\chi\sigma\|_{H^2}^2 + \|p\|_{L^2}^2 \,\mathrm{d}t\\ &\leq C\left(\|\nabla\varphi\|_{L^4(\mathbf{L}^\infty)}^2\|\mathbf{v}\|_{L^4(\mathbf{L}^2)}^2 + \|\mathrm{div}(\mathbf{v})\|_{L^2(L^2)}^2 + \|\mu+\chi\sigma\|_{L^2(H^2)}^2 + \|p\|_{L^2(L^2)}^2\right)\\ &\leq C,\end{aligned}$$

and consequently

$$\|p\|_{L^2(H^2)} \leq C. \tag{6.43}$$

By the non-negativity of $\nu(\cdot)$ and (6.41b) we obtain

$$\mathbf{v} = \frac{1}{\nu(\varphi)}\left(-\nabla p + (\mu+\chi\sigma)\nabla\varphi\right) \quad \text{a.e. in } \Omega_T.$$

Using the assumptions on $\nu(\cdot)$, (6.42) and (6.43), it follows that $\nu(\varphi)^{-1}\left(-\nabla p + (\mu+\chi\sigma)\nabla\varphi\right)$ is bounded in $L^2(\mathbf{H}^1)$. Therefore, we have

$$\|\mathbf{v}\|_{L^2(\mathbf{H}^1)} \leq C.$$

In conjunction with (6.42)-(6.43), this implies

$$\begin{aligned}&\|\varphi\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^4)} + \|\sigma\|_{H^1(H^1)\cap L^\infty(H^2)} + \|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)}\\ &\quad + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)\cap L^4(\mathbf{L}^2_{\mathrm{div}}(\Omega))} + \|p\|_{L^2(H^2)} \leq C\end{aligned}$$

which completes the proof. □

6.3.2 Uniqueness of strong solutions for the Cahn–Hilliard–Darcy model

We have the following result concerning continuous dependence of strong solutions for the Cahn–Hilliard–Darcy system:

Theorem 6.6 (Uniqueness of strong solutions) *Let $(\varphi_i, \mu_i, \sigma_i, \mathbf{v}_i, p_i)$, $i = 1, 2$, be two strong solutions of* (6.6)-(6.7) *in the sense of Definition 6.4 corresponding to initial data $\varphi_{i,0} \in H^2_N$, $i = 1, 2$, and boundary data $\sigma_{i,\infty} \in H^1(H^{\frac{1}{2}}(\partial\Omega))$, $i = 1, 2$. Furthermore, assume that the assumptions of Theorem 6.5 and Assumptions 5.6 hold. Then, the estimate*

$$\begin{aligned}&\sup_{s\in[0,T]} \|\varphi_1(s)-\varphi_2(s)\|_{H^1}^2 + \|\varphi_1-\varphi_2\|_{H^1(0,T;(H^1)^*)\cap L^2(0,T;H^3)}^2 + \|\mu_1-\mu_2\|_{L^2(0,T;H^1)}^2\\ &\quad + \|\sigma_1-\sigma_2\|_{L^2(0,T;H^1)}^2 + \|\mathbf{v}_1-\mathbf{v}_2\|_{L^2(0,T;\mathbf{L}^2)}^2 + \|p_1-p_2\|_{L^2(0,T;H^1)}^2\\ &\leq C\left(\|\varphi_{1,0}-\varphi_{2,0}\|_{H^1}^2 + \|\sigma_{1,\infty}-\sigma_{2,\infty}\|_{L^2(0,T;L^2(\partial\Omega))}^2\right)\end{aligned} \tag{6.44}$$

holds for a positive constant C depending on Ω, T, ε, χ, L_h, L_b, L_f, L_ν, K, k_1, k_2, R_1, R_2, R_3, ρ, ν_0, ν_1, $\|\varphi_i\|_{L^\infty(H^2)\cap L^2(H^3)}$, $\|\mu_i\|_{L^2(H^2)}$, $\|\sigma_i\|_{L^\infty(H^2)}$, $\|\mathbf{v}_2\|_{L^2(\mathbf{H}^1)}$, $\|b_{\mathbf{v}}(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|f_{\mathbf{v}}(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|b_\varphi(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|f_\varphi(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|h(\cdot)\|_{L^\infty(\mathbb{R})}$.
In particular, if $\varphi_{1,0} = \varphi_{2,0}$ and $\sigma_{1,\infty} = \sigma_{2,\infty}$, strong solutions of (6.6)-(6.7) *are unique.*

Proof. Since it has no bearing on the analysis, we set $\varepsilon = 1$ in the following. By C we denote a generic constant that may depend on the same quantities as stated in the theorem. We will frequently use Hölder's and Young's inequalities and we recall the following estimate holding for $i = 1, 2$:

$$\begin{aligned}&\|\varphi_i\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^4)} + \|\sigma_i\|_{H^1(H^1)\cap L^\infty(H^2)} + \|\mu_i\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\operatorname{div}(\varphi_i \mathbf{v}_i)\|_{L^2(L^2)} \\ &\quad + \|\mathbf{v}_i\|_{L^2(\mathbf{H}^1)\cap L^4(\mathbf{L}^2_{\mathrm{div}})} + \|p_i\|_{L^2(H^2)} \leq C. \end{aligned} \tag{6.45}$$

In the following we denote $\Gamma_\varphi(\varphi_i, \sigma_i) \coloneqq \Gamma_{\varphi,i}$, $\Gamma_{\mathbf{v}}(\varphi_i, \sigma_i) \coloneqq \Gamma_{\mathbf{v},i}$, $i = 1, 2$, and $\sigma_\infty \coloneqq \sigma_{1,\infty} - \sigma_{2,\infty}$. Then, the differences $f \coloneqq f_1 - f_2$, $f_i \in \{\varphi_i, \mu_i, \sigma_i, \mathbf{v}_i, p_i\}$, $i = 1, 2$, satisfy the following equations almost everywhere in Ω_T:

$$\operatorname{div}(\mathbf{v}) = \Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}, \tag{6.46a}$$
$$\nu(\varphi_1)\mathbf{v} = -\nabla p + (\mu + \chi\sigma)\nabla\varphi_1 + (\mu_2 + \chi\sigma_2)\nabla\varphi + (\nu(\varphi_2) - \nu(\varphi_1))\mathbf{v}_2, \tag{6.46b}$$
$$-\Delta p = \operatorname{div}\left(\nu(\varphi_1)\mathbf{v} - (\mu + \chi\sigma)\nabla\varphi_1 - (\mu_2 + \chi\sigma_2)\nabla\varphi + (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2\right), \tag{6.46c}$$
$$\partial_t\varphi = \Delta\mu + (\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}) - \varphi\Gamma_{\mathbf{v},1} - \nabla\varphi_1 \cdot \mathbf{v} - \nabla\varphi \cdot \mathbf{v}_2, \tag{6.46d}$$
$$\mu = \psi'(\varphi_1) - \psi'(\varphi_2) - \Delta\varphi - \chi\sigma, \tag{6.46e}$$
$$0 = \Delta\sigma - h(\varphi_1)\sigma - (h(\varphi_1) - h(\varphi_2))\sigma_2. \tag{6.46f}$$

Furthermore, the boundary and initial conditions are given by

$$p = 0, \quad \nabla\varphi \cdot \mathbf{n} = \nabla\mu \cdot \mathbf{n} = 0, \quad \nabla\sigma \cdot \mathbf{n} = K(\sigma_\infty - \sigma) \quad \text{a.e. on } \Sigma_T, \tag{6.47a}$$
$$\varphi(0) = \varphi_{1,0} - \varphi_{2,0} \quad \text{a.e. in } \Omega. \tag{6.47b}$$

We divide the analysis into several steps.

Step 1: Using exactly the same arguments as for (5.38) and (5.41), we have

$$\|\sigma\|_{H^1} \leq C\left(\|\varphi\|_{L^2} + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right), \tag{6.48a}$$
$$\|\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}\|_{L^2} \leq C\left(\|\varphi\|_{L^2} + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right), \tag{6.48b}$$
$$\|\Gamma_{\varphi,1} - \Gamma_{\varphi,2}\|_{L^2} \leq C\left(\|\varphi\|_{L^2} + \|\sigma_\infty\|_{L^2(\partial\Omega)}\right). \tag{6.48c}$$

Multiplying (6.46d) by $-\Delta\varphi$, integrating over Ω and by parts and using (6.46e), (6.47a), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\varphi\|^2_{\mathbf{L}^2} + \|\nabla\Delta\varphi\|^2_{\mathbf{L}^2} &= \int_\Omega (\nabla\varphi_1 \cdot \mathbf{v} + \nabla\varphi \cdot \mathbf{v}_2)\Delta\varphi + \nabla(\psi'(\varphi_1) - \psi'(\varphi_2) - \chi\sigma) \cdot \nabla\Delta\varphi \,\mathrm{d}x \\ &\quad - \int_\Omega \big((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}) - \varphi\Gamma_{\mathbf{v},1}\big)\Delta\varphi \,\mathrm{d}x.\end{aligned}$$

Furthermore, multiplying (6.46b) with $\mathbf{v}$, integrating over Ω and by parts and applying (6.46e), (6.47a), we get

$$\begin{aligned}\nu_0\|\mathbf{v}\|^2_{\mathbf{L}^2} &\leq \int_\Omega (\mu_2 + \chi\sigma_2)\nabla\varphi \cdot \mathbf{v} + p\operatorname{div}(\mathbf{v}) \,\mathrm{d}x + \int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2) - \Delta\varphi)\nabla\varphi_1 \cdot \mathbf{v} \,\mathrm{d}x \\ &\quad + \int_\Omega (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2 \cdot \mathbf{v} \,\mathrm{d}x,\end{aligned}$$

where we used the assumptions on $\nu(\cdot)$. Summing up the last two (in)equalities, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2 \\
&\quad\leq \int_\Omega (\nabla\varphi\cdot\mathbf{v}_2 + \varphi\Gamma_{\mathbf{v},1})\Delta\varphi \,\mathrm{d}x + \int_\Omega \nabla(\psi'(\varphi_1) - \psi'(\varphi_2) - \chi\sigma)\cdot\nabla\Delta\varphi \,\mathrm{d}x \\
&\qquad - \int_\Omega \big((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2})\big)\Delta\varphi \,\mathrm{d}x + \int_\Omega (\mu_2 + \chi\sigma_2)\nabla\varphi\cdot\mathbf{v} + p\,\mathrm{div}(\mathbf{v}) \,\mathrm{d}x \\
&\qquad + \int_\Omega (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2\cdot\mathbf{v} \,\mathrm{d}x + \int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2))\nabla\varphi_1\cdot\mathbf{v} \,\mathrm{d}x. \qquad (6.49)
\end{aligned}$$

We now estimate the terms on the r. h. s. individually. Using Gagliardo–Nirenberg's inequality in 2D, we obtain

$$\|\Delta\varphi\|_{L^4}^2 \leq C\|\Delta\varphi\|_{L^2}\|\Delta\varphi\|_{H^1} \leq C\|\Delta\varphi\|_{L^2}\left(\|\Delta\varphi\|_{L^2} + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}\right) \leq C\left(\|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2\right),$$

which implies

$$\left|\int_\Omega \nabla\varphi\cdot\mathbf{v}_2\Delta\varphi \,\mathrm{d}x\right| \leq \|\nabla\varphi\|_{\mathbf{L}^2}\|\mathbf{v}_2\|_{\mathbf{L}^4}\|\Delta\varphi\|_{L^4} \leq C\|\nabla\varphi\|_{\mathbf{L}^2}^2\|\mathbf{v}_2\|_{\mathbf{L}^4}^2 + \frac{1}{8}\left(\|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2\right).$$

Using the assumptions on Γ_φ and $\Gamma_{\mathbf{v}}$ along with (6.45) and (6.48a)-(6.48c), it holds that

$$\left|\int_\Omega \big((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}) - \varphi\Gamma_{\mathbf{v},1}\big)\Delta\varphi \,\mathrm{d}x\right| \leq C\left(\|\varphi\|_{L^2}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right) + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2.$$

Using the assumptions on $\nu(\cdot)$, applying (6.48a) and the Sobolev embeddings $H^2 \subset L^\infty$, $H^1 \subset L^4$, a straightforward calculation shows that

$$\begin{aligned}
&\left|\int_\Omega (\mu_2 + \chi\sigma_2)\nabla\varphi\cdot\mathbf{v} + (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2\cdot\mathbf{v} - \chi\nabla\sigma\cdot\nabla\Delta\varphi \,\mathrm{d}x\right| \\
&\quad\leq C\left(\|\mu_2 + \chi\sigma_2\|_{H^2}^2 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2 + C\left(\|\varphi\|_{L^2}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right) + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \frac{\nu_0}{4}\|\mathbf{v}\|_{\mathbf{L}^2}^2.
\end{aligned}$$

From the assumptions on $\psi(\cdot)$, the Sobolev embedding $H^2 \subset L^\infty$ and (6.45), we obtain

$$\begin{aligned}
\left|\int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2))\nabla\varphi_1\cdot\mathbf{v} \,\mathrm{d}x\right| &\leq C\int_\Omega (1 + |\varphi_1|^4 + |\varphi_2|^4)|\varphi||\nabla\varphi_1||\mathbf{v}| \,\mathrm{d}x \\
&\leq C\|\varphi_1\|_{H^3}^2\|\varphi\|_{L^2}^2 + \frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2.
\end{aligned}$$

Furthermore, the assumptions on $\psi(\cdot)$ and (6.45) imply

$$\begin{aligned}
\left|\int_\Omega \nabla(\psi'(\varphi_1) - \psi'(\varphi_2))\cdot\nabla\Delta\varphi \,\mathrm{d}x\right| &= \left|\int_\Omega (\psi''(\varphi_1)\nabla\varphi + (\psi''(\varphi_1) - \psi''(\varphi_2))\nabla\varphi_2)\cdot\nabla\Delta\varphi \,\mathrm{d}x\right| \\
&\leq C\int_\Omega |\nabla\varphi||\nabla\Delta\varphi| + (1 + |\varphi_1|^3 + |\varphi_2|^3)|\varphi||\nabla\varphi_2||\nabla\Delta\varphi| \,\mathrm{d}x \\
&\leq C\left(\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\varphi_2\|_{H^3}^2\|\varphi\|_{L^2}^2\right) + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2.
\end{aligned}$$

Finally, using (6.48b) we obtain

$$\left|\int_\Omega p\mathrm{div}(\mathbf{v}) \,\mathrm{d}x\right| \leq \delta\|p\|_{L^2}^2 + C_\delta\left(\|\varphi\|_{L^2}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right)$$

for $\delta > 0$ to be chosen. Employing the last six inequalities in (6.49), we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \frac{3}{4}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \frac{5\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2 \\
&\quad\leq C_\delta\left(1 + \|\varphi_1\|_{H^3}^2 + \|\varphi_2\|_{H^3}^2 + \|\mu_2 + \chi\sigma_2\|_{H^2}^2 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2 \\
&\qquad + C_\delta\|\sigma_\infty\|_{L^2(\partial\Omega)}^2 + \frac{1}{4}\|\Delta\varphi\|_{L^2}^2 + \delta\|p\|_{L^2}^2 \qquad (6.50)
\end{aligned}$$

with $\delta > 0$ to be chosen.

Step 2: Multiplying (6.46d) with φ, integrating over Ω and by parts and using (6.46e), (6.47a), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2 = \int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2) - \chi\sigma)\,\Delta\varphi \,\mathrm{d}x - \int_\Omega (\nabla\varphi_1 \cdot \mathbf{v} + \nabla\varphi\cdot\mathbf{v}_2)\,\varphi\,\mathrm{d}x$$
$$+ \int_\Omega \big((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}) - \varphi\Gamma_{\mathbf{v},1}\big)\,\varphi\,\mathrm{d}x. \qquad (6.51)$$

On account of Young's and Gagliardo–Nirenberg's inequality, we observe that

$$\|\varphi\|_{L^4}^2 \le C\|\varphi\|_{L^2}\|\varphi\|_{H^1} \le C\|\varphi\|_{L^2}\left(\|\varphi\|_{L^2} + \|\nabla\varphi\|_{\mathbf{L}^2}\right) \le C\left(\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{\mathbf{L}^2}^2\right).$$

Then, by the Sobolev embeddings $H^2 \subset L^\infty$ and $\mathbf{H}^1 \subset \mathbf{L}^4$, we infer that

$$\left|\int_\Omega (\nabla\varphi_1\cdot\mathbf{v} + \nabla\varphi\cdot\mathbf{v}_2)\varphi\,\mathrm{d}x\right| \le C\left(1 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2 + \|\varphi_1\|_{H^3}^2\right)\|\varphi\|_{H^1}^2 + \frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2.$$

Using (6.45) and (6.48b)-(6.48c), it is straightforward to show that

$$\left|\int_\Omega \big((\Gamma_{\varphi,1} - \Gamma_{\varphi,2}) - \varphi_2(\Gamma_{\mathbf{v},1} - \Gamma_{\mathbf{v},2}) - \varphi\Gamma_{\mathbf{v},1}\big)\,\varphi\,\mathrm{d}x\right| \le C\left(\|\varphi\|_{L^2}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right).$$

Finally, invoking the assumptions on $\psi(\cdot)$ together with (6.45) and (6.48c) gives

$$\left|\int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2) - \chi\sigma)\,\Delta\varphi\,\mathrm{d}x\right| \le C\int_\Omega \big((1 + |\varphi_1|^4 + |\varphi_2|^4)|\varphi| + |\sigma|\big)|\Delta\varphi|\,\mathrm{d}x$$
$$\le C\left(\|\varphi\|_{L^2}^2 + \|\sigma_\infty\|_{L^2(\partial\Omega)}^2\right) + \frac{1}{4}\|\Delta\varphi\|_{L^2}^2.$$

Employing the last three inequalities in (6.51) and recalling (6.50) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\left(\|\varphi\|_{L^2}^2 + \|\nabla\varphi\|_{\mathbf{L}^2}^2\right) + \frac{1}{2}\left(\|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2\right)$$
$$\le C_\delta\left(1 + \|\varphi_1\|_{H^3}^2 + \|\varphi_2\|_{H^3}^2 + \|\mu_2 + \chi\sigma_2\|_{H^2}^2 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2$$
$$+ C_\delta\|\sigma_\infty\|_{L^2(\partial\Omega)}^2 + \delta\|p\|_{L^2}^2 \qquad (6.52)$$

with $\delta > 0$ to be chosen. It remains to estimate the last term on the r. h. s. of (6.52).

Step 3: Multiplying (6.46c) with p, integrating over Ω and by parts and using (6.47a), we obtain

$$\|\nabla p\|_{\mathbf{L}^2}^2 = \int_\Omega ((\mu + \chi\sigma)\nabla\varphi_1 + (\mu_2 + \chi\sigma_2)\nabla\varphi - \nu(\varphi_1)\mathbf{v} - (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2)\cdot\nabla p\,\mathrm{d}x. \quad (6.53)$$

Since $\nabla\varphi\cdot\mathbf{n} = 0$ a. e. on Σ_T, integrating by parts gives

$$\|\Delta\varphi\|_{L^2}^2 = \int_\Omega |\Delta\varphi|^2\,\mathrm{d}x = -\int_\Omega \nabla\varphi\cdot\nabla\Delta\varphi\,\mathrm{d}x \le \|\nabla\varphi\|_{\mathbf{L}^2}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}.$$

Furthermore, using Gagliardo–Nirenberg's inequality with $j = 1$, $p = q = \infty$, $m = 3$, $r = 2$ yields

$$\|\nabla\varphi_1\|_{\mathbf{L}^\infty} \le C\|\varphi_1\|_{H^3}^{\frac{1}{2}}\|\varphi_1\|_{L^\infty}^{\frac{1}{2}}.$$

Combining the last two inequalities with (6.45), this implies

$$\left|\int_\Omega \Delta\varphi\nabla\varphi_1\cdot\nabla p\,\mathrm{d}x\right| \le \|\Delta\varphi\|_{L^2}\|\nabla\varphi_1\|_{\mathbf{L}^\infty}\|\nabla p\|_{\mathbf{L}^2}$$
$$\le C\|\varphi_1\|_{H^3}\|\nabla\varphi\|_{\mathbf{L}^2}\|\nabla\Delta\varphi\|_{\mathbf{L}^2} + \frac{1}{8}\|\nabla p\|_{\mathbf{L}^2}^2$$
$$\le C\left(\|\varphi_1\|_{H^3}^2\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2\right) + \frac{1}{8}\|\nabla p\|_{\mathbf{L}^2}^2.$$

Invoking the assumptions on $\psi(\cdot)$ and arguing similar as above, we deduce that

$$\left|\int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2))\nabla\varphi_1 \cdot \nabla p \,\mathrm{d}x\right| \leq C\|\varphi_1\|_{H^3}^2 \|\varphi\|_{L^2}^2 + \frac{1}{8}\|\nabla p\|_{\mathbf{L}^2}^2.$$

Employing the last two estimates and the relation (6.46e) for $\mu + \chi\sigma$, we obtain

$$\begin{aligned}\left|\int_\Omega (\mu + \chi\sigma)\nabla\varphi_1 \cdot \nabla p \,\mathrm{d}x\right| &= \left|\int_\Omega (\psi'(\varphi_1) - \psi'(\varphi_2) - \Delta\varphi)\nabla\varphi_1 \cdot \nabla p \,\mathrm{d}x\right| \\ &\leq C\left(\|\varphi_1\|_{H^3}^2 \|\varphi\|_{H^1}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2\right) + \frac{1}{4}\|\nabla p\|_{\mathbf{L}^2}^2.\end{aligned}$$

Due to the Sobolev embedding $H^2 \subset L^\infty$, we infer that

$$\left|\int_\Omega (\mu_2 + \chi\sigma_2)\nabla\varphi \cdot \nabla p \,\mathrm{d}x\right| \leq C\|\mu_2 + \chi\sigma_2\|_{H^2}^2 \|\nabla\varphi\|_{\mathbf{L}^2}^2 + \frac{1}{8}\|\nabla p\|_{\mathbf{L}^2}^2.$$

Using the assumptions on $\nu(\cdot)$ and the Sobolev embedding $H^1 \subset L^4$, we get

$$\left|\int_\Omega \left(\nu(\varphi_1)\mathbf{v} + (\nu(\varphi_1) - \nu(\varphi_2))\mathbf{v}_2\right) \nabla p \,\mathrm{d}x\right| \leq C\left(\|\mathbf{v}\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2 \|\varphi\|_{H^1}^2\right) + \frac{1}{8}\|\nabla p\|_{\mathbf{L}^2}^2.$$

Employing the last three inequalities in (6.53) and using Poincaré's inequality we obtain

$$\|p\|_{H^1}^2 \leq C\left(1 + \|\varphi_1\|_{H^3}^2 + \|\mu_2 + \chi\sigma_2\|_{H^2}^2 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2 + C\left(\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \|\mathbf{v}\|_{\mathbf{L}^2}^2\right). \quad (6.54)$$

Step 4: Choosing δ small enough in (6.52) and using (6.54) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{H^1}^2 + \frac{1}{2}(\|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2) \\ &\leq C\left(1 + \|\varphi_1\|_{H^3}^2 + \|\varphi_2\|_{H^3}^2 + \|\mu_2 + \chi\sigma_2\|_{H^2}^2 + \|\mathbf{v}_2\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2 + C\|\sigma_\infty\|_{L^2(\partial\Omega)}^2.\end{aligned}$$

Integrating this inequality in time from 0 to $t \in (0,T)$, we obtain that

$$\begin{aligned}&\|\varphi(t)\|_{H^1}^2 + \int_0^t \|\Delta\varphi(s)\|_{L^2}^2 + \|\nabla\Delta\varphi(s)\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}(s)\|_{\mathbf{L}^2}^2 \,\mathrm{d}s \\ &\leq C\int_0^t \alpha(s)\|\varphi(s)\|_{H^1}^2 \,\mathrm{d}s + \|\varphi(0)\|_{H^1}^2 + \int_0^t \|\sigma_\infty\|_{L^2(\partial\Omega)}^2 \,\mathrm{d}s,\end{aligned}$$

with

$$\alpha(s) := C\left(1 + \|\varphi_1(s)\|_{H^3}^2 + \|\varphi_2(s)\|_{H^3}^2 + \|\mu_2(s) + \chi\sigma_2(s)\|_{H^2}^2 + \|\mathbf{v}_2(s)\|_{\mathbf{H}^1}^2\right) \in L^1(0,T),$$

where we used (6.45). Therefore, an application of Gronwall's lemma gives

$$\|\varphi(t)\|_{H^1}^2 + \int_0^t \|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2 \,\mathrm{d}s \leq C\left(\|\varphi(0)\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(0,t;L^2(\partial\Omega))}^2\right)$$

for all $t \in (0,T]$. Taking the supremum over all $t \in (0,T]$ and using elliptic regularity theory, this implies

$$\sup_{t\in[0,T]} \|\varphi(t)\|_{H^1}^2 + \|\varphi\|_{L^2(H^3)}^2 + \|\mathbf{v}\|_{L^2(\mathbf{L}^2)}^2 \leq C\left(\|\varphi(0)\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(0,T;L^2(\partial\Omega))}^2\right). \quad (6.55)$$

Step 5: Using (6.55) together with (6.48a) and (6.54), an application of Poincaré's inequality yields

$$\|\sigma\|_{L^2(H^1)}^2 + \|p\|_{L^2(H^1)}^2 \leq C\left(\|\varphi(0)\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(0,T;L^2(\partial\Omega))}^2\right). \quad (6.56)$$

Now, using the assumptions on $\psi(\cdot)$, (6.45) and (6.55), it is straightforward to check that

$$\|\psi'(\varphi_1) - \psi'(\varphi_2)\|_{L^2(H^1)}^2 \leq C\left(\|\varphi(0)\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(0,T;L^2(\partial\Omega))}^2\right).$$

Recalling (6.55)-(6.56) and using the relation (6.46e) for μ yields

$$\|\mu\|_{L^2(H^1)}^2 \leq C\left(\|\varphi(0)\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(0,T;L^2(\partial\Omega))}^2\right).$$

Together with (6.45), (6.48b)-(6.48c) and (6.55)-(6.56), using the equation (6.46e) for $\partial_t\varphi$ gives

$$\|\partial_t\varphi\|_{L^2((H^1)^*)}^2 \leq C\left(\|\varphi(0)\|_{H^1}^2 + \|\sigma_\infty\|_{L^2(0,T;L^2(\partial\Omega))}^2\right).$$

Employing the last two estimates in conjunction with (6.55)-(6.56), the inequality (6.44) is established and the proof is complete. □

6.3.3 A qualitative estimate

In the following we set $\sigma_\infty = 1$ for simplicity, although the estimates in the next theorem can be carried out for sufficiently regular boundary data. We will now establish a qualitative estimate for the difference of strong solutions to (5.1)-(5.2) and (6.6)-(6.7). In particular, this shows that the unique strong solution of the Cahn–Hilliard–Darcy model can be obtained from the zero viscosity limit in the Cahn–Hilliard–Brinkman model.

Theorem 6.7 *Let the assumptions of Theorems 6.5 and 6.6 hold, let $(\varphi_{\eta,\lambda}, \mu_{\eta,\lambda}, \sigma_{\eta,\lambda}, \mathbf{v}_{\eta,\lambda}, p_{\eta,\lambda})$ be the unique strong solution of the Cahn–Hilliard–Brinkman system in the sense of Definition 5.4 according to $\eta(\cdot)$, $\lambda(\cdot)$, and originating from $\varphi_0^{\eta,\lambda} \in H_N^2$, and let $(\varphi_D, \mu_D, \sigma_D, \mathbf{v}_D, p_D)$ be the unique strong solution of the Cahn–Hilliard–Darcy system originating from $\varphi_0 \in H_N^2$. Then, it holds*

$$\begin{aligned}
&\|\varphi_{\eta,\lambda} - \varphi_D\|_{H^1(0,T;(H^1)^*)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^3)}^2 + \|\mu_{\eta,\lambda} - \mu_D\|_{L^2(0,T;H^1)}^2 \\
&\quad + \|\sigma_{\eta,\lambda} - \sigma_D\|_{L^\infty(0,T;H^1)}^2 + \|\mathbf{v}_{\eta,\lambda} - \mathbf{v}_D\|_{L^2(0,T;\mathbf{L}^2)}^2 + \|p_{\eta,\lambda} - p_D\|_{L^2(0,T;L^2)}^2 \\
&\leq C_T\left(\|\varphi_0^{\eta,\lambda} - \varphi_0\|_{H^1}^2 + \|\eta(\cdot)\|_{L^\infty(\mathbb{R})} + \|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}\right)
\end{aligned} \tag{6.57}$$

for a constant C_T depending on Ω, T, ε, χ, L_h, L_b, L_f, L_ν, K, k_1, k_2, R_1, R_2, R_3, ρ, ν_0, ν_1, $\|\varphi_{\eta,\lambda}\|_{L^\infty(H^2)\cap L^2(H^3)}$, $\|\varphi_D\|_{L^\infty(H^2)\cap L^2(H^3)}$, $\|\mu_D\|_{L^2(H^2)}$, $\|\sigma_D\|_{L^\infty(H^2)}$, $\|\mathbf{v}_D\|_{L^2(\mathbf{H}^1)}$, $\|b_{\mathbf{v}}(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|f_{\mathbf{v}}(\cdot)\|_{W^{1,\infty}(\mathbb{R})}$, $\|b_\varphi(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|f_\varphi(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|h(\cdot)\|_{L^\infty(\mathbb{R})}$, $\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}$, and $\|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}$. Moreover, if $\varphi_0^{\eta,\lambda} = \varphi_0$, then

$$\begin{aligned}
&\|\varphi_{\eta,\lambda} - \varphi_D\|_{H^1(0,T;(H^1)^*)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^3)} + \|\mu_{\eta,\lambda} - \mu_D\|_{L^2(0,T;H^1)} \\
&\quad + \|\sigma_{\eta,\lambda} - \sigma_D\|_{L^\infty(0,T;H^1)} + \|\mathbf{v}_{\eta,\lambda} - \mathbf{v}_D\|_{L^2(0,T;\mathbf{L}^2)}^2 + \|p_{\eta,\lambda} - p_D\|_{L^2(0,T;L^2)} \\
&\quad \to 0 \qquad \text{as} \quad \|\eta(\cdot)\|_{C^0(\mathbb{R})} \to 0,\ \|\lambda(\cdot)\|_{C^0(\mathbb{R})} \to 0.
\end{aligned}$$

Proof. For convenience we recall that $(\varphi_D, \mu_D, \sigma_D, \mathbf{v}_D, p_D)$ satisfies

$$\begin{aligned}
\operatorname{div}(\mathbf{v}_D) &= \Gamma_{\mathbf{v}}(\varphi_D, \sigma_D) && \text{a. e. in } \Omega_T, && \text{(6.58a)}\\
\nu(\varphi_D)\mathbf{v}_D &= -\nabla p_D + (\mu_D + \chi\sigma_D)\nabla\varphi_D && \text{a. e. in } \Omega_T, && \text{(6.58b)}\\
\partial_t\varphi_D + \operatorname{div}(\varphi_D\mathbf{v}_D) &= \Delta\mu_D + \Gamma_\varphi(\varphi_D, \sigma_D) && \text{a. e. in } \Omega_T, && \text{(6.58c)}\\
\mu_D &= \psi'(\varphi_D) - \Delta\varphi_D - \chi\sigma_D && \text{a. e. in } \Omega_T, && \text{(6.58d)}\\
0 &= \Delta\sigma_D - h(\varphi_D)\sigma && \text{a. e. in } \Omega_T, && \text{(6.58e)}\\
\nabla\varphi_D\cdot\mathbf{n} = \nabla\mu_D\cdot\mathbf{n} = p &= 0, \quad \nabla\sigma_D\cdot\mathbf{n} = K(1-\sigma_D) && \text{a. e. on } \Sigma_T, && \text{(6.58f)}\\
\varphi_D(0) &= \varphi_0 && \text{a. e. in } \Omega, && \text{(6.58g)}
\end{aligned}$$

and

$$\|\varphi_D\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^4)} + \|\sigma_D\|_{H^1(H^1)\cap L^\infty(H^2)} + \|\mu_D\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\mathrm{div}(\varphi_D \mathbf{v}_D)\|_{L^2(L^2)} + \|\mathbf{v}_D\|_{L^2(\mathbf{H}^1)} + \|p_D\|_{L^2(H^2)} \leq C. \tag{6.59}$$

We denote the differences by $\varphi = \varphi_{\eta,\lambda} - \varphi_D$, $\mu = \mu_{\eta,\lambda} - \mu_D$, $\sigma = \sigma_{\eta,\lambda} - \sigma_D$, $\mathbf{v} = \mathbf{v}_{\eta,\lambda} - \mathbf{v}_D$, $p = p_{\eta,\lambda} - p_D$. Furthermore, we use the notation $\Gamma_{\varphi,D} = \Gamma_\varphi(\varphi_D, \sigma_D)$, $\Gamma_{\mathbf{v},D} = \Gamma_{\mathbf{v}}(\varphi_D, \sigma_D)$, $\Gamma^\lambda_{\varphi,\eta} = \Gamma_\varphi(\varphi_{\eta,\lambda}, \sigma_{\eta,\lambda})$, $\Gamma^\lambda_{\mathbf{v},\eta} = \Gamma_{\mathbf{v}}(\varphi_{\eta,\lambda}, \sigma_{\eta,\lambda})$. Then, the differences fulfil

$$\begin{aligned}
&\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_{\eta,\lambda} \colon \nabla\mathbf{\Phi} + (\lambda(\varphi_{\eta,\lambda})\mathrm{div}(\mathbf{v}_{\eta,\lambda}) - p)\mathrm{div}(\mathbf{\Phi}) + \nu(\varphi_{\eta,\lambda})\mathbf{v}\cdot\mathbf{\Phi} \,\mathrm{d}x \\
&\quad = \int_\Omega (\mu + \chi\sigma)\nabla\varphi_{\eta,\lambda}\cdot\mathbf{\Phi} + (\mu_D + \chi\sigma_D)\nabla\varphi\cdot\mathbf{\Phi} + (\nu(\varphi_D) - \nu(\varphi_{\eta,\lambda}))\mathbf{v}_D\cdot\mathbf{\Phi} \,\mathrm{d}x, && (6.60a) \\
&\langle \partial_t\varphi, \xi\rangle_{H^1} + \int_\Omega (\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v} + \nabla\varphi\cdot\mathbf{v}_D)\,\xi \,\mathrm{d}x + \int_\Omega (\varphi\Gamma^\lambda_{\mathbf{v},\eta} + \varphi_D\mathrm{div}(\mathbf{v}))\xi \,\mathrm{d}x \\
&\quad = -\int_\Omega \nabla\mu\cdot\nabla\xi + \int_\Omega (\Gamma^\lambda_{\varphi,\eta} - \Gamma_{\varphi,D})\xi \,\mathrm{d}x, && (6.60b) \\
&0 = \int_\Omega \nabla\sigma\cdot\nabla\xi + \int_\Omega ((h(\varphi_{\eta,\lambda}) - h(\varphi_D))\sigma_{\eta,\lambda} + h(\varphi_D)\sigma)\xi + \int_{\partial\Omega} K\sigma\xi \,\mathrm{d}\mathcal{H}^{d-1} && (6.60c)
\end{aligned}$$

for a. e. $t \in (0,T)$ and all $\mathbf{\Phi} \in \mathbf{H}^1$, $\xi \in H^1$ as well as

$$\begin{aligned}
\mathrm{div}(\mathbf{v}) &= \Gamma^\lambda_{\mathbf{v},\eta} - \Gamma_{\mathbf{v},D} && \text{a. e. in } \Omega_T, && (6.60d) \\
\mu &= \psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D) - \Delta\varphi - \chi\sigma && \text{a. e. in } \Omega_T, && (6.60e) \\
\nabla\varphi\cdot\mathbf{n} &= \nabla\mu\cdot\mathbf{n} = \nabla\sigma\cdot\mathbf{n} + K\sigma = 0 && \text{a. e. on } \Sigma_T, && (6.60f) \\
\varphi(0) &= \varphi_0^{\eta,\lambda} - \varphi_0 && \text{a. e. in } \Omega. && (6.60g)
\end{aligned}$$

Subtracting $\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D \colon \nabla\mathbf{\Phi} + \lambda(\varphi_{\eta,\lambda})\mathrm{div}(\mathbf{v}_D)\mathrm{div}(\mathbf{\Phi}) \,\mathrm{d}x$ on both sides of (6.60a) and using $\mathbf{D}\mathbf{v}_D \colon \nabla\mathbf{\Phi} = \mathbf{D}\mathbf{v}_D \colon \mathbf{D}\mathbf{\Phi}$, we obtain that

$$\begin{aligned}
&\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v} \colon \nabla\mathbf{\Phi} + (\lambda(\varphi_{\eta,\lambda})\mathrm{div}(\mathbf{v}) - p)\mathrm{div}(\mathbf{\Phi}) + \nu(\varphi_{\eta,\lambda})\mathbf{v}\cdot\mathbf{\Phi} \,\mathrm{d}x \\
&\quad = \int_\Omega \big((\mu + \chi\sigma)\nabla\varphi_{\eta,\lambda} + (\mu_D + \chi\sigma_D)\nabla\varphi + (\nu(\varphi_D) - \nu(\varphi_{\eta,\lambda}))\mathbf{v}_D\big)\cdot\mathbf{\Phi} \,\mathrm{d}x \\
&\quad\quad - \int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D \colon \mathbf{D}\mathbf{\Phi} + \lambda(\varphi_{\eta,\lambda})\mathrm{div}(\mathbf{v}_D)\mathrm{div}(\mathbf{\Phi}) \,\mathrm{d}x. && (6.61)
\end{aligned}$$

In the following we will frequently use Hölder's and Young's inequalities and we divide the analysis into several steps.

Step 1: Choosing $\xi = \sigma$ in (6.60c) and using the non-negativity of $h(\cdot)$ gives

$$\|\nabla\sigma\|^2_{\mathbf{L}^2} + K\|\sigma\|^2_{L^2(\partial\Omega)} \leq \left| \int_\Omega (h(\varphi_{\eta,\lambda}) - h(\varphi_D))\sigma_{\eta,\lambda}\sigma \,\mathrm{d}x \right|.$$

Using (6.59), the Sobolev embedding $H^2 \subset L^\infty$ and the Lipschitz-continuity of $h(\cdot)$, an application of Poincaré's inequality yields

$$\|\sigma\|_{H^1} \leq C\|\varphi\|_{L^2}. \tag{6.62}$$

In particular, using the specific form of $\Gamma_{\mathbf{v}}$ and Γ_φ, by (6.59) we obtain

$$\|\Gamma^\lambda_{\mathbf{v},\eta} - \Gamma_{\mathbf{v},D}\|_{L^2} + \|\Gamma^\lambda_{\varphi,\eta} - \Gamma_{\varphi,D}\|_{L^2} \leq C\|\varphi\|_{L^2}. \tag{6.63}$$

Step 2: We choose $\xi = -\Delta\varphi$ in (6.60b) and use (6.60e) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 = \int_\Omega \big(\varphi\Gamma_{\mathbf{v},\eta}^\lambda + \varphi_D \mathrm{div}(\mathbf{v}) - (\Gamma_{\varphi,\eta}^\lambda - \Gamma_{\varphi,D})\big)\Delta\varphi \,\mathrm{d}x + \int_\Omega (\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v} + \nabla\varphi\cdot\mathbf{v}_D)\Delta\varphi\,\mathrm{d}x + \int_\Omega (\nabla(\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D)) - \chi\nabla\sigma)\cdot\nabla\Delta\varphi\,\mathrm{d}x. \tag{6.64}$$

By Lemma 2.39 there exists a solution $\mathbf{u}\in\mathbf{H}^1$ of

$$\mathrm{div}(\mathbf{u}) = \Gamma_{\mathbf{v},\eta}^\lambda - \Gamma_{\mathbf{v},D} \qquad \text{a. e. in } \Omega,$$
$$\mathbf{u} = \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v},\eta}^\lambda - \Gamma_{\mathbf{v},D}\,\mathrm{d}x\right)\mathbf{n} \qquad \text{a. e. on } \partial\Omega,$$

and using (6.63) it holds that

$$\|\mathbf{u}\|_{\mathbf{H}^1} \le c\|\Gamma_{\mathbf{v},\eta}^\lambda - \Gamma_{\mathbf{v},D}\|_{L^2} \le C\|\varphi\|_{L^2} \tag{6.65}$$

with a constant c depending only on Ω. Choosing $\mathbf{\Phi} = \mathbf{v}-\mathbf{u}$ in (6.61), we infer that

$$\begin{aligned}&\int_\Omega 2\eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2 + \nu(\varphi_{\eta,\lambda})|\mathbf{v}|^2\,\mathrm{d}x\\ &= \int_\Omega \big((\mu+\chi\sigma)\nabla\varphi_{\eta,\lambda} + (\mu_D+\chi\sigma_D)\nabla\varphi + (\nu(\varphi_D)-\nu(\varphi_{\eta,\lambda}))\mathbf{v}_D\big)\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\\ &\quad + \int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}\colon\nabla\mathbf{u} + \nu(\varphi_{\eta,\lambda})\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D\colon\mathbf{D}(\mathbf{v}-\mathbf{u})\,\mathrm{d}x.\end{aligned}$$

Summing up this identity with (6.64) and using

$$\int_\Omega (\mu+\chi\sigma)\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega (-\Delta\varphi + (\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D)))\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v}\,\mathrm{d}x$$

which follows from (6.60e), we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \int_\Omega 2\eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2 + \nu(\varphi_{\eta,\lambda})|\mathbf{v}|^2\,\mathrm{d}x\\ &= \int_\Omega \big(\varphi\Gamma_{\mathbf{v},\eta}^\lambda + \varphi_D\mathrm{div}(\mathbf{v}) - (\Gamma_{\varphi,\eta}^\lambda - \Gamma_{\varphi,D})\big)\Delta\varphi\,\mathrm{d}x + \int_\Omega \nabla\varphi\cdot\mathbf{v}_D\Delta\varphi\,\mathrm{d}x\\ &\quad + \int_\Omega (\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D))\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v}\,\mathrm{d}x + \int_\Omega (\nabla(\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D)) - \chi\nabla\sigma)\cdot\nabla\Delta\varphi\,\mathrm{d}x\\ &\quad + \int_\Omega \big((\mu_D+\chi\sigma_D)\nabla\varphi + (\nu(\varphi_D)-\nu(\varphi_{\eta,\lambda}))\mathbf{v}_D\big)\cdot\big(\mathbf{v}-\mathbf{u}\big)\,\mathrm{d}x - \int_\Omega (\mu+\chi\sigma)\nabla\varphi_{\eta,\lambda}\cdot\mathbf{u}\,\mathrm{d}x\\ &\quad + \int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}\colon\nabla\mathbf{u} + \nu(\varphi_{\eta,\lambda})\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D\colon\mathbf{D}(\mathbf{v}-\mathbf{u})\,\mathrm{d}x.\end{aligned} \tag{6.66}$$

Step 3: We now estimate the terms on the r. h. s. of (6.66) individually. Due to (6.40), (6.59) and (6.63), we get

$$\left|\int_\Omega \big(\varphi\Gamma_{\mathbf{v},\eta}^\lambda + \varphi_D\mathrm{div}(\mathbf{v}) - (\Gamma_{\varphi,\eta}^\lambda - \Gamma_{\varphi,D})\big)\Delta\varphi\,\mathrm{d}x\right| \le C\|\varphi\|_{L^2}^2 + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2. \tag{6.67}$$

For the second term on the r. h. s. of (6.66), we invoke Gagliardo–Nirenberg's inequality and integrate by parts to deduce that

$$\|\Delta\varphi\|_{L^4} \le C\|\Delta\varphi\|_{L^2}^{\frac{1}{2}}\|\Delta\varphi\|_{H^1}^{\frac{1}{2}} \le C\left(\|\Delta\varphi\|_{L^2} + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}\right) \le C\left(\|\nabla\varphi\|_{\mathbf{L}^2} + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}\right).$$

Consequently, by the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^4$ we obtain

$$\left|\int_\Omega \nabla\varphi \cdot \mathbf{v}_D \Delta\varphi \,\mathrm{d}x\right| \leq \|\nabla\varphi\|_{\mathbf{L}^2}\|\mathbf{v}_D\|_{\mathbf{L}^4}\|\Delta\varphi\|_{L^4}$$
$$\leq C\left(1+\|\mathbf{v}_D\|_{\mathbf{H}^1}^2\right)\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2. \tag{6.68}$$

By the assumptions on $\psi(\cdot)$, (6.40), (6.59) and the Sobolev embedding $\mathbf{H}^2 \subset \mathbf{L}^\infty$, we deduce that

$$\left|\int_\Omega (\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D))\nabla\varphi_{\eta,\lambda} \cdot \mathbf{v} \,\mathrm{d}x\right| \leq C\int_\Omega \left(1+|\varphi_{\eta,\lambda}|^4 + |\varphi_D|^4\right)|\varphi||\nabla\varphi_{\eta,\lambda}||\mathbf{v}| \,\mathrm{d}x$$
$$\leq C\|\varphi\|_{L^2}\|\nabla\varphi_{\eta,\lambda}\|_{\mathbf{H}^2}\|\mathbf{v}\|_{\mathbf{L}^2}$$
$$\leq C\|\varphi_{\eta,\lambda}\|_{H^3}^2\|\varphi\|_{L^2}^2 + \frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2. \tag{6.69}$$

Recalling (6.65) and the Sobolev embedding $H^2 \subset L^\infty$, it is straightforward to check that

$$\left|\int_\Omega (\mu_D + \chi\sigma_D)\nabla\varphi \cdot (\mathbf{v}-\mathbf{u}) \,\mathrm{d}x\right| \leq C\left(\|\mu_D + \chi\sigma_D\|_{H^2}^2\right)\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\varphi\|_{L^2}^2 + \frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2. \tag{6.70}$$

Applying (6.62), we have

$$\left|\int_\Omega \chi\nabla\sigma \cdot \nabla\Delta\varphi \,\mathrm{d}x\right| \leq C\|\varphi\|_{L^2}^2 + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2. \tag{6.71}$$

Upon using (6.60e), we can rewrite

$$\int_\Omega (\mu + \chi\sigma)\nabla\varphi_{\eta,\lambda} \cdot \mathbf{u} \,\mathrm{d}x = \int_\Omega (\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D) - \Delta\varphi)\nabla\varphi_{\eta,\lambda} \cdot \mathbf{u} \,\mathrm{d}x.$$

Then, using the assumptions on $\psi(\cdot)$, (6.40), (6.65) and the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^4$, a straightforward calculation yields

$$\left|\int_\Omega (\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D) - \Delta\varphi)\nabla\varphi_{\eta,\lambda} \cdot \mathbf{u} \,\mathrm{d}x\right| \leq C\|\varphi\|_{L^2}^2 + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2,$$

and therefore

$$\left|\int_\Omega (\mu + \chi\sigma)\nabla\varphi_{\eta,\lambda} \cdot \mathbf{u} \,\mathrm{d}x\right| \leq C\|\varphi\|_{L^2}^2 + \frac{1}{8}\|\Delta\varphi\|_{L^2}^2.$$

Moreover, using the assumptions on $\psi(\cdot)$ and the Sobolev embedding $\mathbf{H}^2 \subset \mathbf{L}^\infty$ leads to

$$\left|\int_\Omega \nabla(\psi'(\varphi_{\eta,\lambda}) - \psi'(\varphi_D)) \cdot \nabla\Delta\varphi \,\mathrm{d}x\right|$$
$$\leq \int_\Omega |(\psi''(\varphi_{\eta,\lambda})\nabla\varphi + (\psi''(\varphi_{\eta,\lambda}) - \psi''(\varphi_D))\nabla\varphi_D) \cdot \nabla\Delta\varphi| \,\mathrm{d}x$$
$$\leq C\left(\|\nabla\varphi\|_{\mathbf{L}^2}^2 + \|\varphi_D\|_{H^3}^2\|\varphi\|_{L^2}^2\right) + \frac{1}{8}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2.$$

By (6.65), the assumptions on $\nu(\cdot)$ and the Sobolev embedding $H^1 \subset L^4$, it follows that

$$\left|\int_\Omega (\nu(\varphi_D) - \nu(\varphi_{\eta,\lambda}))\mathbf{v}_D \cdot (\mathbf{v}-\mathbf{u}) \,\mathrm{d}x\right| \leq C\left(1+\|\mathbf{v}_D\|_{\mathbf{H}^1}^2\right)\|\varphi\|_{H^1}^2 + \frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2$$

and

$$\left|\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v} \colon \nabla\mathbf{u} + \nu(\varphi_{\eta,\lambda})\mathbf{v} \cdot \mathbf{u} \,\mathrm{d}x\right| \leq C\left(1+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\right)\|\varphi\|_{L^2}^2$$
$$+ \int_\Omega \eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2 \,\mathrm{d}x + \frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2.$$

Employing the last four inequalities along with (6.67)-(6.71) in (6.66) and using the assumptions on $\nu(\cdot)$, we end up with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\varphi\|_{\mathbf{L}^2}^2+\frac{1}{2}\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2+\int_\Omega \eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x+\frac{\nu_0}{2}\|\mathbf{v}\|_{\mathbf{L}^2}^2\\&\leq C\left(1+\|\varphi_{\eta,\lambda}\|_{H^3}^2+\|\varphi_D\|_{H^3}^2+\|\mathbf{v}_D\|_{\mathbf{H}^1}^2+\|\mu_D+\chi\sigma_D\|_{H^2}^2+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\right)\|\varphi\|_{H^1}^2\\&\quad+\frac{3}{8}\|\Delta\varphi\|_{L^2}^2-\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D\colon\mathbf{D}(\mathbf{v}-\mathbf{u})\,\mathrm{d}x. \qquad (6.72)\end{aligned}$$

Step 4: Choosing $\xi=\varphi$ in (6.60b) and applying (6.60e)-(6.60f) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{L^2}^2+\|\Delta\varphi\|_{L^2}^2&=\int_\Omega((\psi'(\varphi_{\eta,\lambda})-\psi'(\varphi_D)-\chi\sigma)\Delta\varphi\,\mathrm{d}x-\int_\Omega(\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v}+\nabla\varphi\cdot\mathbf{v}_D)\varphi\,\mathrm{d}x\\&\quad-\int_\Omega\left(\varphi\Gamma_{\mathbf{v},\eta}^\lambda+\varphi_D\mathrm{div}(\mathbf{v})-(\Gamma_{\varphi,\eta}^\lambda-\Gamma_{\varphi,D})\right)\varphi\,\mathrm{d}x. \qquad (6.73)\end{aligned}$$

Invoking (6.40), (6.59) and (6.65), we deduce that

$$\left|\int_\Omega\left(\varphi\Gamma_{\mathbf{v},\eta}^\lambda+\varphi_D\mathrm{div}(\mathbf{v})-(\Gamma_{\varphi,\eta}^\lambda-\Gamma_{\varphi,D})\right)\varphi\,\mathrm{d}x\right|\leq C\|\varphi\|_{L^2}^2.$$

From the Sobolev embeddings $H^1\subset L^4$ and $\mathbf{H}^1\subset\mathbf{L}^4$, we obtain

$$\left|\int_\Omega(\nabla\varphi_{\eta,\lambda}\cdot\mathbf{v}+\nabla\varphi\cdot\mathbf{v}_D)\varphi\,\mathrm{d}x\right|\leq C\left(1+\|\mathbf{v}_D\|_{\mathbf{H}^1}^2+\|\varphi_{\eta,\lambda}\|_{H^3}^2\right)\|\varphi\|_{H^1}^2+\frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2.$$

Furthermore, using (6.60f), (6.62) and integrating by parts, we obtain

$$\left|\int_\Omega\chi\sigma\Delta\varphi\,\mathrm{d}x\right|=\left|\int_\Omega\chi\nabla\sigma\cdot\nabla\varphi\right|\leq C\left(\|\varphi\|_{L^2}^2+\|\nabla\varphi\|_{\mathbf{L}^2}^2\right).$$

Finally, by the assumptions on $\psi(\cdot)$, (6.40) and (6.59), we deduce that

$$\left|\int_\Omega(\psi'(\varphi_{\eta,\lambda})-\psi'(\varphi_D))\Delta\varphi\,\mathrm{d}x\right|\leq C\|\varphi\|_{L^2}^2+\frac{1}{8}\|\Delta\varphi\|_{L^2}^2.$$

Employing the last four estimates in (6.73), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{L^2}^2+\frac{7}{8}\|\Delta\varphi\|_{L^2}^2\leq C\left(1+\|\mathbf{v}_D\|_{\mathbf{H}^1}^2+\|\varphi_{\eta,\lambda}\|_{H^3}^2\right)\|\varphi\|_{H^1}^2+\frac{\nu_0}{8}\|\mathbf{v}\|_{\mathbf{L}^2}^2.$$

Summing up this estimate with (6.72), we end up with

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{H^1}^2+\frac{1}{4}\left(\|\Delta\varphi\|_{L^2}^2+\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2+\nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2\right)+\int_\Omega\eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x\\&\quad\leq C\left(1+\|\varphi_{\eta,\lambda}\|_{H^3}^2+\|\varphi_D\|_{H^3}^2+\|\mathbf{v}_D\|_{\mathbf{H}^1}^2+\|\mu_D+\chi\sigma_D\|_{H^2}^2+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\right)\|\varphi\|_{H^1}^2\\&\quad-\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D\colon\mathbf{D}(\mathbf{v}-\mathbf{u})\,\mathrm{d}x. \qquad (6.74)\end{aligned}$$

Step 5: It remains to estimate the last term on the r. h. s. of (6.74). Due to (6.65), we obtain

$$\begin{aligned}\left|\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D\colon\mathbf{D}\mathbf{u}\,\mathrm{d}x\right|&\leq\left\|\sqrt{2\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}_D\right\|_{\mathbf{L}^2}\left\|\sqrt{2\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{u}\right\|_{\mathbf{L}^2}\\&\leq\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\|\mathbf{D}\mathbf{v}_D\|_{\mathbf{L}^2}^2+C\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^2}^2.\end{aligned}$$

Finally, we calculate

$$\begin{aligned}\left|\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_D : \mathbf{D}\mathbf{v}\,\mathrm{d}x\right| &\le \left\|\sqrt{2\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}_D\right\|_{\mathbf{L}^2}\left\|\sqrt{2\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}\right\|_{\mathbf{L}^2}\\ &\le \left\|\sqrt{2\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}_D\right\|_{\mathbf{L}^2}^2 + \frac{1}{4}\left\|\sqrt{2\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}\right\|_{\mathbf{L}^2}^2\\ &\le 2\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\|\mathbf{D}\mathbf{v}_D\|_{\mathbf{L}^2}^2 + \frac{1}{2}\int_\Omega \eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x.\end{aligned}$$

Invoking the last two inequalities in (6.74) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\varphi\|_{H^1}^2 + \frac{1}{4}(\|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2) + \frac{1}{2}\int_\Omega \eta(\varphi_{\eta,\lambda})|\mathbf{D}\mathbf{v}|^2\,\mathrm{d}x\\ &\quad\le \alpha_1(t)\|\varphi\|_{H^1}^2 + \alpha_2(t)\|\eta(\cdot)\|_{L^\infty(\mathbb{R})},\end{aligned}$$

with

$$\begin{aligned}&\alpha_1(t) := C\left(1 + \|\varphi_{\eta,\lambda}\|_{H^3}^2 + \|\varphi_D\|_{H^3}^2 + \|\mathbf{v}_D\|_{\mathbf{H}^1}^2 + \|\mu_D + \chi\sigma_D\|_{H^2}^2 + \|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\right) \in L^1(0,T),\\ &\alpha_2(t) := 3\|\mathbf{D}\mathbf{v}_D(t)\|_{\mathbf{L}^2}^2 \in L^1(0,T),\end{aligned}$$

where we used (6.40) and (6.59). Integrating the last inequality in time from 0 to $s \in (0,T]$ and neglecting the non-negative term $\int_\Omega \eta(\varphi_{\eta,\lambda}(t))|\mathbf{D}\mathbf{v}(t)|^2\,\mathrm{d}x$, we obtain

$$\begin{aligned}&\|\varphi(s)\|_{H^1}^2 + \int_0^s \|\Delta\varphi(t)\|_{L^2}^2 + \|\nabla\Delta\varphi(t)\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2\,\mathrm{d}t\\ &\le \int_0^s 2C\alpha_1(t)\|\varphi(t)\|_{H^1}^2\,\mathrm{d}t + \|\varphi(0)\|_{H^1}^2 + \|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\int_0^s 2\alpha_2(t)\,\mathrm{d}t.\end{aligned} \tag{6.75}$$

We now define

$$\begin{aligned}u(s) &:= \|\varphi(s)\|_{H^1}^2 \in C^0(0,T),\\ v(t) &:= \|\Delta\varphi(t)\|_{L^2}^2 + \|\nabla\Delta\varphi(t)\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2 \in L^1(0,T),\\ \alpha(s) &:= \|\varphi(0)\|_{H^1}^2 + \|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\int_0^s 2\alpha_2(t)\,\mathrm{d}t \in L^1(0,T),\\ \beta(t) &:= 2C\alpha_1(t) \in L^1(0,T),\end{aligned}$$

and we note that α is monotonically increasing. Then, an application of Lemma 2.30 to (6.75) yields

$$\begin{aligned}&\|\varphi(s)\|_{H^1}^2 + \int_0^s \|\Delta\varphi(t)\|_{L^2}^2 + \|\nabla\Delta\varphi(t)\|_{\mathbf{L}^2}^2 + \nu_0\|\mathbf{v}\|_{\mathbf{L}^2}^2\,\mathrm{d}t\\ &\le \left(\|\varphi(0)\|_{H^1}^2 + \|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\int_0^s 2\alpha_2(t)\,\mathrm{d}t\right)\exp\left(\int_0^s \beta(r)\mathrm{d}r\right) \quad \forall s \in (0,T].\end{aligned}$$

Setting

$$C_T := \max\left\{\exp\left(\int_0^T \beta(r)\mathrm{d}r\right),\ \exp\left(\int_0^T \beta(r)\mathrm{d}r\right)\left(\int_0^T 2\alpha_2(t)\,\mathrm{d}t\right)\right\} < \infty,$$

taking the supremum over all $s \in (0,T]$ in the last inequality and using elliptic regularity theory, we get the bound

$$\|\varphi_{\eta,\lambda} - \varphi_D\|_{L^\infty(H^1)\cap L^2(H^3)}^2 + \|\mathbf{v}_{\eta,\lambda} - \mathbf{v}_D\|_{L^2(\mathbf{L}^2)}^2 \le C_T\left(\|\varphi_0^{\eta,\lambda} - \varphi_0\|_{H^1}^2 + \|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\right).$$

In particular, using the equation (6.60e) for μ, recalling (6.62) and possibly enlarging C_T, it holds that

$$\begin{aligned}&\|\varphi_{\eta,\lambda}-\varphi_D\|^2_{L^\infty(H^1)\cap L^2(H^3)}+\|\mu_{\eta,\lambda}-\mu_D\|^2_{L^2(H^1)}+\|\sigma_{\eta,\lambda}-\sigma_D\|^2_{L^\infty(H^1)}+\|\mathbf{v}_{\eta,\lambda}-\mathbf{v}_D\|^2_{L^2(\mathbf{L}^2)}\\&\leq C_T\left(\|\varphi_0^{\eta,\lambda}-\varphi_0\|^2_{H^1}+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\right).\end{aligned}\tag{6.76}$$

Step 6: We recall that

$$\|\sqrt{\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}_{\eta,\lambda}\|_{L^2(\mathbf{L}^2)}+\|\sqrt{\lambda(\varphi_{\eta,\lambda})}\mathrm{div}(\mathbf{v}_{\eta,\lambda})\|_{L^2(\mathbf{L}^2)}\leq C\tag{6.77}$$

with a constant C independent of $\eta(\cdot)$ and $\lambda(\cdot)$. Let $\mathbf{q}\in\mathbf{H}^1$ be a solution of

$$\mathrm{div}(\mathbf{q})=p\quad\text{in }\Omega,\qquad \mathbf{q}=\frac{1}{|\partial\Omega|}\left(\int_\Omega p\,\mathrm{d}x\right)\mathbf{n}\quad\text{on }\partial\Omega$$

such that

$$\|\mathbf{q}\|_{\mathbf{H}^1}\leq c\|p\|_{L^2}.$$

Taking $\mathbf{\Phi}=\mathbf{q}$ in (6.60a) and integrating in time from 0 to T, we obtain

$$\begin{aligned}\|p\|_{L^2(L^2)}&=-\int_0^T\int_\Omega\big((\mu+\chi\sigma)\nabla\varphi_{\eta,\lambda}+(\mu_D+\chi\sigma_D)\nabla\varphi+(\nu(\varphi_D)-\nu(\varphi_{\eta,\lambda}))\mathbf{v}_D\big)\cdot\mathbf{q}\,\mathrm{d}x\,\mathrm{d}t,\\&\quad+\int_0^T\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_{\eta,\lambda}\colon\nabla\mathbf{q}+\lambda(\varphi_{\eta,\lambda})\mathrm{div}(\mathbf{v}_{\eta,\lambda})p+\nu(\varphi_{\eta,\lambda})\mathbf{v}\cdot\mathbf{q}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{6.78}$$

Using (6.40), (6.59) and the Sobolev embedding $\mathbf{H}^1\subset\mathbf{L}^4$, a straightforward calculation yields

$$\begin{aligned}&\left|\int_0^T\int_\Omega\big((\mu+\chi\sigma)\nabla\varphi_{\eta,\lambda}+(\mu_D+\chi\sigma_D)\nabla\varphi+(\nu(\varphi_D)-\nu(\varphi_{\eta,\lambda}))\mathbf{v}_D-\nu(\varphi_{\eta,\lambda})\mathbf{v}\big)\cdot\mathbf{q}\,\mathrm{d}x\,\mathrm{d}t\right|\\&\quad\leq C\left(\|\varphi\|^2_{L^\infty(H^1)\cap L^2(H^2)}+\|\mu+\chi\sigma\|^2_{L^2(L^2)}+\|\mathbf{v}\|^2_{L^2(\mathbf{L}^2)}\right)+\frac{1}{6}\|p\|^2_{L^2(L^2)}.\end{aligned}$$

Furthermore, by (6.77) we infer that

$$\begin{aligned}\left|\int_0^T\int_\Omega 2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_{\eta,\lambda}\colon\nabla\mathbf{q}\,\mathrm{d}x\,\mathrm{d}t\right|&\leq\|2\eta(\varphi_{\eta,\lambda})\mathbf{D}\mathbf{v}_{\eta,\lambda}\|_{L^2(\mathbf{L}^2)}\|\nabla\mathbf{q}\|_{L^2(\mathbf{L}^2)}\\&\leq C\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}\|\sqrt{\eta(\varphi_{\eta,\lambda})}\mathbf{D}\mathbf{v}_{\eta,\lambda}\|^2_{L^2(\mathbf{L}^2)}+\frac{1}{6}\|p\|^2_{L^2(L^2)}\\&\leq C\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}+\frac{1}{6}\|p\|^2_{L^2(L^2)}.\end{aligned}$$

With similar arguments, we deduce

$$\left|\int_0^T\int_\Omega\lambda(\varphi_{\eta,\lambda})\mathrm{div}(\mathbf{v}_{\eta,\lambda})p\,\mathrm{d}x\,\mathrm{d}t\right|\leq C\|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}+\frac{1}{6}\|p\|^2_{L^2(L^2)}.$$

Invoking the last three inequalities in (6.78), we end up with

$$\|p\|^2_{L^2(L^2)}\leq C\left(\|\varphi\|^2_{L^\infty(H^1)\cap L^2(H^2)}+\|\mu+\chi\sigma\|^2_{L^2(L^2)}+\|\mathbf{v}\|^2_{L^2(\mathbf{L}^2)}+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}+\|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}\right).$$

Recalling (6.76) and possibly again enlarging C_T, this implies

$$\|p_{\eta,\lambda}-p_D\|^2_{L^2(L^2)}\leq C_T\left(\|\varphi_0^{\eta,\lambda}-\varphi_0\|^2_{H^1}+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}+\|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}\right).$$

Finally, using (6.40), (6.59) and (6.62)-(6.63), the relation (6.60b) for $\partial_t(\varphi_{\eta,\lambda}-\varphi_D)$ yields

$$\|\partial_t(\varphi_{\eta,\lambda}-\varphi_D)\|^2_{L^2((H^1)^*)}\leq C_T\left(\|\varphi_0^{\eta,\lambda}-\varphi_0\|^2_{H^1}+\|\eta(\cdot)\|_{L^\infty(\mathbb{R})}+\|\lambda(\cdot)\|_{L^\infty(\mathbb{R})}\right).$$

The last two bounds in conjunction with (6.76) yield (6.57), hence the proof is complete. □

7

A tumour growth model with degenerate mobility

In this chapter we analyse a tumour growth model with a degenerate mobility, and we use ideas presented in [62]. In such models diffusive mechanisms are switched off depending on the value of the phase field variable. For the classical Cahn–Hilliard equation, it has been suggested to take a mobility that degenerates in the pure phases $\varphi = \pm 1$ and thus diffusion is restricted to the interfacial region. As an important consequence, the phase field stays in the physical relevant interval $[-1, 1]$. Often, the degeneracy of the mobility is combined with singular potentials of, e. g., logarithmic type, and a typical example is given by

$$m(\varphi) = (1-\varphi^2), \quad \psi_{\log}(\varphi) = \frac{\theta}{2}\big(\ln(1+\varphi)(1+\varphi) + \ln(1-\varphi)(1-\varphi)\big) + \frac{\theta_c}{2}(1-\varphi^2)$$

for positive constants $0 \leq \theta \leq \theta_c$, see, e. g., [62]. Denoting with $\hat{\psi}_{\log}(\varphi) \coloneqq \psi_{\log}(\varphi) - \frac{\theta_c}{2}(1-\varphi^2)$ the convex part of $\psi_{\log}$, we observe that $m(\varphi)\hat{\psi}''_{\log}(\varphi) = \theta$ which plays a central role in the analysis as we will see later. In particular, this property may be used to derive the so-called deep quench limit $\theta \to 0$ which corresponds to the double obstacle potential. In the context of Cahn–Hilliard models describing tumour growth dynamics, the specific form of source terms is crucial. Indeed, we have seen in Chapter 3 that the mobility's degeneracy has, in some sense, to be consistent with the specific form of the source terms. In order to elucidate this observation, we give the following example: using linear kinetics (see (3.31)) and assuming that there is no gain or loss of mass locally (see (3.29)-(3.30)), the equation for $\partial_t \varphi$ is given by

$$\partial_t \varphi + \nabla \varphi \cdot \mathbf{v} = \operatorname{div}(m(\varphi)\nabla\mu) + (\beta - \varphi\alpha)(\mathcal{P}\sigma - \mathcal{A})h(\varphi),$$

where h interpolates between $h(-1) = 0$ and $h(1) = 1$, $\alpha = \bar{\rho}_2^{-1} - \bar{\rho}_1^{-1}$, $\beta = \bar{\rho}_1^{-1} + \bar{\rho}_2^{-1}$, and for positive constants $\mathcal{P}$ and $\mathcal{A}$. In the pure phases $\varphi = \pm 1$ equation (3.28c) formally reads

$$(\pm\alpha - \beta)(\mathcal{P}\sigma - \mathcal{A})h(\pm 1) = \operatorname{div}(m(\pm 1)\nabla\mu). \tag{7.1}$$

In the healthy phase, the left hand side of (7.1) is zero and thus $m(-1) = 0$ does not lead to an inconsistency. However, in the pure tumour region, equation (7.1) may not hold if we assume $m(1) = 0$ since the left hand side of (7.1) only vanishes provided $\mathcal{P}\sigma - \mathcal{A} = 0$. Typical examples for mobilities that degenerate only in the healthy phase are given by

$$m(\varphi) = \max\left(0, \min\left(1, \tfrac{1}{2}(1+\varphi)\right)\right) \quad \text{or} \quad m(\varphi) = \max\left(0, \min\left(1, \tfrac{1}{4}(1+\varphi)^2\right)\right).$$

7.1 Introduction of the model

Our aim is to analyse the following variant of (3.28)

$$\operatorname{div}(\mathbf{v}) = 0 \quad \text{in } \Omega_T, \tag{7.2a}$$
$$-\operatorname{div}(2\eta\mathbf{D}\mathbf{v}) + \nu\mathbf{v} - \nabla p = -\varepsilon\operatorname{div}(\nabla\varphi\otimes\nabla\varphi) \quad \text{in } \Omega_T, \tag{7.2b}$$
$$\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = \operatorname{div}(m(\varphi)\nabla\mu) + g(\varphi,\sigma)h(\varphi) \quad \text{in } \Omega_T, \tag{7.2c}$$
$$\mu = -\varepsilon\Delta\varphi + \varepsilon^{-1}\psi'(\varphi) - \chi_\varphi\sigma \quad \text{in } \Omega_T, \tag{7.2d}$$
$$\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) = \operatorname{div}(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi) - f(\varphi,\sigma)h(\varphi) \quad \text{in } \Omega_T, \tag{7.2e}$$

where the symmetrised velocity gradient is given by

$$\mathbf{D}\mathbf{v} := \tfrac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}).$$

Here, the terms $h(\varphi)g(\varphi,\sigma)$ and $h(\varphi)f(\varphi,\sigma)$ act as source or sink terms. The nutrient free energy density is of the form (3.37), i. e., $N(\varphi,\sigma) = \frac{\chi_\sigma}{2}|\sigma|^2 + \chi_\varphi\sigma(1-\varphi)$, where χ_σ is a nutrient diffusion parameter and χ_φ is a coefficient related to chemotaxis. We denote the partial derivatives of N by

$$N_{,\sigma} = \chi_\sigma\sigma + \chi_\varphi(1-\varphi), \quad N_{,\varphi} = -\chi_\varphi\sigma,$$

and we equip the system with boundary and initial conditions of the form

$$\nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} = \nabla\sigma\cdot\mathbf{n} = 0 \quad \text{on } \Sigma_T, \tag{7.3a}$$
$$\mathbf{v} = \mathbf{0} \quad \text{on } \Sigma_T, \tag{7.3b}$$
$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega. \tag{7.3c}$$

Remark 7.1 (i) We will consider a source term that satisfies $h(\varphi) = 0$ for $\varphi \le -1$ which is consistent with a mobility satisfying $m(-1) = 0$ and a potential with a singularity in $\varphi = -1$. In general, it is sufficient to prescribe $h(-1) = 0$ since, as discussed above, the degenerate mobility guarantees the bound $\varphi \ge -1$ a. e. in Ω_T.

(ii) Equation (7.2a) holds, e. g., in the case of matched pure densities, i. e. $\bar\rho_1 = \bar\rho_2 =: \bar\rho$, and assuming no gain or loss of mass locally. Indeed, this gives (see (3.29)-(3.30))

$$\Gamma_\varphi = \left(\frac{1}{\bar\rho_1} + \frac{1}{\bar\rho_2}\right)\Gamma = \frac{2}{\bar\rho}\Gamma, \qquad \Gamma_{\mathbf{v}} = \left(\frac{1}{\bar\rho_2} - \frac{1}{\bar\rho_1}\right)\Gamma = 0.$$

(iii) Equations (7.2a) and (7.3b) seem to be indispensable for the analysis. Indeed, the Dirichlet condition for $\mathbf{v}$ guarantees that entropy cannot be transported across the boundary of Ω. Furthermore, as a consequence of (7.3b) we require that $\operatorname{div}(\mathbf{v})$ has zero mean for almost all $t \in (0,T)$. This is, as already discussed in the previous chapters, not compatible with a solution dependent source term in (7.2a).

(iv) The standard strategy to prove existence of weak solutions is as follows. First, mobility and potential are regularised and existence of solutions is shown for the system with non-degenerate mobility and regular potential. Then, estimates that are independent of the regularisation parameter are established by a suitable testing procedure. Finally, one recovers solutions for the system with degenerate mobility and singular potential by sending the regularisation parameter to zero.
However, this strategy does not work in our case since solutions for the system with non-degenerate mobility are not regular enough in order to justify an appropriate testing procedure. As a remedy, we will regularize (7.2b) by adding a term $\delta\partial_t\mathbf{v}$ where $\delta > 0$ is

the same regularisation parameter as used for mobility and potential. Then. we derive uniform estimates in δ and we pass to the limit $\delta \to 0$.
Existence of weak solutions for (7.2)-(7.3) with non-degenerate mobility and regular potential will be proven just for the sake of completeness.

7.2 The non-degenerate case

Assumptions 7.2 *Throughout this section, we make the following assumptions.*

(i) The potential $\psi \in C^2(\mathbb{R})$ satisfies

$$|\psi'(t)| \leq C_1(1+|t|), \quad |\psi''(t)| \leq C_2 \quad \psi(t) \geq -C_3 \quad \forall\, t \in \mathbb{R} \tag{7.4}$$

with positive constants C_1, C_2 and C_3.

(ii) The initial data satisfy $\varphi_0 \in H^1$, $\sigma_0 \in L^6$.

(iii) The functions $g, f \colon \mathbb{R}^2 \to \mathbb{R}$ are continuous such that

$$|g(\varphi,\sigma)| \leq C_4(1+|\varphi|+|\sigma|), \qquad |f(\varphi,\sigma)| \leq C_5(1+|\varphi|+|\sigma|) \quad \forall\, \varphi,\sigma \in \mathbb{R} \tag{7.5}$$

for positive constants C_4 and C_5.

(iv) The function $h \colon \mathbb{R} \to \mathbb{R}$ is continuous, non-negative and bounded such that

$$\begin{aligned} h(\varphi) &= 0 && \text{if } \varphi \leq -1, \\ C_6(1+\varphi) \leq h(\varphi) &\leq C_7(1+\varphi) && \text{if } \varphi \in [-1,1], \\ h(\varphi) &\leq C_8 && \text{if } \varphi > 1 \end{aligned}$$

for positive constants C_6, C_7, C_8, and $C_6 \leq C_7$.

(v) For $d = 2, 3$, $\Omega \subset \mathbb{R}^d$ is a bounded domain with C^3-boundary.

Remark 7.3 From Assumptions 7.2, (iv), it follows that h behaves like $(1+\varphi)_+ \coloneqq \max(0, 1+\varphi)$ near $\varphi = -1$. A typical example is given by

$$h(\varphi) \coloneqq \max\left(0, \min\left(\frac{1}{2}(1+\varphi), 1\right)\right).$$

Furthermore, we observe that

$$h(\varphi) \leq h_\infty \quad \forall\, \varphi \in \mathbb{R},$$

where $h_\infty \coloneqq \max\{2C_7, C_8\}$. We refer to Chapter 9 for other examples of source terms that fulfil our assumptions.

The following result treats the case where the mobility is not degenerate.

Proposition 7.4 (non-degenerate mobility) *Let Assumptions 7.2 be fulfilled and let $m \in C^0(\mathbb{R})$ with $m_0 \leq m(s) \leq M_0$ for all $s \in \mathbb{R}$ for positive constants m_0 and M_0. Then, there exists a quadruplet $(\varphi, \sigma, \mu, \mathbf{v})$ with the regularity*

$$\begin{aligned} &\varphi \in H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^3), \quad \sigma \in H^1((H^1)^*) \cap L^\infty(L^6) \cap L^2(H^1), \\ &\mu \in L^4(L^2) \cap L^2(H^1), \quad \mathbf{v} \in L^{\frac{8}{3}}(\mathbf{V}) \cap L^{\frac{8}{5}}(\mathbf{H}^2) \end{aligned}$$

fulfilling the initial conditions together with equations (7.2a)-(7.2c), (7.2e) *in the sense that*

$$\varphi(0) = \varphi_0 \quad a.\,e.\ in\ \Omega, \qquad \sigma(0) = \sigma_0 \quad a.\,e.\ in\ \Omega,$$

and

$$\langle \partial_t \varphi, \xi \rangle_{H^1} = \int_\Omega -m(\varphi)\nabla\mu \cdot \nabla\xi + g(\varphi,\sigma)h(\varphi)\xi + \varphi\mathbf{v}\cdot\nabla\xi \,\mathrm{d}x, \tag{7.6a}$$

$$\langle \partial_t \sigma, \xi \rangle_{H^1} = \int_\Omega -(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi)\cdot\nabla\xi - f(\varphi,\sigma)h(\varphi)\xi + \sigma\mathbf{v}\cdot\nabla\xi \,\mathrm{d}x, \tag{7.6b}$$

$$\int_\Omega 2\eta\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x = \int_\Omega \varepsilon(\nabla\varphi\otimes\nabla\varphi)\colon \nabla\mathbf{u}\,\mathrm{d}x \tag{7.6c}$$

for all $\xi \in H^1$, $\mathbf{u} \in \mathbf{V}$ *and for a. e.* $t \in (0,T)$, *whereas* (7.2d) *and* $(7.3\mathrm{a})_1$ *are fulfilled almost everywhere in their respective sets, i. e.,*

$$\mu = -\varepsilon\Delta\varphi + \varepsilon^{-1}\psi'(\varphi) - \chi_\varphi\sigma \quad a.\,e.\ in\ \Omega_T, \qquad \nabla\varphi\cdot\mathbf{n} = 0 \quad a.\,e.\ on\ \Sigma_T. \tag{7.6d}$$

Moreover, the inequality

$$\begin{aligned} &\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)} + \|\sigma\|_{H^1((H^1)^*)\cap L^\infty(L^6)\cap L^2(H^1)} + \|\mu\|_{L^4(L^2)\cap L^2(H^1)} \\ &\quad + \|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)\cap L^{\frac{8}{5}}(\mathbf{H}^2)} + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathrm{div}(\sigma\mathbf{v})\|_{L^{\frac{8}{7}}(L^{\frac{3}{2}})} \le C \end{aligned} \tag{7.7}$$

is satisfied for a positive constant C *independent of* $(\varphi,\mu,\sigma,\mathbf{v})$.

