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# BETA INVARIANT AND VARIATIONS OF CHAIN THEOREMS

# FOR MATROIDS

DISSERTATION

A Dissertation
presented in partial fulfillment of requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
The University of Mississippi

by

SOOYEON LEE

August 2018

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# ABSTRACT

The beta invariant of a matroid was introduced by Crapo in 1967. We first find the lower bound of the beta invariant of 3-connected matroids with rank $r$ and the matroids which attain the lower bound. Second, we characterize the matroids with beta invariant 5 and 6. For binary matroids, we characterize matroids with beta invariant 7. These results extend earlier work of Oxley. Lastly, we partially answer an open question of chromatic uniqueness of wheels and prove a splitting formula for the beta invariant of generalized parallel connection of two matroids.

Tutte's Wheel-and-Whirl theorem and Seymour's Splitter theorem give, respectively, a constructive and structural view of the 3-connected matroids. Geelen and Whittle proved a chain theorem for sequentially 4-connected matroids and Geelen and Zhou proved a chain theorem for weakly 4-connected matroids. From these theorems, one can obtain a chain theorem for matroids as well. We prove a chain theorem for sequentially 4-connected and weakly 4-connected matroids.

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# 1 INTRODUCTION

In this chapter, we introduce some basic definitions and theorems in matroid theory. A matroid $M$ is an ordered pair $(E, \mathcal{I})$ consisting of a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$ having the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$.

(I3) If $I_1$ and $I_2$ are in $\mathcal{I}$ and $|I_1| < |I_2$, then there exists an element $e \in I_1 - I_2$ such that $I_2 \cup e \in \mathcal{I}$.

Let $M$ be a matroid, $E(M)$, ground set of $M$, and $\mathcal{I}$, collection of independent set of $M$. Then a basis of the matroid $M$ is a maximal independent set in $M$ and the rank of the matroid is the cardinality of a basis of $M$. A subset of $E(M)$ which is not in $\mathcal{I}$ is called dependent. A circuit is a minimal dependent set. For circuit $C_1$ and $C_2$ of $M$, the Circuit Elimination Axiom (referred as CEA in the dissertation) states that if $x \in C_1 \cap C_2$, then there exists another circuit $C_3 \subseteq C_1 \cap C_2 - x$. For a set $X \subseteq E(M)$, the rank of the set $X$ is the cardinality of the basis $B$ of $M|X$ and we denote it as $r_M(X)$ or $r(X)$. The closure of the set $X$, denoted as $cl(X)$ or $cl_M(X) = \{x \in E(M) : r(X \cup x) = r(X)\}$. A set $X \subseteq E(M)$ is called a flat, or is closed, if $cl(X) = X$. If a set $X$ is a flat and $r(F) = r(M) - 1$, then $X$ is called a hyperplane. Let $\mathcal{B}$ be a collection of matroid $M$ and let $\mathcal{B}^*(M) = \{E(M) - B : B \in \mathcal{B}\}$. Then $\mathcal{B}^*(M)$ is the set of basis of a matroid with ground set $E(M)$. This matroid is the dual of the matroid $M$ and is denoted $M^*$. If a set $X$ is independent in $M^*$, then we say the set is coindependent in $M$ as well. Other type of sets, circuit, bases, and hyperplanes are defined in the same manner.

Let $A$ be a $m \times n$ matrix over a field $\mathbb{F}$. Then the matroid obtained by the matrix $A$ is denoted as $M[A]$ and its ground set is the set of column labels of $A$. A set $X$ in $M[X]$ is independent if it is linearly independent in the vector space $V(m, \mathbb{F})$. For example, let $A$ be the matrix shown below.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
A = & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}
\end{array}
$$

Then $M[A]$ consists of the ground set $\{1, 2, 3, 4, 5\}$ and $\mathcal{I}(M[A]) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{1, 4\}, \{2, 4\}, \{5, 4\}\}$. A matroid $M$ is representable over a field $F$ when it is isomorphic to the vector matroid of a matrix over a field $F$. A representable matroid is binary when it is representable over $GF(2)$ and ternary when it is representable over $GF(3)$.

Let $G$ be a graph and let $E$ be the set of edges of $G$ and $\mathcal{I}$ be the set of edges of $G$ which does not contain any cycles. Then $\mathcal{I}$ is the set of independent sets of a matroid on $E$. This type of matroid is graphic, and is called the cycle matroid of $G$ and is denoted $M(G)$. For example, let $G$ be the graph shown in Figure 1.1. Then $M(G)$ consists of ground set $\{1, 2, 3, 4, 5\}$ and $\mathcal{I}(M[A]) = \{\emptyset, \{1\}, \{2\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{1, 4\}, \{2, 4\}, \{5, 4\}\}$.

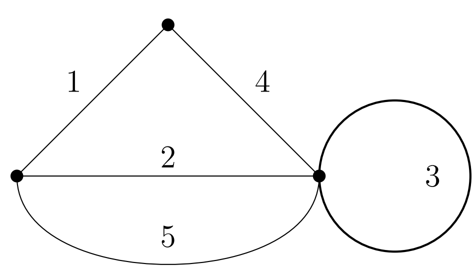


Figure 1.1: A graph $G$ with 5 labeled edges.

For a set $X \subseteq E(M)$, the function $\lambda_M(X) = r(X) + r(E(M) - X) - r(M)$ or equivalently, $\lambda_M(X) = r(X) + r^*(X) - |X|$ is called the connectivity function of $M$. Let $Y = E(M) - X$. For a positive integer $k$, the set $X$ and the pair of the set $(X, Y)$ is called $k-$separating if $\lambda_M(X) < k$. Then the pair $(X, Y)$ is called $k$-separation of $M$ if $\lambda_M(X) < k$ and $|X|$, $|Y| \geq k$. Lastly, $M$ is $n$-connected if $M$ has no $k$-separation for all $1 \leq k < n$.

Given a matroid $M$ and a set $T \subseteq E(M)$, we can delete or contract $T$ from $M$. The matroid obtained by deleting $T$ from $M$ is denoted as $M \backslash T$. This matroid has ground set $E(M) - T$ and the collection of independent set $\mathcal{I}(M \backslash T) = \{I \subseteq E(M) - T : I \in \mathcal{I}(M)\}$. Similarly, the matroid obtained by contracting $T$ from $M$ is denoted as $M/T$, and the collection of independent set of $M/t$ is defined as $\mathcal{I}(M/T) = \{I \subseteq E(M) - T : I \cup B_T \in \mathcal{I}(M)\}$ where $B_T$ is the basis of $M|T$. Deletion and contraction is easier to understand with graphic matroids. On Figure 1.2, we can see $M(G)\backslash e$ and $M(G)/e$.

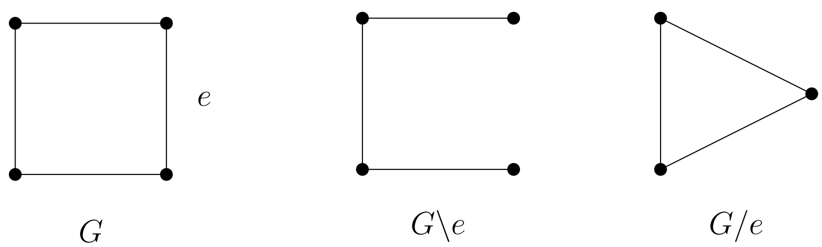


Figure 1.2: Deleting and contracting an element.

Let $M$ and $N$ be matroids such that $M = N \backslash e$. Then $N$ is a single element extension, or an extension of $M$. If $M = N/e$, then $N$ is a single element coextension, or a coextension of $M$. A modular cut of $M$, denoted as $\mathcal{M}$, is a collection of flats of $M$ which satisfies two properties:

(i) If $F \in \mathcal{M}$ and $F \subseteq F'$, then $F \in \mathcal{M}$.

(ii) If $F$ and $F' \in \mathcal{M}$ and $r(F) + r(F') = r(F \cup F') - r(F \cap F')$ then $F \cap F' \in \mathcal{M}$.

Let $M$ be a matroid and $N = M \backslash e$ and $\mathcal{M}$ is the corresponding modular cut corresponding to the extension $N$. Then $N$ is denoted as $M +_{\mathcal{M}} e$. If $X \subseteq E(M)$, then $r_N(X \cup e) = r_M(X)$ if $cl_M(X) \in \mathcal{M}$. Similarly, $r_N(X \cup e) = r_M(X) + 1$ if $cl_M(X) \notin \mathcal{M}$. Lastly, two flats $F$ and $F'$ in (ii) form a modular pair. In this case, we say $(F, F')$ is a modular pair.

This also tells which sets to avoid if we want to find 3-connected extension of $M$ using modular cut. Suppose $N = M +_{\mathcal{M}} e$ for a matroid $M$ and a modular cut $\mathcal{M}$ of $M$. If $\emptyset \in \mathcal{M}$, then $e$ is a loop of $N$. If $a \in E(M)$ and $a \in \mathcal{M}$ then $a$ and $e$ are in parallel in $N$. Therefore if $N$ is 3-connected then $\emptyset$, $\{a\} \notin \mathcal{M}$ for all $a \in E(M)$.

Suppose $M_1$ and $M_2$ are two matroids and $cl_1$ and $cl_2$ be closure operation of the matroid $M_1$ and $M_2$ respectively. Then the generalized parallel connection of two matroids can exist if the following corollary holds:

**Corollary 1.0.1.** *[18] Suppose that $cl_1(T)$ is a modular flat of $M_1$ and every non-loop element of $cl_1(T)$ is parallel to some element of $T$. Then $T$ is fully embedded in $M_1$, so the proper amalgam of $M_1$ and $M_2$ exists.*

If the conditions in the previous corollary hold, then the proper amalgam of $M_1$ and $M_2$ is the generalized parallel connection across $T$ and is denoted by $P_T(M_1, M_2)$. Therefore if $M = P_T(M_1, M_2)$ then $E(M) = E(M_1) \cup E(M_2)$ and $M_1|T = M_2|T$. Also, for every flat $F$ of $M$, we have $r_M(F) = r_M(F \cap E_1) + r_M(F \cap E_2) - r_M(F \cap T)$. When $T$ is a 3-element circuit, then $P_T(M_1, M_2)$ is called the generalized parallel connection of $M_1$ and $M_2$ across a 3-point line. When both $M_1$ and $M_2$ are binary and $|E_1|$, $|E_2| > 6$, then $P_T(M_1, M_2)\backslash T$ is called 3-sum of $M_1$ and $M_2$ and is denoted $M_1 \oplus_3 M_2$. When $T = \{p\}$, then $P_T(M_1, M_2)$ is called parallel connection of $M_1$ and $M_2$ with respect to $p$ and is denoted $P(M_1, M_2)$. Lastly, $P(M_1, M_2)\backslash T = M_1 \oplus_2 M_2$.

Let $W_r$ denote the rank-$r$ wheel graph. The matroid obtained from this graph is often denoted as $W_r$ and is called a wheel. In many literature, this matroid is also noted as $M(W_r)$ as well but in this dissertation, we will note it as $W_r$. A wheel has two types of elements,

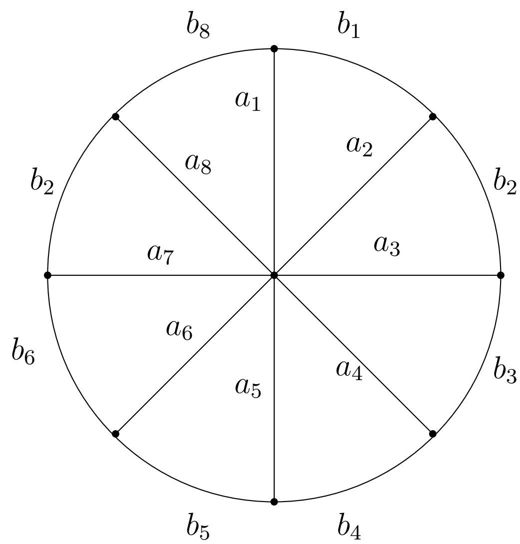


Figure 1.3: Graphic representation of $W_8$.

the spoke, denoted $a_i$, of the wheel and the rim, denoted $b_i$, of the wheel where $1 \leq i \leq r$. Figure 1.3 shows a rank-8 wheel, $W_8$. Let $\mathcal{C}$ be the collection of circuits of $W_r$. Then the set $\{b_1, b_2, \cdots, b_r\} \in \mathcal{C}$. We can see from Figure 1.3 that the set $\{b_1, b_2, \cdots, b_8\}$ forms a cycle of the graph and a circuit of the corresponding matroid. A rank-$r$ whirl, denoted as $W^r$, has the same ground set as a rank-$r$ wheel but $\mathcal{B}(W^r) = \mathcal{B}(W_r) \cup \{b_1, b_2, \cdots, b_r\}$ where $\mathcal{B}(W^r)$ and $\mathcal{B}(W_r)$ are the collection of bases of $W^r$ and $W_r$ respectively. In many literature, a rank-$r$ whirl is also noted as $\mathcal{W}^r$ but in this dissertation, we note it as $W^r$.

Wheels and whirls are important matroids. In a way, these matroids are building blocks of the 3-connected matroids. Also if $W$ is a wheel or a whirl, for any $e \in E(W)$, neither $W \backslash e$ nor $W/e$ is not 3-connected and wheels and whirls are the only matroids with such properties as described in Theorem 1.0.2. Also one can characterize binary and non-binary matroids with wheels and whirls: Every 3-connected binary matroids have a $W_3$-minor and non-binary matroids have a $W^2$-minor.

**Theorem 1.0.2.** *[21] (Tutte's Wheels-and-Whirls Theorem) The following are equivalent for a 3-connected matroid $M$ having at least one element.*
*(i) For every element $e$ of $M$, neither $M\backslash e$ nor $M/e$ is 3-connected.*
*(ii) $M$ has rank at least three and is isomorphic to a wheel or a whirl.*

Another interesting class of matroids is the class of uniform matroids. The uniform matroid $U_{r,n}$ has $n$-elements. The rank of the uniform matroid $U_{r,n}$ is $r$. Lastly, $U_{r,n}$ has a collection of set of bases $\mathcal{B}(U_{r,n}) = \{|X| = r \mid X \subseteq E(U_{r,n})\}$.

On some of the matroids mentioned in this dissertation, we give an appropriate description of the matroid. In the dissertation, we give one of the following representations of the matroid or its dual: Graphic, matrix, geometric representation or a modular cut and the corresponding matroid.

On a geometric representation, elements of matroids are represented by dots. Elements of a rank-2 dependent sets are on a same line and elements of a rank-3 dependent sets are on a same plane. Any 2-points on a line is a rank-2 independent set and any 3-points on a plane but not on a line is a rank-3 independent set.

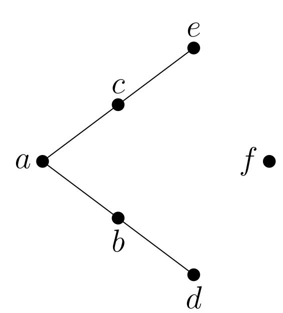


Figure 1.4: Geometric representation of $Q_6$.

For example, consider a geometric representation of a matroid $Q_6$ on the Figure 1.4. The set $\{a, c, e\}$ is a rank-2 circuit. The set $\{a, c, e, f\}$ is on a same plane, thus a rank-4 dependent set. The set $\{a, b, f\}$ is a rank-3 independent set.

The following two theorems tell us 3-connected minors of 3-connected matroids. Restatement of the Splitter Theorem and some application of the theorem can be found in [18].

**Theorem 1.0.3.** *(Seymour's Splitter Theorem [20]) Let $M$ and $N$ be 3-connected matroids such that $N$ is a minor of $M$ with at least four elements and if $N$ is a wheel, then $M$ has no larger wheel as a minor, while if $N$ is a whirl, then $M$ has no longer whirl as a minor. Then there is a sequence $M_0, M_1, \cdots, M_n$ of 3-connected matroids with $M_0 \cong N$ and $M_n \cong M$ such that $M_i$ is a single element deletion or single element contraction of $M_{i+1}$ for all $i$ in $\{0, 1, \cdots, n-1\}$.*

**Theorem 1.0.4.** *[8] Let $N$ be a 3-connected proper minor of a 3-connected matroid $M$ such that $|E(N)| \geq 4$ and $M$ is neither a wheel nor a whirl. Suppose that if $N \cong W^2$, then $M$ has no $W^3$-minor, while if $N \cong W_3$, then $M$ has no $W_4$-minor. Then $M$ has a 3-connected minor $M_1$ and an element $e$ such that $M_1/e$ or $M_1 \backslash e$ is isomorphic to $N$.*

Two theorems imply that if $M$ is a 3-connected matroid, and $W \in \{W_3, W_4, W^2, W^3\}$, then there exists a chain of 3-connected matroids which starts at $W = M_0$ and ends at $M = M_n$ where $M_i$ is a single element contraction or a single element deletion of $M_{i+1}$ for all $i$ in $\{0, 1, \cdots, n-1\}$. Furthermore, if $M$ is a binary matroid with no $W_4$-minor, then $W = W_3$. If $M$ is a binary matroid with a $W_4$-minor, then $W = W_4$. Similarly, if $M$ is a non-binary matroid with no $W^3$-minor, then $W = W^2$. Lastly, if $M$ is a non-binary matroid with a $W^3$-minor, then $W = W^3$. We will use this fact in latter chapters repeatedly.

# 2 BOUNDING THE BETA INVARIANT

In Chapter 2, we give a best possible lower bound of the beta invariant of 3-connected rank $r$ matroids and obtain all such matroids which achieve the lower bound. In Section 1, we give the definitions and results from other literature on the beta invariant. In Section 2, we state and prove the main result.

## 2.1 Introduction

For a matroid $M$, the beta invariant of $M$, denoted as $\beta(M)$, was first introduced by Crapo [9] as follows:

$$\beta(M) = (-1)^{r(M)} \sum_{A \subseteq E(M)} (-1)^{|A|} r(A).$$

The beta invariant satisfies the deletion-contraction formula. That is, if $e$ is neither a loop nor a coloop, then

$$\beta(M) = \beta(M \backslash e) + \beta(M/e). \tag{2.1.1}$$

Crapo also proved the following interesting result.

**Theorem 2.1.1.** *[9] A matroid $M$ with at least two elements is connected if and only if $\beta(M) > 0$. Moreover, $\beta(M) = \beta(M^*)$.*

From this result, we can deduce that if $N$ is a connected single element extension of $M$, then $\beta(N) \geq \beta(M)$. Beta invariant is non-zero if and only if the matroid is connected. There is much information about the matroid that can be obtained by the beta invariant. For example, Brylawski characterized all matroids with beta invariant one.

**Theorem 2.1.2.** *[4] For a connected matroid $M$ on a set of at least two elements, the following statements are equivalent:*

*(i)* $\beta(M) = 1$.

*(ii) $M$ is a series-parallel extension of $U_{1,1}$.*

*(iii) $M$ has no minor isomorphic to $U_{2,4}$ or $W_3$.*

From the previous theorem we can see that the beta invariant of any series-parallel network is 1. Oxley extended Brylawski's result and characterized all matroids with beta invariants 2, 3, and 4. The geometric representation of non-binary matroids $O_7$, $Q_6$, and $F_7^-$ are shown in Figure 2.1. Also matrix representation of binary matroids $F_7$, $S_8$, and $M(K_5 \backslash e)$ are shown in Figure 2.2.

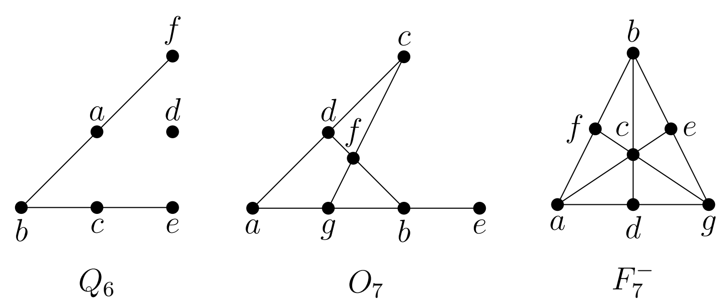


Figure 2.1: Geometric Representation of $Q_6$, $O_7$, and $F_7^-$.

**Theorem 2.1.3.** *[16] Let $M$ be a matroid. Then*

*(i) $\beta(M) = 2$ if and only if $M$ is a series-parallel extension of $U_{2,4}$ or $W_3$.*

*(ii) $\beta(M) = 3$ if and only if $M$ is a series-parallel extension of $U_{2,5}, U_{3,5}, F_7, F_7^*, W_4,$ or $W^3$.*

*(iii) $\beta(M) = 4$ if and only if either*

$$F_7 \begin{bmatrix} & 1 & 0 & 1 & 1 \\ I_3 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 1 & 1 \end{bmatrix} \quad S_8 \begin{bmatrix} & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 \\ I_4 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 \end{bmatrix} \quad K_5 \backslash e \begin{bmatrix} & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Figure 2.2: Matrix representation of $F_7$, $S_8$, and $M(K_5\backslash e)$.

*a) $M$ is a series-parallel extension of one of the matroids $U_{2,6}$, $U_{4,6}$, $W^4$, $W_5$, $Q_6$, $O_7$, $O_7^*$, non-Fano, non-Fano dual, $S_8$, $M(K_5\backslash e)$, or $M^*(K_5\backslash e)$, or*

*b) $M$ is a 2-sum of matroids $M_1$ and $M_2$ each of which is a series-parallel extension of $M(K_4)$ or $U_{2,4}$.*

Oxley also proved the following proposition about matroids with beta invariant $k \geq 1$.

**Proposition 2.1.4.** *[16] Let $M$ be a matroid and suppose that $\beta(M) = k > 1$. Then either*
*(i) $M$ is a series-parallel extension of a 3-connected matroid $N$ such that $\beta(N) = k$, or*
*(ii) $M = M_1 \oplus_2 M_2$, and $\beta(M) = \beta(M_1)\beta(M_2)$ each having $\beta(M_i) < k$ for $i = 1, 2$.*

It is an interesting question to bound the beta invariant of a matroid. In the next result, Oxley showed that in general, the beta invariant is an exponential function of the connectivity of a matroid. He gave a bound for $n$-connected matroids, which is stated in the following theorem.

**Theorem 2.1.5.** *[16] If $M$ is a $n$-connected matroid with $2(n-1)$ elements, then $\beta(M) \geq 2^{n-2}$ for $n \geq 2$.*

Oxley stated that this bound, in general, is weak as $n$ increases. Also, the bound is not sharp even for 4-connected matroids. He also proved in the same paper that for 4-connected matroids, $\beta \geq 10$ unless the matroid is $U_{3,6}$ where $\beta(U_{3,6}) = 6$. Oxley also states that one can use the same proof technique he used to obtain a lower bound for any $n$-connected matroid but this is not an efficient way to do so. Thus, for 3-connected matroids with rank $r$, we give a better lower bound and give all matroids which attains the lower bound. Also, we give a lower bound of beta invariant for rank $r$ non-binary 3-connected matroids as well.

## 2.2 Main Result and Proof

In this section, we state and prove the main result of Chapter 2.

**Theorem 2.2.1.** *Let $M$ be a 3-connected matroid of rank $r$. Then $\beta(M) \geq r(M) - 1$, $\beta(M) = r(M) - 1$ if and only if $M$ is a wheel, $F_7^*$, or the Prism. Moreover, if $M$ is non-binary, then $\beta(M) \geq r(M)$, and $\beta(M) = r(M)$ if and only if $M$ is a whirl, $U_{r,r+2}$ ($r \geq 2$), $(F_7^-)^*$, or $O_7^*$.*

*Proof.* The proof of the theorem is divided into four parts. In Part 1 we state and the lower bound of beta invariant for a general 3-connected matroid and for a 3-connected non-binary matroids. In Part 2, we explain the process of finding all 3-connected matroids such that the beta invariant is one less than the rank and the 3-connected non-binary matroids with beta invariant equal to the rank. In Part 3, we find all 3-connected matroids where the beta invariant is one less than the rank. In Part 4, we find all 3-connected non-binary matroids where the beta invariant equals to the rank.

**Part 1**

In Part 1 we prove that if $M$ is a 3-connected matroid with rank $r$ then $\beta(M) \geq r(M) - 1$. We also prove that if $M$ is 3-connected and non-binary, then $\beta(M) \geq r(M)$.

Suppose $M$ is a 3-connected matroid such that $r(M) = r$. If $M$ is a rank $r$ wheel, then $\beta(M) = r - 1$. Similarly, if $M$ is a rank $r$ whirl, then $\beta(M) = r$. Therefore $\beta(M) \geq r \geq r - 1$ if $M$ is a wheel or a whirl. Now, suppose $M$ is not a wheel or a whirl. Then by Theorem 1.0.4, $M$ has a 3-connected minor $M_1$ and an element $e$ such that $M_1/e$ or $M_1 \backslash e$ is isomorphic to $N$ where $N \in \{W_3, W_4, W^2, W^3\}$. If $M$ is binary, then $N \cong W_3$ or $W_4$ and if $M$ is non-binary, then $N \cong W^2$ or $W^3$. Then by Theorem 1.0.4, there exists a chain of 3-connected matroids from $M$ to $M_1$. Thus we obtain a chain of matroids from $M = M_n$ to $N = M_0$ such that for each $i \in \{0, \cdots, n-1\}$, there exists $e \in E(M_i)$ where $M_i \backslash e$ or $M_i/e \cong M_{i-1}$. Consider the matroid $M_1$ and $M_0$. We know $r(M_1) = r(M_0)$ or $R(M_1) = r(M_0) + 1$. Therefore $r(M_0) \geq r(M_1) - 1$. Also, there exists an element $e \in E(M_1)$ such that $M_1/e$ or $M_1 \backslash e \cong M_0$. Furthermore, because $M_1$ is 3-connected, both $M_1/e$ and $M_1 \backslash e$ are connected and $\beta(M_1/e), \beta(M_1 \backslash e) \geq 1$. Then by the deletion-contraction formula for beta invariant, $\beta(M_1) = \beta(M_1 \backslash e) + \beta(M_1/e) \geq \beta(M_0) + 1$. Furthermore, as $M_0$ is either a wheel or a whirl, we have $\beta(M_0) \geq r(M_0) - 1$ as well. Therefore we have $\beta(M_1) \geq r(M_0) - 1 + 1 \geq r(M_1) - 1$. Now, by applying the same argument to $M_1$ and $M_2$, we can see that $\beta(M_2) \geq \beta(M_1) + 1 \geq r(M_2) - 1$. Therefore by the inductive argument, we have $\beta(M_n) \geq \beta(M_{n-1}) + 1 \geq r(M_n) - 1$ and the first statement of the theorem holds: If $M$ is a 3-connected matroid, then $\beta(M) \geq r(M) - 1$.

For non-binary matroids, if $M$ is a non-binary matroid, then by Theorem 1.0.4 and the Splitter theorem, we have chain of matroid from $M = M_n$ to $N = M_0$ where $N = W^2$ or $W^3$. In this case, because $M_0$ is a whirl, $\beta(M_0) = r(M_0)$. Therefore, $\beta(M_1) \geq \beta(M_0) + 1 = r(M) + 1 \geq r(M_0)$. By repeating the same argument, we have $\beta(M_n) \geq \beta(M_{n-1}) + 1 \geq r(M_n)$. Thus, if $M$ is a 3-connected non-binary matroid, then $\beta(M) \geq r(M)$.