Remark 7.5 As usual for Stokes-like equations, the pressure does not appear in the weak formulation (7.6). Thanks to Lemma 2.40, the pressure can be recovered using (7.6c). Indeed, by Proposition 7.4 we have that

$$-\eta\Delta\mathbf{v} + \nu\mathbf{v} + \varepsilon\mathrm{div}(\nabla\varphi\otimes\nabla\varphi) \in L^{\frac{8}{3}}(\mathbf{V}^*) \cap L^{\frac{8}{5}}(\mathbf{L}^2),$$

and therefore there exists a unique pressure $p \in L^{\frac{8}{3}}(L^2_0) \cap L^{\frac{8}{5}}(H^1)$ satisfying

$$-\eta\Delta\mathbf{v} + \nu\mathbf{v} + \nabla p = -\varepsilon\mathrm{div}(\nabla\varphi\otimes\nabla\varphi) \quad \text{a.e. in } \Omega_T.$$

Thus, we see that (7.6c) holds in the strong sense. In particular, by (7.7) the pressure satisfies

$$\|p\|_{L^{\frac{8}{3}}(L^2_0)\cap L^{\frac{8}{5}}(H^1)} \le C$$

for a constant C independent of $(\varphi,\mu,\sigma,\mathbf{v},p)$.

7.2.1 Construction of approximating solutions

In the following we consider for $\delta > 0$ the system

$$\begin{aligned} \mathrm{div}(\mathbf{v}) &= 0 && \text{in } \Omega_T, && (7.8\mathrm{a}) \\ \delta\partial_t\mathbf{v} - \mathrm{div}(2\eta\mathbf{D}\mathbf{v}) + \nu\mathbf{v} - \nabla p &= (\mu + \chi_\varphi\sigma)\nabla\varphi && \text{in } \Omega_T, && (7.8\mathrm{b}) \\ \partial_t\varphi + \mathrm{div}(\varphi\mathbf{v}) &= \mathrm{div}(m(\varphi)\nabla\mu) + g(\varphi,\sigma)h(\varphi) && \text{in } \Omega_T, && (7.8\mathrm{c}) \\ \mu &= -\varepsilon\Delta\varphi + \varepsilon^{-1}\psi'(\varphi) - \chi_\varphi\sigma && \text{in } \Omega_T, && (7.8\mathrm{d}) \\ \partial_t\sigma + \mathrm{div}(\sigma\mathbf{v}) &= \mathrm{div}(\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi) - f(\varphi,\sigma)h(\varphi) && \text{in } \Omega_T, && (7.8\mathrm{e}) \end{aligned}$$

supplemented with boundary and initial conditions of the form

$$\begin{aligned} &\nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} = \nabla\sigma\cdot\mathbf{n} = 0 && \text{on } \Sigma_T, && (7.9\mathrm{a}) \\ &\mathbf{v} = \mathbf{0} && \text{on } \Sigma_T, && (7.9\mathrm{b}) \\ &\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_{0,\delta}, \quad \mathbf{v}(0) = \mathbf{0} && \text{in } \Omega, && (7.9\mathrm{c}) \end{aligned}$$

where $\sigma_{0,\delta} \in H^2_N$ is the unique solution of

$$-\delta\Delta\sigma_{0,\delta} + \sigma_{0,\delta} = \sigma_0 \quad \text{in } \Omega, \qquad \nabla\sigma_{0,\delta}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{7.9d}$$

Remark 7.6 The modified capillary term on the r. h. s. of (7.8b) simplifies the a priori estimates since the convection term in (7.8c) and the term on the r. h. s. of (7.8b) cancel out within the testing procedure. This is not the case if we use $-\mathrm{div}(\varepsilon(\nabla\varphi\otimes\nabla\varphi))$ as we do not have the formula

$$\int_\Omega -\varepsilon(\nabla\varphi\otimes\nabla\varphi)\colon\nabla\mathbf{v}\,\mathrm{d}x = \int_\Omega (\mu+\chi_\varphi\sigma)\nabla\varphi\cdot\mathbf{v}\,\mathrm{d}x \quad \forall\,\mathbf{u}\in\mathbf{V}$$

on the Galerkin level.

We now prove the following lemma:

Lemma 7.7 (Existence of approximating solutions) *Let $m \in C^0(\mathbb{R})$ with $m_0 \leq m(s) \leq M_0$ for all $s\in\mathbb{R}$ with positive constants m_0, M_0, and let Assumptions 7.2 be fulfilled. Then, there exists a quadruplet $(\varphi_\delta,\sigma_\delta,\mu_\delta,\mathbf{v}_\delta)$ with the regularity*

$$\varphi_\delta \in H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3), \quad \sigma_\delta\in H^1(L^2)\cap L^\infty(H^1)\cap L^2(H^2),$$
$$\mu_\delta\in L^4(L^2)\cap L^2(H^1), \quad \mathbf{v}_\delta\in H^1(L^{\frac{3}{2}})\cap L^\infty(L^2)\cap L^{\frac{16}{5}}(\mathbf{V})\cap L^{\frac{8}{5}}(\mathbf{H}^2)$$

such that the initial conditions and equations (7.8a)-(7.8c), (7.8e) *are fulfilled in the sense that*

$$\varphi_\delta(0)=\varphi_0, \quad \sigma_\delta(0)=\sigma_{0,\delta}, \quad \mathbf{v}_\delta(0)=\mathbf{0} \quad \textit{a. e. in } \Omega,$$

and

$$0 = \langle\partial_t\varphi_\delta,\xi\rangle_{H^1} + \int_\Omega \nabla\varphi_\delta\cdot\mathbf{v}_\delta\,\xi + m(\varphi_\delta)\nabla\mu_\delta\cdot\nabla\xi - g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\xi\,\mathrm{d}x, \tag{7.10a}$$

$$0 = \int_\Omega \delta\partial_t\mathbf{v}_\delta\cdot\mathbf{u} + 2\eta\mathbf{D}\mathbf{v}_\delta\colon\mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta\cdot\mathbf{u} - (\mu_\delta+\chi_\varphi\sigma_\delta)\nabla\varphi_\delta\cdot\mathbf{u}\,\mathrm{d}x \tag{7.10b}$$

for all $\xi\in H^1$, $\mathbf{u}\in\mathbf{V}$, and for a. e. $t\in(0,T)$, whereas (7.8d)-(7.8e) *and* $(7.9\mathrm{a})_1$, $(7.9\mathrm{a})_3$, *are fulfilled almost everywhere in their respective sets, i. e.,*

$$\mu_\delta = -\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi'(\varphi_\delta) - \chi_\varphi\sigma_\delta \quad \textit{a. e. in } \Omega_T, \qquad \nabla\varphi_\delta\cdot\mathbf{n}=0 \quad \textit{a. e. on } \Sigma_T, \tag{7.10c}$$

and

$$\begin{aligned} &\partial_t\sigma_\delta + \nabla\sigma_\delta\cdot\mathbf{v}_\delta = \chi_\sigma\Delta\sigma_\delta - \chi_\varphi\Delta\varphi_\delta - f(\varphi_\delta,\sigma_\delta)h(\varphi_\delta) && \textit{a. e. in } \Omega_T,\\ &\nabla\sigma_\delta\cdot\mathbf{n} = 0 && \textit{a. e. on } \Sigma_T. \end{aligned} \tag{7.10d}$$

Moreover, the estimate

$$\begin{aligned} &\|\varphi_\delta\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)} + \|\sigma_\delta\|_{H^1(L^2)\cap L^\infty(H^1)\cap L^2(H^2)}\\ &\quad + \|\mu_\delta\|_{L^4(L^2)\cap L^2(H^1)} + \|\mathbf{v}_\delta\|_{H^1(L^{\frac{3}{2}})\cap L^\infty(L^2)\cap L^{\frac{16}{5}}(\mathbf{V})\cap L^{\frac{8}{5}}(\mathbf{H}^2)} \leq C \end{aligned} \tag{7.11}$$

is satisfied for a constant C independent of $(\varphi_\delta,\mu_\delta,\sigma_\delta,\mathbf{v}_\delta)$.

Remark 7.8 As before, we can reconstruct the pressure $p_\delta \in L^{\frac{8}{3}}(L^2_0)\cap L^{\frac{8}{5}}(H^1)$ such that

$$\delta\partial_t\mathbf{v} - \mathrm{div}(2\eta\mathbf{D}\mathbf{v}) + \nu\mathbf{v} - \nabla p = (\mu+\chi_\varphi\sigma)\nabla\varphi \quad \text{a. e. in } \Omega_T$$

and

$$\|p_\delta\|_{L^{\frac{8}{3}}(L^2_0)\cap L^{\frac{8}{5}}(H^1)} \leq C$$

holds for a constant C independent of $(\varphi_\delta,\mu_\delta,\sigma_\delta,\mathbf{v}_\delta,p_\delta)$.

Proof of Lemma 7.7. The proof is based on ideas presented in [77] and [83, Theorem 2.1].

Galerkin scheme Let $\{\phi_i\}_{i\in\mathbb{N}}$ be eigenfunctions of the Neumann–Laplace operator, see Chapter 2. Furthermore, let $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$ be the eigenfunctions of the Stokes operator with corresponding eigenvalues $\{\eta_j\}_{j\in\mathbb{N}}$, that means

$$\begin{cases} -\Delta \mathbf{w}_j &= \eta_j \mathbf{w}_j & \text{in } \Omega, \\ \operatorname{div}(\mathbf{w}_j) &= 0 & \text{in } \Omega, \\ \mathbf{w}_j &= \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

It is well-known that $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$ forms (after normalising) an orthonormal Schauder basis in $\mathbf{H}$ which is orthogonal in $\mathbf{V}$ (see, e. g., [90, II.3, Prop. 8, c), p. 135]). We fix $n \in \mathbb{N}$ and put $\mathcal{W}_n := \operatorname{span}\{\phi_1, \dots, \phi_n\}$, $\mathcal{V}_n := \operatorname{span}\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$. Furthermore, we define the projections $\Pi_{\mathcal{W}_n} : L^2 \to \mathcal{W}_n$ and $\Pi_{\mathcal{V}_n} : \mathbf{H} \to \mathcal{V}_n$ by

$$\Pi_{\mathcal{W}_n}\phi = \sum_{i=1}^n (\phi, \phi_i)_{L^2}\phi_i \quad \forall\, \phi \in L^2, \qquad \Pi_{\mathcal{V}_n}\mathbf{w} = \sum_{i=1}^n (\mathbf{w}, \mathbf{w}_i)_{\mathbf{L}^2}\mathbf{w}_i \quad \forall\, \mathbf{w} \in \mathbf{H}.$$

Our aim is to find functions of the form

$$\varphi_{n,\delta}(t,x) = \sum_{i=1}^n a_i^{n,\delta}(t)\phi_i(x), \qquad \mu_{n,\delta}(t,x) = \sum_{i=1}^n b_i^{n,\delta}(t)\phi_i(x),$$
$$\sigma_{n,\delta}(t,x) = \sum_{i=1}^n c_i^{n,\delta}(t)\phi_i(x), \qquad \mathbf{v}_{n,\delta}(t,x) = \sum_{i=1}^n d_i^{n,\delta}(t)\mathbf{w}_i(x),$$

satisfying the approximation problem

$$\int_\Omega \partial_t\varphi_{n,\delta} v \,\mathrm{d}x = \int_\Omega -m(\varphi_{n,\delta})\nabla\mu_{n,\delta}\cdot\nabla v + g(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})v - \nabla\varphi_{n,\delta}\cdot\mathbf{v}_{n,\delta} v \,\mathrm{d}x, \tag{7.12a}$$
$$\int_\Omega \mu_{n,\delta} v \,\mathrm{d}x = \int_\Omega \varepsilon\nabla\varphi_{n,\delta}\cdot\nabla v + \varepsilon^{-1}\psi'(\varphi_{n,\delta})v - \chi_\varphi\sigma_{n,\delta}v \,\mathrm{d}x, \tag{7.12b}$$
$$\int_\Omega \partial_t\sigma_{n,\delta} v \,\mathrm{d}x = \int_\Omega \nabla(\chi_\varphi\varphi_{n,\delta} - \chi_\sigma\sigma_{n,\delta})\cdot\nabla v - f(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})v - \nabla\sigma_{n,\delta}\cdot\mathbf{v}_{n,\delta} v \,\mathrm{d}x \tag{7.12c}$$

for all $v \in \mathcal{W}_n$, and

$$\int_\Omega \partial_t\mathbf{v}_{n,\delta}\cdot\mathbf{u} + \eta\nabla\mathbf{v}_{n,\delta} : \nabla\mathbf{u} + \nu\mathbf{v}_{n,\delta}\cdot\mathbf{u} \,\mathrm{d}x = \int_\Omega (\mu_{n,\delta} + \chi_\varphi\sigma_{n,\delta})\nabla\varphi_{n,\delta}\cdot\mathbf{u} \,\mathrm{d}x \tag{7.12d}$$

for all $\mathbf{u} \in \mathcal{V}_n$. We equip the system with the initial data

$$\varphi_{n,\delta}(0) = \Pi_{\mathcal{W}_n}\varphi_0, \quad \sigma_{n,\delta}(0) = \Pi_{\mathcal{W}_n}\sigma_{0,\delta}, \quad \mathbf{v}_{n,\delta}(0) = \mathbf{0}. \tag{7.13}$$

Furthermore, we define $\mathbf{a}_{n,\delta} := (a_1^{n,\delta}, \dots, a_n^{n,\delta})^\intercal$, $\mathbf{b}_{n,\delta} := (b_1^{n,\delta}, \dots, b_n^{n,\delta})^\intercal$, $\mathbf{c}_{n,\delta} := (c_1^{n,\delta}, \dots, c_n^{n,\delta})^\intercal$ and $\mathbf{d}_{n,\delta} := (d_1^{n,\delta}, \dots, d_n^{n,\delta})^\intercal$. Inserting $v = \phi_j$, $1 \le j \le n$, in (7.12)-(7.12c), $\mathbf{u} = \mathbf{w}_j$, $1 \le j \le n$, in (7.12d) and using the ansatz for the functions $\varphi_{n,\delta}$, $\mu_{n,\delta}$, $\sigma_{n,\delta}$ and $\mathbf{v}_{n,\delta}$, we can rewrite (7.12)-(7.13) as a system of coupled, non-linear ODEs for the unknowns $\mathbf{a}_{n,\delta}$, $\mathbf{c}_{n,\delta}$ and $\mathbf{d}_{n,\delta}$. Owing to the continuity of m, f, g, h and ψ', the Cauchy–Peano theorem ensures that there exists $T_n^* \in (0, \infty]$ such that (7.12)-(7.13) has at least one solution quadruplet

$$(\varphi_{n,\delta}, \mu_{n,\delta}, \sigma_{n,\delta}, \mathbf{v}_{n,\delta}) \in (C^1([0,T_n^*]; \mathcal{W}_n))^3 \times C^1([0,T_n^*]; \mathcal{V}_n).$$

Now, we show a priori estimates for the Galerkin ansatz functions. In particular, this will lead $T_n^* = T$.

A priori estimates Multiplying (7.9d) with $\sigma_{0,\delta}$ and integrating over Ω and by parts, we obtain

$$\delta\|\nabla\sigma_{0,\delta}\|_{\mathbf{L}^2}^2 + \|\sigma_{0,\delta}\|_{L^2}^2 \leq C\|\sigma_0\|_{L^2}^2.$$

Therefore, the continuity of $\Pi_{\mathcal{W}_n}$ on H^1 entails

$$\|\Pi_{\mathcal{W}_n}\sigma_{0,\delta}\|_{H^1} \leq C_0\|\sigma_0\|_{L^2}, \quad \|\Pi_{\mathcal{W}_n}\varphi_0\|_{H^1} \leq \tilde{C}_0\|\varphi_0\|_{H^1} \tag{7.14}$$

for a constant C_0 depending on δ, but not on $n \in \mathbb{N}$, and a constant $\tilde{C}_0$ independent of δ and $n \in \mathbb{N}$.

We choose $v = b_j^{n,\delta}\phi_j$ in (7.12a), $v = -(a_j^{n,\delta})'\phi_j$ in (7.12b), $v = \chi_\sigma c_j^{n,\delta}\phi_j - \chi_\varphi(1 - a_j^{n,\delta}\phi_j)$ in (7.12c) and sum the resulting identities over $j = 1, \dots, n$, to obtain

$$\begin{aligned}
\int_\Omega \partial_t\varphi_{n,\delta}\mu_{n,\delta}\,\mathrm{d}x &= \int_\Omega -m(\varphi_{n,\delta})|\nabla\mu_{n,\delta}|^2 + g(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})\mu_{n,\delta} - \mu_{n,\delta}\nabla\varphi_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x,\\
-\int_\Omega \mu_{n,\delta}\partial_t\varphi_{n,\delta}\,\mathrm{d}x &= \int_\Omega -\varepsilon\nabla\varphi_{n,\delta}\cdot\nabla\partial_t\varphi_{n,\delta} - \varepsilon^{-1}\psi'(\varphi_{n,\delta})\partial_t\varphi_{n,\delta} + \chi_\varphi\sigma_{n,\delta}\partial_t\varphi_{n,\delta}\,\mathrm{d}x,\\
\int_\Omega \partial_t\sigma_{n,\delta}N_{,\sigma}^{n,\delta}\,\mathrm{d}x &= -\int_\Omega |\nabla N_{,\sigma}^{n,\delta}|^2 - f(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})N_{,\sigma}^{n,\delta} - N_{,\sigma}^{n,\delta}\nabla\sigma_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x,
\end{aligned}$$

where $N_{,\sigma}^{n,\delta} = \chi_\sigma\sigma_{n,\delta} + \chi_\varphi(1-\varphi_{n,\delta})$. Adding up the three equations and using the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}N(\varphi_{n,\delta},\sigma_{n,\delta}) = N_{,\sigma}^{n,\delta}\partial_t\sigma_{n,\delta} - \chi_\varphi\sigma_{n,\delta}\partial_t\varphi_{n,\delta}$$

yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi_{n,\delta}|^2 + \varepsilon^{-1}\psi(\varphi_{n,\delta}) + N(\varphi_{n,\delta},\sigma_{n,\delta})\,\mathrm{d}x + \int_\Omega m(\varphi_{n,\delta})|\nabla\mu_{n,\delta}|^2 + |\nabla N_{,\sigma}^{n,\delta}|^2\,\mathrm{d}x\\
&\quad = \int_\Omega g(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})\mu_{n,\delta} - f(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})N_{,\sigma}^{n,\delta}\,\mathrm{d}x\\
&\qquad - \int_\Omega (\mu_{n,\delta}\nabla\varphi_{n,\delta} + N_{,\sigma}^{n,\delta}\nabla\sigma_{n,\delta})\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x.
\end{aligned}$$

Choosing $\mathbf{u} = d_j^{n,\delta}\mathbf{w}_j$ in (7.12d) and summing the resulting identities over $j = 1, \dots, n$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta}{2}\int_\Omega |\mathbf{v}_{n,\delta}|^2\,\mathrm{d}x + \int_\Omega 2\eta|\mathbf{D}\mathbf{v}_{n,\delta}|^2 + \nu|\mathbf{v}_{n,\delta}|^2\,\mathrm{d}x = \int_\Omega (\mu_{n,\delta} + \chi_\varphi\sigma_{n,\delta})\nabla\varphi_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x.$$

Using $\mathrm{div}(\mathbf{v}_{n,\delta}) = 0$ a. e. in Ω and $\mathbf{v}_{n,\delta} = \mathbf{0}$ a. e. on $\partial\Omega$, we deduce

$$\begin{aligned}
-\int_\Omega N_{,\sigma}^{n,\delta}\nabla\sigma_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x &= -\int_\Omega \left(\tfrac{\chi_\sigma}{2}\nabla\left(|\sigma_{n,\delta}|^2\right) + \chi_\varphi(1-\varphi_{n,\delta})\nabla\sigma_{n,\delta}\right)\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x\\
&= -\int_\Omega \chi_\varphi\sigma_{n,\delta}\nabla\varphi_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\mathrm{d}x.
\end{aligned}$$

Employing the last three identities leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi_{n,\delta}|^2 + \varepsilon^{-1}\psi(\varphi_{n,\delta}) + N(\varphi_{n,\delta},\sigma_{n,\delta}) + \frac{\delta}{2}|\mathbf{v}_{n,\delta}|^2\,\mathrm{d}x\\
&\qquad + \int_\Omega m(\varphi_{n,\delta})|\nabla\mu_{n,\delta}|^2 + |\nabla N_{,\sigma}^{n,\delta}|^2 + 2\eta|\mathbf{D}\mathbf{v}_{n,\delta}|^2 + \nu|\mathbf{v}_{n,\delta}|^2\,\mathrm{d}x\\
&\quad = \int_\Omega g(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})\mu_{n,\delta} - f(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})N_{,\sigma}^{n,\delta}\,\mathrm{d}x.
\end{aligned} \tag{7.15}$$

We now estimate the terms on the r. h. s. of (7.15) individually. In order to control the term involving g, we need a bound on the mean $(\mu_{n,\delta})_\Omega$. Taking $v = 1$ in (7.12b) we see that

$$\int_\Omega \mu_{n,\delta} \, \mathrm{d}x = \int_\Omega \varepsilon^{-1}\psi'(\varphi_{n,\delta}) - \chi_\varphi \sigma_{n,\delta} \, \mathrm{d}x.$$

Due to (7.4), this implies

$$\left| \int_\Omega \mu_{n,\delta} \, \mathrm{d}x \right| \leq c_2 \left(1 + \|\varphi_{n,\delta}\|_{L^2} + \|\sigma_{n,\delta}\|_{L^2}\right),$$

where $c_2 = c_2(C_1, \varepsilon, \chi_\varphi, |\Omega|)$. Applying (7.5), we obtain from Hölder's, Young's and Poincaré's inequalities that

$$\begin{aligned}
&\left| \int_\Omega g(\varphi_{n,\delta}, \sigma_{n,\delta}) h(\varphi_{n,\delta}) \mu_{n,\delta} \, \mathrm{d}x \right| \\
&\quad \leq C_4 h_\infty \left(|\Omega|^{\frac{1}{2}} + \|\varphi_{n,\delta}\|_{L^2} + \|\sigma_{n,\delta}\|_{L^2} \right) C_P \left(\|\nabla \mu_{n,\delta}\|_{\mathbf{L}^2} + |(\mu_{n,\delta})_\Omega| \right) \\
&\quad \leq c_3 \left(1 + \|\varphi_{n,\delta}\|_{L^2}^2 + \|\sigma_{n,\delta}\|_{L^2}^2\right) + \frac{m_0}{2} \|\nabla \mu_{n,\delta}\|_{\mathbf{L}^2}^2
\end{aligned}$$

with $c_3 = c_3(c_2, C_4, h_\infty, |\Omega|, C_P, m_0)$. Due to Minkowski's and Young's inequalities, we have

$$\|\chi_\sigma \nabla \sigma_{n,\delta}\|_{\mathbf{L}^2}^2 \leq \left(\|\nabla N_{,\sigma}^{n,\delta}\|_{\mathbf{L}^2} + \|\chi_\varphi \nabla \varphi_{n,\delta}\|_{\mathbf{L}^2} \right)^2 \leq 2\|\nabla N_{,\sigma}^{n,\delta}\|_{\mathbf{L}^2}^2 + 2\|\chi_\varphi \nabla \varphi_{n,\delta}\|_{\mathbf{L}^2}^2.$$

For the term involving f, by Hölder's and Young's inequalities we infer

$$\left| \int_\Omega f(\varphi_{n,\delta}, \sigma_{n,\delta}) h(\varphi_{n,\delta}) (\chi_\sigma \sigma_{n,\delta} - \chi_\varphi(1 - \varphi_{n,\delta})) \, \mathrm{d}x \right| \leq c_1 \left(1 + \|\sigma_{n,\delta}\|_{L^2}^2 + \|\varphi_{n,\delta}\|_{L^2}^2\right),$$

where $c_1 = c_1(C_5, h_\infty, \chi_\varphi, \chi_\sigma, |\Omega|)$. On account of the last three estimates and the assumptions on $m(\cdot)$, (7.15) becomes

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\varepsilon}{2} |\nabla \varphi_{n,\delta}|^2 + \varepsilon^{-1} \psi(\varphi_{n,\delta}) + N(\varphi_{n,\delta}, \sigma_{n,\delta}) + \frac{\delta}{2} |\mathbf{v}_{n,\delta}|^2 \, \mathrm{d}x \\
&\qquad + \int_\Omega \frac{m_0}{2} |\nabla \mu_{n,\delta}|^2 + \frac{\chi_\sigma^2}{2} |\nabla \sigma_{n,\delta}|^2 + 2\eta |\mathbf{D}\mathbf{v}_{n,\delta}|^2 + \nu |\mathbf{v}_{n,\delta}|^2 \, \mathrm{d}x \\
&\quad \leq C \left(1 + \int_\Omega |\varphi_{n,\delta}|^2 + |\nabla \varphi_{n,\delta}|^2 + |\sigma_{n,\delta}|^2 \, \mathrm{d}x \right)
\end{aligned} \tag{7.16}$$

with C depending on the same quantities as c_i, $1 \leq i \leq 3$. Now, we introduce the initial energy

$$c_{0,\delta} \coloneqq \int_\Omega \left(\varepsilon^{-1} \psi(\varphi_0) + \frac{\varepsilon}{2} |\nabla \varphi_0|^2 + \frac{\chi_\sigma}{2} |\sigma_{0,\delta}|^2 + \chi_\varphi \sigma_{0,\delta}(1 - \varphi_0) \right) \, \mathrm{d}x$$

that is bounded due to the assumptions on ψ and the initial data. Integrating (7.16) in time from 0 to $s \in (0, T]$ and using (7.14) along with the assumptions on $\psi(\cdot)$ gives

$$\begin{aligned}
&\int_\Omega \frac{\varepsilon}{2} |\nabla \varphi_{n,\delta}(s)|^2 + \varepsilon^{-1} \psi(\varphi_{n,\delta}(s)) + N(\varphi_{n,\delta}(s), \sigma_{n,\delta}(s)) + \frac{\delta}{2} |\mathbf{v}_{n,\delta}(s)|^2 \, \mathrm{d}x \\
&\quad + \frac{m_0}{2} \|\nabla \mu_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + \frac{\chi_\sigma^2}{2} \|\nabla \sigma_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + 2\eta \|\mathbf{D}\mathbf{v}_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + \nu \|\mathbf{v}_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 \\
&\leq c_{0,\delta} + C \left(s + \|\varphi_{n,\delta}\|_{L^2(0,s;L^2)}^2 + \|\nabla \varphi_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + \|\sigma_{n,\delta}\|_{L^2(0,s;L^2)}^2 \right).
\end{aligned} \tag{7.17}$$

Employing Hölder's and Young's inequalities, we get

$$\begin{aligned}
\left| \int_\Omega \chi_\varphi \sigma_{n,\delta}(s)(1 - \varphi_{n,\delta}(s)) \, \mathrm{d}x \right| &\leq \chi_\varphi \|\sigma_{n,\delta}(s)\|_{L^1} + \chi_\varphi \|\sigma_{n,\delta}(s)\|_{L^2} \|\varphi_{n,\delta}(s)\|_{L^2} \\
&\leq \frac{\chi_\sigma}{4} \|\sigma_{n,\delta}(s)\|_{L^2}^2 + \frac{2\chi_\varphi^2}{\chi_\sigma} \left(|\Omega| + \|\varphi_{n,\delta}(s)\|_{L^2}^2 \right).
\end{aligned} \tag{7.18}$$

Now, the standard strategy to absorb the last term on the r. h. s. of (7.18) is to invoke a lower bound on the potential $\psi(\cdot)$. As we will not be able to guarantee uniform lower bounds for $\psi(\cdot)$ in the degenerate case, we use a different approach.

More precisely, we choose $v = \frac{8\chi_\varphi^2}{\chi_\sigma} a_j^{n,\delta}\phi_j$ in (7.12a) and sum the resulting equations over $j = 1, \dots, n$, to obtain

$$\frac{4\chi_\varphi^2}{\chi_\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\varphi_{n,\delta}|^2 \,\mathrm{d}x = \frac{8\chi_\varphi^2}{\chi_\sigma}\int_\Omega -m(\varphi_{n,\delta})\nabla\mu_{n,\delta}\cdot\nabla\varphi_{n,\delta} + h(\varphi_{n,\delta})g(\varphi_{n,\delta},\sigma_{n,\delta})\varphi_{n,\delta}\,\mathrm{d}x. \quad (7.19)$$

Using Hölder's and Young's inequalities together with the assumptions on $m(\cdot)$, we have

$$\begin{aligned}\frac{8\chi_\varphi^2}{\chi_\sigma}\left|\int_\Omega m(\varphi_{n,\delta})\nabla\mu_{n,\delta}\cdot\nabla\varphi_{n,\delta}\,\mathrm{d}x\right| &\le \frac{8\chi_\varphi^2 M_0}{\chi_\sigma}\|\nabla\mu_{n,\delta}\|_{\mathbf{L}^2}\|\nabla\varphi_{n,\delta}\|_{\mathbf{L}^2}\\ &\le \frac{m_0}{4}\|\nabla\mu_{n,\delta}\|_{\mathbf{L}^2}^2 + c_4\|\nabla\varphi_{n,\delta}\|_{\mathbf{L}^2}^2\end{aligned}$$

with $c_4 = c_4(\chi_\varphi,\chi_\sigma,m_0,M_0)$. Furthermore, by (7.5) and Hölder's and Young's inequalities, we observe that

$$\frac{8\chi_\varphi^2}{\chi_\sigma}\left|\int_\Omega h(\varphi_{n,\delta})g(\varphi_{n,\delta},\sigma_{n,\delta})\varphi_{n,\delta}\,\mathrm{d}x\right| \le c_5\left(1+\|\varphi_{n,\delta}\|_{L^2}^2+\|\sigma_{n,\delta}\|_{L^2}^2\right)$$

with $c_5 = c_5(\chi_\varphi,\chi_\sigma,C_4,h_\infty,|\Omega|)$. Using the assumptions on φ_0, we see that

$$c_6 \coloneqq \frac{4\chi_\varphi^2}{\chi_\sigma}\|\varphi_0\|_{L^2}^2$$

is bounded. Integrating (7.19) in time from 0 to $s \in (0,T]$ and using the last two inequalities yields

$$\begin{aligned}\frac{4\chi_\varphi^2}{\chi_\sigma}\|\varphi_{n,\delta}(s)\|_{L^2}^2 &\le c_6 + \frac{m_0}{4}\|\nabla\mu_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + c_4\|\nabla\varphi_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2\\ &\quad + c_5\left(s+\|\varphi_{n,\delta}\|_{L^2(0,s;L^2)}^2+\|\sigma_{n,\delta}\|_{L^2(0,s;L^2)}^2\right).\end{aligned}$$

Adding this inequality to (7.17) and using (7.18), we get

$$\begin{aligned}&\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi_{n,\delta}(s)|^2 + \varepsilon^{-1}\psi(\varphi_{n,\delta}(s)) + \frac{\chi_\sigma}{4}|\sigma_{n,\delta}(s)|^2 + \frac{2\chi_\varphi^2}{\chi_\sigma}|\varphi_{n,\delta}(s)|^2 + \frac{\delta}{2}|\mathbf{v}_{n,\delta}(s)|^2\,\mathrm{d}x\\ &\quad + \frac{m_0}{4}\|\nabla\mu_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + \frac{\chi_\sigma^2}{2}\|\nabla\sigma_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + 2\eta\|\mathbf{D}\mathbf{v}_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + \nu\|\mathbf{v}_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2\\ &\le C\left(1+\|\varphi_{n,\delta}\|_{L^2(0,s;L^2)}^2 + \|\nabla\varphi_{n,\delta}\|_{L^2(0,s;\mathbf{L}^2)}^2 + \|\sigma_{n,\delta}\|_{L^2(0,s;L^2)}^2\right),\end{aligned}$$

where C depends only on s and the same quantities as $\{c_i\}_{0\le i\le 6}$, but not on $n \in \mathbb{N}$. Together with the estimate

$$\left|\int_\Omega \mu_{n,\delta}\,\mathrm{d}x\right| \le c_2\left(1+\|\varphi_{n,\delta}\|_{L^2}+\|\sigma_{n,\delta}\|_{L^2}\right)$$

and (7.4), a Gronwall argument implies that

$$\begin{aligned}&\operatorname*{ess\,sup}_{s\in(0,T]}\left(\|\psi(\varphi_{n,\delta})(s)\|_{L^1} + \|\varphi_{n,\delta}(s)\|_{H^1}^2 + \|\sigma_{n,\delta}(s)\|_{L^2}^2 + \|\mathbf{v}_{n,\delta}(s)\|_{\mathbf{L}^2}^2\right)\\ &\quad + \int_0^T \|\mu_{n,\delta}\|_{H^1}^2 + \|\nabla\sigma_{n,\delta}\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_{n,\delta}\|_{\mathbf{H}^1}^2\,\mathrm{d}t \le \bar{C}\end{aligned} \quad (7.20)$$

for a positive constant $\bar{C}$ depending on the system parameters, on δ, Ω and T, but not on $n \in \mathbb{N}$.

Higher order estimates With exactly the same arguments as Chapter 5, we obtain that

$$\|\varphi_{n,\delta}\|_{L^2(H^3)} \leq \tilde{C}. \tag{7.21}$$

Using the assumptions on $\psi(\cdot)$ and recalling that $\Pi_{\mathcal{W}_n}$ is the L^2-orthogonal projection onto $\mathcal{W}_n$, this implies

$$\|\Pi_{\mathcal{W}_n}(\psi'(\varphi_{n,\delta}))\|_{L^4(L^2)} \leq C\|\psi'(\varphi_{n,\delta})\|_{L^4(L^2)} \leq C(1+\|\varphi_{n,\delta}\|_{L^4(L^2)}) \leq C.$$

Since $\varphi_{n,\delta} \in H^2_N$, we can calculate

$$\|\Delta\varphi_{n,\delta}\|^2_{L^2} \leq C\|\nabla\varphi_{n,\delta}\|_{\mathbf{L}^2}\|\nabla\Delta\varphi_{n,\delta}\|_{\mathbf{L}^2} \leq C\|\varphi_{n,\delta}\|_{H^1}\|\varphi_{n,\delta}\|_{H^3},$$

which implies

$$\|\Delta\varphi_{n,\delta}\|_{L^4(L^2)} \leq C\|\varphi_{n,\delta}\|^{\frac{1}{2}}_{L^\infty(H^1)}\|\varphi_{n,\delta}\|^{\frac{1}{2}}_{L^2(H^3)} \leq C.$$

From the last three inequalities, we deduce that $\varepsilon^{-1}\Pi_{\mathcal{W}_n}\psi'(\varphi_{n,\delta}) - \varepsilon\Delta\varphi_{n,\delta} \in L^4(L^2)$ with bounded norm which together with (7.12b) leads to

$$\|\mu_{n,\delta} + \chi_\varphi\sigma_{n,\delta}\|_{L^4(L^2)} \leq C.$$

In particular, we obtain that $\mu_{n,\delta}$ is uniformly bounded in $L^4(L^2)$. By (2.4) and Sobolev embedding theory, we have the continuous embeddings $L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^2) \hookrightarrow L^{\frac{8}{3}}(\mathbf{L}^\infty)$ and $H^1 \subset L^6$. Then, it follows that $(\mu_{n,\delta} + \chi_\varphi\sigma_{n,\delta})\nabla\varphi_{n,\delta}$ is bounded uniformly in $L^{\frac{8}{5}}(\mathbf{L}^2) \cap L^2(\mathbf{L}^{\frac{3}{2}})$. By classical regularity theory for the instationary Stokes equation (see, e. g., [90, II.3, Cor. 4, p. 148]), we conclude that

$$\|\mathbf{v}_{n,\delta}\|_{H^1(L^{\frac{3}{2}}) \cap L^{\frac{8}{5}}(\mathbf{H}^2)} \leq C.$$

Applying (2.4) combined with (7.20) and using the last bound, it holds

$$\|\mathbf{v}_{n,\delta}\|_{H^1(L^{\frac{3}{2}}) \cap L^2(\mathbf{W}^{1,\frac{10}{3}}) \cap L^{\frac{8}{5}}(\mathbf{H}^2)} \leq C. \tag{7.22}$$

Now, we derive higher order estimates for the nutrient concentration $\sigma_{n,\delta}$. Choosing $v = \lambda_i c_i^{n,\delta}\phi_i$, $1 \leq i \leq n$, in (7.12c), integrating by parts and summing the resulting equations over $i = 1, \ldots, n$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\nabla\sigma_{n,\delta}|^2 \,\mathrm{d}x + \chi_\sigma\int_\Omega |\Delta\sigma_{n,\delta}|^2 \,\mathrm{d}x = \int_\Omega \big(\chi_\varphi\Delta\varphi_{n,\delta} - f(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})\big)\,\Delta\sigma_{n,\delta}\,\mathrm{d}x \\ + \int_\Omega \nabla\sigma_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\Delta\sigma_{n,\delta}\,\mathrm{d}x.\end{aligned} \tag{7.23}$$

Using the assumptions on f, h and (7.20)-(7.21), an application of Hölder's and Young's inequalities yields

$$\left|\int_\Omega \big(\chi_\varphi\Delta\varphi_{n,\delta} - f(\varphi_{n,\delta},\sigma_{n,\delta})h(\varphi_{n,\delta})\big)\,\Delta\sigma_{n,\delta}\,\mathrm{d}x\right| \leq C(1+\|\Delta\varphi_{n,\delta}\|^2_{L^2}) + \frac{\chi_\sigma}{4}\|\Delta\sigma_{n,\delta}\|^2_{L^2}.$$

With similar arguments and using the Sobolev embedding $\mathbf{W}^{1,\frac{10}{3}} \subset \mathbf{L}^\infty$, we infer that

$$\begin{aligned}\left|\int_\Omega \nabla\sigma_{n,\delta}\cdot\mathbf{v}_{n,\delta}\,\Delta\sigma_{n,\delta}\,\mathrm{d}x\right| &\leq \|\nabla\sigma_{n,\delta}\|_{\mathbf{L}^2}\|\mathbf{v}_{n,\delta}\|_{\mathbf{L}^\infty}\|\Delta\sigma_{n,\delta}\|_{L^2} \\ &\leq C\|\nabla\sigma_{n,\delta}\|^2_{\mathbf{L}^2}\|\mathbf{v}_{n,\delta}\|^2_{\mathbf{W}^{1,\frac{10}{3}}} + \frac{\chi_\sigma}{4}\|\Delta\sigma_{n,\delta}\|^2_{L^2}.\end{aligned}$$

Employing the last two inequalities in (7.23) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\sigma_{n,\delta}|^2 \,\mathrm{d}x + \chi_\sigma\int_\Omega |\Delta\sigma_{n,\delta}|^2 \,\mathrm{d}x \leq C(1+\|\Delta\varphi_{n,\delta}\|^2_{L^2}) + C\|\mathbf{v}_{n,\delta}\|^2_{\mathbf{W}^{1,\frac{10}{3}}}\|\nabla\sigma_{n,\delta}\|^2_{\mathbf{L}^2}.$$

Integrating this inequality in time from 0 to $s \in (0,T]$, using (7.14), (7.21)-(7.22) and elliptic regularity theory, a Gronwall argument yields

$$\|\sigma_{n,\delta}\|_{L^\infty(H^1)\cap L^2(H^2)} \leq C. \tag{7.24}$$

Estimates for the time derivatives and convection terms By (7.20), (7.22), the Sobolev embedding $\mathbf{W}^{1,\frac{10}{3}} \subset \mathbf{L}^\infty$ and Hölder's inequality, we have

$$\|\nabla\varphi_{n,\delta} \cdot \mathbf{v}_{n,\delta}\|_{L^2(L^2)} \leq C\|\nabla\varphi_{n,\delta}\|_{L^\infty(\mathbf{L}^2)}\|\mathbf{v}_{n,\delta}\|_{L^2(\mathbf{L}^\infty)} \leq C\|\varphi_{n,\delta}\|_{L^\infty(H^1)}\|\mathbf{v}_{n,\delta}\|_{L^2(\mathbf{W}^{1,\frac{10}{3}})} \leq C,$$

and therefore

$$\|\mathrm{div}(\varphi_{n,\delta}\mathbf{v}_{n,\delta})\|_{L^2(L^2)} \leq C. \tag{7.25}$$

Furthermore, invoking (7.20), (7.25) along with the assumptions on g and h, for arbitrary $\zeta \in L^2(H^1)$ we have

$$\left|\int_0^T \int_\Omega \partial_t\varphi_{n,\delta}\zeta \,\mathrm{d}x\,\mathrm{d}t\right| = \left|\int_0^T \int_\Omega \partial_t\varphi_{n,\delta}\Pi_{\mathcal{W}_n}\zeta\right| \leq C\int_0^T \|\nabla\mu_{n,\delta}\|_{\mathbf{L}^2}\|\nabla\Pi_{\mathcal{W}_n}\zeta\|_{\mathbf{L}^2}\,\mathrm{d}t$$
$$+ C\int_0^T (1 + \|\mathrm{div}(\varphi_{n,\delta}\mathbf{v}_{n,\delta})\|_{L^2})\,\|\Pi_{\mathcal{W}_n}\zeta\|_{H^1}\,\mathrm{d}t.$$

By Hölder's inequality and the continuity of $\Pi_{\mathcal{W}_n}$ on H^1, we obtain

$$\left|\int_0^T \int_\Omega \partial_t\varphi_{n,\delta}\zeta \,\mathrm{d}x\,\mathrm{d}t\right| \leq C\left(1 + \|\mu_{n,\delta}\|_{L^2(H^1)} + \|\mathrm{div}(\varphi_{n,\delta}\mathbf{v}_{n,\delta})\|_{L^2(L^2)}\right)\|\zeta\|_{L^2(H^1)}.$$

Taking the supremum over all $\zeta \in L^2(H^1)$ and using (7.20), (7.25), we find that

$$\|\partial_t\varphi_{n,\delta}\|_{L^2((H^1)^*)} \leq C.$$

With exactly the same arguments as above, we obtain

$$\|\mathrm{div}(\sigma_{n,\delta}\mathbf{v}_{n,\delta})\|_{L^2(L^2)} \leq C.$$

Then, using the assumptions on f and h, (7.20)-(7.21) and (7.24), it follows that

$$\|\partial_t\sigma_{n,\delta}\|_{L^2(L^2)} \leq C.$$

Summarising the previous estimates, it holds that

$$\begin{aligned}
&\|\varphi_{n,\delta}\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)} + \|\mu_{n,\delta}\|_{L^4(L^2)\cap L^2(H^1)} + \|\sigma_{n,\delta}\|_{H^1(L^2)\cap L^\infty(H^1)\cap L^2(H^2)}\\
&\quad + \|\mathrm{div}(\varphi_{n,\delta}\mathbf{v}_{n,\delta})\|_{L^2(L^2)} + \|\mathrm{div}(\sigma_{n,\delta}\mathbf{v}_{n,\delta})\|_{L^2(L^2)}\\
&\quad + \|\mathbf{v}_{n,\delta}\|_{H^1(\mathbf{L}^{\frac{3}{2}})\cap L^\infty(\mathbf{L}^2)\cap L^{\frac{16}{5}}(\mathbf{V})\cap L^{\frac{8}{5}}(\mathbf{H}^2)} \leq C.
\end{aligned} \tag{7.26}$$

Passing to the limit By standard compactness results (see Lemma 2.36) and compact embeddings in 3D

$$H^{j+1} = W^{j+1,2} \subset\subset W^{j,q}\ \forall\, j \geq 0, j \in \mathbb{Z},\ 1 \leq q < 6,$$

and the compact embedding $L^2 \subset\subset (H^1)^*$, we obtain from (7.26), at least for a subsequence which will again be labelled by n, the weak(-star) convergences

$$\begin{aligned}
\varphi_{n,\delta} &\to \varphi_\delta && \text{weakly-star} && \text{in } H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^3),\\
\sigma_{n,\delta} &\to \sigma_\delta && \text{weakly-star} && \text{in } H^1(L^2) \cap L^\infty(H^1) \cap L^2(H^2),\\
\mu_{n,\delta} &\to \mu_\delta && \text{weakly} && \text{in } L^4(L^2) \cap L^2(H^1),\\
\mathbf{v}_{n,\delta} &\to \mathbf{v}_\delta && \text{weakly-star} && \text{in } H^1(\mathbf{L}^{\frac{3}{2}}) \cap L^\infty(\mathbf{L}^2) \cap L^{\frac{16}{5}}(\mathbf{V}) \cap L^{\frac{8}{5}}(\mathbf{H}^2),\\
\mathrm{div}(\varphi_{n,\delta}\mathbf{v}_{n,\delta}) &\to \xi && \text{weakly} && \text{in } L^2(L^2),\\
\mathrm{div}(\sigma_{n,\delta}\mathbf{v}_{n,\delta}) &\to \theta && \text{weakly} && \text{in } L^2(L^2)
\end{aligned}$$

for some limit functions ξ, $\theta \in L^2(L^2)$, and the strong convergences

$$\begin{aligned} \varphi_{n,\delta} &\to \varphi_\delta \quad \text{strongly} \quad \text{in } C^0([0,T];L^r) \cap L^2(W^{2,r}) \qquad \text{and a.e. in } \Omega_T, \\ \sigma_{n,\delta} &\to \sigma_\delta \quad \text{strongly} \quad \text{in } C^0([0,T];L^r) \cap L^2(W^{1,r}) \qquad \text{and a.e. in } \Omega_T, \\ \mathbf{v}_{n,\delta} &\to \mathbf{v}_\delta \quad \text{strongly} \quad \text{in } C^0([0,T];(\mathbf{H}^1)^*) \end{aligned}$$

for any $r \in [1,6)$.
For the remaining part of the proof, we fix $1 \le j \le n$ and $\zeta \in C_0^\infty(0,T)$. Since $\phi_j \in H^2_N$, $\mathbf{w}_j \in \mathbf{V} \cap \mathbf{H}^2$, we have $\zeta\phi_j \in C^\infty(H^2)$, $\zeta\mathbf{w}_j \in C^\infty(\mathbf{H}^2)$. Then, we can apply the same arguments as in Chapters 4 and 5 to pass to the limit $n \to \infty$. Only for the convection term in (7.12c), we need a more careful argument. Using Gagliardo–Nirenberg's inequality, it holds that

$$\|\nabla\sigma_{n,\delta} - \nabla\sigma_\delta\|_{\mathbf{L}^3} \le \|\sigma_{n,\delta} - \sigma_\delta\|_{H^2}^{\frac{6}{6+r}} \|\sigma_{n,\delta} - \sigma_\delta\|_{L^r}^{\frac{r}{6+r}} \quad \forall\, r \in [1,6).$$

Employing the boundedness of $\sigma_{n,\delta} - \sigma_\delta \in L^2(H^2)$ for all $n \in \mathbb{N}$ and the strong convergence $\sigma_{n,\delta} \to \sigma_\delta$ in $C^0(L^r)$ as $n \to \infty$ for all $r \in [1,6)$, this implies

$$\nabla\sigma_{n,\delta} \to \nabla\sigma_\delta \quad \text{strongly in } L^q(\mathbf{L}^3) \quad \text{as } n \to \infty \quad \forall\, q \in \left[\frac{7}{3}, 4\right).$$

Using the continuous embedding $\mathbf{V} \subset \mathbf{L}^6$ and the weak convergence $\mathbf{v}_{n,\delta} \rightharpoonup \mathbf{v}_\delta$ in $L^{\frac{16}{5}}(\mathbf{V})$ as $n \to \infty$, by the product of weak-strong convergence we obtain

$$\operatorname{div}(\sigma_{n,\delta}\mathbf{v}_{n,\delta}) \to \operatorname{div}(\sigma_\delta\mathbf{v}_\delta) \quad \text{weakly in } L^{\tilde{q}}(L^2) \quad \text{as } n \to \infty \quad \forall\, \tilde{q} \in \left[\frac{112}{83}, \frac{16}{9}\right).$$

By the uniqueness of weak limits, this leads that $\operatorname{div}(\sigma_\delta\mathbf{v}_\delta) = \theta \in L^2(L^2)$ and

$$\int_0^T \int_\Omega \operatorname{div}(\sigma_{n,\delta}\mathbf{v}_{n,\delta})\zeta\phi_j \,\mathrm{d}x \,\mathrm{d}t \to \int_0^T \int_\Omega \operatorname{div}(\sigma_\delta\mathbf{v}_\delta)\zeta\phi_j \,\mathrm{d}x \,\mathrm{d}t \quad \text{as } n \to \infty.$$

Hence, choosing $v = \phi_j$, $1 \le j \le n$, in (7.12a)-(7.12c), $\mathbf{u} = \mathbf{w}_j$, $1 \le j \le n$, in (7.12d), multiplying (7.12a)-(7.12d) with $\zeta \in C_0^\infty(0,T)$, integrating in time from 0 to T and passing to the limit $n \to \infty$, we infer that

$$\begin{aligned} 0 &= \int_0^T \zeta(t) \left(\langle \partial_t\varphi_\delta, \phi_j \rangle_{H^1} + \int_\Omega \nabla\varphi_\delta \cdot \mathbf{v}_\delta\, \phi_j + m(\varphi_\delta)\nabla\mu_\delta \cdot \nabla\phi_j - h(\varphi_\delta)g(\varphi_\delta,\sigma_\delta)\phi_j \,\mathrm{d}x \right) \mathrm{d}t, \\ 0 &= \int_0^T \zeta(t) \left(\int_\Omega \mu_\delta\phi_j - \varepsilon^{-1}\psi'(\varphi_\delta)\phi_j - \varepsilon\nabla\varphi_\delta \cdot \nabla\phi_j + \chi_\varphi\sigma_\delta\phi_j \,\mathrm{d}x \right) \mathrm{d}t, \\ 0 &= \int_0^T \zeta(t) \left(\int_\Omega \partial_t\sigma_\delta\phi_j + \nabla\sigma_\delta \cdot \mathbf{v}_\delta\, \phi_j + (\chi_\sigma\nabla\sigma_\delta - \chi_\varphi\nabla\varphi_\delta) \cdot \nabla\phi_j + h(\varphi_\delta)f(\varphi_\delta,\sigma_\delta)\phi_j \,\mathrm{d}x \right) \mathrm{d}t, \\ 0 &= \int_0^T \zeta(t) \left(\int_\Omega \partial_t\mathbf{v}_\delta \cdot \mathbf{w}_j + 2\eta\mathbf{D}\mathbf{v}_\delta \cdot \nabla\mathbf{w}_j + \nu\mathbf{v}_\delta \cdot \mathbf{w}_j - (\mu_\delta + \chi_\varphi\sigma_\delta)\nabla\varphi_\delta \cdot \mathbf{w}_j \,\mathrm{d}x \right) \mathrm{d}t \end{aligned}$$

holds for arbitrary $\zeta \in C_0^\infty(0,T)$ and all $1 \le j \le n$. As a consequence, we see that $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta)$ satisfies (7.10a)-(7.10b) for $\xi = \phi_j$, $\mathbf{u} = \mathbf{w}_j$, $j \ge 1$, and for a. e. $t \in (0,T)$. As $\{\phi_j\}_{j\in\mathbb{N}}$ is a Schauder basis for H^1 (see Chapter 2) and as $\{\mathbf{w}_j\}_{j\in\mathbb{N}}$ is a Schauder basis for $\mathbf{V}$, we obtain that $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta)$ satisfy (7.10a)-(7.10b) for all $\xi \in H^1$ and $\mathbf{u} \in \mathbf{V}$. Since $\Delta\varphi_\delta$, $\Delta\sigma_\delta \in L^2(L^2)$, a standard argument implies that (7.10c)-(7.10d) and $(7.9\text{a})_1$, $(7.9\text{a})_3$, hold.

Attainment of initial conditions It remains to show that the initial conditions hold. Noting that $\varphi_{n,\delta} \to \varphi_\delta$ strongly in $C^0([0,T];L^2)$ and $\varphi_{n,\delta}(0) \to \varphi_0$ in L^2 as $n \to \infty$, we conclude that

$\varphi_\delta(0) = \varphi_0$. With similar arguments, we deduce that $\sigma_\delta(0) = \sigma_{0,\delta}$ a. e. in Ω. Since $\mathbf{v}_{n,\delta} \to \mathbf{v}_\delta$ strongly in $C^0([0,T];(\mathbf{H}^1)^*)$ and $\mathbf{v}_{n,\delta}(0) = \mathbf{0}$, it holds that

$$\langle \mathbf{v}_\delta(0), \boldsymbol{\xi} \rangle_{\mathbf{H}^1} = \mathbf{0} \quad \forall \boldsymbol{\xi} \in \mathbf{H}^1.$$

Due to the continuous embedding $H^1((\mathbf{H}^1)^*) \cap L^2(\mathbf{H}^1) \subset C^0([0,T];\mathbf{L}^2)$, we observe that $\mathbf{v}_\delta(0) \in \mathbf{L}^2$ and consequently

$$\int_\Omega \mathbf{v}_\delta(0) \cdot \boldsymbol{\xi} \, \mathrm{d}x = \mathbf{0} \quad \forall \boldsymbol{\xi} \in \mathbf{H}^1$$

which implies $\mathbf{v}_\delta(0) = \mathbf{0}$ in $\mathbf{L}^2$ and a. e. in Ω.

Reconstruction of the pressure By standard theory for the instationary Stokes equation (see, e. g., [90, II.3, Cor. 4, p. 148]) and using that $(\mu_\delta + \chi_\varphi \sigma_\delta)\nabla\varphi_\delta \in L^{\frac{8}{5}}(\mathbf{L}^2) \cap L^2(\mathbf{L}^{\frac{3}{2}})$, there exists a unique pressure $p_\delta \in L^{\frac{8}{5}}(H^1) \cap L^2(W^{1,\frac{3}{2}})$ satisfying

$$\int_\Omega p_\delta \, \mathrm{d}x = 0.$$

□

Remark 7.9 Since $\mathbf{v}_\delta \in L^{\frac{16}{5}}(\mathbf{V}) \cap L^{\frac{8}{5}}(\mathbf{H}^2)$, for all $\mathbf{u} \in \mathbf{V}$ and almost every $t \in (0,T)$ it holds that

$$\int_\Omega 2\eta \mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} \, \mathrm{d}x = \int_\Omega 2\eta \mathbf{D}\mathbf{v}_\delta : \nabla\mathbf{u} \, \mathrm{d}x = \int_\Omega \eta \nabla\mathbf{v}_\delta : \nabla\mathbf{u} \, \mathrm{d}x.$$

The first identity follows since

$$\mathbf{D}\mathbf{u} = \nabla\mathbf{u} - \tfrac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^\intercal), \qquad \mathbf{D}\mathbf{v}_\delta : \tfrac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^\intercal) = 0.$$

Taking $\zeta \in C_0^\infty(0,T)$ arbitrary, the second identity follows from integration by parts and the fundamental theorem of calculus of variations, more precisely

$$\begin{aligned}
\int_0^T \int_\Omega \zeta \, 2\eta \mathbf{D}\mathbf{v}_\delta : \nabla\mathbf{u} \, \mathrm{d}x \, \mathrm{d}t &= -\int_0^T \int_\Omega \zeta \, \mathrm{div}(2\eta \mathbf{D}\mathbf{v}_\delta) \cdot \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega -\zeta \, \eta \Delta\mathbf{v}_\delta \cdot \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \zeta \, \eta \nabla\mathbf{v}_\delta : \nabla\mathbf{u} \, \mathrm{d}x \, \mathrm{d}t,
\end{aligned}$$

where we used that $\mathrm{div}(\mathbf{v}_\delta) = 0$ a. e. in Ω_T.

7.2.2 Existence of solutions for non-degenerate mobility

We will now establish the existence of weak solutions for non-degenerate mobility and regular potential.

Proof of Proposition 7.4. Without loss of generality, we assume $\delta \in (0,1)$.

Step 1: We aim to find independent bounds for the initial value $\sigma_{0,\delta}$. Multiplying (7.9d) with $\sigma_{0,\delta}$, integrating over Ω and by parts, we obtain

$$\int_\Omega \delta |\nabla\sigma_{0,\delta}|^2 + |\sigma_{0,\delta}|^2 \, \mathrm{d}x = \int_\Omega \sigma_0 \sigma_{0,\delta} \, \mathrm{d}x.$$

Using Hölder's and Young's inequalities, this implies

$$\sqrt{\delta}\|\sigma_{0,\delta}\|_{H^1} \le C\|\sigma_0\|_{L^2}. \tag{7.27}$$

Furthermore, multiplying (7.9d) with $\sigma_{0,\delta}^5$ and integrating over Ω and by parts gives

$$5\delta\int_\Omega |\nabla\sigma_{0,\delta}|^2|\sigma_{0,\delta}|^4 \,\mathrm{d}x + \int_\Omega |\sigma_{0,\delta}|^6 \,\mathrm{d}x = \int_\Omega \sigma_0\sigma_{0,\delta}^5 \,\mathrm{d}x.$$

Neglecting the non-negative term $5\delta\int_\Omega |\nabla\sigma_{0,\delta}|^2|\sigma_{0,\delta}|^4 \,\mathrm{d}x$ and using Hölder's inequality yields

$$\|\sigma_{0,\delta}\|_{L^6}^6 \le \int_\Omega \sigma_0\sigma_{0,\delta}^5 \,\mathrm{d}x \le \|\sigma_{0,\delta}\|_{L^6}^5\|\sigma_0\|_{L^6} \le \frac{1}{2}\|\sigma_{0,\delta}\|_{L^6}^6 + C\|\sigma_0\|_{L^6}^6.$$

Recalling $\sigma_0 \in L^6$, this implies

$$\|\sigma_{0,\delta}\|_{L^6} \le C\|\sigma_0\|_{L^6} \le C. \tag{7.28}$$

Now, we derive a priori estimates for the solution quadruplet $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta)$ independent of $\delta \in (0,1)$. To this end, using (7.10c), the assumptions on $\psi(\cdot)$ and the regularity $\varphi_\delta \in H^1((H^1)^*) \cap L^2(H^3)$, an application of [123, Lemma 4.1] yields

$$\langle \partial_t\varphi_\delta, \mu_\delta + \chi_\varphi\sigma_\delta + \varphi_\delta\rangle_{H^1} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi(\varphi_\delta) \,\mathrm{d}x \quad \text{for a. e. } t \in (0,T).$$

Now, choosing $\xi = \mu_\delta + \chi_\varphi\sigma_\delta + \varphi_\delta$ in (7.10a) and using the last identity, choosing $\mathbf{u} = \mathbf{v}_\delta$ in (7.10b), multiplying (7.10d) with $D\sigma_\delta$ for $D > 0$, and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi(\varphi_\delta) + \frac{D}{2}|\sigma_\delta|^2 + \frac{\delta}{2}|\mathbf{v}_{n,\delta}|^2\right] \mathrm{d}x \\
&\quad + \int_\Omega m(\varphi_\delta)|\nabla\mu_\delta|^2 + D\chi_\sigma|\nabla\sigma_\delta|^2 + 2\eta|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_{n,\delta}|^2 \,\mathrm{d}x \\
&= \int_\Omega -m(\varphi_\delta)\nabla\mu_\delta\cdot\nabla(\chi_\varphi\sigma_\delta + \varphi_\delta) + D\chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta \,\mathrm{d}x \\
&\quad + \int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)(\mu_\delta + \chi_\varphi\sigma_\delta + \varphi_\delta) - Df(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\sigma_\delta \,\mathrm{d}x
\end{aligned} \tag{7.29}$$

for $D > 0$ to be specified. Now, with similar arguments as in the proof of Lemma 7.7, a straightforward calculation shows that

$$\begin{aligned}
&\left|\int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)(\mu_\delta + \chi_\varphi\sigma_\delta + \varphi_\delta) \,\mathrm{d}x - Df(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\sigma_\delta \,\mathrm{d}x\right| \\
&\quad \le C(m_0, D)\left(1 + \|\sigma_\delta\|_{L^2}^2 + \|\varphi_\delta\|_{L^2}^2\right) + \frac{m_0}{8}\|\nabla\mu_\delta\|_{\mathbf{L}^2}^2.
\end{aligned}$$

Using Hölder's and Young's inequalities, we obtain

$$\left|\int_\Omega D\chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta \,\mathrm{d}x\right| \le \frac{D\chi_\sigma}{4}\|\nabla\sigma_\delta\|_{\mathbf{L}^2}^2 + \frac{D\chi_\varphi^2}{\chi_\sigma}\|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2.$$

With similar arguments, we infer

$$\left|\int_\Omega m(\varphi_\delta)\nabla\mu_\delta\cdot\nabla(\chi_\varphi\sigma_\delta + \varphi_\delta) \,\mathrm{d}x\right| \le \frac{2M_0^2}{m_0}\left(\|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2 + \chi_\varphi^2\|\nabla\sigma_\delta\|_{\mathbf{L}^2}^2\right) + \frac{m_0}{8}\|\nabla\mu_\delta\|_{\mathbf{L}^2}^2.$$

Recalling the assumptions on $m(\cdot)$, plugging in the last three inequalities into (7.29) and choosing

$$D = \frac{4M_0^2\chi_\varphi^2 + m_0}{\chi_\sigma m_0},$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi(\varphi_\delta) + \frac{D}{2}|\sigma_\delta|^2 + \frac{\delta}{2}|\mathbf{v}_{n,\delta}|^2 \,\mathrm{d}x \\ + \int_\Omega \frac{m_0}{2}|\nabla\mu_\delta|^2 + \frac{1}{2}|\nabla\sigma_\delta|^2 + 2\eta|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_{n,\delta}|^2 \,\mathrm{d}x \\ \leq C\left(1 + \|\sigma_\delta\|_{L^2}^2 + \|\varphi_\delta\|_{L^2}^2 + \|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2\right).\end{aligned}$$

Integrating this inequality in time from 0 to $t \in (0,T]$, using (7.4), (7.28), the assumptions on φ_0 and Korn's inequality, a Gronwall argument yields

$$\|\varphi_\delta\|_{L^\infty(H^1)} + \|\sigma_\delta\|_{L^\infty(L^2)\cap L^2(H^1)} + \|\nabla\mu_\delta\|_{L^2(\mathbf{L}^2)} + \sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} \leq C.$$

Multiplying $(7.10\mathrm{c})_1$ with 1, integrating over Ω and by parts, using $(7.10\mathrm{c})_2$ and applying Poincaré's inequality, the last inequality implies

$$\|\mu_\delta\|_{L^2(H^1)} \leq C.$$

Combining the last two estimates, we obtain

$$\|\varphi_\delta\|_{L^\infty(H^1)} + \|\sigma_\delta\|_{L^\infty(L^2)\cap L^2(H^1)} + \|\mu_\delta\|_{L^2(H^1)} + \sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} \leq C. \quad (7.30)$$

Then, with exactly the same arguments as in the proof of Lemma 7.7, it follows that

$$\|\varphi_\delta\|_{L^2(H^3)} + \|\mu_\delta\|_{L^4(L^2)} \leq C. \quad (7.31)$$

Step 2: Next, we want to multiply $(7.10\mathrm{d})_1$ with σ_δ^5 and then integrate by parts. To this end, we have to check that both multiplication and integration by parts can be justified. Using Gagliardo–Nirenberg's inequality we observe that $\sigma_\delta \in L^{10}(L^{10})$, hence $\sigma_\delta^5 \in L^2(L^2)$. Next, using (2.34) we observe that $\nabla\sigma_\delta \in L^4(\mathbf{L}^2(\partial\Omega))$ and $\sigma_\delta \in L^{\frac{20}{3}}(L^{10}(\partial\Omega))$. Hence,

$$\sigma_\delta^5\nabla\sigma_\delta \in L^1(\mathbf{L}^1(\partial\Omega)).$$

A similar argument gives

$$\sigma_\delta^5\nabla\varphi_\delta \in L^1(\mathbf{L}^1(\partial\Omega)).$$

Again applying (2.34), it holds $\sigma_\delta^6 \in L^{\frac{32}{17}}(L^{\frac{32}{31}}(\partial\Omega))$. From Gagliardo–Nirenberg's inequality, we obtain

$$\mathbf{v}_\delta \in L^\infty(\mathbf{L}^2) \cap L^{\frac{8}{5}}(\mathbf{H}^2) \hookrightarrow L^{\frac{32}{15}}(\mathbf{W}^{1,3}).$$

Then, the trace theorem yields

$$\mathbf{v}_\delta \in L^{\frac{32}{15}}(\mathbf{W}^{1,3}) \hookrightarrow L^{\frac{32}{15}}(\mathbf{L}^{32}(\partial\Omega)),$$

and, in particular, we infer

$$\sigma_\delta^6\,\mathbf{v}_\delta \in L^1(\mathbf{L}^1(\partial\Omega)).$$

Hence, multiplying (7.10d) with σ_δ^5, integrating over Ω and by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{6}\int_\Omega |\sigma_\delta|^6 \,\mathrm{d}x + 5\chi_\sigma\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4 \,\mathrm{d}x = 5\chi_\varphi\int_\Omega \nabla\varphi_\delta\cdot\nabla\sigma_\delta\,|\sigma_\delta|^4 \,\mathrm{d}x \\ - \int_\Omega f(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\sigma_\delta^5 \,\mathrm{d}x,\end{aligned} \quad (7.32)$$

where we used $(7.10\mathrm{c})_2$, $(7.10\mathrm{d})_2$ and $\mathbf{v}_\delta = \mathbf{0}$ a. e. on Σ_T. Using the assumptions on f and h, an application of Hölder's and Young's inequalities yields

$$\left|\int_\Omega f(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\sigma_\delta^5 \,\mathrm{d}x\right| \leq C\left(1 + \|\varphi_\delta\|_{L^6}^6 + \|\sigma_\delta\|_{L^6}^6\right).$$

Another application of Hölder's and Young's inequalities along with the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ leads to

$$\begin{aligned}\left|5\chi_\varphi \int_\Omega \nabla\varphi_\delta \cdot \nabla\sigma_\delta\, |\sigma_\delta|^4 \,\mathrm{d}x\right| &\le \frac{5\chi_\sigma}{2}\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4 \,\mathrm{d}x + C\int_\Omega |\nabla\varphi_\delta|^2|\sigma_\delta|^4 \,\mathrm{d}x \\ &\le \frac{5\chi_\sigma}{2}\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4 \,\mathrm{d}x + C\|\nabla\varphi_\delta\|_{\mathbf{L}^6}^2\|\sigma_\delta\|_{L^6}^4 \,\mathrm{d}x \\ &\le \frac{5\chi_\sigma}{2}\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4 \,\mathrm{d}x + C\|\varphi_\delta\|_{H^2}^2\left(1+\|\sigma_\delta\|_{L^6}^6\right).\end{aligned}$$

Invoking the last two estimates and (7.32), we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{6}\int_\Omega |\sigma_\delta|^6 \,\mathrm{d}x + \frac{5\chi_\sigma}{2}\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4 \,\mathrm{d}x \le C\|\varphi_\delta\|_{L^6}^6 + C\left(1+\|\varphi_\delta\|_{H^2}^2\right)\left(1+\|\sigma_\delta\|_{L^6}^6\right).$$

Integrating this inequality in time from 0 to $t \in (0,T]$ and using (7.28) along with (7.31), a Gronwall argument yields

$$\|\sigma_\delta\|_{L^\infty(L^6)} \le C. \tag{7.33}$$

Step 3: We now derive estimates for the time derivatives and the convection terms. For $\zeta \in L^2(H^1)$ arbitrary, by (7.30), (7.33) and the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^3$ it follows

$$\begin{aligned}\left|\int_0^T\int_\Omega \sigma_\delta \mathbf{v}_\delta \cdot \nabla\zeta \,\mathrm{d}x\,\mathrm{d}t\right| &\le C\int_0^T \|\sigma_\delta\|_{L^6}\|\mathbf{v}_\delta\|_{\mathbf{L}^3}\|\nabla\zeta\|_{\mathbf{L}^2}\,\mathrm{d}t \\ &\le C\|\sigma_\delta\|_{L^\infty(L^6)}\|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)}\|\zeta\|_{L^2(H^1)} \\ &\le C\|\zeta\|_{L^2(H^1)},\end{aligned}$$

and therefore

$$\|\mathrm{div}(\sigma_\delta\mathbf{v}_\delta)\|_{L^2((H^1)^*)} \le C.$$

Then, the relation (7.10d) for $\partial_t\sigma_\delta$ yields

$$\|\partial_t\sigma_\delta\|_{L^2((H^1)^*)} \le C.$$

Similarly, using the relation (7.10b) for $\delta\partial_t\mathbf{v}_\delta$ together with (7.30) gives

$$\delta\|\partial_t\mathbf{v}_\delta\|_{L^2((\mathbf{H}^1)^*)} \le C.$$

Furthermore, by the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$, (7.30) and Hölder's inequality, we obtain

$$\|\mathrm{div}(\varphi_\delta\mathbf{v}_\delta)\|_{L^2(L^{\frac{3}{2}})} = \|\nabla\varphi_\delta\cdot\mathbf{v}_\delta\|_{L^2(L^{\frac{3}{2}})} \le C\|\varphi_\delta\|_{L^\infty(H^1)}\|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} \le C.$$

Recalling (7.30) and using the equation (7.10a) for $\partial_t\varphi_\delta$, we conclude

$$\|\partial_t\varphi_\delta\|_{L^2((H^1)^*)} \le C.$$

Employing the last five estimates in conjunction with (7.30)-(7.31) and (7.33), we deduce that

$$\begin{aligned}&\|\varphi_\delta\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)} + \|\sigma_\delta\|_{H^1((H^1)^*)\cap L^\infty(L^6)\cap L^2(H^1)} + \|\mu_\delta\|_{L^4(L^2)\cap L^2(H^1)} \\ &\quad + \sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)} + \delta\|\partial_t\mathbf{v}_\delta\|_{L^2((\mathbf{H}^1)^*)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} \\ &\quad + \|\mathrm{div}(\varphi_\delta\mathbf{v}_\delta)\|_{L^2(L^{\frac{3}{2}})} + \|\mathrm{div}(\sigma_\delta\mathbf{v}_\delta)\|_{L^2((H^1)^*)} \le C.\end{aligned} \tag{7.34}$$

Passing to the limit The approach is based on similar arguments as in the proof of Lemma 7.7. By standard compactness results (see Lemma 2.36) and compact embeddings in 3D

$$H^{j+1} = W^{j+1,2} \subset\subset W^{j,q} \quad \forall\, j \geq 0, j \in \mathbb{Z},\ 1 \leq q < 6,$$

and the compact embedding $L^2 \subset\subset (H^1)^*$, we obtain from (7.26) for a non-relabelled subsequence that

$$\begin{aligned}
\varphi_\delta &\to \varphi && \text{weakly-star} && \text{in } H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^3),\\
\sigma_\delta &\to \sigma && \text{weakly-star} && \text{in } H^1((H^1)^*) \cap L^\infty(L^6) \cap L^2(H^1),\\
\mu_\delta &\to \mu && \text{weakly} && \text{in } L^4(L^2) \cap L^2(H^1),\\
\mathbf{v}_\delta &\to \mathbf{v} && \text{weakly} && \text{in } L^2(\mathbf{H}^1),\\
\operatorname{div}(\varphi_\delta \mathbf{v}_\delta) &\to \theta && \text{weakly} && \text{in } L^2(L^{\frac{3}{2}}),\\
\operatorname{div}(\sigma_\delta \mathbf{v}_\delta) &\to \tau && \text{weakly} && \text{in } L^2((H^1)^*)
\end{aligned}$$

for some limit functions $\theta \in L^2(L^{\frac{3}{2}})$, $\tau \in L^2((H^1)^*)$, and

$$\begin{aligned}
\varphi_\delta &\to \varphi && \text{strongly} && \text{in } C^0([0,T];L^r) \cap L^2(W^{2,r}) && \text{and a.e. in } \Omega_T,\\
\sigma_\delta &\to \sigma && \text{strongly} && \text{in } C^0([0,T];(H^1)^*) \cap L^2(L^r) && \text{and a.e. in } \Omega_T
\end{aligned}$$

for any $r \in [1,6)$. Using weak-star lower semicontinuity of norms and a generalised version of Hölder's inequality, for every $r \in (1,6)$ we obtain

$$\|\sigma_\delta - \sigma\|_{L^r} \leq \|\sigma_\delta - \sigma\|_{L^6}^{\frac{6(r-1)}{5r}} \|\sigma_\delta - \sigma\|_{L^1}^{\frac{6-r}{5r}} \leq C\|\sigma_\delta - \sigma\|_{L^1}^{\frac{6-r}{5r}}.$$

Using the strong convergence $\sigma_\delta \to \sigma$ in $L^2(L^1)$ as $\delta \to 0$, this implies

$$\sigma_\delta \to \sigma \quad \text{in } L^{\frac{10r}{6-r}}(L^r) \quad \text{as } \delta \to 0 \quad \forall\, r \in (1,6).$$

Since $\frac{10r}{6-r} \to \infty$ as $r \to 6$, we conclude that

$$\sigma_\delta \to \sigma \quad \text{strongly in } L^p(L^r) \quad \text{as } \delta \to 0 \quad \forall\, p \in [1,\infty),\ r \in [1,6). \tag{7.35}$$

For the remaining part of the proof, let $\zeta \in C_0^\infty(0,T)$ and $\xi \in H^1$, $\mathbf{u} \in \mathbf{V}$ be arbitrary. We multiply (7.10a)-(7.10b) with ζ, integrate in time and by parts, and use that $\mathbf{v}_\delta(0) = \mathbf{0}$ a. e. in Ω as well as $\zeta(T) = 0$. Moreover, we multiply (7.10c)-(7.10d) with $\zeta\xi$ and integrate over Ω_T and by parts. Then, we obtain

$$\begin{aligned}
0 &= \int_0^T \zeta(t) \left(\langle \partial_t \varphi_\delta, \xi \rangle_{H^1} + \int_\Omega \nabla\varphi_\delta \cdot \mathbf{v}_\delta\, \xi + m(\varphi_\delta)\nabla\mu_\delta \cdot \nabla\xi - g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\xi \,\mathrm{d}x \right) \mathrm{d}t,\\
0 &= -\int_0^T \int_\Omega \zeta'(t)\delta\mathbf{v}_\delta \cdot \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T \zeta(t) \left(\int_\Omega 2\eta\mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta \cdot \mathbf{u} - (\mu_\delta + \chi_\varphi\sigma_\delta)\nabla\varphi_\delta \cdot \mathbf{u} \,\mathrm{d}x \right) \mathrm{d}t,\\
0 &= \int_0^T \zeta(t) \left(\int_\Omega \big(\mu_\delta + \chi_\varphi\sigma_\delta - \varepsilon^{-1}\psi'(\varphi_\delta)\big)\xi - \varepsilon\nabla\varphi_\delta \cdot \nabla\xi \,\mathrm{d}x \right) \mathrm{d}t,\\
0 &= \int_0^T \zeta(t) \left(\int_\Omega \partial_t\sigma_\delta\xi + (\chi_\sigma\nabla\sigma_\delta - \chi_\varphi\nabla\varphi_\delta - \sigma_\delta\mathbf{v}_\delta) \cdot \nabla\xi + f(\varphi_\delta,\sigma_{n,\delta})h(\varphi_\delta)\xi \,\mathrm{d}x \right) \mathrm{d}t.
\end{aligned} \tag{7.36}$$

Now, the arguments for $(7.36)_3$ are exactly the same as in the proof of Lemma 7.7. Using Gagliardo–Nirenberg's inequality, we obtain

$$\|\nabla\varphi_\delta - \nabla\varphi\|_{\mathbf{L}^2} \leq C\|\varphi_\delta - \varphi\|_{L^2}^{\frac{2}{3}} \|\varphi_\delta - \varphi\|_{H^3}^{\frac{1}{3}}.$$

The strong convergence $\varphi_\delta \to \varphi$ in $C^0([0,T];L^2)$ as $\delta \to 0$ and the boundedness of $\varphi_\delta - \varphi$ in $L^2(H^3)$ for all $\delta > 0$ ensure that

$$\nabla\varphi_\delta \to \nabla\varphi \quad \text{strongly in } L^6(\mathbf{L}^2) \quad \text{as } \delta \to 0. \tag{7.37}$$

Then, by the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ and the product of weak-strong convergence, we infer that

$$\nabla\varphi_\delta \cdot \mathbf{v}_\delta \to \nabla\varphi \cdot \mathbf{v} \quad \text{weakly in } L^{\frac{3}{2}}(L^{\frac{3}{2}}) \quad \text{as } \delta \to 0.$$

The uniqueness of weak limits then yields $\operatorname{div}(\varphi\mathbf{v}) = \theta \in L^2(L^{\frac{3}{2}})$. This allows us to pass to the limit in $(7.36)_1$ in a similar manner as in the proof of Lemma 7.7. Using (7.34) and Hölder's inequality, we obtain

$$\begin{aligned}\left|\int_0^T \int_\Omega \zeta'(t)\delta\mathbf{v}_\delta \cdot \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t\right| &\leq C\|\xi'(t)\|_{L^2(0,T)}\sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)}\sqrt{\delta}\|\mathbf{u}\|_{\mathbf{L}^2} \\ &\leq C\|\xi'(t)\|_{L^2(0,T)}\sqrt{\delta}\|\mathbf{u}\|_{\mathbf{L}^2} \to 0 \end{aligned} \tag{7.38}$$

as $\delta \to 0$. Moreover, by (7.37) and the Sobolev embedding $H^1 \subset L^6$, the product of weak-strong convergence yields

$$(\mu_\delta + \chi_\varphi\sigma_\delta)\nabla\varphi_\delta \to (\mu + \chi_\varphi\sigma)\nabla\varphi \quad \text{weakly in } L^{\frac{3}{2}}(\mathbf{L}^{\frac{3}{2}}).$$

Then, we can apply similar arguments as in the proof of Lemma 7.7 to pass to the limit in $(7.36)_2$. It remains to pass to the limit in $(7.36)_4$. We will only present the arguments for the convection term. The remaining terms can be treated in the same way as in the proof of Lemma 7.7. Considering $\zeta\xi$ as an element in $L^2(H^1)$, the weak convergence $\operatorname{div}(\sigma_\delta\mathbf{v}_\delta) \rightharpoonup \tau$ in $L^2((H^1)^*)$ gives

$$\int_0^T \int_\Omega \zeta \operatorname{div}(\sigma_\delta\mathbf{v}_\delta)\xi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \zeta\langle\tau,\xi\rangle_{H^1}\,\mathrm{d}t \quad \text{as } \delta \to 0.$$

Integration by parts yields

$$\int_0^T \int_\Omega \operatorname{div}(\sigma_\delta\mathbf{v}_\delta)\zeta\tau \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_\Omega \zeta\sigma_\delta\mathbf{v}_\delta \cdot \nabla\xi \,\mathrm{d}x\,\mathrm{d}t,$$

and using (7.35) along with $\mathbf{v}_\delta \rightharpoonup \mathbf{v}$ weakly in $L^2(\mathbf{L}^6)$ as $\delta \to 0$, by the product of weak-strong convergence we obtain

$$-\int_0^T \int_\Omega \zeta\sigma_\delta\mathbf{v}_\delta \cdot \nabla\xi \,\mathrm{d}x\,\mathrm{d}t \to -\int_0^T \int_\Omega \zeta\sigma\mathbf{v} \cdot \nabla\xi \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta \to 0.$$

Hence, we conclude that

$$\int_0^T \zeta\langle\tau,\xi\rangle_{H^1}\,\mathrm{d}t = -\int_0^T \int_\Omega \zeta\sigma\mathbf{v} \cdot \nabla\xi \,\mathrm{d}x\,\mathrm{d}t,$$

meaning $\operatorname{div}(\sigma\mathbf{v}) = \tau$ in the sense of distributions. Therefore, we can pass to the limit in (7.36) and use similar arguments as in the proof of Lemma 7.7 to deduce that $(\varphi, \mu, \sigma, \mathbf{v})$ satisfies

$$\begin{aligned} 0 &= \langle\partial_t\varphi,\xi\rangle_{H^1} + \int_\Omega \nabla\varphi \cdot \mathbf{v}\,\xi + m(\varphi)\nabla\mu \cdot \nabla\xi - g(\varphi,\sigma)h(\varphi)\xi \,\mathrm{d}x, \\ 0 &= \int_\Omega 2\eta\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu\mathbf{v} \cdot \mathbf{u} - (\mu + \chi_\varphi\sigma)\nabla\varphi \cdot \mathbf{u} \,\mathrm{d}x, \\ 0 &= \int_\Omega \left(\mu + \chi_\varphi\sigma - \varepsilon^{-1}\psi'(\varphi)\right)\xi - \varepsilon\nabla\varphi \cdot \nabla\xi \,\mathrm{d}x, \\ 0 &= \langle\partial_t\sigma,\xi\rangle_{H^1} + \int_\Omega (\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi) \cdot \nabla\xi + f(\varphi,\sigma)h(\varphi)\xi - \sigma\mathbf{v} \cdot \nabla\xi \,\mathrm{d}x \end{aligned} \tag{7.39}$$

for almost all $t \in (0,T)$ and all $\xi \in H^1$, $\mathbf{u} \in \mathbf{V}$. Choosing $\mathbf{u} = \mathbf{v}$ in $(7.39)_2$, recalling $\varphi \in L^\infty(H^1)$, and applying Korn's, Hölder's and Young's inequalities yields

$$\|\mathbf{v}\|_{\mathbf{H}^1} \leq C\|\mu + \chi_\varphi \sigma\|_{L^3}.$$

By Gagliardo–Nirenberg's inequality, we have $(\mu + \chi_\varphi\sigma) \in L^4(L^2) \cap L^2(H^1) \hookrightarrow L^{\frac{8}{3}}(L^3)$, and consequently

$$\|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)} \leq C.$$

Moreover, since $(\mu + \chi_\varphi\sigma)\nabla\varphi \in L^{\frac{8}{5}}(\mathbf{L}^2)$, classical regularity theory for the Stokes equation gives

$$\|\mathbf{v}\|_{L^{\frac{8}{5}}(\mathbf{H}^2)} \leq C.$$

Together with Gagliardo–Nirenberg's inequality, the last two estimates entail that

$$\|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)\cap L^2(\mathbf{W}^{1,3})\cap L^{\frac{8}{5}}(\mathbf{H}^2)} \leq C. \tag{7.40}$$

Using (7.40) and the boundedness of $\nabla\varphi \in L^8(\mathbf{L}^3)$, $\sigma \in L^2(H^1)$, we deduce that

$$\nabla\varphi \cdot \mathbf{v} \in L^2(L^2), \quad \nabla\sigma \cdot \mathbf{v} \in L^{\frac{8}{7}}(L^{\frac{3}{2}}),$$

hence in particular $\theta = \operatorname{div}(\varphi\mathbf{v}) \in L^2(L^2)$ and $\tau = \operatorname{div}(\sigma\mathbf{v}) \in L^{\frac{8}{7}}(L^{\frac{3}{2}})$. The validity of (7.6d) follows from a standard argument.