For general 3-connected matroid $M$, we have $\beta(M) \geq r(M) - 1$ and for 3-connected non-binary matroid $N$, we have $\beta(N) \geq r(N)$. From this fact, we can deduce that if $M$ is a 3-connected matroid $M$, then $\beta(M) = r(M) - 1$ implies $M$ is binary.

**Part 2**.

In Part 2, we explain the process of finding all 3-connected matroids with beta invariant one less than the rank and 3-connected non-binary matroids with beta invariant equal to the rank. First, suppose $M$ is a 3-connected matroid such that $\beta(M) = r(M) - 1$. In

Part 1, we proved that if $\beta(M) = r(M) - 1$, then $M$ is a binary matroid. If $M$ is a wheel then $\beta(M) = r(M) - 1$ and the theorem holds. Now suppose $M$ is a 3-connected matroid and $M$ is not a wheel. Then by Theorem 1.0.4 and the Splitter theorem, there exists a chain of 3-connected matroid from $W = M_0$ to $M = M_n$ where $W \cong W_3$ or $W_4$. We will first show that for all $i \in \{1, \cdots, n\}$, $M_i$ is a coextension of $M_{i-1}$ and $\beta(M_i) = \beta(M_{i-1}) + 1$. From Part 1 of the proof, we know $\beta(M_i) \geq \beta(M_{i-1}) + 1$ for each $i \in \{1, \cdots, n\}$. Then $\beta(M_n) \geq \beta(M_0) + n$. Now, suppose to the contrary that there are some matroids in the chain where $M_i$ is an extension of $M_{i+1}$. Then there are some matroids in the chain where $r(M_i) = r(M_{i-1})$. Therefore $r(M_n) < r(M_0) + n$. Since $M_0$ is a wheel, $\beta(M_0) = r(M_0) - 1$. Therefore $\beta(M_n) \geq \beta(M_0) + n = r(M_0) + n - 1 > r(M_n) - 1$. A contradiction as $M_n = M$ and $M$ is a 3-connected matroid such that $\beta(M) = r(M) - 1$. Therefore for each matroid in the chain, $M_i$ is a coextension of $M_{i-1}$ and $\beta(M_i) = \beta(M_{i-1}) + 1$. Also, for all $i \in \{0, \cdots, n\}$, we have $\beta(M_i) = r(M_i) - 1$. Thus all matroids in this chain from $M_0$ to $M_n$ are 3-connected and the beta invariant is one less than the rank.

Thus we coextend $W_3$ and $W_4$ by an element and look for any coextensions where the beta invariant increased exactly by one. If there exists a such matroid, then we repeat the process and if there is no such coextension, we stop. Then we can obtain all 3-connected matroids where the beta invariant is one less than its rank. However, beta invariants are invariant under duality and it is sometimes easier to handle extensions than coextensions of a matroid. Thus, instead of coextending matroids from $W_3$ and $W_4$, we will extend the dual of $W_3$ and $W_4$, which are again $W_3$ and $W_4$, and find matroids where the beta invariant increases exactly by one.

For non-binary matroids, the argument and the process of finding all 3-connected non-binary matroids with beta invariant equal to their rank is exactly the same as the binary ones. The only difference from the general case is with non-binary matroids, where we extend the duals of $W^2$ and $W^3$, which are $W^2$ and $W^3$.

Before we proceed with Part 3 of the proof, we can easily check, using the deletion-contraction formula, that the beta invariant of wheel, $F_7^*$, or a prism equals to one less than its rank. And for each of the matroid whirl, $U_{r,r+2}$, $r \geq 2$, $(F_7^-)^*$ and $O_7^*$, the beta invariant equals to its rank.

**Part 3**

Now, we find all 3-connected matroids $M$ such that $\beta(M) = r(M) - 1$ using the procedure explained in Part 2. We first extend $W_3$. By the proof of Part 1, since $\beta(M) = r(M) - 1$ for 3-connected matroids imply $M$ is binary, we look for a 3-connected binary extension of $W_3$ with no $W_4$-minor such that the $\beta = \beta(W_3) + 1 = 3$. We exclude matroids with a $W_4$-minor as we will extend matroids from $W_4$. There are small number of 3-connected binary matroids with 9 or less elements. Most of the matroids in this part are well known as well as their matrix representations. In the dissertation, we use the known list of 3-connected matroids with small number of elements. For binary matroids, we use both known properties of the matroids and the matrix representation to extend and coextend the matroids. The following table was obtained from the paper [11].

| $\|E(M)\|$ | 3-Connected Binary Matroids |
|---|---|
| 6 | $W_3$ |
| 7 | $F_7$, $F_7^*$ |
| 8 | $W_4$, $S_8$, $AG(3,2)$ |
| 9 | $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5\backslash e)$, $Prism$, $P_9$, $P_9^*$, $Z_4$, $Z_4^*$ |

Table 2.1: All 3-connected binary matroids with 6, 7, 8, and 9 elements.

As for the 3-connected binary extension of $W_3$, we only have one extension: $F_7$. Also $\beta(F_7) = \beta(W_3) + 1$ as well. However, there does not exist any 3-connected binary extension of $F_7$. Therefore $W_3$ and $F_7^*$ are the only matroids we can obtain by coextending $W_3$ where the beta invariant is one less than the rank.

Now, we look at the 3-connected binary extensions of $W_4$. By using Table 2.1 and checking the rank, we can see that there are exactly four 3-connected binary extensions of $W_4$: $M^*(K_{3,3})$, $M(K_5\backslash e)$, $P_9$, and $Z_4$. It is not difficult to compute the beta invariant of those matroids and check that $M(K_5\backslash e)$ is the only matroid with $\beta = 4$. Therefore, *Prism*, dual of $M(K_5\backslash e)$, is the only coextension of $W_4$ where the beta invariant is one less than the rank. Now, we need to look at the 3-connected binary extension of $M(K_5\backslash e)$. This is equivalent to adding a vector column to the matrix representation of $M(K_5\backslash e)$ over $GF(2)$. This process is shown on the figure below with the matrix $B$ where $M(B)$ represents a 3-connected binary extension of $M(K_5\backslash e)$. The vector column $f$ of matrix $B$ corresponds to the extended element of $M(K_5\backslash e)$. In each extension the vector column $f = (x_1, x_2, x_3, x_4)^T$, at least two of $x_1$, $x_2$, $x_3$, $x_4$ must be 1 since the extension needs to be 3-connected. Now it is easy to see the only possible choices of $f$ are $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, or $f_6$.

$$
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & f
\end{array}
\qquad
\begin{array}{cccccc}
f_1 & f_2 & f_3 & f_4 & f_5 & f_6
\end{array}
$$

$$
B = \left[\begin{array}{cccc|cccccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & * \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & * \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & * \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & *
\end{array}\right]
\quad
\begin{bmatrix}1\\0\\1\\0\end{bmatrix}
\begin{bmatrix}0\\1\\1\\1\end{bmatrix}
\begin{bmatrix}1\\0\\1\\1\end{bmatrix}
\begin{bmatrix}1\\1\\0\\1\end{bmatrix}
\begin{bmatrix}1\\1\\1\\0\end{bmatrix}
\begin{bmatrix}1\\1\\1\\1\end{bmatrix}
$$

For all $f_1$, $f_2$, ... $f_6$, we have to compute the beta invariant of $M(B)$. To do this, we compute the beta invariant using the matrix representation of the matroid and obtain a lower bound. Since $M(B)\backslash f_1 = M(K_5\backslash e)$, we only need to check the beta invariant of $M(B)/f_1$. The following matrix $B/f$ is a representation of $M(B)/f$.

$$B/f = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{cccc|ccccc} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array} \qquad W = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array}$$

This matrix representation corresponds to a connected matroid with $W_3$-minor by comparing the matrix $B/f$ with $W$, which is a matrix representation of $W_3$. Therefore we have $\beta(M(B)/f_1) \geq \beta(W_3) \geq 2$. We do this for other cases and computed that for each case, $\beta(M(B)/f) \geq 2$.

Then, using computer program SAGE, we confirmed our result by computing the Tutte polynomial of the matroid and then obtaining the beta invariant from the Tutte polynomial. As an example, suppose $f = f_1$. Then, on SAGE, we can construct matroid using matrix representation. Then we can compute the Tutte polynomial and obtain the beta invariant. So if we were to construct a matroid $M(B)$ using $f = f_1$, we obtain the Tutte polynomial of the given matroid, $y^6 + 4y^5 + x^4 + 5xy^3 + 10y^4 + 6x^3 + 10x^2y + 15xy^2 + 15y^3 + 11x^2 + 20xy + 15y^2 + 6x + 6y$. From this, we can see that the beta invariant of this particular extension is 6. Using the same method, we confirmed the beta invariants were at least 6 for all cases.

So, there does not exist a 3-connected binary extension of $M(K_5\backslash e)$ such that the beta invariant increases exactly by one. Therefore the 3-connected binary matroids with $W_4$-minor such that the beta is one less than the rank are: $W_4$ and $Prism$. Hence if $M$ is a 3-connected matroid such that $\beta(M) = r(M) - 1$, then $M$ is isomorphic to a wheel, $F_7^*$ or $Prism$.

**Part 4**

Now we find the 3-connected non-binary matroids $M$ such that $\beta(M) = r(M)$. Again, we use the same procedure used in Part 3 of the proof. We first extend $W^2$ and find all single element extensions of $W^2 = U_{2,4}$ where the $\beta = 3$. There exists only one 3-connected single element extension of $U_{2,4}$, namely, $U_{2,5}$. And $\beta(U_{2,5}) = 3$ as well. With $U_{2,n}$, it is not difficult to compute the beta invariant since for any $e \in E(U_{2,n})$, we can see $U_{2,n}/e \cong U_{1,n-1}$ and $\beta(U_{1,n-1}) = 1$ as it is connected with no $U_{2,4}$-minor. Also, $U_{2,6}$ is the only 3-connected single element extension of $U_{2,5}$ and $\beta(U_{2,6}) = \beta(U_{2,5}) + 1$. We can deduce that $\beta(U_{2,n}) = n - 2 = r(U^*_{2,n})$ for all $n \geq 2$. Therefore $M^* \cong U_{2,n}$ and $M \cong U_{r-2,n}$ for all $n \geq 2$.

Now, we extend $W^3$. Let $N$ be a 3-connected single element extension of $W^3$ such that $\beta(N) = \beta(W^3) + 1 = 4$. Then the beta invariant of $N$ equals to 4 and we already know all 3-connected matroids with beta invariant 4. Furthermore, we know $N$ has 7-elements and the rank is 3. By Theorem 2.1.3, either $N \cong O_7$ or $F_7^-$. Thus $O_7^*$ and $(F_7^-)^*$ are two single element coextension of $W^3$ such that the beta invariant equals to its rank. Now, we need to extend $O_7$ and $F_7^-$. To do so, we use the geometric representation of $O_7$ and $F_7^-$.

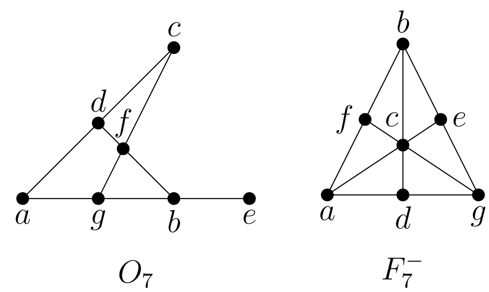


Figure 2.3: Geometric representation of $O_7$ and $F_7^-$.

Suppose $O$ is a 3-connected single-element extension of $O_7$ by an element $x$. Then $O\backslash x = O_7$ and $O/x$ is connected since $O_7$ is 3-connected. Now, we show that $O/x$ has a $U_{2,4}$-minor and thus $\beta(O/x) \geq 2$. Let $cl_O(A)$ denote the closure of a set $A$ in $O$. Then, either $x \notin cl_O(\{a, g, b, e\})$ or $x \in cl_O(\{a, g, b, e\})$. If $x \notin cl_O(\{a, g, b, e\})$ then $r_{O/x}(\{a, g, b, e\}) = 2$ and this implies $O/x|\{a, g, b, e\} \cong U_{2,4}$. Thus in this case, $O/x$ has $U_{2,4}$-minor. Now, suppose $x \in cl_O(\{a, g, b, e\})$. If there exists a circuit $C = \{x, u, v\}$ where $u \in \{c, d, f\}$. Suppose without the loss of generality that $u = c$. If $v = b$ or $e$, then $\{c, b, e\}$ is forced to be a circuit in $O$, and $O_7$, a contradiction. If $v = d$ or $f$, then $\{c, d, b, e\}$ or $\{c, f, b, e\}$ contains a circuit of $O$ and $O_7$, a contradiction as well. Therefore if $x \in cl_O(\{a, g, b, e\})$, then $x$ is not in a rank-2 circuit with any one of the elements $c$, $d$ or $f$. Thus we get only one possible extension as shown in Figure 2.4. It is not difficult to check that by contracting $x$ from $O$, we still obtain $U_{2,4}$-minor. Therefore if $O$ is any 3-connected extension of $O_7$, then $\beta(O) \geq \beta(O_7) + 2$.

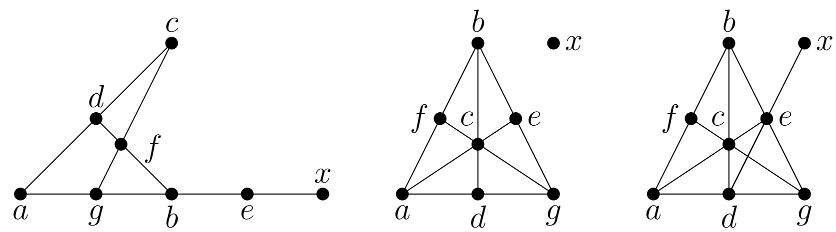


Figure 2.4: Possible extensions of $O_7$ and $F_7^-$.

Now, consider single element extensions of $F_7^-$. Suppose $F$ is a 3-connected single element extension of $F_7^-$ by an element $x$. Suppose $x \in cl_F(C_1)$ where $C_1$ is a rank-2 circuit of $F_7^-$. Without the loss of generality, let $C_1 = \{a, d, g\}$. If $\{b, x, c\}$ is a circuit of $F$ as well, then $x \in cl_F(\{b, c, d\})$. If $\{b, x, g\}$ is a circuit of $F$ as well, then $x \in cl_F(\{b, e, g\})$. Therefore either $x$ is in the closure of exactly one 3-element circuit of $F_7^-$ or $x$ is in more than one 3-element circuit of $F_7^-$. In the first case, suppose $x \in cl_F(\{a, d, g\})$ and $cl_F(\{b, c, d\})$. Then $c \in cl_F(\{a, d, g\})$ as well, a contradiction. In the second case, $F/x$ has a $U_{2,4}$-minor. Now, if $x$ is not be in the closure of any triangles in $F_7^-$, since $F$ is assumed to be 3-connected, $x$ has to be in a triangle with two elements from the set $\{f, e, d\}$ or $x \in cl_O(\{f, d, e\})$ which is equivalent to a free extension. In Figure 2.4, one can see an extension where $\{x, f, e\}$ is a triangle and another one, a free extension. In both cases, one can easily check that if we contract $x$, we still have a $U_{2,4}$-minor. Therefore if $F$ is any 3-connected extension of $F_7^-$, then $\beta(F) \geq \beta(F_7^-) + 2$. Finally we conclude that if $M$ is a 3-connected non-binary matroid such that $\beta(M) = r(M)$, then $M$ is isomorphic to a whirl, $U_{r,r+2}$, $r \geq 2$, $O_7^*$, or $(F_7^-)^*$. $\square$

# 3 MATROIDS WITH SMALL BETA INVARIANT

In this chapter, we study matroids with small beta invariant. In Section 1, we survey the previous research on characterizing matroids with small beta invariants. In Section 2, we prove some lemmas which will be used in Section 3 of this chapter. In Section 3, we state and prove the main results of this chapter which characterizes all matroids with beta invariant 5, and 6, and for binary matroids, 7.

## 3.1 Matroids With Beta Invariant At Most 4

Crapo characterized all matroids with beta invariant 0 and Brylawski characterized all matroids with beta invariant 1. Oxley characterized all matroids with beta invariant 2, 3, and 4. Crapo and Seymour's results are listed in Chapter 2 but we list restatements of their theorems again together with Oxley's result in the following theorem.

**Theorem 3.1.1.** *Let $M$ be a matroid with at least two elements. Then*

*(i) $\beta(M) = 0$ if and only if $M$ is not connected. [9]*

*(ii) $\beta(M) = 1$ if and only if $M$ has no minor isomorphic to $U_{2,4}$ or $M(K_4)$. [4]*

*(iii) $\beta(M) = 2$ if and only if $M$ is a series-parallel extension of $U_{2,4}$ or $M(K_4)$. [16]*

*(iv) $\beta(M) = 3$ if and only if $M$ is a series-parallel extension of $U_{2,5}, U_{3,5}, F_7, F_7^*, M(W_4)$, or $W^3$. [16]*

*(v) $\beta(M) = 4$ if and only if either*

*a) $M$ is a series-parallel extension of one of the matroids $U_{2,6}$, $U_{4,6}$, $W^4$, $M(W_5)$, $Q_6$,*

*$O_7$, $O_7^*$, non-Fano, non-Fano dual, $S_8$, $M(K_5\backslash e)$, or $M^*(K_5\backslash e)$, or*

*b) $M$ is a 2-sum of matroids $M_1$ and $M_2$ each of which is a series-parallel extension of $M(K_4)$ or $U_{2,4}$. [16]*

Furthermore, Benashski, Martin, Moore and Traldi characterized all simple 3-connected graphs with beta invariant 9 or less in [3]. Instead of directly extending and coextending binary matroids with small invariants to find matroids with larger beta invariant, we first compute the beta invariant of the binary matroids with no $P_9$-minor. To compute the beta invariant of the matroids with no $P_9$-minor, we prove a lemma which gives the recursive formula for the beta invariant of the starfish and other classes of matroids with no $P_9$-minor. Note that we can also obtain the list of binary matroids with small beta invariants by extending and coextending $W_3$ and $W_4$ using MACEK and compute the beta invariant by SAGE as well. However, we provide a way to compute without completely relying on the computer programs for binary matroids with no $P_9$-minor.

To classify binary matroids with no $P_9$-minor, we use the following theorems associated with matroid minors. First, the following theorem is result of Ding and Wu [11] which characterized all binary matroids with no $P_9$-minor. A binary matroid starfish is defined in [11]. The simple graphs $K'_{3,n}, K''_{3,n}$ and $K'''_{3,n}$ are obtained from $K_{3,n}$ by adding one, two or three edges in the color class of size three respectively. Take any $t$, $(1 \leq t \leq n)$ disjoint triangles, $T_i$ of $N$ and $t$ copies of $F_7$. Then apply 3-sum operation on $N$ and $F_7$. Any resulting matroid in this class of matroids is called a (multi-legged) starfish. In this paper, when we apply 3-sum operation on $N$ and $F_7$ along the triangles, we will denote it as $N \oplus_3 F_7$ for a one legged starfish and $N \oplus_3 mF_7$ as a $m$-legged starfish for $m \geq 2$. In this paper, we take a closer look at the $X_{10}$, a minor of $Y_{16}$. For a closer study of starfish, $Y_{16}$ and $X_{10}$, see [11]. Matrix representation of $M(X_{10})$ over $GF(2)$ is shown in Figure 3.1.

$$X_{10} = \begin{bmatrix} & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 \\ I_6 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 1 \\ & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 3.1: Matrix representation of $M(X_{10})$.

**Theorem 3.1.2.** *[11] Let $M$ be a binary matroid. Then $M$ is 3-connected having no minor isomorphic to $P_9$ if and only if one of the following is true:*

*(i) $M$ is regular and 3-connected; or*

*(ii) $M$ is a binary spike $Z_r$, $Z_r^*$, $Z_r \backslash y_r$ or $Z_r \backslash t$ for some $r \geq 4$; or*

*(iii) $M$ is a starfish; or*

*(iv) $M$ is one of the 15 internally 4-connected non-regular minors of a 16-element internally 4-connected binary matroid $M(Y_{16})$.*

The following results of Oxley [17] and Seymour [20], respectively, are used in the computation of the beta invariants of matroids with no $P_9$-minor. First, the following are the matrix representation of the binary matroids $R_{10}$, $R_{12}$, $P_9$ and $AG(3,2)$.

$$R_{10} \begin{bmatrix} & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad R_{12} \begin{bmatrix} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ I_6 & 1 & 0 & 0 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad P_9 \begin{bmatrix} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$AG(3,2) \begin{bmatrix} & 0 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 3.2: A matrix representation of $R_{10}$, $R_{12}$, $P_9$ and $AG(3,2)$.

**Theorem 3.1.3.** *Let $M$ be a 3-connected regular matroid. Then $M$ is graphic or cographic, or $M$ is isomoprhic to $R_{10}$, or has a minor isomorphic to $R_{12}$.*

**Proposition 3.1.4.** *Every binary 3-connected single-element extension of $S_8$ is either isomorphic to $P_9$ or has an $AG(3,2)$-minor.*

Then we compute the beta invariant of the matroids with a $P_9$-minor by extending and coextending a column and a row of a matrix representation of $P_9$. We compute the beta invariant of the matroid obtained by contracting the element which was used to extend $P_9$. We check the result using SAGE by constructing the extension and coextension of $P_9$ using the matrix representation and compute the Tutte polynomial to obtain the beta invariant.

For non-binary matroids, as all non-binary matroids have $W^2$ or $W^3$-minor, we extend the matroid and its dual from the small whirls to obtain all the non-binary matroids with beta invariant 5 and 6. Most of this work was done by checking modular cuts and extending them using SAGE.

In Section 3.2, we prove some lemmas which computes the beta invariant of binary matroids with $P_9$-minor and without $P_9$-minor, separately. Also in Section 3.2, we explain how we obtain matroids with $W^2$ and $W^3$-minor and compute the beta invariants using SAGE. In Section 3.3, we list all 3-connected binary matroids with $\beta = 5$, 6 and 7. As for the non-binary matroids, we list all 3-connected non-binary matroids with $\beta = 5$ and 6.

## 3.2 Beta Invariant of Matroids with no $P_9$-minor

First, we introduce the matroids which were not defined in previous sections. If the matroid or the dual is isomorphic to a cycle matroid of a graph, then we give the graph in Figure 3.3. If the matroid and the dual are not graphic, then we give a matrix representation of the matroid over $GF(2)$ in Figure 3.4. For some of the non-binary matroids, we give the modular cut and the matroid which corresponds with it. For example, if $N$ is a 3-connected non-binary matroid with $\beta = 6$, and $N$ is a single element coextension of $M$, then we give a modular cut of $M^*$ and $M^*$ which corresponds with $N^*$. Note that the matroids $T_{10}$, $M(A)$, $M(B)$ and $M(C)$ has both $P_9$ and $P_9^*$ dual as a minor. And $P_{10}^* \cong M^*(K_{2,3}''') \oplus_3 F_7$ has no $P_9$-minor and is a starfish, whereas $P_{10}$ has a $P_9$-minor.

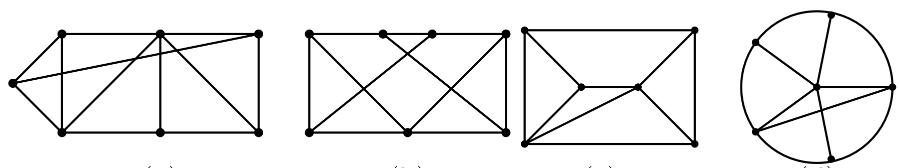


Figure 3.3: A graphic representation of $H_{12}$, $H_{11}$, $Prism + e$, and $W_5 + e$.

$$A = \begin{bmatrix} & 1 & 0 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 \\ I_6 & 0 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 \\ I_5 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 \\ I_5 & 1 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} & 1 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 1 & 1 \\ I_4 & 0 & 0 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} & 1 & 0 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ I_5 & 0 & 1 & 1 & 0 & 0 \\ & 0 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 3.4: A matrix representation of $M(A)$, $M(B)$, $M(C)$, $M(P) = P_{10}$ and $M(T) = T_{10}$ over $GF(2)$.

**Lemma 3.2.1.** *Let $K_{3,n}', K_{3,n}''$ and $K_{3,n}'''$, $n \geq 2$, be a simple graphs obtained from $K_{3,n}$ by adding one, two or three edges in the color class of size three respectively and let $e \in E(F_7) \backslash M^*(E(K_{3,n}))$. Then the following recursive relations hold.*

*(i).* $\beta(M(K_{3,n})) = \beta(M(K_{3,n-1}'')) + \beta(M(K_{3,n-1}')).$

*(ii).* $\beta(M^*(K_{3,n}) \oplus_3 F_7) \backslash e) = \beta(M^*(K_{3,n-1}''')).$

*(iii).* $\beta(M^*(K_{3,n}) \oplus_3 F_7)/e) = \beta(M^*(K_{3,n})).$

*(iv).* $\beta(M^*(K_{3,n}) \oplus_3 2F_7)\backslash e) = \beta(M^*(K'''_{3,n-1}) \oplus_3 F_7)$.

*(v).* $\beta(M^*(K_{3,n}) \oplus_3 2F_7)/e) = \beta(M^*(K_{3,n}) \oplus_3 F_7)$.

*Proof.* Statement (i) can be obtained by applying deletion-contraction formula to $M(K_{3,n})$ immediately and we will omit the proof. It is easily checked that $(M^*(K_{3,n}) \oplus_3 F_7)\backslash e \cong M^*(K'''_{3,n-1})$ and $(M^*(K_{3,n}) \oplus_3 F_7)/e \cong M^*(K_{3,n})$. Hence (ii) and (iii) are true. (iv) and (v) follow from (ii) and (iii). □

In Chapter 4, we provide a splitting formula for computing 3-sum of two matroids. Therefore we can use either the splitting formula or this recursive formula to compute the beta invariant of starfish. Let $M$ be a 3-connected binary matroid with no $P_9$-minor. Using Theorem 3.1.2, and Lemma 3.2.1, we compute the beta invariant of such matroids.

**Lemma 3.2.2.** *Let $M$ be a 3-connected binary matroid with no $P_9$-minor. Then*
*(i) $\beta(M) = 5$ if and only if $M$ is isomorphic to $P_9^*$, $M(K_{3,3})$, $M^*(K_{3,3})$, $W_6$, or $Prism + e$.*
*(ii) $\beta(M) = 6$ if and only if $M$ is isomorphic to $M(K_5)$, $M^*(K_5)$, $M(K'_{3,3})$, $M^*(K'_{3,3})$, $W_5 + e$, $(W_5 + e)^*$, or $W_7$, $AG(3,2)$, or $P_{10}^*$.*
*(iii) $\beta(M) = 7$ if and only if $M$ is isomorphic to $M(K''_{3,3})$, $M^*(K'_{3,3})$, $M(H_{12})$, $M(H_{11})$, $M^*(H_{11})$, $W_8$, $Z_4$, $Z_4^*$, or $M(X_{10})$.*

*Proof.* Let $M$ be a 3-connected binary matroid. It is easily checked that for each matroid $M$ in (i), (ii), or (iii), $\beta(M) = 5$, 6, and 7 respectively, and $M$ has no $P_9$-minor. Now, let $M$ be a 3-connected binary matroid with no $P_9$-minor. Then by Theorem 3.1.2, $M$ is isomorphic to one of the following matroids:

(i) $M$ is regular and 3-connected; or

(ii) $M$ is a binary spike $Z_r$, $Z_r^*$, $Z_r\backslash y_r$ or $Z_r\backslash t$ for some $r \geq 4$; or

(iii) $M$ is a starfish; or

(iv) $M$ is one of the 15 internally 4-connected non-regular minors of a 16-element internally 4-connected binary matroid $M(Y_{16})$.

Case (i). If $M$ is regular, then by Theorem 3.1.3, $M$ is graphic or cographic, or $M$ is isomoprhic to $R_{10}$ or has a $R_{12}$-minor. If $M$ is isomorphic to $R_{10}$, then $\beta(M) = \beta(R_{10}) = 10$. If $M$ has a $R_{12}$-minor, then $\beta(M) \geq \beta(R_{12}) = 14$. Thus if $M$ is a 3-connected regular matroid with $\beta(M) = 5$, 6, or 7, then $M$ is either graphic or cographic. In the paper [3], the authors list all graphs with $\beta = 4$, 5, 6, 7, 8, and 9. To obtain the complete list of the regular matroids with $\beta = 5$, 6, and 7, we only need to add the dual of the non-planar graphs from [3], as the dual matroid has the same beta invariant. We deduce that if $M$ is a 3-connected regular matroid and $\beta(M) = 5$, then $M \cong M(K_{3,3})$, $M^*(K_{3,3})$, $W_6$, or $M(Prism + e)$. If $\beta(M) = 6$, then $M \cong M(K_5)$, $M^*(K_5)$, $M(K'_{3,3})$, $M^*(K'_{3,3})$, $M(W_5+e)$, $M^*(K_5+e)$, or $W_7$. If $\beta(M) = 7$ then $M \cong M(K''_{3,3})$, $M^*(K'_{3,3})$, $M(H_{12})$, $M(H_{11})$, $M^*(H_{11})$, or $W_8$.

Case (ii). If $M$ is a binary spike, then using the matrix representation of the smaller spikes, we computed all of the non-isomorphic smaller spikes. If $M \cong Z_r$, where $r \geq 6$, then $Z_r$ has $Z_5$-minor and thus, $\beta(Z_r) > \beta(Z_5) = 15$ and the beta invariant is too large. Thus we look at the spikes with at most 11 elements. The beta invariants of the spikes with at most 11 elements are listed on Table 3.1.

| $M$ | $Z_4$ | $Z_4\backslash y_4$ | $Z_4\backslash t \cong AG(3,2)$ | $Z_5$ | $Z_5\backslash y_5$ | $Z_5\backslash t$ |
|---|---|---|---|---|---|---|
| $\beta(M)$ | 7 | 4 | 6 | 15 | 8 | 14 |

Table 3.1: The Beta invariant of small spikes.

Therefore if $M$ is a 3-connected binary spike, then $\beta(M) = 6$ if and only if $M \cong AG(3,2)$ and $\beta(M) = 7$ if and only if $M \cong Z_4$ or $Z_4^*$. However, there are no binary spikes with $\beta = 5$.

Case (iii). If $M$ is a starfish, then using Lemma 3.2.1, we give the beta invariants of some of the 3-connected starfish on Table 3.2. Note that $M^*(K''_{2,3}) \oplus_3 F_7 \cong P_9^*$ and $M^*(K'''_{2,3}) \oplus_3 F_7 \cong P_{10}^*$. Because $P_{10}^*$ has a $P_9^*$-minor, $P_{10}$ has a $P_9$-minor and is not listed in Lemma 3.2.2 (ii).

| | | |
|---|---|---|
| $\beta(M^*(K''_{2,3}) \oplus_3 F_7) = 5$ | $\beta(M^*(K''_{2,3}) \oplus_3 2F_7) = 8$ | $\beta(M^*(K_{3,3}) \oplus_3 F_7) = 9$ |
| $\beta(M^*(K'''_{2,3}) \oplus_3 F_7) = 6$ | $\beta(M^*(K'''_{2,3}) \oplus_3 2F_7) = 9$ | |

Table 3.2: The beta invariant of small starfishes.

Since $\beta(M^*(K_{3,3}) \oplus_3 F_7) = 9$, any starfish obtained by adding edges to $M^*(K_{3,n})$ ($n \geq 3$) or adding another leg to $M^*(K_{3,n}) \oplus_3 F_7$ has greater beta invariant and thus has beta invariant at least 10. Also, $M^*(K'''_{2,3}) \oplus_3 2F_7$ is the largest starfish that can be constructed with the cographic matroid $M^*(K_{2,3})$. Therefore if $M$ is a starfish, then $\beta(M) = 5$ if and only if $M \cong P_9^*$. Moreover, $\beta(M) = 6$ if and only if $M \cong P_{10}^*$, and there are no starfish with $\beta = 7$.

Case (iv). If $M$ is one of the 15 internally 4-connected, non-regular minors of 16-element internally 4-connected binary matroid $M(Y_{16})$, then we look at $M(X_{10})$. In the paper [11], the authors note that all of the matroids of this case has a $M(X_{10})$-minor with exception of $F_7$ and $F_7^*$, which have beta invariant 3. However, $\beta(X_{10}) = 7$ and any 3-connected binary matroid with a proper $M(X_{10})$-minor would have greater beta invariant. Thus if $M$ is one of the 15 internally 4-connected, non-regular minors of $M(Y_{16})$, then $\beta(M) = 7$ if and only if $M \cong M(X_{10})$. □

Before introducing the main result of this chapter, we briefly explain how we extend and coextend non-binary matroids from $W^2$ and $W^3$. Note that for any matroid $M$, a modular cut of $M$ corresponds with a single element extension of $M$. Coextension of $M$ corresponds to a extension of the $M^*$. As beta invariant is invariant under the dual operation, to compute the beta invariant of all single element extension and coextension of a matroid

$M$, we will extend $M$ and $M^*$ then compute the extensions. Thus we find modular cuts of $M$ and $M^*$ which corresponds to a 3-connected extension and coextension of $M$.

For small matroids, there are ways to obtain all 3-connected single element extension of a matroid without using a computer. However, in this section, we explain one method to do this using SAGE. For example, suppose we want to find all possible 3-connected single element extensions of $W^3$. Then we first need to find all modular cuts of $W^3$ which corresponds to a 3-connected extension. SAGE contains certain classes of matroids in their program and whirls are one of such matroids. In SAGE, built in matroid $W^3$ has the groundset $\{0, 1, 2, 3, 4, 5\}$. Using SAGE, one can check if a set is a flat or list all flats of different ranks. For example, $\{0, 1, 3\}$ and $\{4, 5\}$ are two flats of $W^3$. Suppose $O$ is an extension of $W^3 = M$ which corresponds to a modular cut of $M$ generated by flats $\{0, 1, 3\}$ and $\{4, 5\}$. Then we first need to extend the matroid using the modular cut generated by flats $\{0, 1, 3\}$ and $\{4, 5\}$ and then compute the Tutte polynomial to obtain the beta invariant. The following command shows the input we typed into the SAGE and the output we obtained from SAGE.

Input:

M = matroids.Whirl(3)

H = [frozenset([0,1,3]), frozenset([4,5])]

O = M.extension('h', H)

O.tutte_polynomial()

Output:

$y^4 + x^3 + xy^2 + 3y^3 + 4x^2 + 5xy + 5y^2 + 4x + 4y$

Next, in order to check if matroids generated by two modular cut of $M$ corresponds to an isomorphic extension of the matroid or not, we use the command *is_isomorphic*. For example, if $P$ is an extension of $W^3$ which corresponds to a modular cut generated by flats $\{0, 2, 5\}$ and $\{3, 4\}$ then we can check if $O$ is isomorphic to $P$ on SAGE as well.

Input :

K = [frozenset([0,2,5]), frozenset([3,4])]

P = M.extension('k', K)

O.is_isomorphic(P)

Output :

True

The symmetry of geometric representation of the matroid is used to reduce the number of modular cuts and corresponding extensions of the single-element extensions. For example, $W^3$ has many symmetries. If $\mathcal{M}_1$ is a modular cut generated by $\{0,1,3\}$ and $\mathcal{M}_2$ is a modular cut generated by $\{0,2,5\}$. Suppose $M_1$ is the matroid which corresponds to $\mathcal{M}_1$ and $M_2$ is the matroid which corresponds to $\mathcal{M}_2$ where in both cases, the element $e$ is the extended element. We can see from Figure 3.5 that $M_1$ and $M_2$ are isomorphic and this is due to the symmetry of $W^3$.

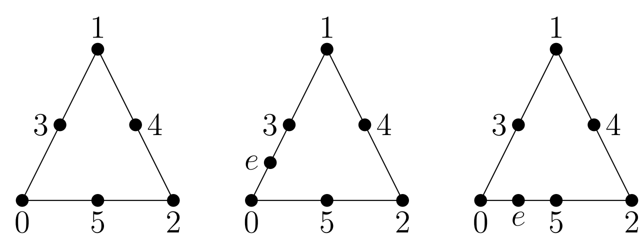


Figure 3.5: A geometric representation of $W^3$, $M_1$, and $M_2$.

## 3.3 Main Results and Proofs

**Theorem 3.3.1.** *Let $M$ be a binary matroid. Then $\beta(M) = 5$ if and only if $M$ is a series-parallel extension of $P_9$, $P_9^*$, $M(K_{3,3})$, $M^*(K_{3,3})$, $W_6$ or $Prism + e$.*

*Proof.* If $M$ is one of the matroids listed in the theorem, then we checked that $\beta(M) = 5$. Now, let $M$ be a binary matroid and $\beta(M) = 5$. By Proposition 2.1.4, $M$ is a series parallel extension of a 3-connected binary matroid $N$ for which $\beta(N) = \beta(M)$. If $N$ has no $P_9$-minor, then it is isomorphic to the one of the matroids listed on Lemma 3.2.2 (i). That is, $N \cong P_9^*$, $M(K_{3,3})$, $M^*(K_{3,3})$, $W_6$ or $Prism + e$. However, $N$ cannot have a proper $P_9$-minor as $\beta(P_9) = 5$. Therefore if $N$ has a $P_9$-minor, then $N \cong P_9$. We conclude that if $M$ is a binary matroid with $\beta(M) = 5$, then $M$ is a series-parallel extension of $P_9$, $P_9^*$, $M(K_{3,3})$, $M^*(K_{3,3})$, $W_6$, or $Prism + e$. □

**Theorem 3.3.2.** *Let $M$ be a binary matroid. Then $\beta(M) = 6$ if and only if either*

*i) $M$ is one of $AG(3,2)$, $M(K_5)$, $M^*(K_5)$, $M(K'_{3,3})$, $M^*(K'_{3,3})$, $W_5 + e$, $(W_5 + e)^*$, $W_7$, $P_{10}$, $P_{10}^*$, or $T_{10}$; or*

*ii) $M$ is 2-sum of matroids $M_1$ and $M_2$ such that $M_1$ is series-parallel extension of $W_3$ and $M_2$ is series-parallel extension of $F_7$, $F_7^*$, or $W_4$.*

*Proof.* If (i) or (ii) holds, then it is easily checked that $\beta(M) = 6$. Suppose $M$ is a binary matroid and $\beta(M) = 6$. Then $M$ is the 2-sum of some 3-connected matroids $M_1$ and $M_2$ such that $1 < \beta(M_1), \beta(M_2) < 6$ and $\beta(M) = \beta(M_1)\beta(M_2)$, or $M$ is a series-parallel extension of 3-connected matroid $N$ for which $\beta(N) = 6$. In the former case, we can assume $\beta(M_1) = 2$ and $\beta(M_2) = 3$ and (ii) holds. So we may assume that $M$ is 3-connected.

The rest of the proof is similar to the proof of the previous theorem. If $N$ has no $P_9$-minor, then $N$ is isomorphic to the one of the matroids listed on Lemma 3.2.2 (ii). In this case, $N \cong AG(3,2)$, $M(K_5)$, $M^*(K_5)$, $M(K'_{3,3})$, $M^*(K'_{3,3})$, $M(W_5 + e)$, $M^*(W_5 + e)$, $W_7$, or $P_{10}^*$.

Now we suppose that $N$ has a $P_9$-minor. We find all such $N$ by extending and coextending $P_9$ one element at a time. This is only possible since $P_9$ is a matroid with relatively small number of elements. We consider all possible binary single element extension of $P_9$ by adding a vector column to the matrix representation of $P_9$. We do the same for

$$X = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{matrix} \\ \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \end{array}$$

Figure 3.6: A matrix representation of $P_9$ over $GF(2)$.

$$X_1 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & e \end{matrix} \\ \left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \quad X_2 = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & e \end{matrix} \\ \left[\begin{array}{cccc|cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Figure 3.7: A matrix representation of a single element extension and a coextension of $P_9$.

the single element coextension of $P_9$ as well. All 3-connected single element extensions and coextensions up to isomorphism with their beta invariants are shown in Table 3.3.

| Vector Column | $\{2,4\}$ | $\{2,3,4\}$ | $\{1,4\}$ |
|---|---|---|---|
| Beta Invariant | 6 | 8 | 7 |

| Vector Row | $\{8,9\}$ | $\{7,9\}$ | $\{7,8,9\}$ | $\{6,7\}$ | $\{6,7,8\}$ | $\{6,7,8,9\}$ | $\{5,7\}$ | $\{5,7,9\}$ |
|---|---|---|---|---|---|---|---|---|
| beta invariant | 9 | 7 | 8 | 6 | 10 | 9 | 7 | 7 |

Table 3.3: The beta invariant of extension and coextension of $P_9$.

In Table 3.3, we give the rows and columns of the added vector where 1's are placed. For example, in this table Vector Column $\{2,4\}$ denote adding a column with 1's on row 2 and 4 as shown below with matrix $X_1$ by column $e$. As for the second table, we have Vector Row $\{8,9\}$. This means first, add a column with all 0's and add a row where we have 1's on column 8, 9, and the column we've added. This is shown below with matrix $X_2$ and a vertex column $e$ and an extra row.

Since $M(X_1)\backslash e \cong P_9$, and the matrix is relatively small, we can try to compute the beta invariant of all single-element extensions and coextensions of $P_9$. We can also compute the beta invariant using SAGE as well.

There exists one extension and one coextension of $P_9$ with $\beta = 6$ which are labeled as $P_{10}$ and $T_{10}$, respectively. Therefore if $M$ is a binary matroid with $\beta = 6$, then either $M$ is a 2-sum of smaller binary matroids with $\beta = 2$ and $\beta = 3$ or $M$ is a series-parallel extension of $AG(3,2)$, $M(K_5)$, $M^*(K_5)$, $M(K'_{3,3})$, $M^*(K'_{3,3})$, $W_5 + e$, $(W_5 + e)^*$, $W_7$, $P_{10}$, $P^*_{10}$, or $T_{10}$. □

**Theorem 3.3.3.** *Let $M$ be a binary matroid. Then $\beta(M) = 7$ if and only if $M$ is a series-parallel extension of one of the fifteen matroids: $M(K''_{3,3})$, $M^*(K''_{3,3})$, $M(H_{12})$, $M(H_{11})$, $M^*(H_{11})$, $W_8$, $Z_4$, $Z^*_4$, $M(X_{10})$, $M^*(X_{10})$, $M(A)$, $M^*(A)$, $M(B)$, $M(C)$, or $M^*(C)$.*

*Proof.* If $M$ is one of the matroids on the theorem then it is easily checked $\beta(M) = 7$. Now, let $M$ be a binary matroid with $\beta(M) = 7$. By Proposition 2.1.4, $M$ is a series parallel extension of a 3-connected binary matroid $N$ for which $\beta(N) = \beta(M)$. If $N$ has no $P_9$-minor, then $N$ is isomorphic to the one of the matroids listed on Lemma 3.2.2 (iii).

Suppose $N$ has a $P_9$-minor. From the proof of Theorem 3.3.3, we already have the list of all single-element extension and coextension of $P_9$ with $\beta = 7$. There is one single-element extension of $P_9$ and three non-isomorphic single-element coextensions. The single element extension of $P_9$ with $\beta = 7$ is isomorphic to $M^*(X_{10})$. As for the single element coextension of $P_9$ with $\beta = 7$, we apply the same process. Let $O_1$, $O_2$, and $O_3$ be matroids obtained after coextending $P_9$ by an element $b$ where we've added a row to the matrix with 1's on $\{7, 9\}$, $\{5, 7\}$ and $\{5, 7, 9\}$ respectively. Using SAGE, we can check that $O^*_1 \cong O_3$ and also, all three matroids are non-isomorphic as well. On the statement of the theorem, $O_1 \cong M(B)$ and $O_2 \cong M(C)$ and $O_3 \cong M^*(B)$. Figure 3.4 shows the matrix representation of the matroids $M(A)$, $M(B)$ and $M(C)$.

There are two single-element extensions and one single-element coextension of $P_9$ with $\beta = 6$: $P_{10}$ and $T_{10}$. We extend and coextend $P_{10}$ and $T_{10}$ using matrix representation of two matroids as well and compute the beta invariant and check for isomorphism. Since these matroids are duals of each other, we give matrix representation of one in Figure 3.3, denoted as $A$.

Therefore if $M$ is a binary matroid with $\beta = 7$, then $M$ is a series-parallel extension of $M(K''_{3,3})$, $M^*(K''_{3,3})$, $M(H_{12})$, $M(H_{11})$, $M^*(H_{11})$, $W_8$, $Z_4$, $Z_4^*$, $M(X_{10})$, $M^*(X_{10})$, $M(A)$, $M^*(A)$, $M(B)$, $M(C)$, or $M^*(C)$. □

Now, we list all non-binary matroids with beta invariant 5 and 6. For 3-connected non-binary matroids with beta invariant 5 (resp. 6), we temporary label matroids that are not well known by $A_i$ (resp. $B_i$). As explained in the end of the Section 2, we obtain these matroids by extensive search using SAGE. Also, in the table below, we give a modular cut and the matroid which generates the matroid. For example, on the table, we have $A_1$, $Q_6$ and $\{a, b, d\}$, $\{e, f\}$. This means $A_1$ is a single element extension of $Q_6$ which corresponds to the moudular cut $\{a, b, d\}$, $\{e, f\}$ of $Q_6$. If the matroid is not self-dual, we give modular cut of the matroid but not its dual. As for the labeling, we will give the label used by SAGE matroid package.

**Theorem 3.3.4.** *Let $M$ be a non-binary matroid. Then $\beta(M) = 5$ if and only if $M$ is a series parallel extension of one of the thirteen matroids: $P_6$, $W^6$, $U_{2,7}$, $U_{5,7}$, $P_7$, $P_7^*$, $A_1$, $A_1^*$, $A_2$, $A_2^*$, $A_3$, $A_3^*$, $A_4$.*

**Theorem 3.3.5.** *Let $M$ be a non-binary matroid. Then $\beta(M) = 6$ if and only if either*

*i) $M$ is one of the thirty three matroids: $W^6$, $U_{3,6}$, $U_{2,8}$, $U_{6,8}$, $B_1$, $B_1^*$, $B_2$, $B_2^*$, $B_3$, $B_3^*$, $B_4$, $B_4^*$, $B_5$, $B_5^*$, $B_6$, $B_6^*$, $B_7$, $B_7^*$, $B_8$, $B_8^*$, $B_9$, $B_{10}$, $B_{11}$, $B_{11}^*$, $B_{12}$, $B_{13}$, $B_{14}$, $B_{15}$, $B_{15}^*$, $B_{16}$, $B_{16}^*$, $B_{17}$, $B_{17}^*$ or*

*ii) $M$ is 2-sum of matroids $M_1$ and $M_2$ such that at least one of $M_1$ and $M_2$ are series-parallel* *extension of a non-binary matroid with beta invariant 2 or 3 and the other, series-parallel extension of a matroid with beta invariant 3 or 2, respectively.*

| $\beta$ | $A_i$ | Minor of $A_i$ | Flats Generating Modular Cut of Given Minor of $A_i$ |
|---|---|---|---|
| 5 | $A_1$ | $Q_6$ | $\{a,b,d\}$, $\{e,f\}$ |
| | $A_2$ | $Q_6$ | $\{a,c\}$, $\{b,f\}$, $\{d,e\}$ |
| | $A_3$ | $O_7^*$ | $\{a,e\}$, $\{c,g\}$ |
| | $A_4$ | $F_7^-$ | $\{a,d\}$, $\{e,b\}$ |
| 6 | $B_1$ | $U_{4,6}$ | $\{0,1\}$, $\{2,3,4\}$ |
| | $B_2$ | $Q_6$ | $\{a,f\}$, $\{d,c\}$ |
| | $B_3$ | $Q_6$ | $\{b,f\}$, $\{a,c\}$ |
| | $B_4$ | $Q_6$ | $\{a,c\}$, $\{d,e\}$ |
| | $B_5$ | $A_1$ | $\{a,b,d,h\}$, $\{c,f\}$ |
| | $B_6$ | $A_2$ | $\{a,e\}$, $\{b,h,f\}$, $\{c,d\}$ |
| | $B_7$ | $O_7$ | $\{e,f\}$, $\{a,d,c\}$ |
| | $B_8$ | $P_7$ | $\{a,b,e\}$, $\{c,d\}$, $\{f,g\}$ |
| | $B_9$ | $A_1^*$ | $\{a,c\}$, $\{d,e\}$, $\{f,b\}$ |
| | $B_{10}$ | $A_2^*$ | $\{f,h\}$, $\{a,b,c,d\}$ |
| | $B_{11}$ | $O_7^*$ | $\{a,b\}$, $\{e,d\}$ |
| | $B_{12}$ | $O_7^*$ | $\{c,f,d\}$, $\{a,b,g\}$ |
| | $B_{13}$ | $O_7^*$ | $\{a,b\}$, $\{c,f,d,g\}$ |
| | $B_{13}$ | $P_7^*$ | $\{c,f\}$, $\{e,b\}$, $\{d,g\}$ |
| | $B_{14}$ | $(F_7^-)^*$ | $\{a,g\}$, $\{d,e,f\}$ |
| | $B_{15}$ | $W^4$ | $\{0,6\}$, $\{2,7\}$, $\{3,4,5\}$ |
| | $B_{16}$ | $W^4$ | $\{0,1,2,4,5\}$, $\{1,2,3,4,,7\}$, $\{0,1,4,6\}$, $\{5,6,7\}$ |
| | $B_{17}$ | $W^4$ | $\{0,1,2,4,5\}$, $\{1,2,3,4,7\}$, $\{0,1,4,6\}$, $\{5,6,7\}$ |

Table 3.4: Some of the 3-connected non-binary matroids with beta invariant 5 and 6.

# 4 CHROMATIC UNIQUENESS AND THE BETA INVARIANT

In this chapter, we partially answer an open question on the chromatic uniqueness of wheels using the beta invariant. Also, we give a splitting formula for computing the beta invariant for generalized parallel connection of two matroids as well. In Section 1, we give history of the question and give definitions and theorems used in this chapter. In Section 2, we state and prove our result on the chromatic uniqueness of wheels. In Section 3, we state and prove a splitting formula for generalized parallel connection of two matroids.

## 4.1 Chromatically Unique Graphs

Let $G$ be a graph. We use $P_G(\lambda)$ to denote its chromatic polynomial. A graph $G$ is chromatically unique if whenever $P_G(\lambda) = P_H(\lambda)$ for a graph $H$, then $G \cong H$. If $G$ is not isomorphic to $H$ but $P_G(\lambda) = P_H(\lambda)$, then $G$ and $H$ are called chromatically equivalent. There are many papers on chromatically unique graphs and chromatically equivalent graphs. In general, it is difficult to determine if a graph is chromatically unique due to the lack of information that can be extracted from the chromatic polynomial of the given graph. For example, given a chromatic polynomial of a graph $G$, one cannot determine if the graph is 3-connected or 2-connected but not 3-connected. One of the conjectures involving chromatic polynomial is on the chromatic uniqueness of the $n$-spoked wheel graph, $W_n$. Chao and Whitehead [5] commented on their paper that the wheels appears to be chromatically unique but they were not able to prove it nor disprove it. However, on a different paper, Chao and

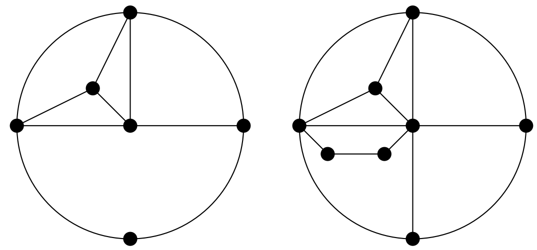


Figure 4.1: $P_G(\lambda) = P_{W_5}(\lambda)$ and $P_H(\lambda) = P_{W_7}(\lambda)$

Whitehead [6] prove that $W_4$ is chromatically unique but $W_5$ is not chromatically unique by providing a chromatically equivalent graph. Xu and Li [22] proved that for even $n \geq 4$, the wheels are chromatically unique. They also made a conjecture on the same paper that not all wheels are chromatically unique.

**Conjecture 4.1.1.** *[22] For every odd $n \geq 9$, the wheel graph $W_n$ is not chromatically unique.*

Xu and Li [22] also provided a graph $G$ which is chromatically equivalent to $W_7$ as well. Figure 4.1 shows two graphs $G$ and $H$ which are chromatically equivalent to $W_5$ and $W_7$, respectively. Note that neither $G$ nor $H$ is 3-connected. These examples show that 3-connectedness might be needed. Read [19] proved that $W_9$ is chromatically unique by generating graphs which has same properties as $W_9$, such as number of triangles, edges and vertices, and comparing the chromatic polynomial using computer. Li and Whitehead [15] provided a proof which does not depend on computer for $W_9$. Al-Rekaby and Khalaf [1] proved that $W_{11}$ is chromatically unique without using computer. Azarija [2], on his Ph.D. thesis, proved that $W_{11}$ and $W_{13}$ are chromatically unique by using a computer. The following are the results on the chromatic uniqueness of the wheels listed in chronological order. Note that some authors use $W_n$ to denote the wheel with $n$ vertices.