Attainment of initial conditions The initial condition for φ is attained since $\varphi_\delta \to \varphi$ strongly in $C^0([0,T];L^2)$ as $\delta \to 0$ and because of $\varphi_\delta(0) = \varphi_0$ a. e. in Ω for all $\delta > 0$. Now, multiplying (7.9d) with $\xi \in H^1$ and integrating over Ω and by parts, we obtain

$$\int_\Omega \delta\nabla\sigma_{0,\delta} \cdot \nabla\xi + \sigma_{0,\delta}\,\xi \,\mathrm{d}x = \int_\Omega \sigma_0\xi \,\mathrm{d}x.$$

Since $\sigma_\delta(0) = \sigma_{0,\delta}$ a. e. in Ω, this implies

$$\int_\Omega \delta\nabla\sigma_{0,\delta} \cdot \nabla\xi \,\mathrm{d}x + \langle\sigma_\delta(0)\,,\xi\rangle_{H^1} = \langle\sigma_0\,,\xi\rangle_{H^1}.$$

Using the strong convergence $\sigma_\delta \to \sigma$ in $C^0([0,T];(H^1)^*)$ as $\delta \to 0$ along with (7.27), passing to the limit $\delta \to 0$ in the last identity yields

$$\langle\sigma(0)\,,\xi\rangle_{H^1} = \langle\sigma_0\,,\xi\rangle_{H^1} \quad \forall\,\xi \in H^1.$$

Due to the continuous embedding $H^1((H^1)^*) \cap L^2(H^1) \hookrightarrow C^0([0,T];L^2)$, this implies

$$\int_\Omega \sigma(0)\xi \,\mathrm{d}x = \int_\Omega \sigma_0\xi \,\mathrm{d}x \quad \forall\,\xi \in H^1,$$

and consequently $\sigma(0) = \sigma_0$ a. e. in Ω. By standard arguments for the Stokes equation, the pressure can be reconstructed and fulfils

$$p \in L^{\frac{8}{3}}(L^2_0) \cap L^{\frac{8}{5}}(H^1), \quad \|p\|_{L^{\frac{8}{3}}(L^2_0)\cap L^{\frac{8}{5}}(H^1)} \leq C. \tag{7.41}$$

Finally, the estimate (7.7) follows from weak(-star) lower semicontinuity of norms along with (7.40) and (7.41). □

7.3 The degenerate case

7.3.1 Introduction of the mathematical setting

In the following let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with $\partial\Omega \in C^3$. We assume that $\psi(\cdot)$ can be decomposed as

$$\psi(\varphi) \coloneqq \psi^1(\varphi) + \psi^2(\varphi) \tag{7.42}$$

with functions ψ^1, ψ^2 where $\psi^2 \in C^2([-1, +\infty))$ satisfies

$$|(\psi^2)''(\varphi)| \le C \quad \forall\, \varphi \in [1, +\infty),$$

and $\psi^1 \colon (-1, +\infty) \to \mathbb{R}$ is convex and of the form

$$(\psi^1)''(\varphi) = \max\big(0, \min\big(\tfrac{1}{2}(1+\varphi), 1\big)\big)^{-p_0} F(\varphi) \quad \text{for some } p_0 \in [1, 2] \tag{7.43}$$

with a C^1-function $F \colon [-1, +\infty) \to \mathbb{R}_0^+$ satisfying

$$\|F\|_{C^1[-1,+\infty)} \le F_0$$

for a positive constant F_0. Hence, ψ is allowed to be singular in the convex part as $\varphi \to -1$. Without loss of generality, we assume that $(\psi^1)'(0) = (\psi^1)(0) = 0$.

We introduce a degenerate mobility $m(\cdot)$ of the form

$$m(\varphi) = \max\big(0, \min\big(\tfrac{1}{2}(1+\varphi), 1\big)\big)^{q_0} \bar{m}(\varphi) \quad \text{with } q_0 \in [1, 2],\ q_0 \ge p_0, \tag{7.44}$$

with p_0 as in (7.43), and a C^1-function $\bar{m} \colon [-1, +\infty) \to \mathbb{R}$ satisfying

$$m_0 \le \bar{m}(\varphi) \le M_0 \quad \forall\, \varphi \in [-1, +\infty), \qquad \|\bar{m}\|_{C^1[-1,+\infty)} \le M_1$$

for positive constants m_0, M_0 and M_1. We extend the definition of $m(\cdot)$ to all of $\mathbb{R}$ by $m(\varphi) = 0$ for $\varphi < -1$.

Finally, we define

$$\Phi : (-1, +\infty) \to \mathbb{R}_0^+$$

by

$$\Phi''(\varphi) = \frac{1}{m(\varphi)}, \quad \Phi'(0) = 0, \quad \Phi(0) = 0. \tag{7.45}$$

Example 7.10 In the following we give two examples for the choice of potentials. We will show a plot of the two potentials in Figure 7.1.

1.) Assuming that cell-cell interactions are attractive in one phase and repulsive in the other phase, so called single-well potentials of Lennard–Jones type are used frequently in the literature. Following, e. g., the works [5, 6], we define

$$\psi^1(\varphi) = \begin{cases} -\frac{1}{2}(1+\varphi^*)\ln(1+\varphi) & \text{for } \varphi > -1, \\ +\infty & \text{else,} \end{cases}$$
$$\psi^2(\varphi) = \tfrac{1}{24}(\varphi-1)^3 + (1+\varphi^*)\big(-\tfrac{1}{16}(\varphi-1)^2 + \tfrac{1}{4}(\varphi-1) + \tfrac{1}{2}\ln(2)\big) \quad \forall\, \varphi \ge -1,$$

where φ^* is the volume fraction at which the cells are at equilibrium. Then, it holds that

$$(\psi^1)''(\varphi) = \frac{(1+\varphi^*)}{2(1+\varphi)^2} \qquad \forall\, \varphi \in (-1, +\infty),$$

and (7.43) is fulfilled with $p_0 = 2$ and $F(\varphi) = \frac{1}{8}(1+\varphi^*)$ if we modify $\psi^1(\cdot)$ for $\varphi \geq 1$ via

$$\psi^1(\varphi) = -\tfrac{1}{2}(1+\varphi^*)\ln(2) - \tfrac{1}{4}(1+\varphi^*)(\varphi-1) + \tfrac{1}{16}(1+\varphi^*)(\varphi-1)^2 \quad \forall\, \varphi \geq 1.$$

Similarly, we have to extend $\psi^2(\cdot)$ in order to fulfil $|(\psi^2)''(\varphi)| \leq C$ for all $\varphi \geq -1$.

2.) We can also use a modified version of the logarithmic potential by setting

$$\psi^1(\varphi) = \begin{cases} \theta\,(\ln(1+\varphi)(1+\varphi) - \varphi) & \text{for } \varphi \geq -1, \\ +\infty & \text{else,} \end{cases} \tag{7.46a}$$

and

$$\psi^2(\varphi) = \frac{\theta_c - \theta\ln(2)}{3}\varphi^3 - \frac{\theta_c}{2}\varphi^2 + \frac{\theta_c + 6\theta - 10\theta\ln(2)}{6} \quad \forall\, \varphi \in [-1, +\infty) \tag{7.46b}$$

for $0 < \theta < \theta_c$. Then, (7.43) is fulfilled with $p_0 = 1$ and $F(\varphi) = \frac{\theta}{2}$. Again, in order to fulfil the assumptions, we need to modify ψ^1 and ψ^2 appropriately.

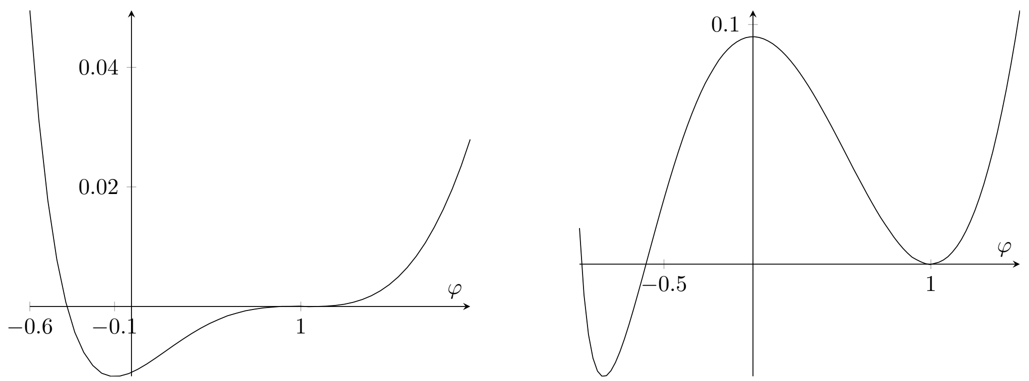

Figure 7.1: On the left a plot of the Lennard–Jones type potential with $\varphi^* = -0.1$, on the right a plot of the logarithmic type potential with $\theta_c = 1.5$ and $\theta = 1$.

7.3.2 The main theorem

The goal of this section is to prove the following theorem:

Theorem 7.11 (degenerate case) *Let Assumptions 7.2,* (ii)-(v)*, be fulfilled. In addition, we assume that* $\varphi_0 \geq -1$ *a. e. in* Ω *and*

$$\int_\Omega \psi(\varphi_0) + \Phi(\varphi_0) \leq C$$

for a positive constant C*. Then, there exists a quadruplet* $(\varphi, \boldsymbol{J}, \sigma, \mathbf{v})$ *satisfying*

a) $\varphi \in H^1((H^1)^*) \cap C([0,T]; L^2) \cap L^\infty(H^1) \cap L^2(H^2)$,

b) $\varphi(0) = \varphi_0$ *in* L^2 *and* $\nabla\varphi \cdot \mathbf{n} = 0$ *a. e. on* Σ_T,

c) $\varphi \geq -1$ *a. e. in* Ω_T,

d) $\sigma \in H^1((H^1)^*) \cap C^0(L^2) \cap L^\infty(L^6) \cap L^2(H^1)$,

e) $\sigma(0) = \sigma_0$ *in* L^2,

f) $\boldsymbol{J} \in L^2(\mathbf{L}^2)$,

g) $\mathbf{v} \in L^2(\mathbf{H}^1)$,

and solving

$$\int_0^T \langle \partial_t \varphi(t), \xi(t) \rangle_{H^1} \,\mathrm{d}t = \int_{\Omega_T} \boldsymbol{J} \cdot \nabla \xi \,\mathrm{d}x \,\mathrm{d}t + \int_{\Omega_T} g(\varphi, \sigma) h(\varphi) \xi - \nabla \varphi \cdot \mathbf{v}\, \xi \,\mathrm{d}x \,\mathrm{d}t, \tag{7.47a}$$

$$\langle \partial_t \sigma, \phi \rangle_{H^1} = \int_\Omega (-\chi_\sigma \nabla \sigma + \chi_\varphi \nabla \varphi + \sigma \mathbf{v}) \cdot \nabla \phi - f(\varphi, \sigma) h(\varphi) \phi \,\mathrm{d}x, \tag{7.47b}$$

$$\int_\Omega 2\eta \mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu \mathbf{v} \cdot \mathbf{u} \,\mathrm{d}x = \int_\Omega \varepsilon \left(\nabla \varphi \otimes \nabla \varphi \right) : \nabla \mathbf{u} \,\mathrm{d}x \tag{7.47c}$$

for almost all $t \in (0,T)$ and all $\xi \in L^2(H^1)$, $\phi \in H^1$, $\mathbf{u} \in \mathbf{V}$, where

$$\boldsymbol{J} = -m(\varphi) \nabla (-\varepsilon \Delta \varphi + \varepsilon^{-1} \psi'(\varphi) - \chi_\varphi \sigma)$$

holds in the sense that

$$\int_{\Omega_T} \boldsymbol{J} \cdot \boldsymbol{\eta} \,\mathrm{d}x \,\mathrm{d}t = -\int_{\Omega_T} \varepsilon \Delta \varphi \operatorname{div}(m(\varphi) \boldsymbol{\eta}) + \varepsilon^{-1} (m \psi'')(\varphi) \nabla \varphi \cdot \boldsymbol{\eta} - \chi_\varphi m(\varphi) \nabla \sigma \cdot \boldsymbol{\eta} \,\mathrm{d}x \,\mathrm{d}t \tag{7.47d}$$

for all $\boldsymbol{\eta} \in L^2(0,T;\mathbf{H}^1) \cap L^\infty(0,T;\mathbf{L}^\infty)$ with $\boldsymbol{\eta} \cdot \mathbf{n} = 0$ a. e. on Σ_T. Furthermore, there exists a unique pressure $p \in L^{\frac{4}{3}}(L^2_0)$ satisfying

$$-\nabla p = -\operatorname{div} \left(2\eta \mathbf{D}\mathbf{v} - \varepsilon (\nabla \varphi \otimes \nabla \varphi) \right) + \nu \mathbf{v} \quad \text{in } L^{\frac{4}{3}}(\mathbf{V}^*).$$

Remark 7.12 In the case $q_0 < 2$ (and so also $p_0 < 2$), the assumption

$$\int_\Omega \psi(\varphi_0) + \Phi(\varphi_0) \leq C$$

imposes no restriction on the initial data since $\psi(\cdot)$ and $\Phi(\cdot)$ are bounded in -1.

7.3.3 Approximation scheme

In the following let $\delta \in (0,1]$. We introduce a positive mobility m_δ by

$$m_\delta(\varphi) \coloneqq \begin{cases} m(-1+\delta) & \text{for } \varphi \leq -1+\delta, \\ m(\varphi) & \text{for } \varphi > -1+\delta, \end{cases}$$

and we define Φ_δ such that $\Phi_\delta''(\varphi) = \frac{1}{m_\delta(\varphi)}$ and $\Phi_\delta'(0) = \Phi_\delta(0) = 0$. In particular, we have $\Phi_\delta(\varphi) = \Phi(\varphi)$ for $\varphi \geq -1+\delta$. The modified potential $\psi_\delta \colon \mathbb{R} \to \mathbb{R}$ is defined by $\psi_\delta \coloneqq \psi_\delta^1 + \psi^2$ where

$$\left(\psi_\delta^1\right)''(\varphi) \coloneqq \begin{cases} \left(\psi^1\right)''(-1+\delta) & \text{for } \varphi \leq -1+\delta, \\ \left(\psi^1\right)''(\varphi) & \text{for } \varphi > -1+\delta, \end{cases}$$

and $\psi_\delta^1(0) = \psi^1(0)$, $\left(\psi_\delta^1\right)'(0) = \left(\psi^1\right)'(0)$. As for Φ we get $\psi_\delta(\varphi) = \psi(\varphi)$ if $\varphi \geq -1+\delta$. Furthermore, we extend ψ^2 to a function on all $\mathbb{R}$ such that $\|\psi^2\|_{C^2(\mathbb{R})} \leq C$.

With this choice for m_δ and ψ_δ, by Lemma 7.7 there exists a weak solution (which will be denoted by $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta, p_\delta))$ of

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= 0 && \text{in } \Omega_T,\\
\delta\partial_t\mathbf{v} - \operatorname{div}(2\eta\mathbf{D}\mathbf{v}) + \nu\mathbf{v} - \nabla p &= (\mu + \chi_\varphi\sigma)\nabla\varphi && \text{in } \Omega_T,\\
\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) &= \operatorname{div}(m_\delta(\varphi)\nabla\mu) + h(\varphi)\, g(\varphi,\sigma) && \text{in } \Omega_T,\\
\mu &= -\varepsilon\Delta\varphi + \varepsilon^{-1}\psi_\delta'(\varphi) - \chi_\varphi\sigma && \text{in } \Omega_T,\\
\partial_t\sigma + \operatorname{div}(\sigma\mathbf{v}) &= \chi_\sigma\Delta\sigma - \chi_\varphi\Delta\varphi - h(\varphi)\, f(\varphi,\sigma) && \text{in } \Omega_T,\\
\nabla\varphi\cdot\mathbf{n} = \nabla\mu\cdot\mathbf{n} &= \nabla\sigma\cdot\mathbf{n} = 0, \quad \mathbf{v} = \mathbf{0} && \text{on } \Sigma_T,
\end{aligned}$$

which fulfils the initial conditions (7.9c)-(7.9d). The weak formulation is given by

$$0 = \langle \partial_t\varphi_\delta\,, \xi\rangle_{H^1} + \int_\Omega \nabla\varphi_\delta\cdot\mathbf{v}_\delta\,\xi + m_\delta(\varphi_\delta)\nabla\mu_\delta\cdot\nabla\xi - g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\xi \,\mathrm{d}x, \tag{7.48a}$$

$$0 = \int_\Omega \delta\partial_t\mathbf{v}_\delta\cdot\mathbf{u} + 2\eta\mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta\cdot\mathbf{u} - (\mu_\delta + \chi_\varphi\sigma_\delta)\nabla\varphi_\delta\cdot\mathbf{u}\,\mathrm{d}x \tag{7.48b}$$

for all $\xi \in H^1$, $\mathbf{u}\in\mathbf{V}$ and for a. e. $t\in(0,T)$, whereas (7.8d)-(7.8e), $(7.9\text{a})_1$ and $(7.9\text{a})_3$, are fulfilled a. e. in their respective sets, i. e.,

$$\mu_\delta = -\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta'(\varphi_\delta) - \chi_\varphi\sigma_\delta \quad \text{a. e. in } \Omega_T, \qquad \nabla\varphi_\delta\cdot\mathbf{n} = 0 \quad \text{a. e. on } \Sigma_T, \tag{7.48c}$$

and

$$\begin{aligned}
\partial_t\sigma_\delta + \nabla\sigma_\delta\cdot\mathbf{v}_\delta &= \chi_\sigma\Delta\sigma_\delta - \chi_\varphi\Delta\varphi_\delta - f(\varphi_\delta,\sigma_\delta)h(\varphi_\delta) && \text{a. e. in } \Omega_T,\\
\nabla\sigma\cdot\mathbf{n}_\delta &= 0 && \text{a. e. on } \Sigma_T.
\end{aligned} \tag{7.48d}$$

Remark 7.13 Due to (7.48c), we see that

$$(\mu_\delta + \chi_\varphi\sigma_\delta)\nabla\varphi_\delta = \nabla\left(\frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta)\right) - \operatorname{div}(\varepsilon\nabla\varphi_\delta\otimes\nabla\varphi_\delta).$$

Therefore, (7.48b) is equivalent to

$$\int_\Omega \delta\partial_t\mathbf{v}_\delta\cdot\mathbf{u} + 2\eta\,\mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta\cdot\mathbf{u}\,\mathrm{d}x = \int_\Omega \varepsilon\,(\nabla\varphi_\delta\otimes\nabla\varphi_\delta) : \nabla\mathbf{u}\,\mathrm{d}x \tag{7.49}$$

for a. e. $t\in(0,T)$ and for all $\mathbf{u}\in\mathbf{V}$.

7.3.4 Some preliminary results

The following lemma will be important to estimate the source terms independent of $\delta\in(0,1]$.

Lemma 7.14 *For all $s\in\mathbb{R}$ it holds that*

$$|h(s)(\psi_\delta^1)'(s)| + |h(s)\Phi_\delta'(s)| \le C(1+|s|) \tag{7.50}$$

with a constant C independent of $\delta\in(0,1]$.

Proof. Let $\delta\in(0,1]$ be arbitrary. In the following we will frequently use the assumptions on $h(\cdot)$, $F(\cdot)$ and $(\psi_\delta^1)'(0) = \Phi_\delta'(0) = 0$. We consider only the case $p_0 = 2$ which corresponds to the highest degree of singularity of $(\psi_\delta^1)''$ and $(\Phi_\delta^1)''$. We distinguish different cases.

(i) For $s \le -1$ we have due to (7.5) that

$$h(s)(\psi_\delta^1)'(s) = 0.$$

(ii) If $s \in (-1, -1+\delta)$ for some $\delta \in (0,1)$, it holds

$$\begin{aligned}|h(s)(\psi_\delta^1)'(s)| &= \left| h(s) \left(\int_s^{-1+\delta} \frac{4F(-1+\delta)}{\delta^2} \,\mathrm{d}t + \int_{-1+\delta}^0 \frac{4F(t)}{(1+t)^2} \,\mathrm{d}t \right) \right| \\ &\leq 4F_0 h(s) \left(-1 + \delta^{-1} + \delta^{-2}|s - (-1+\delta)|\right) \\ &\leq 4F_0 C_7 \delta \left(-1 + 2\delta^{-1}\right) \\ &\leq C\end{aligned}$$

for a positive constant C independent of $\delta \in (0,1]$, where we used that

$$\left(\frac{|s - (-1+\delta)|}{\delta^2} \right) \leq \delta^{-1}, \qquad 0 \leq h(s) \leq C_7 \delta.$$

(iii) In the case $s \in (-1+\delta, 0)$, an easy computation shows

$$|h(s)(\psi_\delta^1)'(s)| \leq h(s) \left| \int_s^0 \frac{4F_0}{(1+t)^2} \,\mathrm{d}t \right| = 4F_0 h(s)(-1 + (1+s)^{-1}).$$

Since $\frac{h(s)}{1+s} \leq C_7$ for $s \in [-1,1]$, this implies that

$$|h(s)(\psi_\delta^1)'(s)| \leq C$$

with C independent of $\delta \in (0,1]$.

(iv) For $s \in [0,1]$, we obtain

$$|h(s)(\psi_\delta^1)'(s)| = h(s) \left| \int_0^s \frac{4F(t)}{(1+t)^2} \,\mathrm{d}t \right| \leq 4F_0 h(s) \left(-(1+s)^{-1} + 1\right) \leq C.$$

(v) Finally, if $s > 1$, we use the case $s \in [0,1]$ to derive that

$$\begin{aligned}|h(s)(\psi_\delta^1)'(s)| &= h(s) \left| \int_0^1 \frac{4F(t)}{(1+t)^2} \,\mathrm{d}t + \int_1^s F(t) \,\mathrm{d}t \right| \\ &\leq h(s) \left(2F_0 + F_0(s-1)\right) \leq C(1+s).\end{aligned}$$

In summary, this shows that

$$|h(s)(\psi_\delta^1)'(s)| \leq C(1+|s|) \quad \forall\, s \in \mathbb{R}$$

with a constant C independent of $\delta \in (0,1]$. Using the assumptions on $\bar{m}(\cdot)$, with exactly the same arguments it follows that

$$|h(s)\Phi_\delta'(s)| \leq C(1+|s|) \quad \forall\, s \in \mathbb{R}$$

with a constant C independent of $\delta \in (0,1]$ which completes the proof. □

The following lemma summarises uniform estimates for the approximating solutions.

Lemma 7.15 (a priori estimates) *There exists* δ_0 *such that for all* $0 < \delta \leq \delta_0$ *the following estimates hold with a constant* C *independent of* δ*:*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0\leq t\leq T} \int_\Omega \left(\frac{1}{2}|\varphi_\delta(t)|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta(t)|^2 + \frac{1}{4}|\sigma_\delta(t)|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta(t)) + \Phi_\delta(\varphi_\delta(t)) + \frac{\delta}{2}|\mathbf{v}_\delta(t)|^2\right) \mathrm{d}x \\
&\quad + \int_0^T \int_\Omega \frac{1}{2} m_\delta(\varphi_\delta)|\nabla\mu_\delta|^2 + \frac{1}{4}|\nabla\sigma_\delta|^2 + \frac{\varepsilon}{2}|\Delta\varphi_\delta|^2 + (\psi_\delta^1)''(\varphi_\delta)|\nabla\varphi_\delta|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \eta|\nabla\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_\delta|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C, && (7.51a)
\end{aligned}$$

$$\operatorname*{ess\,sup}_{0\leq t\leq T} \int_\Omega (-\varphi_\delta(t) - 1)_+^2 \leq C\delta, \tag{7.51b}$$

$$\int_{\Omega_T} |\mathbf{J}_\delta|^2 \leq C \text{ where } \mathbf{J}_\delta := m_\delta(\varphi_\delta)\nabla\mu_\delta. \tag{7.51c}$$

Proof. In the following we denote by C a generic positive constant independent of $\delta \in (0,1]$ which may change its value even within one line. Furthermore, we will frequently use Hölder's and Young's inequalities.

Step 1: Using that $\psi_\delta(\cdot)$ is a quadratic perturbation of a convex functional and invoking [123, Lemma 4.1], for almost every $t \in (0,T)$ it holds

$$\langle \partial_t\varphi_\delta\,, -\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta'(\varphi_\delta) + \varphi_\delta\rangle_{H^1} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta)\,\mathrm{d}x.$$

Then, with exactly the same arguments as in the proof of Proposition 7.4, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta) + \frac{D}{2}|\sigma_\delta|^2 + \frac{\delta}{2}|\mathbf{v}_\delta|^2 \,\mathrm{d}x \\
&\qquad + \int_\Omega m_\delta(\varphi_\delta)|\nabla\mu_\delta|^2 + D\chi_\sigma|\nabla\sigma_\delta|^2 + 2\eta|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_\delta|^2 \,\mathrm{d}x \\
&\quad = \int_\Omega -m_\delta(\varphi_\delta)\nabla\mu_\delta\cdot\nabla(\chi_\varphi\sigma_\delta + \varphi_\delta) + D\chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta \,\mathrm{d}x \\
&\qquad + \int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)(-\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta'(\varphi_\delta) + \varphi_\delta) - Df(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\sigma_\delta \,\mathrm{d}x && (7.52)
\end{aligned}$$

for $D > 0$ to be specified and for almost every $t \in (0,T]$, where we used the expression for $\mu_\delta + \chi_\varphi\sigma_\delta$ given by (7.48c) and the identity

$$\int_\Omega \varphi_\delta\nabla\varphi_\delta\cdot\mathbf{v}_\delta\,\mathrm{d}x = \int_\Omega \nabla\left(|\varphi_\delta|^2\right)\cdot\mathbf{v}_\delta\,\mathrm{d}x = \int_{\partial\Omega}|\varphi_\delta|^2\,\mathbf{v}_\delta\cdot\mathbf{n}\,\mathrm{d}\mathcal{H}^{d-1} - \int_\Omega |\varphi_\delta|^2\operatorname{div}(\mathbf{v}_\delta)\,\mathrm{d}x = 0.$$

The assumptions on $\bar{m}(\cdot)$ guarantee that

$$\begin{aligned}
\left|\int_\Omega m_\delta(\varphi_\delta)\nabla\mu_\delta\cdot\nabla(\chi_\varphi\sigma_\delta + \varphi_\delta)\,\mathrm{d}x\right| &\leq \int_\Omega \sqrt{m_\delta(\varphi_\delta)}|\nabla\mu_\delta|\sqrt{M_0}|\nabla(\chi_\varphi\sigma_\delta + \varphi_\delta)|\,\mathrm{d}x \\
&\leq \frac{1}{4}\int_\Omega m_\delta(\varphi_\delta)|\nabla\mu_\delta|^2\,\mathrm{d}x + 2M_0\left(\chi_\varphi^2\|\nabla\sigma_\delta\|_{\mathbf{L}^2}^2 + \|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2\right).
\end{aligned}$$

Furthermore, it holds that

$$\left|\int_\Omega D\chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta\,\mathrm{d}x\right| \leq \frac{D\chi_\sigma}{2}\|\nabla\sigma_\delta\|_{\mathbf{L}^2}^2 + \frac{D\chi_\varphi^2}{2\chi_\sigma}\|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2.$$

With similar arguments as in the proof of Proposition 7.4 we deduce

$$\begin{aligned}
\left|\int_\Omega Df(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\sigma_\delta\,\mathrm{d}x\right| &\leq C_D\left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\sigma_\delta\|_{L^2}^2\right), \\
\left|\int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\varphi_\delta\,\mathrm{d}x\right| &\leq C\left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\sigma_\delta\|_{L^2}^2\right).
\end{aligned}$$

Employing the last four inequalities in (7.52) and choosing $D = \max\left(1, (1+4M_0\chi_\varphi^2)\chi_\sigma^{-1}\right)$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\tfrac{1}{2}|\varphi_\delta|^2 + \tfrac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta) + \tfrac{1}{2}|\sigma_\delta|^2 + \tfrac{\delta}{2}|\mathbf{v}_\delta|^2\right)\,\mathrm{d}x\\
&\quad + \int_\Omega \tfrac{1}{2}m_\delta(\varphi_\delta)|\nabla\mu_\delta|^2 + \tfrac{1}{2}|\nabla\sigma_\delta|^2 + 2\eta|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_\delta|^2\,\mathrm{d}x\\
&\le C\left(1 + \|\varphi_\delta\|_{H^1}^2 + \|\sigma_\delta\|_{L^2}^2\right) + \int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)(\varepsilon^{-1}\psi_\delta'(\varphi_\delta) - \varepsilon\Delta\varphi_\delta)\,\mathrm{d}x. \qquad (7.53)
\end{aligned}$$

It remains to analyse the last term on the r. h. s. of (7.53). Applying (7.5), we have

$$\left|\int_\Omega \varepsilon\, g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\Delta\varphi_\delta\,\mathrm{d}x\right| \le \gamma\|\Delta\varphi_\delta\|_{L^2}^2 + C_\gamma\left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\sigma_\delta\|_{L^2}^2\right)$$

with $\gamma > 0$ to be chosen later. Due to the assumptions on $\psi^2(\cdot)$ and using Lemma 7.14 for ψ_δ^1 along with (7.5), we obtain

$$\left|\int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\varepsilon^{-1}\psi_\delta'(\varphi_\delta)\,\mathrm{d}x\right| \le C\left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\sigma_\delta\|_{L^2}^2\right).$$

Invoking the last two inequalities in (7.53) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta) + \frac{1}{2}|\sigma_\delta|^2 + \frac{\delta}{2}|\mathbf{v}_\delta|^2\right]\,\mathrm{d}x\\
&\quad + \int_\Omega \frac{1}{2}m_\delta(\varphi_\delta)|\nabla\mu_\delta|^2 + \frac{1}{2}|\nabla\sigma_\delta|^2 + 2\eta|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_\delta|^2\,\mathrm{d}x\\
&\le C_\gamma\left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2 + \|\sigma_\delta\|_{L^2}^2\right) + \gamma\|\Delta\varphi_\delta\|_{L^2}^2. \qquad (7.54)
\end{aligned}$$

Step 2: In the following we aim to derive an estimate for $\Delta\varphi_\delta$ in order to absorb the last term on the r. h. s. of (7.54). Choosing $\Phi_\delta'(\varphi_\delta) \in L^2(H^1)$ as a test function in (7.48a) and invoking [123, Lemma 4.1], we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi_\delta(\varphi_\delta)\,\mathrm{d}x &= -\int_\Omega m_\delta(\varphi_\delta)\nabla(-\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta'(\varphi_\delta) - \chi_\varphi\sigma_\delta)\cdot\nabla\Phi_\delta'(\varphi_\delta)\,\mathrm{d}x\\
&\quad + \int_\Omega g(\varphi_\delta,\sigma_\delta)h(\varphi_\delta)\Phi_\delta'(\varphi_\delta) - \Phi_\delta'(\varphi_\delta)\nabla\varphi_\delta\cdot\mathbf{v}_\delta\,\mathrm{d}x \qquad (7.55)
\end{aligned}$$

for almost every $t \in (0,T)$. Integrating by parts and using $\mathbf{v}_\delta \in L^2(\mathbf{V})$, we see that

$$\int_\Omega \Phi_\delta'(\varphi_\delta)\nabla\varphi_\delta\cdot\mathbf{v}_\delta\,\mathrm{d}x = \int_\Omega \nabla\left(\Phi_\delta(\varphi_\delta)\right)\cdot\mathbf{v}_\delta\,\mathrm{d}x = 0.$$

The identity $\Phi_\delta''(\varphi_\delta) = \frac{1}{m_\delta(\varphi_\delta)}$ and integration by parts yield

$$\begin{aligned}
&\int_\Omega m_\delta(\varphi_\delta)\nabla(-\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta'(\varphi_\delta) - \chi_\varphi\nabla\sigma_\delta)\cdot\nabla\Phi_\delta'(\varphi_\delta)\,\mathrm{d}x\\
&= \int_\Omega (-\varepsilon\nabla\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta''(\varphi_\delta)\nabla\varphi_\delta - \chi_\varphi\nabla\sigma_\delta)\cdot\nabla\varphi_\delta\,\Phi_\delta''(\varphi_\delta)m_\delta(\varphi_\delta)\,\mathrm{d}x\\
&= \int_\Omega \varepsilon|\Delta\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta''(\varphi_\delta)|\nabla\varphi_\delta|^2 - \chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta\,\mathrm{d}x.
\end{aligned}$$

Using the assumptions on $\psi^2(\cdot)$, it holds

$$\left|\int_\Omega \varepsilon^{-1}(\psi^2)''(\varphi_\delta)|\nabla\varphi_\delta|^2 - \chi_\varphi\nabla\sigma_\delta\cdot\nabla\varphi_\delta\,\mathrm{d}x\right| \le \frac{1}{4}\|\nabla\sigma_\delta\|_{\mathbf{L}^2}^2 + C\|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2.$$

For the remaining term on the r. h. s. of (7.55), we apply (7.5) and Lemma 7.14 to obtain

$$\left| \int_\Omega g(\varphi_\delta, \sigma_\delta) h(\varphi_\delta) \Phi_\delta'(\varphi_\delta) \,\mathrm{d}x \right| \leq C \left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\sigma_\delta\|_{L^2}^2\right).$$

Employing the last four (in)equalities in (7.55), we end up with

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Phi_\delta(\varphi_\delta(t)) \,\mathrm{d}x + \int_\Omega \varepsilon |\Delta \varphi_\delta|^2 + (\psi_\delta^1)''(\varphi_\delta) |\nabla \varphi_\delta|^2 \,\mathrm{d}x \\ &\quad \leq C \left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\nabla \varphi_\delta\|_{\mathbf{L}^2}^2 + \|\sigma_\delta\|_{L^2}^2\right) + \frac{1}{4} \|\nabla \sigma_\delta\|_{\mathbf{L}^2}^2 \end{aligned}$$

for almost every $t \in (0,T)$. Adding this inequality to (7.55) and choosing $\gamma = \frac{\varepsilon}{2}$, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} |\varphi_\delta|^2 + \frac{\varepsilon}{2} |\nabla \varphi_\delta|^2 + \varepsilon^{-1} \psi_\delta(\varphi_\delta) + \Phi_\delta(\varphi_\delta) + \frac{1}{2} |\sigma_\delta|^2 + \frac{\delta}{2} |\mathbf{v}_\delta|^2 \right] \mathrm{d}x \\ &\quad + \int_\Omega \frac{1}{2} m_\delta(\varphi_\delta) |\nabla \mu_\delta|^2 + \frac{\varepsilon}{2} |\Delta \varphi_\delta|^2 + (\psi_\delta^1)''(\varphi_\delta) |\nabla \varphi_\delta|^2 + \frac{1}{4} |\nabla \sigma_\delta|^2 + 2\eta |\mathbf{D} \mathbf{v}_\delta|^2 + \nu |\mathbf{v}_\delta|^2 \,\mathrm{d}x \\ &\quad \leq C \left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\nabla \varphi_\delta\|_{\mathbf{L}^2}^2 + \|\sigma_\delta\|_{L^2}^2\right) \end{aligned} \tag{7.56}$$

for almost all $t \in [0,T]$. Next, we notice that $\Phi_\delta(u) \leq \Phi(u)$, $\psi_\delta^1(u) \leq \psi^1(u)$ for δ sufficiently small. Using (7.28) and the Sobolev embedding $H^1 \subset L^6$ along with the assumptions on φ_0 and σ_0, we know that

$$\int_\Omega \frac{1}{2} |\varphi_0|^2 + \frac{\varepsilon}{2} |\nabla \varphi_0|^2 + \varepsilon^{-1} \psi_\delta(\varphi_0) + \Phi_\delta(\varphi_0) + \frac{1}{2} |\sigma_0|^2 \,\mathrm{d}x \leq C.$$

Consequently, integrating (7.56) in time from 0 to $t \in (0,T]$, an application of Gronwall's lemma implies (7.51a).

Step 3: We now prove (7.51b). First observe that the convexity of $\Phi_\delta(\cdot)$ and $\Phi_\delta(0) = \Phi_\delta'(0) = 0$ imply

$$\Phi_\delta(-1+\delta) \geq 0, \quad \Phi_\delta'(-1+\delta) \leq 0.$$

Therefore, for $z \leq -1$ and $\delta \in (0,1)$ we obtain

$$\begin{aligned} \Phi_\delta(z) &= \Phi_\delta(-1+\delta) + \Phi_\delta'(-1+\delta)(z - (-1+\delta)) + \frac{1}{2} \Phi_\delta''(-1+\delta)(z - (-1+\delta))^2 \\ &\geq \frac{1}{2} \Phi''(-1+\delta)(z - (-1+\delta))^2 \\ &\geq \left(\tfrac{2}{\delta}\right)^{p_0} \frac{1}{2\bar{m}(-1+\delta)} (z - (-1+\delta))^2 \\ &\geq \left(\tfrac{2}{\delta}\right)^{p_0} \frac{1}{2M_0} (-z + (-1+\delta))^2 \\ &\geq \left(\tfrac{2}{\delta}\right)^{p_0} \frac{1}{2M_0} (-z-1)^2, \end{aligned}$$

hence, using $\delta^{p_0} \leq \delta$ gives

$$(-z-1)^2 \leq C\delta \Phi_\delta(z) \quad \text{for all } z \leq -1 \text{ and } \delta < 1.$$

Employing (7.51a) we conclude

$$\operatorname*{ess\,sup}_{0 \leq t \leq T} \int_\Omega (-\varphi_\delta(t) - 1)_+^2 \,\mathrm{d}x \leq C\delta \operatorname*{ess\,sup}_{0 \leq t \leq T} \int_\Omega \Phi_\delta(\varphi_\delta(t)) \,\mathrm{d}x \leq C\delta$$

which implies (7.51b). Finally, because of (7.51a), an easy computation shows that

$$\int_0^T \int_\Omega m_\delta(\varphi_\delta)^2 |\nabla \mu_\delta|^2 \,\mathrm{d}x \,\mathrm{d}t \leq C \int_0^T \int_\Omega m_\delta(\varphi_\delta) |\nabla \mu_\delta|^2 \,\mathrm{d}x \,\mathrm{d}t \leq C,$$

and the proof is complete. □

The following lemma will be applied to pass to the limit in the proof of Theorem 7.11.

Lemma 7.16 *Let $\delta \in (0, \delta_0]$ and assume the assumptions of Theorem 7.11 are fulfilled. Then, it holds that*

$$\begin{aligned} &\|\varphi_\delta\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^2)} + \|\sigma_\delta\|_{H^1((H^1)^*)\cap L^\infty(L^6)\cap L^2(H^1)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} \\ &\quad + \sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)} + \|\mathrm{div}(\varphi_\delta \mathbf{v}_\delta)\|_{L^2(\mathbf{L}^{\frac{3}{2}})} + \|\mathrm{div}(\sigma_\delta \mathbf{v}_\delta)\|_{L^2((H^1)^*)} \leq C \end{aligned} \tag{7.57}$$

with a positive constant C independent of $\delta \in (0, \delta_0]$. Furthermore, as $\delta \to 0$ we have (at least for a non-relabelled subsequence)

$$\begin{aligned} \varphi_\delta &\to \varphi && \textit{weakly-star in} && H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^2), && (7.58a)\\ \sigma_\delta &\to \sigma && \textit{weakly-star in} && H^1((H^1)^*) \cap L^\infty(L^6) \cap L^2(H^1), && (7.58b)\\ \mathbf{v}_\delta &\to \mathbf{v} && \textit{weakly in} && L^2(\mathbf{H}^1), && (7.58c)\\ \mathrm{div}(\varphi_\delta \mathbf{v}_\delta) &\to \mathrm{div}(\varphi \mathbf{v}) && \textit{weakly in} && L^2(L^{\frac{3}{2}}), && (7.58d)\\ \mathrm{div}(\sigma_\delta \mathbf{v}_\delta) &\to \mathrm{div}(\sigma \mathbf{v}) && \textit{weakly in} && L^2((H^1)^*), && (7.58e)\\ \boldsymbol{J}_\delta &\to \boldsymbol{J} && \textit{weakly in} && L^2(\mathbf{L}^2), && (7.58f) \end{aligned}$$

and

$$\begin{aligned} \varphi_\delta &\to \varphi \quad \textit{strongly in } C^0([0,T]; L^r) \cap L^2(W^{1,r}) \quad \textit{and a. e. in } \Omega_T, && (7.58g)\\ \sigma_\delta &\to \sigma \quad \textit{strongly in } C^0([0,T]; (H^1)^*) \cap L^p(L^r) \quad \textit{and a. e. in } \Omega_T && (7.58h) \end{aligned}$$

for any $r \in [1, 6)$ and $p \in [1, \infty)$.

Proof. In the following we denote by C a generic constant independent of $\delta \in (0, \delta_0]$. Using (7.51a) and elliptic regularity theory, it follows that

$$\|\varphi_\delta\|_{L^\infty(H^1)\cap L^2(H^2)} \leq C.$$

Due to Korn's inequality (see (2.23)) and (7.51a) we have

$$\|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} + \sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)} \leq C.$$

Invoking the last two inequalities and (7.51a), with exactly the same arguments as in the proof of Proposition 7.4 it follows that

$$\begin{aligned} &\|\varphi_\delta\|_{H^1((\mathbf{H}^1)^*)\cap L^\infty(H^1)\cap L^2(H^2)} + \|\sigma_\delta\|_{H^1((H^1)^*)\cap L^\infty(L^6)\cap L^2(H^1)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} \\ &\quad + \sqrt{\delta}\|\mathbf{v}_\delta\|_{L^\infty(\mathbf{L}^2)} + \|\mathrm{div}(\varphi_\delta \mathbf{v}_\delta)\|_{L^2(L^{\frac{3}{2}})} + \|\mathrm{div}(\sigma_\delta \mathbf{v}_\delta)\|_{L^2((H^1)^*)} \leq C \end{aligned}$$

which implies (7.57).

Recalling (7.51a), (7.57), and arguing as in the proof of Proposition 7.4, we obtain (7.58a)-(7.58c) and (7.58e)-(7.58h). The argument for (7.58d) is slightly different. Indeed, applying (7.57) and reflexive weak compactness arguments, we infer that

$$\mathrm{div}(\varphi_\delta \mathbf{v}_\delta) \to \theta \quad \text{weakly in } L^2(L^{\frac{3}{2}})$$

for some limit function $\theta \in L^2(L^{\frac{3}{2}})$. Integrating by parts, we obtain

$$\|\nabla\varphi_\delta - \nabla\varphi\|_{\mathbf{L}^2}^4 \leq C\|\varphi_\delta - \varphi\|_{L^2}^2 \|\Delta(\varphi_\delta - \varphi)\|_{L^2}^2.$$

Integrating this inequality in time from 0 to T, using (7.57), (7.58g) and weak(-star) lower semicontinuity of norms, this leads to

$$\begin{aligned}\int_0^T \|\nabla\varphi_\delta - \nabla\varphi\|_{\mathbf{L}^2}^4 \,\mathrm{d}t &\le C\int_0^T \|\varphi_\delta - \varphi\|_{L^2}^2 \|\Delta(\varphi_\delta - \varphi)\|_{L^2}^2 \,\mathrm{d}t \\ &\le C\|\varphi_\delta - \varphi\|_{L^\infty(L^2)}^2 \|\varphi_\delta - \varphi\|_{L^2(H^2)}^2 \to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

By the product of weak-strong convergence and (7.58c), this yields

$$\operatorname{div}(\varphi_\delta \mathbf{v}_\delta) \to \operatorname{div}(\varphi\mathbf{v}) \quad \text{weakly in } L^{\frac{4}{3}}(L^{\frac{3}{2}}) \quad \text{as } \delta \to 0.$$

Consequently, by uniqueness of limits we obtain $\operatorname{div}(\varphi\mathbf{v}) = \theta \in L^2(L^{\frac{3}{2}})$ which completes the proof. □

7.3.5 Proof of Theorem 7.11

We divide the analysis into several steps:

Step 1: Passing to the limit in (7.51b) and using (7.58g), we conclude that

$$\varphi \ge -1 \quad \text{a.e. in } \Omega_T. \tag{7.59}$$

Using similar arguments as in the proof of Proposition 7.4 and recalling (7.49), the quadruplet $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta)$ fulfils

$$0 = \int_0^T \left(\langle \partial_t\varphi_\delta, \xi\rangle_{H^1} + \int_\Omega \nabla\varphi_\delta \cdot \mathbf{v}_\delta\, \xi + m_\delta(\varphi_\delta)\nabla\mu_\delta \cdot \nabla\xi - g(\varphi_\delta, \sigma_\delta)h(\varphi_\delta)\xi \,\mathrm{d}x \right) \mathrm{d}t, \tag{7.60a}$$

$$\begin{aligned}0 = &-\int_0^T \int_\Omega \zeta'(t)\delta\mathbf{v}_\delta \cdot \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \zeta \left(\int_\Omega 2\eta\mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta \cdot \mathbf{u} - \varepsilon\big(\nabla\varphi_\delta \otimes \nabla\varphi_\delta\big) : \nabla\mathbf{u} \,\mathrm{d}x \right) \mathrm{d}t,\end{aligned} \tag{7.60b}$$

$$0 = \int_0^T \zeta \left(\int_\Omega \partial_t\sigma_\delta\phi + (\chi_\sigma\nabla\sigma_\delta - \chi_\varphi\nabla\varphi_\delta - \sigma_\delta\mathbf{v}_\delta) \cdot \nabla\phi + f(\varphi_\delta, \sigma_\delta)h(\varphi_\delta)\phi \,\mathrm{d}x \right) \mathrm{d}t \tag{7.60c}$$

for all $\zeta \in C_0^\infty(0,T)$, $\xi \in L^2(H^1)$, $\phi \in H^1$ and $\mathbf{u} \in \mathbf{V}$, where μ_δ is given by

$$\mu_\delta = -\varepsilon^{-1}\Delta\varphi_\delta + \varepsilon\psi_\delta'(\varphi_\delta) - \chi_\varphi\sigma_\delta \qquad \text{a.e. in } \Omega_T. \tag{7.60d}$$

Using Lemma 7.16, with similar arguments as in the proof of Proposition 7.4 it follows that

$$\begin{aligned}\int_0^T \langle \partial_t\varphi, \xi\rangle_{H^1} \,\mathrm{d}t &= \int_0^T \int_\Omega \mathbf{J} \cdot \nabla\xi - \nabla\varphi \cdot \mathbf{v}\xi + g(\varphi,\sigma)h(\varphi)\xi \,\mathrm{d}x\,\mathrm{d}t, \\ \langle \partial_t\sigma, \phi\rangle_{H^1} &= -\int_\Omega (\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi - \sigma\mathbf{v}) \cdot \nabla\phi + f(\varphi,\sigma)h(\varphi)\phi \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for almost all $t \in (0,T)$ and all $\xi \in L^2(H^1)$, $\phi \in H^1$. Due to (7.57) and the continuous embedding $L^\infty(\mathbf{L}^2) \cap L^2(\mathbf{H}^1) \hookrightarrow L^4(\mathbf{L}^3)$, we have that

$$\|\nabla\varphi_\delta \otimes \nabla\varphi_\delta\|_{L^{\frac{4}{3}}(\mathbf{L}^2)} \le C.$$

Using reflexive weak compactness arguments, this means that $\nabla\varphi_\delta \otimes \nabla\varphi_\delta \rightharpoonup \boldsymbol{\theta}$ in $L^{\frac{4}{3}}(\mathbf{L}^2)$ for some $\boldsymbol{\theta} \in L^{\frac{4}{3}}((L^2)^{d\times d})$. Applying (7.58a) and (7.58g), by the product of weak strong convergence we obtain

$$\nabla\varphi_\delta \otimes \nabla\varphi_\delta \to \nabla\varphi \otimes \nabla\varphi \quad \text{weakly in } L^{\frac{4}{3}}(\mathbf{L}^p) \quad \forall p \in (1,2).$$

Then, by uniqueness of weak limits we deduce that $\boldsymbol{\theta} = \nabla\varphi \otimes \nabla\varphi$. Invoking similar arguments as in the proof of Proposition 7.4 and using $\zeta\nabla\mathbf{u} \in C^0([0,T];\mathbf{L}^2)$, we infer that

$$0 = \int_\Omega 2\eta\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu\mathbf{v}\cdot\mathbf{u} - \varepsilon\big(\nabla\varphi \otimes \nabla\varphi\big) \colon \nabla\mathbf{u} \,\mathrm{d}x$$

for almost all $t \in (0,T)$ and all $\mathbf{u} \in \mathbf{V}$.

Step 2: We now identify $\mathbf{J}$. To this end, we pass to the limit in

$$\int_{\Omega_T} \mathbf{J}_\delta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_T} m_\delta(\varphi_\delta)\nabla(-\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi'(\varphi_\delta) - \chi_\varphi\sigma_\delta)\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t, \tag{7.61}$$

where $\boldsymbol{\eta} \in L^2(\mathbf{H}^1) \cap L^\infty(\mathbf{L}^\infty)$ with $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ a. e. on Σ_T. Since $\mathbf{J}_\delta \rightharpoonup \mathbf{J}$ weakly in $L^2(\mathbf{L}^2)$ as $\delta \to 0$, it follows that

$$\int_{\Omega_T} \mathbf{J}_\delta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \mathbf{J}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta \to 0. \tag{7.62}$$

Due to the fact that $\nabla\Delta\varphi_\delta$ may not have a limit in $L^2(\mathbf{L}^2)$, we integrate the first term on the r. h. s. of (7.61) by parts to obtain

$$\int_{\Omega_T} m_\delta(\varphi_\delta)\nabla(-\varepsilon\Delta\varphi_\delta)\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_T} \varepsilon\Delta\varphi_\delta(m_\delta(\varphi_\delta)\,\mathrm{div}(\boldsymbol{\eta}) + m_\delta'(\varphi_\delta)\nabla\varphi_\delta\cdot\boldsymbol{\eta})\,\mathrm{d}x\,\mathrm{d}t. \tag{7.63}$$

By definition of the mobility m_δ, we observe that

$$|m_\delta(z) - m(z)| \le \sup_{-1\le z\le -1+\delta} |m(z)| \to 0 \quad \text{as } \delta \to 0$$

for all $z \in \mathbb{R}$. Hence, it follows that $m_\delta \to m$ uniformly, meaning

$$m_\delta(\varphi_\delta) \to m(\varphi) \quad \text{a. e. in } \Omega_T.$$

Since $\Delta\varphi_\delta \rightharpoonup \Delta\varphi$ weakly in $L^2(L^2)$ and m_δ is uniformly bounded we conclude

$$\int_{\Omega_T} \varepsilon\Delta\varphi_\delta\, m_\delta(\varphi_\delta)\,\mathrm{div}(\boldsymbol{\eta})\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \varepsilon\Delta\varphi\, m(\varphi)\,\mathrm{div}(\boldsymbol{\eta})\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta \to 0. \tag{7.64}$$

To analyse the second term on the r. h. s. of (7.63), we first note that m' is given by

$$m'(u) = \begin{cases} 0 & \text{for } u < -1,\\ q_0\frac{1}{2^{q_0}}(1+u)^{q_0-1}\bar{m}(u) + \big(\frac{1}{2}(1+u)\big)^{q_0}\,\bar{m}'(u) & \text{for } u \in (-1,1),\\ \bar{m}'(u) & \text{for } u > 1. \end{cases}$$

Thus, we observe that $m'(\cdot)$ may be discontinuous in 1, and $m'(\cdot)$ is discontinuous in -1 if $q_0 = 1$ and $\bar{m}(-1) \neq 0$. Therefore, we have to employ a more involved argument to show that

$$m_\delta'(\varphi_\delta)\nabla\varphi_\delta \to m'(\varphi)\nabla\varphi \quad \text{in } L^2(\mathbf{L}^2). \tag{7.65}$$

Using that $\varphi \ge -1$ a. e. in Ω_T, we obtain

$$\begin{aligned}\int_{\Omega_T} |m_\delta'(\varphi_\delta)\nabla\varphi_\delta - m'(\varphi)\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T\cap\{|\varphi|=1\}} |m_\delta'(\varphi_\delta)\nabla\varphi_\delta - m'(\varphi)\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \int_{\Omega_T\cap\{\varphi>-1\}\cap\{\varphi\neq 1\}} |m_\delta'(\varphi_\delta)\nabla\varphi_\delta - m'(\varphi)\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Since $\nabla\varphi = 0$ on the set $\{|\varphi| = 1\}$ (see [92, Lemma 7.7]) we infer

$$\begin{aligned}\int_{\Omega_T\cap\{|\varphi|=1\}} |m'_\delta(\varphi_\delta)\nabla\varphi_\delta - m'(\varphi)\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T\cap\{|\varphi|=1\}} |m'_\delta(\varphi_\delta)\nabla\varphi_\delta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le C\int_{\Omega_T\cap\{|\varphi|=1\}} |\nabla\varphi_\delta|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\to C\int_{\Omega_T\cap\{|\varphi|=1\}} |\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t = 0\end{aligned}$$

as $\delta \to 0$, where we used the boundedness of $m'_\delta(\cdot)$ and the fact that $\nabla\varphi_\delta \to \nabla\varphi$ a. e. on Ω_T. On the set $\{|\varphi| \neq 1\}$ we know that $m'_\delta(\varphi_\delta)\nabla\varphi_\delta \to m'(\varphi)\nabla\varphi$ almost everywhere. On account of Lemma 2.35, this yields

$$\int_{\Omega_T} |m'_\delta(\varphi_\delta)\nabla\varphi_\delta - m'(\varphi)\nabla\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } \delta \to 0$$

which shows (7.65). With similar arguments as for (7.64), we obtain that

$$\int_{\Omega_T} \varepsilon\Delta\varphi_\delta\, m'_\delta(\varphi_\delta)\nabla\varphi_\delta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \varepsilon\Delta\varphi\, m'(\varphi)\nabla\varphi \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta \to 0. \tag{7.66}$$

For the third term on the r. h. s. of (7.61), using the uniform boundedness and continuity of $m_\delta(\cdot)$, the convergence $\varphi_\delta \to \varphi$ a. e. in Ω_T and $\sigma_\delta \rightharpoonup \sigma$ weakly in $L^2(H^1)$ as $\delta \to 0$, we deduce

$$\chi_\varphi \int_{\Omega_T} m_\delta(\varphi_\delta)\nabla\sigma_\delta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \to \chi_\varphi \int_{\Omega_T} m(\varphi)\nabla\sigma \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta \to 0. \tag{7.67}$$

It remains to show that

$$\int_{\Omega_T} \varepsilon^{-1} m_\delta(\varphi_\delta)\psi''_\delta(\varphi_\delta)\nabla\varphi_\delta \cdot \boldsymbol{\eta} \to \int_{\Omega_T} \varepsilon^{-1}(m\psi'')(\varphi)\nabla\varphi \cdot \boldsymbol{\eta} \quad \text{as } \delta \to 0. \tag{7.68}$$

Due to the boundedness and continuity of $(\psi^2)''$ and using similar arguments as above, it follows that

$$m_\delta(\varphi_\delta) \to m(\varphi) \quad \text{a. e. in } \Omega_T, \qquad (\psi^2)''(\varphi_\delta) \to (\psi^2)''(\varphi) \quad \text{a. e. in } \Omega_T.$$

Together with the weak convergence $\varphi_\delta \to \varphi$ in $L^2(H^1)$, this yields

$$\int_{\Omega_T} \varepsilon^{-1} m_\delta(\varphi_\delta)(\psi^2)''(\varphi_\delta)\nabla\varphi_\delta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \varepsilon^{-1} m(\varphi)(\psi^2)''(\varphi)\nabla\varphi \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \tag{7.69}$$

as $\delta \to 0$. For the term involving ψ^1_δ, we first observe that $m_\delta(\psi^1_\delta)''$ is uniformly bounded a. e. in Ω_T. Hence, it is sufficient to show that

$$m_\delta(\varphi_\delta)(\psi^1_\delta)''(\varphi_\delta) \to \big(m(\psi^1)''\big)(\varphi) = \bar{m}(\varphi)F(\varphi) \quad \text{a. e. in } \Omega_T \quad \text{as } \delta \to 0. \tag{7.70}$$

If $\varphi(x,t) > -1+\delta$, the convergence in (7.69) follows from the definition of $m_\delta(\cdot)$ and $\psi^1_\delta(\cdot)$ (recall that $m_\delta(z) = m(z)$ and $\psi^1_\delta(z) = \psi^1(z)$ if $z > -1+\delta$). Thus, let us consider points where $\varphi(x,t) = -1$. We define $k(r) = \max\big(0, \min(\tfrac{1}{2}(1+r), 1)\big)$. Then, for δ with $\varphi_\delta(x,t) \le -1+\delta$ we have

$$\begin{aligned}m_\delta(\varphi_\delta(x,t))(\psi^1_\delta)''(\varphi_\delta(x,t)) &= \bar{m}(-1+\delta)F(-1+\delta)k(-1+\delta)^{q_0-p_0} \\ &\xrightarrow{\delta\to 0} \bar{m}(-1)F(-1)k(-1)^{q_0-p_0} \\ &= \big(m(\psi^1)''\big)\,(\varphi(x,t)).\end{aligned}$$

If $\varphi_\delta(x,t) \geq -1+\delta$ and $\varphi_\delta(x,t) \to -1$, we have

$$\begin{aligned} m_\delta(\varphi_\delta(x,t))(\psi_\delta^1)''(\varphi_\delta(x,t)) &= m(\varphi_\delta(x,t))(\psi^1)''(\varphi_\delta(x,t)) \\ &= \bar{m}(\varphi_\delta(x,t))F(\varphi_\delta(x,t))k(\varphi_\delta(x,t))^{q_0-p_0} \\ &\to \big(m(\psi^1)''\big)\,(\varphi(x,t)). \end{aligned}$$

as $\delta \to 0$, where we used that $\bar{m}$, F, $k \in C^0$. Hence, we have shown (7.70). We remark that the assumption $q_0 \geq p_0$ is essential since otherwise it holds that $k(-1)^{q_0-p_0} = \infty$. Together with the strong convergence $\varphi_\delta \to \varphi$ in $L^2(H^1)$ as $\delta \to 0$, we obtain

$$\int_{\Omega_T} \varepsilon^{-1} m_\delta(\varphi_\delta)(\psi_\delta^1)''(\varphi_\delta)\nabla\varphi_\delta\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \varepsilon^{-1} m(\varphi)(\psi^1)''(\varphi)\nabla\varphi\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta\to 0 \quad (7.71)$$

which proves (7.68). Due to (7.62)-(7.64) and (7.66)-(7.68), we conclude that (7.47d) holds.

Step 3: Attainment of initial conditions follows with exactly the same arguments as in the proof of Proposition 7.4. Moreover, the uniform estimates and weak(-star) lower semicontinuity of norms imply that

$$\mathcal{S} \coloneqq -\mathrm{div}(2\eta\mathbf{D}\mathbf{v} - \varepsilon\,(\nabla\varphi\otimes\nabla\varphi)) + \nu\mathbf{v} \in L^{\frac{4}{3}}(0,T;\mathbf{V}^*).$$

Hence, by Lemma 2.40 there exists a unique pressure $p \in L^{\frac{4}{3}}(0,T;L_0^2)$ satisfying $-\nabla p = \mathcal{S}$ in the sense of distributions which completes the proof.

7.4 The deep quench limit

In classical Cahn–Hilliard models, the deep quench limit is established by sending $\theta \to 0$ for the potential

$$\psi_{\log}(\varphi) = \frac{\theta}{2}\big(\ln(1+\varphi)(1+\varphi) + \ln(1-\varphi)(1-\varphi)\big) + \frac{\theta_c}{2}(1-\varphi^2) \quad \forall\,\varphi\in(-1,1).$$

In this case, one obtains solutions for the so-called double obstacle potential given by

$$\psi_{\mathrm{do}}(r) = \begin{cases} \frac{\theta_c}{2}(1-r^2) & \text{for } |r|\leq 1,\\ +\infty & \text{else.}\end{cases}$$

This is not the case in our situation, since solutions do in general not fulfil $\varphi \leq 1$. However, we may, e. g., consider the potential (7.46). To avoid being too technical, we extend both ψ^1 and ψ^2 (see (7.46a),(7.46b)) quadratically for $r \geq 1$, and without loss of generality we assume that $\theta < \frac{2\theta_c}{4\ln(2)-1}$ in order to ensure that $\psi''(1) > 0$. Then, we define

$$\psi^1(r) \coloneqq \psi_\theta^1(r) = \begin{cases} +\infty & \text{for } r < -1,\\ \theta\,(\ln(1+r)(1+r) - r) & \text{for } |r|\leq 1,\\ \theta\big(2\ln(2)-1\big) + \theta\ln(2)(r-1) + \frac{\theta}{4}(r-1)^2 & \text{for } r > 1,\end{cases}$$

and $\psi^2 \coloneqq \psi_\theta^2$ where ψ_θ^2 is given by

$$\psi_\theta^2(r) = \begin{cases} -\frac{2}{3}(\theta_c + 2\theta\ln(2)) + \theta + (2\theta_c - \theta\ln(2))(r+1) + \frac{-3\theta_c+2\theta\ln(2)}{2}(r+1)^2 & \text{for } r<-1,\\ \frac{\theta_c-\theta\ln(2)}{3}r^3 - \frac{\theta_c}{2}r^2 + \frac{\theta_c+6\theta-10\theta\ln(2)}{6} & \text{for } |r|\leq 1,\\ \theta(1-2\ln(2)) - \theta\ln(2)(r-1) + \frac{\theta_c-2\theta\ln(2)}{2}(r-1)^2 & \text{for } r>1.\end{cases}$$

Then, as $\theta \to 0$, we see that ψ_θ^1 converges formally to $I_{[-1,\infty]}$ defined by

$$I_{[-1,\infty]}(r) = \begin{cases} +\infty & \text{for } r < -1, \\ 0 & \text{else,} \end{cases}$$

and ψ_θ^2 converges formally to ψ_0^2 defined by

$$\psi_0^2(r) = \begin{cases} -\frac{2\theta_c}{3} + 2\theta_c(r+1) - \frac{3\theta_c}{2}(r+1)^2 & \text{for } r < -1, \\ \frac{\theta_c}{3}r^3 - \frac{\theta_c}{2}r^2 + \frac{\theta_c}{6} & \text{for } |r| \leq 1, \\ \frac{\theta_c}{2}(r-1)^2 & \text{for } r \geq 1. \end{cases}$$

We have the following result:

Theorem 7.17 *Let the assumptions of Theorem 7.11 be fulfilled and define ψ^2 as above. Then, there exists a quadruplet $(\varphi, \boldsymbol{J}, \sigma, \mathbf{v})$ satisfying*

a) $\varphi \in H^1((H^1)^*) \cap C([0,T]; L^2) \cap L^\infty(H^1) \cap L^2(H^2)$,

b) $\varphi(0) = \varphi_0$ *in* L^2 *and* $\nabla\varphi \cdot \mathbf{n} = 0$ *a. e. on* Σ_T,

c) $\varphi \geq -1$ *a. e. in* Ω_T,

d) $\sigma \in H^1((H^1)^*) \cap C^0(L^2) \cap L^\infty(L^6) \cap L^2(H^1)$,

e) $\sigma(0) = \sigma_0$ *in* L^2,

f) $\boldsymbol{J} \in L^2(\mathbf{L}^2)$,

g) $\mathbf{v} \in L^2(\mathbf{H}^1)$,

and solving

$$\begin{aligned} \int_0^T \langle \partial_t\varphi(t), \xi(t)\rangle_{H^1} \, \mathrm{d}t &= \int_{\Omega_T} \boldsymbol{J} \cdot \nabla\xi \, \mathrm{d}x \, \mathrm{d}t + \int_{\Omega_T} g(\varphi,\sigma)h(\varphi)\xi - \nabla\varphi \cdot \mathbf{v}\,\xi \, \mathrm{d}x \, \mathrm{d}t, \\ \langle \partial_t\sigma, \phi\rangle_{H^1} &= \int_\Omega (-\chi_\sigma\nabla\sigma + \chi_\varphi\nabla\varphi + \sigma\mathbf{v}) \cdot \nabla\phi - f(\varphi,\sigma)h(\varphi)\phi \, \mathrm{d}x, \\ \int_\Omega 2\eta\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu\mathbf{v}\cdot\mathbf{u} \, \mathrm{d}x &= \int_\Omega \varepsilon\left(\nabla\varphi \otimes \nabla\varphi\right) : \nabla\mathbf{u} \, \mathrm{d}x \end{aligned} \tag{7.72}$$

for almost all $t \in (0,T)$ and all $\xi \in L^2(H^1)$, $\phi \in H^1$, $\mathbf{u} \in \mathbf{V}$, where

$$\boldsymbol{J} = -m(\varphi)\nabla(-\varepsilon\Delta\varphi + \varepsilon^{-1}(\psi^2)'(\varphi) - \chi_\varphi\sigma)$$

holds in the sense that

$$\int_{\Omega_T} \boldsymbol{J}\cdot\boldsymbol{\eta} \, \mathrm{d}x \, \mathrm{d}t = -\int_{\Omega_T} \varepsilon\Delta\varphi \operatorname{div}(m(\varphi)\boldsymbol{\eta}) + \varepsilon^{-1}(m(\psi_0^2)'')(\varphi)\nabla\varphi\cdot\boldsymbol{\eta} - \chi_\varphi m(\varphi)\nabla\sigma\cdot\boldsymbol{\eta} \, \mathrm{d}x \, \mathrm{d}t \tag{7.73}$$

for all $\boldsymbol{\eta} \in L^2(0,T;\mathbf{H}^1) \cap L^\infty(0,T;\mathbf{L}^\infty)$ with $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ a. e. on Σ_T. Furthermore, there exists a unique pressure $p \in L^{\frac{4}{3}}(L_0^2)$ satisfying

$$-\nabla p = -\mathrm{div}\,(2\eta\mathbf{D}\mathbf{v} - \varepsilon(\nabla\varphi \otimes \nabla\varphi)) + \nu\mathbf{v} \quad \text{in } L^{\frac{4}{3}}(\mathbf{V}^*).$$

Proof. Let $\theta \in (0,1)$ and denote by $(\varphi_\theta, \mathbf{J}_\theta, \sigma_\theta, \mathbf{v}_\theta)$ the solution according to Theorem 7.11 corresponding to $\psi_\theta = \psi_\theta^1 + \psi_\theta^2$ with ψ_θ^1, ψ_θ^2, as defined above. From weak(-star) lower semi-continuity of norms, we have, thanks to Lemmas 7.15 and 7.16, the following bounds that are independent of $\theta \in (0,1)$:

$$\begin{aligned}&\|\varphi_\theta\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^2)} + \|\mathbf{J}_\theta\|_{L^2(\mathbf{L}^2)} + \|\sigma_\theta\|_{H^1((H^1)^*)\cap L^\infty(L^6)\cap L^2(H^1)} \\ &\quad + \|\mathbf{v}_\theta\|_{L^2(\mathbf{H}^1)} + \|\mathrm{div}(\varphi_\theta \mathbf{v}_\theta)\|_{L^2(\mathbf{L}^{\frac{3}{2}})} + \|\mathrm{div}(\sigma_\theta \mathbf{v}_\theta)\|_{L^2((H^1)^*)} \le C. \end{aligned} \tag{7.74}$$

Furthermore, as $\theta \to 0$ it holds (at least for a non-relabelled subsequence) that

$$\begin{aligned}
\varphi_\theta &\to \varphi && \text{weakly-star in } && H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^2), && (7.75a)\\
\sigma_\theta &\to \sigma && \text{weakly-star in } && H^1((H^1)^*) \cap L^\infty(L^6) \cap L^2(H^1), && (7.75b)\\
\mathbf{v}_\theta &\to \mathbf{v} && \text{weakly in } && L^2(\mathbf{H}^1), && (7.75c)\\
\mathrm{div}(\varphi_\theta \mathbf{v}_\theta) &\to \mathrm{div}(\varphi \mathbf{v}) && \text{weakly in } && L^2(L^{\frac{3}{2}}), && (7.75d)\\
\mathrm{div}(\sigma_\theta \mathbf{v}_\theta) &\to \mathrm{div}(\sigma \mathbf{v}) && \text{weakly in } && L^2((H^1)^*), && (7.75e)\\
\mathbf{J}_\theta &\to \mathbf{J} && \text{weakly in } && L^2(\mathbf{L}^2), && (7.75f)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_\theta &\to \varphi \quad \text{strongly in } C^0([0,T];L^r) \cap L^2(W^{1,r}) && \text{and a.e. in } \Omega_T, && (7.75g)\\
\sigma_\theta &\to \sigma \quad \text{strongly in } C^0([0,T];(H^1)^*) \cap L^p(L^r) && \text{and a.e. in } \Omega_T && (7.75h)
\end{aligned}$$

for any $r \in [1,6)$ and $p \in [1,\infty)$. Due to the bound $\varphi_\theta \ge -1$ a.e. in Ω_T for all $\theta \in (0,1)$, we obtain that $\varphi \ge -1$ a.e. in Ω_T. Then, with exactly the same arguments as above, we can pass to the limit $\theta \to 0$ to obtain (7.72). It remains to pass to the limit in

$$\begin{aligned}\int_{\Omega_T} \mathbf{J}_\theta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = &-\int_{\Omega_T} \varepsilon \Delta\varphi_\theta \,\mathrm{div}(m(\varphi_\theta)\boldsymbol{\eta}) + \varepsilon^{-1}\tfrac{\theta}{2} k(\varphi_\theta)^{q_0-1}\bar{m}(\varphi_\theta)\nabla\varphi_\theta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_{\Omega_T} \varepsilon^{-1}\big(m(\psi_\theta^2)''\big)(\varphi_\theta)\nabla\varphi_\theta \cdot \boldsymbol{\eta} - \chi_\varphi m(\varphi_\theta)\nabla\sigma_\theta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{7.76}$$

where $k(\varphi_\theta) \coloneqq \max\Big(0, \min\big(1, \tfrac{1}{2}(1+\varphi_\theta)\big)\Big)$. Invoking the boundedness of k and $\bar{m}$ together with $q_0 \ge 1$ and (7.74), we obtain

$$\int_{\Omega_T} \varepsilon^{-1}\tfrac{\theta}{2} k(\varphi_\theta)^{q_0-1}\bar{m}(\varphi_\theta)\nabla\varphi_\theta \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad \text{as } \theta \to 0.$$

The remaining terms in (7.76) can be treated with exactly the same arguments as above, and we deduce in the limit $\theta \to 0$ that

$$\int_{\Omega_T} \mathbf{J} \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_T} \varepsilon \Delta\varphi \,\mathrm{div}(m(\varphi)\boldsymbol{\eta}) + \varepsilon^{-1}\big(m(\psi_0^2)''\big)(\varphi)\nabla\varphi \cdot \boldsymbol{\eta} - \chi_\varphi m(\varphi)\nabla\sigma \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t.$$

Finally, attainment of the initial conditions can be shown as before which completes the proof. □

7.5 Further applications

Now, we consider the model (3.28) supplemented with (7.3) and with the choices $\Gamma_{\mathbf{v}} = 0$, $\Gamma_\varphi = \Gamma_\sigma = P(\varphi)(\chi_\sigma \sigma + \chi_\varphi(1-\varphi) - \mu)$ where

$$P(\varphi) = \begin{cases} (1-\varphi^2) & \text{for } |\varphi| \le 1, \\ 0 & \text{else.} \end{cases}$$

Moreover, we take $m(\varphi) = P(\varphi)$ and

$$\psi_\theta = \psi_{\log}(\varphi) = \frac{\theta}{2}\big(\ln(1+\varphi)(1+\varphi) + \ln(1-\varphi)(1-\varphi)\big) + \frac{\theta_c}{2}(1-\varphi^2) \quad \forall\,\varphi \in (-1,1).$$

Finally, we set

$$\psi^1 := \psi^1_\theta = \frac{\theta}{2}\big(\ln(1+\varphi)(1+\varphi) + \ln(1-\varphi)(1-\varphi)\big), \quad \psi^2 = \frac{\theta_c}{2}(1-\varphi^2).$$

Then, we have the following result:

Proposition 7.18 (degenerate case) *Let Assumptions 7.2,* (ii),(v)*, be fulfilled, let m and P be defined as above and let $\theta \in (0,1)$. In addition, we assume that $|\varphi_0| \le 1$ a. e. in Ω and*

$$\int_\Omega \psi_\theta(\varphi_0) + \Phi(\varphi_0) \le C$$

for a positive constant C. Then, there exists a quadruplet $(\varphi, \boldsymbol{J}, \sigma, \mathbf{v})$ satisfying

a) $\varphi \in H^1((H^1)^*) \cap C([0,T];L^2) \cap L^\infty(H^1) \cap L^2(H^2)$,

b) $\varphi(0) = \varphi_0$ *in* L^2 *and* $\nabla\varphi \cdot \mathbf{n} = 0$ *a. e. on* Σ_T,

c) $|\varphi| \le 1$ *a. e. in* Ω_T,

d) $\sigma \in H^1((H^1)^*) \cap L^\infty(L^6) \cap L^2(H^1)$,

e) $\sigma(0) = \sigma_0$ *in* L^2,

f) $\mathbf{J} \in L^2(\mathbf{L}^2)$,

g) $\mathbf{v} \in L^2(\mathbf{H}^1)$,

and solving

$$\begin{aligned}
0 &= \int_0^T \langle \partial_t\varphi(t), \xi(t)\rangle_{H^1}\,\mathrm{d}t - \int_{\Omega_T} (\mathbf{J} - \nabla\varphi\cdot\mathbf{v})\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\Omega_T} P(\varphi)\big((\chi_\sigma+\chi_\varphi)\sigma + \chi_\varphi(1-\varphi) + \varepsilon\Delta\varphi\big)\xi - \varepsilon^{-1}(P\psi_\theta')(\varphi)\xi\,\mathrm{d}x\,\mathrm{d}t, \\
0 &= \langle \partial_t\sigma, \phi\rangle_{H^1} + \int_\Omega (\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi - \sigma\mathbf{v})\cdot\nabla\phi\,\mathrm{d}x \\
&\quad + \int_\Omega P(\varphi)\big((\chi_\sigma+\chi_\varphi)\sigma + \chi_\varphi(1-\varphi) + \varepsilon\Delta\varphi\big)\phi - \varepsilon^{-1}(P\psi_\theta')(\varphi)\phi\,\mathrm{d}x, \\
0 &= \int_\Omega 2\eta\mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{u} + \nu\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \varepsilon\,(\nabla\varphi\otimes\nabla\varphi) : \nabla\mathbf{u}\,\mathrm{d}x
\end{aligned} \tag{7.77}$$

for almost all $t \in (0,T)$ and all $\xi \in L^2(H^1)$, $\phi \in H^1$, $\mathbf{u} \in \mathbf{V}$, where

$$\boldsymbol{J} = -m(\varphi)\nabla(-\varepsilon\Delta\varphi + \varepsilon^{-1}\psi_\theta'(\varphi) - \chi_\varphi\sigma)$$

holds in the sense that

$$\int_{\Omega_T} \boldsymbol{J}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_T} \varepsilon\Delta\varphi\,\mathrm{div}(m(\varphi)\boldsymbol{\eta}) + \varepsilon^{-1}(m\psi_\theta'')(\varphi)\nabla\varphi\cdot\boldsymbol{\eta} - \chi_\varphi m(\varphi)\nabla\sigma\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \tag{7.78}$$

for all $\boldsymbol{\eta} \in L^2(0,T;\mathbf{H}^1) \cap L^\infty(0,T;\mathbf{L}^\infty)$ with $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ a. e. on Σ_T. Furthermore, there exists a unique pressure $p \in L^2(L^2_0)$ satisfying

$$-\nabla p = -\mathrm{div}\,(2\eta\mathbf{D}\mathbf{v} - \varepsilon(\nabla\varphi\otimes\nabla\varphi)) + \nu\mathbf{v} \quad \textit{in } L^2(\mathbf{V}^*).$$

Remark 7.19 We split the source term in the weak formulation (7.77) since $\psi_\theta'(r) \to \pm\infty$ as $r \to \pm 1$, while the product $P\psi_\theta'$ given by

$$\left(P\psi_\theta'\right)(r) = \tfrac{\theta}{2}\left(\ln(1+\varphi)(1+\varphi)(1-\varphi) - \ln(1-\varphi)(1-\varphi)(1+\varphi)\right) - \theta_c\varphi(1-\varphi^2)$$

is bounded on $[-1,1]$ and satisfies $\left(P\psi_\theta'\right)(r) \to 0$ as $r \to \pm 1$.