**Theorem 4.1.2.** *(1) ([6], 1978) $W_5$ is not chromatically unique.*

*(2) ([22], 1984) For every even $n \geq 4$, the wheel graph $W_n$ is chromatically unique.*

*(3) ([22], 1984) $W_7$ is not chromatically unique.*

*(4) ([19], 1988), ([15], 1992) $W_9$ is chromatically unique.*

*(5) ([1], 2014), ([2], 2016) $W_{11}$ and $W_{13}$ are chromatically unique.*

## 4.2 Beta Invariant and Other Polynomial Invariants

An attractive property of the beta invariant is its relation with other polynomial invariants of the matroid. Using this relation, we prove the chromatic uniqueness of the wheel for 3-connected graphs and matroids. The characteristic (or chromatic) polynomial of the matroid $M$, denoted $P(M, \lambda)$, is defined by

$$P(M; \lambda) = \sum_{A \subseteq E(M)} (-1)^{|A|} \lambda^{r(M)-r(A)}. \tag{4.2.1}$$

When a matroid $M$ is the cycle matroid of a graph $G$, then the chromatic polynomial of the graph $G$ can be obtained from the characteristic polynomial of the $M(G)$. Let $\omega(G)$ be the number of the components of $G$. Then, the chromatic polynomial of $G$, denoted $P_G(\lambda)$ is defined as follows:

$$P_G(\lambda) = \lambda^{\omega(G)} P(M(G); \lambda). \tag{4.2.2}$$

The beta invariant of the matroid $M$ is related to the $P(M; \lambda)$ by the following identity

$$\beta(M) = (-1)^{r(M)+1} \frac{dP(M; \lambda)}{d\lambda} \bigg|_{\lambda=1}. \tag{4.2.3}$$

Let $T_G(x, y)$ be the Tutte polynomial of the graph $G$ with at least two edges. Then $\beta(M(G))$ is the coefficient of either $x$ or $y$ in $T_G(x, y)$. From the Tutte polynomial of $G$, we can also obtain the flow polynomial of the graph $G$, denoted $Q_G(u)$. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Then

$$Q_G(x) = (-1)^{m-n+1} T_G(0, 1-x). \tag{4.2.4}$$

Moreover, the beta invariant of $G$ is given by the absolute value of the coefficient of $t = 1-x$ of the flow polynomial after we make the substitution $t = 1-x$.

One of the bounds of the beta invariant we consider is the upper bound on the number of the elements of the 3-connected matroids with the same beta invariant. Using this bound, we show the chromatic uniqueness of the wheel as well as the other polynomial uniqueness properties. Not only that, for certain beta invariants, we also obtain a lower bound on the number of elements and this gives uniqueness of polynomial invariants as well.

First, we state two proposition to help with the proof of the main result.

**Proposition 4.2.1.** *Let $M$ and $N$ be simple matroids such that $P(M;\lambda) = P(N;\lambda)$. Then $r(M) = r(N)$ and $|E(M)| = |E(N)|$.*

**Proposition 4.2.2.** *[12] Let $G$ and $H$ be connected cosimple graphs with at least one edge such that $Q_G(x) = Q_H(x)$. Then $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$.*

Now, using the two propositions, and another propositions which we will prove in the next section, we will prove the following theorem and corollaries in the next section.

**Theorem 4.2.3.** *Let $M$ be a 3-connected matroid such that $|E(M)| = |E(W_n)|$ and $\beta(M) = \beta(W_n)$. Then $M \cong W_n$.*

**Corollary 4.2.4.** *Let $M$ be a 3-connected matroid such that $M$ and $W_n$ have the same chromatic polynomial: $P(M;\lambda) = P(W_n;\lambda)$. Then $M \cong W_n$.*

**Corollary 4.2.5.** *Let $G$ be a 3-connected graph such that $M$ and $W_n$ have the same chromatic polynomial: $P_G(\lambda) = P_{W_n}(\lambda)$. Then $G \cong W_n$.*

**Corollary 4.2.6.** *Let $G$ be a 3-connected graph such that $G$ and $W_n$ have the same flow polynomial: $Q_G(x) = Q_{W_n}(x)$. Then $G \cong W_n$.*

**Corollary 4.2.7.** *Let $M$ be a 3-connected matroid such that $M$ and $W_n$ have the same flow polynomial: $Q_M(x) = Q_{W_n}(x)$. Then $M \cong W_n$.*

## 4.3 First Main Result and Proof

**Theorem 4.3.1.** *Let $\mathcal{M}_n$ be the set of 3-connected matroids with $\beta = n$, $n > 1$. Then $|E(M)| = max\{|E(N)| : N \in \mathcal{M}_n\}$ if and only if $M \cong W_{n+1}$.*

*Proof.* Assume to the contrary that $M \in \mathcal{M}_n$, $M \ncong W_{n+1}$, and $|E(M)| = max\{|E(N)| : N \in \mathcal{M}_n\}$. We will show that $|E(W_{n+1})| > |E(M)|$. Note that $M \ncong W^n$ since $|E(W_{n+1})| > |E(W^n)|$ but $\beta(W_{n+1}) = \beta(W^n) = n$. Let $W$ be the largest wheel or whirl minor of $M$, where $r(W) = r$. By Theorem 1.0.3, there exists a sequence of 3-connected matroids $M_0, \cdots, M_k$, such that $M_0 \cong W$ and $M_k \cong M$. Since $M$ is not a wheel or a whirl, $k > 0$ and $|E(M)| = |E(W)| + k = 2r + k$.

Since $W \cong M_0$, if $W$ is a wheel, then $\beta(M_0) = r(M_0) - 1$. If $W$ is a whirl, then $\beta(M_0) = r(M_0)$. Let $e \in E(M_{i+1}) \backslash E(M_i)$ for some $0 \le i < k$. Then by the deletion-contraction formula for the beta invariant, $\beta(M_{i+1}) = \beta(M_{i+1} \backslash e) + \beta(M_{i+1}/e)$. Since $M_{i+1}$ is 3-connected, both $M_{i+1} \backslash e$ and $M_{i+1}/e$ are connected and both $\beta(M_{i+1} \backslash e), \beta(M_{i+1}/e) \ge 1$. Also either $M_{i+1} \backslash e \cong M_i$ or $M_{i+1}/e \cong M_i$. Combined with the deletion contraction formula, for each $i \in \{0, 1, \cdots, k-1\}$,

$$\beta(M_{i+1}) \ge \beta(M_i) + 1. \tag{4.3.1}$$

Thus, by applying induction on (4.3.1),

$$\beta(M) = \beta(M_k) \ge \beta(M_0) + k. \tag{4.3.2}$$

If $W$ is a rank $r$ whirl, then $\beta(M_0) = r$ and by (4.3.2), and $\beta(M_k) = n \geq r + k$. Similarly, if $W$ is a rank $r$ wheel, $\beta(M_0) = r - 1$ and $\beta(M) = \beta(M_k) \geq r - 1 + k$. Therefore $\beta(M) = \beta(M_k) \geq r - 1 + k$. Let $s \in \mathbb{N}$ such that $r + s = n + 1$. Then $\beta(W_{n+1}) = r + s - 1$ and since $n \geq r - 1 + k$, we have that $s \geq k > 0$. Therefore, $|E(W_{n+1})| = 2r + 2s > 2r + k = |E(M)|$. Thus $\beta(W_{n+1}) = \beta(M)$ but $|E(W_{n+1})| > |E(M)|$, a contradiction. □

Theorem 4.2.3 is an immediate consequence of Theorem 4.3.1 as wheels have the maximum number of elements among all 3-connected matroids with a fixed beta invariant. Now we prove Corollaries 4.2.4, 4.2.5, 4.2.6 and 4.2.7.

*Proof.* Suppose $M$ is a 3-connected matroid such that $P(M; \lambda) = P(W_n; \lambda)$. This implies that $\beta(M) = \beta(W_n)$ and by Proposition 4.2.1, $|E(M)| = |E(W_n)|$ as well. Therefore, by Theorem 4.2.3, $M \cong W_n$. If $G$ is a 3-connected graph such that $P_G(\lambda) = P_{W_n}(\lambda)$, then $|E(G)| = |E(W_n)|$ and $\beta(M(G)) = \beta(W_n)$, where $M(G)$ is a cycle matroid of the graph $G$. Thus $M(G) \cong M(W_n)$. As both $G$ and $W_n$ are 3-connected, $G \cong W_n$. By (4.2.4), the beta invariant is the coefficient of $t$ in $Q_G(x)$ after we make the substitution $t = 1 - x$. Hence if $Q_G(x) = Q_{W_n}(x)$, we have that $\beta(M(G)) = \beta(W_n)$. Then again, by applying Proposition 4.2.2 and Theorem 4.2.3, we conclude that $G \cong W_n$. Thus Corollary 4.2.6 holds. We omit the similar proof of Corollary 4.2.7. □

It is already proven that for $n \geq 3$, the wheel graph is completely determined by its Tutte polynomial [10]. It is somewhat striking that if we know the number of elements of a 3-connected matroid, then only one coefficient of the Tutte polynomial, (the coefficient of $x$ or $y$, which equals to the beta invariant), determines if $M$ is a wheel or not.

## 4.4 The Generalized Parallel Connection of Two Matroids

In this section, we will restate some of the definitions and properties related to generalized parallel connection of two matroids. Also we give some previous results on the beta invariant of special case of generalized parallel connection of two matroids as well. Suppose $M_1$ and $M_2$ are two matroids and $cl_1$ and $cl_2$ be closure operation of the matroid $M_1$ and $M_2$ respectively. Then the generalized parallel connection of two matroids can exist if the following corollary holds:

**Lemma 4.4.1.** *[18] Suppose that $cl_1(T)$ is a modular flat of $M_1$ and every non-loop element of $cl_1(T)$ is parallel to some element of $T$. Then $T$ is fully embedded in $M_1$, so the proper amalgam of $M_1$ and $M_2$ exists.*

If the conditions in the previous Lemma hold, then the proper amalgam of $M_1$ and $M_2$ is the generalized parallel connection across $T$ and is denoted by $P_T(M_1, M_2)$. Therefore if $M = P_T(M_1, M_2)$ then $E(M) = E(M_1) \cup E(M_2)$ and $M_1|T = M_2|T$. Also, for every flat $F$ of $M$, we have $r_M(F) = r_M(F \cap E_1) + r_M(F \cap E_2) - r_M(F \cap T)$. The generalized parallel connection of two matroids has some interesting properties regarding minors which are shown in the following proposition.

**Proposition 4.4.2.** *[18] The generalized parallel connection has the following properties:*
*(i) $P_T(M_1, M_2)|E_1 = M_1$ and $P_T(M_1, M_2)|E_2 = M_2$.*
*(ii) If $e \in E_1 - T$, then $P_T(M_1, M_2)\backslash e = P_T(M_1\backslash e, M_2)$.*
*(iii) If $e \in E_1 - cl_1(T)$, then $P_T(M_1, M_2)/e = P_T(M_1/e, M_2)$.*

The proposition states that if there is an element $e \in E_1$ such that $e \notin cl_1(T)$, then $cl_1(T)$ is still a modular flat of $M_1\backslash e$ and $M_1/e$. We will use this proposition later in the dissertation. When $T$ is a specific set, then the generalized parallel connection of two matroids are often denoted differently. For example, when $T$ is a triangle, then $P_T(M_1, M_2)$ is called the generalized parallel connection of $M_1$ and $M_2$ across a 3-point line. When both $M_1$ and $M_2$ are binary and $|E_1|$, $|E_2| > 6$, then $P_T(M_1, M_2)\backslash T$ is called 3-sum of $M_1$ and $M_2$ and is denoted $M_1 \oplus_3 M_2$. When $T = \{p\}$, then $P_T(M_1, M_2)$ is called parallel connection of

$M_1$ and $M_2$ with respect to $p$ and is denoted $P(M_1, M_2)$ and $P(M_1, M_2)\backslash T$ is called 2-sum of $M_1$ and $M_2$ and is denoted $M_1 \oplus_2 M_2$. For $M = P(M_1, M_2)$, Brylawski proved the following result:

**Theorem 4.4.3.** *[4] Suppose that $p$ is neither a loop nor a coloop of $M_1$ and $M_2$, then $\beta(P(M_1, M_2)) = \beta(M_1)\beta(M_2)$.*

In the next section, as a corollary, we prove that if $M = P_T(M_1, M_2)$ where $T$ is a triangle, then $\beta(M) = \beta(M_1)\beta(M_2)$. This result, with Brylawski's result proves that the beta invariant cannot distinguish 3-connected matroids from the 2-connected matroids as $P(M_1, M_2)$ is 2-connected but not 3-connected if $M_1$ and $M_2$ are 3-connected and $P_T(M_1, M_2)$ is 3-connected if $T$ is a triangle and $M_1$ and $M_2$ are 3-connected. Therefore the connectivity condition in Corollary 4.2.4 can not be dropped.

## 4.5 The Second Main Result and its Proof

In this section, we give some results on the computation of the beta invariant of a generalized parallel connection across a 3-point line and as a corollary, the 3-sum of two binary matroids.

**Theorem 4.5.1.** *Let $M_1$ and $M_2$ be matroids and $M = P_T(M_1, M_2)$, the generalized parallel connection of $M_1$ and $M_2$ across $T$. If $cl_1(T)$ is a modular flat of $M_1$ such that $\beta(cl_1(T)) \neq 0$, then $\beta(M) = \frac{\beta(M_1)\beta(M_2)}{\beta(cl_1(T))}$.*

*Proof.* Let $M_1$ and $M_2$ be matroids and $M = P_T(M_1, M_2)$, the generalized parallel connection of $M_1$ and $M_2$ across $T$. Since $cl_1(T)$ is a modular flat of $M_1$ where $\beta(cl_1(T)) \neq 0$, we can see that $M|cl_1(T) = cl_1(T)$ is connected as the beta invariant is zero if and only if the matroid is not connected. Also, by the definition of generalized parallel connection of $M_1$ and $M_2$ across $T$, every elements of $cl_1(T) - T$ are parallel to some elements of $T$. Therefore

$\beta(cl_1(T)) = \beta(T)$ and there are no loops in $M_1$ as well. If there are loops in $M_2$, then same loops will be loops in $M$ as well. In such case, $\beta(M) = \beta(M_2) = 0$ and the theorem holds as well.

We prove the theorem by induction on the elements of $E(M) - E(M_2)$. For the base case, consider the matroid $P_T(cl_1(T), M_2)$. This matroid exists as $cl_1(T)$ is a modular flat of itself. Then $P_T(cl_1(T), M_2)$ is isomorphic to adding elements parallel to $T$ in $M_1$ to $M_2$ as only elements in $cl_1(T) - T$ are elements in parallel with $T$ in $M_1$. Since beta invariant does not change by adding elements in parallel, $\beta(P_T(cl_1(T), M_2)) = \beta(P_T(T, M_2)) = \beta(M_2)$. Thus the statement holds as $\beta(P_T(cl_1(T), M_2)) = \frac{\beta(cl_1(T))\beta(M_2)}{\beta(cl_1(T))} = \beta(M_2)$.

Suppose the theorem holds for all matroid $N$ such that $N$ is a minor of $M_1$ and $cl_1(T) \subset E(N) \subset E(M)$. If such a minor does not exist, then either $cl_1(T) = E(M)$ or $M_1$ is a single element extension or coexension of $cl_1(T)$. If $cl_1(T) = M_1$, we are back to the base case and the theorem holds. If there exists an element $e \in E(M) - cl_1(T)$ such that $M_1 \backslash e$ or $M_1/e \cong cl_1(T)$, then $e$ has to be a loop or coloop of $M_1$. However, we showed in the beginning of the proof that $M_1$ has no loops and thus, $e$ has to be a coloop of $M_1$ and $M$. Then $\beta(M_1^*) = 0 = \beta(M_1)$ and $\beta(M) = 0$. Therefore the theorem holds. Thus, suppose that for any $N$, a proper minor of $M_1$ such that $cl_1(T) \subset E(N) \subset E(M_1)$, the theorem holds and $\beta(P_T(N, M_2)) = \frac{\beta(N)\beta(M_2)}{\beta(cl_1(T))}$.

Now, pick an element $e \in E(M_1) - cl_1(T)$. Such element exists as $cl_1(T) \ncong M_1$. Then by Proposition 4.4.2, $P_T(M_1 \backslash e, M_2) = M \backslash e$ and $P_T(M_1/e, M_2) = M/e$. Therefore, by applying the deletion-contraction formula and the induction step,

$$
\begin{aligned}
\beta(M) &= \beta(M \backslash e) + \beta(M/e) \\
&= \beta(P_T(M_1 \backslash e, M_2)) + \beta(P_T(M_1/e, M_2)) \\
&= \frac{\beta(M_1 \backslash e)\beta(M_2)}{\beta(cl_{M_1}(T))} + \frac{\beta(M_1/e)\beta(M_2)}{\beta(cl_{M_1}(T))} \\
&= \frac{(\beta(M_1 \backslash e) + \beta(M_1/e))\beta(M_2)}{\beta(cl_{M_1}(T))}
\end{aligned}
$$

$$= \frac{\beta(M_1)\beta(M_2)}{\beta(cl_{M_1}(T))}.$$

In either case, if $\beta(cl_1(T)) \neq 0$, then $\beta(M) = \frac{\beta(M_1)\beta(M_2)}{\beta(cl_1(T))}$. □

Then the following result follows immediately when $T$ is a triangle of $M_1$ since $\beta(T) = 1$.

**Corollary 4.5.2.** *Let $M_1$ and $M_2$ be matroids and $M = P_T(M_1, M_2)$ be the generalized parallel connection of $M_1$ and $M_2$ across a 3-point line $T = \{p, s, q\}$. Then $\beta(M) = \beta(M_1)\beta(M_2)$.*

Also, we obtain the following corollary by applying deletion-contraction formula to Corollary 4.5.2.

**Corollary 4.5.3.** *Let $M_1$ and $M_2$ be two binary matroids and $M = M_1 \oplus_3 M_2$. Then $\beta(M) = \beta(M_1)\beta(M_2) - \beta(M_1/s)\beta(M_2/s) - \beta(M_1/p)\beta(M_2/p) - \beta(M_1/q)\beta(M_2/q)$.*

# 5 CHAIN THEOREMS

In this chapter, we introduce known results on chain theorems for matroids. In Section 1, we state the known results and in Section 2, we introduce questions which rises with chain theorems.

## 5.1 Some Known Chain Theorems for Matroids

For a connected matroid, Tutte proved the following result in [21].

**Theorem 5.1.1.** *[21] (Tutte's Theorem) Let $M$ be a connected matroid and $e$ be an element of $M$. Then either $M\backslash e$ or $M/e$ is connected.*

Then for 3-connected matroids, he proved the following theorem as well.

**Theorem 5.1.2.** *[21] (Tutte's Wheels and Whirls Theorem) If $M$ is a 3-connected matroid and, for every element $e$, neither $M\backslash e$ nor $M/e$ is 3-connected, then $M$ is a wheel or a whirl or rank at least three.*

Using Tutte's Wheels and Whirls theorem, we can obtain the following chain theorem.

**Theorem 5.1.3.** *(Tutte's Chain Theorem) Let $M$ be a 3-connected matroid other than a wheel or a whirl. Then there is a chain of 3-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0$ is either a wheel or a whirl; $M_n = M$, and $M_{i+1}$ is a single-element extension or co-extension of $M_i$ for $i = 0, 1, \ldots, n-1$.*

Let $\mathcal{N}$ be a class of matroids, and $M$ be a matroid. We say there is an $\mathcal{N}M$-chain if there is chain of matroids $M_0, M_1, \ldots, M_n$ such that $M_0 \in \mathcal{N}$, $M_n = M$, and each $M_i$ is a minor of $M_{i+1}$ for all $i = 0, 1, \ldots, n-1$.

**Theorem 5.1.4.** *(Restatement of Tutte's Wheels and Whirls Theorem) Let $M$ be a 3-connected matroid other than a wheel or a whirl. Then there is a $\mathcal{W}M$ chain of 3-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0$ is either a wheel or a whirl, $M_n = M$, and each $M_{i+1}$ is a single element extension or coextension of $M_i$ for all $i = 0, 1, \ldots, n-1$.*

Note that in this case, the set $\mathcal{W}$ contains all wheels and whirls and thus has infinite cardinality. Which implies unless there is some upper bound restriction, we might need to check all possible extensions and coextensions of all wheels and whirls in certain cases when the theorem is used. On the other hand, Coullard and Oxley proved the following result which restrictions this set $\mathcal{W}$ into a set containing four matroids: $W_3$, $W_4$, $W^2$ and $W^3$.

**Theorem 5.1.5.** *[8] Let $N$ be a 3-connected proper minor of a 3-connected matroid $M$ such that $|E(N)| \geq 4$ and $M$ is not a wheel or a whirl. Suppose that if $N \cong W^2$, then $M$ has no $W^3$-minor, while if $N \cong M(W_3)$, then $M$ has no $M(W_4)$-minor. Then $M$ has a 3-connected minor $M_1$ and an element $e$ such that $M_1/e$ or $M_1\backslash e$ is isomorphic to $N$.*

Here we give a restatement of Theorem 5.1.5.

**Corollary 5.1.6.** *(Restatement of Theorem 5.1.5) Let $M$ be a 3-connected matroid other than a wheel or a whirl. Denote $\mathcal{N} = \{W_3, W_4, W^2, W^3\}$. Then there is a $\mathcal{N}M$ chain of 3-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0 \in \mathcal{N}$, $M_n = M$, and each $M_{i+1}$ is a single element extension or coextension of $M_i$ for all $i = 0, 1, \ldots, n-1$.*

Note that all of the theorems so far applies to 3-connected matroids. The natural next step is to increase connectivity. However, there does not exist a splitter theorem for 4-connected matroids and thus, a chain theorem, which depends on a splitter theorem type results, does not exist as well. However, there are results for sequentially 4-connected matroids by Geelen and Whittle [13] and weakly 4-connected matroids by Geelen and Zhou [14]. First, let us give definition for sequentially 4-connected matroids and weakly 4-connected matroids.

Let $(A, B)$ be a $k$-separation of a matroid $M$. Then $(A, B)$ is called sequential if the elements of $A$ can be ordered $(a_1, \cdots, a_m)$ such that $\{a_1, \cdots, a_i\}$ is $k$-separating for $i = 1, \cdots, m$. It is non-sequential if neither $(A, B)$ nor $(B, A)$ is sequential. A matroid $M$ is sequentially 4-connected if $M$ is 3-connected and has no non-sequential 3-separations. In paper [7], authors give an equivalent definition for sequential separations. A set $A$ of $E$ is fully closed if $A$ is both closed and coclosed in $M$. The full closure $fcl(A)$ of $A$ is the intersection of all fully closed sets that contain $A$. If $M$ is $k$-connected, then a $k$-separation $(A, B)$ is sequential if $fcl(A)$ or $fcl(B)$ is $E$.

A matroid $M$ is internally 4-connected if $M$ is 3-connected and for each 3-separation $(X, Y)$ of $M$, either $|X| \leq 3$ or $|Y| \leq 3$. A matroid $M$ is weakly 4-connected if $M$ is 3-connected and for each 3-separation $(X, Y)$ of $M$, either $|X| \leq 4$ or $|Y| \leq 4$. Thus internally 4-connected matroids are also weakly 4-connected.

**Theorem 5.1.7.** *[13] If $M$ be a sequentially 4-connected matroid that is neither a wheel nor a whirl, then $M$ has an element $x$ such that $M\backslash x$ or $M/x$ is sequentially 4-connected.*

Like the 3-connected case, the following is the restatement of the previous theorem.

**Theorem 5.1.8.** *(Restatement of Theorem 5.1.7) Let $M$ be a sequentially 4-connected matroid. Then there is a $\mathcal{WM}$ chain of sequentially 4-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0$ is either a wheel or a whirl, $M_n = M$, and each $M_{i+1}$ is a single element extension or coextension of $M_i$ for all $i = 0, 1, \ldots, n-1$.*

Note that the set $\mathcal{W}$ contains infinite number of wheels and whirls. We have similar results for weakly 4-connected matroids. The following theorem are Geelen and Zhou's result and the restatement of their result.

**Theorem 5.1.9.** *[14] Let $M$ be a weakly 4-connected matroid with $|E(M)| \geq 7$. Then either*

*1. there exists $e \in E(M)$ such that $M \backslash e$ or $M/e$ is weakly 4-connected,*

*2. $M$ has a 4-element 3-separating set $A$ with elements $c$, $d \in A$ such that $M \backslash d/c$ is weakly 4-connected,*

*3. $M$ or $M^*$ is isomorphic to the cycle matroid of a ladder, or*

*4. $|E(M)| = 12$ and $M$ is a trident.*

Let $\mathcal{L}$ be the class of planar ladders and Mӧbius ladders and their dual matroids, and $\mathcal{T}$ be the class of 12 elements tridents where a trident is a 12-element rank-6 matroid whose ground set is the union of three disjoint 4-element 3-separating sets of rank 3.

**Theorem 5.1.10.** *(Restatement of Theorem 5.1.9) Let $M$ be a weakly 4-connected matroid with $|E(M)| \geq 7$. Then there is a $(\mathcal{L} \cup \mathcal{T})M$ chain of weakly 4-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0 \in \mathcal{L} \cup \mathcal{T}$, $M_n = M$, and for all $i = 0, 1, \ldots, n-1$, either each $M_{i+1}$ is a single element extension or coextension of $M_i$, or $M_{i+1}$ has a 4-element 3-separating set $A_i$ with elements $c_i$, $d_i \in A_i$ such that $M_{i+1} \backslash d_i/c_i = M_i$.*

Again, in both sequentially 4-connected matroids and weakly 4-connected matroids, the chain starts (or ends) at $\mathcal{W}$ or $\mathcal{L}$ where both sets have infinite size. Now, we have to ask if we can replace set of matroids with infinite cardinality with a finite set. In other words, we want a theorem that is similar to Coullard and Oxley's. For example, can all non-wheel and non-whirl sequentially 4-connected matroids be constructed from a finite number of matroids? In Chapter 5 and 6, we answer theses questions for sequentially 4-connected matroids and weakly 4-connected matroids respectively.

# 6 SEQUENTIALLY 4-CONNECTED MATROIDS

In this chapter, we prove a chain theorem for sequentially 4-connected matroids. In Section 1, we give known results on a splitting theorem for sequentially 4-connected matroids and provide motivation for our work. Then in Section 2, we prove lemmas which are used in the proof of the main result. In Section 3, we prove the main result of this chapter.

## 6.1 Motivation and Main Result

Geelen and Whittle's result on the sequentially 4-connected matroids give us a $\mathcal{W}M$-chain of sequentially 4-connected matroids as shown in the following theorem.

**Theorem.** *(Restatement of Theorem 5.1.7) Let $M$ be a sequentially 4-connected matroid. Then there is a $\mathcal{W}M$-chain of sequentially 4-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0$ is either a wheel or a whirl, $M_n = M$, and each $M_{i+1}$ is a single element extension or coextension of $M_i$ for all $i = 0, 1, \ldots, n-1$.*

However, the set $\mathcal{W}$ contains wheel and whirls of all rank and thus, has an infinite cardinality. In Section 3, we prove that for sequentially 4-connected matroids, we have a $\mathcal{W}'M$-chain of sequentially 4-connected matroids where $\mathcal{W}' = \{W_3, W_4, W^2, W^3\}$. The following theorem is the main result of this chapter.

**Theorem 6.1.1.** *Let $M$ be a sequentially 4-connected matroid such that $|E(M)| \geq 4$ that is neither a wheel nor a whirl and let $\mathcal{W}' = \{W_3, W_4, W^2, W^3\}$. Then there is a $\mathcal{W}'M$-chain of* *sequentially 4-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0 \in \mathcal{W}'$, $M_n = M$, and each $M_{i+1}$ is a single element extension or coextension of $M_i$ for all $i = 0, 1, \ldots, n-1$.*

The following are two theorems from the paper [13]. We will later use the two theorems in the lemma which will be used to prove the main theorem in this chapter.

**Theorem 6.1.2.** *[13] If $T$ is a triangle in an internally 4-connected matroid $M$, then either*
*(i) there exists $t \in T$ such that $M\backslash t$ is sequentially 4-connected, or*
*(ii) $M$ has at most 11 elements, and there exists an element $y$ of $M$ sch that $M/y$ is sequentially 4-connected.*

**Theorem 6.1.3.** *[13] Let $M$ be a sequentially 4-connected matroid with a sequential 3-separation $(A, B)$, where $|A| \geq 4$. Assume that the elements of $A$ are ordered $(a_1, \cdots, a_k)$. Let $A_i$ denote $\{a_i, \cdots, a_i\}$, and let $B_i$ denote $\{a_i \cdots, a_k\} \cup B$. For $i \geq 3$, if $a_i \in cl(A_i) \cap cl(B_{i+1})$ and $M\backslash a_i$ is 3-connected, then $M\backslash a_i$ is sequentially 4-connected.*

## 6.2 Lemmas

In this section, we prove some lemmas which will be used to prove the main theorem. Let $M$ be a simple, cosimple matroid and $S$ be a subset of $E(M)$ having at least three elements. Then $S$ is a fan in $M$ if there is an ordering $(s_1, s_2, \cdots, s_n)$ of the elements of $S$ such that, for all $i$ in $\{1, 2, \cdots, n-2\}$,
(i) $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle or a triad; and
(ii) when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triangle, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triad; and, when $\{s_i, s_{i+1}, s_{i+2}\}$ is a triad, $\{s_{i+1}, s_{i+2}, s_{i+3}\}$ is a triangle.

**Lemma 6.2.1.** *Let $M = W + e$ be a 3-connected matroid such that $W = W_k$ or $W^k$, $(k \geq 3)$. Then for any 3-separation $(X \cup e, Y)$ of $M$ where $|X|$, $|Y| \geq 3$, both $X$ and $Y$ are fans of $W$.*

*Proof.* Since $M$ is a single extension of $W$, $r_M(X) = r_W(X)$ and $r_M(E(M)) = r_W(E(W))$ for all $X \subseteq E(W)$. Therefore we won't distinguish between $r_M$ and $r_W$ in the proof. Let $(X \cup e, Y)$ be a 3-separation of $M$ where $|X|$, $|Y| \geq 3$. As $M$ is 3-connected, $r(X \cup e) + r(Y) = r(M) + 2$ where $|X \cup e| \geq 4$ and $|Y| \geq 3$. If $r(X \cup e) = r(X) + 1$, then $r(X) + r(Y) = r(W) + 1$. Also, $|X|$ and $|Y| \geq 3$ implies that $(X, Y)$ is a 2-separation of $W$, which is not possible as $W$ is 3-connected.

If $r(X \cup e) = r(X)$, then $r(X) + r(Y) = r(W) + 2$. Also $r(X) + r(Y) = r(cl(X)) + r(cl(Y)) \geq r(cl(X) \cup cl(Y)) + r(cl(X) \cap cl(Y)) = r(M) + r(cl(X) \cap cl(Y))$. Therefore $r(cl(X) \cap cl(Y)) \leq 2$. Suppose $X$ is not a fan of $W$. Then as $|X|$, $|Y| \geq 3$, there exists at least three elements $a$, $b$, $c$, such that $\{a, b, c\}$ is not a triangle and $\{a, b, c\} \subseteq cl(X) \cap cl(Y)$. Thus $r(cl(X) \cap cl(Y)) \geq 3$ which contradicts the fact that $r(cl(X) \cap cl(Y)) \leq 2$. Therefore $X$ is a fan and consequently $Y$ is as well. □

**Theorem 6.2.2.** *Suppose that $M = W + e$, where $W = W_k (k \geq 5)$ or $W^k (k \geq 4)$, is sequentially 4-connected. Then there exists $a \in E(W)$ such that either $M/a$ or $M \backslash e$ is sequentially 4-connected and is not isomorphic to a wheel or a whirl.*

*Proof.* We prove the theorem by proving three claims. Also, for the notations for the elements of $W$, refer to the Figure 6.1.

**6.2.3.** *One of the triad $T_i = \{b_i, a_i, b_{i+1}\}$ of $W$ is no longer a triad of $M$.*

Assume to the contrary that $T_i$ is a triad of $W$ and $M$ for all $i \in \{1, 2, \cdots, k\}$. As $M$ is 3-connected, there exists a circuit $C$ such that $e$, $b_1 \in C$. Since $e$, $b_1 \in C$ and $T_2 = \{b_1, a_2, b_2\}$ is a triad of $M$, by orthogonality, either $b_2$ or $a_2 \in C$. Thus there exists $j$, $1 \leq j \leq k-1$ such that either (1) $\{e, b_1, b_2, \cdots, b_j, a_{j+1}\} \subseteq C$, or (2) $\{e, b_1, b_2, \cdots, b_k\} \subseteq C$.

If (1) holds, then as $T_1 = \{b_1, a_1, b_k\}$ is a triad, by orthogonality, either $b_k \in C$ or $a_1 \in C$. If there exists $m$, $k > m > j+1$ such that $a_m \in C$, then $\{e, b_1, \cdots, b_j, a_{j+1}, b_k, \cdots, b_m, a_m\} \subseteq C$. But this is not possible as $\{a_{j+1}, b_j, \cdots, b_m, a_m\} \subset C$ is a circuit of $W$ and also a circuit

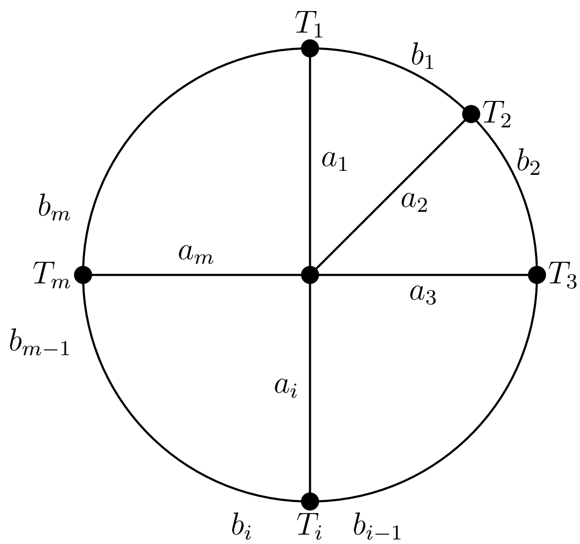


Figure 6.1: Labels of elements, $a_i$ and $b_i$, and tridents, $T_i$, of $W$.

of $M$ as well. Thus $\{e, b_1, b_2, \cdots, b_k, a_{j+1}\} \subset C$. However $\{b_1, b_2, \cdots, b_k\}$ is a circuit in $W_k$ and $\{b_1, b_2, \cdots, b_k, a_{j+1}\}$ is a circuit in $W^k$, (1) is not possible.

If (2) holds, by orthogonality and as (1) is not possible, $C = \{e, b_1, b_2, \cdots, b_k\}$. However, $\{b_1, b_2, \cdots, b_k\}$ is a circuit in $W_k$, thus it must be that $W = W^k$ and $C$ is a circuit in $M$. Then by the circuit elimination axiom, $\{e, a_1, b_2, \cdots, b_k\}$ contains a circuit $C_1$. However $T_2 = \{b_1, a_2, b_2\}$ is a triad of $M$ by assumption, $C_2 \cap T_2 = \{b_2\}$, which is not possible. Therefore there is at least one triad $T_i$ of $W$ is no longer a triad in $M$.

**6.2.4.** *If $T_i = \{b_i, b_{i-1}, a_i\}$ is not a triad in $M$, then $M \backslash a_i$ is 3-connected.*

Suppose that $M \backslash a_i$ is not 3-connected and there exists a 2-separation $(X, Y)$ of $M \backslash a_i$ such that $r(X)+r(Y) = r(M \backslash a_i)+1$ and $|X|, |Y| \geq 2$. If $|X| = 2$, then $r(X)+r^*(X)-|X| = 1$ and $r(X) + r^*(X) = 3$. As $M$ is 3-connected, $r(X) = 2$ and thus $r^*(X) = 1$. Then $X \cup a_i$ has to be a triad of $M$ since $M$ is 3-connected. However by assumption $a_i$ is not in any triad of $M$ and $T_i \cup e = \{b_{i-1}, a_i, b_i, e\}$ is a 4-element cocircuit. Therefore $|X|, |Y| \geq 3$.

Since $M$ is 3-connected, and $|X \cup a_i|$, $|Y| \geq 3$, we must have $r(X \cup a_i) + r(Y) = r(X) + r(Y \cup a_i) = r(M) + 2$. As $r(M \backslash a_i) = r(M)$ and $r(X) + r(Y) = r(M \backslash a_i) + 1$, we can deduce that $r(Y \cup a_i) = r(Y) + 1$ and thus $(X \cup a_i, Y)$ and $(X, Y \cup a_i)$ are 3-separatin of $M$. B y Lemma 6.2.1, if $e \in X$, $X \cup a_i - e$, $Y$, $X - e$, $Y \cup a_i$ are fan of $W$. Therefore $r(Y \cup a_i) = r(Y)$ which is a contradiction.

**6.2.5.** *There exists an element $x \neq e$ such that either $M \backslash x$ is sequentially 4-connected or $M/e$ is sequentially 4-connected.*

First, suppose that at least one of the triad of $W$ is a triad of $M$. Then $M$ contains 4-element fan which contains a triad of $M$. Let $F$ be a maximal fan containing the 4-element fan and $a$ be the end of the fan $F$. Then by step 1, $a$ is in a triad of $W$ but not in a triad of $M$ and $a$ is a spoke of $W$ as well. Also, $e \notin F$ as $e$ is not in any triad of $M$. If there exists triad $T$ of $M$ such that $e \in T$, there always exists a triangle $C$ of $M$ such that $|T \cap C| = 1$, which contradicts orthogonality of matroid. As $a \in T_j$ such that $T_j$ is not a triad of $M$ but is a triad of $W$, by step 2, $M \backslash a$ is 3-connected.

Relabel the elements of the fan such that it starts at a spoke $a_1$ and end at the spoke $a_t = a$. Thus $F = \{a_1, b_1, a_2, b_2, \cdots, a_t\}$. As $F$ is a maximal fan, at the ends of $F$, either or both $a_1$ and $a_t$ are not contained in a triad of $M$ and $r^*_M(F) = r^*_W(F)$. Thus $r(F) + r^*(F) - |F| = t + (t+1) - (2t-1) = 2$ and $a \in cl(F) \cap cl(E(M) - F)$ as well. Thus by Theorem 6.1.3, $M \backslash a$ is sequentially 4-connected as $a$ is in guts of $(F - a, E(M) - F \cup a)$. Also, $M \backslash a$ is not isomorphic to a wheel or a whirl.

Suppose to the contrary that $M \backslash a$ a wheel or a whirl. Note that $a = a_i$, a spoke of the wheel or a whirl matroid. Without the loss of generality, suppose $i = 2$. Since $\{a_1, b_1, a_2\}$ and $\{a_2, b_2, a_3\}$ is a triangle of $W$, when we delete $a_2$, elements $a_1, b_1, e$ have to be in a triangle and $b_2, a_3, e$ has to be in a triangle of $M \backslash a_2$. Also, as $M$ is 3-connected, $\{e, a_2\}$ is not a 2-element circuit. Then, $\{e, a_1, a_2, b_1\}$ and $\{a_2, a_3, b_2, e\}$ are isomorphic to $U_{2,4}$. Then, $\{b_1, a_2, b_2\}$ and

$\{b_1, a_2, e\}$ is a triangle in $M$ which implies $r_M(\{b_1, b_2, a_2, e\}) = 2$. Now, we choose $a_2$ such that $\{b_1, b_2, a_2\}$ is no longer a triad in $M$. Therefore $\{b_1, b_2, a_2, e\}$ is a 4-element cocircuit in $M$. Thus $r^*(\{b_1, b_2, e, a_2\}) = 3$. Then $r_M(\{b_1, b_2, e, a_2\}) + r^*_M\{b_1, b_2, e, a_2\} - |\{b_1, b_2, e, a_2\} = 2 + 3 - 4 = 1$. However, this is not possible as $M$ is 3-connected. Therefore $M\backslash a_2$ is not isomorphic to a wheel or a whirl.

If all of the triads of $W$ are not triads of $M$, then we first show that $M$ is internally 4-connected. Let $(X \cup e, Y)$ be a 3-separation of $M$ and $|X \cup e|$, $|Y| \geq 4$. Then $r(X \cup e) + r(Y) = r(M) + 2 = r(W) + 2$ as $M$ is an extension of $W$. If $r(X \cup e) = r(X) + 1$, then $r(X \cup e) + r(Y) = r(X) + 1 + r(Y) = r_W(X) + 1 + r_W(Y) = r(W) + 2$. Then $(X, Y)$ is a 2-separation of $W$, contradicting the connectivity of $W$. Thus $r(X \cup e) = r(X)$ and $(X, Y)$ is a 3-separation of $W$. By Lemma 6.2.1, $X$ and $Y$ are fan of $W$. As $Y$ is a fan of $W$, and the elements of wheels are either rims or spokes, there are three possible cases for $Y$:

$Y = Y_1 = \{a_1, b_1, \cdots, b_{t-1}, a_t\}$

$Y = Y_2 = \{a_1, b_1, \cdots, a_{t-1}, b_{t-1}\}$

$Y = Y_3 = \{b_1, a_2, b_2, \cdots, a_{t-1}, b_{t-1}\}$

Note that in each case, $r_M(Y) = r_W(Y)$ as $M$ is a single element extension of $W$. Also, because none of the triads of $W$ are triads of $M$, $r^*_M(Y) = r^*_W(Y) + 1$. Therefore in the first case, $r(Y_1) + r^*(Y_1) - |Y_1| = t + t + 2 - (2t - 1) = 3$. In the second case, $r(Y_2) + r^*(Y_2) - |Y_2| = t + t + 1 - (2t - 2) = 3$ and lastly, in third, $r(Y_3) + r^*(Y_3) - |Y_3| = t + 2 + t + 2 - (2t - 1) = 3$. Thus in each case, if $Y$ is a fan of size 4 or greater, $r(Y) + r^*(Y) - |Y| \neq 2$ which contradicts the assumption that $(X, Y)$ is a 3-separation of $M$ and $W$.

Then by Theorem 6.1.2, for each triangle $T$ in $M$, there exists $t \in T$ such that $M\backslash t$ is sequentially 4-connected or $|E(M)| \leq 11$ and there exists $y \in E(M)$ such that $y$ is not in any triangle of $M$ and such that $M/y$ is sequentially 4-connected. As $e$ is only element in $E(M)$ not in any triangle, $M/e$ is sequentially 4-connected and as $r(M/e) = r(M) - 1 = r(W) - 1$, we can see that $M/e$ is not a wheel or a whirl. Also, $M\backslash t$ is not isomorphic to a wheel or a whirl matroid as well. Suppose $M\backslash t$ ia isomorphic to a wheel or a whirl. Before we state this, remember $M$ is an extension where non of the triads from a wheel (or whirl) is also a triad. Let $T_i = \{b_i, b_{i+1}, a_{i+1}\}$ be a triad in $W$. If $H_i = E(W) - T_i$ then $cl_M(H_i) = H_i$ and $T_i \cup e$ is a 4-element cocircuit in $M$ for all $i$. Suppose $t = a_i$ or $b_i$. Then $t \in H_{i+2}$ and $t \notin T_{i+2} = \{b_{i+1}, b_{i+2}, a_{i+2}\}$. Also, $cl_W(H_{i+2} - t) = H_{i+2}$ implies $cl_M(H_{i+2} - t) = cl_M(H_{i+2}) = H_{i+2}$. Thus $cl_{M\backslash t}(H_{i+2} - t) = cl_M(H_{i+2} - t) - t = H_{i+2} - t$. Also, $H_{i+2}$ is a hyperplane in $M\backslash t$ as $r_M(H_{i+2}) = r_{M\backslash t}(H_{i+2})$ and $cl_{M\backslash t}(H_{i+2} - t) = H_{i+2} - t$. Which implies $T_{i+2} \cup e$ is a 4-element cocircuit in $M\backslash t$, which is not possible as $M\backslash t$ is supposed to be isomorphic to a wheel or a whirl implies $T_{i+2}$ is supposed to be a 3-element cocircuit. Therefore $M\backslash t$ is not a wheel or a whirl. This completes the proof. □

## 6.3 Proof of Main Result

Now, we prove the main result of this chapter. We state this theorem again.

**Theorem.** *Let $M$ be a sequentially 4-connected matroid such that $|E(M)| \geq 4$ that is neither a wheel nor a whirl and let $\mathcal{W}' = \{W_3, W_4, W^2, W^3\}$. Then there is a $\mathcal{W}'M$-chain of sequentially 4-connected matroids $M_0, M_1, \ldots, M_n$ such that $M_0 \in \mathcal{W}'$, $M_n = M$, and each $M_{i+1}$ is a single element extension or coextension of $M_i$ for all $i = 0, 1, \ldots, n-1$.*

*Proof.* Let $M$ be a sequentially 4-connected matroid that is neither a wheel nor a whirl. Then by Theorem 5.1.7, there exists $x \in E(M)$ such that either $M\backslash x$ or $M/x$ is sequentially 4-connected. Let $W \in \mathcal{W}'$ be a minor of $M$ such that if $M$ has a $W_3$-minor, then $M$ has no $W_4$-minor. If $M$ has a $W^2$-minor, then $M$ has no $W^3$-minor. Let $M_n = M$ and $M_{n-1} \cong M/x$ where $M_n$ is sequentially 4-connected and $x \in E(M)$. If $M_i \cong W_k$ ($k \geq 5$) or $M_i \cong W^k$ ($k \geq 4$), for some $i \geq 1$, then $M_{i-1}$ is either a single element extension or coextension of $M_i$. Without loss of generality, suppose the former. Then $M_{i+1} \cong W_k + e$ or $M_{i+1} \cong W^k + e$ and by Theorem 6.2.2, there exists $a \in E(M_{i+1})$ such that either $M\backslash a$ or $M/e$ is sequentially 4-connected and is not isomorphic to a wheel or a whirl. We continue this each time $M_j \notin \mathcal{W}$. Eventually, $M_0 = W$ for some $W \in \mathcal{W}'$ and the chain stops. □

# 7 WEAKLY 4-CONNECTED MATROIDS

In this chapter, we prove a chain theorem for weakly 4-connected matroids. In Section 1, we give known results on a splitting theorem for weakly 4-connected matroids and provide motivation for our work. Then in Section 2, we prove lemmas which are used in the proof of the main result. In Section 3, we prove the main result of this chapter.

## 7.1 Motivation and Main Result

As we have seen in Chapter 5, Geelen and Zhou's result on the weakly 4-connected matroids gives us a $(\mathcal{L} \cup \mathcal{T})M$-chain of weakly 4-connected matroids. The set $\mathcal{T}$ contains finite number of matroids, but the set $\mathcal{L}$ contains all planar and Möbius ladders and their duals and thus, have infinite cardinality.

Before stating the main theorem of this chapter, we give definitions of some matroids in the theorem. In the paper [14], the authors give the definition of planar ladders and Möbius ladders. See Figure 7.1 for the graphic representation of the matroids and the label used throughout this chapter. There are two types of ladders: planar ladders and Möbius ladders. In this dissertation, we write cycle matroid of a planar ladder as a planar ladder and the cycle matroid of a Möbius ladder as a Möbius ladder. In the paper [14], authors refer to both ladders as a cycle matroid of a ladder. Also we denote the planar ladder with $3n$ elements as $L_n$ and the Möbius ladder with $3n$ elements as $L^n$ for $n \geq 3$. It is not difficult to check that both planar ladder and Möbius ladder are internally 4-connected.

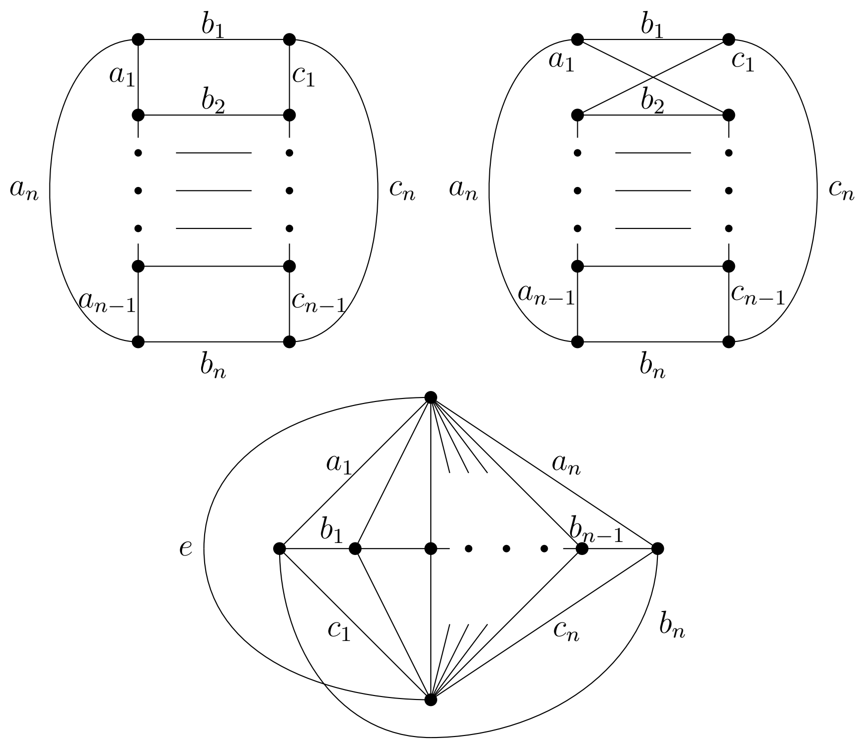


Figure 7.1: $L_n$, $L_n^*$, and $P_n$ with labels

The definition of the trident is given in [14]. A trident is a weakly 4-connected rank-6 matroid $M$ on 12 elements such that $E(M)$ can be partitioned into three 4-element rank-3 3-separating sets.

Lastly, let $P_n$ denote the unique internally 4-connected single element extension of the dual of a planar ladder, $L_n^*$, by an element $e$ where $\{a_i, c_i, e\}$ is a circuit for all $i \in \{1, \cdots, n\}$. Similarly, let $P^n$ denote the unique internally 4-connected single element extension of a Möbius ladder, $(L^n)^*$ by an element $e$ where $\{a_i, c_i, e\}$ is a circuit for all $i \in \{1, \cdots, n\}$. Note that $P_n^*$ (resp. $(P^n)^*$) is also a single element coextension of $L_n$ (resp. $L^n$). In Lemma 7.2.4 and 7.2.9, we prove we prove that these two classes of matroids are uniquely defined. The following is the main result of this chapter.

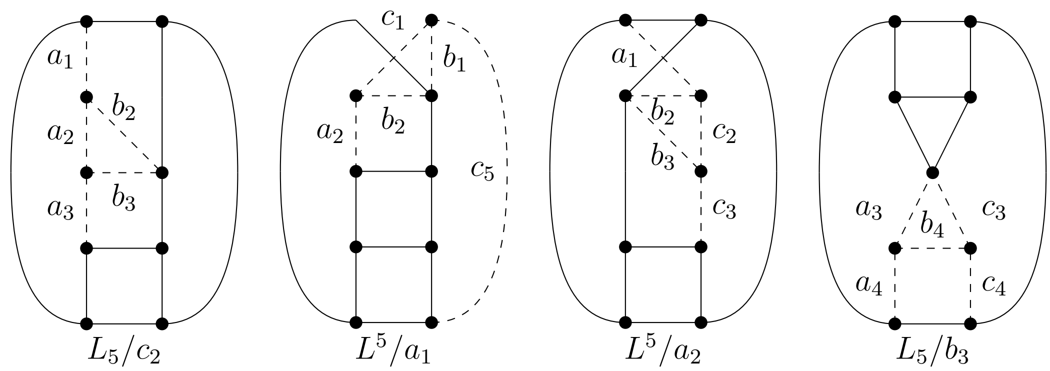


Figure 7.2: Examples of 3-separations $(X, Y)$ of $L_5$ and $L^5$ where $|X| = 5$.

**Theorem 7.1.1.** *Let $M$ be a weakly 4-connected matroid with $|E(M)| \geq 7$ such that $M$ or $M^*$ is not isomorphic to the cycle matroid of a ladder($L_n$ and $L^n$), $P_n$, $P^n$, or a trident. Then there exists a weakly 4-connected $W'M$-chain where $W' = M_0 \in \{W_3, W^3, W^2, trident\}$, and $M = M_n$ and for each $i \in \{0, \cdots, n\}$,*

*1. there exists $e \in E(M_i)$ such that $M_i \backslash e \cong M_{i-1}$ or $M_i/e \cong M_{i-1}$ is weakly 4-connected, or*

*2. $M_i$ has a 4-element 3-separating set $A$ with elements $c$, $d \in A$ such that $M_i \backslash d/c \cong M_{i-1}$ is weakly 4-connected.*

## 7.2 Lemmas

For Lemmas 7.2.2, 7.2.3, 7.2.4, and 7.2.7 let $X_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subset E(L_n)$, and $Y_i = E(L_n) - (X_i \cup a_i)$ for $i \in \{1, \cdots, n\}$. Similarly, let $X_i' = \{b_i, b_{i+1}, a_{i-1}, a_i, a_{i+1}\} \subset E(L_n)$ and $Y_i' = E(L_n) - (X_i' \cup c_i)$. Define $X_i$, $X_i'$, $Y_i$, and $Y_i'$ for $L_n^*$ in the same manner. For the Möbius ladder and the dual, let $X_i$, $X_i'$, $Y_i$ and $Y_i'$ be defined in a same manner as $L_n$ for $i \in \{3, \cdots, n\}$. For $i = 1$ and 2, let $X_1 = \{c_1, a_2, c_n, b_1, b_2\}$ and $X_2 = \{a_1, c_2, c_3, b_2, b_3\}$ for $L^n$ and the dual. Define $Y_i$ and $Y_i'$ for $L^n$ and the dual in the same manner.