Proof. The proof follows with slight modifications of the arguments in the proof of Theorem 7.11. We will formally sketch the arguments that can be deduced rigorously with the same arguments as used above.

Step 1: First, we replace g, f and h by Γ_φ, Γ_σ in (7.10a), (7.10d), and we regularize m and ψ as above. Furthermore, we denote by $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta)$ the solutions to the corresponding system. Then, we take $\xi = \mu_\delta + 4\frac{\chi_\varphi^2}{\chi_\sigma}\varphi_\delta$ in (7.10a), $\mathbf{u} = \mathbf{v}_\delta$ in (7.10b), we multiply (7.10d) with $\chi_\sigma\sigma_\delta + \chi_\varphi(1-\varphi_\delta)$ and we use (7.10c) to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{2\chi_\varphi^2}{\chi_\sigma}|\varphi_\delta|^2 + \frac{\varepsilon}{2}|\nabla\varphi_\delta|^2 + \varepsilon^{-1}\psi_\delta(\varphi_\delta) + N(\varphi_\delta,\sigma_\delta) + \frac{\delta}{2}|\mathbf{v}_\delta|^2 \,\mathrm{d}x \\
&\quad + \int_\Omega m_\delta(\varphi_\delta)|\nabla\mu_\delta|^2 + |\nabla N_{\sigma,\delta}|^2 + 2\eta|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_\delta|^2 + P(\varphi_\delta)(\chi_\sigma\sigma_\delta + \chi_\varphi(1-\varphi_\delta) - \mu_\delta)^2 \,\mathrm{d}x \\
&\quad = 4\frac{\chi_\varphi^2}{\chi_\sigma}\int_\Omega -m_\delta(\varphi_\delta)\nabla\mu_\delta\cdot\varphi_\delta + P(\varphi_\delta)(\chi_\sigma\sigma_\delta + \chi_\varphi(1-\varphi_\delta) - \mu_\delta)\varphi_\delta \,\mathrm{d}x,
\end{aligned} \tag{7.79}$$

where

$$N(\varphi_\delta,\sigma_\delta) = \frac{\chi_\sigma}{2}|\sigma_\delta|^2 + \chi_\varphi\sigma_\delta(1-\varphi_\delta), \quad N_{\sigma,\delta} := \chi_\sigma\sigma_\delta + \chi_\varphi(1-\varphi_\delta).$$

Using Hölder's and Young's inequalities along with the assumptions on $m(\cdot)$ and $P(\cdot)$, we obtain

$$\begin{aligned}
\frac{\chi_\sigma^2}{2}\|\nabla\sigma_\delta\|_{\mathbf{L}^2}^2 - C\|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2 &\le \int_\Omega |\nabla N_{\sigma,\delta}|^2 \,\mathrm{d}x, \\
\left|4\frac{\chi_\varphi^2}{\chi_\sigma}\int_\Omega -m(\varphi_\delta)\nabla\mu_\delta\cdot\nabla\varphi_\delta \,\mathrm{d}x\right| &\le \frac{1}{2}\int_\Omega m(\varphi_\delta)|\nabla\mu_\delta|^2 \,\mathrm{d}x + C\|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2, \\
\left|4\frac{\chi_\varphi^2}{\chi_\sigma}\int_\Omega P(\varphi_\delta)(N_{\sigma,\delta} - \mu_\delta)\varphi_\delta \,\mathrm{d}x\right| &\le \frac{1}{2}\int_\Omega P(\varphi_\delta)(N_{\sigma,\delta} - \mu_\delta)^2 \,\mathrm{d}x + C\|\varphi_\delta\|_{L^2}^2, \\
\left|\int_\Omega \chi_\varphi\sigma_\delta(t)(1-\varphi_\delta(t)) \,\mathrm{d}x\right| &\le \frac{3\chi_\sigma}{8}\|\sigma_\delta(t)\|_{L^2}^2 + C\left(1 + \tfrac{\chi_\varphi^2}{\chi_\sigma}\|\varphi_\delta(t)\|_{L^2}^2\right),
\end{aligned}$$

where the last inequality holds for all $t \in (0,T)$. Integrating (7.79) in time from 0 to $t \in (0,T]$, and using the last four inequalities along with the non-negativity of ψ_δ, a Gronwall argument yields

$$\begin{aligned}
&\|\varphi_\delta\|_{L^\infty(H^1)} + \|\sigma_\delta\|_{L^\infty(L^2)\cap L^2(H^1)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} + \|\sqrt{\delta}\mathbf{v}_\delta\|_{L^\infty(L^2)} + \|\psi_\delta(\varphi_\delta)\|_{L^\infty(L^1)} \\
&\quad + \|\sqrt{m_\delta(\varphi_\delta)}\nabla\mu_\delta\|_{L^2(\mathbf{L}^2)} + \|\sqrt{P(\varphi_\delta)}(\chi_\sigma\sigma_\delta + \chi_\varphi(1-\varphi_\delta) - \mu_\delta)\|_{L^2(L^2)} \le C.
\end{aligned} \tag{7.80}$$

Step 2: Now, we will derive entropy estimates. Taking $\xi = \Phi_\delta'(\varphi_\delta)$ in (7.10a) and using (7.10c), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Phi_\delta(\varphi_\delta) \,\mathrm{d}x + \int_\Omega \varepsilon|\Delta\varphi_\delta|^2 + \varepsilon^{-1}(\psi_\delta^1)''(\varphi_\delta)|\nabla\varphi_\delta|^2 \,\mathrm{d}x + \int_\Omega P(\varphi_\delta)\varepsilon^{-1}(\psi_\delta^1)'(\varphi_\delta)\Phi_\delta'(\varphi_\delta) \,\mathrm{d}x \\
&= \int_\Omega P(\varphi_\delta)\big((\chi_\sigma + \chi_\varphi)\sigma_\delta + \chi_\varphi(1-\varphi_\delta) + \varepsilon\Delta\varphi_\delta - \varepsilon^{-1}(\psi^2)'(\varphi_\delta)\big)\Phi_\delta'(\varphi_\delta) \,\mathrm{d}x \\
&\quad + \int_\Omega \chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta - (\psi^2)''(\varphi_\delta)|\nabla\varphi_\delta|^2 \,\mathrm{d}x.
\end{aligned} \tag{7.81}$$

Next, we observe that

$$P(r)\Phi_\delta'(r) = \begin{cases} (1-r^2)\left(\frac{\ln(2-\delta)-\ln(\delta)}{2} + \frac{(r-(1-\delta))}{\delta(2-\delta)}\right) & \text{for } r \in (1-\delta,1), \\ (1-r^2)\left(\frac{1}{2}(\ln(1+r)-\ln(1-r))\right) & \text{for } |r| \le 1-\delta, \\ (1-r^2)\left(\frac{\ln(\delta)-\ln(2-\delta)}{2} + \frac{(r-(-1+\delta))}{\delta(2-\delta)}\right) & \text{for } r \in (-1, 1-\delta), \\ 0 & \text{else.} \end{cases} \tag{7.82}$$

Since $\delta\ln(\delta) \to 0$ as $\delta \to 0$, this implies that $|P(\varphi_\delta)\Phi_\delta'(\varphi_\delta)| \le C$ for a constant C independent of $\delta \in (0,1)$. Then, using (7.80) and the specific form of ψ^2, we can estimate the r. h. s. of (7.81) by

$$\begin{aligned} &\left|\int_\Omega P(\varphi_\delta)\big((\chi_\sigma+\chi_\varphi)\sigma_\delta + \chi_\varphi(1-\varphi_\delta) + \varepsilon\Delta\varphi_\delta - \varepsilon^{-1}(\psi^2)'(\varphi_\delta)\big)\Phi_\delta'(\varphi_\delta)\,\mathrm{d}x\right| \\ &\quad + \left|\int_\Omega \chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta - (\psi^2)''(\varphi_\delta)|\nabla\varphi_\delta|^2\,\mathrm{d}x\right| \\ &\le C\left(1 + \|\sigma_\delta\|_{H^1}^2 + \|\varphi_\delta\|_{H^1}^2\right) + \frac{\varepsilon}{2}\|\Delta\varphi_\delta\|_{L^2}^2. \end{aligned} \tag{7.83}$$

Now, using the convexity of $\psi_\delta^1(\cdot)$ and $\Phi_\delta(\cdot)$ along with the non-negativity of $P(\cdot)$ and $(\psi_\delta^1)'(0) = \Phi_\delta'(0) = 0$, we obtain

$$P(r)\varepsilon^{-1}(\psi_\delta^1)'(r)\Phi_\delta'(r) \begin{cases} \ge 0 & \text{for } |r| < 1, \\ = 0 & \text{for } |r| \ge 1. \end{cases}$$

Hence, we can neglect the last term on the l. h. s. of (7.81). Integrating (7.81) in time from 0 to $t \in (0,T)$, using (7.81) and $\Phi_\delta(r) \le \Phi(r)$ for δ small enough, a Gronwall argument along with (7.80) and elliptic regularity theory yields

$$\begin{aligned} &\|\varphi_\delta\|_{L^\infty(H^1)\cap L^2(H^2)} + \|\sigma_\delta\|_{L^\infty(L^2)\cap L^2(H^1)} + \|\mathbf{v}_\delta\|_{L^2(\mathbf{H}^1)} + \|\sqrt{\delta}\mathbf{v}_\delta\|_{L^\infty(L^2)} \\ &\quad + \|\psi_\delta(\varphi_\delta)\|_{L^\infty(L^1)} + \|\Phi_\delta(\varphi_\delta)\|_{L^\infty(L^1)} + \|\big((\psi_\delta^1)''\big)^{1/2}\nabla\varphi_\delta\|_{L^2(\mathbf{L}^2)} \\ &\quad + \|(m_\delta(\varphi_\delta))^{1/2}\nabla\mu_\delta\|_{L^2(\mathbf{L}^2)} + \|\sqrt{P(\varphi_\delta)}(\chi_\sigma\sigma_\delta + \chi_\varphi(1-\varphi_\delta) - \mu_\delta)\|_{L^2(L^2)} \le C. \end{aligned} \tag{7.84}$$

In order to obtain the $L^\infty(L^6)$-bound for σ_δ, we need a modified argument. Multiplying (7.10c) with σ_δ^5, integrating over Ω and by parts and using (7.10d) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{6}\|\sigma_\delta\|_{L^6}^6 + 5\chi_\sigma\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4\,\mathrm{d}x \\ &= \int_\Omega 5\chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta|\sigma_\delta|^4 - P(\varphi_\delta)((\chi_\sigma+\chi_\varphi)\sigma_\delta + \chi_\varphi(1-\varphi_\delta))\sigma_\delta^5\,\mathrm{d}x \\ &\quad + \int_\Omega P(\varphi_\delta)(\varepsilon^{-1}\psi_\delta'(\varphi_\delta) - \varepsilon\Delta\varphi_\delta)\sigma_\delta^5\,\mathrm{d}x. \end{aligned} \tag{7.85}$$

For the first term on the r. h. s. of (7.85), we can argue similar as above to obtain

$$\begin{aligned} &\left|\int_\Omega 5\chi_\varphi\nabla\varphi_\delta\cdot\nabla\sigma_\delta|\sigma_\delta|^4 - P(\varphi_\delta)((\chi_\sigma+\chi_\varphi)\sigma_\delta + \chi_\varphi(1-\varphi_\delta))\sigma_\delta^5\,\mathrm{d}x\right| \\ &\quad \le C\left(1+\|\varphi_\delta\|_{H^2}^2\right)\left(1+\|\sigma_\delta\|_{L^6}^6\right) + \chi_\sigma\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4\,\mathrm{d}x. \end{aligned}$$

Moreover, using the boundedness of the product $P\psi_\delta'$ yields

$$\left|\int_\Omega P(\varphi_\delta)\varepsilon^{-1}\psi_\delta'(\varphi_\delta)\sigma_\delta^5\,\mathrm{d}x\right| \le C\left(1+\|\sigma_\delta\|_{L^6}^6\right).$$

Now, observing that $|\nabla(\sigma_\delta^3)| = 3|\nabla\sigma_\delta||\sigma_\delta|^2$ and using the boundedness of $P(\cdot)$ along with the Sobolev embedding $H^1 \subset L^6$, we obtain

$$\begin{aligned}\left|\int_\Omega P(\varphi_\delta)\varepsilon\Delta\varphi_\delta\,\sigma_\delta^5\,\mathrm{d}x\right| &\le C\|\Delta\varphi_\delta\|_{L^2}\|\sigma_\delta\|_{L^6}^2\|\sigma_\delta^3\|_{L^6}\\ &\le C\|\Delta\varphi_\delta\|_{L^2}\|\sigma_\delta\|_{L^6}^2\left(\|\sigma_\delta^3\|_{L^2} + \|\nabla(\sigma_\delta^3)\|_{\mathbf{L}^2}\right)\\ &\le C\left(1+\|\varphi_\delta\|_{H^2}^2\right)\left(1+\|\sigma_\delta\|_{L^6}^6\right) + \chi_\sigma\int_\Omega |\nabla\sigma_\delta|^2|\sigma_\delta|^4\,\mathrm{d}x.\end{aligned}$$

Employing the last three inequalities in (7.85) and neglecting the term $3\chi_\sigma\int_\Omega|\nabla\sigma_\delta|^2|\sigma_\delta|^4\,\mathrm{d}x$ which is non-negative, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{6}\|\sigma_\delta\|_{L^6}^6 \le C\left(1+\|\varphi_\delta\|_{H^2}^2\right)\left(1+\|\sigma_\delta\|_{L^6}^6\right).$$

Then, applying (7.84), a Gronwall argument yields

$$\|\sigma_\delta\|_{L^\infty(L^6)} \le C,$$

and with similar arguments as before (see also [81]) we deduce

$$\|\partial_t\varphi_\delta\|_{L^2((H^1)^*)} + \|\partial_t\sigma_\delta\|_{L^2((H^1)^*)} + \|\mathrm{div}(\varphi_\delta\mathbf{v}_\delta)\|_{L^2(L^2)} + \|\mathrm{div}(\sigma_\delta\mathbf{v}_\delta)\|_{L^{\frac{4}{3}}((H^1)^*)} \le C. \quad (7.86)$$

Finally, with exactly the same arguments as in [62], it follows that

$$\int_\Omega (|\varphi_\delta|-1)_+^2\,\mathrm{d}x \le C\delta \quad \text{for all } \delta>0 \text{ small enough} \quad (7.87)$$

and

$$\int_{\Omega_T} |\mathbf{J}_\delta|^2\,\mathrm{d}x\,\mathrm{d}t \le C \quad \text{where } \mathbf{J}_\delta = -m_\delta(\varphi_\delta)\nabla(-\varepsilon\Delta\varphi_\delta + \varepsilon^{-1}\psi_\delta'(\varphi_\delta) - \chi_\varphi\sigma_\delta). \quad (7.88)$$

Passing to the limit As before, we can extract non-relabelled subsequences to obtain for $\delta\to 0$ that

$$\begin{aligned}\varphi_\delta &\to \varphi && \text{weakly-star in } && H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^2),\\ \sigma_\delta &\to \sigma && \text{weakly-star in } && H^1((H^1)^*)\cap L^\infty(L^6)\cap L^2(H^1),\\ \mathbf{v}_\delta &\to \mathbf{v} && \text{weakly in } && L^2(\mathbf{H}^1),\\ \mathrm{div}(\varphi_\delta\mathbf{v}_\delta) &\to \mathrm{div}(\varphi\mathbf{v}) && \text{weakly in } && L^2(L^{\frac{3}{2}}),\\ \mathrm{div}(\sigma_\delta\mathbf{v}_\delta) &\to \mathrm{div}(\sigma\mathbf{v}) && \text{weakly in } && L^2((H^1)^*),\\ \mathbf{J}_\delta &\to \mathbf{J} && \text{weakly in } && L^2(\mathbf{L}^2),\end{aligned}$$

and

$$\begin{aligned}\varphi_\delta &\to \varphi \quad \text{strongly in } C^0([0,T];L^r)\cap L^2(W^{1,r}) && \text{and a.e. in } \Omega_T,\\ \sigma_\delta &\to \sigma \quad \text{strongly in } C^0([0,T];(H^1)^*)\cap L^2(L^r) && \text{and a.e. in } \Omega_T\end{aligned}$$

for any $r\in[1,6)$ and $p\in[1,\infty)$. Then, we can pass to the limit with exactly the same arguments as before, except from the source terms. Indeed, inserting the expression for μ_δ, multiplying with $\xi\phi$ where $\xi\in C_0^\infty(0,T)$ and $\phi\in H^1$, and integrating over Ω_T, the resulting terms are given by

$$\pm\int_{\Omega_T} P(\varphi_\delta)\big((\chi_\sigma+\chi_\varphi)\sigma_\delta + \chi_\varphi(1-\varphi_\delta) + \varepsilon\Delta\varphi_\delta - \varepsilon^{-1}(\psi^2)'(\varphi_\delta) - \varepsilon^{-1}(\psi_\delta^1)'(\varphi_\delta)\big)\xi\phi\,\mathrm{d}x\,\mathrm{d}t. \quad (7.90)$$

Since $\varphi_\delta \to \varphi$ weakly in $L^2(H^2)$ and a. e. in Ω_T and $\sigma_\delta \rightharpoonup \sigma$ weakly in $L^2(H^1)$, using the linearity of $(\psi^2)'(\cdot)$ yields

$$\begin{aligned}&\int_{\Omega_T} P(\varphi_\delta)\big((\chi_\sigma+\chi_\varphi)\sigma_\delta+\chi_\varphi(1-\varphi_\delta)+\varepsilon\Delta\varphi_\delta-\varepsilon^{-1}(\psi^2)'(\varphi_\delta)\big)\big)\xi\phi \,\mathrm{d}x\,\mathrm{d}t\\ &\to \int_{\Omega_T} P(\varphi)\big((\chi_\sigma+\chi_\varphi)\sigma+\chi_\varphi(1-\varphi)+\varepsilon\Delta\varphi-\varepsilon^{-1}(\psi^2)'(\varphi)\big)\big)\xi\phi\,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta\to 0. \qquad (7.91)\end{aligned}$$

For the term involving $(\psi_\delta^1)'$ we need a more subtle argument. First, using the expression for $(\psi_\delta^1)''$ and the fact that $(\psi_\delta^1)'(0)=0$, it follows that $P(r)(\psi_\delta^1)'(r)=\theta P(r)(\Phi_\delta^1)'(r)$ which is given by (7.82). Then, it is easy to check that $P(\psi_\delta^1)'$ is uniformly bounded and therefore, it suffices to show that

$$P(\varphi_\delta)(\psi_\delta^1)'(\varphi_\delta)\to P(\varphi)(\psi^1)'(\varphi) \quad \text{a. e. in } \Omega_T. \qquad (7.92)$$

For points $(x,t)\in\Omega_T$ where $|\varphi(x,t)|<1$, this follows from the definition of $(\psi_\delta^1)'$ since $(\psi_\delta^1)'(z)=(\psi^1)'(z)$ for $|z|<1-\delta$. Therefore, we consider points $(x,t)\in\Omega_T$ where $|\varphi(x,t)|=1$. Without loss of generality, we can assume $\varphi_\delta(x,t)\to 1=\varphi(x,t)$. Following the arguments in [62], we first consider δ with $\varphi_\delta(x,t)\geq 1-\delta$. Then, it follows that

$$\begin{aligned}P(\varphi_\delta(x,t))(\psi_\delta^1)'(\varphi_\delta(x,t)) &= (1-\varphi_\delta(x,t)^2)\theta\left(\tfrac{\ln(2-\delta)-\ln(\delta)}{2}+\tfrac{(\varphi_\delta(x,t)-(1-\delta))}{\delta(2-\delta)}\right)\\ &\leq \delta(2-\delta)\theta\left(\tfrac{\ln(2-\delta)-\ln(\delta)}{2}+\tfrac{1}{\delta(2-\delta)}\right)\\ &\to 0=\left(P(\psi^1)'\right)(1) \quad \text{as } \delta\to 0,\end{aligned}$$

where we used that $\delta\ln(\delta)\to 0$ as $\delta\to 0$. For $\varphi_\delta(x,t)\leq 1-\delta$ and $\varphi_\delta(x,t)\to 1$, we have

$$\begin{aligned}P(\varphi_\delta(x,t))(\psi_\delta^1)'(\varphi_\delta(x,t)) &= \tfrac{\theta}{2}(1-\varphi_\delta(x,t))(1+\varphi_\delta(x,t))\ln(1+\varphi_\delta(x,t))\\ &\quad -\tfrac{\theta}{2}(1+\varphi_\delta(x,t))(1-\varphi_\delta(x,t))\ln(1-\varphi_\delta(x,t))\\ &\to 0=\left(P(\psi^1)'\right)(1) \quad \text{as } \delta\to 0,\end{aligned}$$

where we used again that $x\ln(x)\to 0$ for $x\to 0$. Putting all the arguments together, we can pass to the limit in the source terms.

For the pressure, the argument is slightly different. Using that $|\varphi|\leq 1$ a. e. in Ω_T, we obtain that $\|\varphi\|_{L^\infty(\Omega_T)}\leq C$. Then, using Gagliardo–Nirenberg's inequality gives

$$\int_{\Omega_T}|\nabla\varphi\otimes\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t\leq C\int_0^T\|\nabla\varphi\|_{\mathbf{L}^4}^4\,\mathrm{d}t\leq C\int_0^T\|\varphi\|_{L^\infty}^2\|\varphi\|_{H^2}^2\,\mathrm{d}t\leq C\|\varphi\|_{L^2(H^2)}^2\leq C.$$

Therefore, we have $\|\nabla\varphi\otimes\nabla\varphi\|_{L^2(\mathbf{L}^2)}\leq C$ and, in particular,

$$-\mathrm{div}\,(2\eta\mathbf{D}\mathbf{v}-\varepsilon(\nabla\varphi\otimes\nabla\varphi))+\nu\mathbf{v} \quad \text{in } L^2(\mathbf{V}^*).$$

Consequently, the pressure satisfies $p\in L^2(L_0^2)$ and

$$-\nabla p=-\mathrm{div}\,(2\eta\mathbf{D}\mathbf{v}-\varepsilon(\nabla\varphi\otimes\nabla\varphi))+\nu\mathbf{v} \quad \text{in } L^2(\mathbf{V}^*),$$

which completes the proof. □

Since all the estimates deduced above are independent of $\theta\in(0,1)$, we have the following result:

Proposition 7.20 (deep quench limit) *Let the assumptions of Proposition 7.18 be fulfilled. Then, there exists a quadruplet* $(\varphi,\boldsymbol{J},\sigma,\mathbf{v})$ *satisfying* *a)* $\varphi \in H^1((H^1)^*) \cap C([0,T];L^2) \cap L^\infty(H^1) \cap L^2(H^2)$,

b) $\varphi(0) = \varphi_0$ *in* L^2 *and* $\nabla\varphi \cdot \mathbf{n} = 0$ *a. e. on* Σ_T,

c) $|\varphi| \leq 1$ *a. e. in* Ω_T,

d) $\sigma \in H^1((H^1)^*) \cap L^\infty(L^6) \cap L^2(H^1)$,

e) $\sigma(0) = \sigma_0$ *in* L^2,

f) $\boldsymbol{J} \in L^2(\mathbf{L}^2)$,

g) $\mathbf{v} \in L^2(\mathbf{H}^1)$,

and solving

$$\begin{aligned}
0 &= \int_0^T \langle \partial_t\varphi(t), \xi(t)\rangle_{H^1}\,\mathrm{d}t - \int_{\Omega_T} (\mathbf{J} - \nabla\varphi\cdot\mathbf{v})\cdot\nabla\xi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\Omega_T} P(\varphi)\big((\chi_\sigma + \chi_\varphi)\sigma + \chi_\varphi(1-\varphi) + \varepsilon\Delta\varphi + \varepsilon^{-1}\theta_c\varphi\big)\xi\,\mathrm{d}x\,\mathrm{d}t, \\
0 &= \langle \partial_t\sigma, \phi\rangle_{H^1} + \int_\Omega (\chi_\sigma\nabla\sigma - \chi_\varphi\nabla\varphi - \sigma\mathbf{v})\cdot\nabla\phi\,\mathrm{d}x \\
&\quad + \int_\Omega P(\varphi)\big((\chi_\sigma + \chi_\varphi)\sigma + \chi_\varphi(1-\varphi) + \varepsilon\Delta\varphi + \varepsilon^{-1}\theta_c\varphi\big)\phi\,\mathrm{d}x, \\
0 &= \int_\Omega 2\eta\mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{u} + \nu\mathbf{v}\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \varepsilon\,(\nabla\varphi\otimes\nabla\varphi) : \nabla\mathbf{u}\,\mathrm{d}x
\end{aligned} \tag{7.93}$$

for almost all $t \in (0,T)$ *and all* $\xi \in L^2(H^1)$, $\phi \in H^1$, $\mathbf{u} \in \mathbf{V}$, *where*

$$\boldsymbol{J} = -m(\varphi)\nabla(-\varepsilon\Delta\varphi - \varepsilon^{-1}\theta_c\varphi - \chi_\varphi\sigma)$$

holds in the sense that

$$\int_{\Omega_T} \boldsymbol{J}\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t = -\int_{\Omega_T} \varepsilon\Delta\varphi\,\mathrm{div}(m(\varphi)\boldsymbol{\eta}) - \varepsilon^{-1}\theta_c m(\varphi)\nabla\varphi\cdot\boldsymbol{\eta} - \chi_\varphi m(\varphi)\nabla\sigma\cdot\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \tag{7.94}$$

for all $\boldsymbol{\eta} \in L^2(0,T;\mathbf{H}^1) \cap L^\infty(0,T;\mathbf{L}^\infty)$ *with* $\boldsymbol{\eta}\cdot\mathbf{n} = 0$ *a. e. on* Σ_T. *Furthermore, there exists a unique pressure* $p \in L^2(L^2_0)$ *satisfying*

$$-\nabla p = -\mathrm{div}\,(2\eta\mathbf{D}\mathbf{v} - \varepsilon(\nabla\varphi\otimes\nabla\varphi)) + \nu\mathbf{v} \quad \textit{in } L^2(\mathbf{V}^*).$$

Proof. This follows from the estimates in Proposition 7.18 and with the same arguments as in the proof of Theorem 7.17. □

Remark 7.21 Since $\mathbf{v}$ satisfies a homogeneous Dirichlet boundary condition and due to Korn's inequality, all the results in this chapter hold true for the case $\nu = 0$ which corresponds to a Stokes equation for the velocity $\mathbf{v}$.

Furthermore, by scaling the viscosity appropriately, i. e., choosing $\eta = \delta$, when passing to the limit $\delta \to 0$ one recovers a Darcy law for the velocity. In the limit $\delta \to 0$, the convection terms can be treated similarly as in [81, Sec. 5] and for the velocity field we refer to the arguments in [21, Sec. 7]. However, we point out that solutions of the Cahn–Hilliard–Darcy model are less regular than for the Cahn–Hilliard–Stokes model.

8

A tumour growth model with singular potentials

In this chapter we will consider the model studied in Chapters 5 and 6, but now with singular instead of regular potentials. Classical examples are given by the logarithmic potential

$$\psi_{\log}(r) \coloneqq \tfrac{\theta}{2}\left(\ln(1+r)(1+r) + \ln(1-r)(1-r)\right) + \tfrac{\theta_c}{2}(1-r^2) \quad \forall\, r \in (-1,1)$$

for constants $0 < \theta < \theta_c$ and the double obstacle potential

$$\psi_{\mathrm{do}}(r) \coloneqq I_{[-1,1]}(r) + \tfrac{1}{2}(1-r^2) \quad \forall\, r \in [-1,1], \quad I_{[-1,1]}(r) = \begin{cases} 0 & \text{for } r \in [-1,1], \\ +\infty & \text{else.} \end{cases}$$

These kind of potentials are quite popular since they force the phase field to stay in between the physical bounds $\varphi \in [-1,1]$. Although they have been studied quite extensive for the classical Cahn–Hilliard equation, contributions for models with source terms are rather rare in the literature. Indeed, these causes several difficulties since bounds for the source terms are quite delicate to establish, and specific assumptions on the source terms have to be imposed. In particular, the property of mass conservation is lost which plays a crucial role in the analysis. We remark that the problem we study is not only important for tumour growth dynamics, but also for, e. g., the inpainting problem for image reconstruction (see, e. g., [88]) and the Cahn–Hilliard–Oono (see, e. g., [93]) equation which has applications in mathematical biology.

We study the following system of equations

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= \Gamma_{\mathbf{v}}(\varphi,\sigma) && \text{in } \Omega_T, && (8.1a)\\
-\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu\mathbf{v} &= (\mu + \chi\sigma)\nabla\varphi && \text{in } \Omega_T, && (8.1b)\\
\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) &= \Delta\mu + \Gamma_{\varphi}(\varphi,\sigma) && \text{in } \Omega_T, && (8.1c)\\
\mu &= \psi'(\varphi) - \Delta\varphi - \chi\sigma && \text{in } \Omega_T, && (8.1d)\\
0 &= \Delta\sigma - h(\varphi)\sigma && \text{in } \Omega_T, && (8.1e)
\end{aligned}$$

where the viscous stress tensor $\mathbf{T}$ and the symmetrised velocity gradient are given by

$$\mathbf{T}(\mathbf{v},p) := 2\eta(\varphi)\mathbf{D}\mathbf{v} + \lambda(\varphi)\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I}, \quad \mathbf{D}\mathbf{v} := \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}).$$

We furnish (8.1) with the initial and boundary conditions

$$\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = 0 \qquad \text{on } \Sigma_T, \tag{8.2a}$$

$$\nabla\sigma\cdot\mathbf{n} = K(1-\sigma) \qquad \text{on } \Sigma_T, \tag{8.2b}$$

$$\mathbf{T}(\mathbf{v},p)\mathbf{n} = \mathbf{0} \qquad \text{on } \Sigma_T, \tag{8.2c}$$

$$\varphi(0) = \varphi_0 \qquad \text{in } \Omega. \tag{8.2d}$$

To establish our results, we need the following

Assumptions 8.1 *Throughout this chapter we make the following assumptions:*

(A1) *for* $d\in\{2,3\}$, $\Omega\subset\mathbb{R}^d$ *is a bounded domain with* C^3*-boundary.*

(A2) *the positive constants* ν, K *and the non-negative constant* χ *are fixed.*

(A3) *the viscosities* $\eta,\lambda\in C^2(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ *satisfy*

$$\eta_0\le\eta(t)\le\eta_1, \quad 0\le\lambda(t)\le\lambda_0 \quad \forall\, t\in\mathbb{R}$$

for positive constants η_0, η_1 *and a non-negative constant* λ_0, *and the function* $h\in C^0(\mathbb{R})\cap L^\infty(\mathbb{R})$ *is non-negative.*

(A4) *the source terms* $\Gamma_{\mathbf{v}}$ *and* Γ_φ *are of the form*

$$\Gamma_{\mathbf{v}}(\varphi,\sigma) = b_{\mathbf{v}}(\varphi)\sigma + f_{\mathbf{v}}(\varphi), \quad \Gamma_\varphi(\varphi,\sigma) = b_\varphi(\varphi)\sigma + f_\varphi(\varphi), \tag{8.3}$$

where $b_{\mathbf{v}}, f_{\mathbf{v}}\in C^{0,1}(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ *and* $b_\varphi, f_\varphi\in C^0(\mathbb{R})\cap L^\infty(\mathbb{R})$.

It can be easily checked that the assumptions on $b_{\mathbf{v}}$ and $f_{\mathbf{v}}$ are enough in order to prove Theorem 5.5. Indeed, the more restrictive assumption in Assumptions 5.1 is only needed in order to show the existence of strong solutions (cf. Theorem 5.11). The Galerkin ansatz in the proof of Theorem 5.5 for velocity and pressure can be refined with a similar argument as in the proof of Theorem 4.4 and therefore the Lipschitz continuity of $b_{\mathbf{v}}$ and $f_{\mathbf{v}}$ is enough.

8.1 Main results

8.1.1 The time-dependent problem for Brinkman's law

We begin with a suitable notion of weak solutions for the model with the double obstacle potential ψ_{do} and the logarithmic potential $\psi_{\log}$.

Definition 8.2 A quintuple $(\varphi,\mu,\sigma,\mathbf{v},p)$ is a weak solution to the CHB system (8.1)-(8.2) with the double obstacle potential ψ_{do} if the following properties hold:

(a) the functions satisfy

$$\varphi\in H^1(0,T;(H^1)^*)\cap L^\infty(0,T;H^1)\cap L^2(0,T;H^2_N), \quad \mu\in L^2(0,T;H^1),$$
$$\sigma\in L^\infty(0,T;H^2), \quad \mathbf{v}\in L^2(0,T;\mathbf{H}^1), \quad p\in L^2(0,T;L^2)$$

with $\varphi(0)=\varphi_0$ a. e. in Ω.

(b) equation (8.1a) holds a. e. in Ω_T, while (5.5a), (5.5b) and (5.5d) are satisfied for a. e. $t\in(0,T)$ and for all $\mathbf{\Phi}\in\mathbf{H}^1$ and $\Phi\in H^1$.

(c1) for a. e. $t \in (0,T)$, $\varphi(t) \in \mathcal{K} := \{f \in H^1 : |f| \leq 1 \text{ a. e. in } \Omega\}$ and

$$\int_\Omega (\mu + \chi\sigma + \varphi)(\zeta - \varphi) - \nabla\varphi \cdot \nabla(\zeta - \varphi) \, \mathrm{d}x \leq 0 \quad \forall\, \zeta \in \mathcal{K}. \tag{8.4}$$

We say that $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is a weak solution to (8.1)-(8.2) with the logarithmic potential $\psi_{\log}$ if properties (a) and (b) hold along with

(c2) $|\varphi(x,t)| < 1$ for a. e. $(x,t) \in \Omega \times (0,T)$ and for a. e. $t \in (0,T)$,

$$\int_\Omega (\mu + \chi\sigma - \psi'_{\log}(\varphi))\zeta - \nabla\varphi \cdot \nabla\zeta \, \mathrm{d}x = 0 \quad \forall\, \zeta \in H^1. \tag{8.5}$$

Our first result concerns the existence of weak solutions to the CHB system (8.1)-(8.2) with singular potentials.

Theorem 8.3 *Suppose Assumptions 8.1,* (A1)-(A4) *hold along with*

(B1) *the source terms* $\Gamma_{\mathbf{v}}$ *and* Γ_φ *are of the form* (8.3) *with* $f_{\mathbf{v}} \in C^{0,1}([-1,1])$, $f_\varphi \in C^0([-1,1])$, $b_{\mathbf{v}} \in C^{0,1}([-1,1])$, $b_\varphi \in C^0([-1,1])$ *satisfying*

$$b_{\mathbf{v}}(\pm 1) = b_\varphi(\pm 1) = 0, \quad f_\varphi(1) - f_{\mathbf{v}}(1) < 0, \quad f_\varphi(-1) + f_{\mathbf{v}}(-1) > 0. \tag{8.6}$$

(B2) *the initial datum* φ_0 *belongs to* $\mathcal{K}$.

Then, there exists a weak solution $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *to* (8.1)-(8.2) *with the double obstacle potential* ψ_{do} *in the sense of Definition 8.2 and, in addition,* $\sigma \in [0,1]$ *almost everywhere in* Ω_T.

In addition, suppose the following assumptions are satisfied:

(C1) *there exists a constant* c *such that for any* $0 < \delta \ll 1$,

$$|b_\varphi(s)| \leq c\delta, \quad |b_{\mathbf{v}}(s)| \leq c\delta \quad \text{for all } s \in [-1, -1+\delta] \cup [1-\delta, 1].$$

(C2) *it holds* $b_\varphi(s)\log(\frac{1+s}{1-s}) \in C^0([-1,1])$ *and* $b_{\mathbf{v}}(s)\log(\frac{1+s}{1-s}) \in C^0([-1,1])$.

(C3) *the initial condition* $\varphi_0 \in H^1(\Omega)$ *satisfies* $|\varphi_0(x)| < 1$ *for a. e.* $x \in \Omega$.

Then, there exists a weak solution $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *to* (8.1)-(8.2) *with the logarithmic potential* $\psi_{\log}$ *in the sense of Definition 8.2. Furthermore, for a. e.* $t \in (0,T)$*, it holds that*

$$\begin{aligned} &\int_\Omega \psi_{\log}(\varphi(t)) + \tfrac{1}{2}|\nabla\varphi(t)|^2 \, \mathrm{d}x + \int_0^t \int_\Omega |\nabla\mu|^2 + 2\eta(\varphi)|\mathbf{D}\mathbf{v}|^2 + \nu|\mathbf{v}|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \leq \int_0^t \int_\Omega -\chi\nabla\mu \cdot \nabla\sigma + (\Gamma_\varphi - \varphi\Gamma_{\mathbf{v}})(\psi'_{\log}(\varphi) - \Delta\varphi) \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \int_0^t \int_\Omega 2\eta(\varphi)\mathbf{D}\mathbf{v} \colon \mathbf{D}\mathbf{u} + \nu\mathbf{v} \cdot \mathbf{u} \, \mathrm{d}x - (\psi'_{\log}(\varphi) - \Delta\varphi)\nabla\varphi \cdot \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \int_\Omega \psi_{\log}(\varphi_0) + \tfrac{1}{2}|\nabla\varphi_0|^2 \, \mathrm{d}x, \end{aligned} \tag{8.7}$$

where $\mathbf{u}$ *is the unique solution to the divergence problem in Lemma 2.39 with data* $f = \Gamma_{\mathbf{v}}(\varphi, \sigma)$ *and* $\mathbf{a} = \frac{1}{|\partial\Omega|}\big(\int_\Omega \Gamma_{\mathbf{v}}(\varphi,\sigma) \, \mathrm{d}x\big)\mathbf{n}$*, and* $\sigma \in [0,1]$ *almost everywhere in* Ω_T.

Example 8.4 We now give a biologically relevant example for the source terms $\Gamma_{\mathbf{v}}$ and Γ_φ that satisfy (8.6). Following the arguments in Chapter 3, in a domain Ω occupied by both tumour cells and healthy cells, we denote by ρ_1 the actual mass density of the healthy cells per unit volume in Ω and by $\bar{\rho}_1$ the (constant) mass density of the healthy cells occupying the whole of Ω. Then, it follows that $\rho_1 \in [0, \bar{\rho}_1]$ and the volume fraction of the healthy cells can be defined as the ratio $u_1 = \frac{\rho_1}{\bar{\rho}_1} \in [0,1]$. Let ρ_2, $\bar{\rho}_2$ and u_2 be the actual mass density of the tumour cells per unit volume in Ω, the (constant) mass density of the tumour cells occupying the whole of Ω, and the volume fraction of the tumour cells, respectively. Assuming there is no external volume compartment aside from the tumour and healthy cells, we have $u_1 + u_2 = 1$. Then, for some function $\Gamma(\varphi, \sigma)$ one obtains

$$\Gamma_{\mathbf{v}} = \alpha\Gamma, \quad \Gamma_\varphi = \rho_S \Gamma, \quad \alpha := \frac{1}{\bar{\rho}_2} - \frac{1}{\bar{\rho}_1}, \ \rho_S := \frac{1}{\bar{\rho}_1} + \frac{1}{\bar{\rho}_2}.$$

We choose

$$\Gamma(\varphi, \sigma) := \mathcal{P}(1-\varphi^2)\sigma - \mathcal{A}\varphi \quad \text{ for } \varphi \in [-1,1]$$

where $\mathcal{P}, \mathcal{A} > 0$ are the constant proliferation and apoptosis rates, so that proliferation occurs only at the interface region $\{-1 < \varphi < 1\}$. Furthermore, we have

$$b_{\mathbf{v}}(\varphi) = \alpha\mathcal{P}(1-\varphi^2), \quad b_\varphi(\varphi) = \rho_S \mathcal{P}(1-\varphi^2), \quad f_{\mathbf{v}}(\varphi) = -\alpha\mathcal{A}\varphi, \quad f_\varphi(\varphi) = -\rho_S\mathcal{A}\varphi,$$

where $b_{\mathbf{v}}(\pm 1) = b_\varphi(\pm 1) = 0$ and

$$f_\varphi(1) - f_{\mathbf{v}}(1) = -\mathcal{A}\frac{2}{\bar{\rho}_1} < 0, \quad f_\varphi(-1) + f_{\mathbf{v}}(-1) = \mathcal{A}\frac{2}{\bar{\rho}_2} > 0.$$

It is also clear that $b_\varphi(s) = \rho_S\mathcal{P}(1-s^2)$ satisfies (C1) and (C2).

8.1.2 The stationary problem without flow

We will also consider the stationary problem without flow, i. e., equations (8.1d)-(8.1e) posed in Ω and

$$0 = \Delta\mu + \Gamma_\varphi(\varphi, \sigma) \qquad \text{in } \Omega, \tag{8.8}$$

together with the boundary conditions (8.2a)-(8.2b) posed on $\partial\Omega$. For the stationary problem with flow, we refer to [59].

Definition 8.5 A triplet (φ, μ, σ) is a weak solution to the stationary system with the double obstacle potential ψ_{do} if the following properties hold:

(d) the functions satisfy

$$\varphi \in H^2_N, \quad \mu \in H^2_N, \quad \sigma \in H^2.$$

(e) equations (8.1e), (8.8) hold a. e. in Ω while (8.2a)-(8.2b) hold a. e. on $\partial\Omega$.

(f1) (8.4) holds along with $\varphi \in \mathcal{K} = \{f \in H^1 : |f| \le 1 \text{ a. e. in } \Omega\}$.

We say that (φ, μ, σ) is a weak solution to the stationary system with the logarithmic potential $\psi_{\log}$ if properties (d) and (e) hold along with

(f2) (8.5) holds along with $|\varphi(x)| < 1$ for a. e. $x \in \Omega$.

Proposition 8.6 *Under Assumptions 8.1,* (A1)-(A4) *and* (B1) *with* $\mathbf{f_v}(\cdot) = b_{\mathbf{v}}(\cdot) = 0$*, there exists a weak solution* (φ, μ, σ) *to the stationary model with double obstacle potential* ψ_{do} *in the sense of Definition 8.5. If, in addition,* (C1) *and* (C2) *hold, then there exists a weak solution* $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *to the stationary model with logarithmic potential* $\psi_{\log}$ *in the sense of Definition 8.5. Moreover, it holds that* $\sigma \in [0,1]$ *almost everywhere in* Ω*.*

8.1.3 The time-dependent problem for Darcy's law

By setting the viscosities $\eta(\cdot)$ and $\lambda(\cdot)$ to zero, the CHB model (8.1)-(8.2) reduces to a Cahn–Hilliard–Darcy (CHD) model consisting of (8.1a), (8.1c)-(8.1e) and the Darcy law

$$\mathbf{v} = -\frac{1}{\nu}\Big(\nabla p - (\mu + \chi\sigma)\nabla\varphi\Big) \qquad \text{in } \Omega_T, \tag{8.9}$$

furnished with the initial-boundary conditions (8.2a)-(8.2b), (8.2d) together with the Dirichlet boundary condition

$$p = 0 \qquad \text{on } \Sigma_T. \tag{8.10}$$

We begin with a notion of weak solutions for the CHD model with singular potentials.

Definition 8.7 A quintuple $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is a weak solution to the CHD system (8.1a), (8.1c)-(8.1e), (8.9), (8.2a)-(8.2b), (8.2d), (8.10) with the double obstacle potential ψ_{do} if property (c1) from Definition 8.2 holds along with:

(g) the functions satisfy

$$\begin{aligned}
&\varphi \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2_N) \cap W^{1,\frac{8}{5}}(0,T;(H^1)^*), \quad \mu \in L^2(0,T;H^1),\\
&\sigma \in L^\infty(0,T;H^2), \quad \mathbf{v} \in L^2(0,T;\mathbf{L}^2), \quad p \in L^2(0,T;L^2) \cap L^{\frac{8}{5}}(0,T;H^1_0)
\end{aligned}$$

with $\varphi(0) = \varphi_0$ a. e. in Ω.

(h) for a. e. $t \in (0,T)$ and for all $\mathbf{\Phi} \in \mathbf{H}^1$, $\chi \in H^1_0$ and $\Phi \in H^1$, (5.5b) and (5.5d) are satisfied along with

$$\begin{aligned}
0 &= \int_\Omega (\nu\mathbf{v} - (\mu + \chi\sigma)\nabla\varphi) \cdot \mathbf{\Phi} - p\,\mathrm{div}(\mathbf{\Phi})\,\mathrm{d}x,\\
0 &= \int_\Omega \frac{1}{\nu}(\nabla p - (\mu + \chi\sigma)\nabla\varphi) \cdot \nabla\chi - \Gamma_{\mathbf{v}}(\varphi,\sigma)\chi\,\mathrm{d}x.
\end{aligned} \tag{8.11}$$

We say that $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is a weak solution to the CHD system (8.1a), (8.1c)-(8.1e), (8.9), (8.2a)-(8.2b), (8.2d), (8.10) with the logarithmic potential $\psi_{\log}$ if property (c2) in Definition 8.2 holds along with properties (g) and (h).

Remark 8.8 The variational equality $(8.11)_1$ comes naturally from (5.5a) when we neglect the viscosities $\eta(\varphi)$ and $\lambda(\varphi)$. Meanwhile, the variational equality $(8.11)_2$ arises from the weak formulation of the elliptic problem obtained from taking the divergence of Darcy's law (8.9) in conjunction with the equation (8.1a) and the boundary condition (8.10).

Theorem 8.9 *Under* (A1)-(A3), (B1) *and* (B2)*, there exists a weak solution* $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *to the CHD model with double obstacle potential* ψ_{do} *in the sense of Definition 8.7 and, in addition,* $\sigma \in [0,1]$ *almost everywhere in* Ω_T*.*
Under (A1)-(A3), (B1), (B2), (C1)-(C3)*, there exists a weak solution* $(\varphi, \mu, \sigma, \mathbf{v}, p)$ *to the CHD model with logarithmic potential* $\psi_{\log}$ *in the sense of Definition 8.7 and* $\sigma \in [0,1]$ *almost everywhere in* Ω_T*. Furthermore, for a. e.* $t \in (0,T)$ *the inequality* (8.7) *holds with* $\eta(\varphi) \equiv 0$*.*

8.2 The Brinkman model – Proof of Theorem 8.3

The standard procedure is to approximate the singular potentials with a sequence of regular potentials, employ Theorem 5.5 to obtain approximate solutions, derive uniform estimates and pass to the limit.

8.2.1 Approximation potentials and their properties

Double obstacle potential

We point out that in order to use Theorem 5.5 the approximate potential should at least belong to $C^2(\mathbb{R})$. We fix $\delta > 0$ which serves as the regularisation parameter, and we define

$$\hat{\beta}_{\mathrm{do},\delta}(r) = \begin{cases} \frac{1}{2\delta}\left(r - \left(1 + \frac{\delta}{2}\right)\right)^2 + \frac{\delta}{24} & \text{for } r \geq 1 + \delta, \\ \frac{1}{6\delta^2}(r-1)^3 & \text{for } r \in (1, 1+\delta), \\ 0 & \text{for } |r| \leq 1, \\ -\frac{1}{6\delta^2}(r+1)^3 & \text{for } r \in (-1-\delta, -1), \\ \frac{1}{2\delta}\left(r + \left(1 + \frac{\delta}{2}\right)\right)^2 + \frac{\delta}{24} & \text{for } r \leq -1 - \delta. \end{cases} \tag{8.12}$$

Formally, it is easy to see that $\hat{\beta}_{\mathrm{do},\delta}(r) \to \mathbb{I}_{[-1,1]}(r)$ as $\delta \to 0$, and so

$$\psi_{\mathrm{do},\delta}(r) := \hat{\beta}_{\mathrm{do},\delta}(r) + \frac{1}{2}(1 - r^2) \tag{8.13}$$

will serve as our approximation for the double obstacle potential. Let $\beta_{\mathrm{do},\delta}(r) = \hat{\beta}'_{\mathrm{do},\delta}(r) = (r + \psi'_{\mathrm{do},\delta}(r)) \in C^1(\mathbb{R})$ denote the derivative of the convex part $\hat{\beta}_{\mathrm{do},\delta}$:

$$\beta_{\mathrm{do},\delta}(r) = \begin{cases} \frac{1}{\delta}\left(r - \left(1 + \frac{\delta}{2}\right)\right) & \text{for } r \geq 1 + \delta, \\ \frac{1}{2\delta^2}(r-1)^2 & \text{for } r \in (1, 1+\delta), \\ 0 & \text{for } |r| \leq 1, \\ -\frac{1}{2\delta^2}(r+1)^2 & \text{for } r \in (-1-\delta, -1), \\ \frac{1}{\delta}\left(r + \left(1 + \frac{\delta}{2}\right)\right) & \text{for } r \leq -1 - \delta. \end{cases} \tag{8.14}$$

Then, it is clear that $\beta_{\mathrm{do},\delta}$ is Lipschitz continuous with $0 \leq \beta'_{\mathrm{do},\delta}(r) \leq \frac{1}{\delta}$ for all $r \in \mathbb{R}$.

Proposition 8.10 *Let $\hat{\beta}_{\mathrm{do},\delta}$ and $\psi_{\mathrm{do},\delta}$ be defined as above. Then, there exist positive constants C_0 and C_1 such that for all $r \in \mathbb{R}$ and for all $\delta \in (0, 1/4)$,*

$$\psi_{\mathrm{do},\delta}(r) \geq C_0 |r|^2 - C_1, \tag{8.15a}$$

$$\delta \beta_{\mathrm{do},\delta}(r)^2 \leq 2\hat{\beta}_{\mathrm{do},\delta}(r) \leq \delta(\beta_{\mathrm{do},\delta}(r))^2 + 1, \tag{8.15b}$$

$$\delta(\beta'_{\mathrm{do},\delta}(r))^2 \leq \beta'_{\mathrm{do},\delta}(r). \tag{8.15c}$$

Proof. As $\psi_{\mathrm{do},\delta}$ is bounded for $|r| \leq 1 + \delta$, it suffices to show that (8.15a) holds for $|r| > 1 + \delta$. By Young's inequality it is clear that for $r > 1 + \delta$ with $\delta \in (0, 1/4)$,

$$\psi_{\mathrm{do},\delta}(r) \geq 2\left(r - \left(1 + \frac{\delta}{2}\right)\right)^2 - r^2 \geq C_0|r|^2 - C_1,$$

and a similar assertion holds also for $r < -1 - \delta$. This establishes (8.15a).

From the definitions of $\hat{\beta}_{\mathrm{do},\delta}$ and $\beta_{\mathrm{do},\delta}$ we see that if $r \in (1, 1+\delta)$

$$\delta\beta^2_{\mathrm{do},\delta}(r) = \frac{1}{4\delta^3}(r-1)^4 \leq \frac{1}{4\delta^2}(r-1)^3 \leq 2\hat{\beta}_{\mathrm{do},\delta}(r) \leq \frac{\delta}{3} \leq 1 + \delta\beta_{\mathrm{do},\delta}(r)^2,$$

and if $r > 1+\delta$

$$\delta\beta_{\mathrm{do},\delta}(r)^2 = \tfrac{1}{\delta}\left(r - \left(1+\tfrac{\delta}{2}\right)\right)^2 \leq 2\hat{\beta}_{\mathrm{do},\delta}(r) \leq \tfrac{1}{\delta}\left(r - \left(1+\tfrac{\delta}{2}\right)\right)^2 + 1 \leq \delta\beta_{\mathrm{do},\delta}(r)^2 + 1.$$

Similar assertions also hold for the cases $r \in (-1-\delta, -1)$ and $r < -1-\delta$ which then yield (8.15b).

A straightforward computation shows

$$\begin{aligned}
\beta'_{\mathrm{do},\delta}(r) = \delta(\beta'_{\mathrm{do},\delta}(r))^2 &= \begin{cases} \frac{1}{\delta} & \text{for } |r| \geq 1+\delta, \\ 0 & \text{for } |r| \leq 1, \end{cases} \\
\delta(\beta'_{\mathrm{do},\delta}(r))^2 &= \frac{1}{\delta^3}(r-1)^2 \leq \frac{1}{\delta^2}(r-1) = \beta'_{\mathrm{do},\delta}(r) \quad \text{for } r \in (1, 1+\delta), \\
\delta(\beta'_{\mathrm{do},\delta}(r))^2 &= \frac{1}{\delta^3}(-(r+1))^2 \leq -\frac{1}{\delta^2}(r+1) = \beta'_{\mathrm{do},\delta}(r) \quad \text{for } r \in (-1-\delta, -1),
\end{aligned}$$

and so (8.15c) is established. □

Aside from approximating the singular potential, it would be necessary to extend the source functions $b_{\mathbf{v}}, b_\varphi, f_{\mathbf{v}}$ and f_φ from $[-1,1]$ to the whole real line. Since the solution variable φ is supported in $[-1,1]$ (see (c1) of Definition 8.2), the particular form of extensions outside $[-1,1]$ does not play a significant role and we have the flexibility to choose extensions that would easily lead to uniform estimates. Hence, unless stated otherwise we assume that $b_{\mathbf{v}}$, b_φ, $f_{\mathbf{v}}$ and f_φ can be extended to $\mathbb{R}$ such that $f_\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f_{\mathbf{v}} \in C^{0,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $b_\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $b_{\mathbf{v}} \in C^{0,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, and fulfil

$$b_{\mathbf{v}}(r) = 0, \quad b_\varphi(r) = 0 \qquad \forall\, |r| \geq 1, \tag{8.16}$$

$$r(f_\varphi(r) - f_{\mathbf{v}}(r)r) < 0 \qquad \forall\, |r| > 1. \tag{8.17}$$

The latter implies that $f_\varphi(r) - f_{\mathbf{v}}(r)r$ is strictly negative (resp. positive) for $r > 1$ (resp. $r < -1$). For the functions stated in Example 8.4, we can consider following extensions: For $r \in \mathbb{R}$, we set

$$\begin{aligned}
&b_{\mathbf{v}}(r) = \alpha \max(0, P(1-r^2)) \quad b_\varphi(r) = \rho_S \max(0, P(1-r^2)), \\
&f_\varphi(r) = \max(-\rho_S A, \min(\rho_S A, -\rho_S A r)),
\end{aligned}$$

and

$$f_{\mathbf{v}}(r) = \begin{cases} -A\alpha & r \geq 1, \\ -A\alpha r & |r| \leq 1, \\ A\alpha e^{(r+1)} & r \leq -1, \end{cases} \text{ if } \alpha < 0, \quad f_{\mathbf{v}}(r) = \begin{cases} -A\alpha e^{(1-r)} & r \geq 1, \\ -A\alpha r & |r| \leq 1, \\ A\alpha & r \leq -1, \end{cases} \text{ if } \alpha > 0.$$

It is clear that (8.16) is fulfilled. For $\alpha = 0$, we see that $f_{\mathbf{v}}(r) = 0$ and so $f_\varphi(r) - f_{\mathbf{v}}(r)r = f_\varphi(r)$ satisfies (8.17). For $\alpha > 0$, if $r \leq -1$ we see that $f_\varphi(r) - f_{\mathbf{v}}(r)r = A(\rho_S - \alpha r) > 0$ and if $r \geq 1$ we see that $f_\varphi(r) - f_{\mathbf{v}}(r)r$ attains its maximum at $r = 1$ with value $A(\alpha - \rho_S) < 0$. Similarly, for $\alpha < 0$, if $r \geq 1$ we see that $f_\varphi(r) - f_{\mathbf{v}}(r)r = A(\alpha r - \rho_S) < 0$ and if $r \leq -1$ we see that $f_\varphi(r) - f_{\mathbf{v}}(r)r$ attains its minimum at $r = -1$ with value $A(\rho_S - \alpha) > 0$. Hence, the extensions fulfil (8.17). Then, we can employ Theorem 5.5 to deduce the existence of a quintuple $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta, p_\delta)$ to (8.1)-(8.2) with $\psi'_{\mathrm{do},\delta}$ replacing ψ' and source terms $b_{\mathbf{v}}$, b_φ, $f_{\mathbf{v}}$ and f_φ modified as above. Uniform estimates will be derived in the next section and then we can pass to the limit $\delta \to 0$ to infer the existence of a weak solution to (8.1) with the double obstacle potential in the sense of Definition 8.2.

Logarithmic potential

For $\delta \in (0,1)$ we define

$$\psi_{\log,\delta}(r) = \begin{cases} \psi_{\log}(1-\delta) + \psi'_{\log}(1-\delta)\big(r-(1-\delta)\big) \\ \quad +\frac{1}{2}\psi''_{\log}(1-\delta)\big(r-(1-\delta)\big)^2 & \text{for } r \geq 1-\delta, \\ \psi_{\log}(r) & \text{for } |r| \leq 1-\delta, \\ \psi_{\log}(\delta-1) + \psi'_{\log}(\delta-1)\big(r-(\delta-1)\big) \\ \quad +\frac{1}{2}\psi''_{\log}(\delta-1)\big(r-(\delta-1)\big)^2 & \text{for } r \leq -1+\delta, \end{cases} \tag{8.18}$$

with convex part

$$\hat{\beta}_{\log,\delta}(r) \coloneqq \psi_{\log,\delta}(r) - \frac{\theta_c}{2}(1-r^2) \quad \forall\, r \in \mathbb{R}. \tag{8.19}$$

As before, we define $\beta_{\log,\delta}(r) = \hat{\beta}'_{\log,\delta}(r)$ for all $r \in \mathbb{R}$ and

$$\hat{\beta}_{\log}(r) \coloneqq \psi_{\log}(r) - \frac{\theta_c}{2}(1-r^2), \quad \beta_{\log}(r) \coloneqq \hat{\beta}'_{\log}(r) \quad \forall\, r \in (-1,1).$$

Proposition 8.11 *Let $\hat{\beta}_{\log,\delta}$ and $\psi_{\log,\delta}$ be defined as above. Then, there exist positive constants $C_0,\dots,C_3$, such that for all $r \in \mathbb{R}$ and for all $0 < \delta \leq \min(1, \theta/(4\theta_c))$, it holds*

$$\psi_{\log,\delta}(r) \geq C_0|r|^2 - C_1, \tag{8.20a}$$

$$\hat{\beta}_{\log,\delta}(r) \geq \theta(4\delta)^{-1}(|r|-1)_+^2, \tag{8.20b}$$

$$\delta\beta_{\log,\delta}(r)^2 \leq 2\theta\hat{\beta}_{\log,\delta}(r) + C_2 \leq C_3\left(\delta(\beta_{\log,\delta}(r))^2 + 1\right), \tag{8.20c}$$

$$\delta(\beta'_{\log,\delta}(r))^2 \leq \theta\beta'_{\log,\delta}(r), \tag{8.20d}$$

where $(|r|-1)_+ \coloneqq \max(0, |r|-1)$.

Proof. For $r \geq 1-\delta$ with $\delta \leq \frac{\theta}{4\theta_c}$, a short calculation shows that

$$\hat{\beta}_{\log}(1-\delta) > 0, \quad \beta_{\log}(1-\delta)(r-(1-\delta)) \geq 0,$$
$$\beta'_{\log}(1-\delta)(r-(1-\delta))^2 \geq 2\theta_c(r-(1-\delta))^2.$$

Then, it is easy to see that (8.20a) holds with the help of Young's inequality. Analogously, using $\hat{\beta}_{\log}(\delta-1) > 0$, $\beta'_{\log}(\delta-1) = \beta'_{\log}(1-\delta)$ and $\beta_{\log}(\delta-1)(r-(\delta-1)) \geq 0$ for $r \leq -1+\delta$, we infer that (8.20a) also holds for $r \leq -1+\delta$ with $\delta \leq \frac{\theta}{4\theta_c}$. Meanwhile, for $|r| \leq 1-\delta$ the non-negativity of $\hat{\beta}_{\log}$ yields

$$\psi_{\log}(r) \geq \frac{\theta_c}{2}(1-r^2) \geq C_0|r|^2 - C_1 \quad \forall\, |r| \leq 1-\delta.$$

This completes the proof of (8.20a).
Using the non-negativity of $\hat{\beta}_{\log,\delta}$ implies (8.20b) for all $r \in \mathbb{R}$ with $|r| \leq 1$. Now, let $r > 1$. Then, from the definition of $\hat{\beta}_{\log,\delta}$ it follows with similar arguments as before that

$$\hat{\beta}_{\log,\delta}(r) \geq \frac{1}{2}\beta'_{\log}(1-\delta)(r-(1-\delta))^2 \geq \frac{\theta}{2\delta(2-\delta)}(r-1)^2 \geq \frac{\theta}{4\delta}(r-1)^2.$$

Similarly, it holds that $\hat{\beta}_{\log,\delta}(r) \geq \frac{\theta}{4\delta}(|r|-1)^2$ for $r < -1$ which shows (8.20b).
For (8.20c) we first observe that $\delta\beta^2_{\log,\delta}(0) = 0 = \hat{\beta}_{\log,\delta}(0)$ and for $\delta \in (0,1]$

$$\beta'_{\log,\delta}(s) \leq \frac{\theta}{\delta(2-\delta)} \leq \frac{\theta}{\delta}, \quad \beta_{\log,\delta}(s) \geq 0 \quad \forall\, s \in [0, 1-\delta],$$

$$0 < \delta\beta'_{\log,\delta}(s) = \frac{\delta\theta}{1-s^2} \leq \theta, \quad \beta_{\log,\delta}(s) \leq 0 \quad \forall\, s \in [-1+\delta, 0],$$

which implies

$$[\delta(\beta_{\log,\delta}(s))^2]' = 2\delta\beta_{\log,\delta}(s)\beta'_{\log,\delta}(s) \leq 2\theta\beta_{\log,\delta}(s) \quad \forall\, s \in [0, 1-\delta],$$
$$[\delta(\beta_{\log,\delta}(s))^2]' = 2\delta\beta_{\log,\delta}(s)\beta'_{\log,\delta}(s) \geq 2\theta\beta_{\log,\delta}(s) \quad \forall\, s \in [-1+\delta, 0].$$

Integrating the first inequality from 0 to $r \in (0, 1-\delta]$ and the second inequality from $r \in [-1+\delta, 0)$ to 0 yields

$$\delta(\beta_{\log,\delta}(r))^2 \leq 2\theta\hat{\beta}_{\log,\delta}(r) \quad \forall\, r \in [-1+\delta, 1-\delta].$$

Taking note that $\hat{\beta}_{\log,\delta}(r)$ is bounded on $[-1+\delta, 1-\delta]$ uniformly in $\delta \in (0,1]$, we easily infer the upper bound $2\theta\hat{\beta}_{\log,\delta}(r) \leq C_2(\delta(\beta_{\log,\delta}(r))^2 + 1)$ for some positive constant $C_2 > 0$ holding for all $r \in [-1+\delta, 1-\delta]$. Meanwhile, a direct calculation shows that for $r \geq 1-\delta$ and $\delta \in (0,1]$ we have

$$\begin{aligned}
\delta\beta_{\log,\delta}(r)^2 &= \frac{\theta^2}{\delta(2-\delta)^2}(r-(1-\delta))^2 + \frac{\theta^2}{2-\delta}\log\Big(\frac{2-\delta}{\delta}\Big)(r-(1-\delta)) + \frac{\theta^2}{4}\delta\log\Big(\frac{2-\delta}{\delta}\Big)^2 \\
&\leq 2\theta\hat{\beta}_{\log,\delta}(r) + \frac{\theta^2}{4}\delta\log\Big(\frac{2-\delta}{\delta}\Big)^2 - 2\theta\hat{\beta}_{\log}(1-\delta) \leq 2\theta\hat{\beta}_{\log,\delta}(r) + C_1, \\
\frac{\theta}{2}\hat{\beta}_{\log,\delta}(r) &= \frac{\theta}{2}\hat{\beta}_{\log}(1-\delta) + \frac{\theta^2}{4}\log\Big(\frac{2-\delta}{\delta}\Big)(r-(1-\delta)) + \frac{\theta^2}{2(2-\delta)\delta}(r-(1-\delta))^2 \\
&\leq \delta\beta_{\log,\delta}(r)^2 + \frac{\theta}{2}\hat{\beta}_{\log}(1-\delta) - \frac{\theta^2}{4}\delta\log\Big(\frac{2-\delta}{\delta}\Big)^2 \leq \delta\beta_{\log,\delta}(r)^2 + C
\end{aligned}$$

on account of $\frac{1}{2} \leq \frac{1}{2-\delta} \leq 1$, the positivity of $\log((2-\delta)/\delta)(r-(1-\delta))$ and the boundedness of $\hat{\beta}_{\log}(1-\delta)$ and $\delta\log((2-\delta)/\delta)^2$ for $\delta \in (0,1]$. An analogous calculation leads to similar inequalities for $r \leq -1+\delta$, and thus (8.20c) is established.

For (8.20d) a straightforward calculation using $\frac{1}{2-\delta} \leq 1$ gives

$$\delta(\beta'_{\log,\delta}(r))^2 = \frac{\theta^2}{(2-\delta)^2\delta} \leq \theta\beta'_{\log,\delta}(r) \quad \forall\, |r| \geq 1-\delta,$$
$$\delta(\beta'_{\log,\delta}(r))^2 = \frac{\delta\theta^2}{(1-r^2)^2} \leq \frac{\theta^2}{(2-\delta)(1-r^2)} \leq \theta\beta'_{\log,\delta}(r) \quad \forall\, |r| \leq 1-\delta.$$

This completes the proof. □

Once again, we extend the source functions $b_{\mathbf{v}}$, b_φ, $f_{\mathbf{v}}$ and f_φ from $[-1,1]$ to the whole real line in a way that satisfies (8.16) and additionally

$$r f_{\mathbf{v}}(r) - f_\varphi(r) \begin{cases} > 0 & \text{for } r \in [1, r_0], \\ = 0 & \text{for } |r| \geq 2r_0, \\ < 0 & \text{for } r \in [-r_0, -1] \end{cases} \tag{8.21}$$

with smooth interpolation in $[r_0, 2r_0]$ and $[-2r_0, r_0]$ for some fixed constant $r_0 \in (1,2)$. For instance, with the functions $f_{\mathbf{v}}$ and f_φ introduced in Example 8.4, we can take as extensions

$$f_{\mathbf{v}}(r) = \begin{cases} -\alpha A\frac{1}{r} & \text{for } |r| \geq 1, \\ -\alpha A r & \text{for } |r| \leq 1, \end{cases} \quad f_\varphi(r) = \begin{cases} -\alpha A & \text{for } r \geq 2r_0, \\ \frac{A(\rho_S-\alpha)}{2r_0-1}(r-2r_0) - \alpha A & \text{for } r \in [1, 2r_0], \\ -\rho_S A r & \text{for } |r| \leq 1, \\ \frac{A(\rho_S+\alpha)}{2r_0-1}(r+2r_0) - \alpha A & \text{for } r \in [-2r_0, -1], \\ -\alpha A & \text{for } r \leq -2r_0, \end{cases}$$

with $r_0 = \frac{3}{2}$. Then, using $\rho_S - \alpha > 0$ it is clear that $r f_{\mathbf{v}}(r) - f_\varphi(r)$ fulfils (8.21).

8.2.2 Existence of approximate solutions

To unify our analysis, we use the notation

$$\psi_\delta = \begin{cases} \psi_{\mathrm{do},\delta} & \text{for } \psi_{\mathrm{do}}, \\ \psi_{\log,\delta} & \text{for } \psi_{\log}, \end{cases} \qquad \Theta_c = \begin{cases} 1 & \text{for } \psi_{\mathrm{do}}, \\ \theta_c & \text{for } \psi_{\log}, \end{cases}$$

and denote by $\hat{\beta}_\delta$ the convex part of ψ_δ. Employing Propositions 8.10 and 8.11 and using that ψ_δ has quadratic growth, we see that (A5), (ii) is satisfied, and by Theorem 5.5, for every $\delta \in (0,1)$ we infer the existence of a weak solution quintuple $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta, p_\delta)$ to (8.1)-(8.2) with ψ'_δ in (8.1d). More precisely, it holds that

$$\operatorname{div}(\mathbf{v}_\delta) = \Gamma_{\mathbf{v}}(\varphi_\delta, \sigma_\delta) \qquad \text{a.e. in } \Omega_T, \tag{8.22a}$$

$$\mu_\delta = \psi'_\delta(\varphi_\delta) - \Delta\varphi_\delta - \chi\sigma_\delta \qquad \text{a.e. in } \Omega_T, \tag{8.22b}$$

and

$$0 = \int_\Omega \mathbf{T}(\mathbf{v}_\delta, p_\delta) \colon \nabla\mathbf{\Phi} + \nu\mathbf{v}_\delta \cdot \mathbf{\Phi} - (\mu_\delta + \chi\sigma_\delta)\nabla\varphi_\delta \cdot \mathbf{\Phi} \,\mathrm{d}x, \tag{8.22c}$$

$$0 = \langle \partial_t\varphi_\delta, \zeta\rangle_{H^1} + \int_\Omega \nabla\mu_\delta \cdot \nabla\zeta + (\nabla\varphi_\delta \cdot \mathbf{v}_\delta + \varphi_\delta\Gamma_{\mathbf{v}}(\varphi_\delta, \sigma_\delta) - \Gamma_\varphi(\varphi_\delta, \sigma_\delta))\zeta \,\mathrm{d}x, \tag{8.22d}$$

$$0 = \int_\Omega \nabla\sigma_\delta \cdot \nabla\zeta + h(\varphi_\delta)\sigma_\delta\zeta \,\mathrm{d}x + \int_{\partial\Omega} K(\sigma_\delta - 1)\zeta \,\mathrm{d}\mathcal{H}^{d-1} \tag{8.22e}$$

for a.e. $t \in (0,T)$ and for all $\mathbf{\Phi} \in \mathbf{H}^1$ and $\zeta \in H^1$.

8.2.3 Uniform estimates

We first state the following lemma:

Lemma 8.12 *Let $\beta_{\log,\delta}$ denote the derivative of* (8.19)*, and let $b_{\mathbf{v}} \in C^{0,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $b_\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f_{\mathbf{v}} \in C^{0,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, and $f_\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be given such that* (8.16), (C1), (C2) *and* (8.21) *are satisfied. Then, there exists $\delta_0 > 0$ and a positive constant C independent of $\delta \in (0, \delta_0)$ such that for all $\delta \in (0, \delta_0)$, $s \in \mathbb{R}$ and $r \in \mathbb{R}$,*

$$(r\Gamma_{\mathbf{v}}(r,s) - \Gamma_\varphi(r,s))\beta_{\log,\delta}(r) \geq -C(1 + |s| + |r|). \tag{8.23}$$

Proof. We define

$$G(r,s) = s(b_{\mathbf{v}}(r)r - b_\varphi(r))\beta_{\log,\delta}(r).$$

Then, due to (8.16), $G(r,s) = 0$ for $s \in \mathbb{R}$ and $|r| \geq 1$. Using (C1), we have for $\delta \in (0,1)$, $s \in \mathbb{R}$ and $r \in [1-\delta, 1]$ that

$$\begin{aligned} |G(r,s)| &\leq |s| \left(|b_{\mathbf{v}}(r)| + |b_\varphi(r)|\right) \beta_{\log,\delta}(r) \leq 2|s|c\delta\beta_{\log,\delta}(r) \\ &= 2|s|c\delta \left(\tfrac{\theta}{2} \log \tfrac{2-\delta}{\delta} + \tfrac{\theta}{\delta(2-\delta)}\left(r - (1-\delta)\right)\right) \leq C|s|, \end{aligned}$$

where we used that $|\delta \log \delta| \leq C$ for $\delta \in (0,1)$. Consequently, we have that $G(r,s) \geq -C|s|$ for $\delta \in (0,1)$, $s \in \mathbb{R}$ and $r \in [1-\delta, 1]$. A similar assertion holds for $r \in [-1, -1+\delta]$. Lastly, for $|r| \leq 1-\delta$, we use (C2) to deduce that

$$\begin{aligned} |G(r,s)| &\leq |s| \left(|r||b_{\mathbf{v}}(r)\beta_{\log,\delta}(r)| + |b_\varphi(r)\beta_{\log,\delta}(r)|\right) \\ &\leq |s| \left(\max_{r \in [-1,1]} \left|b_{\mathbf{v}}(r) \log \tfrac{1+r}{1-r}\right| + \max_{r \in [-1,1]} \left|b_\varphi(r) \log \tfrac{1+r}{1-r}\right|\right) \leq C|s|, \end{aligned}$$

and consequently $G(r,s) \geq -C|s|$ for all $|r| \leq 1-\delta$ and all $s \in \mathbb{R}$. Therefore, for all $\delta > 0$, $s \in \mathbb{R}$ and $r \in \mathbb{R}$, it holds that $G(r,s) \geq -C|s|$. Next, we define

$$H(r) = r f_{\mathbf{v}}(r) - f_{\varphi}(r).$$

By continuity of $H(r)$ and invoking (8.21), we can find a constant $\delta_0 \in (0, r_0 - 1)$ such that

$$\begin{aligned} H(r) &> 0 \quad \text{for } r \in (1-\delta_0, 1+\delta_0), \\ H(r) &< 0 \quad \text{for } r \in (-1-\delta_0, -1+\delta_0). \end{aligned}$$

Then, it is clear that for $|r| \geq 2r_0$ it holds $H(r) = 0$ thanks to (8.21) satisfied by the extensions of $f_{\mathbf{v}}$ and f_{φ}. Meanwhile, for any $\delta \in (0, \delta_0)$ we see that if $|r| \leq 1-\delta_0 < 1-\delta$, then

$$|\beta_{\log,\delta}(r)| = \left| \log \tfrac{1+r}{1-r} \right| \leq \log \tfrac{2-\delta_0}{\delta_0}, \quad |H(r)| \leq C(1+|r|),$$

which implies that

$$H(r)\beta_{\log,\delta}(r) \geq -C(1+|r|) \quad \text{for } |r| \leq 1-\delta_0.$$

On the other hand, as $r_0 > 1$, for $r \in [-2r_0, -1+\delta_0] \cup [1-\delta_0, 2r_0]$ we use that $\beta_{\log,\delta}(r)$ and $H(r)$ have the same sign, so that their product $H(r)\beta_{\log,\delta}(r)$ is non-negative. Hence, combining with the above analysis for the function G, we obtain the assertion (8.23). $\square$

In the following we derive uniform estimates for $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta, p_\delta)$ in δ, and denote by C a generic constant independent of δ which may change its value even within one line.