Note that $(X_i, Y_i)$, $(X_i - c_{i+1}, Y_i \cup c_{i+1})$, and $(X_i - c_{i-1}, Y_i \cup c_{i-1})$ are the only 3-separation of $L_n/a_i$ where both sides of the separation have cardinality at least 4 for all $i \in \{1, \cdots, n\}$ and $n \geq 4$. The same holds for the Möbius ladder except, when $i = 1$, $(X_1, Y_1)$, $(X_1 - c_n, Y_1 \cup c_n)$, $(X_1 - a_2, Y_1 \cup c_2)$ are the 3-separations of $L^n/a_1$ where both sides of the separation have cardinality at least 4. For both $L_n/b_i$ and $L^n/b_i$, if $X$ is a 3-separation, then $\{a_i, c_i\} \subset X$. Figure 7.2 shows 3-separations of $L_5/c_2$, $L^5/a_1$, $L^5/a_2$ and $L_5/c_3$ where one of the sides of the separation have cardinality 5. For $L_5/c_2$, the set $\{a_1, a_2, a_3, b_2, b_3\} = X'_1$ and $E(L_5) - (X_1 \cup a_1) = Y'_1$. Throughout this section, we refer to the closure of the set $X_i$ and $Y_i$ of the cycle matroid of a ladder and the dual. Note that because of the symmetry, if $(X_i, Y_i)$ is a 3-separation of $L_n/a_i$, then $(X'_i, Y'_i)$ is a 3-separation of $L_n/c_i$. Many of the proofs in this section are similar due to the symmetry of the cycle matroid of a ladder. Also, a planar ladder and a Möbius ladder are very similar and we also use this similarity to use a proof used for a planar ladder to a Möbius ladder case. Let us first introduce a proposition that will be used in several lemmas in this section.

**Proposition 7.2.1.** *([18], P.300) Let $e$ be an element of a matroid $M$. Suppose that $M \backslash e$ is $n$-connected but $M$ is not. Then either $e$ is a coloop of $M$, or $M$ has a circuit that contains $e$ and has fewer than $n$ elements.*

Now, we state and prove the lemmas which will be used to prove the main result of this chapter.

**Lemma 7.2.2.** *If $M$ is a 3-connected single element extension of $L_n$, $n \geq 4$, then $M$ is weakly 4-connected. Moreover, there exists an element $a \in E(L_n)$ such that both $M/a$ and $(M/a)^*$ are weakly 4-connected and are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$.*

*Proof.* Let $\mathcal{M}$ be a modular cut of $L_n$, $n \geq 4$ and let $M$ be a 3-connected extension of $L_n$ such that $M = L_n +_{\mathcal{M}} e$. Then $|E(M)| = |E(L_n)| + 1 = 3n + 1$ and $r(M) = r(L_n) = 2n - 1$.
**Claim 1:** $M$ is weakly 4-connected.

*Proof.* Suppose $M$ is not weakly 4-connected. Since $M$ is a 3-connected single element extension of $L_n$, there exists a 3-separation $(X \cup e, Y)$ in $L_n$ where $|X \cup e|$, $|Y| \geq 5$. However, this separation induces either a 2-separation or a 3-separation, $(X, Y)$ in $L_n$. Since $|X| \geq 4$ and $|Y| \geq 5$, and $L_n$ is internally 4-connected, this is not possible. Therefore $M$ is weakly 4-connected. □

Now, suppose $M/a$ is not weakly 4-connected for some $a_i \in E(L_n)$. Since $M$ is weakly 4-connected, $M/a$ is always 2-connected. In Claim 2a, we suppose $M/a$ is 2-connected but not 3-connected. In Claim 2b, we suppose $M/a$ is 3-connected but not weakly 4-connected. In Claim 2a and 2b, we see how connectivity of $M/a$ effects $M$ and $L_n/a$.

**Claim 2a:** If $M/a$ is 2-connected but not 3-connected for some $a \in E(L_n)$, then there exists $x \in E(L_n) - a$ such that $\{e, a, x\}$ is a triangle in $M$.

*Proof.* Let $a \in E(L_n)$ such that $M/a$ is 2-connected but not 3-connected. However $M/a \backslash e = M \backslash e/a = L_n/a$ is 3-connected. Then by Proposition 7.2.1, there exists a circuit containing $e$ and has size fewer than 3 in $M/a$. Let $\{e, x\}$ be a 2-element circuit in $M/a$ where $x \in E(L_n) - a$. Since $M$ is weakly 4-connected, $\{e, x\}$ is not a 2-element circuit in $M$. Therefore $\{e, x, a\}$ has to be a 3-element circuit, a triangle, in $M$. □

**Claim 2b:** If $M/a$ is 3-connected but not weakly 4-connected, then a 3-separation $(X \cup e, Y)$, $|X \cup e|$, $|Y| \geq 5$, in $M/a$ induces a 3-separation $(X, Y)$ in $L_n/a$.

*Proof.* Let $M/a$ be a 3-connected but not weakly 4-connected matroid and $(X \cup e, Y)$ be a 3-separation of $M/a$ such that $|X \cup e|$ and $|Y| \geq 5$. Then $\lambda_M(X \cup e) = 2$ or 3. If $\lambda_M(X \cup e) = 2$ then $(X \cup e, Y \cup a)$ is a 3-separation of $M$ where $|X \cup e|$, $|Y \cup a| \geq 5$. However, by Claim 1, $M$ is weakly 4-connected and if $(X \cup e, Y \cup a)$ is a 3-separation, then $|X \cup e|$ or $|Y \cup a| \leq 4$. A contradiction. Therefore $\lambda_M(X \cup e) = 3$ and the following statement holds:

(i) $\lambda_M(Y) = r_M(X \cup e \cup a) + r_M(Y) - r(M) = 3$ and $a \in cl_M(Y)$.

The sets $X \cup a$ and $Y$ have cardinality at least 4 and the matroid $L_n$ is weakly 4-connected. Therefore $\lambda_{L_n}(Y) \neq 2$. Thus following statement holds.

(ii) $\lambda_{L_n}(Y) = r_{L_n}(X \cup a) + r_{L_n}(Y) - r(L_n) = 3$ and $e \in cl_M(X \cup a)$.

Since $M$ is a single element extension of $L_n$ and $a \in cl_M(Y)$, by (i), $a \in cl_{L_n}(Y) = cl_M(Y) - e$. Therefore, $r_{L_n}(Y) = r_{L_n}(Y \cup a)$. Also, by (ii), $r_M(X \cup e \cup a) = r_M(X \cup a)$. Therefore $r_{L_n}(X \cup a) + r_{L_n}(Y \cup a) - r(L_n) = 3$ as well. If $r_{L_n}(X \cup a) \neq r_{L_n}(X)$ then $r_{L_n}(X) + r_{L_n}(Y \cup a) - r(L_n) = 2$, a contradiction as $L_n$ is internally 4-connected and $|X|$, $|Y| \geq 4$. Therefore the following statement holds.

(iii) $\lambda_{L_n}(X) = r_{L_n}(X) + r_{L_n}(Y \cup a) - r(L_n) = 3$ and $a \in cl(X)$.

By (i) and (iii), we can deduce that $a \in cl_{L_n}(X)$ and $cl_{L_n}(Y)$. Therefore $r_{L_n/a}(X) = r_{L_n}(X \cup a) - 1 = r_{L_n}(X) - 1$ and similarly, $r_{L_n/a}(Y) = r_{L_n}(Y) - 1$ as well. Therefore $\lambda_{L_n/a} = 2$ and $(X, Y)$ is a 3-separation in $L_n/a$. □

In Claim 3, by using Claim 2a and 2b, we provide necessary conditions for $M/a_i$ to be weakly 4-connected for some $a_i \in E(L_n)$.

**Claim 3:** If there exists $i \in \{1, \cdots, n\}$ such that $cl_{L_n}(X_i)$, $cl_{L_n}(Y_i) \notin \mathcal{M}$ (resp. $cl_{L_n}(X_i')$, $cl_{L_n}(Y_i') \notin \mathcal{M}$) for some $i \in \{1, \cdots, n\}$, then $M/a_i$ (resp. $M/c_i$) is weakly 4-connected.

*Proof.* Suppose to the contrary that $cl_{L_n}(X_i)$, $cl_{L_n}(Y_i) \notin \mathcal{M}$ but $M/a_i$ is not weakly 4-connected. Then either $M/a_i$ is not 3-connected or $M/a_i$ is 3-connected but not weakly 4-connected. First, if $M/a_i$ is 2-connected but not 3-connected, then by Claim 2a, $\{e, a_i, x\}$ must be a triangle in $M$ for some $x \in E(L_n)$ where $x \neq a_i$. Therefore $\{a_i, x\} \in \mathcal{M}$, as otherwise, $\{e, a_i, x\}$ would not be a triangle in $M$. Since $X_i \cup Y_i \cup a_i = E(L_n)$ and $a_i \neq x$, we have $a_i \in cl_{L_n}(X_i)$ and $cl_{L_n}(Y_i)$ and $x \in X_i$ or $x \in Y_i$. Suppose $x \in X_i$. Then $\{a_i, x\} \subset cl_{L_n}(X_i)$. This implies $cl_{L_n}(X_i) \in \mathcal{M}$, a contradiction since we assumed that $cl_{L_n}(X) \notin \mathcal{M}$. Therefore $M/a_i$ must be 3-connected but not weakly 4-connected.

If $M\backslash a_i$ is 3-connected but not weakly 4-connected, then there exists a 3-separation $(X \cup e, Y)$ in $M/a_i$ such that $|X \cup e|, |Y| \geq 5$. From Claim 2b, we know $e \in cl_M(X \cup a)$ and $a \in cl_{L_n}(X)$. This implies $a \in cl_M(X)$ and $e \in cl_M(X \cup a) = cl_M(X)$. Thus $cl_{L_n}(X) \notin \mathcal{M}$. Since $(X, Y)$ induces a 3-separation in $L_n/a_i$, $X$ or $Y$ equals to the following sets: $X_i$, $X_i - c_{i-1}$, or $X_i - c_{i+1}$. If $X = X_i$, then, $cl_{L_n^*}(X_i) \in \mathcal{M}$, a contradiction. If $X = X_i - c_{i-1}$ or $X_i - c_{i+1}$, again, $cl_{L_n}(X_i - c_{i-1})$ or $cl_{L_n}(X_i - c_{i+1}) \in \mathcal{M}$. However, $X_i - c_{i-1}$, $X_i - c_{i+1} \subset X_i$ and since $\mathcal{M}$ is a modular cut, this implies $X_i \in \mathcal{M}$, a contradiction as well. Therefore $M/a_i$ must be weakly 4-connected. □

By Claim 3, if there exists $i \in \{1, \cdots, n\}$ such that $cl_{L_n}(X_i) \in \mathcal{M}$, $cl_{L_n}(Y_i) \notin \mathcal{M}$ then $M/a_i$ is weakly 4-connected. By symmetry, if $cl_{L_n}(X_i')$, $cl_{L_n}(Y_i') \notin \mathcal{M}$, then $M/c_i$ is weakly 4-connected. Now, we need to show that there exists $i \in \{1, \cdots, n\}$ such that $cl_{L_n}(X_i)$, $cl_{L_n}(Y_i) \notin \mathcal{M}$. We prove this in multiple steps. In Claim 4, we prove that there exists $i$ such that $cl_{L_n}(X_i) \notin \mathcal{M}$. Then in Claim 5, we prove that if there exists $i$ such that $cl_{L_n}(X_i) \notin \mathcal{M}$ but $cl_{L_n}(X_{i-1}) \in \mathcal{M}$, then there exists $a_j$ or $c_j \in E(L_n)$ such that $M/a_j$ or $M/c_j$ is weakly 4-connected. In Claim 6, we prove that if $cl_{L_n}(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$, then there exists $a_j \in E(L_n)$ or $c_j \in E(L_n)$ such that $M/a_j$ or $M/c_j$ is weakly 4-connected.

In the rest of the proof, for $cl_{L_n}(X)$, we omit $L_n$ and write $cl(X)$. When we take the closure of a set in a different matroid, say $M$, we will denote it properly as $cl_M(X)$ in the proof. For Lemma 7.2.2, Claims 4, 5, and 6, to see how different flats in the modular cut create modular pairs, we provide an example with $L_5$ in each claim. Elements in the sets in modular cuts will be drawn in thick lines and the elements in the intersection will be drawn in dashed lines. Also for all $i \in \{3, \cdots, n\}$, the closure of a set $X_i$ in $L_n$ and $L^n$ are equal. Therefore we choose specific sets in some of the proofs of the claims in Lemma 7.2.2 so that we can apply the same proof in other lemmas.

**Claim 4:** There exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Since $L_n$ is symmetric, without loss of generality, suppose $cl(X_2)$, $cl(X_3)$, $cl(X_4) \in \mathcal{M}$. Note that $r(cl(X_i)) = 5$ for all $i$. Also, $r(cl(X_3) \cup cl(X_4)) = r(cl(\{a_3, a_4, b_3, b_4, b_5, c_2, c_3, c_4, c_{4+1}\})) = 7$. If $L_n = L_4$, then $c_{4+1} = c_1$. If $n \geq 5$, then we have $c_{4+1} = c_5$. And $r(cl(X_3) \cap cl(X_4)) = r(cl(b_4, c_3, c_4)) = 3$. Then ($cl(X_3)$,$cl(X_4)$) is a modular pair as $r(cl(X_3)) + r(cl(X_4)) = r(cl(X_3) \cup cl(X_4)) + r(cl(X_3) \cap cl(X_4)) = 10$. Therefore, $cl(X_3) \cap cl(X_4) = \{b_4, c_3, c_4\} \in \mathcal{M}$. Also, ($\{b_4, c_3, c_4\}$, $cl(X_2)$) is a modular pair as $r(cl(X_2) \cup \{b_4, c_3, c_4\}) = r(\{a_2, b_2, b_3, c_1, c_2, c_3, c_4\}) = 7$ and $r(cl(X_2) \cap \{b_4, c_3, c_4\}) = r(\{c_3\}) = 1$. Therefore $\{b_4, c_3, c_4\} \cap cl(X_2) = \{c_3\} \in \mathcal{M}$. This implies that $e$ is in parallel with $c_3$ in $M$, which is a contradiction since $M$ is weakly 4-connected. Therefore there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$.

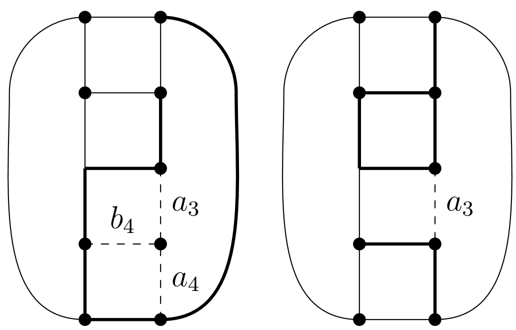


Figure 7.3: Example of Claim 4 with $L_5$.

□

**Claim 5:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$ and $cl(X_{i-1}) \in \mathcal{M}$, then $M/a_i$, $M/c_i$, or $M/a_{i-3}$ is weakly 4-connected.

*Proof.* Suppose that there exists some $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$ and $cl(X_{i-1}) \in \mathcal{M}$. Because $L_n$ is symmetric, assume without loss of generality that $i = 4$. Thus $cl(X_4) \notin \mathcal{M}$ but $cl(X_3) \in \mathcal{M}$. If $cl(Y_4) \notin \mathcal{M}$, then by Claim 3, $M/a_4$ is weakly 4-connected and we are done. Therefore, suppose that $cl(Y_4) \in \mathcal{M}$. Then $r(cl(X_3)) = 5$ and $r(cl(Y_4)) = 2n-3$. Also

$r(cl(X_3) \cup cl(Y_4)) = r(L_n) = 2n-1$ and $r(cl(X_3) \cap cl(Y_4)) = r(\{a_3, b_3, c_2\}) = 3$. Therefore $(cl(X_3)$,$cl(Y_4))$ is a modular pair and thus $\{a_3, b_3, c_2\} \in \mathcal{M}$.

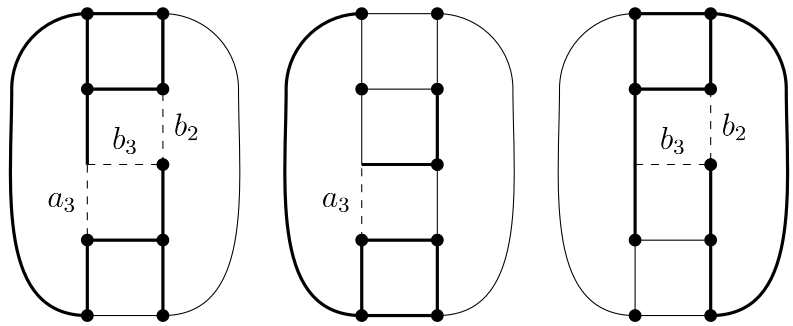


Figure 7.4: Example of Claim 4 with $L_5$.

We now show that $M/c_4$ or $M/a_1$ is weakly 4-connected. We need to show that $cl(X'_4)$, $cl(Y'_4) \notin \mathcal{M}$ or $cl(X_1)$, $cl(Y_1) \notin \mathcal{M}$. Suppose to the contrary that both $M/c_4$ and $M/a_1$ are not weakly 4-connected. Since $M/c_4$ is not weakly 4-connected, $cl(X'_4)$ or $cl(Y'_4) \in \mathcal{M}$. Similarly, since $M/a_1$ is not weakly 4-connected, $cl(X_1)$ or $cl(Y_1) \in \mathcal{M}$. If $cl(X'_4) \in \mathcal{M}$ then $(cl(X'_4)$,$\{a_3, b_3, c_2\})$ is a modular pair. Thus $cl(X'_4) \cap \{a_3, b_3, c_2\} = \{a_3\} \in \mathcal{M}$. This contradicts the fact that $M$ is weakly 4-connected. Therefore $cl(Y'_4)$ must be an element of $\mathcal{M}$. Also, $(cl(Y'_4)$,$\{a_3, b_3, c_2\})$ is a modular pair so $\{b_3, c_2\} \in \mathcal{M}$. Similarly, if $cl(X_1) \in \mathcal{M}$ then $(cl(X_1)$, $\{a_3, b_3, c_2\})$ is a modular pair, which implies $cl(X_1) \cap \{a_3, b_3, c_2\} = \{c_2\} \in \mathcal{M}$ as well. Therefore $cl(Y_1)$ must be an element of $\mathcal{M}$. This implies $(cl(Y_1)$,$\{a_3, b_3, c_2\})$ is a modular pair and thus, $\{a_3, b_3\} \in \mathcal{M}$. Therefore $\{b_3, c_2\}$,$\{a_3, b_3\} \in \mathcal{M}$. However, the two sets form a modular pair. Therefore, the intersection, $\{b_3\}$ must be in $\mathcal{M}$. This is a contradiction as $M$ is weakly 4-connected. Therefore $cl(Y'_4)$ or $cl(Y_1) \notin \mathcal{M}$. By assumption, $cl(X'_4)$, $cl(X_1) \notin \mathcal{M}$. Therefore $M/c_4$ or $M/a_1$ is weakly 4-connected by Claim 3. □

**Claim 6:** If $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$, then there exists $j$, $k \in \{1, \cdots, n\}$ such that $M/a_j$ or $M/c_k$ is weakly 4-connected.

*Proof.* Let $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. If there exists $j \in \{1, \cdots, n\}$ such that $cl(Y_j) \notin \mathcal{M}$, then by Claim 3, $M/a_j$ is weakly 4-connected and we are done. Thus, suppose that $cl(Y_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. We first show that in this case, $\bigcap_{i=3}^{n} cl(Y_i) \in \mathcal{M}$. Now for each $k \in \{4, \cdots, n\}$ the following statements hold.

(i) $r(Y_i) = 2n - 3 = r - 2$ for all $i$.

(ii) $\bigcap_{i=3}^{k} cl(Y_i) = E(L_n) - \{b_3, \cdots, b_{k+1}, c_2, \cdots, c_{k+1}\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i)) = r - 2 - (k-3) = r - k + 1$.

(iii) $(\bigcap_{i=3}^{k} cl(Y_i)) \cup cl(Y_{k+1}) = E(L_n) - \{b_{k+1}, c_k, c_{k+3}\}$ and $r((\bigcap_{i=3}^{k} cl(Y_i)) \cup cl(Y_{k+1})) = r - 1$.

(iv) $(\bigcap_{i=3}^{k} cl(Y_i)) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1} cl(Y_i)$ and $r((\bigcap_{i=3}^{k} cl(Y_i)) \cap cl(Y_{k+1})) = r - 2 - (k+1-3) = r - k$.

From (i)-(iv), we can deduce that $(\bigcap_{i=3}^{k} cl(Y_i), cl(Y_{k+1}))$ is a modular pair. Therefore the intersection of two sets, $(\bigcap_{i=3}^{k} cl(Y_i)) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1} cl(Y_i) \in \mathcal{M}$. Therefore $\bigcap_{i=3}^{n} cl(Y_i) = \{a_1, \cdots, a_n, b_2\}$ must be an element of $\mathcal{M}$. If there exists $i \in \{1, \cdots, n\}$ such that $cl(X'_i) \notin$

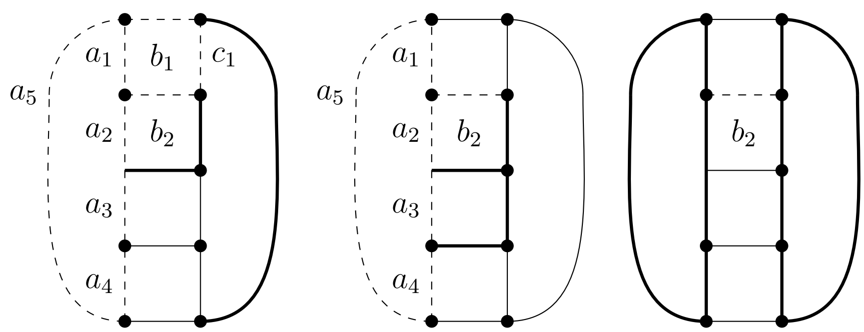


Figure 7.5: Example of Claim 6 with $L_5$.

$\mathcal{M}$ but $cl(X'_{i-1}) \in \mathcal{M}$, then by Claim 5, $M/c_i$ or $M/c_{i-3}$ is weakly 4-connected and we are done. If $cl(X'_i) \notin \mathcal{M}$ for all $i$, then we look at $cl(Y'_j)$. If there exists $j$ such that $cl(Y'_j) \notin \mathcal{M}$ then $M/c_j$ is weakly 4-connected and we are done. If $cl(Y'_i) \in \mathcal{M}$ for all $i$, then like $cl(Y_i)$ case, $\bigcap_{i=3}^{n} cl(Y'_i) = \{c_1, \cdots, c_n, b_2\} \in \mathcal{M}$. However, $\{a_1, \cdots, a_n, b_2\} \in \mathcal{M}$ as well and $(\{a_1, \cdots, a_n, b_2\}, \{c_1, \cdots, c_n, b_2\})$ is a modular pair. Therefore their intersection,

$\{b_2\} \in \mathcal{M}$, a contradiction as $M$ is weakly 4-connected. Therefore if $cl(Y_i) \in \mathcal{M}$ for all $i$ then there must exist some $k$ such that $cl(Y_k') \notin \mathcal{M}$. Then $cl(X_k')$ and $cl(Y_k') \notin \mathcal{M}$. Thus by Claim 3, $M/c_k$ is weakly 4-connected. Therefore in all possible cases, there exists some $a_j$ or $c_k$ such that $M/a_j$ or $M/c_k$ is weakly 4-connected. □

Note that in Claim 6, we can also use $\bigcap_{i=1}^{n} cl(Y_i) \in \mathcal{M}$ and prove $(\bigcap_{i=1}^{n} cl(Y_i), \bigcap_{i=1}^{n} cl(Y_i'))$ is a modular pair. In this case, $\emptyset \in \mathcal{M}$, which contradicts the fact that $M$ is weakly 4-connected. However, to use the this proof in Claim 6 of Lemma 7.2.3, we use $\bigcap_{i=3}^{n} cl(Y_i)$.

In all possible cases, we have proven that there exists some $j \in \{1, \cdots, n\}$ such that $M/a_j$ or $M/c_j$ is weakly 4-connected. Suppose $M/a_j$ is weakly 4-connected. Then, by comparing number of elements and the rank of $M/a_j$, we can deduce that $M/a_j$ and $(M/a_j)^*$ are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$. □

Now, we prove a similar lemma for a Möbius ladder.

**Lemma 7.2.3.** *If $M$ is a 3-connected single element extension of $L^n$, $n \geq 4$, then $M$ is weakly 4-connected. Moreover there exists an element $a \in E(L^n)$ such that $M/a$ and $(M/a)^*$ are weakly 4-connected and is not isomorphic to $L_m$, $L^m$, $P_m$ or $P^m$ for $m \geq 4$.*

*Proof.* Let $\mathcal{M}$ be a modular cut of $L^n$, $n \geq 4$ and let $M$ be a 3-connected single element extension of $L^n$ such that $M = L^n +_{\mathcal{M}} e$. Let $r = 2n - 1$ denote the rank of $L^n$.

**Claim 1:** $M$ is weakly 4-connected.

*Proof.* Suppose $M$ is not weakly 4-connected. Since $M$ is a 3-connected single element extension of $L^n$, there exists a 3-separation $(X \cup e, Y)$ where $|X \cup e|, |Y| \geq 5$. However, this separation induces either a 2-separation or a 3-separation, $(X, Y)$ in $L^n$. This is not possible since $L^n$ is internally 4-connected. Therefore $M$ is weakly 4-connected. □

Now, suppose $M/a$ is not weakly 4-connected for some $a \in E(L^n)$. Since $M$ is weakly 4-connected, $M/a$ is always 2-connected. In Claim 2a, we suppose $M/a$ is 2-connected but not 3-connected. In Claim 2b, we suppose $M/a$ is 3-connected but not weakly 4-connected.

**Claim 2a:** If $M/a$ is 2-connected but not 3-connected, then there exists $x \in E(L^n) - a$ such that $\{e, a, x\}$ is a triangle in $M$.

**Claim 2b:** If $M/a$ is 3-connected but not weakly 4-connected, then a 3-separation $(X \cup e, Y)$ in $M/a$ where $|X \cup e|$, $|Y| \geq 5$, induces a 3-separation $(X, Y)$ in $L^n/a$.

Since $L^n$ is internally 4-connected and $M$ is weakly 4-connected, we can apply the proof of Claim 2a and 2b from 7.2.2 as both proofs only uses connectivity argument and does not depend on the structure of the cycle matroid of a ladder at all. Now, the next claim provides necessary conditions for $M/a_i$ to be weakly 4-connected.

**Claim 3:** If there exists $i$ such that $cl_{L^n}(X_i)$, $cl_{L^n}(Y_i) \notin \mathcal{M}$ (resp. $cl_{L^n}(X_i')$, $cl_{L^n}(Y_i') \notin \mathcal{M}$) for some $i \in \{1, \cdots, n\}$, then $M/a_i$ (resp. $M/c_i$) is weakly 4-connected.

Since $L^n$ is internally 4-connected and because we defined $X_i$ and $Y_i$ of $L^n$ such that they correspond with the sets $X_i$ and $Y_i$ of $L_n$, the proof of Claim 3 from Lemma 7.2.2 works on this claim as well.