Nutrient estimates Choosing $\zeta = \sigma_\delta$ in (8.22e) and using the non-negativity of $h(\cdot)$ leads to

$$\int_\Omega |\nabla \sigma_\delta|^2 \, \mathrm{d}x + K \int_{\partial\Omega} |\sigma_\delta|^2 \, \mathrm{d}\mathcal{H}^{d-1} \leq K \int_{\partial\Omega} \sigma_\delta \, \mathrm{d}\mathcal{H}^{d-1} \leq \frac{K}{2} \|\sigma_\delta\|^2_{L^2(\partial\Omega)} + \frac{K}{2} |\partial\Omega|,$$

from which we deduce that σ_δ is uniformly bounded in $L^\infty(0,T;H^1)$. Elliptic regularity additionally yields

$$\|\sigma_\delta\|_{L^\infty(0,T;H^2)} \leq C. \tag{8.24}$$

Meanwhile, choosing $\zeta = -(\sigma_\delta)_- := -\max(0, -\sigma_\delta)$ shows that

$$\|\nabla(\sigma_\delta)_-\|^2_{L^2} + K\|(\sigma_\delta)_-\|^2_{L^2(\partial\Omega)} \leq -K \int_{\partial\Omega} (\sigma_\delta)_- \, \mathrm{d}\mathcal{H}^{d-1} \leq 0$$

on account of the fact that $(\sigma_\delta)_- \geq 0$. Hence, we deduce that $(\sigma_\delta)_- \equiv 0$ a. e. in Ω_T and as a consequence, σ_δ is non-negative a. e. in Ω_T. Similarly, choosing $\zeta = (\sigma_\delta - 1)_+ := \max(\sigma_\delta - 1, 0)$, we have

$$\|\nabla(\sigma_\delta - 1)_+\|^2_{\mathbf{L}^2} + K\|(\sigma_\delta - 1)_+\|^2_{L^2(\partial\Omega)} = -\int_\Omega h(\varphi_\delta)\sigma_\delta(\sigma_\delta - 1)_+ \, \mathrm{d}x \leq 0$$

on account of the non-negativity of $h(\cdot)$ and $\sigma_\delta(\sigma_\delta - 1)_+$. As before, this gives $(\sigma_\delta - 1)_+ \equiv 0$ a. e. in Ω_T and consequently $\sigma_\delta \leq 1$ a. e. in Ω_T. All together, it holds that $\sigma_\delta \in [0,1]$ a. e. in Ω_T. Furthermore, by the continuous embedding $H^2 \hookrightarrow L^\infty$ and the assumptions on $b_{\mathbf{v}}$, b_φ, $f_{\mathbf{v}}$ and f_φ, we have

$$\|\Gamma_\varphi\|_{L^\infty(0,T;L^\infty)} + \|\Gamma_{\mathbf{v}}\|_{L^\infty(0,T;L^\infty)} \leq C. \tag{8.25}$$

Estimates from energy identity Thanks to (8.25), there exists a solution $\mathbf{u} \in \mathbf{H}^1$ to the problem

$$\operatorname{div}(\mathbf{u}) = \Gamma_{\mathbf{v}} \quad \text{in } \Omega, \qquad \mathbf{u} = \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v}} \, \mathrm{d}x\right)\mathbf{n} =: \mathbf{a} \quad \text{on } \partial\Omega, \tag{8.26}$$

satisfying the estimate

$$\|\mathbf{u}\|_{\mathbf{W}^{1,p}} \leq C\|\Gamma_{\mathbf{v}}\|_{L^p} \leq C \quad \forall\, p \in (1,\infty). \tag{8.27}$$

Technically, we should stress the dependence of $\mathbf{u}$ on δ, but in light of the uniform estimate (8.27) we infer that $\mathbf{u}_\delta$ is bounded in $L^\infty(0,T;\mathbf{W}^{1,p})$ for any $p \in (1,\infty)$. Henceforth, we drop the index δ and reuse the variable $\mathbf{u}$.

Choosing $\mathbf{\Phi} = \mathbf{v}_\delta - \mathbf{u}$ in (8.22c), $\zeta = \mu_\delta + \chi\sigma_\delta$ in (8.22d), using (8.22b) and summing the resulting identities, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi_\delta(\varphi_\delta) + \frac{1}{2}|\nabla\varphi_\delta|^2 \, \mathrm{d}x + \int_\Omega |\nabla\mu_\delta|^2 + 2\eta(\varphi_\delta)|\mathbf{D}\mathbf{v}_\delta|^2 + \nu|\mathbf{v}_\delta|^2 \, \mathrm{d}x \\
&\quad = \int_\Omega -\chi\nabla\mu_\delta\cdot\nabla\sigma_\delta + (\Gamma_\varphi - \varphi_\delta\Gamma_{\mathbf{v}})(\beta_\delta(\varphi_\delta) - \Theta_c\varphi_\delta - \Delta\varphi_\delta) \, \mathrm{d}x \\
&\qquad + \int_\Omega 2\eta(\varphi_\delta)\mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta\cdot\mathbf{u} \, \mathrm{d}x - \int_\Omega (\beta_\delta(\varphi_\delta) - \Theta_c\varphi_\delta - \Delta\varphi_\delta)\nabla\varphi_\delta\cdot\mathbf{u} \, \mathrm{d}x,
\end{aligned} \tag{8.28}$$

where we used the fact that ψ_δ is a quadratic perturbation of a convex function $\hat{\beta}_\delta$, in conjunction with [123, Lem. 4.1] to obtain the time derivative of the energy. By Young's inequality and the estimates (8.24), (8.25) and (8.27), we find that

$$\begin{aligned}
&\left|\int_\Omega 2\eta(\varphi_\delta)\mathbf{D}\mathbf{v}_\delta : \mathbf{D}\mathbf{u} + \nu\mathbf{v}_\delta\cdot\mathbf{u} - \chi\nabla\mu_\delta\cdot\nabla\sigma_\delta - (\Gamma_\varphi - \varphi_\delta\Gamma_{\mathbf{v}} - \nabla\varphi_\delta\cdot\mathbf{u})(\Theta_c\varphi_\delta + \Delta\varphi_\delta) \, \mathrm{d}x\right| \\
&\quad \leq \|\sqrt{\eta(\varphi_\delta)}\mathbf{D}\mathbf{v}_\delta\|_{\mathbf{L}^2}^2 + \frac{\nu}{2}\|\mathbf{v}_\delta\|_{\mathbf{L}^2}^2 + \frac{1}{4}\|\nabla\mu_\delta\|_{\mathbf{L}^2}^2 + 2\varepsilon\|\Delta\varphi_\delta\|_{L^2}^2 + C(1 + \|\varphi_\delta\|_{H^1}^2)
\end{aligned} \tag{8.29}$$

for a positive constant ε yet to be determined. It remains to control the two terms with $\beta_\delta(\varphi_\delta)$. Integrating by parts and employing (8.26) leads to

$$\begin{aligned}
\int_\Omega \beta_\delta(\varphi_\delta)\nabla\varphi_\delta\cdot\mathbf{u} \, \mathrm{d}x &= \int_\Omega \nabla(\hat{\beta}_\delta(\varphi_\delta))\cdot\mathbf{u} \, \mathrm{d}x \\
&= \frac{1}{|\partial\Omega|}\left(\int_\Omega \Gamma_{\mathbf{v}} \, \mathrm{d}x\right)\int_{\partial\Omega}\hat{\beta}_\delta(\varphi_\delta) \, \mathrm{d}\mathcal{H}^{d-1} - \int_\Omega \hat{\beta}_\delta(\varphi_\delta)\Gamma_{\mathbf{v}} \, \mathrm{d}x.
\end{aligned} \tag{8.30}$$

Using (8.27) and the relation $\hat{\beta}_\delta(r) = \psi_\delta(r) - \frac{\Theta_c}{2}(1-r^2)$, we obtain

$$\begin{aligned}
\left|\int_\Omega \hat{\beta}_\delta(\varphi_\delta)\Gamma_{\mathbf{v}} \, \mathrm{d}x\right| &= \left|\int_\Omega \left(\psi_\delta(\varphi_\delta) - \tfrac{\Theta_c}{2}(1-\varphi_\delta^2)\right)\Gamma_{\mathbf{v}} \, \mathrm{d}x\right| \\
&\leq C\left(1 + \|\psi_\delta(\varphi_\delta)\|_{L^1} + \|\varphi_\delta\|_{L^2}^2\right).
\end{aligned} \tag{8.31}$$

Meanwhile, using (8.15b), (8.15c), (8.20c), (8.20d) and the trace theorem (with constant c_{tr}), it follows that

$$\begin{aligned}
\|\hat{\beta}_\delta(\varphi_\delta)\|_{L^1(\partial\Omega)} &\leq C + C_3\|\sqrt{\delta}\,\beta_\delta(\varphi_\delta)\|_{L^2(\partial\Omega)}^2 \\
&\leq C_3c_{\mathrm{tr}}^2\left(\|\sqrt{\delta}\,\beta_\delta(\varphi_\delta)\|_{L^2}^2 + \|\sqrt{\delta}\beta_\delta'(\varphi_\delta)\nabla\varphi_\delta\|_{\mathbf{L}^2}^2\right) + C \\
&\leq C_3c_{\mathrm{tr}}^2\left(C_2\|\hat{\beta}_\delta(\varphi_\delta)\|_{L^1} + C_4\int_\Omega \beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 \, \mathrm{d}x\right) + C \\
&\leq C\left(1 + \|\varphi_\delta\|_{L^2}^2 + \|\psi_\delta(\varphi_\delta)\|_{L^1}\right) + C_3c_{\mathrm{tr}}^2C_4\int_\Omega \beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 \, \mathrm{d}x.
\end{aligned} \tag{8.32}$$

To deal with the remaining term we need to distinguish between $\beta_{\mathrm{do},\delta}$ and $\beta_{\log,\delta}$. From the definition (8.14) of $\beta_{\mathrm{do},\delta}$ and (8.16)-(8.17), for any $s \geq 0$ and $r \in \mathbb{R}$,

$$\beta_{\mathrm{do},\delta}(r)(\Gamma_\varphi(r,s) - r\Gamma_{\mathbf{v}}(r,s)) \begin{cases} = 0 & \text{for } |r| \leq 1 \text{ and } s \geq 0, \\ < 0 & \text{for } |r| > 1 \text{ and } s \geq 0 \end{cases}$$

which implies that

$$\int_\Omega (\Gamma_\varphi(\varphi_\delta, \sigma_\delta) - \varphi_\delta \Gamma_{\mathbf{v}}(\varphi_\delta, \sigma_\delta))\beta_{\mathrm{do},\delta}(\varphi_\delta) \, \mathrm{d}x \leq 0.$$

Meanwhile, for $\beta_{\log,\delta}$, we use (8.23) and (8.24) to obtain

$$\int_\Omega (\Gamma_\varphi(\varphi_\delta, \sigma_\delta) - \varphi_\delta \Gamma_{\mathbf{v}}(\varphi_\delta, \sigma_\delta))\beta_{\log,\delta}(\varphi_\delta) \, \mathrm{d}x \leq C\Big(1 + \|\varphi_\delta\|_{L^1}\Big).$$

Therefore, we find that

$$\int_\Omega (\Gamma_\varphi(\varphi_\delta, \sigma_\delta) - \varphi_\delta \Gamma_{\mathbf{v}}(\varphi_\delta, \sigma_\delta))\beta_\delta(\varphi_\delta) \, \mathrm{d}x \leq C\big(1 + \|\varphi_\delta\|_{L^2}\big),$$

and so, when substituting (8.29)-(8.32) into (8.28), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \psi_\delta(\varphi_\delta) + \frac{1}{2}|\nabla\varphi_\delta|^2 \, \mathrm{d}x + \int_\Omega \frac{3}{4}|\nabla\mu_\delta|^2 + \eta(\varphi_\delta)|\mathbf{D}\mathbf{v}_\delta|^2 + \frac{\nu}{2}|\mathbf{v}_\delta|^2 \, \mathrm{d}x \\
&\quad \leq C\Big(1 + \|\psi_\delta(\varphi_\delta)\|_{L^1} + \|\varphi_\delta\|_{L^2}^2 + \|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2\Big) \\
&\qquad + 2\varepsilon\|\Delta\varphi_\delta\|_{L^2}^2 + C_3 c_{\mathrm{tr}}^2 C_4 \int_\Omega \beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 \, \mathrm{d}x.
\end{aligned}$$

Testing (8.22b) with $-A\Delta\varphi_\delta$ for some positive constant A, integrating by parts and using $\nabla\varphi_\delta \cdot \mathbf{n} = 0$ on $\partial\Omega$ and (8.24) yields

$$\begin{aligned}
A\int_\Omega |\Delta\varphi_\delta|^2 + \beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 \, \mathrm{d}x &= A\int_\Omega \nabla(\mu_\delta + \chi\sigma_\delta)\cdot\nabla\varphi_\delta - |\nabla\varphi_\delta|^2 \, \mathrm{d}x \\
&\leq C\left(1 + \|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2\right) + \frac{1}{4}\|\nabla\mu_\delta\|_{L^2}^2.
\end{aligned}$$

Then, summing up the last two inequalities and choosing $A > C_3 c_{\mathrm{tr}}^2 C_4$ and $\varepsilon < \frac{A}{4}$ yields

$$\begin{aligned}
&\frac{d}{dt}\left(\|\psi_\delta(\varphi_\delta)\|_{L^1} + \|\nabla\varphi_\delta\|_{L^2}^2\right) - C\left(\|\psi_\delta(\varphi_\delta)\|_{L^1} + \|\nabla\varphi_\delta\|_{L^2}^2\right) \\
&\quad + \|\nabla\mu_\delta\|_{L^2}^2 + \|(\beta_\delta'(\varphi_\delta))^{1/2}\nabla\varphi_\delta\|_{\mathbf{L}^2}^2 \\
&\quad + \|\Delta\varphi_\delta\|_{L^2}^2 + \|(\eta(\varphi_\delta))^{1/2}\mathbf{D}\mathbf{v}_\delta\|_{L^2}^2 + \nu\|\mathbf{v}_\delta\|_{L^2}^2 \\
&\leq C. \qquad (8.33)
\end{aligned}$$

Before applying a Gronwall argument, we first note that for the double obstacle potential, invoking the assumption (B2) implies $\hat{\beta}_{\mathrm{do},\delta}(\varphi_0) = 0$, and for the logarithmic potential, the assumption (C3) implies there exists $\delta_1 > 0$ such that $|\varphi_0(x)| \leq 1 - \delta_1$ for a. e. $x \in \Omega$, and so $\hat{\beta}_{\log,\delta}(\varphi_0)$ is uniformly bounded. Hence, for $0 < \delta < \min(1, \theta/(4\theta_c), \delta_0, \delta_1) =: \delta_*$, we see that

$$\|\hat{\beta}_\delta(\varphi_0)\|_{L^1} \leq C.$$

Integrating (8.33) in time from 0 to $s \in (0,T]$, using (8.15a), (8.20a), Korn's inequality and elliptic regularity theory, we deduce the uniform estimate

$$\begin{aligned}
&\|\psi_\delta(\varphi_\delta)\|_{L^\infty(0,T;L^1)} + \|\varphi_\delta\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^2)} + \|\nabla\mu_\delta\|_{L^2(0,T;\mathbf{L}^2)} \\
&\quad + \|(\beta_\delta'(\varphi_\delta))^{1/2}\nabla\varphi_\delta\|_{L^2(0,T;\mathbf{L}^2)} + \|\mathbf{v}_\delta\|_{L^2(0,T;\mathbf{H}^1)} \leq C. \qquad (8.34)
\end{aligned}$$

Returning to (8.32), we readily infer that

$$\|\hat{\beta}_\delta(\varphi_\delta)\|_{L^1(0,T;L^1(\partial\Omega))} \leq C, \tag{8.35}$$

while by the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ and (8.34) it follows that

$$\int_0^T \|\nabla\varphi_\delta \cdot \mathbf{v}_\delta\|_{L^{3/2}}^2 \, \mathrm{d}t \leq \int_0^T \|\nabla\varphi_\delta\|_{\mathbf{L}^2}^2 \|\mathbf{v}_\delta\|_{\mathbf{L}^6}^2 \, \mathrm{d}t \leq \|\varphi_\delta\|_{L^\infty(0,T;H^1)}^2 \|\mathbf{v}_\delta\|_{L^2(0,T;\mathbf{H}^1)}^2 \leq C.$$

A similar argument together with (8.25) shows that $\varphi_\delta \Gamma_{\mathbf{v}}$ is bounded in $L^2(0,T;L^{\frac{3}{2}})$. Then, from (5.5b) we obtain

$$\|\partial_t\varphi_\delta\|_{L^2(0,T;(H^1)^*)} + \|\mathrm{div}(\varphi_\delta\mathbf{v}_\delta)\|_{L^2(0,T;L^{3/2})} \leq C. \tag{8.36}$$

Furthermore, we find that the mean value $(\varphi_\delta)_\Omega$ satisfies

$$|\partial_t(\varphi_\delta)_\Omega| = \frac{1}{|\Omega|}\left| \int_\Omega \Gamma_\varphi(\varphi_\delta,\sigma_\delta) - \varphi_\delta\Gamma_{\mathbf{v}}(\varphi_\delta,\sigma_\delta) - \nabla\varphi_\delta \cdot \mathbf{v}_\delta \, \mathrm{d}x \right| \in L^2(0,T),$$

and so

$$\|(\varphi_\delta)_\Omega\|_{H^1(0,T)} \leq C. \tag{8.37}$$

In particular, by the fundamental theorem of calculus, it holds that

$$|(\varphi_\delta)_\Omega(r) - (\varphi_\delta)_\Omega(s)| = \left| \int_s^r \partial_t(\varphi_\delta)_\Omega(t) \, \mathrm{d}t \right| \leq C|r-s|^{\frac{1}{2}}. \tag{8.38}$$

8.2.4 Estimates for the mean value of the chemical potential

In order to pass to the limit $\delta \to 0$ rigorously, it remains to derive uniform estimates for μ_δ, $\beta_\delta(\varphi_\delta)$ and p_δ in $L^2(0,T;L^2)$. To do so we appeal to the method introduced in [88] which involves first deducing that the limit φ of φ_δ has mean value strictly in the open interval $(-1,1)$ for all times. We first state a useful auxiliary result.

Proposition 8.13 *For* $\delta \in (0,1)$*, let* $\beta_{\log,\delta}$ *denote the derivative of* (8.19)*. Then, there exist positive constants* c_1 *and* c_2 *independent of* δ *such that*

$$r\beta_{\log,\delta}(r) \geq |\beta_{\log,\delta}(r)| - c_1|r| - c_2 \quad \forall\, r \in \mathbb{R}. \tag{8.39}$$

Remark 8.14 The estimate (8.39) is more refined than commonly stated estimates of the form

$$r\beta_{\log,\delta}(r) \geq \tilde{c}_0|\beta_{\log,\delta}(r)| - \tilde{c}_1 \quad \forall\, r \in \mathbb{R} \tag{8.40}$$

with positive constant $\tilde{c}_0$ and non-negative constant $\tilde{c}_1$ that are independent of δ, provided δ is sufficiently small, cf. [36, (2.12)], in which the constant $\tilde{c}_0$ is usually not quantified.

Proof of Proposition 8.13. From the definition of $\hat{\beta}_{\log,\delta}$ in (8.19), we infer that for $r \geq 1-\delta$, $\delta \in (0,1)$,

$$\begin{aligned}
\beta_{\log,\delta}(r)r &= \beta_{\log}'(1-\delta)(r-(1-\delta))r + \beta_{\log}(1-\delta)r \\
&\geq \beta_{\log}'(1-\delta)(r-(1-\delta)) + \beta_{\log}(1-\delta) - \delta\beta_{\log}'(1-\delta)(r-(1-\delta)) - \delta\beta_{\log}(1-\delta) \\
&= \beta_{\log,\delta}(r) - \tfrac{\theta}{2-\delta}(r-(1-\delta)) - \delta\log\tfrac{2-\delta}{\delta} \\
&\geq \beta_{\log,\delta}(r) - \theta(r-(1-\delta)) - c
\end{aligned}$$

for some positive constant c independent of $\delta \in (0,1)$. In a similar fashion, for $r \le -1+\delta$, $\delta \in (0,1)$, we have

$$\begin{aligned} \beta_{\log}(\delta-1)r &\ge -\beta_{\log}(\delta-1) + \delta\beta_{\log}(\delta-1) \ge |\beta_{\log}(\delta-1)| - c, \\ \beta'_{\log}(\delta-1)(r-(\delta-1))r &\ge -\beta'_{\log}(\delta-1)(r-(\delta-1)) + \delta\beta'_{\log}(\delta-1)(r-(\delta-1)) \\ &= |\beta'_{\log}(\delta-1)(r-(\delta-1))| + \tfrac{\theta}{2-\delta}(r-(\delta-1)), \end{aligned}$$

and when combined this yields (8.39). For the remaining case $|r| \le 1-\delta$, we employ the fact that $\beta_{\log,\delta}(r) = \beta_{\log}(r)$ and

$$\lim_{r\to 1^-} (1-r)\beta_{\log}(r) = 0, \quad \lim_{r\to 1^+} (r+1)\beta_{\log}(r) = 0$$

to infer the existence of a constant $c>0$ independent of $\delta \in (0,1)$ such that

$$\beta_{\log}(r)(r-1) \ge -c \quad \text{for } 0 \le r < 1, \quad \beta_{\log}(r)(r+1) \ge -c \quad \text{for } -1 < r \le 0.$$

Hence, for $\delta \in (0,1)$ it holds

$$\beta_{\log,\delta}(r)r = \beta_{\log}(r)r \ge |\beta_{\log}(r)| - c = |\beta_{\log,\delta}(r)| - c \quad \forall\, |r| \le 1-\delta.$$

This completes the proof. □

Now, using reflexive weak compactness arguments (Aubin–Lions theorem) and Lemma 2.36, for $\delta \to 0$ along a non-relabelled subsequence, we infer that

$$\begin{aligned} \varphi_\delta &\to \varphi && \text{weakly-star} && \text{in } H^1(0,T;(H^1)^*) \cap L^\infty(0,T;H^1) \cap L^2(0,T;H^2), \\ \varphi_\delta &\to \varphi && \text{strongly} && \text{in } C^0([0,T];L^r) \cap L^2(0,T;W^{1,r}) \text{ and a.e. in } \Omega_T, \\ \sigma_\delta &\to \sigma && \text{weakly-star} && \text{in } L^\infty(0,T;H^2), \\ \nabla\mu_\delta &\to \boldsymbol{\xi} && \text{weakly} && \text{in } L^2(0,T;\mathbf{L}^2), \\ \mathbf{v}_\delta &\to \mathbf{v} && \text{weakly} && \text{in } L^2(0,T;\mathbf{H}^1), \\ \operatorname{div}(\varphi_\delta \mathbf{v}_\delta) &\to \theta && \text{weakly} && \text{in } L^2(0,T;L^{\frac{3}{2}}) \end{aligned} \tag{8.41}$$

for some limit functions $\boldsymbol{\xi} \in L^2(0,T;\mathbf{L}^2)$, $\theta \in L^2(0,T;L^{\frac{3}{2}})$ and for all $r \in [1,6)$. The interpolation inequality $\|f\|_{H^1} \le C\|f\|_{L^2}^{1/2}\|f\|_{H^2}^{1/2}$, the boundedness of $\varphi_\delta - \varphi$ in $L^2(0,T;H^2)$ and the strong convergence $\varphi_\delta \to \varphi$ in $L^\infty(0,T;L^2)$ also allow us to deduce that $\varphi_\delta \to \varphi$ strongly in $L^4(0,T;H^1)$. Consequently, for an arbitrary test function $\lambda \in L^4(0,T;L^3)$ it holds that $\lambda\varphi_\delta \to \lambda\varphi$ strongly in $L^2(0,T;L^2)$ and $\lambda\nabla\varphi_\delta \to \lambda\nabla\varphi$ strongly in $L^2(0,T;L^{\frac{6}{5}})$. Using the weak convergence of $\mathbf{v}_\delta \rightharpoonup \mathbf{v}$ in $L^2(0,T;\mathbf{H}^1)$ and the product of weak-strong convergence we obtain

$$\int_0^T \int_\Omega \operatorname{div}(\varphi_\delta \mathbf{v}_\delta)\lambda \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \operatorname{div}(\varphi\mathbf{v})\lambda \,\mathrm{d}x\,\mathrm{d}t \quad \text{as } \delta \to 0. \tag{8.42}$$

This implies $\operatorname{div}(\varphi_\delta\mathbf{v}_\delta) \to \operatorname{div}(\varphi\mathbf{v})$ weakly in $L^{\frac{4}{3}}(0,T;L^{\frac{3}{2}})$ as $\delta \to 0$. Since $L^2(0,T;L^{\frac{3}{2}}) \subset L^{\frac{4}{3}}(0,T;L^{\frac{3}{2}})$, by uniqueness of weak limits we obtain $\operatorname{div}(\varphi\mathbf{v}) = \theta$. Using the assumption on Γ_φ and the above convergences, we can pass to the limit in (5.5b) to obtain

$$\langle \partial_t\varphi, \zeta\rangle_{H^1} + \int_\Omega \operatorname{div}(\varphi\mathbf{v})\zeta \,\mathrm{d}x = \int_\Omega -\boldsymbol{\xi}\cdot\nabla\zeta + \Gamma_\varphi(\varphi,\sigma)\zeta \,\mathrm{d}x \tag{8.43}$$

for a.e. $t \in (0,T)$ and for all $\zeta \in H^1$. Technically, one would multiply (5.5b) with a function $\kappa \in C_c^\infty(0,T)$, integrate the resulting product over $(0,T)$, pass to the limit $\delta \to 0$ and then recover (8.43) with the fundamental lemma of calculus of variations.

Now, for the obstacle potential, the uniform boundedness of $\psi_\delta(\varphi_\delta)$ in $L^1(0,T;L^1)$ and (8.15b) imply $\sqrt{\delta}\beta_{\mathrm{do},\delta}(\varphi_\delta)$ is uniformly bounded in $L^2(0,T;L^2)$, and so $\delta\beta_{\mathrm{do},\delta}(\varphi_\delta) \to 0$ strongly in $L^2(0,T;L^2)$. However, from the definition of $\beta_{\mathrm{do},\delta}$ we have

$$\delta\beta_{\mathrm{do},\delta}(r) \to g(r) := \begin{cases} r-1 & \text{if } r \geq 1 \\ 0 & \text{if } |r| \leq 1 \\ r+1 & \text{if } r \leq -1 \end{cases} \qquad \text{as } \delta \to 0, \tag{8.44}$$

which implies that (see [19, Proof of Thm. 2.2])

$$|\varphi| \leq 1 \quad \text{a.e. in } \Omega_T. \tag{8.45}$$

For the logarithmic potential, we use (8.20b) and the uniform boundedness of $\hat{\beta}_{\log,\delta}(\varphi_\delta)$ in $L^1(0,T;L^1)$ to obtain that

$$\int_{\Omega_T} (|\varphi_\delta| - 1)_+^2 \,\mathrm{d}x\,\mathrm{d}t \leq C\delta.$$

Since $\varphi_\delta \to \varphi$ a.e. in Ω_T and strongly in $L^2(L^2)$, passing to the limit $\delta \to 0$ in the last inequality also implies (8.45). From this we claim that $\varphi_\Omega(t) \in (-1,1)$ for all $t \in (0,T)$. Indeed, choosing $\zeta = 1$ in (8.43) leads to

$$\langle \partial_t \varphi, 1 \rangle_{H^1} + \int_\Omega \nabla\varphi \cdot \mathbf{v} \,\mathrm{d}x = \int_\Omega \Gamma_\varphi(\varphi,\sigma) - \varphi\Gamma_{\mathbf{v}}(\varphi,\sigma)\,\mathrm{d}x \quad \text{for a.e. } t \in (0,T). \tag{8.46}$$

Suppose to the contrary there exists a time $t_* \in (0,T)$ such that $\varphi_\Omega(t_*) = 1$ and (8.46) holds. Due to (8.45), this implies $\varphi(t_*,x) \equiv 1$ a.e. in Ω and thus $\nabla\varphi(t_*,x) \equiv \mathbf{0}$ a.e. in Ω. Using (8.46) and (8.16)-(8.17), we obtain

$$\langle \partial_t \varphi(t_*), 1 \rangle_{H^1} = \int_\Omega f_\varphi(1) - f_{\mathbf{v}}(1)\,\mathrm{d}x < 0.$$

Hence, by continuity of $t \mapsto (\varphi(t))_\Omega$, the mean value $(\varphi(t))_\Omega$ must be strictly decreasing in a neighbourhood of t_*, i.e., $(\varphi(t))_\Omega > 1$ for $t < t_*$ which contradicts (8.45). Using a similar argument and the assumption $f_\varphi(-1) + f_{\mathbf{v}}(-1) > 0$ leads to the conclusion that $(\varphi(t))_\Omega > -1$ for all $t \in (0,T)$. In particular, $(\varphi(t))_\Omega \in (-1,1)$ for all $t \in (0,T)$.

This allows us to derive uniform estimates on the mean value of μ_δ. Integrating (8.22b) and taking the modulus on both sides gives

$$\left| \int_\Omega \mu_\delta(t)\,\mathrm{d}x \right| \leq \int_\Omega |\beta_\delta(\varphi_\delta(t))| + \Theta_c|\varphi_\delta(t)| + \chi|\sigma_\delta(t)|\,\mathrm{d}x \tag{8.47}$$

for a.e. $t \in (0,T)$. Using (8.39) and the fact

$$|\beta_{\mathrm{do},\delta}(r)| \leq r\beta_{\mathrm{do},\delta}(r) \quad \text{ for all } r \in \mathbb{R} \tag{8.48}$$

(which unfortunately does not hold for $\beta_{\log,\delta}$, hence the necessity of Proposition 8.13), we deduce that (suppressing the t-dependence)

$$\begin{aligned} \left| \int_\Omega \mu_\delta \,\mathrm{d}x \right| &\leq \int_\Omega |\beta_{\mathrm{do},\delta}(\varphi_\delta)| + |\beta_{\log,\delta}(\varphi_\delta)| + \Theta_c|\varphi_\delta| + \chi|\sigma_\delta|\,\mathrm{d}x, \\ &\leq \int_\Omega \varphi_\delta\beta_{\mathrm{do},\delta}(\varphi_\delta) + \varphi_\delta\beta_{\log,\delta}(\varphi_\delta) + (c_1 + \Theta_c)|\varphi_\delta| + \chi|\sigma_\delta| + c_2\,\mathrm{d}x \\ &= \int_\Omega \varphi_\delta\beta_\delta(\varphi_\delta) + (c_1 + \Theta_c)|\varphi_\delta| + \chi|\sigma_\delta| + c_2\,\mathrm{d}x. \end{aligned}$$

Together with the identity

$$\|\nabla\varphi_\delta(t)\|_{\mathbf{L}^2}^2 + \int_\Omega \beta_\delta(\varphi_\delta(t))\varphi_\delta(t) - \chi\sigma_\delta(t)\varphi_\delta(t)\,\mathrm{d}x = \int_\Omega \mu_\delta(t)\varphi_\delta(t) + \Theta_c|\varphi_\delta(t)|^2\,\mathrm{d}x$$

obtained from testing (8.22b) with $\zeta = \varphi_\delta$, we see that

$$\left|\int_\Omega \mu_\delta(t)\,\mathrm{d}x\right| \le \int_\Omega \mu_\delta(t)\varphi_\delta(t)\,\mathrm{d}x + C\left(1 + \|\varphi_\delta(t)\|_{L^2}^2 + \|\sigma_\delta(t)\|_{L^2}^2\right). \tag{8.49}$$

Now, let $f_\delta \in H^2_N \cap L^2_0$ be the unique solution to the Neumann–Laplace problem

$$\begin{cases} -\Delta f_\delta = \varphi_\delta(t) - (\varphi_\delta(t))_\Omega & \text{in } \Omega, \\ \nabla f_\delta \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.50}$$

Using Poincaré's inequality, it holds that

$$\|f_\delta\|_{H^1} \le C\|\nabla\varphi_\delta(t)\|_{\mathbf{L}^2}. \tag{8.51}$$

Testing (8.22d) with f_δ, integrating by parts and rearranging yields

$$\begin{aligned}\int_\Omega \mu_\delta(t)\varphi_\delta(t) = &-\langle\partial_t\varphi_\delta(t), f_\delta\rangle_{H^1} - \int_\Omega \big(\mathrm{div}(\varphi_\delta(t)\mathbf{v}_\delta(t)) - \Gamma_\varphi(\varphi_\delta(t),\sigma_\delta(t))\big) f_\delta\,\mathrm{d}x \\ &+ ((\varphi_\delta(t))_\Omega - (\varphi(t))_\Omega + (\varphi(t))_\Omega)\int_\Omega \mu_\delta(t)\,\mathrm{d}x.\end{aligned}$$

Plugging in this identity into (8.49) and rearranging again, we deduce that

$$\begin{aligned}&\left(1 - |(\varphi(t))_\Omega| - \sup_{t\in(0,T)}|(\varphi_\delta(t) - \varphi(t))_\Omega|\right)\left|\int_\Omega \mu_\delta(t)\,\mathrm{d}x\right| \\ &\quad\le C\left(\|\sigma_\delta(t)\|_{L^2}^2 + \|\varphi_\delta(t)\|_{L^2}^2\right) - \langle\partial_t\varphi_\delta(t), f_\delta\rangle_{H^1} \\ &\qquad - \int_\Omega \big(\mathrm{div}(\varphi_\delta(t)\mathbf{v}_\delta(t)) - \Gamma_\varphi(\varphi_\delta(t),\sigma_\delta(t))\big) f_\delta\,\mathrm{d}x\end{aligned} \tag{8.52}$$

for a. e. $t \in (0,T)$. Recalling (8.37)-(8.38), we have the equiboundedness and equicontinuity of $\{(\varphi_\delta)_\Omega\}_{\delta\in(0,1)}$ so that by the Arzelà–Ascoli theorem,

$$(\varphi_\delta(t))_\Omega \to (\varphi(t))_\Omega \quad \text{strongly in } C^0([0,T]) \quad \text{as } \delta \to 0$$

along a non-relabelled subsequence. Then, one can find an index $\delta_3 \in (0,1)$ such that for all $\delta < \min(\delta_3,\delta_*) =: \delta_4$, where δ_* is defined after (8.33), it holds

$$\sup_{t\in(0,T)}|(\varphi_\delta(t) - \varphi(t))_\Omega| \le \tfrac{1}{2}\sup_{t\in(0,T)}(1 - |(\varphi(t))_\Omega|).$$

Since $|(\varphi(t))_\Omega| < 1$ for all $t \in (0,T)$ and φ_Ω is continuous on $[0,T]$, the prefactor on the left-hand side of (8.52) is bounded away from 0 uniformly in t. As the right-hand side of (8.52) is uniformly bounded in $L^2(0,T)$ by previously established uniform estimates, we obtain that $\{(\mu_\delta)_\Omega\}_{\delta\in(0,\delta_4)}$ is bounded in $L^2(0,T)$, and the Poincaré inequality gives

$$\|\mu_\delta\|_{L^2(0,T;L^2)} \le C. \tag{8.53}$$

Let us mention that if instead of (8.39) we employ the less refined estimate (8.40), we arrive at

$$\varphi_\delta\beta_\delta(\varphi_\delta) \ge \min(1,\tilde{c}_0)|\beta_\delta(\varphi_\delta)| - \tilde{c}_1,$$

and ultimately

$$\left(1 - \frac{|(\varphi(t))_\Omega|}{\min(1,\tilde{c}_0)}\right) \left| \int_\Omega \mu_\delta(t) \, \mathrm{d}x \right| \leq C.$$

Since $\tilde{c}_0$ is usually not quantified, we may not be able to rule out the situation where $\tilde{c}_0 < 1$ which may imply that the prefactor $1 - \frac{|(\varphi(t))_\Omega|}{\min(1,\tilde{c}_0)}$ is negative.

The uniform estimate (8.53) for μ_δ allows us to infer further estimates for $\beta_\delta(\varphi_\delta)$ and p_δ. Indeed, testing (8.22b) with $\beta_\delta(\varphi_\delta)$ yields

$$\|\beta_\delta(\varphi_\delta)\|_{L^2}^2 + \int_\Omega \beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 \, \mathrm{d}x = \int_\Omega (\varphi_\delta + \mu_\delta + \chi\sigma_\delta)\beta_\delta(\varphi_\delta) \, \mathrm{d}x.$$

Integrating this identity in time from 0 to T, using the non-negativity of $\beta_\delta'(\cdot)$, (8.34) and (8.53), it follows that

$$\|\beta_{\mathrm{do},\delta}(\varphi_\delta)\|_{L^2(0,T;L^2)} + \|\beta_{\log,\delta}(\varphi_\delta)\|_{L^2(0,T;L^2)} \leq C. \tag{8.54}$$

For the pressure p_δ, we invoke Lemma 2.39 to deduce the existence of a solution $\mathbf{q}_\delta \in \mathbf{H}^1$ to the problem

$$\begin{cases} \mathrm{div}(\mathbf{q}_\delta) = p_\delta & \text{in } \Omega, \\ \mathbf{q}_\delta = \frac{1}{|\partial\Omega|}\left(\int_\Omega p_\delta \, \mathrm{d}x\right)\mathbf{n} & \text{on } \partial\Omega, \end{cases}$$

such that

$$\|\mathbf{q}_\delta\|_{\mathbf{H}^1} \leq C\|p_\delta\|_{L^2} \tag{8.55}$$

for a positive constant C depending only on Ω. Then, testing (8.22c) with $\mathbf{\Phi} = \mathbf{q}_\delta$ yields

$$\begin{aligned} \|p_\delta\|_{L^2}^2 \leq {} & 2\sqrt{\eta_1}\|\eta^{1/2}(\varphi_\delta)\mathbf{D}\mathbf{v}_\delta\|_{\mathbf{L}^2}\|\mathbf{D}\mathbf{q}_\delta\|_{\mathbf{L}^2} + \lambda_1\|\Gamma_{\mathbf{v}}(\varphi_\delta,\sigma_\delta)\|_{L^2}\|p_\delta\|_{L^2} \\ & + \nu\|\mathbf{v}_\delta\|_{\mathbf{L}^2}\|\mathbf{q}_\delta\|_{\mathbf{L}^2} + \|(\mu_\delta + \chi\sigma_\delta)\|_{L^3}\|\nabla\varphi_\delta\|_{\mathbf{L}^2}\|\mathbf{q}_\delta\|_{\mathbf{L}^6}. \end{aligned}$$

Applying Young's inequality and using the uniform estimates (8.24), (8.34), (8.53) and (8.55) leads to

$$\|p_\delta\|_{L^2(0,T;L^2)} \leq C. \tag{8.56}$$

8.2.5 Passing to the limit

Let us first consider the double obstacle case. In addition to the convergence statements in (8.41), we further obtain

$$\begin{aligned} \mu_\delta \to \mu \quad & \text{weakly} \quad \text{in } L^2(0,T;H^1), \\ \beta_{\mathrm{do},\delta}(\varphi_\delta) \to \tau \quad & \text{weakly} \quad \text{in } L^2(0,T;L^2), \\ p_\delta \to p \quad & \text{weakly} \quad \text{in } L^2(0,T;L^2) \end{aligned}$$

for some limit function $\tau \in L^2(0,T;L^2)$. Moreover, due to (8.53) we have $\boldsymbol{\xi} = \nabla\mu$ which allows us to fully recover (5.5b) in the limit. To obtain (5.5a), (5.5d) and (8.1a) in the limit, the arguments are exactly the same as in Chapter 5. It remains to show (8.4) is recovered in the limit $\delta \to 0$ from (8.22b). By arguing as in [85, Sec. 5.2], using the weak convergence $\beta_{\mathrm{do},\delta}(\varphi_\delta) \rightharpoonup \tau$ in $L^2(0,T;L^2)$, the strong convergence $\varphi_\delta \to \varphi$ in $L^2(0,T;L^2)$, and the maximal monotonicity of the subdifferential $\partial\mathbb{I}_{[-1,1]}$ we can infer τ is an element of the set $\partial\mathbb{I}_{[-1,1]}(\varphi)$ which implies that for any $\zeta \in \mathcal{K}$ and a. e. $t \in (0,T)$,

$$\int_\Omega \tau(t)(\zeta - \varphi(t))\,\mathrm{d}x \le 0.$$

Hence, testing (8.22b) (where $\psi_\delta = \psi_{\mathrm{do},\delta}$) with $\zeta - \varphi$ and passing to the limit $\delta \to 0$ allows us to recover (8.4). This completes the proof of Theorem 8.3 for the double obstacle potential.

For the logarithmic case, the additional estimate for $\beta_{\log,\delta}(\varphi_\delta)$ in $L^2(0,T;L^2)$ allow us to infer, by the arguments in [36, Sec. 4] or [94, Sec. 3.3], that the limit φ satisfies the tighter bounds

$$|\varphi(x,t)| < 1 \quad \text{a. e. in } \Omega_T.$$

Furthermore, by the almost everywhere convergence of φ_δ to φ we have $\beta_{\log,\delta}(\varphi_\delta) \to \beta_{\log}(\varphi)$ a. e. in Ω_T.
Meanwhile, the inequality (8.7) is obtained from integrating (8.28) over $(0,t)$ for $t \in (0,T)$ and then passing to the limit with the compactness assertions (8.41), weak lower semicontinuity, and Fatou's lemma. This completes the proof of Theorem 8.3 for the logarithmic potential.

8.3 The Darcy model (Proof of Theorem 8.9)

We can adapt most of the arguments and estimates from the proof of Theorem 8.3. The main idea is to consider a weak solution quintuple $(\varphi_\delta, \mu_\delta, \sigma_\delta, \mathbf{v}_\delta, p_\delta)$ to the Cahn–Hilliard–Brinkman model (8.1)-(8.2) with stress tensor

$$\mathbf{T}_\delta(\mathbf{v}_\delta, p_\delta) := 2\delta \mathbf{D} v_\delta + \delta \mathrm{div}(\mathbf{v}_\delta)\mathbf{I} - p_\delta \mathbf{I},$$

where we have set $\eta(\cdot) = \lambda(\cdot) = \delta$. Proceeding as in the proof of Theorem 8.3 we obtain the uniform estimates (8.24), (8.25) and

$$\begin{aligned}&\|\psi_\delta(\varphi_\delta)\|_{L^\infty(0,T;L^1)} + \|\varphi_\delta\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^2)} + \|\nabla\mu_\delta\|_{L^2(0,T;\mathbf{L}^2)} \\ &\quad + \|\sqrt{\beta_\delta'(\varphi_\delta)}\nabla\varphi_\delta\|_{L^2(0,T;\mathbf{L}^2)} + \|\mathbf{v}_\delta\|_{L^2(0,T;\mathbf{L}^2_{\mathrm{div}})} + \sqrt{\delta}\|\mathbf{D}\mathbf{v}_\delta\|_{L^2(0,T;\mathbf{L}^2)} \le C,\end{aligned} \tag{8.57}$$

where in the above ψ_δ and β_δ denote the approximations to either singular potentials and the derivatives of the corresponding convex part. Multiplying (8.22b) with $-\Delta\varphi_\delta$, using the convexity of $\hat{\beta}_\delta$ and arguing as above, it holds that

$$\|\Delta\varphi_\delta\|_{L^2}^2 \le C\|\nabla(\mu_\delta + \chi\sigma_\delta)\|_{\mathbf{L}^2} \in L^2(0,T),$$

and by elliptic regularity we infer

$$\|\varphi_\delta\|_{L^4(0,T;H^2)} \le C. \tag{8.58}$$

Moreover, by the Gagliardo–Nirenberg inequality we find that

$$\int_0^T \|\nabla\varphi_\delta \cdot \mathbf{v}_\delta\|_{L^{\frac{6}{5}}}^{\frac{8}{5}}\,\mathrm{d}t \le C\|\varphi_\delta\|_{L^\infty(0,T;H^1)}^{\frac{4}{5}}\|\varphi_\delta\|_{L^4(0,T;H^2)}^{\frac{4}{5}}\|\mathbf{v}_\delta\|_{L^2(0,T;\mathbf{L}^2)}^{\frac{8}{5}} \le C,$$

so that from (5.5b) and previous uniform estimates we arrive at

$$\|\partial_t\varphi_\delta\|_{L^{\frac{8}{5}}(0,T;(H^1)^*)} + \|\nabla\varphi_\delta\cdot\mathbf{v}_\delta\|_{L^{\frac{8}{5}}(0,T;L^{\frac{6}{5}})} \le C, \tag{8.59a}$$

$$\|\partial_t\varphi_\delta + \mathrm{div}(\varphi_\delta\mathbf{v}_\delta)\|_{L^2(0,T;(H^1)^*)} + \|(\varphi_\delta)_\Omega\|_{W^{1,\frac{8}{5}}(0,T)} \le C, \tag{8.59b}$$

$$|(\varphi_\delta)_\Omega(r) - (\varphi_\delta)_\Omega(s)| \le C|r-s|^{\frac{3}{8}} \quad \forall\, r,s \in (0,T). \tag{8.59c}$$

Let us mention that the sum $\partial_t \varphi_\delta + \operatorname{div}(\varphi_\delta \mathbf{v}_\delta)$ has better temporal integrability than each of its constituents, a fact which will play an important role for deriving uniform estimates for $(\mu_\delta)_\Omega$ below.

By reflexive weak compactness arguments and Lemma 2.36, for $\delta \to 0$ along a non-relabelled subsequence, it holds for any $r \in [1,6)$ that

$$\begin{aligned}
\varphi_\delta &\to \varphi && \text{weakly-star} && \text{in } W^{1,\frac{8}{5}}(0,T;(H^1)^*) \cap L^\infty(0,T;H^1) \cap L^4(0,T;H^2), \\
\varphi_\delta &\to \varphi && \text{strongly} && \text{in } C^0([0,T];L^r) \cap L^4(0,T;W^{1,r}) \quad \text{and a.e. in } \Omega_T, \\
\sigma_\delta &\to \sigma && \text{weakly-star} && \text{in } L^\infty(0,T;H^2), \\
\mathbf{v}_\delta &\to \mathbf{v} && \text{weakly} && \text{in } L^2(0,T;\mathbf{L}^2), \\
\operatorname{div}(\varphi_\delta \mathbf{v}_\delta) &\to \theta && \text{weakly} && \text{in } L^{\frac{8}{5}}(0,T;L^{\frac{6}{5}})
\end{aligned}$$

for a limit function $\theta \in L^{\frac{8}{5}}(0,T;L^{\frac{6}{5}})$. The identification $\theta = \operatorname{div}(\varphi \mathbf{v})$ follows analogously as in Section 8.2.4 where the assertion (8.42) now holds for arbitrary $\lambda \in L^4(0,T;L^6)$ by the strong convergence $\nabla \varphi_\delta \to \nabla \varphi$ in $L^4(0,T;\mathbf{L}^3)$ and the weak convergence $\mathbf{v}_\delta \to \mathbf{v}$ in $L^2(0,T;\mathbf{L}^2)$.

In order to obtain uniform estimates for the chemical potential μ_δ in $L^2(0,T;L^2)$, we again follow the argument in Section 8.2.4. Namely, we pass to the limit $\delta \to 0$ in (5.5b) to obtain (8.43), and use the uniform boundedness of $\psi_\delta(\varphi_\delta)$ in $L^1(0,T;L^1)$ from (8.57) to obtain that the limit φ satisfies the pointwise bound (8.45). Choosing $\zeta = 1$ in (8.43) leads to (8.46) and we obtain by a contradiction argument that $(\varphi(t))_\Omega \in (-1,1)$ for all $t \in (0,T)$.

Let us consider the double obstacle case where $\psi_\delta = \psi_{\mathrm{do},\delta}$, and define $f_\delta \in H^2_N \cap L^2_0$ as the unique solution to (8.50) satisfying (8.51). Then, the right-hand side of (8.52) can be estimated as

$$\begin{aligned}
\text{RHS} \leq\; & C\Big(\|\sigma_\delta(t)\|_{L^2}^2 + \|\varphi_\delta(t)\|_{L^2}^2 \Big) + \|\Gamma_\varphi(\varphi_\delta(t), \sigma_\delta(t))\|_{L^2} \|f_\delta\|_{L^2} \\
& + \|\partial_t \varphi_\delta(t) + \operatorname{div}(\varphi_\delta(t)\mathbf{v}_\delta(t))\|_{(H^1)^*} \|f_\delta\|_{H^2}
\end{aligned} \tag{8.60}$$

which is bounded in $L^2(0,T)$ by (8.59b). This modification allows us to infer that $(\mu_\delta)_\Omega$ is uniformly bounded in $L^2(0,T)$, whereas simply using (8.59a) would only give the uniform boundedness of $(\mu_\delta)_\Omega$ in $L^{\frac{8}{5}}(0,T)$. Hence, we recover the uniform $L^2(0,T;L^2)$-estimate (8.53) for μ_δ and also (8.54) for $\beta_{\mathrm{do},\delta}(\varphi_\delta)$.

The argument for the logarithmic potential is entirely analogous, as the only modification is (8.60), and thus we skip the details. To complete the proof of Theorem 8.9 we still require uniform estimates on the pressure p_δ. From the above paragraphs we have

$$\|\mu_\delta\|_{L^2(0,T;H^1)} + \|\beta_\delta(\varphi_\delta)\|_{L^2(0,T;L^2)} \leq C. \tag{8.61}$$

From Lemma 2.39 there exists a solution $\mathbf{q}_\delta \in \mathbf{H}^1$ to the problem

$$\begin{cases}
\operatorname{div}(\mathbf{q}_\delta) = p_\delta & \text{in } \Omega, \\
\mathbf{q}_\delta = \frac{1}{|\partial\Omega|}\Big(\int_\Omega p_\delta \,\mathrm{d}x \Big)\mathbf{n} & \text{on } \partial\Omega,
\end{cases}$$

satisfying for a positive constant C depending only on Ω the estimate

$$\|\mathbf{q}_\delta\|_{\mathbf{H}^1} \leq C \|p_\delta\|_{L^2}. \tag{8.62}$$

Then, testing (8.22c) with $\mathbf{\Phi} = \mathbf{q}_\delta$ yields

$$\begin{aligned}
\|p_\delta\|_{L^2}^2 \leq\; & 2\delta \|\mathbf{D}\mathbf{v}_\delta\|_{\mathbf{L}^2} \|\mathbf{D}\mathbf{q}_\delta\|_{\mathbf{L}^2} + \delta \|\Gamma_{\mathbf{v}}(\varphi_\delta, \sigma_\delta)\|_{L^2} \|p_\delta\|_{L^2} \\
& + \nu \|\mathbf{v}_\delta\|_{\mathbf{L}^2} \|\mathbf{q}_\delta\|_{\mathbf{L}^2} + \|\mu_\delta + \chi \sigma_\delta\|_{L^3} \|\nabla \varphi_\delta\|_{\mathbf{L}^2} \|\mathbf{q}_\delta\|_{\mathbf{L}^6}.
\end{aligned}$$

Applying Young's inequality and using the uniform estimates (8.24), (8.57), (8.61) and (8.62) leads to

$$\|p_\delta\|_{L^2(0,T;L^2)} \leq C. \tag{8.63}$$

Then, in addition to the above compactness assertions, we further deduce that

$$\begin{aligned} \mu_\delta &\to \mu \quad \text{weakly} \quad \text{in } L^2(0,T;H^1), \\ p_\delta &\to p \quad \text{weakly} \quad \text{in } L^2(0,T;L^2). \end{aligned}$$

The arguments to recover (5.5b), (5.5d) and (8.4) (resp. (8.5)) for the double obstacle (resp. logarithmic) case in the limit $\delta \to 0$ proceed as in the proof of Theorem 8.3, whereas recovery of $(8.11)_1$, $(8.11)_2$, the improved regularity $p \in L^{\frac{8}{5}}(0,T;H^1)$ and the boundary condition (8.10) follow from similar arguments as outlined in Chapter 6.

8.4 Proof of Proposition 8.6 – Stationary solutions

As with the time-dependent case, we extend b_φ and f_φ from $[-1,1]$ to $\mathbb{R}$ such that $f_\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $b_\varphi \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is non-negative and fulfil (8.16), (8.17) and (8.21) with $f_{\mathbf{v}}(\cdot) \equiv 0$, $\mathbf{b}_{\mathbf{v}}(\cdot) \equiv 0$.

8.4.1 Basics for nonlinear monotone operators

Since the proof of Proposition 8.6 is based on the theory of nonlinear monotone operators, we need the following definition that can be found in, e. g., [139, Def. 25.2., 26.1., 27.5.]:

Definition 8.15 Let X be a real Banach space and let $A\colon X \to X^*$ be an operator.

(i) We call A *monotone* iff

$$\langle Au - Av\,, u - v\rangle_X \geq 0 \quad \forall\, u, v \in X.$$

(ii) We call A *hemicontinuous* iff the real function

$$t \mapsto \langle A(u + tv)\,, w\rangle_X$$

is continuous on $[0,1]$ for all u, v, $w \in X$.

(iii) We call A *strongly continuous* iff

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{as } n \to \infty \Longrightarrow Au_n \to Au \quad \text{in } X^* \quad \text{as } n \to \infty.$$

(iv) We call A *coercive* iff

$$\lim_{\|u\|_X \to +\infty} \frac{\langle Au\,, u\rangle_X}{\|u\|_X} = +\infty.$$

(v) We call A *pseudomonotone* iff

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{as } n \to \infty \quad \text{and} \quad \limsup_{n\to\infty} \langle Au_n\,, u_n - u\rangle_X \leq 0$$

implies

$$\langle Au\,, u - w\rangle_X \leq \liminf_{n\to\infty} \langle Au_n\,, u_n - w\rangle_X \quad \forall\, w \in X.$$

Furthermore, we need the following lemma, see [139, Prop. 27.6.(f), Thm. 27.A] for a proof:

Lemma 8.16 *Let X be a real Banach space and let A, $B\colon X \to X^*$ be operators. Then, the following statements hold true:*

(i) if A is monotone and hemicontinuous and B is strongly continuous, then $A + B$ is pseudomonotone.

(ii) if A is pseudomonotone, bounded and coercive, then, for each $b \in X^*$ there exists $u \in X$ such that $Au = b$ in X^*.

8.4.2 Approximation scheme

We consider a smooth function $\hat{g} : \mathbb{R} \to [0,1]$ such that $g(r) = 1$ for $r \geq 3$ and $g(r) = 0$ for $r \leq 2$, and define $F : L^2(\Omega) \to \mathbb{R}$ as

$$F(v) := C_F \hat{g}\big(\tfrac{1}{|\Omega|}\|v\|_{L^2}^2\big) \quad \text{for } v \in L^2(\Omega),$$

where C_F is a positive constant to be specified later. We reuse the notation ψ_δ to mean $\psi_{\mathrm{do},\delta}$ for the double obstacle and $\psi_{\log,\delta}$ for the logarithmic potential. Furthermore, we denote by $\gamma(r,s)$ the function

$$\gamma(r,s) := -\Gamma_\varphi(r,s).$$

Then, we seek for a solution $\varphi_\delta \in H^2_N$ of the approximating system

$$\begin{cases} \sqrt{\delta}\beta_\delta(\varphi_\delta) + F(\varphi_\delta)\varphi_\delta + \Delta(\Delta\varphi_\delta - \psi_\delta'(\varphi_\delta) + \chi\sigma_\delta) = -\gamma(\varphi_\delta,\sigma_\delta) & \text{in } \Omega, \\ \nabla\varphi_\delta \cdot \mathbf{n} = \nabla(\Delta\varphi_\delta + \chi\sigma_\delta)\cdot\mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.64}$$

where $\sigma_\delta \in H^2$ is the unique non-negative solution to the nutrient subsystem

$$\begin{cases} 0 = \Delta\sigma_\delta - h(\varphi_\delta)\sigma_\delta & \text{in } \Omega, \\ \nabla\sigma_\delta \cdot \mathbf{n} = K(1-\sigma_\delta) & \text{on } \partial\Omega. \end{cases} \tag{8.65}$$

We aim to use pseudomonotone operator theory, akin to the methodology used in [88], to deduce the existence of at least one solution $\varphi_\delta \in H^2_N$ to (8.64) for each $\delta \in (0,1)$. Then, we derive enough uniform estimates to pass to the limit $\delta \to 0$ in order to prove Proposition 8.6.

8.4.3 Preparatory result

For $u \in H^2_N$, let σ_u denote the unique solutions to the nutrient subsystem (8.65) corresponding to $\varphi_\delta = u$.

Lemma 8.17 *For each $\delta \in (0,1)$, the operator $\mathcal{A} : H^2_N \to (H^2_N)^*$ defined as*

$$\langle \mathcal{A}u, \zeta\rangle_{H^2_N} := \int_\Omega -\chi\nabla\sigma_u \cdot \nabla\zeta + \gamma(u,\sigma_u)\zeta \,\mathrm{d}x$$

is strongly continuous.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^2_N$ be a sequence of functions such that $u_n \rightharpoonup u$ in H^2_N and denote by σ_n the corresponding unique solutions to the nutrient subsystem (8.65) where $\varphi_\delta = u_n$. Then, we easily infer that

$$\|\sigma_n\|_{H^2} \leq C, \quad \sigma_n \in [0,1] \quad \text{a.e. in } \Omega$$

for a positive constant C independent of n. Hence, for fixed $\delta \in (0,1)$, there exist a function $\sigma_u \in H^2(\Omega)$, such that along a non-relabelled subsequence, $\sigma_n \rightharpoonup \sigma_u$ in H^2 as $n \to \infty$. It is clear that σ_u is the unique solution to (8.65) corresponding to u. By Rellich's theorem and the assumptions on f_φ, b_φ, it is easy to see that

$$\int_\Omega -\chi \nabla \sigma_n \cdot \nabla \zeta + \gamma(u_n, \sigma_n)\zeta \, \mathrm{d}x \to \int_\Omega -\chi \nabla \sigma_u \cdot \nabla \zeta + \gamma(u, \sigma_u)\zeta \, \mathrm{d}x \quad \forall \, \zeta \in H^2_N.$$

This shows that $\mathcal{A}$ is strongly continuous. □

8.4.4 Existence of approximate solutions

We fix $\delta \in (0,1)$, and define operators $A_1, A_2 : H^2_N \to (H^2_N)^*$ by

$$\begin{aligned} \langle A_1 u, \zeta \rangle_{H^2_N} &:= \int_\Omega \sqrt{\delta}\beta_\delta(u)\zeta + \Delta u \, \Delta \zeta \, \mathrm{d}x \\ \langle A_2 u, \zeta \rangle_{H^2_N} &:= \int_\Omega \big(F(u)u + \gamma(u, \sigma_u)\big)\zeta + \nabla(\psi'_\delta(u) - \chi\sigma_u) \cdot \nabla \zeta \, \mathrm{d}x. \end{aligned}$$

Then, $\varphi_\delta \in H^2_N$ is a weak solution to (8.64) if $\langle (A_1 + A_2)\varphi_\delta, \zeta \rangle_{H^2_N} = 0$ for all $\zeta \in H^2_N$.
Since β'_δ is bounded and β_δ has sublinear growth, we deduce that the operator A_1 is monotone and hemicontinuous. On the other hand, Lemma 8.17 together with the continuity and sublinear growth of ψ'_δ, and the continuity and boundedness of F imply that A_2 is strongly continuous. Then, by Lemma 8.16 the sum $A = A_1 + A_2$ is a pseudomonotone operator. We now claim that A is additionally coercive on H^2_N. Indeed, using the estimate $\|\sigma_u\|_{H^2} \le C$, the assumptions on $\mathbf{b}_\varphi$, f_φ along with Hölder's and Young's inequalities, it follows

$$\begin{aligned} \langle Au, u \rangle_{H^2_N} &= \int_\Omega F(u)|u|^2 + \sqrt{\delta}\beta_\delta(u)u + |\Delta u|^2 + \beta'_\delta(u)|\nabla u|^2 - \Theta_c |\nabla u|^2 \, \mathrm{d}x \\ &\quad + \int_\Omega -\chi \nabla u \cdot \nabla \sigma_u + u\gamma(u, \sigma_u)\big) \, \mathrm{d}x \\ &\ge \int_\Omega F(u)|u|^2 + \sqrt{\delta}\beta_\delta(u)u + \tfrac{1}{2}|\Delta u|^2 + \tfrac{1}{2}\beta'_\delta(u)|\nabla u|^2 \, \mathrm{d}x \\ &\quad - C\left(1 + \|u\|^2_{L^2}\right) \end{aligned} \tag{8.66}$$

for a positive constant C independent of u and δ. Recalling that $F(u) = C_F$ for $\|u\|^2_{L^2} > 3|\Omega|$, and so, choosing $C_F = 2C$ gives

$$\langle Au, u \rangle_{H^2_N} \ge \int_\Omega \sqrt{\delta}\beta_\delta(u)u + \tfrac{1}{2}C_F|u|^2 + \tfrac{1}{2}|\Delta u|^2 \, \mathrm{d}x - C \ge c\|u\|^2_{H^2} - C$$

for $\|u\|^2_{L^2} \ge 3|\Omega|$ which in turn implies coercivity of A.

Invoking Lemma 8.16, we deduce for every $\delta \in (0,1)$ the existence of a solution $\varphi_\delta \in H^2_N$ to $A\varphi_\delta = 0$ in $(H^2_N)^*$. Setting

$$\mu_\delta = -\Delta\varphi_\delta + \psi'_\delta(\varphi_\delta) - \chi\sigma_{\varphi_\delta} \tag{8.67}$$

we see that the equation $A\varphi_\delta = 0$ in $(H^2_N)^*$ implies

$$\int_\Omega \mu_\delta \Delta\zeta \, \mathrm{d}x = \int_\Omega f_\delta \zeta \, \mathrm{d}x \quad \forall \, \zeta \in H^2_N \tag{8.68}$$

with right-hand side

$$f_\delta := \sqrt{\delta}\beta_\delta(\varphi_\delta)\varphi_\delta + F(\varphi_\delta)\varphi_\delta - \gamma(\varphi_\delta, \sigma_\delta).$$

Thanks to the regularity $\varphi_\delta \in H^2_N$, $\sigma_\delta \in H^2$, and the linear growth of β_δ, we easily infer that $f_\delta \in L^2(\Omega)$. On the other hand, choosing $\zeta = 1$ in (8.68) implies that $f_\delta \in L^2_0(\Omega)$. Then, by arguing as in [88, Sec. 3.1], we obtain that $\mu_\delta \in H^2_N$ for all $\delta \in (0,1)$. Again using elliptic regularity theory, (8.67) and the boundedness of $\sigma_\delta \in H^2$, we infer that $\varphi_\delta \in H^4$. A comparison argument in (8.67) implies that $\nabla(\Delta\varphi_\delta + \chi\sigma_{\varphi_\delta}) \cdot \mathbf{n} = 0$ a. e. on $\partial\Omega$.

8.4.5 Uniform estimates

From (8.64) and (8.68), the pair $(\varphi_\delta, \mu_\delta) \in H^2_N \times H^2_N$ satisfies

$$0 = \int_\Omega \big(F(\varphi_\delta)\varphi_\delta + \sqrt{\delta}\beta_\delta(\varphi_\delta) + \gamma(\varphi_\delta, \sigma_\delta) - \Delta\mu_\delta\big)\zeta \, \mathrm{d}x, \tag{8.69a}$$

$$0 = \int_\Omega \big(\beta_\delta(\varphi_\delta) - \Theta_c\varphi_\delta - \mu_\delta - \chi\sigma_\delta - \Delta\varphi_\delta\big)\zeta \, \mathrm{d}x \tag{8.69b}$$

for all $\zeta \in L^2$. Returning to the proof of the coercivity of the operator A, replacing u with φ_δ in (8.66) gives

$$\int_\Omega F(\varphi_\delta)|\varphi_\delta|^2 + \sqrt{\delta}\beta_\delta(\varphi_\delta)\varphi_\delta + \tfrac{1}{2}\beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 + \tfrac{1}{2}|\Delta\varphi_\delta|^2 \le \tfrac{1}{2}C_F\|\varphi_\delta\|^2_{L^2} + C, \tag{8.70}$$

where we used that $A\varphi_\delta = 0$ in $(H^2_N)^*$. If $\|\varphi_\delta\|^2_{L^2} \ge 3|\Omega|$, then as before we have

$$\int_\Omega \sqrt{\delta}\beta_\delta(\varphi_\delta)\varphi_\delta + \beta_\delta'(\varphi_\delta)|\nabla\varphi_\delta|^2 \, \mathrm{d}x + \|\varphi_\delta\|^2_{H^2} \le C. \tag{8.71}$$

If $\|\varphi_\delta\|^2_{L^2} < 3|\Omega|$, then adding $\|\varphi_\delta\|^2_{L^2}$ to both sides of (8.70) and neglecting the non-negative term $F(\varphi_\delta)|\varphi_\delta|^2$ on the left-hand side yields the uniform estimate (8.71). Hence, $\{\varphi_\delta\}_{\delta\in(0,1)}$ is bounded in H^2_N, and along a non-relabelled subsequence, it holds that

$$\varphi_\delta \rightharpoonup \varphi, \quad \sigma_\delta \rightharpoonup \sigma_\varphi \quad \text{in } H^2 \quad \text{as } \delta \to 0,$$

where σ_φ is the unique solution to the nutrient subsystem (8.65) with data φ. Convexity of $\hat{\beta}_\delta$ and $\hat{\beta}_\delta(0) = 0$ imply the inequality

$$\hat{\beta}_\delta(s) \le \beta_\delta(s)s \quad \text{for all } s \in \mathbb{R}.$$

For the double obstacle potential, we use (8.15b) and (8.71) to deduce that

$$\delta^2 \int_\Omega |\beta_{\mathrm{do},\delta}(\varphi_\delta)|^2 \, \mathrm{d}x \le 2\delta \int_\Omega \hat{\beta}_{\mathrm{do},\delta}(\varphi_\delta) \, \mathrm{d}x \le 2\delta \int_\Omega \beta_\delta(\varphi_\delta)\varphi_\delta \, \mathrm{d}x \le C\sqrt{\delta}.$$

Hence, $\delta\beta_{\mathrm{do},\delta}(\varphi_\delta) \to 0$ in L^2 as $\delta \to 0$, and by (8.44) we deduce that the limit φ satisfies

$$|\varphi| \le 1 \quad \text{a. e. in } \Omega.$$

For the logarithmic potential, we use (8.20b), (8.71) and the inequality $\hat{\beta}_\delta(s) \le \beta_\delta(s)s$ for all $s \in \mathbb{R}$ to deduce that

$$\theta \int_\Omega (|\varphi_\delta| - 1)^2_+ \, \mathrm{d}x \le 4\delta \int_\Omega \hat{\beta}_{\log,\delta}(\varphi_\delta) \, \mathrm{d}x \le 4\delta \int_\Omega \beta_{\log,\delta}(\varphi_\delta)\varphi_\delta \, \mathrm{d}x \le C\sqrt{\delta}.$$

Since $\varphi_\delta \to \varphi$ strongly in L^2 and a. e. in Ω, the limit $\delta \to 0$ yields $|\varphi| \le 1$ a. e. in Ω. In particular, we have $\|\varphi\|^2_{L^2} \le |\Omega|$. Using the norm convergence $\|\varphi_\delta\|^2_{L^2} \to \|\varphi\|^2_{L^2}$, we then infer the existence of $\delta_5 > 0$ such that $\|\varphi_\delta\|^2_{L^2} \le 2|\Omega|$ for $\delta \in (0, \delta_5)$. Consequently, $F(\varphi_\delta) = 0$ for $\delta \in (0, \delta_5)$, and in the sequel we will neglect the term $F(\varphi_\delta)\varphi_\delta$.

Choosing $\zeta = -\Delta\mu_\delta$ in (8.69b), $\zeta = \beta_\delta(\varphi_\delta)$ and also $\zeta = -\Delta\varphi_\delta$ in (8.69a) yields after summation and integrating by parts that

$$\begin{aligned}\|\nabla\mu_\delta\|_{\mathbf{L}^2}^2 + \sqrt{\delta}\|\beta_\delta(\varphi_\delta)\|_{L^2}^2 &\leq \int_\Omega \gamma(\varphi_\delta,\sigma_\delta)(\Delta\varphi_\delta - \beta_\delta(\varphi_\delta)) - \nabla(\Theta_c\varphi_\delta + \chi\sigma_\delta)\cdot\nabla\mu_\delta \,\mathrm{d}x \\ &\leq C + \frac{1}{2}\|\nabla\mu_\delta\|_{\mathbf{L}^2}^2 \end{aligned} \tag{8.72}$$

on account of the boundedness of φ_δ and σ_δ in H^2, and with the same estimates for the term $\gamma(\varphi_\delta,\sigma_\delta)\beta_\delta(\varphi_\delta)$ as above. Consequently, it holds

$$\|\nabla\mu_\delta\|_{\mathbf{L}^2}^2 + \sqrt{\delta}\|\beta_\delta(\varphi_\delta)\|_{L^2}^2 \leq C \tag{8.73}$$

which implies

$$\|\sqrt{\delta}\beta_\delta(\varphi_\delta)\|_{L^2}^2 \leq C\sqrt{\delta} \to 0 \text{ as } \delta \to 0,$$

and so $\sqrt{\delta}\beta_\delta(\varphi_\delta) \to 0$ in $L^2(\Omega)$. Then, choosing $\zeta = 1$ in (8.69a), using that $\mu_\delta \in H_N^2$ and passing to the limit $\delta \to 0$ yields

$$0 = \int_\Omega \gamma(\varphi,\sigma)\,\mathrm{d}x. \tag{8.74}$$

Then, we infer from (8.6) and (8.74) that the limit φ has mean value $\varphi_\Omega \in (-1,1)$. Indeed, substituting $\varphi = 1$ or -1 in (8.74) leads to a contradiction on account of (8.6), and as $|\varphi| \leq 1$ a. e. in Ω, we have that $\varphi_\Omega \in (-1,1)$.

Arguing as in the time-dependent case we can derive a uniform estimate on the mean value of μ_δ, and consequently

$$\|\mu_\delta\|_{H^1} + \|\beta_{\log,\delta}(\varphi_\delta)\|_{L^2} + \|\beta_{\mathrm{do},\delta}(\varphi_\delta)\|_{L^2} \leq C,$$

where the boundedness of $\beta_{\log,\delta}(\varphi_\delta)$ in $L^2(\Omega)$ implies the tighter bounds

$$|\varphi| < 1 \quad \text{a. e. in } \Omega.$$

8.4.6 Passing to the limit

In (8.69a) we take $\zeta \in H^1$ and integrate by parts to get

$$0 = \int_\Omega \big(\sqrt{\delta}\beta_\delta(\varphi_\delta) - \Gamma_{\varphi_\delta}\big)\zeta + \nabla\mu_\delta\cdot\nabla\zeta\,\mathrm{d}x \quad \forall\,\zeta \in H^1.$$

Passing to the limit $\delta \to 0$ then yields

$$\int_\Omega \nabla\mu\cdot\nabla\zeta\,\mathrm{d}x = \int_\Omega \Gamma_\varphi(\varphi,\sigma)\zeta\,\mathrm{d}x \quad \forall\,\zeta \in H^1.$$

Since this holds for all $\zeta \in H^1$ and since $\Gamma_\varphi(\varphi,\sigma) \in L^2$, we deduce that $\mu \in H_N^2$ with bounded norm and (8.8) holds. Meanwhile, (8.4) or (8.5) can be recovered in the limit $\delta \to 0$ from (8.69b) in a fashion similar to the time-dependence case, as with the recovery of (8.1e) and (8.2b), and thus the triplet (φ,μ,σ) is a stationary solution in the sense of Definition 8.5. Moreover, from the above estimates and weak lower semicontinuity of norms, we know that

$$\|\varphi\|_{H^2} + \|\mu\|_{H^2} + \|\sigma\|_{H^2} \leq C \tag{8.75}$$

which completes the proof.

9

An optimal control problem

In this chapter we analyse an optimal control problem for tumour growth, i. e., we aim to minimise a certain cost functional under constraints which are given by a system of PDEs describing the evolution of the tumour.

Optimal control problems may give valuable insights into the response of cancer cells to drug therapy and they may serve as a basis to design patient specific treatment strategies. Although this treatments may not fully eliminate the tumour, they can help to reduce the tumour to a certain size which is ideally suited for further treatments like, e. g., surgery. Mathematically, we describe the therapeutic objectives by including a final treatment target φ_f and a desired tumour evolution φ_d in the cost functional via the terms

$$\frac{\alpha_0}{2}\int_\Omega |\varphi(T)-\varphi_f|^2 \,\mathrm{d}x + \frac{\alpha_1}{2}\int_{\Omega_T} |\varphi-\varphi_d|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where α_0 and α_1 are non-negative constants which will be specified later.

Moreover, we aim to analyse the influence of cytotoxic drugs, i. e., drugs that specifically detect cancer cells and kill them while not causing too much harm to the surrounding normal tissue. Well known examples are anti-cancer T cells or natural killer cells (NK cells), see, e. g., [109, Chap. 20]. The influence of cytotoxic drugs can be modelled by an additional term $-u\mathbb{h}(\varphi)$ where u is the dose of a certain type of cytotoxic drugs and $\mathbb{h}(\cdot)$ is a function that interpolates between $\mathbb{h}(-1)=0$ and $\mathbb{h}(1)=1$.

Although cytotoxic drugs mainly affect cancer cells, they can still cause harm to the patient. It has been reported in [109, Chap. 6.6.1] that the cytotoxic therapy may cause diseases like, e. g., myelodysplastic syndrome (MDS) and acute myeloid (AML). In order to avoid those harmful consequences, we extend the cost functional by incorporating an additional term

$$\frac{\kappa}{2}\int_{\Omega_T} |u|^2 \,\mathrm{d}x\,\mathrm{d}t \qquad \text{for } \kappa\geq 0$$

that penalises high drug doses administered to the patient.

In the following we will carefully introduce the mathematical setting for the optimal control problem. The results are based on [57].

9.1 Introduction of the optimal control problem

We study an optimal control problem with the state system given by

$$
(\mathrm{CHB})\begin{cases}
\operatorname{div}(\mathbf{v}) = (\mathcal{P}\sigma - \mathcal{A})\mathbb{h}(\varphi) & \text{in } \Omega_T, & (9.1a)\\
-\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu\mathbf{v} = (\mu + \chi\sigma)\nabla\varphi & \text{in } \Omega_T, & (9.1b)\\
\partial_t\varphi + \operatorname{div}(\varphi\mathbf{v}) = m\Delta\mu + (\mathcal{P}\sigma - \mathcal{A} - u)\mathbb{h}(\varphi) & \text{in } \Omega_T, & (9.1c)\\
\mu = -\varepsilon\Delta\varphi + \varepsilon^{-1}\psi'(\varphi) - \chi\sigma & \text{on } \Omega_T, & (9.1d)\\
-\Delta\sigma + \mathbb{h}(\varphi)\sigma = \mathcal{B}(\sigma_B - \sigma) & \text{in } \Omega_T, & (9.1e)\\
\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = \nabla\sigma\cdot\mathbf{n} = 0 & \text{on } \Sigma_T, & (9.1f)\\
\mathbf{T}(\mathbf{v},p)\mathbf{n} = 0 & \text{in } \Sigma_T, & (9.1g)\\
\varphi(0) = \varphi_0 & \text{in } \Omega, & (9.1h)
\end{cases}
$$

where the viscous stress tensor is defined by

$$\mathbf{T}(\mathbf{v},p) := 2\eta\mathbf{D}\mathbf{v} + \lambda\operatorname{div}(\mathbf{v})\mathbf{I} - p\mathbf{I},$$

and the symmetric velocity gradient is given by

$$\mathbf{D}\mathbf{v} := \tfrac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}).$$

The model (9.1) is a modification of (5.1)-(5.2) with constant viscosities, permeability and mobility.
In (9.1e), we have an additional term $\mathcal{B}(\sigma_B - \sigma)$ describing the nutrient supply of a pre-existing vasculature. By σ_B we denote the nutrient concentration in the pre-existing vasculature and $\mathcal{B}$ is a positive constant related to the blood-tissue transfer. Hence, the term $\mathcal{B}(\sigma_B - \sigma)$ models the nutrient supply from the blood vessels if $\sigma_B > \sigma$ and the nutrient transport away from the domain for $\sigma_B < \sigma$ and acts to balance the differences in nutrient concentration between the tumour and its vascular system. In the avascular growth case it holds $\mathcal{B} = 0$ and we refer to, e. g., [25, 84, 137] for more details regarding this term. Furthermore, the term $-u\mathbb{h}(\varphi)$ in (9.1c) models the elimination of tumour cells by cytotoxic drugs and the function u will act as our control. Since it does not play any role in the analysis, we set $\varepsilon = 1$.

We investigate the following distributed optimal control problem:

$$\text{Minimize}\quad I(\varphi,u) := \frac{\alpha_0}{2}\|\varphi(T) - \varphi_f\|_{L^2}^2 + \frac{\alpha_1}{2}\|\varphi - \varphi_d\|_{L^2(\Omega_T)}^2 + \frac{\kappa}{2}\|u\|_{L^2(\Omega_T)}^2$$

subject to the control constraint

$$u \in \mathbb{U} := \left\{u \in L^2(L^2) \;\middle|\; a(x,t) \le u(x,t) \le b(x,t) \text{ for almost every } (x,t) \in \Omega_T\right\}$$

for box-restrictions $a, b \in L^2(L^2)$ and the state system (CHB). Here, α_0, α_1 and κ are non-negative constants such that $\alpha_0 + \alpha_1 + \kappa > 0$.

The optimal control problem can be interpreted as the search for a strategy how to supply a medication such that a desired evolution φ_d and a therapeutic target φ_f are achieved in the best possible way without causing harm to the patient (expressed by both the control constraint and the last term in the cost functional). The ratio between the parameters α_0, α_1 and κ can be adjusted according to the importance of the individual therapeutic targets. In general, it is possible to include additional terms in the cost functional, see for example [58].

In the case when $\mathbb{h}(-1) = 0$, the term $-u\mathbb{h}(\varphi)$ models the elimination of tumour cells by a supply of cytotoxic drugs represented by the control u. This specific control term has been investigated in [58] and also in [84] where a simpler model was studied in which the influence of the velocity $\mathbf{v}$ is neglected. However, in some situations it may be more reasonable to control, for instance, the evolution at the interface and one has to use a different form for $\mathbb{h}(\cdot)$, see Remark 9.2 below. Therefore, we allow $\mathbb{h}(\cdot)$ to be rather general.

We now give a short overview on this chapter.
In Section 9.2, we prove the existence of a control-to-state operator that maps any admissible control $u \in \mathbb{U}$ onto a corresponding unique strong solution of the state equation (CHB). Furthermore, we show that this control-to-state operator is Lipschitz-continuous, Fréchet differentiable and satisfies a weak compactness property. In particular, we establish the fundamental requirements for calculus of variations.

In Section 9.3, we investigate the adjoint system. Its solution, that is called the adjoint state or the costate, is an important tool in optimal control theory as it provides a better description of optimality conditions. We prove the existence of a control-to-costate operator which maps any admissible control onto its corresponding adjoint state. Then, we show that this control-to-costate operator is Lipschitz continuous and Fréchet differentiable.

Eventually, in Section 9.4, we investigate the above optimal control problem. First, we show that there exists at least one globally optimal solution. After that, we establish first-order necessary conditions for local optimality. These conditions are of great importance for possible numerical implementations as they provide the foundation for many computational optimization methods. We also present a second-order sufficient condition for strict local optimality, a globality criterion for critical controls and a uniqueness result for the optimal control on small time intervals.

9.1.1 Preliminaries

First, we introduce the function spaces

$$
\begin{aligned}
\mathcal{V}_1 &:= \big(H^1(L^2) \cap L^\infty(H^2) \cap L^2(H^4)\big) \times \big(L^\infty(L^2) \cap L^2(H^2)\big) \times L^\infty(H^2) \times L^8(\mathbf{H}^2) \times L^8(H^1),\\
\mathcal{V}_2 &:= L^8(\mathbf{L}^2) \times L^8(H^1) \times L^2(L^2) \times \big(L^\infty(L^2) \cap L^2(H^2)\big) \times L^\infty(L^2),\\
V_3 &:= \big(H^1((H^1)^*) \cap L^\infty(H^1) \cap L^2(H^3)\big) \times L^2(H^1) \times L^2(H^2) \times L^2(\mathbf{H}^2) \times L^2(H^1),\\
\mathcal{V}_4 &:= H^1 \times L^2(L^{6/5}) \times L^2(H^1) \times L^2(L^2) \times L^2(\mathbf{H}^1) \times L^2(\mathbf{L}^2)
\end{aligned}
$$

endowed with their standard norms.

Assumptions 9.1 *For the rest of this chapter, we make the following assumptions.*

(A1) *The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is bounded with C^4-boundary $\partial\Omega$. Moreover, the initial datum $\varphi_0 \in H^2_N$ and $\sigma_B \in C([0,T]; L^2)$ are given functions.*

(A2) *The constants T, η, ν, m, $\mathcal{B}$ are positive and the constants $\mathcal{P}$, $\mathcal{A}$, λ, χ are non-negative.*

(A3) *The non-negative function $\mathbb{h}$ belongs to $C^3_b(\mathbb{R})$, i. e., $\mathbb{h}$ is bounded, three times continuously differentiable and its first, second and third-order derivatives are bounded. Without loss of generality, we assume that $|\mathbb{h}| \le 1$.*

(A4) *The function ψ is the smooth double-well potential, i. e., $\psi(s) := \frac{1}{4}\left(s^2 - 1\right)^2$ for all $s \in \mathbb{R}$.*

Remark 9.2 (a) In principle, it would be possible to consider more general potentials $\psi(\cdot)$. However, since the double-well potential is the classical choice for Cahn–Hilliard type equations (apart from singular potentials like the logarithmic or double-obstacle potential) and to avoid being too technical, we focus on the above choice for ψ.

(b) For the function $\mathbb{h}(\cdot)$, there are two choices which are quite popular in the literature. In, e. g., [82, 87], the choice for $\mathbb{h}$ is given by

$$\mathbb{h}(\varphi) = \max\left(0, \min\left(1, \tfrac{1}{2}(1+\varphi)\right)\right) \quad \forall\, \varphi \in \mathbb{R},$$

satisfying $\mathbb{h}(-1) = 0$, $\mathbb{h}(1) = 1$. Other authors preferred to assume that $\mathbb{h}$ is only active on the interface, i. e., for values of φ between -1 and 1, which motivates functions of the form

$$\mathbb{h}(\varphi) = \max\left(0, \tfrac{1}{2}\left(1-\varphi^2\right)\right) \quad \text{or} \quad \mathbb{h}(\varphi) = \tfrac{1}{2}\left(\cos\left(\pi \min\left(1, \max\left(\varphi, -1\right)\right)\right) + 1\right),$$

see, e. g., [103, 106]. Surely, we would have to use regularised versions of these choices to fulfil (A3).

9.2 The control-to-state operator and its properties

We consider the system (CHB) as presented at the beginning. The first step is to define a set of controls that are admissible for our problem. Then, we show that each of these admissible controls induces a unique strong solution (the so-called **state**) of the system (CHB). Thus, we can define a control-to-state operator which maps any admissible control onto its corresponding state. We show that this operator has several important properties that are essential for calculus of variations: it is Lipschitz-continuous, Fréchet-differentiable and weakly compact in some suitable sense.

9.2.1 The set of admissible controls

The set of admissible controls is defined as follows:

Definition 9.3 Let $a, b \in L^2(L^2)$ be arbitrary fixed functions with $a \leq b$ almost everywhere in Ω_T. Then, the set

$$\mathbb{U} := \left\{ u \in L^2(L^2) \;\middle|\; a(x,t) \leq u(x,t) \leq b(x,t) \text{ for almost every } (x,t) \in \Omega_T \right\}$$

is referred to as the **set of admissible controls**. Its elements are called **admissible controls**.

Note that this box-restricted set of admissible controls $\mathbb{U}$ is a non-empty, bounded subset of the Hilbert space $L^2(L^2)$ since for all $u \in \mathbb{U}$,

$$\|u\|_{L^2(L^2)} < \|a\|_{L^2(L^2)} + \|b\|_{L^2(L^2)} + 1 =: R.$$

This means that

$$\mathbb{U} \subsetneq \mathbb{U}_R \quad \text{with} \quad \mathbb{U}_R := \left\{ u \in L^2(L^2) \;\middle|\; \|u\|_{L^2(L^2)} < R \right\}.$$

Obviously, the set $\mathbb{U}$ is also convex and closed in $L^2(L^2)$. Therefore, it is weakly sequentially compact (see [135, Thm. 2.11]).

9.2.2 Strong solutions and uniform bounds

We can show that the system (CHB) has a unique strong solution for every control $u \in \mathbb{U}_R$:

Proposition 9.4 *Let $u \in \mathbb{U}_R$ be arbitrary. Then, there exists a strong solution quintuple $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u) \in \mathcal{V}_1$ of (CHB) in the sense that $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ fulfils (CHB) almost everywhere in the respective sets. Moreover, every strong solution $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ satisfies the bounds*

$$\|(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)\|_{\mathcal{V}_1} \leq C_1 \tag{9.2}$$

for a constant $C_1 > 0$ independent of $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u, u)$.

Proof. The assertion follows with slight modifications in the proofs of Theorems 5.5 and 5.11. The estimates can be derived rigorously within a Galerkin scheme where the control u has to be approximated by a sequence $\{u_n\}_{n\in\mathbb{N}} \in C^0([0,T];L^2)$ such that $u_n \to u$ in $L^2(L^2)$. For a better readability, we drop the index n and we sketch the main differences to the proofs of Theorems 5.5 and 5.11 in the following:

Step 1: Testing (9.1e) with σ, using (9.1f), the non-negativity of $\mathbb{h}(\cdot)$ and Hölder's and Young's inequalities, we obtain

$$\int_\Omega |\nabla\sigma|^2 \,\mathrm{d}x + \mathcal{B}\int_\Omega |\sigma|^2 \mathrm{d}x \leq \frac{\mathcal{B}}{2}\int_\Omega |\sigma|^2 + |\sigma_B|^2 \,\mathrm{d}x,$$

meaning

$$\|\sigma\|_{H^1} \leq C\|\sigma_B\|_{L^2}. \tag{9.3}$$

Testing (9.1c) with $\mu + \chi\sigma$, it turns out that we have to estimate an additional term given by $-\int_\Omega u\mathbb{h}(\varphi)(\mu+\chi\sigma)\,\mathrm{d}x$. Using Hölder's, Young's and Poincaré's inequalities, we obtain

$$\begin{aligned}\left|\int_\Omega u\mathbb{h}(\varphi)(\mu+\chi\sigma)\,\mathrm{d}x\right| &= \left|\int_\Omega u\mathbb{h}(\varphi)\big((\mu+\chi\sigma-(\mu_\Omega-\chi\sigma_\Omega)\big)\,\mathrm{d}x + (\mu_\Omega+\chi\sigma_\Omega)\int_\Omega u\mathbb{h}(\varphi)\,\mathrm{d}x\right| \\ &\leq \frac{1}{4\delta}\|\mathbb{h}(\varphi)\|_{L^\infty}^2\|u\|_{L^2}^2 + \delta C_P^2\|\nabla(\mu+\chi\sigma)\|_{\mathbf{L}^2}^2 + |\mu_\Omega+\chi\sigma_\Omega|\,\|\mathbb{h}(\varphi)\|_{L^\infty}\|1\|_{L^2}\|u\|_{L^2}\end{aligned} \tag{9.4}$$

for $\delta > 0$ arbitrary, where C_P is the constant arising in Poincaré's inequality. Testing (9.1d) with 1, using (9.1f) and the assumptions on $\psi(\cdot)$, we obtain

$$|\mu_\Omega + \chi\sigma_\Omega| \leq C\left(1 + \|\psi(\varphi)\|_{L^1}\right).$$

Applying this inequality and (9.3) in (9.4), using the boundedness of $\mathbb{h}(\cdot)$ and Young's inequality, we obtain

$$\left|\int_\Omega u\mathbb{h}(\varphi)(\mu+\chi\sigma)\,\mathrm{d}x\right| \leq C_\delta\left(1+\|u\|_{L^2}^2\right)\left(1+\|\psi(\varphi)\|_{L^1}\right) + C_\delta\|\sigma_B\|_{L^2}^2 + 2\delta C_P^2\|\nabla\mu\|_{\mathbf{L}^2}^2.$$

Then, the first two terms on the right-hand side of this inequality can be controlled via a Gronwall argument, whereas the last term can be absorbed into the left-hand side of an energy identity. Then, with exactly the same arguments as in the proofs of Theorems 5.5 and 5.11 it follows that

$$\begin{aligned}&\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^4(H^2)\cap L^2(H^3)} + \|\sigma\|_{L^4(H^1)} + \|\mu\|_{L^2(H^1)\cap L^4(L^2)} \\ &\quad + \|\mathrm{div}(\varphi\mathbf{v})\|_{L^2(L^2)} + \|\mathbf{v}\|_{L^{\frac{8}{3}}(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \leq C\end{aligned} \tag{9.5}$$

with a constant C independent of $n \in \mathbb{N}$

Step 2: Now, we establish higher order estimates. Using elliptic regularity theory, the assumptions on $\mathbb{h}(\cdot)$ and σ_B, (9.1e)-(9.1g), (9.3) and (9.5), it is easy to check that

$$\|\sigma\|_{L^\infty(H^2)} \leq C. \tag{9.6}$$

Together with the boundedness of $\mathbb{h}(\cdot)$ and the Sobolev embedding $H^2 \subset L^\infty$, this implies

$$\|\sigma\|_{L^\infty(L^\infty)} + \|\mathrm{div}(\mathbf{v})\|_{L^\infty(L^\infty)} + \|(\mathcal{P}\sigma - \mathcal{A})\mathbb{h}(\varphi)\|_{L^\infty(L^\infty)} \le C. \tag{9.7}$$

Testing (9.1c) with $\Delta^2\varphi$, (9.1d) with $m\Delta^3\varphi$, integrating by parts and summing the resulting identities, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi|^2 \,\mathrm{d}x + m\int_\Omega |\Delta^2\varphi|^2 \,\mathrm{d}x &= \int_\Omega \big((\mathcal{P}\sigma - \mathcal{A} - u)\mathbb{h}(\varphi) - \mathrm{div}(\varphi\mathbf{v})\big)\Delta^2\varphi \,\mathrm{d}x \\ &\quad + \int_\Omega m\Delta(\psi'(\varphi) - \chi\sigma)\Delta^2\varphi \,\mathrm{d}x. \end{aligned}\tag{9.8}$$

Here, we used Corollary 9.5 below to deduce that

$$\int_\Omega \partial_t\varphi\, \Delta^2\varphi \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi|^2 \,\mathrm{d}x \quad \forall\, t \in (0,T).$$

Due to Hölder's and Young's inequalities and (9.5)-(9.7), the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ and elliptic estimates, it follows that

$$\begin{aligned}\left|\int_\Omega \big((\mathcal{P}\sigma - \mathcal{A} - u)\mathbb{h}(\varphi) - \mathrm{div}(\varphi\mathbf{v}) - m\chi\Delta\sigma\big)\Delta^2\varphi \,\mathrm{d}x\right| &\le C\left(1 + \|u\|_{L^2}^2\right) + C\|v\|_{\mathbf{H}^1}^2\left(1 + \|\Delta\varphi\|_{L^2}^2\right) \\ &\quad + \frac{m}{4}\|\Delta^2\varphi\|_{L^2}^2.\end{aligned}$$

Now, we observe that

$$\Delta(\psi'(\varphi)) = \psi'''(\varphi)|\nabla\varphi|^2 + \psi''(\varphi)\Delta\varphi.$$

Using Hölder's, Young's and Gagliardo–Nirenberg's inequalities, the assumptions on $\psi(\cdot)$, elliptic regularity theory and (9.5), we obtain

$$\begin{aligned}\left|\int_\Omega m\Delta(\psi'(\varphi))\Delta^2\varphi \mathrm{d}x\right| &\le C\left(1 + \|\varphi\|_{L^\infty}^4 + \|\varphi\|_{L^\infty}^2\|\nabla\varphi\|_{\mathbf{L}^3}^2\right)\left(1 + \|\Delta\varphi\|_{L^2}^2\right) + \frac{m}{2}\|\Delta^2\varphi\|_{L^2}^2 \\ &\le C\left(1 + \|\varphi\|_{H^3}^2\right)\left(1 + \|\Delta\varphi\|_{L^2}^2\right) + \frac{m}{2}\|\Delta^2\varphi\|_{L^2}^2.\end{aligned}$$

Invoking the last two inequalities in (9.8), recalling (9.5) and using elliptic regularity theory, a Gronwall argument yields

$$\|\varphi\|_{L^\infty(H^2)} + \|\varphi\|_{L^2(H^4)} \le C.$$

Then, using the equation for μ given by (9.1d) yields

$$\|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} \le C.$$

Employing the relation (9.1c) for $\partial_t\varphi$ gives

$$\|\partial_t\varphi\|_{L^2(L^2)} \le C.$$

Using the previous estimates, the assumptions on $\mathbb{h}(\cdot)$ and Gagliardo–Nirenberg's inequality, it is easy to check that

$$\|(\mathcal{P}\sigma - \mathcal{A})\mathbb{h}(\varphi)\|_{L^8(H^1)} \le C, \qquad \|(\mu + \chi\sigma)\nabla\varphi\|_{L^8(\mathbf{L}^2)} \le C.$$

Due to Proposition 2.50, this implies

$$\|\mathbf{v}\|_{L^8(\mathbf{H}^2)} + \|p\|_{L^8(H^1)} \le C$$

which completes the proof. □

The following corollary shows that the φ-component of a strong solution quintuple has a representative that is continuous in Ω_T.

Corollary 9.5 *Let $u \in \mathbb{U}_R$ and $\varphi_0 \in H^2_N(\Omega)$ be arbitrary and let $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ denote the strong solution of the system (CHB). Then, φ_u satisfies*

$$\varphi_u \in C([0,T]; H^2_N), \quad \varphi_u \in C(\overline{\Omega_T}) \quad \textit{with} \quad \|\varphi_u\|_{C([0,T];H^2)\cap C(\overline{\Omega_T})} \leq C_2$$

for some constant $C_2 > 0$ independent of φ_u and u.

Proof. We define the functional $\mathcal{J}\colon L^2 \to \mathbb{R}$ by

$$\mathcal{J}(v) = \begin{cases} \frac{1}{2}\int_\Omega |\Delta v|^2 + |v|^2 \,\mathrm{d}x & \text{if } v \in H^2_N, \\ +\infty & \text{else.} \end{cases}$$

Since $\frac{1}{2}\int_\Omega |\Delta v|^2 + |v|^2 \,\mathrm{d}x$ defines a norm on $H^2_N \subset L^2$, it is straightforward to check that $\mathcal{J}$ is convex, lower semi-continuous and proper on L^2. Define $\eta = \Delta^2\varphi_u + \varphi_u \in L^2(L^2)$. We claim that $\eta(t) \in \partial\mathcal{J}(\varphi_u(t))$ for a.e. $t \in (0,T)$ which is equivalent to

$$\mathcal{J}(\varphi_u(t)) + \int_\Omega \eta(t)(y - \varphi_u(t)) \,\mathrm{d}x \leq \mathcal{J}(y) \quad \forall\, y \in L^2 \text{ and a.e. } t \in (0,T). \tag{9.9}$$

If $y \in L^2 \backslash H^2_N$, this is trivial. If $y \in H^2_N$, integrating by parts and using φ_u, $\Delta\varphi_u \in H^2_N$ for a.e. $t \in (0,T)$, after rearranging we observe that (9.9) is equivalent to

$$\int_\Omega \Delta\varphi_u(t)\Delta y + \varphi_u(t) y \,\mathrm{d}x \leq \mathcal{J}(y) + \mathcal{J}(\varphi_u(t)) \quad \forall\, y \in H^2_N \text{ and a.e. } t \in (0,T).$$

Using the definition of $\mathcal{J}$ together with Hölder's and Young's inequality, this implies (9.9). Applying [23, Lemma 3.3], we deduce that $\mathcal{J}(\varphi_u(\cdot)) \in C^0([0,T])$ and

$$\mathcal{J}(\varphi_u(t)) = \int_0^t \partial_t\varphi_u(\Delta^2\varphi_u + \varphi_u) \,\mathrm{d}x + \mathcal{J}(\varphi_0) \quad \forall\, t \in [0,T].$$

Hence, using elliptic regularity theory, $\varphi_0 \in H^2_N$ and (9.2), it follows that $\varphi_u \in C^0(H^2)$ and

$$\|\varphi_u\|_{C^0(H^2)} \leq C.$$

As H^2 is continuously embedded in $C(\overline{\Omega})$, it directly follows that $\varphi_u \in C(\overline{\Omega_T})$ with

$$\|\varphi_u\|_{C(\overline{\Omega_T})} = \|\varphi_u\|_{L^\infty(L^\infty)} \leq C_0 \|\varphi_u\|_{L^\infty(H^2)} \leq C_0 C_1 =: C_6$$

for some constant $C_0 \geq 0$ independent of φ_u and u. This means that the second assertion is established. □

Furthermore, we can show that any control $u \in \mathbb{U}_R$ induces a unique strong solution of the system (CHB):

Theorem 9.6 *Let $u \in \mathbb{U}_R$ and $\varphi_0 \in H^2_N(\Omega)$ be arbitrary and let $(\varphi_u, \mu_u, \mathbf{v}_u, \sigma_u, p_u)$ denote the corresponding strong solution as given by Proposition 9.4. Then, this strong solution is unique.*

Proof. Let $u, \tilde{u} \in \mathbb{U}_R$ be arbitrary and let C denote a generic non-negative constant that may change its value from line to line. For brevity, we set

$$(\varphi, \mu, \mathbf{v}, \sigma, p) := (\varphi_u, \mu_u, \mathbf{v}_u, \sigma_u, p_u) - (\varphi_{\tilde{u}}, \mu_{\tilde{u}}, \mathbf{v}_{\tilde{u}}, \sigma_{\tilde{u}}, p_{\tilde{u}}),$$

where $(\varphi_u, \mu_u, \mathbf{v}_u, \sigma_u, p_u)$ and $(\varphi_{\tilde{u}}, \mu_{\tilde{u}}, \mathbf{v}_{\tilde{u}}, \sigma_{\tilde{u}}, p_{\tilde{u}})$ are strong solutions of (CHB) to the controls u and $\tilde{u}$. In particular, this means that both strong solutions satisfy the initial condition (9.1h), i. e., $\varphi_u(\cdot, 0) = \varphi_{\tilde{u}}(\cdot, 0) = \varphi_0$ holds almost everywhere in Ω.

Then, the following equations are satisfied:

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= \mathcal{P}\sigma \mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_{\tilde{u}} - \mathcal{A})(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})) && \text{in } \Omega_T, && (9.10a)\\
-\operatorname{div}(\mathbf{T}(\mathbf{v}, p)) + \nu \mathbf{v} &= (\mu + \chi\sigma)\nabla\varphi_u + (\mu_{\tilde{u}} + \chi\sigma_{\tilde{u}})\nabla\varphi && \text{in } \Omega_T, && (9.10b)\\
\partial_t \varphi + \operatorname{div}(\varphi_u \mathbf{v}) + \operatorname{div}(\varphi \mathbf{v}_{\tilde{u}}) &= m\Delta\mu - (u\mathbb{h}(\varphi_u) - \tilde{u}\mathbb{h}(\varphi_{\tilde{u}})) && && \\
&\quad + \mathcal{P}\sigma \mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_{\tilde{u}} - \mathcal{A})(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})) && \text{in } \Omega_T, && (9.10c)\\
\mu &= -\Delta\varphi + (\psi'(\varphi_u) - \psi'(\varphi_{\tilde{u}})) - \chi\sigma && \text{in } \Omega_T, && (9.10d)\\
-\Delta\sigma + \mathcal{B}\sigma + \mathbb{h}(\varphi_u)\sigma &= -\sigma_{\tilde{u}}(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})) && \text{in } \Omega_T, && (9.10e)\\
\nabla\mu \cdot \mathbf{n} = \nabla\varphi \cdot \mathbf{n} = \nabla\sigma \cdot \mathbf{n} &= 0 && \text{on } \Sigma_T, && (9.10f)\\
\mathbf{T}(\mathbf{v}, p)\mathbf{n} &= 0 && \text{on } \Sigma_T, && (9.10g)\\
\varphi(0) &= 0 && \text{in } \Omega. && (9.10h)
\end{aligned}$$

Our aim is to show that $\|(\varphi, \mu, \mathbf{v}, \sigma, p)\|_{\mathcal{V}_1} = 0$ if $u = \tilde{u}$. The argumentation is split into two steps.

Step 1: First, we show that the following inequalities hold: for any $\delta > 0$ and all $u, \tilde{u} \in \mathbb{U}$,

$$\left|\left(u\mathbb{h}(\varphi_u) - \tilde{u}\mathbb{h}(\varphi_{\tilde{u}}), \varphi\right)_{L^2}\right| \le C\,\|u - \tilde{u}\|_{L^2}^2 + C\,\|\varphi\|_{L^2}^2 + C\,\|\tilde{u}\|_{L^2}\|\varphi\|_{H^1}^2, \tag{9.11}$$

$$\begin{aligned}
\left|\left(u\mathbb{h}(\varphi_u) - \tilde{u}\mathbb{h}(\varphi_{\tilde{u}}), \Delta\varphi\right)_{L^2}\right| &\le C\delta^{-1}\,\|u - \tilde{u}\|_{L^2}^2 + C\delta^{-1}\,\|\tilde{u}\|_{L^2}^2\|\varphi\|_{H^1}^2 \\
&\quad + 2\delta\|\Delta\varphi\|_{L^2}^2 + \delta\|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2.
\end{aligned} \tag{9.12}$$

To prove (9.11) and (9.12) we use that $u\mathbb{h}(\varphi_u) - \tilde{u}\mathbb{h}(\varphi_{\tilde{u}}) = (u - \tilde{u})\mathbb{h}(\varphi_u) + \tilde{u}\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})\big)$. From $\|\mathbb{h}(\varphi_u)\|_{L^\infty} \le C$ we deduce that

$$|((u - \tilde{u})\mathbb{h}(\varphi_u), \varphi)_{L^2}| \le C\,\left(\|u - \tilde{u}\|_{L^2}^2 + \|\varphi\|_{L^2}^2\right),$$

and, by Young's inequality with $\delta > 0$,

$$|((u - \tilde{u})\mathbb{h}(\varphi_u), \Delta\varphi)_{L^2}| \le C\delta^{-1}\,\|u - \tilde{u}\|_{L^2}^2 + \delta\|\Delta\varphi\|_{L^2}^2.$$

Moreover, we have

$$\left|\left(\tilde{u}\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})\big), \varphi\right)_{L^2}\right| \le C\,\|\tilde{u}\|_{L^2}\,\|\varphi\|_{L^4}^2 \le C\,\|\tilde{u}\|_{L^2}\,\|\varphi\|_{H^1}^2,$$

and, using Gagliardo–Nirenberg's and Young's inequalities together with the Sobolev embedding $H^1 \subset L^p$, $p \in [1, 6]$, we obtain that

$$\begin{aligned}
\left|\left(\tilde{u}\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})\big), \Delta\varphi\right)_{L^2}\right| &\le C\,\|\tilde{u}\|_{L^2}\,\|\varphi\|_{L^6}\,\|\Delta\varphi\|_{L^3} \\
&\le C_\delta\|\tilde{u}\|_{L^2}^2\|\varphi\|_{H^1}^2 + \delta\left(\|\Delta\varphi\|_{L^2}^2 + \|\nabla\Delta\varphi\|_{\mathbf{L}^2}^2\right)
\end{aligned}$$

for $\delta > 0$ to be chosen. Invoking the last four estimates, we obtain (9.11)-(9.12). Multiplying (9.10e) with σ, integrating by parts and using (9.10f), it follows that

$$\int_\Omega |\nabla\sigma|^2 + \mathcal{B}|\sigma|^2 + \mathbb{h}(\varphi_u)|\sigma|^2\,\mathrm{d}x = -\int_\Omega \sigma_{\tilde{u}}(h(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}}))\sigma\,\mathrm{d}x.$$

Using the assumptions on $\mathbb{h}(\cdot)$, Proposition 9.4 and Hölder's and Young's inequalities, it is therefore easy to check that

$$\|\sigma\|_{H^1} \le C\|\varphi\|_{L^2}.$$

Then, we can follow the arguments in the proof of Theorem 5.7 to deduce that

$$\begin{aligned}&\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^1)\cap L^2(H^3)} + \|\mu\|_{L^2(H^1)} + \|\sigma\|_{L^2(H^1)} + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \\ &\quad \leq C\|u-\tilde{u}\|_{L^2(L^2)}. \end{aligned} \tag{9.13}$$

Step 2: We now prove higher order estimates. Using elliptic regularity theory, Proposition 9.4, (9.13) and the assumptions on $\mathbb{h}(\cdot)$, it is easy to check that

$$\|\sigma\|_{L^\infty(H^2)} \leq C\|u-\tilde{u}\|_{L^2(L^2)}. \tag{9.14}$$

Multiplying (9.10c) with $\Delta^2\varphi$ and inserting the expression for μ given by (9.10d), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi|^2\,\mathrm{d}x + m\int_\Omega |\Delta^2\varphi|^2\,\mathrm{d}x &= \int_\Omega \big(\mathcal{P}\sigma\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_{\tilde{u}} - \mathcal{A})(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}}))\big)\Delta^2\varphi\,\mathrm{d}x \\ &\quad - \int_\Omega \big(\mathrm{div}(\varphi_u\mathbf{v}) + \mathrm{div}(\varphi\mathbf{v}_{\tilde{u}})\big)\Delta^2\varphi\,\mathrm{d}x \\ &\quad + m\int_\Omega \Delta\big((\psi'(\varphi_u) - \psi'(\varphi_{\tilde{u}})) - \chi\sigma\big)\Delta^2\varphi\,\mathrm{d}x \\ &\quad - \int_\Omega (u\mathbb{h}(\varphi_u) - \tilde{u}\mathbb{h}(\varphi_{\tilde{u}}))\Delta^2\varphi\,\mathrm{d}x, \end{aligned} \tag{9.15}$$

where we used that, for almost every $t \in (0,T)$,

$$\int_\Omega \partial_t\varphi\,\Delta^2\varphi\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi|^2\,\mathrm{d}x \quad \forall\,\varphi \in H^2(L^2)\cap L^2(H^4), \quad \nabla\varphi\cdot\mathbf{n} = \nabla\Delta\varphi\cdot\mathbf{n} = 0 \text{ on } \partial\Omega.$$

Using Proposition 9.4 and (9.13)-(9.14) together with Hölder's and Young's inequalities, it follows that

$$\begin{aligned}&\left|\int_\Omega \big(\mathcal{P}\sigma\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_{\tilde{u}} - \mathcal{A})(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde{u}})) - \mathrm{div}(\varphi_u\mathbf{v}) - \mathrm{div}(\varphi\mathbf{v}_{\tilde{u}}) - m\chi\Delta\sigma\big)\Delta^2\varphi\,\mathrm{d}x\right| \\ &\quad \leq C\left(\|\mathbf{v}\|_{\mathbf{H}^1}^2 + \|\varphi\|_{H^1}^2\|\mathbf{v}_{\tilde{u}}\|_{\mathbf{H}^2}^2 + \|\sigma\|_{H^2}^2 + \|\varphi\|_{L^2}^2\right) + \tfrac{m}{8}\|\Delta^2\varphi\|_{L^2}^2. \end{aligned} \tag{9.16}$$

Using the Sobolev embedding $H^2 \subset L^\infty$, Proposition 9.4, (9.13), the assumptions on $\mathbb{h}(\cdot)$ and the elliptic estimate

$$\|\varphi\|_{H^2} \leq C\left(\|\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2}\right) \quad \forall\,\varphi \in H^2_N,$$

we obtain

$$\begin{aligned}\left|\int_\Omega (u\mathbb{h}(\varphi_u)\tilde{u} - \mathbb{h}(\varphi_{\tilde{u}}))\Delta^2\varphi\,\mathrm{d}x\right| &= \left|\int_\Omega \big((u-\tilde{u})\mathbb{h}(\varphi_u) + \tilde{u}(\mathbb{h}(\varphi_u) - h(\varphi_{\tilde{u}}))\big)\Delta^2\varphi\,\mathrm{d}x\right| \\ &\leq C\left(\|u-\tilde{u}\|_{L^2} + \|\tilde{u}\|_{L^2}(\|\varphi\|_{L^2} + \|\Delta\varphi\|_{L^2})\right)\|\Delta^2\varphi\|_{L^2} \\ &\leq C\left(\|u-\tilde{u}\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2\left(\|\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2\right)\right) \\ &\quad + \tfrac{m}{8}\|\Delta^2\varphi\|_{L^2}^2. \end{aligned} \tag{9.17}$$

Now, we observe that

$$\begin{aligned}\Delta(\psi'(\varphi_u) - \psi'(\varphi_{\tilde{u}})) &= \psi''(\varphi_u)\Delta\varphi + \Delta\varphi_{\tilde{u}}(\psi''(\varphi_u) - \psi''(\varphi_{\tilde{u}})) \\ &\quad + \psi'''(\varphi_u)\left(\nabla\varphi_u + \nabla\varphi_{\tilde{u}}\right)\left(\nabla\varphi_u - \nabla\varphi_{\tilde{u}}\right) + \left(\psi'''(\varphi_u) - \psi'''(\varphi_{\tilde{u}})\right)|\nabla\varphi_{\tilde{u}}|^2. \end{aligned}$$

Due to the assumptions on $\psi(\cdot)$ and because of Proposition 9.4, it is straightforward to check that

$$\int_\Omega |\psi''(\varphi_u)\Delta\varphi|^2 + |\Delta\varphi_{\tilde{u}}(\psi''(\varphi_u) - \psi''(\varphi_{\tilde{u}}))|^2\,\mathrm{d}x \leq C\left(\|\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2\right),$$

where we used the Sobolev embedding $H^2 \subset L^\infty$ and elliptic regularity theory. With similar argument, using the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$ and the assumptions on $\psi(\cdot)$, we obtain

$$\int_\Omega |\psi'''(\varphi_u)\,(\nabla\varphi_u + \nabla\varphi_{\tilde{u}})\,(\nabla\varphi_u - \nabla\varphi_{\tilde{u}})\,|^2 + |\,(\psi'''(\varphi_u) - \psi'''(\varphi_{\tilde{u}}))\,|^2 |\nabla\varphi_{\tilde{u}}|^4 \,\mathrm{d}x \\ \leq C\left(\|\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2\right).$$

From the last two inequalities we obtain

$$\|\Delta(\psi'(\varphi_u) - \psi'(\varphi_{\tilde{u}}))\|_{L^2}^2 \leq C\left(\|\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2\right). \tag{9.18}$$

Therefore, we have

$$\left| m \int_\Omega \Delta\big((\psi'(\varphi_u) - \psi'(\varphi_{\tilde{u}}))\big)\Delta^2\varphi \,\mathrm{d}x \right| \leq C\left(\|\varphi\|_{L^2}^2 + \|\Delta\varphi\|_{L^2}^2\right) + \frac{m}{8}\|\Delta^2\varphi\|_{L^2}^2. \tag{9.19}$$

Employing (9.16)-(9.19) in (9.15), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\Delta\varphi|^2 \,\mathrm{d}x + m\int_\Omega |\Delta^2\varphi|^2 \,\mathrm{d}x \leq C\left(\|\varphi\|_{L^2}^2 + \|\tilde{u}\|_{L^2}^2\|\varphi\|_{L^2}^2 + \|\mathbf{v}_{\tilde{u}}\|_{\mathbf{H}^2}^2\|\varphi\|_{H^1}^2\right) \\ + C\left(\|\sigma\|_{H^2}^2 + \|\mathbf{v}\|_{\mathbf{H}^1}^2 + \|u - \tilde{u}\|_{L^2}^2\right) \\ + C\left(1 + \|\tilde{u}\|_{L^2}^2\right)\|\Delta\varphi\|_{L^2}^2.$$

Invoking Proposition 9.4 and eq. (9.13) and using elliptic regularity theory, a Gronwall argument yields

$$\|\varphi\|_{H^1((H^1)^*)\cap L^\infty(H^2)\cap L^2(H^3)} + \|\Delta\varphi\|_{L^2(H^2)} + \|\mu\|_{L^2(H^1)} + \|\sigma\|_{L^\infty(H^2)} + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \\ \leq C\|u - \tilde{u}\|_{L^2(L^2)}. \tag{9.20}$$

Using (9.18) and (9.20), invoking equation (9.10d) implies

$$\|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} \leq C\|u - \tilde{u}\|_{L^2(L^2)}.$$

Together with elliptic regularity theory and (9.10d), (9.10f), we therefore obtain

$$\|\varphi\|_{L^2(H^4)} \leq C\|u - \tilde{u}\|_{L^2(L^2)}.$$

From the previous two bounds along with the relation (9.10c) for $\partial_t\varphi$, we infer that

$$\|\partial_t\varphi\|_{L^2(L^2)} \leq C\|u - \tilde{u}\|_{L^2(L^2)}.$$

Using the last three estimates and (9.20), we obtain

$$\|\varphi\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^4)} + \|\mu\|_{L^\infty(L^2)\cap L^2(H^2)} + \|\sigma\|_{L^\infty(H^2)} + \|\mathbf{v}\|_{L^2(\mathbf{H}^1)} + \|p\|_{L^2(L^2)} \\ \leq C\|u - \tilde{u}\|_{L^2(L^2)}. \tag{9.21}$$

Together with the assumptions on $\mathbb{h}(\cdot)$ and Gagliardo–Nirenberg's inequality, it follows that

$$\|\mathrm{div}(\mathbf{v})\|_{L^8(H^1)} \leq C\|u - \tilde{u}\|_{L^2(L^2)}, \quad \|(\mu + \chi\sigma)\nabla\varphi_u + (\mu_{\tilde{u}} + \chi\sigma_{\tilde{u}})\nabla\varphi\|_{L^8(\mathbf{L}^2)} \leq C\|u - \tilde{u}\|_{L^2(L^2)}.$$

Then, an application of Proposition 2.50 yields

$$\|\mathbf{v}\|_{L^8(\mathbf{H}^2)} + \|p\|_{L^8(H^1)} \leq C\|u - \tilde{u}\|_{L^2(L^2)}.$$

Together with (9.21), this implies that

$$\|(\varphi, \mu, \sigma, \mathbf{v}, p)\|_{\mathcal{V}_1} \leq C\|u - \tilde{u}\|_{L^2(L^2)}. \tag{9.22}$$

Hence, setting $u = \tilde{u}$ completes the proof. □

Due to Proposition 9.4 and Theorem 9.6, we can define an operator that maps any control $u \in \mathbb{U}_R$ onto its corresponding state.

Definition 9.7 For any $u \in \mathbb{U}_R$ we write $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ to denote the corresponding unique strong solution of (CHB) given by Proposition 9.4. Then, the operator

$$\mathbb{S}: \mathbb{U}_R \to \mathcal{V}_1, \quad u \mapsto \mathbb{S}(u) := (\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$$

is called the **control-to-state** operator.

Remark 9.8 The control-to-state operator is defined not only for admissible controls but for all controls in $\mathbb{U}_R$. This will be especially important in subsection 3.4 because Fréchet differentiability is merely defined for open subsets of $L^2(L^2)$. Unlike the open ball $\mathbb{U}_R$, the set $\mathbb{U}$ is closed and its interior is empty. Therefore, it makes sense to investigate the control-to-state operator on the open superset $\mathbb{U}_R$ instead.

In the following we establish some properties of the control-to-state operator that are essential for the treatment of optimal control problems.

9.2.3 Lipschitz continuity

The proof of Theorem 9.6 does actually provide more than uniqueness of strong solutions of (CHB). In fact, we have showed the Lipschitz-continuity of the control-to-state operator.

Corollary 9.9 *The control-to-state operator* $\mathbb{S}\colon \mathbb{U}_R \to \mathcal{V}_1$ *is Lipschitz continuous, i. e., for all* $u, \tilde{u} \in \mathbb{U}_R$ *it holds*

$$\|\mathbb{S}(u) - \mathbb{S}(\tilde{u})\|_{\mathcal{V}_1} \leq L_1 \|u - \tilde{u}\|_{L^2(L^2)} \tag{9.23}$$

for a positive constant $L_1 > 0$ *independent of* u *and* $\tilde{u}$.

Proof. The assertion follows directly from (9.22). □

9.2.4 A weak compactness property

As the control-to-state operator is nonlinear, the following result will be essential to prove existence of an optimal control:

Lemma 9.10 *Suppose that* $(u_k)_{k\in\mathbb{N}} \subset \mathbb{U}$ *converges weakly in* $L^2(L^2)$ *to some limit* $\bar{u} \in \mathbb{U}$. *Then, as* $k \to \infty$,

$$\begin{aligned}
&\varphi_{u_k} \rightharpoonup \varphi_{\bar{u}} \quad \text{in } H^1(L^2) \cap L^2(H^4), && \varphi_{u_k} \to \varphi_{\bar{u}} \quad \text{in } C\big([0,T];W^{1,r}\big) \cap C\big(\overline{\Omega_T}\big), \quad r \in [1,6),\\
&\mu_{u_k} \rightharpoonup \mu_{\bar{u}} \quad \text{in } L^2(H^2), && \mathbf{v}_{u_k} \rightharpoonup \mathbf{v}_{\bar{u}} \quad \text{in } L^2(\mathbf{H}^2),\\
&\sigma_{u_k} \rightharpoonup \sigma_{\bar{u}} \quad \text{in } L^2(H^2), && p_{u_k} \rightharpoonup p_{\bar{u}} \quad \text{in } L^2(H^1)
\end{aligned}$$

after extraction of a subsequence, where the limit $(\varphi_{\bar{u}}, \mu_{\bar{u}}, \sigma_{\bar{u}}, \mathbf{v}_{\bar{u}}, p_{\bar{u}})$ *is the strong solution of (CHB) to the control* $\bar{u} \in \mathbb{U}$.

Proof. Using the uniform bounds that were established in Proposition 9.4 and standard compactness arguments, we can conclude that there exist functions φ, $\mathbf{v}$, μ, σ and p having the desired regularity such that

$$\varphi_{u_k} \overset{*}{\rightharpoonup} \varphi \quad \text{in } H^1(L^2) \cap L^\infty(H^2) \cap L^2(H^4), \quad \mu_{u_k} \rightharpoonup \mu \quad \text{in } L^2(H^2),$$
$$\sigma_{u_k} \rightharpoonup \sigma \quad \text{in } L^2(H^2), \quad \mathbf{v}_{u_k} \rightharpoonup \mathbf{v} \quad \text{in } L^2(H^2), \quad p_{u_k} \rightharpoonup p \quad \text{in } L^2(H^1)$$

up to a subsequence. The Aubin–Lions lemma (see Lemma 2.36) implies that $H^1(L^2) \cap L^\infty(H^2)$ is compactly embedded in the space $C\big([0,T];W^{1,r}\big)$, $r \in [1,6)$, and thus the convergence $\varphi_{u_k} \to \varphi$ in $C\big([0,T];W^{1,r}\big)$, $r \in [1,6)$, directly follows after subsequence extraction. In particular, by the Sobolev embedding $W^{1,r} \subset C^0(\overline{\Omega})$, $r \in (3,6)$, we obtain that $\varphi_{u_k} \to \varphi$ in $C^0(\overline{\Omega_T})$, whence

$$\psi'(\varphi_{u_k}) \to \psi'(\varphi) \quad \text{and} \quad \mathbb{h}(\varphi_{u_k}) \to \mathbb{h}(\varphi) \quad \text{in } C^0(\overline{\Omega_T}) \quad \text{as } k \to \infty. \tag{9.24}$$

It remains to show that the quintuple $(\varphi, \mathbf{v}, \mu, \sigma, p)$ is a strong solution of the system (CHB) according to the control u which means it is equal to $(\varphi_{\bar u}, \mu_{\bar u}, \sigma_{\bar u}, \mathbf{v}_{\bar u}, p_{\bar u})$. Due to the above convergence properties, all linear terms in (CHB) are converging weakly in $L^2(L^2)$ to their respective limit. The nonlinear terms must be treated individually. From (9.24), we can easily conclude that

$$\sigma_{u_k}\mathbb{h}(\varphi_{u_k}) \rightharpoonup \sigma\mathbb{h}(\varphi) \quad \text{and} \quad u_k\mathbb{h}(\varphi_{u_k}) \rightharpoonup \bar u\mathbb{h}(\varphi) \quad \text{in } L^2(\Omega_T) \quad \text{as } k \to \infty$$

since $\|\mathbb{h}(\varphi)\|_{L^\infty(\Omega_T)} \le C$, $\|u_k\|_{L^2(L^2)} < R$ and $\|\sigma_{u_k}\|_{L^2(L^2)} \le C_4$. Recalling that $\varphi_{u_k} \to \varphi$ in $C^0([0,T];W^{1,r}) \cap C^0(\overline{\Omega_T})$ and $\mathbf{v}_{u_k} \rightharpoonup \mathbf{v}$ in $L^2(H^2)$ as $k \to \infty$, by the product of weak-strong convergence it follows that

$$\operatorname{div}(\varphi_{u_k}\mathbf{v}_{u_k}) \rightharpoonup \operatorname{div}(\varphi\mathbf{v}) \quad \text{in } L^2(\Omega_T) \quad \text{as } k \to \infty.$$

Now, let $\zeta \in C_0^\infty(\Omega_T)$ be arbitrary. Then, since $C_0^\infty(\Omega_T) \subset L^2(\Omega_T)$, we obtain

$$\begin{aligned}
&\int_{\Omega_T} \big(\partial_t\varphi - \operatorname{div}(\varphi\mathbf{v}) - m\Delta\mu - (\mathcal{P}\sigma - \mathcal{A} - \bar u)\mathbb{h}(\varphi)\big)\,\zeta \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \lim_{k\to\infty} \int_{\Omega_T} \big(\partial_t\varphi_{u_k} - \operatorname{div}(\varphi_{u_k}\mathbf{v}_{u_k}) - m\Delta\mu_{u_k} - (\mathcal{P}\sigma_{u_k} - \mathcal{A} - u_k)\mathbb{h}(\varphi_{u_k})\big)\,\zeta\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}$$

and consequently

$$\partial_t\varphi - \operatorname{div}(\varphi\mathbf{v}) = m\Delta\mu + (\mathcal{P}\sigma - \mathcal{A} - \bar u)\mathbb{h}(\varphi) \quad \text{a.e. in } \Omega_T.$$

We proceed analogously with the remaining equations of (CHB). This proves that $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is a strong solution of the system (CHB) to the control $\bar u$ and thus, because of uniqueness, we have $(\varphi, \mu, \sigma, \mathbf{v}, p) = (\varphi_{\bar u}, \mu_{\bar u}, \sigma_{\bar u}, \mathbf{v}_{\bar u}, p_{\bar u})$ almost everywhere in Ω_T. □

Remark 9.11 This result actually means weak compactness of the control-to-state operator restricted to $\mathbb{U}$ since any bounded sequence in $\mathbb{U}$ has a weakly convergent subsequence according to the Banach–Alaoglu theorem. However, this property can not be considered as weak continuity as the extraction of a subsequence is necessary.

9.2.5 The linearised system

We want to show that the control-to-state operator is also Fréchet differentiable on the open ball $\mathbb{U}_R$ (and therefore especially on its strict subset $\mathbb{U}$). Since the Fréchet derivative is a linear approximation of the control-to-state operator at some certain point $u \in \mathbb{U}_R$, it will be given by the linearisation of (CHB)

$$
\text{(LIN)}\begin{cases}
\operatorname{div}(\mathbf{v}) = \mathcal{P}\sigma\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi + F_1 & \text{in } \Omega_T, \quad (9.25\text{a})\\
-\operatorname{div}(\mathbf{T}(\mathbf{v},p)) + \nu\mathbf{v} = (\mu_u + \chi\sigma_u)\nabla\varphi + (\mu + \chi\sigma)\nabla\varphi_u + \mathbf{F} & \text{in } \Omega_T, \quad (9.25\text{b})\\
\partial_t\varphi + \operatorname{div}(\varphi_u\mathbf{v} + \varphi\mathbf{v}_u) = m\Delta\mu + (\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi & \\
\qquad + \mathcal{P}\sigma\mathbb{h}(\varphi_u) + F_2 & \text{in } \Omega_T, \quad (9.25\text{c})\\
\mu = -\Delta\varphi + \psi''(\varphi_u)\varphi - \chi\sigma + F_3 & \text{in } \Omega_T, \quad (9.25\text{d})\\
-\Delta\sigma + \mathcal{B}\sigma + \mathbb{h}(\varphi_u)\sigma = -\mathbb{h}'(\varphi_u)\varphi\sigma_u + F_4 & \text{in } \Omega_T, \quad (9.25\text{e})\\
\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = \nabla\sigma\cdot\mathbf{n} = 0 & \text{on } \Sigma_T, \quad (9.25\text{f})\\
\mathbf{T}(\mathbf{v},p)\mathbf{n} = 0 & \text{on } \Sigma_T, \quad (9.25\text{g})\\
\varphi(0) = 0 & \text{in } \Omega, \quad (9.25\text{h})
\end{cases}
$$

where $F_i : \Omega_T \to \mathbb{R}$, $1 \le i \le 4$ and $\mathbf{F} : \Omega_T \to \mathbb{R}^3$ are given functions that will be specified later on. A strong solution of this linearised system is defined as follows:

Definition 9.12 Let $u \in \mathbb{U}_R$ be arbitrary. Then, a quintuple $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is called a strong solution of (LIN) if it lies in $\mathcal{V}_1$ and satisfies (LIN) almost everywhere in the respective sets.

We now establish existence and uniqueness of strong solutions to the linearised system.

Proposition 9.13 *Let $u \in \mathbb{U}_R$ be any control and let $(\varphi, \mu, \sigma, \mathbf{v}, p)$ denote its corresponding state. Moreover, let $(\mathbf{F}, F_1, F_2, F_3, F_4) \in \mathcal{V}_2$ be arbitrary. Then, the system (LIN) has a unique strong solution $(\varphi, \mu, \sigma, \mathbf{v}, p) \in \mathcal{V}_1$. Moreover, it holds that*

$$
\|(\varphi, \mu, \sigma, \mathbf{v}, p)\|_{\mathcal{V}_1} \le C\|(\mathbf{F}, F_1, F_2, F_3, F_4)\|_{\mathcal{V}_2} \tag{9.26}
$$

for a constant $C > 0$ independent of $(\varphi, \mu, \sigma, \mathbf{v}, p, u)$.

Proof. The prove is divided into several steps.

Step 1: Galerkin approximation First, we remark that technically we would have to approximate the given functions $(\mathbf{F}, F_1, F_2, F_3, F_4) \in \mathcal{V}_2$ and $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u) \in \mathcal{V}_1$ with functions that are continuous in time. This could be done with similar arguments as in the proof of Proposition 9.4. To keep the notation clear, we will omit the corresponding approximation parameter in the following.

We construct approximate solutions by applying a Galerkin approximation with respect to φ and μ and at the same time solve for σ, $\mathbf{v}$ and p in the corresponding whole function spaces. As Galerkin basis for φ and μ, we will use the eigenfunctions of the Neumann-Laplace operator $\{w_i\}_{i\in\mathbb{N}}$ and we choose $w_1 = 1$. We fix $k \in \mathbb{N}$ and define $\mathcal{W}_k := \operatorname{span}\{w_1, ..., w_k\}$. Our aim is to find functions of the form

$$
\varphi_k(t,x) = \sum_{i=1}^k a_i^k(t) w_i(x), \quad \mu_k(t,x) = \sum_{i=1}^k b_i^k(t) w_i(x)
$$

satisfying for all $v \in \mathcal{W}_k$ the following approximation problem

$$\int_\Omega \partial_t \varphi_k v \, \mathrm{d}x = \int_\Omega -m\nabla\mu_k \cdot \nabla v + \big((\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi_k + \mathcal{P}\sigma_k \mathbb{h}(\varphi_u) + F_2\big) v \, \mathrm{d}x - \int_\Omega \big(\mathrm{div}(\varphi_u \mathbf{v}_k) + \mathrm{div}(\varphi_k \mathbf{v}_u)\big) v \, \mathrm{d}x, \tag{9.27a}$$

$$\int_\Omega \mu_k v \, \mathrm{d}x = \int_\Omega \nabla\varphi_k \cdot \nabla v + \big(\psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3\big) v \, \mathrm{d}x, \tag{9.27b}$$

$$\varphi_k(0,\cdot) = 0, \tag{9.27c}$$

where the nutrient concentration σ_k is defined as the unique strong solution of

$$0 = -\Delta\sigma_k + \mathcal{B}\sigma_k + \mathbb{h}'(\varphi_u)\varphi_k\sigma_u + \mathbb{h}(\varphi_u)\sigma_k - F_4 \quad \text{in } \Omega, \tag{9.27d}$$

$$\nabla\sigma_k \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{9.27e}$$

and the velocity $\mathbf{v}_k$ and the pressure p_k are defined as the strong solutions of

$$-\mathrm{div}(\mathbf{T}(\mathbf{v}_k, p_k)) + \nu\mathbf{v}_k = (\mu_u + \chi\sigma_u)\nabla\varphi_k + (\mu_k + \chi\sigma_k)\nabla\varphi_u + \mathbf{F} \quad \text{in } \Omega, \tag{9.27f}$$

$$\mathrm{div}(\mathbf{v}_k) = \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k + F_1 \quad \text{in } \Omega, \tag{9.27g}$$

$$\mathbf{T}(\mathbf{v}_k, p_k)\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{9.27h}$$

Using the continuous embedding $H^2_N \hookrightarrow L^\infty$, the assumptions on $\mathbb{h}(\cdot)$, $\mathbf{F}$, F_1 and Proposition 9.4, it is straightforward to verify that $\big((\mu_u + \chi\sigma_u)\nabla\varphi_k + (\mu_k + \chi\sigma_k)\nabla\varphi_u + \mathbf{F}\big) \in \mathbf{L}^2$ and $\big(\mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k + F_1\big) \in H^1$. Therefore, by Lemma 2.49, we obtain that $(\mathbf{v}_k, p_k) \in \mathbf{H}^2 \times H^1$ and (9.27f)-(9.27h) are fulfilled almost everywhere in the respective sets. After some straightforward calculations, it can be verified that (9.27) is equivalent to a linear system of k ODEs in the k unknowns $(a_1^k, ..., a_k^k)^\intercal =: \mathbf{a}_k$. Due to the assumptions on $\psi(\cdot)$, $\mathbb{h}(\cdot)$, the stability of (9.27d)-(9.27e) and (9.27f)-(9.27h) under perturbations, and Proposition 9.4, the theory of ODEs (see Lemma 2.29) yields the existence of a unique $\mathbf{a}_k \in W^{1,1}([0,T_k^*];\mathbb{R}^k)$ for each $k \in \mathbb{N}$ on some maximal existence interval T_k^* which may depend on $k \in \mathbb{N}$. Then, we first define σ_k as the unique strong solution of (9.27d)-(9.27e) and then $\mathbf{b}_k := (b_1^k, ..., b_k^k)^\intercal$ using (9.27b). Hence, the Galerkin scheme yields the existence of a unique solution triple $(\varphi_k, \mu_k) \in (W^{1,1}([0,T_k^*]; H^2_N \cap H^4))^2$, $\sigma_k \in L^2(0,T_k^*;H^2)$. Finally, we can define $(\mathbf{v}_k, p_k)$ as the solution of the subsystem (9.27f)-(9.27h) and, with similar arguments as above, it follows that $\mathbf{v}_k(t) \in \mathbf{H}^2$ and $p_k(t) \in H^1$ for almost every $t \in [0,T_k^*)$. We remark that $(\varphi_k, \mu_k) \in (C^0([0,T_k^*]; H^2_N \cap H^4))^2$ and (9.27a)-(9.27b), (9.27d)-(9.27e), (9.27f)-(9.27h) are fulfilled almost everywhere in $(0,T_k^*)$.

Step 2: In the following we establish a priori estimates for the solutions of (9.27a)-(9.27h). In particular, the uniform estimates will guarantee that $T_k^* = T$ for each $k \in \mathbb{N}$. We use a generic constant C which may change it's value from one line to another, but has to be independent of $k \in \mathbb{N}$, and we frequently use Hölder's and Young's inequalities.

Applying Lemma 2.39, there exists a solution $\mathbf{w}_k \in \mathbf{H}^1$ of

$$\mathrm{div}(\mathbf{w}_k) = \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k + F_1 \quad \text{in } \Omega,$$

$$\mathbf{w_k} = \left(\frac{1}{|\partial\Omega|}\int_\Omega \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k + F_1 \, \mathrm{d}x\right)\mathbf{n} \quad \text{on } \partial\Omega,$$

satisfying

$$\|\mathbf{w}_k\|_{\mathbf{H}^1} \leq C\|\mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k + F_1\|_{L^2}. \tag{9.28}$$

Then, multiplying (9.27f) with $\mathbf{v}_k - \mathbf{w}_k$, choosing $v = a_i^k(\lambda_i w_i + w_i)$ in (9.27a), $v = m a_i^k \lambda_i(\lambda_i w_i + w_i)$ in (9.27b), summing the resulting identities over $i = 1, \ldots, k$, integrating by parts and adding the resulting equations, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\varphi_k|^2 + |\nabla\varphi_k|^2 \,\mathrm{d}x + m\int_\Omega |\Delta\varphi_k|^2 + |\nabla\Delta\varphi_k|^2 \,\mathrm{d}x + \int_\Omega 2\eta|\mathbf{D}\mathbf{v}_k|^2 + \nu|\mathbf{v}_k|^2 \,\mathrm{d}x \\
&= m\int_\Omega \nabla(\psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3)\cdot\nabla\Delta\varphi_k \,\mathrm{d}x + \int_\Omega (\operatorname{div}(\varphi_u\mathbf{v}_k) + \operatorname{div}(\varphi_k\mathbf{v}_u) - F_2)\Delta\varphi_k \,\mathrm{d}x \\
&\quad - \int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi_k + \mathcal{P}\sigma_k\mathbb{h}(\varphi_u)\big)\Delta\varphi_k - m(\psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3)\Delta\varphi_k \,\mathrm{d}x \\
&\quad + \int_\Omega \big((\mu_u + \chi\sigma_u)\nabla\varphi_k + (\mu_k + \chi\sigma_k)\nabla\varphi_u + \mathbf{F}\big)\cdot(\mathbf{v}_k - \mathbf{w}_k) + 2\eta\mathbf{D}\mathbf{v}_k : \nabla\mathbf{w}_k + \nu\mathbf{v}_k\cdot\mathbf{w}_k \,\mathrm{d}x \\
&\quad + \int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi_k + \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + F_2 - \operatorname{div}(\varphi_u\mathbf{v}_k - \varphi_k\mathbf{v}_u)\big)\varphi_k \,\mathrm{d}x, \qquad (9.29)
\end{aligned}$$

where we used (9.27g)-(9.27h). In what follows, we will estimate the terms on the right-hand side of (9.29) individually.
Due to the boundedness of $\psi''(\varphi_u), \psi'''(\varphi_u) \in L^\infty(\Omega_T)$ and $\nabla\varphi_u \in L^\infty(\mathbf{L}^6)$ and the Sobolev embedding $H^1 \subset L^3$, we calculate

$$\begin{aligned}
&\left| m\int_\Omega \nabla(\psi''(\varphi_u)\varphi_k)\cdot\nabla\Delta\varphi_k \,\mathrm{d}x\right| = \left|\int_\Omega (\psi'''(\varphi_u)\varphi_k\nabla\varphi_u + \psi''(\varphi_u)\nabla\varphi_k)\cdot\nabla\Delta\varphi_k \,\mathrm{d}x\right| \\
&\quad \le C\left(\|\varphi_k\|_{L^3}\|\nabla\varphi_u\|_{\mathbf{L}^6} + \|\nabla\varphi_k\|_{\mathbf{L}^2}\right)\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2} \le C\|\varphi_k\|_{H^1}^2 + \frac{m}{16}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^2. \qquad (9.30a)
\end{aligned}$$

For the next two terms, we obtain

$$\left|\int_\Omega m\nabla(F_3 - \chi\sigma_k)\cdot\nabla\Delta\varphi_k \,\mathrm{d}x\right| \le C\left(\|\nabla F_3\|_{\mathbf{L}^2}^2 + \|\nabla\sigma_k\|_{\mathbf{L}^2}^2\right) + \frac{m}{16}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^2. \qquad (9.30b)$$

Since $\varphi_k \in H^2_N$, we know that $\|\Delta\varphi_k\|_{L^2} \le \|\nabla\varphi_k\|_{\mathbf{L}^2}^{1/2}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^{1/2}$. Applying the boundedness of $\varphi_u \in L^\infty(H^2)$, we conclude that

$$\begin{aligned}
&\left|\int_\Omega \operatorname{div}(\varphi_u\mathbf{v}_k)\Delta\varphi_k \,\mathrm{d}x\right| = \left|\int_\Omega (\nabla\varphi_u\cdot\mathbf{v}_k + \varphi_u\operatorname{div}(\mathbf{v}_k))\Delta\varphi_k \,\mathrm{d}x\right| \le C\|\varphi_u\|_{H^2}\|\mathbf{v}_k\|_{\mathbf{H}^1}\|\Delta\varphi_k\|_{L^2} \\
&\quad \le \delta_1\|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + C\|\nabla\varphi_k\|_{\mathbf{L}^2}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2} \le \delta_1\|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + \frac{m}{16}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^2 + C\|\nabla\varphi_k\|_{\mathbf{L}^2}^2 \qquad (9.30c)
\end{aligned}$$

with $\delta_1 > 0$ to be chosen later. Using the Sobolev embeddings $H^1 \subset L^3$, $\mathbf{H}^1 \subset \mathbf{L}^6$ and $\mathbf{H}^2 \subset \mathbf{L}^\infty$, we infer that

$$\begin{aligned}
&\left|\int_\Omega \operatorname{div}(\varphi_k\mathbf{v}_u)\Delta\varphi_k \,\mathrm{d}x\right| \le C\left(\|\nabla\varphi_k\|_{\mathbf{L}^2}\|\mathbf{v}_u\|_{\mathbf{L}^\infty} + \|\varphi_k\|_{L^3}\|\operatorname{div}(\mathbf{v}_u)\|_{L^6}\right)\|\Delta\varphi_k\|_{L^2} \\
&\quad \le C\|\mathbf{v}_u\|_{\mathbf{H}^2}\|\varphi_k\|_{H^1}\|\Delta\varphi_k\|_{L^2} \le C\|\mathbf{v}_u\|_{\mathbf{H}^2}^2\|\varphi_k\|_{H^1}^2 + \frac{m}{16}\|\Delta\varphi_k\|_{L^2}^2. \qquad (9.30d)
\end{aligned}$$

Since $\mathbb{h}(\varphi_u)$, $\mathbb{h}'(\varphi_u)$, $\psi''(\varphi_u) \in L^\infty(\Omega_T)$, $\sigma_u \in L^\infty(L^6)$ with bounded norm, we easily obtain that

$$\begin{aligned}
&\left|\int_\Omega \big(-F_2 - (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k - \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + m(\psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3)\big)\Delta\varphi_k \,\mathrm{d}x\right| \\
&\quad \le C\left(\|\varphi_k\|_{H^1}^2 + \|\sigma_k\|_{L^2}^2 + \|F_2\|_{L^2}^2 + \|F_3\|_{L^2}^2\right) + \frac{m}{16}\|\Delta\varphi_k\|_{L^2}^2. \qquad (9.30e)
\end{aligned}$$

With similar arguments and using the Sobolev embedding $H^1 \subset L^4$, it follows that

$$\begin{aligned}
&\left|\int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi_k + \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + F_2\big)\varphi_k \,\mathrm{d}x\right| \\
&\quad \le C\left(1 + \|u\|_{L^2}^2\right)\|\varphi_k\|_{H^1}^2 + C\left(\|\sigma_k\|_{L^2}^2 + \|F_2\|_{L^2}^2\right). \qquad (9.30f)
\end{aligned}$$

Again using the boundedness of $\mathbb{h}'(\varphi_u) \in L^\infty(\Omega_T)$, the Sobolev embedding $H^1 \subset L^6$ and Gagliardo–Nirenberg's inequality, we calculate

$$\begin{aligned}\left|\int_\Omega u\mathbb{h}'(\varphi_u)\varphi_k\Delta\varphi_k \,\mathrm{d}x\right| &\le C\|u\|_{L^2}\|\varphi_k\|_{L^6}\|\Delta\varphi_k\|_{L^3}\\ &\le C\|u\|_{L^2}\|\varphi_k\|_{L^6}\|\Delta\varphi_k\|_{L^2}^{1/2}\left(\|\Delta\varphi_k\|_{L^2}^{\frac{1}{2}} + \|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^{1/2}\right)\\ &\le C_{\delta_2,\delta_3}\|u\|_{L^2}^2\|\varphi_k\|_{L^6}^2 + \delta_2\|\Delta\varphi_k\|_{L^2}^2 + \delta_3\|\Delta\varphi_k\|_{L^2}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}\\ &\le C_{\delta_2,\delta_3}\|u\|_{L^2}^2\|\varphi_k\|_{L^6}^2 + (\delta_2+\delta_3)\|\Delta\varphi_k\|_{L^2}^2 + \tfrac{\delta_3}{4}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^2\end{aligned}$$

with $\delta_2, \delta_3 > 0$ arbitrary. Then, choosing δ_2, δ_3 sufficiently small, we conclude that

$$\left|\int_\Omega u\mathbb{h}'(\varphi_u)\varphi_k\Delta\varphi_k \,\mathrm{d}x\right| \le C\|u\|_{L^2}^2\|\varphi_k\|_{L^6}^2 + \frac{m}{16}\left(\|\Delta\varphi_k\|_{L^2}^2 + \|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^2\right). \tag{9.30g}$$

Due to the Sobolev embeddings $H^1 \subset L^p$, $\mathbf{H}^1 \subset \mathbf{L}^p$, $p \in [1,6]$, and the boundedness of $\varphi_u \in L^\infty(H^1)$, we obtain

$$\begin{aligned}\left|\int_\Omega \big(\mathrm{div}(\varphi_u\mathbf{v}_k) + \mathrm{div}(\varphi_k\mathbf{v}_u)\big)\varphi_k \,\mathrm{d}x\right| &\le C\left(\|\varphi_u\|_{H^1}\|\mathbf{v}_k\|_{\mathbf{H}^1}\|\varphi_k\|_{H^1} + \|\mathbf{v}_u\|_{\mathbf{H}^1}\|\varphi_k\|_{H^1}^2\right)\\ &\le \delta_4\|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + C_{\delta_4}\left(1 + \|\mathbf{v}_u\|_{\mathbf{H}^1}^2\right)\|\varphi_k\|_{H^1}^2\end{aligned} \tag{9.30h}$$

with $\delta_4 > 0$ to be chosen later. Next, we apply the Sobolev embeddings $H^1 \subset L^p$, $\mathbf{H}^1 \subset \mathbf{L}^p$ for $p \in [1,6]$, $H^2 \subset L^\infty$ and the boundedness of $\varphi_u \in L^\infty(H^2)$ to get

$$\begin{aligned}&\left|\int_\Omega \big((\mu_u + \chi\sigma_u)\nabla\varphi_k + (\mu_k + \chi\sigma_k)\nabla\varphi_u + \mathbf{F}_2\big)\cdot(\mathbf{v}_k - \mathbf{w}_k)\,\mathrm{d}x\right|\\ &\quad\le C\left(\|\mu_u + \chi\sigma_u\|_{H^2}\|\varphi_k\|_{H^1} + \|\mu_k + \chi\sigma_k\|_{L^2} + \|\mathbf{F}\|_{\mathbf{L}^2}\right)\|\mathbf{v}_k - \mathbf{w}_k\|_{\mathbf{H}^1}\\ &\quad\le C_{\delta_5}\left(\|\mu_u + \chi\sigma_u\|_{H^2}^2\|\varphi_k\|_{H^1}^2 + \|\mu_k + \chi\sigma_k\|_{L^2}^2 + \|\mathbf{F}\|_{\mathbf{L}^2}^2 + \|\mathbf{w}_k\|_{\mathbf{H}^1}^2\right) + \delta_5\|\mathbf{v}_k\|_{\mathbf{H}^1}^2\end{aligned} \tag{9.30i}$$

with $\delta_5 > 0$ to be chosen later. We recall that the L^2-orthogonal projection $\mathbb{P}_k$ onto $\mathcal{W}_k$ is continuous on H^1. Consequently, choosing $v = (b_i^k + \chi(\sigma_k, w_i)_{L^2})w_i$ in (9.27b), summing the resulting identities over $i = 1, \cdots, k$, using the boundedness of $\psi''(\varphi_u) \in L^\infty(\Omega_T)$ and the elliptic estimate $\|\Delta\varphi_k\|_{L^2} \le \|\nabla\varphi_k\|_{\mathbf{L}^2}^{1/2}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^{1/2}$, it follows that

$$\begin{aligned}\|\mu_k + \chi\sigma_k\|_{L^2}^2 &\le \|\Delta\varphi_k\|_{L^2}\|\mu_k + \chi\sigma_k\|_{L^2} + C(\|\varphi_k\|_{L^2} + \|F_2\|_{L^2})\|\mu_k + \chi\sigma_k\|_{L^2}\\ &\le \|\nabla\varphi_k\|_{\mathbf{L}^2}^{1/2}\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^{1/2}\|\mu_k + \chi\sigma_k\|_{L^2} + C(\|\varphi_k\|_{L^2} + \|F_2\|_{L^2})\|\mu_k + \chi\sigma_k\|_{L^2}\\ &\le \left(\delta_6\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2} + \frac{1}{4\delta_6}\|\nabla\varphi_k\|_{\mathbf{L}^2} + C\left(\|\varphi_k\|_{L^2} + \|F_2\|_{L^2}\right)\right)\|\mu_k + \chi\sigma_k\|_{L^2},\end{aligned}$$

and therefore

$$\|\mu_k + \chi\sigma_k\|_{L^2} \le \left(\delta_6\|\nabla\Delta\varphi_k\|_{\mathbf{L}^2} + \frac{1}{4\delta_6}\|\nabla\varphi_k\|_{\mathbf{L}^2} + C\left(\|\varphi_k\|_{L^2} + \|F_2\|_{L^2}\right)\right) \tag{9.30j}$$

for $\delta_6 > 0$ arbitrary. For the remaining term on the right-hand side of (9.29), we obtain

$$\left|\int_\Omega 2\eta\mathbf{D}\mathbf{v}_k : \nabla\mathbf{w}_k + \nu\mathbf{v}_k\cdot\mathbf{w}_k \,\mathrm{d}x\right| \le C\|\mathbf{w}_k\|_{\mathbf{H}^1}^2 + \delta_7\|\mathbf{v}_k\|_{\mathbf{H}^1}^2 \tag{9.30k}$$

for $\delta_7 > 0$ to be chosen. Employing the bounds (9.30) in (9.29), using Korn's inequality and chosing δ_i, $i \in \{1,4,5,6,7\}$, sufficiently small, we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_k\|_{H^1}^2 + \|\Delta\varphi_k\|_{L^2}^2 + \|\nabla\Delta\varphi_k\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_k\|_{\mathbf{H}^1}^2\\ &\quad\le \beta(t)\|\varphi_k(t)\|_{H^1}^2 + C\left(\|\sigma_k\|_{H^1}^2 + \|\mathbf{w}_k\|_{\mathbf{H}^1}^2 + \|\nabla F_3\|_{\mathbf{L}^2}^2 + \|\mathbf{F}\|_{\mathbf{L}^2}^2 + \sum_{i=2}^3\|F_i\|_{L^2}^2\right),\end{aligned} \tag{9.31}$$

where $\beta(t) \coloneqq C\left(1 + \|\mathbf{v}_u(t)\|^2_{\mathbf{H}^2} + \|\mu_u(t) + \chi\sigma_u(t)\|^2_{H^2} + \|u(t)\|^2_{L^2}\right)$. Due to the definition of $\mathbb{U}_R$ and using Proposition 9.4, it follows that $\beta \in L^1(0,T)$. From the boundedness of $\mathbb{h}(\varphi_u), \mathbb{h}'(\varphi_u) \in L^\infty(\Omega_T)$, $\sigma_u \in L^\infty(L^6)$ and due to (9.28), we infer that

$$\|\mathbf{w}_k\|_{\mathbf{H}^1} \leq C\left(\|\sigma_k\|_{L^2} + \|\varphi_k\|_{H^1} + \|F_1\|_{L^2}\right). \tag{9.32}$$

Multiplying (9.27d) with σ_k, integrating by parts and using (9.27e), the boundedness of $\mathbb{h}'(\varphi_u) \in L^\infty(\Omega_T)$, $\sigma_u \in L^\infty(L^6)$ and the non-negativity of $\mathbb{h}(\cdot)$ yields

$$\|\nabla\sigma_k\|^2_{\mathbf{L}^2} + \mathcal{B}\|\sigma_k\|^2_{L^2} = \left|\int_\Omega \left(F_4 - \mathbb{h}'(\varphi_u)\varphi_k\sigma_u\right)\sigma_k \,\mathrm{d}x\right| \leq \delta_8\|\sigma_k\|^2_{H^1} + C_{\delta_8}\left(\|\varphi_k\|^2_{L^2} + \|F_4\|^2_{L^2}\right)$$

for $\delta_8 > 0$ arbitrary. Choosing δ_8 sufficiently small, this implies that

$$\|\sigma_k\|_{H^1} \leq C\left(\|\varphi_k\|_{L^2} + \|F_4\|_{L^2}\right). \tag{9.33}$$

Applying (9.32)-(9.33) to (9.31), we end up with

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_k\|^2_{H^1} + \|\Delta\varphi_k\|^2_{L^2} + \|\nabla\Delta\varphi_k\|^2_{\mathbf{L}^2} + \|\mathbf{v}_k\|^2_{\mathbf{H}^1} \\
&\quad \leq \beta(t)\|\varphi_k\|^2_{H^1} + C\left(\|\nabla F_3\|^2_{\mathbf{L}^2} + \|\mathbf{F}\|^2_{\mathbf{L}^2} + \sum_{i=1}^4 \|F_i\|^2_{L^2}\right).
\end{aligned}$$

Recalling (9.27c) and using elliptic regularity theory, an application of Gronwall's lemma gives

$$\|\varphi_k\|_{L^\infty(H^1)\cap L^2(H^3)} + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)} \leq C\left(\|\nabla F_3\|_{L^2(\mathbf{L}^2)} + \|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^4 \|F_i\|_{L^2(L^2)}\right). \tag{9.34}$$

Step 3: Using (9.30j) and (9.33), from (9.34) we obtain

$$\|\sigma_k\|_{L^2(H^1)} + \|\mu_k\|_{L^2(L^2)} \leq C\left(\|\nabla F_3\|_{L^2(\mathbf{L}^2)} + \|\mathbf{F}\|_{L^2(\Omega_T;\mathbb{R}^3)} + \sum_{i=1}^4 \|F_i\|_{L^2(\Omega_T)}\right). \tag{9.35}$$

Now, choosing $v = \lambda_i b_i^k w_i$ in (9.27b), summing the resulting identities over $i = 1, \ldots, k$, and integrating by parts, we have

$$\begin{aligned}
\|\nabla\mu_k\|^2_{\mathbf{L}^2} &= \left|\int_\Omega \nabla\big(-\Delta\varphi_k + \psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3\big)\cdot\nabla\mu_k \,\mathrm{d}x\right| \\
&\leq \tfrac{1}{2}\left(\|\nabla\left(-\Delta\varphi_k + \psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3\right)\|^2_{\mathbf{L}^2} + \|\nabla\mu_k\|^2_{\mathbf{L}^2}\right)
\end{aligned}$$

which implies

$$\|\nabla\mu_k\|^2_{\mathbf{L}^2} \leq \left(\|\nabla\left(-\Delta\varphi_k + \psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3\right)\|^2_{\mathbf{L}^2}\right).$$

Integrating this inequality in time from 0 to T and using (9.34)-(9.35), we obtain

$$\|\nabla\mu_k\|_{L^2(\mathbf{L}^2)} \leq C\left(\|\nabla F_3\|_{L^2(\mathbf{L}^2)} + \|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^4 \|F_i\|_{L^2(L^2)}\right). \tag{9.36}$$

Step 4: To get an estimate for the pressure, we test (9.27f) with $\mathbf{q}_k \in \mathbf{H}^1$ where $\mathbf{q}_k$ satisfies

$$\operatorname{div}(\mathbf{q}_k) = p_k \quad \text{in } \Omega, \quad \mathbf{q}_k = \left(\frac{1}{|\Omega|}\int_\Omega p_k \,\mathrm{d}x\right)\mathbf{n} \quad \text{on } \partial\Omega, \quad \text{and} \quad \|\mathbf{q}_k\|_{\mathbf{H}^1} \leq C\|p_k\|_{L^2}. \tag{9.37}$$

Therefore, using the boundedness of $\mu_u + \chi\sigma_u \in L^\infty(L^2)$, $\nabla\varphi_u \in L^\infty(\mathbf{H}^1)$, we obtain that

$$\|p_k\|^2_{L^2} \leq C\left(\|\mathbf{v}_k\|^2_{\mathbf{H}^1} + \|\nabla\varphi_k\|^2_{\mathbf{L}^3} + \|\mu_k + \chi\sigma_k\|^2_{H^1} + \|\mathbf{F}\|^2_{\mathbf{L}^2}\right).$$

Integrating this inequality in time from 0 to T and using (9.34)-(9.36), we get

$$\|p_k\|_{L^2(L^2)} \le C\left(\|\nabla F_3\|_{L^2(\mathbf{L}^2)} + \|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^{4}\|F_i\|_{L^2(L^2)}\right). \tag{9.38}$$

Summarising (9.34)-(9.38) gives

$$\begin{aligned}&\|\varphi_k\|_{L^\infty(H^1)\cap L^2(H^3)} + \|\mu_k\|_{L^2(H^1)} + \|\sigma_k\|_{L^2(H^1)} + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)} + \|p_k\|_{L^2(L^2)}\\ &\quad\le C\left(\|\nabla F_3\|_{L^2(\mathbf{L}^2)} + \|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^{4}\|F_i\|_{L^2(L^2)}\right).\end{aligned} \tag{9.39}$$

Step 5: We want to establish higher order estimates for φ_k, μ_k and σ_k. With (9.27d)-(9.27e) and elliptic regularity theory, it follows that

$$\|\sigma_k\|_{H^2} \le C\left(\|\mathbb{h}'(\varphi_u)\varphi_k\sigma_u\|_{L^2} + \|F_4\|_{L^2}\right).$$

Due to the assumptions on $\mathbb{h}(\cdot)$, using Proposition 9.4 and (9.39) implies

$$\|\sigma\|_{L^\infty(H^2)} \le C\left(\|\nabla F_3\|_{L^2(\mathbf{L}^2)} + \|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^{3}\|F_i\|_{L^2(L^2)} + \|F_4\|_{L^\infty(L^2)}\right). \tag{9.40}$$

Now, choosing $v = a_i^k\lambda_i^2 w_i$ in (9.27a), $v = -ma_i^k\lambda_i^3 w_i$ in (9.27b), integrating by parts and summing the resulting equations over $j = 1, \dots, k$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi_k|^2\,\mathrm{d}x + m\int_\Omega |\Delta^2\varphi_k|^2\,\mathrm{d}x &= \int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi_k + \mathcal{P}\sigma\mathbb{h}(\varphi_u) + F_2\big)\Delta^2\varphi\,\mathrm{d}x\\ &\quad - \int_\Omega \big(\operatorname{div}(\varphi_u\mathbf{v}_k) + \operatorname{div}(\varphi_k\mathbf{v}_u)\big)\Delta^2\varphi_k\,\mathrm{d}x\\ &\quad + m\int_\Omega \Delta\big(\psi''(\varphi_u)\varphi_k - \chi\sigma_k + F_3\big)\Delta^2\varphi_k\,\mathrm{d}x.\end{aligned} \tag{9.41}$$

Using Proposition 9.4, the assumptions on $\mathbb{h}(\cdot)$, (9.39)-(9.40) and the Sobolev embeddings $H^1 \subset L^6$, $\mathbf{H}^1 \subset \mathbf{L}^6$, $\mathbf{H}^2 \subset \mathbf{L}^\infty$, we have

$$\begin{aligned}&\left|\int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi_k + \mathcal{P}\sigma_k\mathbb{h}(\varphi_u) + F_2 - \operatorname{div}(\varphi_u\mathbf{v}) + \operatorname{div}(\varphi_k\mathbf{v}_u)\big)\Delta^2\varphi_k\,\mathrm{d}x\right|\\ &\quad\le C\left(1 + \|\mathbf{v}_u\|_{\mathbf{H}^2}^2\right)\|\varphi_k\|_{H^1}^2 + C\left(\|\sigma_k\|_{L^2}^2 + \|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + \|F_2\|_{L^2}^2\right) + \frac{m}{8}\|\Delta^2\varphi_k\|_{L^2}^2.\end{aligned} \tag{9.42a}$$

Furthermore, it is straightforward to check that

$$\left|m\int_\Omega \Delta\big(-\chi\sigma_k + F_3\big)\Delta^2\varphi_k\,\mathrm{d}x\right| \le C\left(\|\sigma_k\|_{H^2}^2 + \|F_3\|_{H^2}^2\right) + \frac{m}{8}\|\Delta^2\varphi_k\|_{L^2}^2. \tag{9.42b}$$

Now, using elliptic regularity theory, the Sobolev embedding $H^2 \subset L^\infty$, the assumptions on $\mathbb{h}(\cdot)$ and Proposition 9.4, we calculate

$$\begin{aligned}\left|\int_\Omega u\mathbb{h}'(\varphi_u)\varphi_k\Delta^2\varphi_k\,\mathrm{d}x\right| &\le C\|u\|_{L^2}\|\varphi_k\|_{H^2}\|\Delta^2\varphi_k\|_{L^2}\\ &\le C\|u\|_{L^2}^2\left(\|\varphi_k\|_{L^2}^2 + \|\Delta\varphi_k\|_{L^2}^2\right) + \frac{m}{8}\|\Delta^2\varphi_k\|_{L^2}^2.\end{aligned} \tag{9.42c}$$

Next, we observe that

$$\Delta(\psi''(\varphi_u)\varphi_k) = \psi^{(4)}(\varphi_u)|\nabla\varphi_u|^2\varphi_k + \psi'''(\varphi_u)\Delta\varphi_u\varphi_k + 2\psi'''(\varphi_u)\nabla\varphi_u\cdot\nabla\varphi_k + \psi''(\varphi_u)\Delta\varphi_k.$$

Using the Sobolev embeddings $H^1 \subset L^6$, $H^2 \subset L^\infty$, $\mathbf{H}^1 \subset \mathbf{L}^6$, the assumptions on $\psi(\cdot)$, Proposition 9.4 and elliptic regularity theory again, we obtain

$$\|\Delta(\psi''(\varphi_u)\varphi_k)\|_{L^2}^2 \leq C\left(\|\varphi\|_{L^2}^2 + \|\Delta\varphi_k\|_{L^2}^2\right). \tag{9.42d}$$

Consequently,

$$\begin{aligned}\left| m\int_\Omega \Delta(\psi''(\varphi_u)\varphi_k)\Delta^2\varphi_k \,\mathrm{d}x \right| &\leq C\|\Delta(\psi''(\varphi_u)\varphi_k)\|_{L^2}^2 + \frac{m}{8}\|\Delta^2\varphi_k\|_{L^2}^2 \\ &\leq C\left(\|\varphi_k\|_{L^2}^2 + \|\Delta\varphi_k\|_{L^2}^2\right) + \frac{m}{8}\|\Delta^2\varphi_k\|_{L^2}^2. \end{aligned} \tag{9.42e}$$

Employing the estimates (9.42) in (9.41), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega |\Delta\varphi_k|^2 \,\mathrm{d}x + \frac{m}{2}\int_\Omega |\Delta^2\varphi_k|^2 \,\mathrm{d}x \\ &\quad\leq C\left(1+\|\mathbf{v}_u\|_{\mathbf{H}^2}^2\right)\|\varphi\|_{H^1}^2 + C\left(\|\sigma_k\|_{H^2}^2 + \|\mathbf{v}_k\|_{\mathbf{H}^1}^2 + \|F_2\|_{L^2}^2 + \|F_3\|_{H^2}^2\right) \\ &\qquad + C\left(1+\|u\|_{L^2}^2\right)\left(\|\varphi_k\|_{L^2}^2 + \|\Delta\varphi_k\|_{L^2}^2\right).\end{aligned}$$

Integrating this inequality in time from 0 to T, using (9.39)-(9.40), Proposition 9.4 and elliptic regularity theory, a Gronwall arguments yields

$$\begin{aligned}&\|\varphi_k\|_{L^\infty(H^2)\cap L^2(H^3)} + \|\Delta^2\varphi_k\|_{L^2(L^2)} + \|\mu_k\|_{L^2(H^1)} + \|\sigma_k\|_{L^\infty(H^2)} + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)} + \|p_k\|_{L^2(L^2)} \\ &\quad \leq C\left(\|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^2 \|F_i\|_{L^2(L^2)} + \|F_3\|_{L^2(H^2)} + \|F_4\|_{L^\infty(L^2)}\right).\end{aligned} \tag{9.43}$$

Now, using elliptic regularity theory, (9.43), the relation (9.27b) for μ_k yields

$$\|\mu_k\|_{L^2(H^2)} \leq C\left(\|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^2 \|F_i\|_{L^2(L^2)} + \|F_3\|_{L^2(H^2)} + \|F_4\|_{L^\infty(L^2)}\right). \tag{9.44}$$

Furthermore, using Proposition 9.4, the assumptions on $\psi(\cdot)$ and (9.43), using (9.27b) yields

$$\|\mu_k\|_{L^\infty(L^2)} \leq C\,C_F, \tag{9.45}$$

where

$$C_F := \left(\|\mathbf{F}\|_{L^2(\mathbf{L}^2)} + \sum_{i=1}^2 \|F_i\|_{L^2(L^2)} + \|F_3\|_{L^\infty(L^2)\cap L^2(H^2)} + \|F_4\|_{L^\infty(L^2)}\right).$$

Invoking the relation (9.27b) for $\partial_t\varphi_k$ together with (9.43)-(9.44) gives

$$\|\partial_t\varphi_k\|_{L^2(L^2)} \leq C\,C_F. \tag{9.46}$$

Summarising (9.43)-(9.46), it holds that

$$\begin{aligned}&\|\varphi_k\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^3)} + \|\Delta^2\varphi_k\|_{L^2(L^2)} + \|\mu_k\|_{L^\infty(L^2)\cap L^2(H^2)} \\ &\quad + \|\sigma_k\|_{L^\infty(H^2)} + \|\mathbf{v}_k\|_{L^2(\mathbf{H}^1)} + \|p_k\|_{L^2(L^2)} \leq C\,C_F.\end{aligned} \tag{9.47}$$

Step 6: Now, we prove higher order estimates for $\mathbf{v}_k$ and p_k. Using Proposition 9.4, the assumptions on $\mathbb{h}(\cdot)$ and (9.34), a straightforward calculation shows that

$$\|\mathcal{P}\sigma\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi\|_{L^8(H^1)} \leq C\,C_F.$$

Using Gagliardo–Nirenberg's inequality, we have the continuous embedding $L^\infty(L^2)\cap L^2(H^2) \hookrightarrow L^8(L^3)$ which, together with Proposition 9.4, (9.47) and the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$, implies that

$$\|(\mu_u + \chi\sigma_u)\nabla\varphi_k + (\mu_k + \chi\sigma_k)\nabla\varphi_u\|_{L^8(\mathbf{L}^2)} \leq C\,C_F.$$

Invoking the last two inequalities in conjunction with (9.47), an application of Lemma 2.49 to (9.27f)-(9.27h) yields

$$\begin{aligned}&\|\varphi_k\|_{H^1(L^2)\cap L^\infty(H^2)\cap L^2(H^3)} + \|\Delta^2\varphi_k\|_{L^2(L^2)} + \|\mu_k\|_{L^\infty(L^2)\cap L^2(H^2)}\\ &\quad + \|\sigma_k\|_{L^\infty(H^2)} + \|\mathbf{v}_k\|_{L^8(\mathbf{H}^2)} + \|p_k\|_{L^8(H^1)} \leq C\,C_F\|(\mathbf{F}, F_1, F_2, F_3, F_4)\|_{\mathcal{V}_2}.\end{aligned} \tag{9.48}$$

Step 7: Due to (9.36), we can pass to the limit in the Galerkin scheme to deduce that (LIN) holds. The initial condition is attained due to the compact embedding $H^1((H^1)^*) \cap L^\infty(H^1) \hookrightarrow C([0,T]; L^2)$ (see Lemma 2.36). Moreover, the estimate (9.48) holds for the solution quintuple $(\varphi, \mu, \sigma, \mathbf{v}, p)$ due to weak-star lower semicontinuity of norms. Therefore, we can apply elliptic regularity theory in (9.25d) to deduce that

$$\|\varphi\|_{L^2(H^4)} \leq C\,C_F\|(\mathbf{F}, F_1, F_2, F_3, F_4)\|_{\mathcal{V}_2}.$$

Together with (9.48) for $(\varphi, \mu, \sigma, \mathbf{v}, p)$, this implies

$$\|(\varphi, \mu, \sigma, \mathbf{v}, p)\|_{\mathcal{V}_1} \leq C\|(\mathbf{F}, F_1, F_2, F_3, F_4)\|_{\mathcal{V}_2}$$

which shows (9.26). Finally, uniqueness follows from linearity of the system together with (9.26). Indeed, it can be checked that all the estimates carried out above can also be deduced on the continuous level where, instead of testing (9.27b) with $m\Delta^3\varphi_k$, the relation (9.25d) for μ is used. This completes the proof. □

9.2.6 Fréchet differentiability

Now, the last result can be used to prove Fréchet differentiability of the control-to-state operator.