We now prove that there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i)$ and $cl(Y_i) \in \mathcal{M}$ or $cl(X_i')$ and $cl(Y_i') \in \mathcal{M}$. The flow of the rest of the proof is same as Lemma 7.2.2. Because $L_n$ and $L^n$ are similar, we use the proof of Claim 4, 5, and 6 to prove the claims. However, $cl_{L_n}(X_i) \neq cl_{L^n}(X_i)$ for $i = 1$ and 2. Thus in some cases, we add extra explanations when necessary. Lastly, in the rest of the proof, for $cl_{L^n}(X)$, we omit $L^n$ and write $cl(X)$.

**Claim 4:** There exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Then $cl(X_2)$, $cl(X_3)$, and $cl(X_4) \in \mathcal{M}$. Note that $cl_{L_n}(X_3) \cap cl_{L_n}(X_4) = \{a_3, b_3, c_2\}$ and $cl_{L^n}(X_3) \cup$

$cl_{L^n}(X_4) = \{a_3, b_3, c_2\}$. Because the structure of two matroids are similar, part of the proofs from Lemma 7.2.2 can be applied in the proof of Lemma 7.2.3. Thus, using the proof of the Lemma 7.2.2 Claim 4, ($cl(X_3)$,$cl(X_4)$) is a modular pair and their intersection, $cl(X_3) \cap cl(X_4) = \{b_4, c_3, c_4\} \in \mathcal{M}$. However, we can not use the same proof from Claim 4 of Lemma 7.2.2 for $cl(X_2)$ and $\{b_4, c_3, c_4\}$. For $L_n$, we have $cl_{L_n}(X_2) = \{b_2, b_3, c_1, c_2, c_3\}$ but $cl_{L^n}(X_2) = \{a_1, c_2, c_3, b_2, b_3\}$. However, $r(cl(X_2) \cup \{b_4, c_3, c_4\}) = r(\{a_1, a_2, b_2, b_3, c_2, c_3, c_4\}) = 7$ and $r(cl(X_2) \cap \{b_4, c_3, c_4\}) = r(\{c_3\}) = 1$. Therefore ($cl(X_2)$,$\{b_4, c_3, c_4\}$) is a modular pair as well and $\{c_3\} \in \mathcal{M}$. This is a contradiction as $M$ is weakly 4-connected. □

**Claim 5:** If there exists $i$ such that $cl(X_i) \notin \mathcal{M}$ but $cl(X_{i-1}) \in \mathcal{M}$, then $M/a_i$, $M/c_i$ or $M/a_{i-3}$ is weakly 4-connected if $i \geq 4$ and $M/c_1$, $M/c_2$, $M/c_3$, $M/c_n$ or $M/a_{n-1}$ is weakly 4-connected if $i = 1$, 2 or 3.

*Proof.* Suppose there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$ but $cl(X_{i-1}) \in \mathcal{M}$. If $cl(Y_i) \notin \mathcal{M}$ then we are done. So suppose $cl(Y_i) \in \mathcal{M}$. Now, if $i \geq 4$, then because $cl_{L_n}(X_i) = cl_{L^n}(X_i)$, proof of Claim 5 from the previous lemma works in this claim as well. However, we still need to check for the case where $i = 1$, 2 and 3.

If $cl(X_1)$, $cl(Y_2) \in \mathcal{M}$ and $cl(X_2) \notin \mathcal{M}$, then ($cl(X_1)$,$cl(Y_2)$) is a modular pair. Therefore $\{a_2, b_1, c_1, c_n\} \in \mathcal{M}$. Let $\{a_2, b_1, c_1, c_n\} = Z$. We will show that either $M/c_1$, $M/c_2$ or $M/c_3$ is weakly 4-connected. Suppose to the contrary that $M/c_1$, $M/c_2$ and $M/c_3$ are not weakly 4-connected. For each case, $cl(X_i') \in \mathcal{M}$ or $cl(Y_i') \in \mathcal{M}$. If $cl(X_i') \in \mathcal{M}$, then $(cl(X_i'), Z)$ is a modular pair. If $cl(Y_i') \in \mathcal{M}$, then $(cl(Y_i'), Z)$ is a modular pair. Then the following statements hold:

If $cl(X_1')$ and $Z \in \mathcal{M}$ then $\{b_1, c_1\} \in \mathcal{M}$.

If $cl(Y_1')$ and $Z \in \mathcal{M}$ then $\{c_1, c_n, a_2\} \in \mathcal{M}$.

If $cl(X_2')$ and $Z \in \mathcal{M}$ then $\{b_1, c_n\} \in \mathcal{M}$.

If $cl(Y_2')$ and $Z \in \mathcal{M}$ then $\{a_2, c_1\} \in \mathcal{M}$.

If $cl(X'_3)$ and $Z \in \mathcal{M}$ then $\{a_2\} \in \mathcal{M}$.

If $cl(Y'_3)$ and $Z \in \mathcal{M}$ then $\{b_1, c_1, c_n\} \in \mathcal{M}$.

If $cl(X'_1)$ and $cl(Y'_2) \in \mathcal{M}$ then $(\{b_1, c_1\}, \{b_1, c_n\})$ is a modular pair and $\{b_1\}$ is forced to be in $\mathcal{M}$. Following this idea, the following statements hold:

If $cl(X'_1)$ and $cl(Y'_2) \in \mathcal{M}$, then $\{b_1\} \in \mathcal{M}$.

If $cl(X'_1)$ and $cl(X'_2) \in \mathcal{M}$, then $\{c_1\} \in \mathcal{M}$.

If $cl(Y'_1)$ and $cl(Y'_2) \in \mathcal{M}$, then $\{c_n\} \in \mathcal{M}$.

If $cl(Y'_1)$ and $cl(X'_2) \in \mathcal{M}$, then $\{a_2, c_1\} \in \mathcal{M}$.

Because neither $M/c_1$ nor $M/c_2$ is weakly 4-connected, $cl(X'_1) \in \mathcal{M}$ or $cl(Y'_1) \in \mathcal{M}$. Similarly, $cl(X'_2) \in \mathcal{M}$ or $cl(Y'_2) \in \mathcal{M}$. However, the only case which does not induce a contradiction is when $cl(Y'_1)$, $cl(X'_2)$, $\{a_2, c_1\} \in \mathcal{M}$. Since $M/c_3$ is not weakly 4-connected, $cl(X'_3) \in \mathcal{M}$ or $cl(Y'_3) \in \mathcal{M}$. If $\{a_2, c_1\}$ and $cl(X'_3) \in \mathcal{M}$ then $(\{a_2, c_1\}, cl(X'_3))$ is a modular pair and $\{a_2\} \in \mathcal{M}$. This is a contradiction since $M$ is weakly 4-connected. If $\{a_2, c_1\}$ and $cl(Y'_3) \in \mathcal{M}$ then $(\{a_2, c_1\}, cl(Y'_3))$ is a modular pair and $\{c_1\} \in \mathcal{M}$, a contradiction. Therefore if $M/c_1$, $M/c_2$ and $M/c_3$ are not weakly 4-connected, then $M$ is not weakly 4-connected, a contradiction. Thus at least one of $M/c_1$, $M/c_2$ or $M/c_3$ is weakly 4-connected.

If $cl(X_2)$, $cl(Y_3) \in \mathcal{M}$ but $cl(X_3) \notin \mathcal{M}$, then $(cl(Y_3), cl(X_2))$ is a modular pair. Therefore $\{a_1, a_2, b_2\} \in \mathcal{M}$. This time, let $\{a_1, a_2, b_2\} = Z$. Now, we will prove that $M/c_2$ or $M/c_3$ is weakly 4-connected. Suppose to the contrary that neither $M/c_2$ nor $M/c_3$ is weakly 4-connected. If $cl(X'_2) \in \mathcal{M}$ then $(Z, cl(Y'_2))$ is a modular pairs and $\{c_2\} \in \mathcal{M}$. This is a contradiction since $M$ is weakly 4-connected. If $cl(X'_3) \in \mathcal{M}$, then $(Z, cl(X'_3))$ is a modular pair and $\{a_2\} \in \mathcal{M}$. This is a contradiction as well. Thus $cl(X'_2)$, $cl(Y'_3) \in \mathcal{M}$. Therefore $M/c_2$ and $M/c_3$ are not weakly 4-connected. However, these sets form a modular pair with $Z$. This would force $\{b_2, a_2\}$, $\{a_1, b_2\} \in \mathcal{M}$, respectively. Then again, $(\{b_2, a_2\}, \{a_1, b_2\})$ is a modular pair and thus $\{b_2\} \in \mathcal{M}$, a contradiction. Therefore $M/c_2$ or $M/c_3$ must be weakly 4-connected.

Lastly, if $cl(X_n)$ and $cl(Y_1) \in \mathcal{M}$ but $cl(X_1) \notin \mathcal{M}$, then ($cl(Y_1)$,$cl(X_1)$) is a modular pair and $\{a_n, b_n, c_{n-1}\} \in \mathcal{M}$. Using the same reasoning as the previous proofs, either $M/c_n$ or $M/a_{n-1}$ is weakly 4-connected. In all possible cases noted in the claim, there exists some $a \in E(L^n)$ such that $M/a$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$, then there exists $j$, $k \in \{1, \cdots, n\}$ such that $M/a_j$ or $M/c_k$ is weakly 4-connected.

*Proof.* Since $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$, if there exists $j \in \{1, \cdots, n\}$ such that $cl(Y_j) \notin \mathcal{M}$, then we are done as $M/a_j$ would be weakly 4-connected. Now, suppose that $cl(Y_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Then $\bigcap_{i=3}^{n} cl(Y_i) \in \mathcal{M}$. Note that for all $i \in \{3, \cdots, n\}$, we have $cl_{L_n}(X_i) = cl_{L^n}(X_i)$. Therefore part of the proof from the Claim 6 from Lemma 7.2.2 holds here as well and $\bigcap_{i=3}^{n} cl(Y_i) = \{a_1, \cdots, a_n, b_2\} \in \mathcal{M}$. Similarly, if $cl(Y_i') \in \mathcal{M}$ for all $i$, then $\{c_1, \cdots, c_n, b_2\} \in \mathcal{M}$. Then two sets $\{a_1, \cdots, a_n, b_2\}$ and $\{c_1, \cdots, c_n, b_2\}$ form a modular pair and $\{b_n\} \in \mathcal{M}$. This contradicts the fact that $M$ is weakly 4-connected. Therefore there must be some $k$ such that $cl(Y_k') \notin \mathcal{M}$. Therefore if $cl(X_i)$, $cl(X_i') \notin \mathcal{M}$ then there exists some $k$ such that $cl(X_k')$, $cl(Y_k') \notin \mathcal{M}$. Therefore $M/c_k$ would be weakly 4-connected.

Now suppose there exists $i$ such that $cl(X_i') \notin \mathcal{M}$ but $cl(X_{i-2}) \in \mathcal{M}$. If $cl(Y_i') \notin \mathcal{M}$, then we are done. If $cl(Y_i') \in \mathcal{M}$ then by Claim 5, there exists some $a \in E(L^n)$ such that $M/a$ is weakly 4-connected. Therefore in all cases, there exists $j$, $k \in \{1, \cdots, n\}$ such that $M/a_j$ or $M/c_k$ is weakly 4-connected. □

In all possible cases, we have proven that there exists some $j \in \{1, \cdots, n\}$ such that $M/a_j$ or $M/c_j$ is weakly 4-connected. Suppose $M/a_j$ is weakly 4-connected. Then by comparing the number of elements and the rank of $M/a_j$, we can deduce that neither $M/a_j$ nor $(M/a_j)^*$ are isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$. □

Next, we prove slightly different lemmas for the dual of a planar ladder and the dual of a Möbius ladder. In Lemma 7.2.4, we prove that if $M$ is a 3-connected single element extension of $L_n^*$, then there exists $a \in E(L_n^*)$ such that $(M\backslash a)$ and $(M\backslash a)^*$ are weakly 4-connected and is not isomorphic to cycle matroid of a ladder or $M \cong P_n$. Then in Lemma 7.2.5, we look at all 3-connected single element extension of $P_n$. If $N$ is a 3-connected single element extension of $P_n$, then we prove that there exists an element $a \in E(L_n^*)$ such that $N/a$ and $(N/a)^*$ is weakly 4-connected and are not isomorphic to $L_m$, $L^m$, $P_m$ and $P^m$ for all $m \geq 4$. In Lemma 7.2.6, we look at all 3-connected single element extension of $(P_n^*)$ and prove a similar result. In Lemmas 7.2.7, 7.2.8, and 7.2.9, we prove similar results for Möbius ladder.

**Lemma 7.2.4.** *If $M$ is a 3-connected single element extension of $L_n^*$, $n \geq 4$, then $M$ is weakly 4-connected. Moreover either,*

*i) there exists an element $a \in E(L_n^*)$ such that $M\backslash a$ and $(M\backslash a)^*$ are weakly 4-connected and not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$, for $m \geq 4$, or*

*ii) $M \cong P_n$.*

*Proof.* Let $\mathcal{M}$ be a modular cut of $L_n^*$, $n \geq 4$ and let $M$ be a 3-connected extension of $L_n^*$ such that $M = L_n^* +_{\mathcal{M}} e$. Then $|E(M)| = |E(L_n^*) + 1| = 3n + 1$ and $r(M) = r(L_n^*) = n + 1$.

**Claim 1:** $M$ is weakly 4-connected.

As $L_n$ is internally 4-connected, the dual, $L_n^*$ is also internally 4-connected as well. Since $M$ is a 3-connected single element extension of $L_n^*$, by the proof of Claim 1 of Lemma 7.2.2, $M$ is weakly 4-connected.

Now, suppose $M\backslash a$ is not weakly 4-connected for some $a \in E(L_n^*)$. Since $M$ is weakly 4-connected, $M\backslash a$ is always 2-connected. In Claim 2a, we prove that $M\backslash a$ is 3-connected. In Claim 2b, we suppose $M\backslash a$ is 3-connected but not weakly 4-connected.

**Claim 2a:** If $M\backslash a$ is 3-connected.

*Proof.* Suppose $M\backslash a$ is not 3-connected. Since $M$ is weakly 4-connected, $M\backslash a$ is 2-connected. Furthermore, $M\backslash a\backslash e = M\backslash e\backslash a = L_n^*\backslash a$ is 3-connected for $a \in E(L_n^*)$. Then by Proposition 7.2.1, there exists a circuit containing $e$ and has size fewer than 3 in $M\backslash a$. Let $\{e, x\}$ be a 2-element circuit in $M\backslash a$ where $x \in E(L_n^*) - a$. However, any circuit in $M\backslash a$ is also a circuit in $M$. If $\{e, x\}$ is a circuit in $M$, then $M$ is not weakly 4-connected. This is a contradiction. Therefore $M\backslash a$ is 3-connected. □

**Claim 2b:** $M\backslash a$ is 3-connected but not weakly 4-connected, then a 3-separation $(X \cup e, Y)$, $|X \cup e|$, $|Y| \geq 5$, in $M\backslash a$ induces a 3-separation $(X, Y)$ in $L_n^*\backslash a$.

*Proof.* Suppose $M\backslash a$ is 3-connected but not weakly 4-connected and $(X \cup e, Y)$ is a 3-separation in $M\backslash a$ where $|X \cup e|$, $|Y| \geq 5$. Since $M$ is weakly 4-connected, $\lambda_M(X \cup e) \neq 2$. Otherwise $(X \cup e, Y \cup a)$ induces a 3-separation $(X, Y)$ in $M$. Therefore the following statement holds:

i) $\lambda_M(X \cup e) = r_M(X \cup e) + r_M(Y \cup a) - r(M) = 3$ and $a \notin cl_M(Y)$.

Then $\lambda_{L_n^*}(Y) = 3$ because $L_n^*$ is internally 4-connected and $|X| \geq 4$, $|Y| \geq 5$. Therefore, the following statement holds:

ii) $\lambda_{L_n^*}(Y \cup a) = r_{L_n^*}(X) + r_{L_n^*}(Y \cup a) - r(L_n^*) = 3$ and $e \in cl_M(X)$.

By (i), $a \notin cl_M(Y)$ implies that $a \notin cl_{L_n^*}(Y)$. Therefore $r_{L_n^*}(Y \cup a) = r_{L_n^*}(Y) + 1$. Finally, the last statement holds:

(iii) $\lambda_{L_n^*\backslash a}(X) = r_{L_n^*\backslash a}(X) + r_{L_n^*\backslash a}(Y) - r(L_n^*\backslash a) = 2$.

Therefore if $M\backslash a$ is not weakly 4-connected, then the 3-separation $(X \cup e, Y)$, $|X \cup e|$, $|Y| \geq 5$, induces a 3-separation in $L_n^*\backslash a$. □

**Claim 3:** If $cl_{L_n^*}(X_i)$, $cl_{L_n^*}(Y_i) \notin \mathcal{M}$ (resp. $cl_{L_n^*}(X_i')$, $cl_{L_n^*}(Y_i') \notin \mathcal{M}$), then $M\backslash a_i$ (resp. $M\backslash c_i$)) is weakly 4-connected.

*Proof.* Suppose to the contrary that there exists $i \in \{1, \cdots, n\}$ such that $cl_{L_n^*}(X_i)$, $cl_{L_n^*}(Y_i) \notin \mathcal{M}$ but $M \backslash a_i$ is not weakly 4-connected. If $M \backslash a_i$ is not weakly 4-connected, then by Claim 2a, $M \backslash a_i$ must be 3-connected but not weakly 4-connected. Thus, there exists a 3-separation $(X \cup e, Y)$ in $M \backslash a_i$ such that $|X \cup e|$, $|Y| \geq 5$. From Claim 2b, two statements hold:

(i) $e \in cl_M(X)$, which implies $cl_{L_n^*}(X) \in \mathcal{M}$.

(ii) $(X, Y)$ induces a 3-separation in $L_n^* \backslash a_i$.

From (ii), we know $(X, Y)$ induces a 3-separation in $L_n^* \backslash a_i$. However, $L_n^* \backslash a_i = (L_n / a_i)^*$ and thus $(X, Y)$ must induce a 3-separation in $L_n / a_i$ as well. Without loss of generality, suppose $X = X_i$, $X_i - c_{i-1}$, or $X_i - c_{i+1}$. If $X = X_i$, then by (i), $cl_{L_n^*}(X_i) \in \mathcal{M}$, a contradiction. If $X = X_i - c_{i-1}$ or $X_i - c_{i+1}$, then by (i), $cl_{L_n^*}(X_i - c_{i-1})$ or $cl_{L_n^*}(X_i - c_{i+1}) \in \mathcal{M}$. However, $X_i - c_{i-1}$, $X_i - c_{i+1} \subset X_i$ and since $\mathcal{M}$ is a modular cut, this implies $X_i \in \mathcal{M}$. This is a contradiction. Therefore $M \backslash a_i$ must be weakly 4-connected. □

The flow of the proof is similar to the proof of Lemma 7.2.2. In $L_n$, we have $a_i \in cl_{L_n}(X_i)$ and $a_i \in cl_{L_n}(Y_i)$ for all $i \in \{1, \cdots, n\}$. In $L_n^*$, we have $a_i \notin cl_{L_n^*}(X_i)$ and $a_i \notin cl_{L_n^*}(Y_i)$. Thus, there are some differences between the proof of Lemma 7.2.2 and Lemma 7.2.4. In the rest of the proof, for $cl_{L_n^*}(X)$, we omit $L_n^*$ and write $cl(X)$. When we take closure of a set in a different matroid, say $M$, we will denote it properly as $cl_M(X)$ in the proof.

**Claim 4:** There exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Then $cl(X_2)$, $cl(X_3)$ and $cl(X_4) \in \mathcal{M}$. Note that $r(cl(X_i)) = 3$ for all $i$. Also $r(cl(X_2) \cup cl(X_3)) = r(cl(\{b_1, b_2, b_3, c_1, c_2, c_3, c_4\})) = 4$. Lastly, $r(cl(X_2) \cap cl(X_3)) = r(\{b_3, c_2, c_3\}) = 2$. Therefore $(cl(X_2),cl(X_3))$ is a modular pair and thus, $\{b_3, c_2, c_3\} \in \mathcal{M}$. Also, $(\{b_3, c_2, c_3\},cl(X_4))$ is a modular pair as well and $\{b_3, c_2, c_3\} \cap cl(X_4) = \{c_3\} \in \mathcal{M}$. This is a contradiction since $M$ is weakly 4-connected. Therefore there exist some $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$. □

**Claim 5:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$ but $cl(X_{i-1}) \in \mathcal{M}$, then $M \backslash c_{i-1}$ is weakly 4-connected.

*Proof.* Suppose to the contrary that there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$ but $cl(X_{i-1})$, $cl(Y_i) \in \mathcal{M}$ but $M \backslash c_{i-1}$ is not weakly 4-connected. Then $(cl(X_{i-1}), cl(Y_i))$ is a modular pair and thus $cl(X_{i-1}) \cap cl(Y_i) = \{b_{i-1}, c_{i-2}, c_{i-1}\} \in \mathcal{M}$. Since $M \backslash c_{i-1}$ is not weakly 4-connected, $cl(X'_{i-1}) \in \mathcal{M}$ or $cl(Y'_{i-1}) \in \mathcal{M}$. If $cl(X'_{i-1}) \in \mathcal{M}$, then $(cl(X'_{i-1}),\{b_{i-1}, c_{i-2}, c_{i-1}\})$ is a modular pair. Therefore $cl(X'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{b_{i-1}\} \in \mathcal{M}$, which implies $M$ is not weakly 4-connected which leads to a contradiction. If $cl(Y'_{i-1}) \in \mathcal{M}$, then $(cl(Y'_{i-1}),\{b_{i-1}, c_{i-2}, c_{i-1}\})$ is a modular pair. Therefore $cl(Y'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{c_{i-1}\} \in \mathcal{M}$, which also leads to a contradiction. Therefore $cl(X'_{i-1})$, $cl(Y'_{i-1}) \notin \mathcal{M}$ and $M \backslash c_{i-1}$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ then either there exists $j$ such that $cl(Y_j) \notin \mathcal{M}$ or $M \cong P_n$ where $P_n \backslash a$ is not weakly 4-connected for all $a \in E(L^*_n)$.

*Proof.* If there exists $cl(Y_j) \notin \mathcal{M}$ for some $j \in \{1, \cdots, n\}$ then we are done as $cl(X_j)$, $cl(Y_j) \notin \mathcal{M}$ implies $M \backslash a_j$ is weakly 4-connected. Note that for $L^*_n$, we have $cl(Y_i) = cl(Y'_i) = E(L^*_n) - \{a_i, b_i, b_{i+1}, c_i\}$ for all $i \in \{1, \cdots, n\}$. This implies that if $cl(Y_i) \in \mathcal{M}$ then $cl(Y'_i) \in \mathcal{M}$.

Suppose that $cl(Y_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Then the following statements holds for $k \geq 3$:

(i) $\bigcap_{i=3}^{k} cl(Y_i) = E(L^*_n) - \{a_2, \cdots, a_k, b_2, \cdots, b_{k+1}, c_2, \cdots, c_k\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i)) = r - k + 1$.

(ii) $\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1}) = E(L^*_n) - c_{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r$.

(iii) $\bigcap_{i=3}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r - k$.

From (i), (ii), and (iii) we can see that for each $k \in \{3, \cdots, n\}$, $\bigcap_{i=3}^{k} cl(Y_j)$ and $cl(Y_{k+1})$ is a modular pair and thus, $\bigcap_{i=2}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1} cl(Y_i) \in \mathcal{M}$. Therefore $\bigcap_{i=2}^{n} cl(Y_j) = \{a_1, c_1\} \in \mathcal{M}$. This applies for any intersection of $n-1$ number of $cl(Y_i)$ due to the symmetry of $L_n^*$. Thus $\{a_i, c_i\} \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. In this case, $M \cong P_n$. Therefore for all $i \in \{1, \cdots, n\}$, neither $M \backslash a_i$ nor $M \backslash c_i$ are weakly 4-connected. It is also not too difficult to check that $M \backslash b_i$ is not weakly 4-connected. Therefore if $cl(X_i) \notin \mathcal{M}$ for all $i$, then either there exists $j \in \{1, \cdots, n\}$ such that $M/a_j$ is weakly 4-connected or $M \cong P_n$. $\square$

By Claim 6, $P_n \backslash a$ is not weakly 4-connected for all $a \in E(L_n^*)$. Furthermore, if $P_n = L_n^* +_{\mathcal{M}} e$, then $\{a_i, c_i\} \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ as well. Now, we prove that $P_n$ is the only weakly 4-connected single element extension of $L_n^*$ such that $P_n \backslash a$ is not weakly 4-connected for all $a \in E(L_n^*)$ and $\{a_i, c_i, e\}$ is a triangle.

**Claim 7:** $P_n$ is the only weakly 4-connected single element extension of $L_n^*$ such that $P_n \backslash a$ is not weakly 4-connected for all $a \in E(L_n^*)$.

*Proof.* Suppose that $\mathcal{M}$ is a modular cut of $L_n^*$ such that $cl(Y_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ and $M \cong L_n^* +_{\mathcal{M}} e$. Then by Claim 6, $\{a_i, c_i\} \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ and $M \cong P_n$. We will prove that there does not exist any other modular cut of $L_n^*$, say $\mathcal{M}_1$, such that $\{a_i, c_i\}$ for all $i \in \{1, \cdots, n\}$ and $\mathcal{M}_1 \neq \mathcal{M}$. Let $M_1 = L_n^* +_{\mathcal{M}_1} e$. We will show that $M_1$ is not 3-connected and thus, Claim 7 holds.

Let $F$ be a flat of $L_n^*$ such that $F \in \mathcal{M}_1$ but $F \notin \mathcal{M}$. Then $\{a_i, c_i\} \nsubseteq F$ for all $i \in \{1, \cdots, n\}$. If $\{a_i, c_i\} \subseteq F$ for some $i \in \{1, \cdots, n\}$, then $F \in \mathcal{M}$ because $\mathcal{M}$ is a modular cut. Therefore $|\{a_i, c_i\} \cap F| = 1$ or 0 for all $i$.

Suppose there exists $i \in \{1, \cdots, n\}$ such that $|\{a_i, c_i\} \cap F| = 1$. Assume without loss of generality that $a_i \in F$ and $c_i \notin F$. Then $r(F \cup \{a_i, c_i\}) = r(F)$ or $r(F) + 1$. If $r(F \cup \{a_i, c_i\}) = r(F) + 1$, then $(F, \{a_i, c_i\})$ is a modular pair, and $F \cap \{a_i, c_i\} = \{a_i\} \in \mathcal{M}_1$. This implies $M_1$ is not 3-connected. If $r(F \cup \{a_i, c_i\}) = r(F)$, then $\{a_i, c_i\} \subseteq F$. This contradicts our assumption that $\{a_i, c_i\} \nsubseteq F$. Therefore there does not exist $i$ such that $|\{a_i, c_i\} \cap F| = 1$. Thus $\{a_i, c_i\} \cap F = \emptyset$ for all $i$. This implies that $F \subseteq \{b_1, \cdots, b_n\}$. Since $F$ is a flat, $r(F \cup a_i) = r(F) + 1$ and $r(F \cup c_i) = r(F) + 1$ for all $i$. If $r(F \cup \{a_i, c_i\}) = r(F) + 1$, then there exists a circuit $C \subseteq F \cup \{a_i, c_i\}$ such that $\{a_i, c_i\} \subseteq C$. However, there does not exist a circuit in $L_n^*$ where the circuit only contains $\{a_i, c_i\}$ and some elements from the set $\{b_1, \cdots, b_n\}$. Therefore $r(F \cup \{a_i, c_i\}) = 2$. Then $(F, \{a_i, c_i\})$ is a modular pair. This implies $F \cap \{a_i, c_i\} = \emptyset \in \mathcal{M}_1$, and $M_1$ is not 3-connected. Therefore Claim 7 is true. □

By Claim 1, $P_n$ is weakly 4-connected. Now, we prove $P_n$ is also internally 4-connected.