Proposition 9.14 *The following statements hold:*

(i) *the control-to-state operator $\mathbb{S}$ is Fréchet differentiable on $\mathbb{U}_R$, i. e., for any $u \in \mathbb{U}_R$ there exists a unique bounded linear operator*

$$\mathbb{S}'(u) : L^2(L^2) \to \mathcal{V}_1, \quad h \mapsto \mathbb{S}'(u)[h] = \big(\varphi_u', \mu_u', \mathbf{v}_u', \sigma_u', p_u'\big)[h]$$

such that

$$\frac{\|\mathbb{S}(u+h) - \mathbb{S}(u) - \mathbb{S}'(u)[h]\|_{\mathcal{V}_1}}{\|h\|_{L^2(L^2)}} \to 0 \quad \text{as} \quad \|h\|_{L^2(L^2)} \to 0.$$

For any $u \in \mathbb{U}_R$ and $h \in L^2(L^2)$, the Fréchet derivative $\big(\varphi_u', \mu_u', \mathbf{v}_u', \sigma_u', p_u'\big)[h]$ is the unique strong solution of the system (LIN) with

$$F_1, F_3, F_4 = 0, \quad \mathbf{F} = \mathbf{0} \quad \text{and} \quad F_2 = -h\,\mathbb{h}(\varphi_u).$$

(ii) *the Frechet-derivative is Lipschitz continuous, i. e., for any $u, \tilde{u} \in \mathbb{U}_R$, it holds that*

$$\|\mathbb{S}'(u) - \mathbb{S}'(\tilde{u})\|_{\mathcal{L}(L^2(L^2);\mathcal{V}_1)} \leq L_2\|u - \tilde{u}\|_{L^2(L^2)} \tag{9.49}$$

with a constant $L_2 > 0$ independent of u and $\tilde{u}$.

Proof. Let C denote a generic non-negative constant which may change its value from line to line.

Proof of (i)**:** To prove Fréchet differentiability we must consider the difference

$$(\varphi, \mu, \sigma, \mathbf{v}, p) := (\varphi_{u+h}, \mu_{u+h}, \mathbf{v}_{u+h}, \sigma_{u+h}, p_{u+h}) - (\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$$

for some arbitrary $u \in \mathbb{U}_R$ and $h \in L^2(L^2)$ with $u+h \in \mathbb{U}_R$. Therefore, we assume that $\|h\|_{L^2(L^2)} < \delta$ for some sufficiently small $\delta > 0$. Now, we Taylor expand the nonlinear terms in (CHB) to pick out the linear contributions. We obtain that

$$\begin{aligned} \mathbb{h}(\varphi_{u+h}) - \mathbb{h}(\varphi_u) &= \mathbb{h}'(\varphi_u)\varphi + \mathcal{R}_1, \\ \sigma_{u+h}\mathbb{h}(\varphi_{u+h}) - \sigma_u \mathbb{h}(\varphi_u) &= \sigma \mathbb{h}(\varphi_u) + \sigma_u \mathbb{h}'(\varphi_u)\varphi + \mathcal{R}_2, \\ (u+h)\mathbb{h}(\varphi_{u+h}) - u\mathbb{h}(\varphi_u) &= u\mathbb{h}'(\varphi_u)\varphi + h\mathbb{h}(\varphi_u) + \mathcal{R}_3, \\ (\mu_{u+h} + \chi\sigma_{u+h})\nabla\varphi_{u+h} - (\mu_u + \chi\sigma_u)\nabla\varphi_u &= (\mu_u + \chi\sigma_u)\nabla\varphi + (\mu + \chi\sigma)\nabla\varphi_u + \mathcal{R}_4, \\ \operatorname{div}(\varphi_{u+h}\mathbf{v}_{u+h}) - \operatorname{div}(\varphi_u \mathbf{v}_u) &= \operatorname{div}(\varphi \mathbf{v}_u) + \operatorname{div}(\varphi_u \mathbf{v}) + \mathcal{R}_5, \\ \psi'(\varphi_{u+h}) - \psi'(\varphi_u) &= \psi''(\varphi_u)\varphi + \mathcal{R}_6, \end{aligned}$$

where the nonlinear remainders are given by

$$\begin{aligned} &\mathcal{R}_1 := \tfrac{1}{2}\mathbb{h}''(\zeta)(\varphi_{u+h} - \varphi_u)^2, \\ &\mathcal{R}_2 := (\sigma_{u+h} - \sigma_u)(\mathbb{h}(\varphi_{u+h}) - \mathbb{h}(\varphi_u)) + \tfrac{1}{2}\sigma_u \mathbb{h}''(\zeta)(\varphi_{u+h} - \varphi_u)^2, \\ &\mathcal{R}_3 := \tfrac{1}{2}u\, \mathbb{h}''(\zeta)(\varphi_{u+h} - \varphi_u)^2 + h\big(\mathbb{h}(\varphi_{u+h}) - \mathbb{h}(\varphi_u)\big), \\ &\mathcal{R}_4 := \big[(\mu_{u+h} - \mu_u) + \chi(\sigma_{u+h} - \sigma_u)\big](\nabla\varphi_{u+h} - \nabla\varphi_u), \\ &\mathcal{R}_5 := \operatorname{div}\big[(\varphi_{u+h} - \varphi_u)(\mathbf{v}_{u+h} - \mathbf{v}_u)\big], \\ &\mathcal{R}_6 := \tfrac{1}{2}\psi'''(\xi)(\varphi_{u+h} - \varphi_u)^2 \end{aligned}$$

with $\zeta = \vartheta\varphi_{u+h} + (1-\vartheta)\varphi_u$ and $\xi = \theta\varphi_{u+h} + (1-\theta)\varphi_u$ for some $\vartheta, \theta \in [0,1]$. This means that the difference $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is the strong solution of (LIN) with

$$F_1 = \mathcal{P}\mathcal{R}_2 - \mathcal{A}\mathcal{R}_1, \quad \mathbf{F} = \mathcal{R}_4, \quad F_2 = \mathcal{P}\mathcal{R}_2 - \mathcal{A}\mathcal{R}_1 - \mathcal{R}_3 - \mathcal{R}_5 - h\,\mathbb{h}(\varphi_u), \quad F_3 = \mathcal{R}_6, \quad F_4 = -\mathcal{R}_2.$$

By a simple computation, one can show that these functions have the desired regularity. Now, we write $(\varphi_u^h, \mu_u^h, \mathbf{v}_u^h, \sigma_u^h, p_u^h)$ to denote the strong solution of (LIN) with

$$F_1, F_3, F_4 = 0, \quad \mathbf{F} = \mathbf{0} \quad \text{and} \quad F_2 = -h\,\mathbb{h}(\varphi_u),$$

and $(\varphi_{\mathcal{R}}^h, \mu_{\mathcal{R}}^h, \mathbf{v}_{\mathcal{R}}^h, \sigma_{\mathcal{R}}^h, p_{\mathcal{R}}^h)$ to denote the strong solution of (LIN) with

$$F_1 = \mathcal{P}\mathcal{R}_2 - \mathcal{A}\mathcal{R}_1, \ \ \mathbf{F} = \mathcal{R}_4, \ \ F_2 = \mathcal{P}\mathcal{R}_2 - \mathcal{A}\mathcal{R}_1 - \mathcal{R}_3 - \mathcal{R}_5, \ \ F_3 = \mathcal{R}_6, \ \ F_4 = -\mathcal{R}_2. \tag{9.50}$$

Because of linearity of the system (LIN) and uniqueness of its solution, it follows that

$$\begin{aligned} (\varphi_{\mathcal{R}}^h, \mu_{\mathcal{R}}^h, \mathbf{v}_{\mathcal{R}}^h, \sigma_{\mathcal{R}}^h, p_{\mathcal{R}}^h) = (\varphi_{u+h}, \mu_{u+h}, \mathbf{v}_{u+h}, \sigma_{u+h}, p_{u+h}) - (\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u) \\ - (\varphi_u^h, \mu_u^h, \mathbf{v}_u^h, \sigma_u^h, p_u^h). \end{aligned} \tag{9.51}$$

We conclude from Proposition 9.4 that ζ and ξ are uniformly bounded. This yields

$$\|\psi^{(i)}(\zeta)\|_{L^\infty(\Omega_T)} \le C \quad \forall\, 1 \le i \le 4, \qquad \text{and} \quad \|\mathbb{h}^{(j)}(\zeta)\|_{L^\infty(\Omega_T)} \le C \quad \forall\, 1 \le j \le 3.$$

Moreover, since $\mathbb{h}(\cdot)$ is Lipschitz continuous, it holds that

$$\|\mathbb{h}(\varphi_{u+h}) - \mathbb{h}(\varphi_u)\|_{L^\infty(\Omega_T)} \le C\, \|\varphi_{u+h} - \varphi_u\|_{L^\infty(\Omega_T)} \le C\, \|h\|_{L^2(L^2)}.$$

Together with the Lipschitz estimates from Corollary 9.9 we obtain that

$$\begin{aligned} \|\mathcal{R}_i\|_{L^2(L^2)} &\le C\, \|h\|_{L^2(L^2)}^2, \quad i \in \{1,2,3,6\}, \\ \|\mathcal{R}_i\|_{L^\infty(L^2)} &\le C\, \|h\|_{L^2(L^2)}^2, \quad i \in \{1,2,6\}. \end{aligned}$$

Moreover, we have

$$\begin{aligned}\|\nabla(\varphi_{u+h}-\varphi_u)\cdot(\mathbf{v}_{u+h}-\mathbf{v}_u)\|_{L^2(L^2)} &\le \|\nabla(\varphi_{u+h}-\varphi_u)\|_{L^\infty(L^3)}\,\|\mathbf{v}_{u+h}-\mathbf{v}_u\|_{L^2(L^6)}\\ &\le C\,\|\varphi_{u+h}-\varphi_u\|_{L^\infty(H^1)}\,\|\mathbf{v}_{u+h}-\mathbf{v}_u\|_{L^2(H^1)}, \end{aligned}\tag{9.52}$$

and then Corollary 9.9 yields

$$\begin{aligned}&\|\mathcal{R}_5\|_{L^2(L^2)}\\ &\quad\le C\|\varphi_{u+h}-\varphi_u\|_{L^\infty(\Omega_T)}\|\mathbf{v}_{u+h}-\mathbf{v}_u\|_{L^2(H^1)}+C\|\nabla(\varphi_{u+h}-\nabla\varphi_u)\cdot(\mathbf{v}_{u+h}-\mathbf{v}_u)\|_{L^2(L^2)}\\ &\quad\le C\,\|h\|^2_{L^2(L^2)}.\end{aligned}$$

Due to the continuous embedding $L^\infty(\mathbf{H}^1)\cap L^2(\mathbf{H}^3)\hookrightarrow L^8(\mathbf{L}^\infty)$ resulting from Gagliardo–Nirenberg's inequality, an application of Corollary 9.9 gives

$$\|\mathcal{R}_4\|_{L^8(\mathbf{L}^2)}\le C\|h\|^2_{L^2(L^2)}.\tag{9.53}$$

Furthermore, we have

$$\begin{aligned}&\|\nabla\mathcal{R}_6\|_{L^2(\mathbf{L}^2)}\\ &\quad\le C\,\|\nabla\xi\|_{L^\infty(L^6)}\,\|\varphi_{u+h}-\varphi_u\|^2_{L^\infty(L^6)}+C\,\|\nabla(\varphi_{u+h}-\varphi_u)\|_{L^2(L^3)}\,\|\varphi_{u+h}-\varphi_u\|_{L^\infty(L^6)}\\ &\quad\le C\,\|\nabla\xi\|_{L^\infty(\mathbf{H}^1)}\,\|\varphi_{u+h}-\varphi_u\|^2_{L^\infty(H^1)}+C\,\|\varphi_{u+h}-\varphi_u\|_{L^2(H^2)}\,\|\varphi_{u+h}-\varphi_u\|_{L^\infty(H^1)}\\ &\quad\le C\,\|h\|^2_{L^2(L^2)}\end{aligned}$$

and

$$\|\Delta\mathcal{R}_6\|_{L^2(L^2)}\le C\,\Big(1+\|\xi\|^2_{L^\infty(H^2)}\Big)\,\|\varphi_{u+h}-\varphi_u\|^2_{L^\infty(H^2)}\le C\,\|h\|^2_{L^2(L^2)}.$$

From the last two inequalities and elliptic regularity theory, we infer that

$$\|\mathcal{R}_6\|_{L^2(H^2)}\le C\,\|h\|^2_{L^2(L^2)}.$$

Now, we first observe that

$$\begin{aligned}\|\nabla\mathcal{R}_1\|_{L^8(\mathbf{L}^2)} &\le \|\mathbb{h}^{(3)}(\zeta)\nabla\zeta\,(\varphi_{u+h}-\varphi_u)^2\|_{L^8(\mathbf{L}^2)}+\|2\mathbb{h}''(\zeta)\nabla(\varphi_{u+h}-\varphi_u)\,(\varphi_{u+h}-\varphi_u)\|_{L^8(\mathbf{L}^2)}\\ &\le C\|\nabla\zeta\|_{L^\infty(\mathbf{L}^6)}\|\varphi_{u+h}\|^2_{L^\infty(L^6)}+C\|\nabla(\varphi_{u+h}-\varphi_u)\|_{L^\infty(\mathbf{L}^6)}\|\varphi_{u+h}-\varphi_u\|_{L^\infty(L^3)}\\ &\le C\,\|h\|^2_{L^2(L^2)}.\end{aligned}$$

With similar arguments, it follows that

$$\begin{aligned}\|\nabla\big(\sigma_u\mathbb{h}''(\zeta)(\varphi_{u+h}-\varphi_u)^2\big)\|_{L^8(\mathbf{L}^2)} &\le \|\nabla\sigma_u\mathbb{h}''(\zeta)(\varphi_{u+h}-\varphi_u)^2\|_{L^8(\mathbf{L}^2)}+\|2\sigma_u\nabla\mathcal{R}_1\|_{L^8(\mathbf{L}^2)}\\ &\le C\|\nabla\sigma_u\|_{L^\infty(L^6)}\|\varphi_{u+h}\|^2_{L^\infty(L^6)}+C\|\sigma_u\|_{L^\infty(\Omega_T)}\|\nabla\mathcal{R}_1\|_{L^8(\mathbf{L}^2)}\\ &\le C\,\|h\|^2_{L^2(L^2)}.\end{aligned}$$

From the Lipschitz-continuity of $\mathbb{h}'(\cdot)$, we deduce that

$$\begin{aligned}\|\nabla\big((\sigma_{u+h}-\sigma_u)(\mathbb{h}(\varphi_{u+h})-\mathbb{h}(\varphi_u))\big)\|_{L^8(\mathbf{L}^2)} &\le C\|\sigma_{u+h}-\sigma_u\|_{L^\infty(H^2)}\|\varphi_{u+h}-\varphi_u\|_{L^\infty(H^2)}\\ &\le C\,\|h\|^2_{L^2(L^2)}.\end{aligned}$$

The last two inequalities imply

$$\|\mathcal{R}_i\|_{L^8(H^1)}\le C\,\|h\|^2_{L^2(L^2)},\quad i\in\{1,2\}.$$

This finally yields

$$\|(\mathbf{F}, F_1, F_2, F_3, F_4)\|_{\mathcal{V}_2} \le C\,\|h\|^2_{L^2(L^2)},$$

where F_i denote the functions given by (9.50). Hence, due to (9.26) we obtain that

$$\|(\varphi^h_{\mathcal{R}}, \mu^h_{\mathcal{R}}, \sigma^h_{\mathcal{R}}, \mathbf{v}^h_{\mathcal{R}}, p^h_{\mathcal{R}})\|_{\mathcal{V}_1} \le C\,\|h\|^2_{L^2(L^2)}$$

which completes the proof of (i).

Proof of (ii)**:** In the following we write

$$(\varphi, \mu, \sigma, \mathbf{v}, p) := \big(\varphi'_u, \mu'_u, \sigma'_u, \mathbf{v}'_u, p'_u\big)[h] - \big(\varphi'_{\tilde u}, \mu'_{\tilde u}, \sigma'_{\tilde u}, \mathbf{v}'_{\tilde u}, p'_{\tilde u}\big)[h].$$

Then, using the mean value theorem, a long but straightforward calculation shows that

$$\begin{aligned}
\operatorname{div}(\mathbf{v}) &= \mathcal{P}\sigma\mathbb{h}(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi + F_1 && \text{in } \Omega_T, && (9.54a)\\
-\operatorname{div}(\mathbf{T}(\mathbf{v}, p)) + \nu\mathbf{v} &= (\mu_u + \chi\sigma_u)\nabla\varphi + (\mu + \chi\sigma)\nabla\varphi_u + \mathbf{F} && \text{in } \Omega_T, && (9.54b)\\
\partial_t\varphi + \operatorname{div}(\varphi_u\mathbf{v}) + \operatorname{div}(\varphi\mathbf{v}_u) &= m\Delta\mu + (\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\varphi && && \\
&\quad + \mathcal{P}\sigma\mathbb{h}(\varphi_u) + F_2 && \text{in } \Omega_T, && (9.54c)\\
\mu &= -\Delta\varphi + \psi''(\varphi_u)\varphi - \chi\sigma + F_3 && \text{in } \Omega_T, && (9.54d)\\
-\Delta\sigma + \mathcal{B}\sigma + \mathbb{h}'(\varphi_u)\varphi\sigma_u + \mathbb{h}(\varphi_u)\sigma &= F_4 && \text{in } \Omega_T, && (9.54e)\\
\nabla\mu\cdot\mathbf{n} = \nabla\varphi\cdot\mathbf{n} = \nabla\sigma\cdot\mathbf{n} &= 0 && \text{on } \Sigma_T, && (9.54f)\\
\mathbf{T}(\mathbf{v}, p)\mathbf{n} &= 0 && \text{on } \Sigma_T, && (9.54g)\\
\varphi(0) &= 0 && \text{in } \Omega, && (9.54h)
\end{aligned}$$

where

$$\begin{aligned}
F_1 &= \mathcal{P}\sigma'_{\tilde u}[h](\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde u})) + (\mathcal{P}\sigma_{\tilde u} - \mathcal{A})\varphi'_{\tilde u}[h]\mathbb{h}'(\xi)(\varphi_u - \varphi_{\tilde u}) + \mathcal{P}\varphi'_{\tilde u}[h](\sigma_u - \sigma_{\tilde u})\mathbb{h}'(\varphi_u),\\
\mathbf{F} &= \varphi'_{\tilde u}[h]\big((\mu_u + \chi\sigma_u) - (\mu_{\tilde u} + \chi\sigma_{\tilde u})\big) + (\mu'_{\tilde u}[h] + \chi\sigma'_{\tilde u}[h])(\nabla\varphi_u - \nabla\varphi_{\tilde u}),\\
F_2 &= F_1 - \varphi'_{\tilde u}[h](u - \tilde u)\mathbb{h}'(\varphi_u) - \tilde u\mathbb{h}'(\xi)(\varphi_u - \varphi_{\tilde u}) - h(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde u}))\\
&\quad - \operatorname{div}((\varphi_u - \varphi_{\tilde u})\mathbf{v}'_{\tilde u}[h]) - \operatorname{div}(\varphi'_{\tilde u}[h](\mathbf{v}_u - \mathbf{v}_{\tilde u})),\\
F_3 &= \varphi'_{\tilde u}[h]\psi'''(\xi)(\varphi_u - \varphi_{\tilde u}),\\
F_4 &= -\sigma'_{\tilde u}[h](\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\tilde u})) - \varphi'_{\tilde u}[h]\big((\sigma_u - \sigma_{\tilde u})\mathbb{h}'(\varphi_u) + \sigma_{\tilde u}\mathbb{h}'(\xi)(\varphi_u - \varphi_{\tilde u})\big).
\end{aligned}$$

Using the Lipschitz-continuity of $h(\cdot)$ together with (9.2), (9.23) and (9.26), a straightforward calculation shows that

$$\begin{aligned}
\|F_i\|_{L^2(L^2)} &\le C\|h\|_{L^2(L^2)}\|u - \tilde u\|_{L^2(L^2)}, \quad 1 \le i \le 4,\\
\|\mathbf{F}\|_{L^2(\mathbf{L}^2)} &\le C\|h\|_{L^2(L^2)}\|u - \tilde u\|_{L^2(L^2)}.
\end{aligned} \tag{9.55}$$

With similar arguments, it follows that

$$\|F_1\|_{L^8(L^2)} \le C\|h\|_{L^2(L^2)}\|u - \tilde u\|_{L^2(L^2)}.$$

Using the assumptions on $\mathbb{h}(\cdot)$, (9.2), (9.23) and (9.26), we obtain

$$\|\nabla F_1\|_{L^8(\mathbf{L}^2)} \le C\|h\|_{L^2(L^2)}\|u - \tilde u\|_{L^2(L^2)}.$$

Now, using the continuous embedding $L^\infty(H^1) \cap L^2(H^3) \hookrightarrow L^8(L^\infty)$ resulting from Gagliardo–Nirenberg's inequality along with (9.23) and (9.26) yields

$$\|\mathbf{F}\|_{L^8(\mathbf{L}^2)} \le C\|h\|_{L^2(L^2)}\|u - \tilde u\|_{L^2(L^2)}.$$

From the Lipschitz-continuity of $\mathbb{h}(\cdot)$ and the boundedness of $\mathbb{h}'(\cdot)$, applying (9.2), (9.23) and (9.26) yields

$$\begin{aligned}\|F_4\|_{L^\infty(L^2)} &\leq C\|\sigma'_{\tilde{u}}[h]\|_{L^\infty(L^2)}\|\varphi_u - \varphi_{\tilde{u}}\|_{L^\infty(\Omega_T)} \\ &\quad + C\|\varphi'_{\tilde{u}}[h]\|_{L^\infty(L^2)}\left(\|\sigma_u - \sigma_{\tilde{u}}\|_{L^\infty(\Omega_T)} + \|\varphi_u - \varphi_{\tilde{u}}\|_{L^\infty(\Omega_T)}\right) \\ &\leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)}.\end{aligned}$$

Invoking the last four inequalities and (9.55), we obtain

$$\|\mathbf{F}\|_{L^8(\mathbf{L}^2)} + \|F_1\|_{L^8(H^1)} + \|F_2\|_{L^2(L^2)} + \|F_4\|_{L^\infty(L^2)} \leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)}. \tag{9.56}$$

It remains to estimate the term F_3. Using the boundedness of $\psi'''(\xi) \in L^\infty(\Omega_T)$, (9.23) and (9.26), we deduce that

$$\|F_3\|_{L^\infty(L^2)} \leq C\|\varphi'_{\tilde{u}}[h]\|_{L^\infty(\Omega_T)}\|\varphi_u - \varphi_{\tilde{u}}\|_{L^\infty(\Omega_T)} \leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)}.$$

Using the assumptions on $\psi(\cdot)$ and the Sobolev embeddings $H^1 \subset L^p$, $\mathbf{H}^1 \subset \mathbf{L}^p$, $p \in [1, 6]$, thanks to (9.2), (9.23) and (9.26) we have

$$\begin{aligned}&\|\nabla F_3\|_{L^2(\mathbf{L}^2)} \\ &\quad = \|\nabla\varphi'_{\tilde{u}}[h]\psi'''(\xi)(\varphi_u - \varphi_{\tilde{u}}) + \varphi'_{\tilde{u}}[h]\psi^{(4)}(\xi)\nabla\xi(\varphi_u - \varphi_{\tilde{u}}) + \varphi'_{\tilde{u}}[h]\psi'''(\xi)\nabla(\varphi_u - \varphi_{\tilde{u}})\|_{L^2(\mathbf{L}^2)} \\ &\quad \leq C\|\varphi'_{\tilde{u}}[h]\|_{L^\infty(H^2)}\|\varphi_u - \varphi_{\tilde{u}}\|_{L^\infty(H^2)} \\ &\quad \leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)}.\end{aligned}$$

With similar arguments, it follows that

$$\|\Delta F_3\|_{L^2(L^2)} \leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)}.$$

Invoking the last three inequalities together with (9.55) and elliptic regularity, we obtain

$$\|F_3\|_{L^\infty(L^2)\cap L^2(H^2)} \leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)}.$$

Together with (9.55), an application of Proposition 9.13 yields

$$\|(\varphi, \mu, \sigma, \mathbf{v}, p)\|_{\mathcal{V}_1} \leq C\|h\|_{L^2(L^2)}\|u - \tilde{u}\|_{L^2(L^2)},$$

hence (9.49) holds. This completes the proof. □

Remark 9.15 Since the Fréchet derivative $\mathbb{S}'(u)$ maps again into the space $\mathcal{V}_1$ and is also continuous with respect to the operator norm on $\mathcal{L}(L^2(L^2); \mathcal{V}_1)$, we conjecture that the procedure of Proposition 9.14 can be repeated arbitrarily often provided that ψ, $\mathbb{h}$, φ_0, σ_B and $\partial\Omega$ are smooth. Then, it was possible to show that the control-to-state operator is actually smooth.

Assuming that the control-to-state operator was at least twice continuously Fréchet differentiable, we could use this property to derive an alternative second-order sufficient condition for local optimality. However, we decided to use a different approach which is based on Fréchet differentiability of the control-to-costate operator (see below) as we prefer the resulting optimality condition.

9.3 The adjoint system and its properties

In optimal control theory, it is a standard approach to use **adjoint variables** to express the optimality conditions suitably. They are given by the **adjoint system** which can be derived by formal Lagrangian technique. It consists of the following equations:

$$
\text{(ADJ)}\begin{cases}
\operatorname{div}(\mathbf{w}) = 0 & \text{in } \Omega_T, & (9.57a)\\
-\eta\Delta\mathbf{w} + \nu\mathbf{w} = -\nabla q + \varphi_u\nabla\phi & \text{in } \Omega_T, & (9.57b)\\
\partial_t\phi + \nabla\phi\cdot\mathbf{v}_u = -(\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}'(\varphi_u)\phi - \mathbb{h}'(\varphi_u)\sigma_u\rho\tau & & \\
\qquad - \psi''(\varphi_u)\tau + \Delta\tau + \nabla(\mu_u + \chi\sigma_u)\cdot\mathbf{w} & & \\
\qquad + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)q - \alpha_1(\varphi_u - \varphi_d) & \text{in } \Omega_T, & (9.57c)\\
\tau = \nabla\varphi_u\cdot\mathbf{w} + m\Delta\phi & \text{in } \Omega_T, & (9.57d)\\
\Delta\rho - \mathcal{B}\rho = \mathbb{h}(\varphi_u)\rho - \chi\tau + \mathcal{P}\mathbb{h}(\varphi_u)(\phi - q) + \chi\nabla\varphi_u\cdot\mathbf{w} & \text{in } \Omega_T, & (9.57e)\\
\nabla\phi\cdot\mathbf{n} = \nabla\rho\cdot\mathbf{n} = 0 & \text{on } \Sigma_T, & (9.57f)\\
0 = (2\eta\mathbf{D}\mathbf{w} - q\mathbf{I} + \varphi_u\phi\mathbf{I})\mathbf{n} & \text{on } \Sigma_T, & (9.57g)\\
\nabla\tau\cdot\mathbf{n} = \phi\mathbf{v}_u\cdot\mathbf{n} - (\mu_u + \chi\sigma_u)\mathbf{w}\cdot\mathbf{n} & \text{on } \Sigma_T, & (9.57h)\\
\phi(T) = \alpha_0(\varphi_u(T) - \varphi_f) & \text{in } \Omega. & (9.57i)
\end{cases}
$$

9.3.1 Existence and uniqueness of weak solutions

A weak solution of this system is referred to as an **adjoint state** or **costate** and is defined as follows:

Definition 9.16 Let $u \in \mathbb{U}_R$ be any control and let $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ denote its corresponding state. Then, $(\phi, \tau, \rho, \mathbf{w}, q)$ is called a weak solution of the adjoint system (ADJ) if

$$\phi \in H^1\big((H^1)^*\big)\cap L^\infty(H^1)\cap L^2(H^3), \quad \tau \in L^2(H^1), \quad \rho \in L^2(H^1), \quad \mathbf{w} \in L^2\big(\mathbf{H}^1\big), \quad q \in L^2(L^2)$$

such that

$$\phi(T) = \alpha_0(\varphi_u(T) - \varphi_f) \quad \text{a.e. in } \Omega, \tag{9.58a}$$

$$\operatorname{div}(\mathbf{w}) = 0 \quad \text{a.e. in } \Omega_T, \tag{9.58b}$$

and

$$\int_\Omega \mathbf{T}(\mathbf{w}, q)\colon \nabla\tilde{\mathbf{w}} + \nu\mathbf{w}\cdot\tilde{\mathbf{w}}\,\mathrm{d}x = -\int_\Omega \phi\nabla\varphi_u\cdot\tilde{\mathbf{w}} + \phi\varphi_u\operatorname{div}(\tilde{\mathbf{w}})\,\mathrm{d}x, \tag{9.58c}$$

$$
\begin{aligned}
\big\langle\partial_t\phi, \tilde{\phi}\big\rangle_{H^1} = &-\int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)(\phi - q) - \mathbb{h}'(\varphi_u)(u\phi - \sigma_u\rho)\big)\,\tilde{\phi}\,\mathrm{d}x\\
&+ \int_\Omega \big(\phi\operatorname{div}(\mathbf{v}_u) - \psi''(\varphi_u)\tau - \alpha_1(\varphi_u - \varphi_d)\big)\,\tilde{\phi}\,\mathrm{d}x\\
&+ \int_\Omega \big(\phi\mathbf{v}_u - (\mu_u + \chi\sigma_u)\mathbf{w} - \nabla\tau\big)\cdot\nabla\tilde{\phi}\,\mathrm{d}x,
\end{aligned} \tag{9.58d}
$$

$$\int_\Omega \tau\tilde{\tau}\,\mathrm{d}x = \int_\Omega \nabla\varphi_u\cdot\mathbf{w}\tilde{\tau} - m\nabla\phi\cdot\nabla\tilde{\tau}\,\mathrm{d}x, \tag{9.58e}$$

$$
\begin{aligned}
-\int_\Omega \nabla\rho\cdot\nabla\tilde{\rho} + \mathcal{B}\rho\tilde{\rho}\,\mathrm{d}x = &\int_\Omega \big(-\chi\tau + \mathcal{P}\mathbb{h}(\varphi_u)\phi + \chi\nabla\varphi_u\cdot\mathbf{w}\big)\,\tilde{\rho}\,\mathrm{d}x\\
&+ \int_\Omega \big(-\mathcal{P}\mathbb{h}(\varphi_u)q + \mathbb{h}(\varphi_u)\rho\big)\,\tilde{\rho}\,\mathrm{d}x
\end{aligned} \tag{9.58f}
$$

for a.e. $t \in (0, T)$ and all $\tilde{\phi}$, $\tilde{\tau}$, $\tilde{\rho} \in H^1$, $\tilde{\mathbf{w}} \in \mathbf{H}^1$ where $\mathbf{T}(\mathbf{w}, q) := 2\eta\mathbf{D}\mathbf{w} - q\mathbf{I}$.

To prove existence and uniqueness of solutions for (ADJ), we will use the following lemma:

Lemma 9.17 *Let $u \in \mathbb{U}_R$ be any control and let $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ denote its corresponding state. Furthermore, let $(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2) \in \mathcal{V}_4$ be arbitrary. Then, there exists a unique solution $(\phi, \tau, \rho, \mathbf{w}, q) \in \mathcal{V}_3$ solving*

$$\operatorname{div}(\mathbf{w}) = 0 \qquad \textit{a. e. in } \Omega_T, \tag{9.59a}$$
$$-\eta\Delta\mathbf{w} + \nu\mathbf{w} = -\nabla q + \varphi_u\nabla\phi + \mathbf{G}_1 \qquad \textit{a. e. in } \Omega_T, \tag{9.59b}$$
$$\tau = \nabla\varphi_u\cdot\mathbf{w} + m\Delta\phi + G_2 \qquad \textit{a. e. in } \Omega_T, \tag{9.59c}$$
$$\Delta\rho - \mathcal{B}\rho - \mathbb{h}(\varphi_u)\rho = -\chi\tau + \mathcal{P}\mathbb{h}(\varphi_u)\phi + \chi\nabla\varphi_u\cdot\mathbf{w} - \mathcal{P}\mathbb{h}(\varphi_u)q + G_3 \qquad \textit{a. e. in } \Omega_T, \tag{9.59d}$$
$$\nabla\phi\cdot\mathbf{n} = \nabla\rho\cdot\mathbf{n} = 0 \qquad \textit{a. e. on } \Sigma_T, \tag{9.59e}$$
$$0 = (2\eta\mathbf{D}\mathbf{w} - q\mathbf{I} + \varphi_u\phi\mathbf{I})\mathbf{n} \qquad \textit{a. e. on } \Sigma_T, \tag{9.59f}$$
$$\phi(T) = G_0 \qquad \textit{a. e. in } \Omega, \tag{9.59g}$$

and

$$\begin{aligned}\langle\partial_t\phi, \tilde\phi\rangle_{H^1} = &-\int_\Omega \left((\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)(\phi - q) - \mathbb{h}'(\varphi_u)(u\phi - \sigma_u\rho)\right)\tilde\phi\,\mathrm{d}x\\ &+\int_\Omega \left(\phi\operatorname{div}(\mathbf{v}_u) - \psi''(\varphi_u)\tau - G_1\right)\tilde\phi\,\mathrm{d}x\\ &+\int_\Omega \left(\phi\mathbf{v}_u - (\mu_u + \chi\sigma_u)\mathbf{w} - \nabla\tau + \mathbf{G}_2\right)\cdot\nabla\tilde\phi\,\mathrm{d}x\end{aligned} \tag{9.59h}$$

for a. e. $t \in (0,T)$ and all $\tilde\phi \in H^1$. In addition, it holds that

$$\|(\phi, \tau, \rho, \mathbf{w}, q)\|_{\mathcal{V}_3} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4} \tag{9.60}$$

for a constant $C > 0$ independent of $(\phi, \tau, \rho, \mathbf{w}, q, u)$.

Proof of Lemma 9.17. We will only show a priori estimates for the solutions of (9.59). The justification can be carried out rigorously within a Galerkin scheme similar as in the proof of Proposition 9.13. In particular, equation (9.59h) is satisfied by the Galerkin solutions with the duality product replaced by the L^2-scalar-product and G_0 replaced by $\mathbb{P}_k G_0$ where $\mathbb{P}_k$ denotes the L^2-orthogonal projection onto the k-dimensional subspaces spanned by the eigenfunctions of the Neumann–Laplace operator. In the following we will suppress the subscript k.

Hölder's and Young's inequalities will be frequently used as well as a generic constant C which does not depend on the approximating solutions deduced within the Galerkin scheme. The approach will be split into several steps.

Step 1: We define $\pi := q - \varphi_u\phi$. Then, from (9.59a)-(9.59b), (9.59f), we see that $(\mathbf{w}, \pi)$ is for almost every $t \in (0,T)$ a solution of

$$\begin{aligned} -\eta\Delta\mathbf{w} + \nu\mathbf{w} + \nabla\pi &= \mathbf{f} && \text{a. e. in } \Omega,\\ \operatorname{div}(\mathbf{w}) &= 0 && \text{a. e. in } \Omega,\\ (2\eta\mathbf{D}\mathbf{w} - \pi\mathbf{I})\mathbf{n} &= \mathbf{0} && \text{a. e. on } \partial\Omega,\end{aligned}$$

where $\mathbf{f} := -\phi\nabla\varphi_u + \mathbf{G}$. Applying Lemma 2.49, we obtain (for a. e. $t \in (0,T)$)

$$\|\mathbf{w}\|_{\mathbf{H}^2} + \|\pi\|_{H^1} \le C\|\mathbf{f}\|_{L^2}.$$

In particular, by the definition of π and $\mathbf{f}$ and using that

$$\|\phi\nabla\varphi_u\|_{\mathbf{L}^2} \le C\|\phi\varphi_u\|_{H^1},$$

we have

$$\|\mathbf{w}\|_{\mathbf{H}^2} + \|q\|_{H^1} \le C\left(\|\phi\varphi_u\|_{H^1} + \|\mathbf{G}_1\|_{\mathbf{L}^2}\right).$$

Hence, we have to estimate the first term on the right-hand side of this equation. Using the boundedness of $\varphi_u \in L^\infty(H^2) \cap L^\infty(\Omega_T)$ and the Sobolev embedding $H^1 \subset L^3(\Omega)$, we obtain

$$\begin{aligned}\|\varphi_u\phi\|_{H^1} &\leq C\left(\|\varphi_u\phi\|_{L^2} + \|\phi\nabla\varphi_u\|_{\mathbf{L}^2} + \|\varphi_u\nabla\phi\|_{\mathbf{L}^2}\right)\\ &\leq C\left(\|\varphi_u\|_{L^\infty}\|\phi\|_{L^2} + \|\nabla\varphi_u\|_{\mathbf{L}^6}\|\phi\|_{L^3} + \|\varphi_u\|_{L^\infty}\|\nabla\phi\|_{\mathbf{L}^2}\right)\\ &\leq C\|\phi\|_{H^1}.\end{aligned}$$

Employing the last two bounds, we infer that

$$\|\mathbf{w}\|_{\mathbf{H}^2} + \|q\|_{H^1} \leq C\left(\|\phi\|_{H^1} + \|\mathbf{G}_1\|_{\mathbf{L}^2}\right). \tag{9.61}$$

Step 2: Choosing $\tilde{\tau} = \chi\rho$ in (9.59c), $\tilde{\rho} = -\rho$ in (9.59d), integrating by parts, using (9.59e) and summing the resulting identities, we obtain

$$\begin{aligned}\int_\Omega |\nabla\rho|^2 \,\mathrm{d}x + \mathcal{B}\int_\Omega |\rho|^2 \,\mathrm{d}x + \int_\Omega \mathbb{h}(\varphi_u)|\rho|^2 \mathrm{d}x = -\int_\Omega \big(\mathcal{P}\mathbb{h}(\varphi_u)\phi - \mathcal{P}\mathbb{h}(\varphi_u)q + G_3\big)\rho \,\mathrm{d}x\\ + \chi\int_\Omega G_2\rho - m\nabla\phi\cdot\nabla\rho \,\mathrm{d}x.\end{aligned}$$

Using the boundedness of $\mathbb{h}(\varphi_u) \in L^\infty(\Omega_T)$, the non-negativity of $\mathbb{h}(\cdot)$ and (9.61), this implies that

$$\|\rho\|_{H^1} \leq C\left(\|\phi\|_{H^1} + \|G_2\|_{L^2} + \|G_3\|_{L^2} + \|\mathbf{G}_1\|_{\mathbf{L}^2}\right). \tag{9.62}$$

Choosing $\tilde{\tau} = \tau$ in (9.59c), integrating by parts, using the boundedness of $\nabla\varphi_u \in L^\infty(\mathbf{L}^3)$ and (9.61), we obtain

$$\begin{aligned}\|\tau\|_{L^2}^2 &= \left|\int_\Omega \big(\nabla\varphi_u\cdot\mathbf{w} + m\Delta\phi + G_2\big)\tau \,\mathrm{d}x\right|\\ &\leq C\left(\|\nabla\varphi_u\|_{\mathbf{L}^3}\|\mathbf{w}\|_{\mathbf{L}^6} + \|\Delta\phi\|_{L^2} + \|G_2\|_{L^2}\right)\|\tau\|_{L^2}\\ &\leq C\left(\|\phi\|_{L^2} + \|\mathbf{G}_1\|_{\mathbf{L}^2} + \|\Delta\phi\|_{L^2} + \|G_2\|_{L^2}\right)\|\tau\|_{L^2},\end{aligned}$$

where we applied (9.59e). Consequently, we have

$$\|\tau\|_{L^2} \leq C\left(\|\phi\|_{L^2} + \|\Delta\phi\|_{L^2} + \|\mathbf{G}_1\|_{\mathbf{L}^2} + \|G_2\|_{L^2}\right). \tag{9.63}$$

Step 3: Choosing $\tilde{\phi} = \Delta\phi - \phi$ in (9.59h), integrating by parts, inserting the equation for $\tilde{\tau}$ given by (9.59c) and summing the resulting identities, we obtain

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi\|_{L^2}^2 + \|\nabla\phi\|_{\mathbf{L}^2}^2\right) + m\left(\|\Delta\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{\mathbf{L}^2}^2\right)\\ &= \int_\Omega \big((\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)(\phi - q) + \mathbb{h}'(\varphi_u)\sigma_u\rho + \psi''(\varphi_u)\tau + G_1\big)\big(\phi - \Delta\phi\big) \,\mathrm{d}x\\ &\quad + \int_\Omega \big(\mathrm{div}(\mathbf{v}_u) + u\mathbb{h}'(\varphi_u)\big)\phi(\Delta\phi - \phi) + \big(\phi\mathbf{v}_u - (\mu_u + \chi\sigma_u)\mathbf{w} + \mathbf{G}_2\big)\cdot(\nabla\Delta\phi - \nabla\phi) \,\mathrm{d}x\\ &\quad - \int_\Omega \big(\nabla(\nabla\varphi_u\cdot\mathbf{w}) + \nabla G_2\big)\cdot\nabla\Delta\phi + \big(\nabla\varphi_u\cdot\mathbf{w} + G_2\big)\Delta\phi \,\mathrm{d}x.\end{aligned} \tag{9.64}$$

Using the boundedness of $\mathbb{h}'(\varphi_u), \psi''(\varphi_u) \in L^\infty(\Omega_T)$, $\sigma_u \in L^\infty(L^6)$ and (9.61)-(9.62), we calculate

$$\begin{aligned}&\left|\int_\Omega \big(-\mathcal{A}\mathbb{h}'(\varphi_u)(\phi - q) + \mathbb{h}'(\varphi_u)\sigma_u\rho + G_1\big)\big(\phi - \Delta\phi\big) \,\mathrm{d}x\right|\\ &\leq C\left(\|\phi\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|q\|_{L^2}^2 + \|G_1\|_{L^{\frac{6}{5}}}^2\right) + \frac{m}{16}\left(\|\Delta\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{\mathbf{L}^2}^2\right)\\ &\leq C\left(\|\phi\|_{H^1}^2 + \|G_1\|_{L^{\frac{6}{5}}}^2 + \|G_3\|_{L^2}^2 + \|G_2\|_{L^2}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right)\\ &\quad + \frac{m}{16}\left(\|\Delta\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{\mathbf{L}^2}^2\right).\end{aligned} \tag{9.65a}$$

Using the boundedness of $\mathbb{h}'(\varphi_u) \in L^\infty(\Omega_T)$, $\sigma_u \in L^\infty(\Omega_T)$ and (9.61) yields

$$\begin{aligned}\left|\int_\Omega \mathcal{P}\sigma_u \mathbb{h}'(\varphi_u)(\phi - q)(\phi - \Delta\phi)\,\mathrm{d}x\right| &\le C\left(\|\phi\|_{L^2} + \|q\|_{L^2}\right)\left(\|\phi\|_{L^2} + \|\Delta\phi\|_{L^2}\right) \\ &\le C\left(\|\phi\|_{H^1}^2 + \|q\|_{L^2}^2\right) + \frac{m}{16}\|\Delta\phi\|_{L^2}^2 \\ &\le C\left(\|\phi\|_{H^1}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right) + \frac{m}{16}\|\Delta\phi\|_{L^2}^2. \end{aligned} \tag{9.65b}$$

Furthermore, using the boundedness of $\psi''(\varphi_u) \in L^\infty(\Omega_T)$, (9.63) and the inequality

$$\|\Delta\phi\|_{L^2}^2 \le \|\nabla\phi\|_{\mathbf{L}^2}\|\nabla\Delta\phi\|_{\mathbf{L}^2} \quad \forall\, \phi \in H^3, \quad \nabla\phi\cdot\mathbf{n} = 0 \quad \text{a.e. on } \partial\Omega,$$

we obtain

$$\begin{aligned}\left|\int_\Omega \psi''(\varphi_u)\tau(\phi - \Delta\phi)\,\mathrm{d}x\right| &\le C\left(\|\phi\|_{L^2}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2 + \|G_2\|_{L^2}^2\right) + C\|\Delta\phi\|_{L^2}^2 \\ &\le C\left(\|\phi\|_{L^2}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2 + \|G_2\|_{L^2}^2\right) + C\|\nabla\phi\|_{\mathbf{L}^2}\|\nabla\Delta\phi\|_{\mathbf{L}^2} \\ &\le C\left(\|\phi\|_{H^1}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2 + \|G_2\|_{L^2}^2\right) + \frac{m}{16}\|\nabla\Delta\phi\|_{\mathbf{L}^2}^2. \end{aligned} \tag{9.65c}$$

From the Sobolev embeddings $H^1 \subset L^3$, $\mathbf{H}^1 \subset \mathbf{L}^6$, we obtain

$$\begin{aligned}\left|\int_\Omega \mathrm{div}(\mathbf{v}_u)\phi(\Delta\phi - \phi)\,\mathrm{d}x\right| &\le \|\mathrm{div}(\mathbf{v}_u)\|_{L^6}\|\phi\|_{L^3}\left(\|\phi\|_{L^2} + \|\Delta\phi\|_{L^2}\right) \\ &\le C\left(1 + \|\mathbf{v}_u\|_{\mathbf{H}^2}^2\right)\|\phi\|_{H^1}^2 + \frac{m}{16}\|\Delta\phi\|_{L^2}^2. \end{aligned} \tag{9.65d}$$

Using the Sobolev embeddings $H^1 \subset L^6$, $\mathbf{H}^1 \subset \mathbf{L}^3$, $\mathbf{H}^2 \subset \mathbf{L}^\infty$, (9.61) and the boundedness of $\mu_u + \chi\sigma_u \in L^\infty(L^2)$, we calculate

$$\begin{aligned}&\left|\int_\Omega \left(\phi\mathbf{v}_u - (\mu_u + \chi\sigma_u)\mathbf{w} + \mathbf{G}_2\right)\cdot(\nabla\Delta\phi - \nabla\phi)\,\mathrm{d}x\right| \\ &\quad\le \left(\|\phi\|_{L^6}\|\mathbf{v}_u\|_{\mathbf{L}^3} + \|\mu_u + \chi\sigma_u\|_{L^2}\|\mathbf{w}\|_{\mathbf{L}^\infty} + \|\mathbf{G}_2\|_{\mathbf{L}^2}\right)\left(\|\nabla\phi\|_{\mathbf{L}^2} + \|\nabla\Delta\phi\|_{\mathbf{L}^2}\right) \\ &\quad\le C\left(1 + \|\mathbf{v}_u\|_{\mathbf{H}^1}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right)\|\phi\|_{H^1}^2 + C\left(\|\mathbf{G}_1\|_{\mathbf{L}^2}^2 + \|\mathbf{G}_2\|_{\mathbf{L}^2}\right) + \frac{m}{16}\|\nabla\Delta\phi\|_{\mathbf{L}^2}^2. \end{aligned} \tag{9.65e}$$

With similar arguments and using the boundedness of $\nabla\varphi_u \in L^\infty(\mathbf{L}^6)$, we obtain

$$\begin{aligned}&\left|\int_\Omega \nabla G_2\cdot\nabla\Delta\phi + \left(\nabla\varphi_u\cdot\mathbf{w} + G_2\right)\Delta\phi\,\mathrm{d}x\right| \\ &\quad\le \|\nabla G_2\|_{\mathbf{L}^2}\|\nabla\Delta\phi\|_{\mathbf{L}^2} + \left(\|\nabla\varphi_u\|_{\mathbf{L}^6}\|\mathbf{w}\|_{\mathbf{L}^3} + \|G_2\|_{L^2}\right)\|\Delta\phi\|_{L^2} \\ &\quad\le C\left(\|\phi\|_{H^1}^2 + \|G_2\|_{H^1}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right) + \frac{m}{16}\left(\|\Delta\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{\mathbf{L}^2}^2\right). \end{aligned} \tag{9.65f}$$

Now, with exactly the same arguments as used for (9.30g), we get

$$\left|\int_\Omega u\mathbb{h}'(\varphi_u)\phi(\Delta\phi - \phi)\,\mathrm{d}x\right| \le C\left(1 + \|u\|_{L^2}^2\right)\|\phi\|_{H^1}^2 + \frac{m}{16}\left(\|\Delta\phi\|_{L^2}^2 + \|\nabla\Delta\phi\|_{\mathbf{L}^2}^2\right). \tag{9.65g}$$

It remains to analyse the term

$$\int_\Omega \nabla(\nabla\varphi_u\cdot\mathbf{w})\cdot\nabla\Delta\phi\,\mathrm{d}x = \int_\Omega (\nabla^2\varphi_u\mathbf{w})\cdot\nabla\Delta\phi + (\nabla\mathbf{w}^\intercal\nabla\varphi_u)\cdot\nabla\Delta\phi\,\mathrm{d}x.$$

For the first term, we apply the Sobolev embedding $\mathbf{H}^2 \subset \mathbf{L}^\infty$, the boundedness of $\varphi_u \in L^\infty(H^2)$ and (9.61) to obtain

$$\begin{aligned}\left|\int_\Omega (\nabla^2\varphi_u\mathbf{w})\cdot\nabla\Delta\phi\,\mathrm{d}x\right| &\le \|\nabla^2\varphi_u\|_{\mathbf{L}^2}\|\mathbf{w}\|_{\mathbf{L}^\infty}\|\nabla\Delta\phi\|_{\mathbf{L}^2} \\ &\le C\left(\|\phi\|_{H^1} + \|\mathbf{G}_1\|_{\mathbf{L}^2}\right)\|\nabla\Delta\phi\|_{\mathbf{L}^2} \\ &\le C\left(\|\phi\|_{H^1}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right) + \frac{m}{16}\|\nabla\Delta\phi\|_{\mathbf{L}^2}^2. \end{aligned} \tag{9.65h}$$

With similar arguments and using the Sobolev embeddings $\mathbf{H}^1 \subset \mathbf{L}^6$, $\mathbf{H}^1 \subset \mathbf{L}^3$, we infer that

$$\left|\int_\Omega (\nabla \mathbf{w}^\intercal \nabla \varphi_u) \cdot \nabla \Delta \phi \,\mathrm{d}x\right| \le \|\nabla \varphi_u\|_{\mathbf{L}^6} \|\mathbf{w}\|_{\mathbf{H}^2} \|\nabla \Delta \phi\|_{\mathbf{L}^2}$$

$$\le C\left(\|\phi\|_{H^1}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right) + \frac{m}{16}\|\nabla \Delta \phi\|_{\mathbf{L}^2}^2. \tag{9.65i}$$

Using the estimates (9.65) in (9.64), we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\phi\|_{L^2}^2 + \|\nabla \phi\|_{\mathbf{L}^2}^2\right) + \frac{m}{2}\left(\|\Delta \phi\|_{L^2}^2 + \|\nabla \Delta \phi\|_{\mathbf{L}^2}^2\right) \le \beta_1(t)\|\phi(t)\|_{H^1}^2 + \beta_2(t),$$

where

$$\beta_1(t) := C\left(1 + \|\mathbf{v}_u(t)\|_{\mathbf{H}^2}^2 + \|(\mu_u + \chi\sigma_u)(t)\|_{H^1}^2 + \|\mathbf{G}_1(t)\|_{\mathbf{L}^2}^2 + \|u(t)\|_{L^2}^2 + \|\varphi_u(t)\|_{H^4}^2\right),$$
$$\beta_2(t) := C\left(\|G_1\|_{L^{\frac{6}{5}}}^2 + \|G_2\|_{H^1}^2 + \|G_3\|_{L^2}^2 + \|\mathbf{G}_1\|_{\mathbf{L}^2}^2\right).$$

Due to Proposition 9.4, it is easy to check that $\beta_1,\ \beta_2 \in L^1(0,T)$. Therefore, integrating the last inequality in time from $s \in (0,T)$ to T and using that $\varphi_u \in C^0([0,T];H^1)$ with bounded norm, a Gronwall argument yields

$$\|\phi\|_{L^\infty(H^1)\cap L^2(H^3)} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4}. \tag{9.66}$$

Together with (9.61)-(9.63), this implies

$$\|\tau\|_{L^2(\Omega_T)} + \|\rho\|_{L^2(H^1)} + \|\mathbf{w}\|_{L^2(\mathbf{H}^2)} + \|q\|_{L^2(H^1)} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4}. \tag{9.67}$$

Step 4: We now take $\tilde{\tau} = -\Delta\tau$ in (9.59c) and integrate by parts to get

$$\|\nabla \tau\|_{\mathbf{L}^2}^2 = \int_\Omega \left((\nabla^2 \varphi_u \mathbf{w}) + (\nabla \mathbf{w}^T \nabla \varphi_u) + m\nabla\Delta\phi + \nabla G_2\right) \cdot \nabla \tau \,\mathrm{d}x. \tag{9.68}$$

For the last two terms on the right-hand side of this identity, we easily obtain

$$\left|\int_\Omega (m\nabla \Delta\phi + \nabla G_2) \cdot \nabla \tau \,\mathrm{d}x\right| \le C\left(\|\nabla\Delta\phi\|_{\mathbf{L}^2}^2 + \|G_2\|_{H^1}^2\right) + \frac{1}{4}\|\nabla\tau\|_{\mathbf{L}^2}^2.$$

For the other terms, we use the Sobolev embeddings $\mathbf{H}^1 \subset \mathbf{L}^6$, $\mathbf{H}^1 \subset \mathbf{L}^3$, $\mathbf{H}^2 \subset \mathbf{L}^\infty$, and the boundedness of $\varphi_u \in L^\infty(H^2)$ to deduce

$$\left|\int_\Omega \left((\nabla^2\varphi_u \mathbf{w}) + (\nabla \mathbf{w}^\intercal \nabla\varphi_u)\right) \cdot \nabla\tau \,\mathrm{d}x\right| \le C\|\mathbf{w}\|_{\mathbf{H}^2}^2 + \frac{1}{4}\|\nabla\tau\|_{\mathbf{L}^2}^2.$$

Invoking the last two inequalities in (9.68), we obtain

$$\|\nabla\tau\|_{\mathbf{L}^2}^2 \le C\left(\|\nabla\Delta\phi\|_{\mathbf{L}^2}^2 + \|G_2\|_{H^1}^2 \|\mathbf{w}\|_{\mathbf{H}^2}^2\right).$$

Integrating this inequality in time from 0 to T, using the boundedness of $\varphi_u \in L^2(H^4)$ and (9.66)- (9.67), we infer that

$$\|\nabla\tau\|_{L^2(\mathbf{L}^2)} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4}. \tag{9.69}$$

The relation for $\partial_t\phi$ given by (9.59h) together with (9.66)-(9.67) and (9.69) gives

$$\|\partial_t\phi\|_{L^2((H^1)^*)} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4} \tag{9.70}$$

Using elliptic regularity theory in (9.59d)-(9.59e), we obtain

$$\|\rho\|_{H^2} \le C\|\mathbb{h}(\varphi_u)\rho - \chi\tau + \mathcal{P}\mathbb{h}(\varphi_u)\phi + \chi\nabla\varphi_u\cdot\mathbf{w} - \mathcal{P}\mathbb{h}(\varphi_u)q + G_3\|_{L^2}.$$

Invoking the boundedness of $\nabla\varphi_u \in L^\infty(\mathbf{L}^3)$ and the Sobolev embedding $\mathbf{H}^1 \subset \mathbf{L}^6$, we calculate

$$\|\nabla\varphi_u \cdot \mathbf{w}\|_{L^2} \le C\|\nabla\varphi_u\|_{\mathbf{L}^3}\|\mathbf{w}\|_{\mathbf{L}^6} \le C\|\mathbf{w}\|_{\mathbf{H}^1}.$$

Therefore, using (9.66)-(9.67) and the boundedness of $\mathbb{h}(\varphi_u) \in L^\infty(\Omega_T)$ yields

$$\|\rho\|_{L^2(H^2)} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4}.$$

Together with (9.66)-(9.67) and (9.69)-(9.70), this implies

$$\|(\phi, \tau, \rho, \mathbf{w}, q)\|_{\mathcal{V}_3} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4}.$$

Step 5: Employing the last estimate, we can pass to the limit in the weak formulation of (9.59a)-(9.59h) to obtain the existence of solutions. In particular, we infer that (9.59a)-(9.59f) are fulfilled almost everywhere in the respective sets. We notice that (9.59g) is fulfilled due to the compact embedding $H^1((H^1)^*) \cap L^\infty(H^1) \subset C([0,T]; L^2)$, see Lemma 2.36. Moreover, the estimate (9.60) results from the weak(-star) lower semicontinuity of norms. Finally, uniqueness follows by linearity of the system and because of (9.60). □

Corollary 9.18 *Let $u \in \mathbb{U}_R$ be any control and let $(\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u)$ denote its corresponding state. Then, there exists a unique weak solution $(\phi_u, \tau_u, \rho_u, \mathbf{w}_u, q_u) \in \mathcal{V}_3$ of (ADJ) in the sense of Definition 9.16.*

Proof. This follows from an application of Lemma 9.17 with the following choices:

$$\mathbf{G}_1 = \mathbf{0}, \quad \mathbf{G}_2 = \mathbf{0}, \quad G_0 = \alpha_0(\varphi_u(T) - \varphi_f), \quad G_1 = \alpha_1(\varphi_u - \varphi_d), \quad G_2 = 0, \qquad G_3 = 0.$$

Since $\varphi_u \in C(H^2)$, it follows that $\varphi_u(T) \in H^1$ with bounded norm. Hence, it is easy to check that $(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2) \in \mathcal{V}_4$ with bounded norm. Moreover, using (9.59a)-(9.59g), it is straightforward to check that (9.58a)-(9.58c) and (9.58e)-(9.58f) are fulfilled. This completes the proof. □

Similar to the definition of the control-to-state operator, we can define an operator that maps any control $u \in \mathbb{U}_R$ onto its corresponding adjoint state.

Definition 9.19 We define the **control-to-costate operator** $\mathbb{A}\colon \mathbb{U}_R \to \mathcal{V}_3$ as the operator assigning to every $u \in \mathbb{U}_R$ the unique weak solution $(\phi_u, \tau_u, \rho_u, \mathbf{w}_u, q_u) \in \mathcal{V}_3$ of the adjoint system (ADJ).

9.3.2 Lipschitz continuity

In the following we show that the control-to-costate operator is Lipschitz-continuous.

Proposition 9.20 *For all $u, \tilde{u} \in \mathbb{U}_R$, it holds that*

$$\|\mathbb{A}(\tilde{u}) - \mathbb{A}(u)\|_{\mathcal{V}_3} \le L_3\|\tilde{u} - u\|_{L^2(L^2)} \tag{9.71}$$

with a constant L_3 independent of u and $\tilde{u}$.

Proof. We first define

$$(\phi, \tau, \rho, \mathbf{w}, q) := \mathbb{A}(\tilde{u}) - \mathbb{A}(u) = (\phi_{\tilde{u}}, \tau_{\tilde{u}}, \rho_{\tilde{u}}, \mathbf{w}_{\tilde{u}}, q_{\tilde{u}}) - (\phi_u, \tau_u, \rho_u, \mathbf{w}_u, q_u)$$

and introduce the variable

$$\pi := q - \phi_{\tilde{u}}(\varphi_{\tilde{u}} - \varphi_u).$$

Then, the quintuple $(\phi, \tau, \rho, \mathbf{w}, \pi)$ fulfils (9.59) with

$$\begin{aligned}
\mathbf{G}_1 &= -\phi_{\tilde{u}} \nabla(\varphi_{\tilde{u}} - \varphi_u),\\
\mathbf{G}_2 &= \phi_{\tilde{u}}(\mathbf{v}_{\tilde{u}} - \mathbf{v}_u) - \mathbf{w}_{\tilde{u}}\big((\mu_{\tilde{u}} + \chi\sigma_{\tilde{u}}) - (\mu_u + \chi\sigma_u)\big),\\
G_0 &= \alpha_0(\varphi_{\tilde{u}}(T) - \varphi_u(T)),\\
G_1 &= \phi_{\tilde{u}}[\big(\mathcal{P}(\sigma_{\tilde{u}} - \sigma_u) - (\tilde{u} - u)\big)\mathbb{h}'(\varphi_u)] + \phi_{\tilde{u}}(\mathcal{P}\sigma_{\tilde{u}} - \mathcal{A} - \tilde{u})\big(\mathbb{h}'(\varphi_{\tilde{u}}) - \mathbb{h}'(\varphi_u)\big)\\
&\quad + \rho_{\tilde{u}}\big(\mathbb{h}'(\varphi_u)(\sigma_{\tilde{u}} - \sigma_u) + \sigma_{\tilde{u}}(\mathbb{h}'(\varphi_{\tilde{u}}) - \mathbb{h}'(\varphi_u))\big) + (\psi''(\varphi_{\tilde{u}}) - \psi''(\varphi_u))\tau_{\tilde{u}}\\
&\quad - q_{\tilde{u}}[\mathcal{P}(\sigma_{\tilde{u}} - \sigma_u)\mathbb{h}'(\varphi_u) + (\mathcal{P}\sigma_{\tilde{u}} - \mathcal{A})(\mathbb{h}'(\varphi_{\tilde{u}}) - \mathbb{h}'(\varphi_u))] - (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\phi_{\tilde{u}}(\varphi_{\tilde{u}} - \varphi_u)\\
&\quad - \phi_{\tilde{u}}\mathrm{div}(\mathbf{v}_{\tilde{u}} - \mathbf{v}_u) + \alpha_1(\varphi_{\tilde{u}} - \varphi_u),\\
G_2 &= \nabla(\varphi_{\tilde{u}} - \varphi_u) \cdot \mathbf{w}_{\tilde{u}},\\
G_3 &= \rho_{\tilde{u}}(\mathbb{h}(\varphi_{\tilde{u}}) - \mathbb{h}(\varphi_u)) + \mathcal{P}\phi_{\tilde{u}}(\mathbb{h}(\varphi_{\tilde{u}}) - \mathbb{h}(\varphi_u)) + \chi\nabla(\varphi_{\tilde{u}} - \varphi_u) \cdot \mathbf{w}_{\tilde{u}}\\
&\quad - \mathcal{P}\mathbb{h}(\varphi_u)\phi_{\tilde{u}}(\varphi_{\tilde{u}} - \varphi_u) - \mathcal{P}q_{\tilde{u}}(\mathbb{h}(\varphi_{\tilde{u}}) - \mathbb{h}(\varphi_u)).
\end{aligned}$$

Using (9.2), (9.23) and the mean value theorem, it is easy to check that

$$\|\psi''(\varphi_{\tilde{u}}) - \psi''(\varphi_u)\|_{L^\infty(L^\infty)} + \|\mathbb{h}'(\varphi_{\tilde{u}}) - \mathbb{h}'(\varphi_u)\|_{L^\infty(L^\infty)} \le C\|\varphi_{\tilde{u}} - \varphi_u\|_{L^\infty(L^\infty)}.$$

Then, using Proposition 9.4, Corollaries 9.9 and 9.18, a straightforward calculation shows that

$$\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4} \le C\|\tilde{u} - u\|_{L^2(L^2)}.$$

Consequently, the estimate (9.60) implies that

$$\|(\phi, \tau, \rho, \mathbf{w}, \pi)\|_{\mathcal{V}_3} \le C\|(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)\|_{\mathcal{V}_4} \le C\|\tilde{u} - u\|_{L^2(L^2)}.$$

Recalling the definitions of π and $\mathcal{V}_3$, it remains to show that

$$\|\phi_{\tilde{u}}(\varphi_{\tilde{u}} - \varphi_u)\|_{L^2(H^1)} \le C\|\tilde{u} - u\|_{L^2(L^2)}.$$

However, this is another easy consequence of Corollaries 9.9 and 9.18. Therefore, it follows that

$$\|(\phi, \tau, \rho, \mathbf{w}, q)\|_{\mathcal{V}_3} \le C\|\tilde{u} - u\|_{L^2(L^2)}$$

which completes the proof. □

9.3.3 Fréchet differentiability

We can also show that the control-to-costate operator is continuously Fréchet differentiable.

Proposition 9.21 *The following statements hold:*

(i) *the control-to-costate operator $\mathbb{A}$ is Frechét-differentiable on $\mathbb{U}_R$, i. e., for any $u \in \mathbb{U}_R$ there exists a unique bounded, linear operator*

$$\mathbb{A}'(u)\colon L^2(L^2) \to \mathcal{V}_3, \quad h \mapsto \mathbb{A}'(u)[h] = (\phi'_u, \tau'_u, \rho'_u, \mathbf{w}'_u, q'_u)[h]$$

such that

$$\frac{\|\mathbb{A}(u+h) - \mathbb{A}(u) - \mathbb{A}'(u)[h]\|_{\mathcal{V}_3}}{\|h\|_{L^2(L^2)}} \to 0 \quad \textit{as} \quad \|h\|_{L^2(L^2)} \to 0.$$

For any $\bar{u} \in \mathbb{U}$ and $h \in L^2(L^2)$, the Fréchet-derivative $(\phi'_u, \tau'_u, \rho'_u, \mathbf{w}'_u, q'_u)[h]$ is the unique solution of (9.59) *with*

$$
\begin{aligned}
G_0 &= \alpha_0 \varphi'_u[h](T),\\
G_1 &= (\mathcal{P}\sigma_u - \mathcal{A} - u)\mathbb{h}''(\varphi_u)\varphi'_u[h]\phi_u + \mathbb{h}'(\varphi_u)(\mathcal{P}\sigma'_u[h] - h)\phi_u\\
&\quad + \big(\mathbb{h}'(\varphi_u)\sigma'_u[h] + \mathbb{h}''(\varphi_u)\varphi'_u[h]\sigma_u\big)\rho_u + \psi^{(3)}(\varphi_u)\varphi'_u[h]\tau_u\\
&\quad - \big(\mathcal{P}\sigma'_u[h]\mathbb{h}'(\varphi_u) + (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}''(\varphi_u)\varphi'_u[h]\big)q_u - \phi_u \operatorname{div}(\mathbf{v}'_u[h]) + \alpha_1\varphi'_u[h],\\
G_2 &= \nabla\varphi'_u[h]\cdot \mathbf{w}_u,\\
G_3 &= \mathbb{h}'(\varphi_u)\varphi'_u[h]\rho_u + \mathcal{P}\mathbb{h}'(\varphi_u)\varphi'_u[h]\phi_u + \chi\nabla\varphi'_u[h]\cdot\mathbf{w}_u - \mathcal{P}\mathbb{h}'(\varphi_u)\varphi'_u[h]q_u,\\
\mathbf{G}_1 &= \varphi'_u[h]\nabla\phi_u,\\
\mathbf{G}_2 &= \phi_u\mathbf{v}'_u[h] - (\mu'_u[h] + \chi\sigma'_u[h])\mathbf{w}_u,
\end{aligned}
$$

and (9.59f) *replaced by*

$$(2\eta\mathbf{D}\mathbf{w} - q\mathbf{I} + \varphi_u\phi\mathbf{I} + \varphi'_u[h]\phi_u\mathbf{I})\mathbf{n} = 0 \quad \textit{a. e. on } \Sigma_T. \tag{9.72}$$

(ii) *the Frechet-derivative is Lipschitz continuous, i. e., for any $u, \tilde{u} \in \mathbb{U}_R$ it holds that*

$$\|\mathbb{A}'(u) - \mathbb{A}'(\tilde{u})\|_{\mathcal{L}(L^2(L^2);\mathcal{V}_3)} \le L_4\|u - \tilde{u}\|_{L^2(L^2)} \tag{9.73}$$

with a constant $L_4 > 0$ independent of u and $\tilde{u}$.

Proof. The proof proceeds similarly to the proof of Proposition 9.14.

Proof of (i)**:** Existence of a solution to (9.59) with the above choices for G_0, G_1, G_2, G_3, $\mathbf{G}_1$ and $\mathbf{G}_2$ follows from a simple pressure reformulation argument. Indeed, let us define

$$
\begin{aligned}
&\tilde{\mathbf{G}}_1 = -\phi_u\nabla\varphi'_u[h], \quad \tilde{G}_1 = G_1 - (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\varphi'_u[h]\phi_u,\\
&\tilde{G}_3 = G_3 - \mathcal{P}\mathbb{h}(\varphi_u)\varphi'_u[h]\phi_u, \quad \tilde{G}_0 = G_0, \quad \tilde{G}_2 = G_2, \quad \tilde{\mathbf{G}}_2 = \mathbf{G}_2.
\end{aligned}
$$

By Propositions 9.4 and 9.14 and Lemma 9.17, we can check that $(\tilde{G}_0, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_3) \in \mathcal{V}_4$ with bounded norm. Therefore, there exists a unique weak solution $(\phi, \tau, \rho, \mathbf{w}, \pi) \in \mathcal{V}_3$ of (9.59) with $(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2) = (\tilde{G}_0, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_3) \in \mathcal{V}_4$. We now define

$$q = \pi + \varphi'_u[h]\phi_u.$$

Using Proposition 9.4 and Lemma 9.17, it holds that $\varphi'_u[h]\phi_u \in L^2(H^1)$ with bounded norm. Therefore, the quintuple that $(\phi, \tau, \rho, \mathbf{w}, q) \in \mathcal{V}_3$ is a weak solution of (9.59) with $(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2)$ as above and (9.59f) replaced by (9.72). Uniqueness of solutions of this system follows due to linearity of the system and estimate (9.60).

In the following we define

$$
\begin{aligned}
(\phi^h_{\mathcal{R}}, \tau^h_{\mathcal{R}}, \rho^h_{\mathcal{R}}, \mathbf{w}^h_{\mathcal{R}}, q^h_{\mathcal{R}}) &= (\phi_{u+h}, \tau_{u+h}, \rho_{u+h}, \mathbf{w}_{u+h}, q_{u+h}) - (\phi_u, \tau_u, \rho_u, \mathbf{w}_u, q_u)\\
&\quad - (\phi'_u[h], \tau'_u[h], \rho'_u[h], \mathbf{w}'_u[h], q'_u[h]).
\end{aligned}
$$

Moreover, we recall that the definition of $(\varphi^h_{\mathcal{R}}, \mu^h_{\mathcal{R}}, \mathbf{v}^h_{\mathcal{R}}, \sigma^h_{\mathcal{R}}, p^h_{\mathcal{R}})$ given by (9.51). Then, we can check that $(\phi^h_{\mathcal{R}}, \tau^h_{\mathcal{R}}, \rho^h_{\mathcal{R}}, \mathbf{w}^h_{\mathcal{R}}, q^h_{\mathcal{R}})$ is the solution of (9.59) with

$$
\begin{aligned}
G_0 &= \alpha_0 \varphi^h_{\mathcal{R}}(T),\\
G_1 &= (\mathcal{P}\sigma_u - \mathcal{A} - u)\left[\mathbb{h}''(\varphi_u)\varphi^h_{\mathcal{R}} + \frac{1}{2}\mathbb{h}^{(3)}(\xi)(\varphi_{u+h}-\varphi_u)^2\right]\phi_u + \mathcal{P}\sigma^h_{\mathcal{R}}\mathbb{h}'(\varphi_u)\phi_u\\
&\quad + [\mathcal{P}(\sigma_{u+h}-\sigma_u) - h](\mathbb{h}'(\varphi_{u+h}) - \mathbb{h}'(\varphi_u))\phi_u\\
&\quad + (\phi_{u+h}-\phi_u)\left[(\mathcal{P}\sigma_u - \mathcal{A} - u)(\mathbb{h}'(\varphi_{u+h}) - \mathbb{h}'(\varphi_u)) + \big(\mathcal{P}(\sigma_{u+h}-\sigma_u) - h\big)\mathbb{h}'(\varphi_{u+h})\right]\\
&\quad + \rho_u\left[\mathbb{h}''(\varphi_u)\varphi^h_{\mathcal{R}}\sigma_u + \frac{1}{2}\mathbb{h}^{(3)}(\xi)(\varphi_{u+h}-\varphi_u)^2\sigma_u\right]\\
&\quad + \rho_u\left[\mathbb{h}'(\varphi_u)\sigma^h_{\mathcal{R}} + (\mathbb{h}'(\varphi_{u+h}) - \mathbb{h}'(\varphi_u))(\sigma_{u+h}-\sigma_u)\right]\\
&\quad + (\rho_{u+h}-\rho_u)\left[(\mathbb{h}'(\varphi_{u+h}) - \mathbb{h}'(\varphi_u))\sigma_u + \mathbb{h}'(\varphi_{u+h})(\sigma_{u+h}-\sigma_u)\right]\\
&\quad + \tau_u\left[\psi^{(3)}(\varphi_u)\varphi^h_{\mathcal{R}} + \psi^{(4)}(\xi)(\varphi_{u+h}-\varphi_u)^2\right] + (\tau_{u+h}-\tau_u)(\psi''(\varphi_{u+h}) - \psi''(\varphi_u))\\
&\quad - (\mathcal{P}\sigma_u - \mathcal{A})\left[\mathbb{h}''(\varphi_u)\varphi^h_{\mathcal{R}} + \frac{1}{2}\mathbb{h}^{(3)}(\xi)(\varphi_{u+h}-\varphi_u)^2\right]q_u - \mathcal{P}\sigma^h_{\mathcal{R}}\mathbb{h}'(\varphi_u)q_u\\
&\quad - \mathcal{P}(\sigma_{u+h}-\sigma_u)(\mathbb{h}'(\varphi_{u+h}) - \mathbb{h}'(\varphi_u))q_u\\
&\quad - (q_{u+h}-q_u)\left[(\mathcal{P}\sigma_u - \mathcal{A})(\mathbb{h}'(\varphi_{u+h}) - \mathbb{h}'(\varphi_u)) + \mathcal{P}(\sigma_{u+h}-\sigma_u)\mathbb{h}'(\varphi_{u+h})\right]\\
&\quad - \phi_u \operatorname{div}(\mathbf{v}^h_{\mathcal{R}}) - (\phi_{u+h}-\phi_u)\operatorname{div}(\mathbf{v}_{u+h}-\mathbf{v}_u) + \alpha_1\varphi^h_{\mathcal{R}},\\
G_2 &= \nabla\varphi^h_{\mathcal{R}}\cdot\mathbf{w}_u + \nabla(\varphi_{u+h}-\varphi_u)\cdot(\mathbf{w}_{u+h}-\mathbf{w}_u),\\
G_3 &= [\mathbb{h}'(\varphi_u)\varphi^h_{\mathcal{R}} + \frac{1}{2}\mathbb{h}''(\xi)(\varphi_{u+h}-\varphi_u)^2]\rho_u + (\mathbb{h}(\varphi_{u+h}) - \mathbb{h}(\varphi_u))(\rho_{u+h}-\rho_u)\\
&\quad + \mathcal{P}[\mathbb{h}'(\varphi_u)\varphi^h_{\mathcal{R}} + \frac{1}{2}\mathbb{h}''(\xi)(\varphi_{u+h}-\varphi_u)^2](\phi_u - q_u)\\
&\quad + \mathcal{P}(\mathbb{h}(\varphi_{u+h}) - \mathbb{h}(\varphi_u))[(\phi_{u+h}-\phi_u) - (q_{u+h}-q_u)]\\
&\quad + \chi\nabla\varphi^h_{\mathcal{R}}\cdot(\mathbf{w}_{u+h}-\mathbf{w}_u) + \chi\nabla(\varphi_{u+h}-\varphi_u)\cdot(\mathbf{w}_{u+h}-\mathbf{w}_u),\\
\mathbf{G}_1 &= \varphi^h_{\mathcal{R}}\nabla\phi_u + (\varphi_{u+h}-\varphi_u)\nabla(\phi_{u+h}-\phi_u),\\
\mathbf{G}_2 &= \phi_u\mathbf{v}^h_{\mathcal{R}} + (\phi_{u+h}-\phi_u)(\mathbf{v}_{u+h}-\mathbf{v}_u) - (\mu^h_{\mathcal{R}} + \chi\sigma^h_{\mathcal{R}})\mathbf{w}_u\\
&\quad - [(\mu_{u+h} + \chi\sigma_{u+h}) - (\mu_u + \chi\sigma_u)](\mathbf{w}_{u+h}-\mathbf{w}_u),
\end{aligned}
$$

and (9.59f) replaced by

$$
\big(2\eta\mathbf{D}\mathbf{w} - q\mathbf{I} + \varphi_u\phi\mathbf{I} + \varphi^h_{\mathcal{R}}\phi_u\mathbf{I} + (\varphi_{u+h}-\varphi_u)(\phi_{u+h}-\phi_u)\mathbf{I}\big)\,\mathbf{n} = 0 \quad \text{a.e. on } \Sigma_T. \tag{9.74}
$$

We now introduce a new pressure

$$
\pi^h_{\mathcal{R}} = q^h_{\mathcal{R}} + \varphi^h_{\mathcal{R}}\phi_u + (\varphi_{u+h}-\varphi_u)(\phi_{u+h}-\phi_u)
$$

and we define

$$
\begin{aligned}
\tilde{\mathbf{G}}_1 &= -\phi_u\nabla\varphi^h_{\mathcal{R}} - (\phi_{u+h}-\phi_u)\nabla(\varphi_{u+h}-\varphi_u),\\
\tilde{G}_1 &= G_1 - (\mathcal{P}\sigma_u - \mathcal{A})\mathbb{h}'(\varphi_u)\left[\varphi^h_{\mathcal{R}}\phi_u + (\varphi_{u+h}-\varphi_u)(\phi_{u+h}-\phi_u)\right],\\
\tilde{G}_3 &= G_3 - \mathcal{P}\mathbb{h}(\varphi_u)\left[\varphi^h_{\mathcal{R}}\phi_u + (\varphi_{u+h}-\varphi_u)(\phi_{u+h}-\phi_u)\right],\\
\tilde{G}_0 &= G_0, \quad \tilde{G}_2 = G_2, \quad \tilde{\mathbf{G}}_2 = \mathbf{G}_2.
\end{aligned}
$$

Then, we can check that $(\phi^h_{\mathcal{R}}, \tau^h_{\mathcal{R}}, \rho^h_{\mathcal{R}}, \mathbf{w}^h_{\mathcal{R}}, \pi^h_{\mathcal{R}})$ solves (9.59) with $(G_0, G_1, G_2, G_3, \mathbf{G}_1, \mathbf{G}_2) = (\tilde{G}_0, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_2)$.

Using Propositions 9.4 and 9.14, Corollaries 9.9 and 9.18, and Lemma 9.17, it can be checked that

$$\|(\phi^h_{\mathcal{R}}, \tau^h_{\mathcal{R}}, \rho^h_{\mathcal{R}}, \mathbf{w}^h_{\mathcal{R}}, \pi^h_{\mathcal{R}})\|_{\mathcal{V}_3} \leq C\|(\tilde{G}_0, \tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{\mathbf{G}}_1, \tilde{\mathbf{G}}_2)\|_{\mathcal{V}_4} \leq C\|h\|^2_{L^2(L^2)}.$$

Employing this inequality together with Corollaries 9.9 and 9.18, Proposition 9.14 and Lemma 9.17, recalling the definition of $\mathcal{V}_3$ and the expression for $\pi^h_{\mathcal{R}}$, it follows that

$$\begin{aligned}\|q^h_{\mathcal{R}}\|_{L^2(H^1)} &\leq \|\pi^h_{\mathcal{R}}\|_{L^2(H^1)} + \|\phi_u\|_{L^\infty(H^1)}\|\varphi^h_{\mathcal{R}}\|_{L^\infty(H^2)} + \|\phi_{u+h} - \phi_u\|_{L^\infty(H^1)}\|\varphi_{u+h} - \varphi_u\|_{L^\infty(H^2)} \\ &\leq C\|h\|^2_{L^2(L^2)}.\end{aligned}$$

In summary, we obtain

$$\|(\phi^h_{\mathcal{R}}, \tau^h_{\mathcal{R}}, \rho^h_{\mathcal{R}}, \mathbf{w}^h_{\mathcal{R}}, q^h_{\mathcal{R}})\|_{\mathcal{V}_3} \leq C\|h\|^2_{L^2(L^2)}.$$

Proof of (ii)**:** Since the operator $\mathbb{S}'(\cdot)[h]\colon \mathbb{U}_R \to \mathcal{V}_1$ is Lipschitz-continuous for all $h \in L^2(L^2)$, the proof follows with similar arguments as the proof of Proposition 9.20. □

9.4 The optimal control problem

In this section we analyse the optimal control problem that was motivated in the introduction: We intend to minimize the ***cost functional***

$$I(\varphi, u) := \frac{\alpha_0}{2}\|\varphi(T) - \varphi_f\|^2_{L^2} + \frac{\alpha_1}{2}\|\varphi - \varphi_d\|^2_{L^2(L^2)} + \frac{\kappa}{2}\|u\|^2_{L^2(L^2)}$$

subject to the following conditions:

- u is an admissible control, i. e., $u \in \mathbb{U}$,
- $(\varphi, \mu, \sigma, \mathbf{v}, p)$ is a strong solution of the system (CHB) to the control u.

Using the control-to-state operator we can formulate this optimal control problem alternatively as

$$\text{Minimize} \quad J(u) \quad \text{s. t.} \quad u \in \mathbb{U}, \tag{9.75}$$

where the ***reduced cost functional*** J is defined by

$$J(u) := I\big([\mathbb{S}(u)]_1, u\big) = I(\varphi_u, u) \quad \forall\, u \in \mathbb{U} \tag{9.76}$$

with $[\mathbb{S}(u)]_1$ denoting the first component of the control-to-state operator. A globally/locally optimal control of this optimal control problem is defined as follows:

Definition 9.22 Let $\bar{u} \in \mathbb{U}$ be any admissible control.

(a) We call $\bar{u}$ a **(globally) optimal control** of the problem (9.75) if $J(\bar{u}) \leq J(u)$ for all $u \in \mathbb{U}$.

(b) We call $\bar{u}$ a **locally optimal control** of the problem (9.75) if there exists some $\delta > 0$ such that $J(\bar{u}) \leq J(u)$ for all $u \in \mathbb{U}$ with $\|u - \bar{u}\|_{L^2(L^2)} < \delta$.

In this case, $\mathbb{S}(\bar{u})$ is called the corresponding **globally/locally optimal state**.

9.4.1 Existence of a globally optimal control

Of course, the optimal control problem (9.75) does only make sense if there exists at least one globally optimal solution. This is established by the following theorem:

Theorem 9.23 *The optimization problem* (9.75) *possesses a globally optimal solution.*

Proof. This result can be proved by the direct method of calculus of variations. Obviously, the functional J is bounded from below by zero. Therefore, the infimum $m := \inf_{u\in\mathbb{U}} J(u)$ exists and we can find a minimizing sequence $(u_k) \subset \mathbb{U}$ with $J(u_k) \to m$ as $k \to \infty$. As the set $\mathbb{U}$ is weakly sequentially compact, there exists $\bar{u} \in \mathbb{U}$ such that $u_k \rightharpoonup \bar{u}$ in $L^2(L^2)$ after extraction of a subsequence. Now, according to Lemma 9.10 we obtain that

$$\begin{aligned}
&\varphi_{u_k} \rightharpoonup \varphi_{\bar{u}} \quad \text{in } H^1(L^2) \cap L^2(H^4), && \varphi_{u_k} \to \varphi_{\bar{u}} \quad \text{in } C\big([0,T]; W^{1,r}\big) \cap C\big(\overline{\Omega_T}\big), \quad r \in [1,6),\\
&\mu_{u_k} \rightharpoonup \mu_{\bar{u}} \quad \text{in } L^2(H^2), && \mathbf{v}_{u_k} \rightharpoonup \mathbf{v}_{\bar{u}} \quad \text{in } L^2(H^2),\\
&\sigma_{u_k} \rightharpoonup \sigma_{\bar{u}} \quad \text{in } L^2(H^2), && p_{u_k} \rightharpoonup p_{\bar{u}} \quad \text{in } L^2(H^1)
\end{aligned}$$

after another subsequence extraction (in particular, it follows that $\varphi_{u_k}(T) \to \varphi_{\bar{u}}(T)$ in L^2). Furthermore, Lemma 9.10 yields that

$$\mathbb{S}(\bar{u}) = (\varphi_{\bar{u}}, \mu_{\bar{u}}, \sigma_{\bar{u}}, \mathbf{v}_{\bar{u}}, p_{\bar{u}}),$$

hence $(\bar{u}, \mathbb{S}(\bar{u}))$ is an admissible control-state pair. From the weak lower semicontinuity of the cost functional J we can conclude that

$$J(\bar{u}) \leq \liminf_{k\to\infty} J(u_k) = \lim_{k\to\infty} J(u_k) = m,$$

and $J(\bar{u}) = m$ immediately follows by the definition of the infimum. This means that $\bar{u}$ is a globally optimal control with corresponding state $\mathbb{S}(\bar{u}) = (\varphi_{\bar{u}}, \mu_{\bar{u}}, \sigma_{\bar{u}}, \mathbf{v}_u, p_{\bar{u}})$. □

9.4.2 First-order necessary conditions for local optimality

Obviously, Theorem 9.23 does not provide uniqueness of the globally optimal control $\bar{u}$. As the control-to-state operator is nonlinear we cannot expect the cost functional to be convex. Therefore, it is possible that the optimization problem has several locally optimal controls or even several globally optimal controls. In the following, since numerical methods will (in general) only detect local minimizers, our goal is to characterize locally optimal controls by necessary optimality conditions.

Since the control-to-state operator is Fréchet differentiable according to Proposition 9.14, Fréchet differentiability of the cost functional easily follows by the chain rule. If $\bar{u} \in \mathbb{U}$ is a locally optimal control, it must hold that $J'(\bar{u})[u - \bar{u}] \geq 0$ for all $u \in \mathbb{U}$. The Fréchet derivative $J'(\bar{u})$ can be described by means of the so-called adjoint state that was introduced above.

In the following we characterize locally optimal controls of (9.75) by necessary conditions which are particularly important for computational methods. The adjoint variables can be used to express the variational inequality in a very concise form.

Theorem 9.24 *Let $\bar{u} \in \mathbb{U}$ be a locally optimal control of the minimization problem* (9.75)*. Then, $\bar{u}$ satisfies the* ***variational inequality***

$$J'(\bar{u})[u - \bar{u}] = \int_{\Omega_T} \big[\kappa\bar{u} - \phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}})\big](u - \bar{u}) \,\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \textit{for all}\, u \in \mathbb{U}. \tag{9.77}$$

Proof. In Proposition 9.14 we have shown that the control-to-state operator is Fréchet differentiable with respect to the norm on $\mathcal{V}_1$. Fréchet differentiability of the reduced cost functional J immediately follows. Its derivative can be computed by the chain rule. Hence, if $\bar{u}$ is a locally optimal control, the following inequality must hold:

$$0 \le J'(\bar{u})[u-\bar{u}] = \alpha_0 \int_\Omega (\varphi_{\bar{u}}(T) - \varphi_f)\varphi'_{\bar{u}}[u-\bar{u}](T)\,\mathrm{d}x + \alpha_1 \int_{\Omega_T} (\varphi_{\bar{u}} - \varphi_d)\varphi'_{\bar{u}}[u-\bar{u}]\,\mathrm{d}x \\ + \int_{\Omega_T} \kappa\bar{u}(u-\bar{u})\,\mathrm{d}x. \tag{9.78}$$

Therefore, it remains to show that the sum of the first two terms on the right-hand side of (9.78) is equal to $-\int_{\Omega_T} \phi_{\bar{u}} \mathbb{h}(\varphi_{\bar{u}})(u-\bar{u})\,\mathrm{d}x$. For brevity and to reduce the amount of indices we write $h := u - \bar{u}$ and

$$(\varphi, \mu, \sigma, \mathbf{v}, p) := (\varphi_{\bar{u}}, \mu_{\bar{u}}, \sigma_{\bar{u}}, \mathbf{v}_{\bar{u}}, p_{\bar{u}}), \qquad (\phi, \tau, \rho, \mathbf{w}, q) := (\phi_{\bar{u}}, \tau_{\bar{u}}, \rho_{\bar{u}}, \mathbf{w}_{\bar{u}}, q_{\bar{u}}),$$
$$(\tilde{\varphi}, \tilde{\mu}, \tilde{\sigma}, \tilde{\mathbf{v}}, \tilde{p}) := (\varphi'_{\bar{u}}[u-\bar{u}],\, \mu'_{\bar{u}}[u-\bar{u}],\, \sigma'_{\bar{u}}[u-\bar{u}],\, \mathbf{v}'_{\bar{u}}[u-\bar{u}],\, p'_{\bar{u}}[u-\bar{u}]).$$

In the following the strategy is to test the weak formulations of the linearised system (which produces the Fréchet derivative) with the adjoint variables. Using ϕ as a test function in (9.25c) with $F_2 = -h\,\mathbb{h}(\varphi)$ yields

$$0 = \int_0^T \langle \partial_t \tilde{\varphi}, \phi \rangle_{H^1}\,\mathrm{d}t + \int_{\Omega_T} m\nabla\tilde{\mu}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} \big[\mathrm{div}(\varphi\tilde{\mathbf{v}}) + \mathrm{div}(\tilde{\varphi}\mathbf{v}) - (\mathcal{P}\sigma - \mathcal{A} - u)\mathbb{h}'(\varphi)\tilde{\varphi} + h\,\mathbb{h}(\varphi) - \mathcal{P}\mathbb{h}(\varphi)\tilde{\sigma}\big]\,\phi\,\mathrm{d}x\,\mathrm{d}t. \tag{9.79}$$

Since both $\tilde{\varphi}$ and ϕ lie in $H^1\big((H^1)^*\big) \cap L^2(H^1)$ integration by parts with respect to t is permitted. We obtain

$$\int_0^T \langle \partial_t \tilde{\varphi}, \phi \rangle_{H^1}\,\mathrm{d}t = \alpha_0 \int_\Omega \tilde{\varphi}(T)\big(\varphi(T) - \varphi_f\big)\,\mathrm{d}x - \int_0^T \langle \partial_t \phi, \tilde{\varphi} \rangle_{H^1}\,\mathrm{d}t$$

because of the initial condition $\tilde{\varphi}(0) = 0$ and the final condition $\phi(T) = \alpha_0(\varphi(T) - \varphi_f)$ which are satisfied almost everywhere in Ω. The term $\partial_t\phi$ can be replaced using the weak formulation (9.58d) tested with $\tilde{\varphi}$. We obtain that

$$0 = \alpha_0 \int_\Omega \tilde{\varphi}(T)\big(\varphi(T) - \varphi_f\big)\,\mathrm{d}x + \int_{\Omega_T} \nabla\tau\cdot\nabla\tilde{\varphi} + (\mu + \chi\sigma)\nabla\tilde{\varphi}\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} \mathbb{h}'(\varphi)\sigma\rho\tilde{\varphi} + \psi''(\varphi)\tau\tilde{\varphi} - (\mathcal{P}\sigma - \mathcal{A})\mathbb{h}'(\varphi)q\tilde{\varphi} + \alpha_1(\varphi - \varphi_d)\tilde{\varphi}\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\Omega_T} \phi\,\mathrm{div}(\varphi\tilde{\mathbf{v}}) - \mathcal{P}\mathbb{h}(\varphi)\tilde{\sigma}\phi + h\mathbb{h}(\varphi)\phi + m\nabla\tilde{\mu}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.$$

Since $\mathrm{div}(\mathbf{w}) = 0$ almost everywhere in Ω_T, we have $\mathbf{T}(\tilde{\mathbf{v}}, \tilde{p}) : \nabla\mathbf{w} = 2\eta\mathbf{D}\tilde{\mathbf{v}} : \nabla\mathbf{w} = 2\eta\mathbf{D}\mathbf{w} : \nabla\tilde{\mathbf{v}}$. This identity, the weak formulation (9.25b) tested with $\mathbf{w}$ and the weak formulation (9.58c) tested with $\tilde{\mathbf{v}}$ can be used to deduce that

$$0 = \int_{\Omega_T} \mathbf{T}(\tilde{\mathbf{v}}, \tilde{p}) : \nabla\mathbf{w} + \nu\tilde{\mathbf{v}}\mathbf{w} - (\mu + \chi\sigma)\nabla\tilde{\varphi}\cdot\mathbf{w} - (\tilde{\mu} + \chi\tilde{\sigma})\nabla\varphi\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} \big[q\,\mathrm{div}(\tilde{\mathbf{v}}) - \phi\nabla\varphi\cdot\tilde{\mathbf{v}} - \varphi\phi\,\mathrm{div}(\tilde{\mathbf{v}}) - (\mu + \chi\sigma)\nabla\tilde{\varphi}\cdot\mathbf{w} - (\tilde{\mu} + \chi\tilde{\sigma})\nabla\varphi\cdot\mathbf{w}\big]\,\mathrm{d}x\,\mathrm{d}t.$$

Proceeding similarly with the remaining linearised equations and adjoint variables gives

$$
\begin{aligned}
0 &= \int_{\Omega_T} \tilde{\mu}\tau - \psi''(\varphi)\tilde{\varphi}\tau + \chi\tilde{\sigma}\tau - \nabla\tilde{\varphi}\cdot\nabla\tau \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T} \nabla\varphi\cdot\mathbf{w}\tilde{\mu} - m\nabla\phi\cdot\nabla\tilde{\mu} - \psi''(\varphi)\tilde{\varphi}\tau + \chi\tilde{\sigma}\tau - \nabla\tilde{\varphi}\cdot\nabla\tau \,\mathrm{d}x\,\mathrm{d}t, \\
0 &= \int_{\Omega_T} -q\,\mathrm{div}(\tilde{\mathbf{v}}) + \mathcal{P}\tilde{\sigma}\mathbb{h}(\varphi)q + (\mathcal{P}\sigma - \mathcal{A})\mathbb{h}'(\varphi)\tilde{\varphi}q \,\mathrm{d}x\,\mathrm{d}t, \\
0 &= -\int_{\Omega_T} \nabla\tilde{\sigma}\cdot\nabla\rho + \mathcal{B}\tilde{\sigma}\rho + \mathbb{h}'(\varphi)\tilde{\varphi}\sigma\rho + \mathbb{h}(\varphi)\tilde{\sigma}\rho \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T} \mathcal{P}\mathbb{h}(\varphi)\phi\tilde{\sigma} + \chi\nabla\varphi\cdot\mathbf{w}\tilde{\sigma} - \mathcal{P}\mathbb{h}(\varphi)q\tilde{\sigma} - \chi\tau\tilde{\sigma} - \mathbb{h}'(\varphi)\tilde{\varphi}\sigma\rho \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Adding up the the last five identities, we ascertain that a large number of terms cancels out. We obtain

$$
0 = \alpha_0 \int_{\Omega} \big(\varphi(T) - \varphi_f\big)\tilde{\varphi}(T)\,\mathrm{d}x + \alpha_1 \int_{\Omega_T} (\varphi - \varphi_d)\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \mathbb{h}(\varphi)\phi h \,\mathrm{d}x\,\mathrm{d}t.
$$

Together with (9.78) this completes the proof. □

As our set of admissible controls is a box-restricted subset of $L^2(L^2)$, a locally optimal control $\bar{u}$ can also be characterized by a projection of $\frac{1}{\kappa}\,\phi_{\bar{u}}\,\mathbb{h}(\varphi_{\bar{u}})$ onto the set $\mathbb{U}$ provided that $\kappa > 0$.

Corollary 9.25 *Let $\bar{u} \in \mathbb{U}$ be a locally optimal control of the minimization problem* (9.75) *and let $\kappa > 0$. Then, $\bar{u}$ is given implicitly by the* ***projection formula***

$$
\bar{u}(x,t) = \mathbb{P}_{[a(x,t),b(x,t)]}\left(\frac{1}{\kappa}\,\phi_{\bar{u}}(x,t)\,\mathbb{h}\big(\varphi_{\bar{u}}(x,t)\big)\right) \quad \text{for almost all } (x,t) \in \Omega_T, \tag{9.80}
$$

where the projection $\mathbb{P}$ is defined by

$$
\mathbb{P}_{[c,d]}(s) = \max\big\{c, \min\{d, s\}\big\}
$$

for any c, d, $s \in \mathbb{R}$ with $c \le d$. This constitutes another necessary condition for local optimality that is equivalent to condition (9.77).

Since this is a well-known inference of the necessary optimality condition provided by the variational inequality, we omit the proof. For a similar proof we refer to [135, pp. 71-73].

Remark 9.26 The necessary optimality conditions (9.77) and (9.80) are equivalent (cf. [135, pp. 71-73]).

9.4.3 A second-order sufficient condition for strict local optimality

We also want to establish a sufficient condition for (strict) local optimality. Since the control-to-state operator $\mathbb{S} : \mathbb{U}_R \to \mathcal{V}_1$ and the control-to-costate operator $\mathbb{A} : \mathbb{U}_R \to \mathcal{V}_2$ are continuously Fréchet differentiable, so is the cost functional J due to chain rule.

Therefore, we can easily establish a sufficient condition for strict local optimality: Let $\bar{u} \in \mathbb{U}$ satisfy the variational inequality (9.77) (or the projection formula (9.80) respectively) and we assume that $J''(\bar{u})$ is positive definite, i. e.,

$$
J''(\bar{u})[h,h] > 0 \tag{9.81}
$$

for all directions $h \in L^2(L^2) \setminus \{0\}$. Then, $\bar{u}$ is a strict local minimizer of J on the set $\mathbb{U}$.

However, this condition is far too restrictive as it suffices to require (9.81) only for a certain class of critical directions. Such a condition for optimal control problems with general semilinear elliptic or parabolic PDE constraints was firstly established in [34]. Meanwhile, it can also be found, for instance, in the textbook [135, pp. 245-248]. We proceed similarly and define the cone of critical directions as follows:

Definition 9.27 Let $\bar{u} \in \mathbb{U}$. We say that $h \in L^2(L^2)$ is a critical direction if the following condition is satisfied:

$$\text{for allmost all } (x,t) \in \Omega_T: \quad h(x,t) \begin{cases} \geq 0, & \bar{u}(x,t) = a(x,t), \\ \leq 0, & \bar{u}(x,t) = b(x,t), \\ = 0, & \kappa\bar{u}(x,t) - \phi_{\bar{u}}(x,t)\,\mathbb{h}(\varphi_{\bar{u}})(x,t) \neq 0. \end{cases} \tag{9.82}$$

We define the **cone of critical directions** as

$$\mathbf{C}(\bar{u}) \coloneqq \big\{h \in L^2(L^2)\colon h \text{ satisfies condition (9.82)}\big\} \tag{9.83}$$

Now, we can use the cone $\mathbf{C}(\bar{u})$ to formulate a sufficient condition for strict local optimality.

Theorem 9.28 *Let $\bar{u} \in \mathbb{U}$ be any control satisfying the variational inequality* (9.77) *and let $\kappa > 0$. Moreover, we assume that $J''(\bar{u})[h,h] > 0$ which is equivalent to*

$$\int_{\Omega_T} \big(\phi'_{\bar{u}}[h]\,\mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\,\mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h]\big)\, h \,\mathrm{d}x\,\mathrm{d}t < \kappa\|h\|^2_{L^2(L^2)} \quad \textit{for all } h \in \mathbf{C}(\bar{u}) \setminus \{0\}. \tag{9.84}$$

Then, $\bar{u}$ satisfies a quadratic growth condition, i. e., there exist $\delta, \theta > 0$ such that for all $u \in \mathbb{U}$ with $\|u - \bar{u}\|_{L^2(L^2)} < \delta$,

$$J(u) \geq J(\bar{u}) + \frac{\theta}{2}\|u - \bar{u}\|^2_{L^2(L^2)}. \tag{9.85}$$

In particular, this means that $\bar{u}$ is a strict local minimizer of the functional J on the set $\mathbb{U}$.

For the proof of Theorem 9.28, we need the subsequent lemma:

Lemma 9.29 *The following statements hold true:*

(i) *for any sequences $(u_k) \subset \mathbb{U}$ and $(h_k) \subset L^2(L^2)$ with $u_k \to \bar{u}$ and $h_k \rightharpoonup h$ in $L^2(L^2)$ as $k \to \infty$, it holds that*

$$J'(u_k)[h_k] \to J'(\bar{u})[h] \qquad \textit{as } k \to \infty.$$

(ii) *for any sequence $(h_k) \subset L^2(L^2)$ with $h_k \rightharpoonup h$ in $L^2(L^2)$ as $k \to \infty$, it holds that*

$$\mathbf{B}(h_k, h_k) \to \mathbf{B}(h,h) \qquad \textit{as } k \to \infty$$

up to subsequence extraction, where the bilinear form $\mathbf{B}$ is defined by

$$\mathbf{B}: L^2(L^2) \times L^2(L^2) \to \mathbb{R}, \quad (h_1, h_2) \mapsto \int_{\Omega_T} \big(\phi'_{\bar{u}}[h_1]\,\mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\,\mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h_1]\big)\, h_2 \,\mathrm{d}x\,\mathrm{d}t.$$

Proof. **Proof of (i):** Let $(u_k) \subset \mathbb{U}$ and $(h_k) \subset L^2(L^2)$ with $u_k \to \bar{u}$ and $h_k \rightharpoonup h$ in $L^2(L^2)$ be arbitrary. Recall that the first-order Fréchet derivative of J is given by

$$J'(u)[h] = \int_{\Omega_T} \big[\kappa u - \phi_u\,\mathbb{h}(\varphi_u)\big]\, h \,\mathrm{d}x\,\mathrm{d}t, \qquad u \in \mathbb{U},\, h \in L^2(L^2).$$

Since $\kappa\bar{u} - \phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}})$ lies in $L^2(L^2)$ it directly follows that $J'(\bar{u})[h_k] \to J'(\bar{u})[h]$. Furthermore, we can use the Lipschitz estimates from Corollary 9.9 and Proposition 9.20 to conclude that

$$\kappa u_k - \phi_{u_k}\, \mathbb{h}(\varphi_{u_k}) \;\to\; \kappa\bar{u} - \phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}}) \quad \text{in } L^2(L^2) \quad \text{as } k \to \infty.$$

Now, since (h_k) is uniformly bounded in $L^2(L^2)$, we obtain that

$$\big|J'(u_k)[h_k] - J'(\bar{u})[h_k]\big| \le \big\|\kappa u_k - \phi_{u_k}\, \mathbb{h}(\varphi_{u_k}) - \kappa\bar{u} + \phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}})\big\|_2 \,\big\|h_k\big\|_2 \to 0 \quad \text{as } k \to \infty.$$

Consequently,

$$J'(u_k)[h_k] - J'(\bar{u})[h] = J'(u_k)[h_k] - J'(\bar{u})[h_k] + J'(\bar{u})[h_k] - J'(\bar{u})[h] \to 0 \quad \text{as } k \to \infty$$

which proves (i).

Proof of (ii)**:** The proof is very similar to the proof of (i). Let $(h_k) \subset L^2(L^2)$ with $h_k \rightharpoonup h$ in $L^2(L^2)$ be any sequence. As $\phi'_{\bar{u}}[h]\, \mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\, \mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h]$ lies in $L^2(L^2)$, we have $\mathbf{B}(h, h_k) \to \mathbf{B}(h, h)$. Moreover, due to Propositions 9.14 and 9.21 and the compact embeddings

$$H^1(L^2) \cap L^\infty(H^2) \hookrightarrow C(\overline{\Omega_T}) \quad \text{and} \quad H^1((H^1)^*) \cap L^2(H^3) \hookrightarrow L^2(L^2),$$

we obtain that

$$\|\varphi'_{\bar{u}}[h_k] - \varphi'_{\bar{u}}[h]\|_{L^\infty(L^\infty)} \to 0 \quad \text{and} \quad \|\phi'_{\bar{u}}[h_k] - \phi'_{\bar{u}}[h]\|_{L^2(L^2)} \to 0 \quad \text{as } k \to \infty$$

after extraction of a subsequence. Hence, we can conclude that

$$\phi'_{\bar{u}}[h_k]\, \mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\, \mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h_k] \;\to\; \phi'_{\bar{u}}[h]\, \mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\, \mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h] \qquad \text{in } L^2(L^2) \text{ as } k \to \infty$$

by means of Hölder's inequality. As (h_k) is a bounded sequence in $L^2(L^2)$, it follows that

$$\begin{aligned}
&\big|\mathbf{B}(h_k, h_k) - \mathbf{B}(h, h_k)\big| \\
&\quad \le \big\|\phi'_{\bar{u}}[h_k]\, \mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\, \mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h_k] - \phi'_{\bar{u}}[h]\, \mathbb{h}(\varphi_{\bar{u}}) - \phi_{\bar{u}}\, \mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h]\big\|_2 \,\big\|h_k\big\|_2 \;\to\; 0
\end{aligned}$$

as $k \to \infty$, and thus

$$\mathbf{B}(h_k, h_k) - \mathbf{B}(h, h) = \mathbf{B}(h_k, h_k) - \mathbf{B}(h, h_k) + \mathbf{B}(h, h_k) - \mathbf{B}(h, h) \to 0 \quad \text{as } k \to \infty,$$

which proves (ii). ☐

Now, we can proceed with the proof of Theorem 9.28.

Proof of Theorem 9.28. The second-order Fréchet derivative of J is given by

$$J''(\bar{u})[h_1, h_2] = \kappa \left(h_1, h_2\right)_{L^2(L^2)} - \int_{\Omega_T} \left(\phi'_{\bar{u}}[h_1]\, \mathbb{h}(\varphi_{\bar{u}}) + \phi_{\bar{u}}\, \mathbb{h}'(\varphi_{\bar{u}})\varphi'_{\bar{u}}[h_1]\right) h_2 \,\mathrm{d}x\,\mathrm{d}t \tag{9.86}$$

for all $h_1, h_2 \in L^2(L^2)$. Thus, condition (9.84) is equivalent to

$$J''(\bar{u})[h, h] > 0 \qquad \text{for all } h \in \mathbf{C}(\bar{u}) \setminus \{0\}. \tag{9.87}$$

Following the strategy presented in [34] and applied in [108], we now argue by contradiction. Assume that condition (9.85) was not satisfied. Then, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathbb{U}\backslash\{\bar{u}\}$ such that

$$\bar{u}_k \to \bar{u} \quad \text{in } L^2(L^2) \quad \text{as } k \to \infty, \quad J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(L^2)} > J(u_k) \quad \forall\, k \in \mathbb{N}. \tag{9.88}$$

Moreover, we define

$$d_k := \|u_k - \bar{u}\|_{L^2(L^2)}, \quad h_k := \frac{1}{d_k}(u_k - \bar{u}) \quad \forall\, k \in \mathbb{N}.$$

Since $\|h_k\|_{L^2(L^2)} = 1$ for all $k \in \mathbb{N}$, by weak reflexive compactness we can extract a subsequence (again labelled by k) such that, as $k \to \infty$, $h_k \rightharpoonup h$ in $L^2(L^2)$ for some $h \in L^2(L^2)$.

Step 1: We claim that $J'(\bar{u})[h] = 0$. Using the mean value theorem, we obtain

$$J(u_k) = J(\bar{u}) + d_k J'(v_k)[h_k]$$

for an intermediate point $v_k \in L^2(L^2)$ between $\bar{u}$ and u_k. Rearranging and invoking (9.88) yields

$$J'(v_k)[h_k] = \frac{1}{d_k}\left(J(u_k) - J(\bar{u})\right) < \frac{1}{kd_k}\|u_k - \bar{u}\|_{L^2(L^2)} = \frac{1}{k}\|u_k - \bar{u}\|_{L^2(L^2)}.$$

Since $u_k \to \bar{u}$ in $L^2(L^2)$ as $k \to \infty$, it also holds that $v_k \to \bar{u}$ in $L^2(L^2)$ as $k \to \infty$, since v_k is an intermediate point between $\bar{u}$ and u_k. Then, invoking Lemma 9.29, (i), and the last inequality, we obtain

$$J'(\bar{u})[h] = \lim_{k\to\infty} J'(v_k)[h_k] \leq \lim_{k\to\infty} \frac{1}{k}\|u_k - \bar{u}\|_{L^2(L^2)} = 0.$$

For the reverse inequality, we use (9.77) to deduce that

$$J'(\bar{u})[h_k] = \frac{1}{d_k}\int_{\Omega_T} \left[\kappa\bar{u} - \phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}})\right](u_k - \bar{u})\, \mathrm{d}x\, \mathrm{d}t \geq 0.$$

Again using Lemma 9.29, (i), taking the limit $k \to \infty$ in this inequality yields $J'(\bar{u})[h] \geq 0$, and therefore $J'(\bar{u})[h] = 0$.

Step 2: We claim that $h \in \mathbf{C}(\bar{u})$. To this end, we define

$$\mathbf{K}(\bar{u}) := \left\{ u \in L^2(L^2) \;\middle|\; \begin{array}{c} \text{For almost all } (x,t) \in \Omega_T: \\ u(x,t) \geq 0 \text{ if } \bar{u}(x,t) = a(x,t) \;\wedge\; u(x,t) \leq 0 \text{ if } \bar{u}(x,t) = b(x,t). \end{array} \right\},$$

which is a closed and convex subset of $L^2(L^2)$. Therefore, $\mathbf{K}(\bar{u})$ is also weakly closed. By definition of $\mathbb{U}$, it follows that $u_k - \bar{u} \in \mathbf{K}(\bar{u})$ for all $k \in \mathbb{N}$ and therefore the same holds for h_k. Since $\mathbf{K}(\bar{u})$ is weakly closed, this implies $h \in \mathbf{K}(\bar{u})$.
Now, let us consider points $(x,t) \in \Omega_T$ where $\kappa\bar{u}(x,t) - \phi_{\bar{u}}(x,t)\mathbb{h}(\varphi_{\bar{u}}(x,t)) > 0$ or equivalently $\bar{u}(x,t) > \frac{1}{\kappa}\phi_{\bar{u}}(x,t)\mathbb{h}(\varphi_{\bar{u}}(x,t))$. Then, by the projection formula (9.80) we get $\bar{u}(x,t) = a(x,t)$ and along with $h \in \mathbf{K}(\bar{u})$, this yields $h(x,t) \geq 0$. In the case $\kappa\bar{u}(x,t) - \phi_{\bar{u}}(x,t)\mathbb{h}(\varphi_{\bar{u}}(x,t)) < 0$, we can argue analogously to obtain $h(x,t) \leq 0$. In particular, $(\kappa\bar{u}(x,t) - \phi_{\bar{u}}(x,t)\mathbb{h}(\varphi_{\bar{u}}(x,t)))h(x,t) \geq 0$ holds for almost all $(x,t) \in \Omega_T$. Moreover, from the first step we obtain

$$\int_{\Omega_T} (\kappa\bar{u} - \phi_{\bar{u}}\mathbb{h}(\varphi_{\bar{u}}))h\, \mathrm{d}x\, \mathrm{d}t = J'(\bar{u})[h] = 0.$$

Since the integrand is non-negative for almost all $(x,t) \in \Omega_T$, we deduce that

$$(\kappa\bar{u}(x,t) - \phi_{\bar{u}}(x,t)\mathbb{h}(\varphi_{\bar{u}}(x,t)))h(x,t) = 0 \quad \text{for almost all } (x,t) \in \Omega_T.$$

Consequently, for almost all $(x,t) \in \Omega_T$ it holds $h(x,t) = 0$ if $\kappa\bar{u}(x,t) - \phi_{\bar{u}}(x,t)\mathbb{h}(\varphi_{\bar{u}}(x,t)) \neq 0$. Together with the fact that $h \in \mathbf{K}(\bar{u})$, this implies $h \in \mathbf{C}(\bar{u})$.

Step 3: We claim that $h = 0$. Recalling (9.87), it suffices to show that $J''(\bar{u})[h,h] \leq 0$. A second-order Taylor expansion shows that

$$\begin{aligned} J(u_k) &= J(\bar{u}) + d_k J'(\bar{u})[h_k] + \frac{d_k^2}{2} J''(w_k)[h_k, h_k] \\ &= J(\bar{u}) + d_k J'(\bar{u})[h_k] + \frac{d_k^2}{2} J''(\bar{u})[h_k, h_k] + \frac{d_k^2}{2}\left(J''(w_k)[h_k,h_k] - J''(\bar{u})[h_k,h_k]\right) \quad \forall\, k \in \mathbb{N}, \end{aligned}$$

where $w_k \in L^2(L^2)$ is an intermediate point between u_k and $\bar{u}$. After rearranging, this gives

$$J''(\bar{u})[h_k,h_k] = \frac{2}{d_k^2}(J(u_k) - J(\bar{u})) - \frac{2}{d_k}J'(\bar{u})[h_k] - \left(J''(w_k)[h_k,h_k] - J''(\bar{u})[h_k,h_k]\right).$$

For the second term on the r. h. s., the variational inequality (9.77) yields

$$\frac{2}{d_k} J'(\bar{u})[h_k] = \frac{2}{d_k^2} J'(\bar{u})[u_k - \bar{u}] \geq 0.$$

Invoking (9.88), we have

$$\frac{2}{d_k^2}(J(u_k) - J(\bar{u})) < \frac{2}{kd_k^2}\|u_k - \bar{u}\|^2 = \frac{2}{k}.$$

Moreover, using Propositions 9.14 and 9.21 along with $\|h_k\|_{L^2(L^2)} = 1$ for all $k \in \mathbb{N}$, it follows that

$$|J''(w_k)[h_k, h_k] - J''(\bar{u})[h_k, h_k]| \leq \|w_k - \bar{u}\|^2_{L^2(L^2)} \leq d_k^2,$$

where we used that w_k is an intermediate point between u_k and $\bar{u}$. From the last four inequalities, we obtain

$$J''(\bar{u})[h_k, h_k] < \frac{2}{k} + d_k^2 \to 0 \quad \text{as } k \to \infty, \quad \text{i. e. } \limsup_{k\to\infty} J''(\bar{u})[h_k, h_k] \leq 0. \tag{9.89}$$

Now, using $h_k \rightharpoonup h$ weakly in $L^2(L^2)$ along with Lemma 9.29, (ii), (9.86) and weak lower semi-continuity of norms, we obtain

$$\begin{aligned} J''(\bar{u})[h, h] = \kappa\|h\|^2_{L^2(L^2)} - \mathbf{B}(h, h) &\leq \kappa \liminf_{k\to\infty}\|h_k\|^2_{L^2(L^2)} + \liminf_{k\to\infty}(-B(h_k, h_k)) \\ &\leq \limsup_{k\to\infty} J''(\bar{u})[h_k, h_k] \leq 0, \end{aligned}$$

where B is defined as in Lemma 9.29. Using (9.87) and $h \in \mathbf{C}(\bar{u})$, this implies $h = 0$.

Step 4: Using (9.86), (9.89), Lemma 9.29, (ii), and the fact that $h_k \rightharpoonup h = 0$ in $L^2(L^2)$ and $d_k \to 0$ as $k \to \infty$, it follows that

$$\kappa\|h_k\|^2_{L^2(L^2)} < \frac{2}{k} + d_k^2 + \mathbf{B}(h_k, h_k) \to \mathbf{B}(h, h) = 0 \quad \text{as } k \to \infty.$$

Therefore, we obtain $h_k \to 0$ strongly in $L^2(L^2)$. Since $\|h_k\|_{L^2(L^2)} = 1$ for all $k \in \mathbb{N}$, this is a contradiction which completes the proof. □

9.4.4 A condition for global optimality of critical controls

Even if a control $\bar{u} \in \mathbb{U}$ satisfies the sufficient optimality condition from Theorem 9.28 it is not clear whether this control is globally optimal. However, we will establish a globality criterion for controls which satisfy the variational inequality or the equivalent projection formula. In the following these controls will be referred to as **critical controls**.

The technique we are using was firstly introduced in [7] for optimal control problems constrained by a general semilinear elliptic PDE of second order. Recently it has also been adapted for optimal control of the obstacle problem, see [8]. Our globality condition will be proved similarly and reads as follows:

Theorem 9.30 *Suppose that $\alpha_1 > 0$ and $\kappa > 0$ and let C_1 and L_1 denote the constants from Proposition 9.4 and Corollary 9.9. Moreover, we set*

$$r := \sup_{u\in\mathbb{U}}\|\varphi_u\|_\infty \leq C_1.$$

We assume that the control $\bar{u} \in \mathbb{U}$ satisfies the variational inequality (9.77) *(or the projection formula* (9.80)*) and that one of the following conditions holds:*

(G1) *it holds that*

$$\frac{\kappa}{2} \geq \Big[\|(\mathcal{P}\sigma_{\bar{u}} - \mathcal{A})(\phi_{\bar{u}} - q_{\bar{u}}) - \sigma_{\bar{u}}\rho_{\bar{u}} - \bar{u}\phi_{\bar{u}}\|_{L^1(L^1)} \, \|\mathbb{h}''\|_{L^\infty(\mathbb{R})} \, L_1^2 + \|\tau_{\bar{u}}\|_{L^1(L^1)} \, \|\psi'''\|_{L^\infty([-r,r])} L_1^2 + \|\phi_{\bar{u}}\|_{L^2(L^2)} \, \|\mathbb{h}'\|_{L^\infty(\mathbb{R})} \, L_1\Big]. \tag{9.90}$$

(G2) *there exists a real number* $\theta > 0$ *such that*

$$2\kappa\theta \geq \|\phi_{\bar{u}}\|^2_{L^\infty(L^\infty)} \, \|\mathbb{h}''\|_{L^\infty(\mathbb{R})} \tag{9.91a}$$

and

$$\frac{\alpha_1}{2} \geq \|(\mathcal{P}\sigma_{\bar{u}} - \mathcal{A})(\phi_{\bar{u}} - q_{\bar{u}}) - \sigma_{\bar{u}}\rho_{\bar{u}} - \bar{u}\phi_{\bar{u}}\|_{L^\infty(L^\infty)}\|\mathbb{h}''\|_{L^\infty(\mathbb{R})} + \|\tau_{\bar{u}}\|_{L^\infty(L^\infty)}\|\psi'''\|_{L^\infty([-r,r])} + \theta\|\phi_{\bar{u}}\|^2_{L^\infty(L^\infty)}\|\mathbb{h}'\|^2_{L^\infty(\mathbb{R})}. \tag{9.91b}$$

Then, $\bar{u}$ *is a globally optimal control of problem* (9.75).

In addition, the globally optimal control $\bar{u}$ *is unique if one of the following conditions holds:*

(U1) *condition* (G1) *is satisfied and* (9.90) *holds with "*$>$*" instead of "*$\geq$*",*

(U2) *condition* (G2) *is satisfied and* (9.91a) *holds with "*$>$*" instead of "*$\geq$*".*

Remark 9.31

(a) Of course, for the double-well potential ψ, we have $\psi'''(s) = 6s$ and thus

$$\|\psi'''\|_{L^\infty([-r,r])} = 6r.$$

(b) The conditions (G1) and (G2) will be satisfied if the adjoint variables $\phi_{\bar{u}}$, $\tau_{\bar{u}}$, $\rho_{\bar{u}}$ and $q_{\bar{u}}$ are sufficiently small in the occurring norms.

(c) Since the state and adjoint variables are sufficiently regular, the right-hand side of (9.90) is at least always finite. However, is seems very difficult to verify the condition (G1) by numerical methods as the Lipschitz constant L_1 which depends on the domain Ω has to be determined.

(d) Condition (G2) has the advantage that all occurring quantities except for $\|\psi'''\|_{L^\infty([-r,r])}$ can be computed very easily. However, the constant r can hardly be determined explicitly.

To overcome this disadvantage, one can use a modified version ψ_δ of the double-well potential such that $\psi_\delta \in C^{3,1}(\mathbb{R})$ with $\psi = \psi_\delta$ on $[-\delta, \delta]$ for some $\delta > 1$ and ψ_δ''' bounded and Lipschitz continuous with a constant L_δ. It is not difficult to see that all other results in this chapter remain true after this replacement (cf. Remark 9.2(a)).

Of course, if $\delta > r$ the values of the state and costate variables will not change if ψ is replaced by ψ_δ. Various numerical results for the Cahn–Hilliard equation have shown that $1 \leq r \ll 2$ can be expected, i. e., r is usually very close to one (see, e. g., [101]).

We will show a possible construction of such a potential ψ_δ in the following example.

Example. Let us consider the function ψ_δ given by

$$\psi_\delta(s) = \begin{cases} \psi(\delta) + \psi'(\delta)(s-\delta) + \frac{1}{2}\psi''(\delta)(s-\delta)^2 + \frac{1}{6}\psi'''(\delta)(s-\delta)^3 & \text{for } s > \delta, \\ \psi(s) & \text{for } |s| \leq \delta, \\ \psi(\delta) + \psi'(-\delta)(s+\delta) + \frac{1}{2}\psi''(-\delta)(s+\delta)^2 + \frac{1}{6}\psi'''(-\delta)(s+\delta)^3 & \text{for } s < -\delta \end{cases}$$

for $\delta \geq 1$. Then, it is easy to check that $\psi_\delta \in C^{3,1}(\mathbb{R})$ with Lipschitz constant $L_\delta = 6$ and

$$\|\psi_\delta'''\|_{L^\infty(\mathbb{R})} = |\psi'''(\pm\delta)| = 6\delta.$$

Furthermore, ψ_δ fulfils the Assumptions of, e. g., [58], and thus the results in this chapter remain true after replacing ψ with ψ_δ. The Lipschitz continuity of ψ_δ''' is needed in order to establish Corollary 9.9. We plot the function ψ_δ for $\delta \in \{1, 1.25\}$, see Figure 9.1.

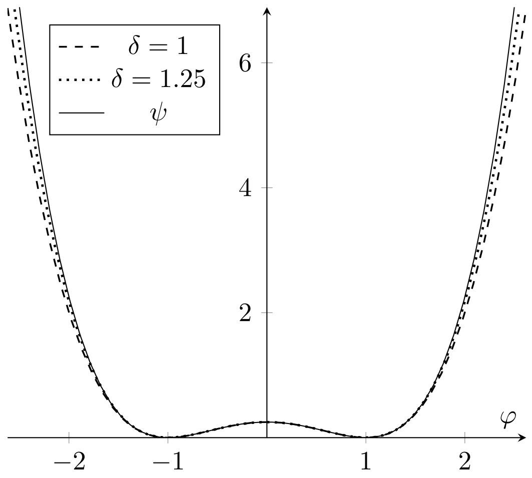

Figure 9.1: Plot of ψ_δ for $\delta \in \{1, 1.25\}$ and comparison with the double-well potential.

We will now present the proof of our theorem

Proof of Theorem 9.30. To prove global optimality of the control $\bar{u}$, we intend to show that $J(u) - J(\bar{u}) \geq 0$ for all $u \in \mathbb{U} \setminus \{\bar{u}\}$. The proof is divided into three steps.

Step 1: Let $u \in \mathbb{U}$ be arbitrary. Recall that

$$\big(\kappa\bar{u}, u - \bar{u}\big)_{L^2(L^2)} \geq -\big(\phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}}), u - \bar{u}\big)_{L^2(L^2)}$$

due to the variational inequality (9.77). Then, by a straightforward computation, we obtain that

$$J(u) - J(\bar{u}) \geq \frac{\alpha_0}{2}\|\varphi_u(T) - \varphi_{\bar{u}}(T)\|_{L^2}^2 + \frac{\alpha_1}{2}\|\varphi_u - \varphi_{\bar{u}}\|_{L^2(L^2)}^2 + \frac{\kappa}{2}\|u - \bar{u}\|_{L^2(L^2)}^2 + \mathcal{R}, \quad (9.92)$$

where

$$\mathcal{R} := \alpha_0\big(\varphi_{\bar{u}}(T) - \varphi_f, \varphi_u(T) - \varphi_{\bar{u}}(T)\big)_{L^2} + \alpha_1\big(\varphi_{\bar{u}} - \varphi_d, \varphi_u - \varphi_{\bar{u}}\big)_{L^2(L^2)} - \big(\phi_{\bar{u}}\, \mathbb{h}(\varphi_{\bar{u}}), u - \bar{u}\big)_{L^2(L^2)}.$$

Our aim is to show that

$$|\mathcal{R}| \leq \frac{\alpha_1}{2}\|\varphi_u - \varphi_{\bar{u}}\|_{L^2(L^2)}^2 + \frac{\kappa}{2}\|u - \bar{u}\|_{L^2(L^2)}^2 \quad \text{for all } u \in \mathbb{U} \setminus \{\bar{u}\} \quad (9.93)$$

if condition (G1) or condition (G2) is fulfilled. Then, (9.92) yields $J(u) \geq J(\bar{u})$ and global optimality of $\bar{u}$ directly follows.

Step 2: The idea is to express the remainder $\mathcal{R}$ by the adjoint variables. For brevity, we write

$$(\tilde{\varphi}, \tilde{\mu}, \tilde{\sigma}, \tilde{\mathbf{v}}, \tilde{p}) := (\varphi_u, \mu_u, \sigma_u, \mathbf{v}_u, p_u) - (\varphi_{\bar{u}}, \mu_{\bar{u}}, \sigma_{\bar{u}}, \mathbf{v}_{\bar{u}}, p_{\bar{u}}).$$

This means that $(\tilde\varphi, \tilde\mu, \tilde\sigma, \tilde{\mathbf{v}}, \tilde p)$ is a solution of (9.10). In the following, the strategy is to test the equations of the system (9.10) with the adjoint variables. Testing (9.10c) with $\phi_{\bar u}$ and integrating by parts with respect to t yields

$$\begin{aligned}
0 &= \int_{\Omega_T} \Big[\partial_t\tilde\varphi + \mathrm{div}(\varphi_u\tilde{\mathbf{v}}) + \mathrm{div}(\tilde\varphi\mathbf{v}_{\bar u}) - m\Delta\tilde\mu - \mathcal{P}\tilde\sigma\mathbb{h}(\varphi_u) - (\mathcal{P}\sigma_{\bar u} - \mathcal{A})\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u})\big) \\
&\qquad\qquad + \big(u\mathbb{h}(\varphi_u) - \bar u\mathbb{h}(\varphi_{\bar u})\big)\Big]\,\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T} \mathrm{div}(\varphi_u\tilde{\mathbf{v}})\phi_{\bar u} + \mathrm{div}(\tilde\varphi\mathbf{v}_{\bar u})\phi_{\bar u} - m\Delta\tilde\mu\,\phi_{\bar u} - \mathcal{P}\tilde\sigma\mathbb{h}(\varphi_u)\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_{\Omega_T} (\mathcal{P}\sigma_{\bar u} - \mathcal{A})\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u})\big)\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T} u\mathbb{h}(\varphi_u)\phi_{\bar u} - \bar u\mathbb{h}(\varphi_{\bar u})\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tilde\varphi(T)\,\phi_{\bar u}(T)\,\mathrm{d}x - \int_0^T \langle\partial_t\phi_{\bar u}, \tilde\varphi\rangle_{H^1}\,\mathrm{d}t.
\end{aligned}$$

Now, the term $\langle\partial_t\phi_{\bar u}, \tilde\varphi\rangle_{H^1}$ can be expressed by (9.58d) with test function $\tilde\phi = \tilde\varphi$. We obtain that

$$\begin{aligned}
0 &= \int_{\Omega_T} \big(\mathrm{div}(\varphi_u\tilde{\mathbf{v}}) - \mathcal{P}\tilde\sigma\mathbb{h}(\varphi_u) - (\mathcal{P}\sigma_{\bar u} - \mathcal{A})\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u})\big) + u\mathbb{h}(\varphi_u) - \bar u\mathbb{h}(\varphi_{\bar u})\big)\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T} -m\Delta\tilde\mu\,\phi_{\bar u} + (\mathcal{P}\sigma_{\bar u} - \mathcal{A} - \bar u)\mathbb{h}'(\varphi_{\bar u})\phi_{\bar u}\tilde\varphi + \mathbb{h}'(\varphi_{\bar u})\sigma_{\bar u}\rho_{\bar u}\tilde\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T} \alpha_1(\varphi_{\bar u} - \varphi_d)\tilde\varphi + (\mu_{\bar u} + \chi\sigma_{\bar u})\mathbf{w}_{\bar u}\cdot\nabla\tilde\varphi + \nabla\tau_{\bar u}\cdot\nabla\tilde\varphi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_{\Omega_T} \psi''(\varphi_{\bar u})\tau_{\bar u}\tilde\varphi - (\mathcal{P}\sigma_{\bar u} - \mathcal{A})\mathbb{h}'(\varphi_{\bar u})q_{\bar u}\tilde\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \tilde\varphi(T)\,\phi_{\bar u}(T)\,\mathrm{d}x. \qquad (9.94)
\end{aligned}$$

Furthermore, testing (9.10b) with $\mathbf{w}_{\bar u}$ yields

$$\begin{aligned}
0 &= \int_{\Omega_T} -\mathrm{div}\big(T(\tilde{\mathbf{v}}, \tilde p)\big)\cdot\mathbf{w}_{\bar u} + \nu\tilde{\mathbf{v}}\cdot\mathbf{w}_{\bar u} - (\tilde\mu + \chi\tilde\sigma)\nabla\varphi_u\cdot\mathbf{w}_{\bar u} - (\mu_{\bar u} + \chi\sigma_{\bar u})\nabla\tilde\varphi\cdot\mathbf{w}_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_{\Omega_T} 2\eta\mathbf{D}\mathbf{w}_{\bar u} : \nabla\tilde{\mathbf{v}} + \nu\tilde{\mathbf{v}}\cdot\mathbf{w}_{\bar u} - (\tilde\mu + \chi\tilde\sigma)\nabla\varphi_u\cdot\mathbf{w}_{\bar u} - (\mu_{\bar u} + \chi\sigma_{\bar u})\nabla\tilde\varphi\cdot\mathbf{w}_{\bar u}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

due to the definition of $T(\tilde{\mathbf{v}}, \tilde p)$ and the fact that $\mathrm{div}(\mathbf{w}_{\bar u}) = 0$. The term $2\eta\mathbf{D}\mathbf{w}_{\bar u} : \nabla\tilde{\mathbf{v}}$ can be expressed by choosing $\tilde{\mathbf{w}} = \tilde{\mathbf{v}}$ in (9.58c). Thus,

$$0 = \int_{\Omega_T} \mathrm{div}(\tilde{\mathbf{v}})q_{\bar u} - \phi_{\bar u}\nabla\varphi_u\cdot\tilde{\mathbf{v}} - \phi_{\bar u}\varphi_u\mathrm{div}(\tilde{\mathbf{v}}) - (\tilde\mu + \chi\tilde\sigma)\nabla\varphi_u\cdot\mathbf{w}_{\bar u} - (\mu_{\bar u} + \chi\sigma_{\bar u})\nabla\tilde\varphi\cdot\mathbf{w}_{\bar u}\,\mathrm{d}x\,\mathrm{d}t.$$

Proceeding similarly with the other equations of (9.10) gives

$$\begin{aligned}
0 &= \int_{\Omega_T} \nabla\varphi_{\bar u}\cdot\mathbf{w}_{\bar u}\,\tilde\mu + m\Delta\tilde\mu\,\phi_{\bar u} - \nabla\tilde\varphi\cdot\nabla\tau_{\bar u} - \big(\psi'(\varphi_u) - \psi'(\varphi_{\bar u})\big)\tau_{\bar u} + \chi\tilde\sigma\tau_{\bar u}\,\mathrm{d}x\,\mathrm{d}t, \\
0 &= \int_{\Omega_T} -\mathrm{div}(\tilde{\mathbf{v}})q_{\bar u} + \mathcal{P}\tilde\sigma\mathbb{h}(\varphi_u)q_{\bar u} + (\mathcal{P}\sigma_u - \mathcal{A})\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u})\big)q_{\bar u}\,\mathrm{d}x\,\mathrm{d}t, \\
0 &= \int_{\Omega_T} \mathcal{P}\tilde\sigma\mathbb{h}(\varphi_u)(\phi_{\bar u} - q_{\bar u}) - \chi\tau_{\bar u}\tilde\sigma + \chi\tilde\sigma\nabla\varphi_u\cdot\mathbf{w}_{\bar u} - \sigma_{\bar u}\rho_{\bar u}\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u})\big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Adding up the last four identities and (9.94), we ascertain that a large number of terms cancels out. We end up with

$$
\begin{aligned}
0 = &\int_\Omega \phi_{\bar u}(T)\tilde\varphi(T)\,\mathrm{d}x + \alpha_1 \int_{\Omega_T} (\varphi_{\bar u} - \varphi_d)\tilde\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} (u-\bar u)\mathbb{h}(\varphi_u)\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T} (\mathcal{P}\sigma_{\bar u} - \mathcal{A})\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u}) - \mathbb{h}'(\varphi_u)\tilde\varphi\big)(q_{\bar u} - \phi_{\bar u})\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T} \sigma_{\bar u}\rho_{\bar u}\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u}) - \mathbb{h}'(\varphi_u)\tilde\varphi\big)\,\mathrm{d}x\,\mathrm{d}t \\
&- \int_{\Omega_T} \big(\psi'(\varphi_u) - \psi'(\varphi_{\bar u}) - \psi''(\varphi_{\bar u})\tilde\varphi\big)\tau_{\bar u}\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T} \bar u\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u}) - \mathbb{h}'(\varphi_{\bar u})\tilde\varphi\big)\phi_{\bar u} + (u-\bar u)\big(\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u})\big)\phi_{\bar u}\,\mathrm{d}x\,\mathrm{d}t. \qquad (9.95)
\end{aligned}
$$

Since $\phi_{\bar u}(T) = \alpha_0(\varphi_{\bar u}(T) - \varphi_f)$, the first three terms on the right-hand side of (9.95) are equal to $\mathcal{R}$. Moreover, using Taylor expansion, we can find $\zeta, \xi, \theta \in [0,1]$ such that

$$
\begin{aligned}
\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u}) &= \mathbb{h}'(\varphi_\zeta)\tilde\varphi^2 && \text{with} \quad \varphi_\zeta = \varphi_{\bar u} + \zeta(\varphi_u - \varphi_{\bar u}), \\
\mathbb{h}(\varphi_u) - \mathbb{h}(\varphi_{\bar u}) - \mathbb{h}'(\varphi_u)\tilde\varphi &= \mathbb{h}''(\varphi_\xi)\tilde\varphi^2 && \text{with} \quad \varphi_\xi = \varphi_{\bar u} + \xi(\varphi_u - \varphi_{\bar u}), \\
\psi'(\varphi_u) - \psi'(\varphi_{\bar u}) - \psi''(\varphi_u)\tilde\varphi &= \psi'''(\varphi_\theta)\tilde\varphi^2 && \text{with} \quad \varphi_\theta = \varphi_{\bar u} + \theta(\varphi_u - \varphi_{\bar u}).
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
\mathcal{R} = &\int_{\Omega_T} \Big[(\mathcal{P}\sigma_{\bar u} - \mathcal{A})(\phi_{\bar u} - q_{\bar u}) - \sigma_{\bar u}\rho_{\bar u} - \bar u\phi_{\bar u}\Big]\mathbb{h}''(\varphi_\xi)\tilde\varphi^2\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T} \tau_{\bar u}\psi'''(\varphi_\theta)\tilde\varphi^2\,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} (u-\bar u)\phi_{\bar u}\mathbb{h}'(\varphi_\zeta)\tilde\varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Step 3: Now, we will use the identity for $\mathcal{R}$ to prove estimates in the fashion of (9.93). A simple computation gives

$$
\begin{aligned}
|\mathcal{R}| \le \Big[&\|(\mathcal{P}\sigma_{\bar u} - \mathcal{A})(\phi_{\bar u} - q_{\bar u}) - \sigma_{\bar u}\rho_{\bar u} - \bar u\phi_{\bar u}\|_{L^1(\Omega_T)}\|\mathbb{h}''\|_{L^\infty(\mathbb{R})}\, L_1^2 \\
&+ \|\tau_{\bar u}\|_{L^1(\Omega_T)}\,\|\psi'''\|_{L^\infty([-r,r])}\, L_1^2 + \|\phi_{\bar u}\|_{L^2(\Omega_T)}\|\mathbb{h}'\|_{L^\infty(\mathbb{R})}\, L_1\Big]\,\|u - \bar u\|^2_{L^2(\Omega_T)}
\end{aligned}
$$

since $\|\varphi_\theta\|_{L^\infty(\Omega_T)} \le r$. Furthermore, using Young's inequality with θ, the remainder $\mathcal{R}$ can also be bounded by

$$
\begin{aligned}
|\mathcal{R}| \le \Big[&\|(\mathcal{P}\sigma_{\bar u} - \mathcal{A})(\phi_{\bar u} - q_{\bar u}) - \sigma_{\bar u}\rho_{\bar u} - \bar u\phi_{\bar u}\|_{L^\infty(\Omega_T)}\|\mathbb{h}''\|_{L^\infty(\mathbb{R})} \\
&+ \|\tau_{\bar u}\|_{L^\infty(\Omega_T)}\|\psi'''\|_{L^\infty([-r,r])} + \theta\|\phi_{\bar u}\|^2_{L^\infty(\Omega_T)}\|\mathbb{h}'\|^2_{L^\infty(\mathbb{R})}\Big]\|\tilde\varphi\|^2_{L^2(\Omega_T)} \\
&+ \frac{1}{4\theta}\|\phi_{\bar u}\|^2_{L^\infty(\Omega_T)}\|\mathbb{h}'\|^2_{L^\infty(\mathbb{R})}\|u - \bar u\|^2_{L^2(\Omega_T)}.
\end{aligned}
$$

Hence, if condition (G1) or (G2) is satisfied, we can use one of the previous two estimates to conclude that

$$
|\mathcal{R}| \le \frac{\alpha_1}{2}\|\tilde\varphi\|^2_{L^2(L^2)} + \frac{\kappa}{2}\|u - \bar u\|^2_{L^2(L^2)},
$$

and inequality (9.92) implies that $\bar u$ is a globally optimal control. If, in addition, either condition (U1) or (U2) is satisfied, it even holds that

$$
|\mathcal{R}| < \frac{\alpha_1}{2}\|\tilde\varphi\|^2_{L^2(L^2)} + \frac{\kappa}{2}\|u - \bar u\|^2_{L^2(L^2)}.
$$

Then, (9.92) implies that

$$
J(u) > J(\bar u) \quad \text{for all } u \in \mathbb{U} \setminus \{\bar u\},
$$

and uniqueness of the globally optimal control $\bar u$ follows. □

9.4.5 Uniqueness of the optimal control on small time intervals

Finally, we present a condition on T which ensures uniqueness of the optimal control. A similar result was established, e. g., in [107, 108]. The idea behind the approach is as follows. If we choose the final time T sufficiently small, the state equation will differ only slightly from its linearisation. In the case $\kappa > 0$, a linearised state equation would produce a strictly convex cost functional and the corresponding optimal control would be unique. If T is small enough, we can expect that this property transfers to our problem. On the other hand, if the parameter κ is large, the strictly convex part of the cost functional J will be more dominant. Thus, it is not surprising that the size of the time interval on which the optimal control is unique will also depend on κ.

In our theorem, we use the following notation: for any $p \in [1,6]$, let $c_\Omega(p) \geq 0$ denote a constant for which Sobolev's inequality

$$\|v\|_{L^p(\Omega)} \leq c_\Omega(p)\,\|v\|_{H^1(\Omega)}, \quad \text{for all } v \in H^1(\Omega)$$

is satisfied.

Theorem 9.32 *Suppose that $\kappa > 0$ and let $\bar{u} \in \mathbb{U}$ be a locally optimal control of problem (9.75). Let $p, q \in [3,6]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ be arbitrary.*

Moreover, we assume that

$$T < \left(\frac{\sqrt{3}\,\kappa}{2\big(L_3 + \sqrt{2} L_1 c_\Omega(p)\, c_\Omega(q) \|\phi_{\bar{u}}\|_{L^\infty(H^1)}\, \|\mathbb{h}'\|_{L^\infty(\mathbb{R})}\big)} \right)^{4/3}. \tag{9.96}$$

Then, $\bar{u}$ is the unique locally optimal control.

Proof. Let us assume that there exists another locally optimal control u. Then, it holds that

$$\begin{aligned}\|\varphi_u(t) - \varphi_{\bar{u}}(t)\|_{L^2}^2 &\leq 2\int_0^t \|\partial_s\varphi_u(s) - \partial_s\varphi_{\bar{u}}(s)\|_{L^2}\, \|\varphi_u(s) - \varphi_{\bar{u}}(s)\|_{L^2}\, \mathrm{d}s \\ &\leq 2\sqrt{t}\,\|\varphi_u - \varphi_{\bar{u}}\|_{H^1(L^2)}\, \|\varphi_u - \varphi_{\bar{u}}\|_{L^\infty(L^2)} \leq 2L_1^2\sqrt{t}\,\|u - \bar{u}\|_{L^2(L^2)}^2.\end{aligned}$$

Integrating by parts, we also obtain the estimate

$$\begin{aligned}\|\nabla\varphi_u(t) - \nabla\varphi_{\bar{u}}(t)\|_{L^2}^2 &\leq 2\int_0^t \|\partial_s\varphi_u(s) - \partial_s\varphi_{\bar{u}}(s)\|_{L^2}\, \|\Delta\varphi_u(s) - \Delta\varphi_{\bar{u}}(s)\|_{L^2}\, \mathrm{d}s \\ &\leq 2\sqrt{t}\,\|\varphi_u - \varphi_{\bar{u}}\|_{H^1(L^2)}\, \|\varphi_u - \varphi_{\bar{u}}\|_{L^\infty(H^2)} \leq 2L_1^2\sqrt{t}\,\|u - \bar{u}\|_{L^2(L^2)}^2.\end{aligned}$$

Consequently, we have

$$\|\varphi_u - \varphi_{\bar{u}}\|_{L^2(H^1)} \leq \frac{2\sqrt{2}}{\sqrt{3}}\, L_1\, T^{3/4}\, \|u - \bar{u}\|_{L^2(L^2)}. \tag{9.97}$$

In the same fashion, we can derive the estimate

$$\begin{aligned}\|\phi_u(t) - \phi_{\bar{u}}(t)\|_{L^2}^2 &= 2\int_t^T \langle \partial_s\phi_u(s) - \partial_s\phi_{\bar{u}}(s), \phi_u(s) - \phi_{\bar{u}}(s)\rangle_{H^1}\, \mathrm{d}s \\ &\leq 2\sqrt{t}\,\|\phi_u - \phi_{\bar{u}}\|_{H^1((H^1)^*)}\, \|\phi_u - \phi_{\bar{u}}\|_{L^\infty(H^1)} \leq 2L_3^2\sqrt{T-t}\,\|u - \bar{u}\|_{L^2(L^2)}^2,\end{aligned}$$

and thus

$$\|\phi_u - \phi_{\bar{u}}\|_{L^2(L^2)} \leq \frac{2}{\sqrt{3}}\, L_3\, T^{3/4}\, \|u - \bar{u}\|_{L^2(L^2)}. \tag{9.98}$$

Furthermore, we know from Corollary 9.25 that both u and $\bar{u}$ satisfy the projection formula (9.80). A straightforward computation yields

$$|u(x,t)-\bar{u}(x,t)| \le \frac{1}{\kappa}\|\mathbb{h}\|_{L^\infty(\mathbb{R})}|\phi_u(x,t)-\phi_{\bar{u}}(x,t)| + |\phi_{\bar{u}}(x,t)|\,\|\mathbb{h}'\|_{L^\infty(\mathbb{R})}|\varphi_u(x,t)-\varphi_{\bar{u}}(x,t)|$$

for almost all $(x,t)\in\Omega_T$, and, recalling that $\|\mathbb{h}\|_{L^\infty(\mathbb{R})}\le 1$ and using (9.97) along with (9.98), we conclude that

$$\begin{aligned}\|u-\bar{u}\|_{L^2(L^2)} &\le \frac{1}{\kappa}\|\phi_u-\phi_{\bar{u}}\|_{L^2(L^2)} + \frac{1}{\kappa}\|\phi_{\bar{u}}\|_{L^\infty(L^p)}\,\|\mathbb{h}'\|_{L^\infty(\mathbb{R})}\,\|\varphi_u-\varphi_{\bar{u}}\|_{L^2(L^q)}\\ &\le \frac{1}{\kappa}\|\phi_u-\phi_{\bar{u}}\|_{L^2(L^2)} + \frac{1}{\kappa}c_\Omega(p)\|\phi_{\bar{u}}\|_{L^\infty(H^1)}\,\|\mathbb{h}'\|_{L^\infty(\mathbb{R})}\,c_\Omega(q)\|\varphi_u-\varphi_{\bar{u}}\|_{L^2(H^1)}\\ &\le \frac{2}{\sqrt{3}\kappa}\,T^{3/4}\left(L_3+\sqrt{2}L_1c_\Omega(p)\,c_\Omega(q)\|\phi_{\bar{u}}\|_{L^\infty(H^1)}\,\|\mathbb{h}'\|_{L^\infty(\mathbb{R})}\right)\|u-\bar{u}\|_{L^2(L^2)}.\end{aligned}$$

However, if (9.96) is satisfied we have

$$\frac{2}{\sqrt{3}\kappa}\,T^{3/4}\left(L_3+\sqrt{2}L_1c_\Omega(p)\,c_\Omega(q)\|\phi_{\bar{u}}\|_{L^\infty(H^1)}\,\|\mathbb{h}'\|_{L^\infty(\mathbb{R})}\right) < 1.$$

Therefore, the above inequality can hold true only if $\|u-\bar{u}\|_{L^2(L^2)}=0$ which means uniqueness of the locally optimal control. □

Remark 9.33 We can also interpret (9.96) as a condition for κ. Indeed, for arbitrary but fixed $T>0$, condition (9.96) is fulfilled provided $\kappa>0$ is large enough. This means that, for a sufficiently high penalisation of the medication dose, a locally optimal control $\bar{u}\in\mathbb{U}$ of problem (9.75) is unique.

The above theorem in particular yields the following:

Corollary 9.34 *Suppose that $T>0$ and $\kappa>0$ satisfy the assumption* (9.96) *of Theorem 9.32. Then, there exists a unique globally optimal control $\bar{u}$ of problem* (9.75).

In this case, each of the equivalent necessary optimality conditions (9.77) *and* (9.80) *is a sufficient condition for global optimality.*

Proof. Theorem 9.23 ensures the existence of at least one globally optimal control $\bar{u}\in\mathbb{U}$. Of course, $\bar{u}$ is also locally optimal. Hence, since assumption (9.96) holds, Theorem 9.32 implies that $\bar{u}$ is the unique locally optimal control. It follows immediately that $\bar{u}$ is the unique globally optimal control.

Moreover, $\bar{u}$ satisfies the equivalent necessary optimality conditions (9.77) and (9.80). Because of Theorem 9.32 it is also the only control satisfying these conditions. Hence, (9.77) (or (9.80) respectively) is a sufficient condition for global optimality. □

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"I've always believed that if you put in the work, the results will come."

- Michael Jordan

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