**Claim 8:** $P_n$ is internally 4-connected.

*Proof.* Suppose to the contrary that $P_n$ is not internally 4-connected. Since $P_n$ is weakly 4-connected by Claim 1, there exists a 3-separation $(X \cup e, Y)$ such that $|X \cup e|, |Y| \geq 4$. Since $L_n^*$ is internally 4-connected, $r_{L_n^*}(X) + r_{L_n^*}(Y) - r(L_n^*) = 2$. This implies that $e \in cl_{P_n}(X)$ and $|X| = 3$. Thus $X$ is one of the triangles in $L_n^*$. However, $e \in cl_{P_n}(X)$ implies $cl(X) \in \mathcal{M}$ and $\{a_i, c_i\} \subseteq X$ for some $i \in \{1, \cdots, n\}$. This contradicts the assumption that $X$ is a triangle in $L_n^*$ as $\{a_i, c_i, x\}$ can not be a triangle in $L_n^*$ for any $x \in E(L_n^*)$ for all $i \in \{1, \cdots, n\}$. Therefore $P_n$ is internally 4-connected. □

Suppose $M$ is a weakly 4-connected single element extension of $L_n^*$ such that $M \ncong P_n$. Then there exists element $a \in E(L_n^*)$ such that $M \backslash a$ is weakly 4-connected. If $M \backslash a$ or $(M \backslash a)^*$ is isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for some $n \geq 4$, it must be the case that $M \backslash a \cong L_n^*$ or $(L^n)^*$. If $m \neq n$, then the rank or the number of element does not match. Suppose there exists $a_i \in E(L_n^*)$ such that $M \backslash a_i$ is weakly 4-connected and $M \backslash a_i \cong L_n^*$. Since $\{a_{i-1}, a_i, b_i\}$ and $\{a_i, a_{i+1}, b_{i+1}\}$ is a triangle in $L_n^*$, if $M \backslash a_i \cong L_n^*$, then $\{a_{i-1}, e, b_i\}$ and $\{e, a_{i+1}, b_{i+1}\}$ is a triangle in $M \backslash a_i$ and $M$. Then $cl(\{a_{i-1}, b_i\}) = \{a_{i-1}, a_i, b_i\}$ and $cl(\{a_{i+1}, b_{i+1}\}) = \{a_i, a_{i+1}, b_i\} \in \mathcal{M}$. However, two sets are modular pairs. Therefore $\{a_i\} \in \mathcal{M}$, a contradiction as $M$ is weakly 4-connected. Therefore if $M \backslash a_i$ is weakly 4-connected, $M \backslash a_i \neq L_n^*$. For the same reason, $M \backslash a_i \neq (L^n)^*$ as well. □

Now, we prove results similar to Lemma 7.2.2 and 7.2.4 for $P_n$. As for the coextension of $P_n$, we look at the extension of $P_n^*$ instead.

**Lemma 7.2.5.** *If $N$ is a 3-connected extension of $P_n$, $n \geq 4$, then $N$ is weakly 4-connected. Moreover, there exists an element $a \in E(L_n)$ such that $N\backslash a$ and $(N\backslash a)^*$ are weakly 4-connected and are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$.*

*Proof.* Let $\mathcal{N}$ be a modular cut of $P_n$, $n \geq 4$ and let $N$ be a 3-connected extension of $P_n$ such that $N = P_n +_{\mathcal{N}} f$. Let $r$, denote the rank of $P_n$. Then $r(N) = r(P_n) = r(L_n^*) = n+1$. For Lemmas 7.2.5 and 7.2.6, let $X_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E(L_n^*) \subseteq E(P_n)$, for all $i \in \{1, \cdots, n\}$. And let $Y_i = E(P_n) - (X_i \cup a_i)$. Also, let $X_i' = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E(L_n^*)$ and $Y_i' = E(L_n^*) - (X_i' \cup a_i)$.

**Claim 1:** $N$ is weakly 4-connected.

*Proof.* Since $P_n$ is internally 4-connected, if $N$ is not weakly 4-connected,then there exists a 3-separation, $(X \cup f, Y)$ in $N$ such that $|X \cup f|, |Y| \geq 5$. However, this separation induces a 3-separation $(X, Y)$ in $P_n$ where $|X| \geq 4$ and $|Y| \geq 5$, a contradiction as $P_n$ is internally 4-connected. Therefore $N$ is weakly 4-connected. □

Suppose $N\backslash a$ is not weakly 4-connected for some $a = a_i \in E(L_n^*)$. Since $N$ is weakly 4-connected, either $N\backslash a$ is 2-connected but not 3-connected or $N\backslash a$ is 3-connected but not weakly 4-connected.

**Claim 2a:** If $N\backslash a$ is 3-connected.

*Proof.* Note that $N\backslash a$ is 2-connected but $N\backslash a\backslash f = N\backslash f\backslash a = P_n\backslash a$. From Lemma 7.2.4 Claim 2a, we already proved that for any weakly 4-connected extension of $L_n^*$, any single element deletion is still 3-connected. Therefore $P_n\backslash a$ is 3-connected as well. Then by Proposition 7.2.1, there exists a circuit containing $f$ and has size fewer than 3 in $N\backslash a$. Let $\{f, x\}$ be a 2-element circuit in $N\backslash a$ where $x \in E(P_n) - a$. However, any circuit in $N\backslash a$ is also a circuit in $P_n$ as well and we have a contradiction. If $\{e, x\}$ is a circuit in $P_n$, then $P_n$ can not be internally 4-connected. Therefore $N\backslash a$ is 3-connected. $\square$

**Claim 2b:** If $N\backslash a$ is 3-connected but not weakly 4-connected, then a 3-separation $(X \cup f, Y)$, where $|X \cup f|$, $|Y| \geq 5$, in $N\backslash a$ induces a 3-separation $(X - e, Y - e)$ in $L_n^*\backslash a$.

*Proof.* Suppose $N\backslash a$ is 3-connected but not weakly 4-connected and $(X \cup f, Y)$ is a 3-separation in $N\backslash a$ such that $|X \cup f|$, $|Y| \geq 5$. Since $N$ is weakly 4-connected, $\lambda_N(X \cup f) \neq 2$ as $(X \cup f, Y \cup a)$ induces a 3-separation and $|X \cup f| \geq 5$, $|Y \cup a| \geq 6$. Therefore, the following statement holds:

(i) $\lambda_N(X \cup f) = r_N(X \cup f) + r_N(Y \cup a) - r(N) = 3$ and $a \notin cl_N(Y)$.

Then $\lambda_{P_n}(Y \cup a) = 3$ because $P_n$ is internally 4-connected and $|X| \geq 4$, $|Y \cup a| \geq 6$. Therefore, the following statement holds:

(ii) $\lambda_{P_n}(Y \cup a) = r_{P_n}(X) + r_{P_n}(Y \cup a) - r(P_n) = 3$ and $f \in cl(X)$.

By (i), $a \notin cl_N(Y)$ and this implies $a \notin cl_{P_n}(Y)$. Therefore $r_{P_n}(Y \cup a) = r_{P_n}(Y) + 1$. Finally, the last statement holds:

(iii) $\lambda_{P_n\backslash a}(X) = r_{P_n\backslash a}(X) + r_{P_n\backslash a}(Y) - r(P_n\backslash a) = 2$.

Note that $(X, Y)$ is a 3-separation of $P_n\backslash a$ where $|X| \geq 4$ and $|Y| \geq 5$. If $|X| = 4$, then $P_n\backslash a$ is weakly 4-connected. This contradicts Lemma 7.2.4 as $P_n\backslash a$ is not weakly 4-connected for all $a \in E(L^n)$. Therefore $|X| \geq 5$. Since $|X|$, $|Y| \geq 5$ and $e \in X$ or $e \in Y$, we have $|X - e|$, $|Y - e| \geq 4$. In this case, by Lemma 7.2.4 Claim 2b, $(X - e, Y - e)$ induces a 3-separation in $L_n^*$. $\square$

**Claim 3:** If $cl(X_i)$, $cl(Y_i) \notin \mathcal{N}$ (resp. $cl(X_i')$, $cl(Y_i') \notin \mathcal{N}$), then $N\backslash a_i$ (resp. $N\backslash c_i$) is weakly 4-connected.

*Proof.* Suppose to the contrary that there exists $i \in \{1, \cdots, n\}$ such that $cl_{P_n}(X_i)$, $cl_{P_n}(Y_i) \notin \mathcal{N}$ but $N\backslash a_i$ is not weakly 4-connected. If $N\backslash a_i$ is not weakly 4-connected, then by Claim

2a, $N\backslash a_i$ must be 3-connected but not weakly 4-connected. Thus, there exists a 3-separation $(X \cup f, Y)$ in $N\backslash a_i$ such that $|X \cup f|$, $|Y| \geq 5$. From Claim 2b, two statements hold:

(i) $f \in cl_{P_n}(X)$, which implies $cl_{P_n}(X) \in \mathcal{N}$.

(ii) $(X - e, Y - e)$ induces a 3-separation in $L_n^*\backslash a_i$ where $|X - e|$, $|Y - e| \geq 4$.

From (ii), we know $(X - e, Y - e)$ induces a 3-separation in $L_n^*\backslash a_i$. Without loss of generality, suppose $X - e = X_i$, $X_i - c_{i-1}$, or $X_i - c_{i+1}$. If $X - e = X_i$, then by (i), $cl_{P_n}(X_i) \in \mathcal{N}$, a contradiction. If $X - e = X_i - c_{i-1}$ or $X_i - c_{i+1}$, again, by (i), $cl_{P_n}(X_i - c_{i-1})$ or $cl_{P_n}(X_i - c_{i+1}) \in \mathcal{N}$. However, $X_i - c_{i-1}$, $X_i - c_{i+1} \subset X_i$ and since $\mathcal{N}$ is a modular cut, this implies $cl_{P_n}(X_i) \in \mathcal{N}$, a contradiction as well. Therefore $N\backslash a_i$ must be weakly 4-connected. □

In the rest of the proof of Lemma 7.2.5, when we write $cl(X)$, we mean $cl_{P_n}(X)$. When we take closure of a set in a different matroid, say $N$, we will denote it properly as $cl_N(X)$ in the proof.

**Claim 4:** There exists $i$ such that $cl(X_i) \notin \mathcal{N}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Then $cl(X_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Therefore $cl(X_2)$, $cl(X_3)$ and $cl(X_4) \in \mathcal{M}$. Note that $f \notin cl(X_i)$ for all $i$ since $cl(X_i)$ does not contain $a_i$ and $\{a_i, c_i, f\}$ is a triangle for all $i \in \{1, \cdots, n\}$ in $P_n$. Therefore $cl(X_i) = cl_{L_n^*}(X_i)$ and we can apply the proof of the Claim 4 of Lemma 7.2.4 to prove this claim. □

**Claim 5:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1}) \in \mathcal{N}$, then $N/c_{i-1}$ is weakly 4-connected.

*Proof.* Suppose there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1}) \in \mathcal{N}$. Then $(cl(X_{i-1}), cl(Y_i))$ is a modular pair and thus, $cl(X_{i-1}) \cap cl(Y_i) = \{b_{i-1}, c_{i-2}, c_{i-1}\} \in \mathcal{N}$. Now, suppose to the contrary that $N\backslash c_{i-1}$ is not weakly 4-connected. Then $cl(X'_{i-1}) \in \mathcal{N}$ or

$cl(Y'_{i-1}) \in \mathcal{N}$. If $cl(X'_{i-1}) \in \mathcal{N}$, then ($cl(X'_{i-1})$,$\{b_{i-1}, c_{i-2}, c_{i-1}\}$) is a modular pair. Therefore $cl(X'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{b_{i-1}\} \in \mathcal{N}$, which implies $N$ is not weakly 4-connected. This is a contradiction. If $cl(Y'_{i-1}) \in \mathcal{N}$, then ($cl(Y'_{i-1})$,$\{b_{i-1}, c_{i-2}, c_{i-1}\}$) is a modular pair. Therefore $cl(Y'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{c_{i-1}\} \in \mathcal{N}$, which also creates a contradiction. Therefore both $cl(X'_{i-1})$, $cl(Y'_{i-1}) \notin \mathcal{N}$ and thus $N \backslash c_{i-1}$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{N}$ for all $i \in \{1, \cdots, n\}$, then there exists $j$ such that $cl(Y_j) \in \mathcal{N}$ and $N \backslash a_j$ is weakly 4-connected.

*Proof.* Suppose to the contrary that $cl(Y_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Then, $r(Y_i) = r - 1$ and the following statements hold.

(i) $\bigcap_{i=3}^{k} cl(Y_i) = E(P_n) - \{a_2, \cdots, a_k, b_2, \cdots, b_{k+1}, c_2, \cdots, c_k\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i)) = r - k + 1$.

(ii) $\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1}) = E(L_n^*) - c_{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r$.

(iii) $\bigcap_{i=3}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r - k$.

From (i), (ii), and (iii) we can see that for each $k \in \{3, \cdots, n\}$, ($\bigcap_{i=3}^{k} cl(Y_j)$,$cl(Y_{k+1})$) is a modular pair and thus, $\bigcap_{i=2}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1} cl(Y_i) \in \mathcal{N}$. Therefore $\bigcap_{i=2}^{n} cl(Y_j) = \{a_1, c_1, e\} \in \mathcal{N}$ as well. Now, consider the set $\{a_1, c_1, e\}$ and $cl(Y_1)$. Two sets form a modular pair and thus their intersection $\{e\} \in \mathcal{N}$. This is a contradiction as $N$ is weakly 4-connected. Therefore there exists some $j \in \{1, \cdots, n\}$ such that $cl(Y_j) \notin \mathcal{N}$ and $N \backslash a_j$ is weakly 4-connected. □

Suppose $N \backslash a_i$ is weakly 4-connected. Then we need to make sure $N \backslash a_i$ and $(N \backslash a_i)^*$ are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$. By comparing the rank and the number of elements, we only need to check if $N \backslash a_i$ is not isomorphic to $P_n$ or $P^n$. Note that $\{a_i, a_{i+1}, b_{i+1}\}$ is a triangle for all $i \in \{1, \cdots, n\}$ in $P_n$. Also, every element $a_i$ is in two such triangles as $\{a_{i-1}, a_i, b_i\}$ is also a triangle. Therefore if $N \backslash a_i \cong$, then $\{f, a_{i+1}, b_{i+1}\}$ and $\{a_{i-1}, f, b_i\}$ must be two triangles in $N \backslash a_i$. Since $N$ is an extension of $P_n$, the two set $\{f, a_{i+1}, b_{i+1}\}$ and $\{a_{i-1}, f, b_i\}$ must be also a triangle in $N$ as well. If $\{a_i, f\}$ is not in parallel, by applying the CEA to $\{a_i, a_{i+1}, b_{i+1}\}$ and $\{f, a_{i+1}, b_{i+1}\}$ we see that $\{a_{i+1}, f, b_{i+1}\}$ must be a triangle in $N$ as well. This implies that $\{a_{i+1}, b_{i+1}\} \in \mathcal{N}$ and by the construction of $P_n$, we also have $\{a_{i+1}, c_{i+1}\} \in \mathcal{N}$ as well. Two sets form a modular pair and this forces $\{a_{i+1}\} \in \mathcal{N}$, a contradiction as $N$ is weakly 4-connected matroid. Therefore if $N/a_i \cong P_n$, then $\{a_i, f\}$ is in parallel, which again, is a contradiction. Thus, $N\backslash a_i \not\cong P_n$. For the same reason, $N\backslash a_i \not\cong P^n$ as well. Furthermore, if $N\backslash a_i$ has a $L_n^*$-minor, then $N\backslash a_i$ must be one of weakly 4-connected extension of $L_n^*$ which is not isomorphic to $P_n$ and in this case, by Lemma 7.2.4, there exists an element $b \in E(L_n^*)$ such that $N\backslash a_i\backslash b$ is weakly 4-connected. This complete the proof of Lemma 7.2.5. □

**Lemma 7.2.6.** *If $N$ is a 3-connected extension of $P_n^*$, then $N$ is weakly 4-connected and there exists an element $a \in E(L_n)$ such that $N/a$ and $(N/a)^*$ are weakly 4-connected and are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$.*

*Proof.* Let $\mathcal{N}$ be a modular cut of $P_n^*$, $n \geq 4$ and let $N$ be a 3-connected extension of $P_n^*$ such that $N = P_n^* +_{\mathcal{N}} e$. Note that $P_n^*$ is a coextension of $L_n$ and $N^*$ is a coextension of $P_n$. Let $r$, denote the rank of $P_n^*$. For this lemma, let $X_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E(L_n) \subseteq E(P_n^*)$, for all $i \in \{1, \cdots, n\}$. And let $Y_i = E(P_n^*) - (X_i \cup a_i)$. Also, let $X_i' = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E(L_n)$ and $Y_i' = E(L_n) - (X_i' \cup a_i)$.

**Claim 1:** $N$ is weakly 4-connected.

*Proof.* Suppose $N$ is not weakly 4-connected. Therefore there exists a 3-separation, $(X \cup f, Y)$ in $N$ where $|X \cup f|, |Y| \geq 5$. This separation induces either a 2-separation or a 3-separation $(X, Y)$ in $P_n^*$. However $P_n$ is internally 4-connected. Therefore $P_n^*$ is internally 4-connected as well, and thus can not have a 3-separation where both sides of the separation have cardinality at least 4. Therefore $N$ is weakly 4-connected. □

Now, suppose $N/a$ is not weakly 4-connected for some $a = a_i \in E(L_n^*)$. Then either $N/a$ is 2-connected but not 3-connected, or it is 3-connected but not weakly 4-connected.

**Claim 2a:** If $N/a$ is 2-connected but not 3-connected, then there exists $x \in E(L_n^*) - a$ such that $\{f, a, x\}$ is a triangle in $P_n$.

In the proof of Claim 2a of Lemma 7.2.2, we only used the fact that $L_n$ is internally 4-connected and $M$ is weakly 4-connected single element extension of $M$. We did not use any properties of $L_n$ in the proof. Thus, we can use the same proof in this claim as well.

**Claim 2b:** If $N/a$ is 3-connected but not weakly 4-connected for $a = a_i \in E(L_n)$, then a 3-separation $(X \cup f, Y)$ where $|X \cup f|$, $|Y| \geq 5$, in $N/a$ induces a 3-separation $(X - e, Y - e)$ in $L_n/a$.

*Proof.* Suppose $N/a$ is not weakly 4-connected and $(X \cup f, Y)$ is a 3-separation where $|X \cup f|$, $|Y| \geq 5$. Then, $\lambda_{N/a}(X \cup f) = 2$ and $\lambda_N(X \cup f) = 3$. If $\lambda_N(X \cup f) = 2$, then $(X \cup f, Y \cup a)$ is a 3-separation of $N$ where $|X \cup f|$, $|Y \cup a| \geq 5$. Since $N$ is weakly 4-connected, this is a contradiction. Therefore $\lambda_N(X \cup f) = 3$ and the following statement holds:

(i) $\lambda_N(Y) = r_N(X \cup f \cup a) + r_N(Y) - r(N) = 3$ and $a \in cl_N(Y)$.

Since $|X \cup a|$, $|Y| \geq 4$ and $P_n^*$ is internally 4-connected, $\lambda_{P_n^*}(Y) \neq 2$. Thus following statement holds:

(ii) $\lambda_{P_n^*}(Y) = r_{P_n^*}(X \cup a) + r_{P_n^*}(Y) - r(P_n^*) = 3$ and $f \in cl_N(X \cup a)$.

Since $N$ is a single element extension of $P_n^*$ and $a \in cl_N(Y)$, by (i), $a \in cl_{P_n^*}(Y) = cl_N(Y) - e$. Therefore, $r_{P_n^*}(Y) = r_{P_n^*}(Y \cup a)$. Also, by (i), $r_{P_n^*}(X \cup e \cup a) = r_{P_n^*}(X \cup a)$. Therefore $r_{P_n^*}(X \cup a) + r_{P_n^*}(Y \cup a) - r(P_n^*) = 3$ as well. If $r_{P_n^*}(X \cup a) \neq r_{P_n^*}(X)$, then $r_{P_n^*}(X) + r_{P_n^*}(Y \cup a) - r(P_n^*) = 2$ where $|X|$, $|Y| \geq 4$. This is a a contradiction as $P_n^*$ is internally 4-connected. Therefore the following statement holds:

(iii) $\lambda_{P_n^*}(X) = r_{P_n^*}(X) + r_{P_n^*}(Y \cup a) - r(P_n^*) = 3$ and $a \in cl_{P_n^*}(X)$.

By (i) and (iii), we can deduce that $a \in cl_{P_n^*}(X)$ and $cl_{P_n^*}(Y)$. Therefore $r_{P_n^*/a}(X) = r_{P_n^*}(X \cup a) - 1 = r_{P_n^*}(X) - 1$ and similarly, $r_{P_n^*/a}(Y) = r_{P_n^*}(Y) - 1$ as well. Therefore $\lambda_{P_n^*/a}(X) = 2$ and $(X, Y)$ is a 3-separation in $P_n^*/a$ such that $|X| \geq 4$ and $|Y| \geq 5$. Note that $P_n^*/a = (P_n\backslash a)^*$, and in Lemma 7.2.4, we already proved that $P_n\backslash a$ is not weakly 4-connected for all $a \in E(L_n^*)$. Therefore if $|X| = 4$, then we obtain a contradiction. Therefore $|X| \geq 5$ and $(X, Y)$ is a 3-separation of $P_n^*/a$ where $|X|$, $|Y| \geq 5$. Also, $(X, Y)$ is a 3-separation of $P_n\backslash a$. From Lemma 7.2.4, this separation induces a 3-separation $(X - e, Y - e)$ in $L_n^*\backslash a$ and this, in turn, induces a 3-separation $(X - e, Y - e)$ in $L_n/a$. □

Using Claim 2a and 2b, we provide necessary conditions for $N/a_i$ to be weakly 4-connected in Claim 3.

**Claim 3:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i)$, $cl(Y_i) \notin \mathcal{N}$ (resp. $cl(X_i')$, $cl(Y_i') \notin \mathcal{N}$), then $N/a_i$ (resp. $N/c_i$) is weakly 4-connected.

*Proof.* Suppose to the contrary that $cl(X_i)$, $cl(Y_i) \notin \mathcal{N}$ but $N/a_i$ is not weakly 4-connected. Then $N/a_i$ is not 3-connected or $N/a_i$ is 3-connected but not weakly 4-connected. First, if $N/a_i$ is 2-connected but not 3-connected, then by Claim 2a, $\{f, a_i, x\}$ must be a triangle in $N$ for some $x \in E(P_n^*)$ where $x \neq a_i$. Therefore $\{a_i, x\} \in \mathcal{N}$, as otherwise, $\{f, a_i, x\}$ would not be a triangle in $N$. Note that $a_i \in cl(X_i)$ and $a_i \in cl(Y_i)$. Also $x \in cl(X_i)$ or $x \in cl(Y_i)$ as $cl(X_i) \cup cl(Y_i) = E(L_n^*)$. Without the loss of generality, suppose $x \in X_i$. Then $\{a_i, x\} \subset cl(X_i)$ which implies $cl(X_i) \in \mathcal{N}$ a contradiction. Therefore $N/a_i$ must be 3-connected.

If $N/a_i$ is 3-connected but not weakly 4-connected, then there exists a 3-separation $(X \cup f, Y)$ in $N/a_i$ such that $|X \cup f|$, $|Y| \geq 5$. From Claim 2b, we know $e \in cl_N(X \cup a)$ and $a \in cl_{P_n^*}(X)$. This implies $a \in cl_N(X)$ and $e \in cl_N(X \cup a) = cl_N(X)$. Thus $cl_{P_n^*}(X) \notin \mathcal{N}$. Since $(X - e, Y - e)$ induces a 3-separation in $L_n/a_i$, either $X - e$ or $Y$ is $X_i$, $(X_i - c_{i-1})$, or $(X_i - c_{i+1})$. If $X - e = X_i$, then, $cl_{L_n^*}(X_i) \in \mathcal{N}$, a contradiction. If $X - e = X_i - c_{i-1}$ or $X - e = X_i - c_{i+1}$, again, $cl_{L_n^*}(X_i - c_{i-1}) \in \mathcal{N}$ or $cl_{L_n^*}(X_i - c_{i+1}) \in \mathcal{N}$. However, $X_i - c_{i-1}$, $X_i - c_{i+1} \subset X_i$ and since $\mathcal{N}$ is a modular cut, this implies $X_i \in \mathcal{N}$, a contradiction as well. Therefore $N/a_i$ must be weakly 4-connected. □

In the rest of the proof of Lemma 7.2.6, when we write $cl(X)$, we mean $cl_{P_n^*}(X)$. When we take the closure of a set in a different matroid, say $N$, we will denote it properly as $cl_N(X)$ in the proof.

**Claim 4:** There exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Therefore $cl(X_2)$, $cl(X_3)$ and $cl(X_4) \in \mathcal{N}$. Note that $r(cl(X_i)) = 5$ for all $i$. Also, $r(cl(X_2) \cup cl(X_3)) = r(cl(\{a_2, a_3, b_2, b_3, b_4, c_1, c_2, c_3, c_4\})) = 7$. Lastly, $r(cl(X_2) \cap cl(X_3)) = r(cl(\{b_3, c_2, c_3\})) = 3$. Then $(cl(X_2),cl(X_3))$ is a modular pair and thus $cl(X_2) \cap cl(X_3) = \{b_3, c_2, c_3\} \in \mathcal{N}$. Also, $(\{b_3, c_2, c_3\},cl(X_4))$ is a modular pair and their intersection is in $\mathcal{N}$ as well. However, $\{b_3, c_2, c_3\} \cap cl(X_4) = \{c_4\}$. This implies that $f$ is in parallel with $c_4$ in $N$, which is a contradiction since $N$ weakly 4-connected. Therefore there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$. □

**Claim 5:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1}) \in \mathcal{N}$, then $N/a_i$, $N/c_i$, or $N/a_{i-3}$ is weakly 4-connected.

*Proof.* Now suppose that there exists some $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1}) \in \mathcal{N}$. Without loss of generality suppose that $i = 4$. Thus $cl(X_4) \notin \mathcal{N}$ but $cl(X_3) \in \mathcal{N}$. If $cl(Y_4) \notin \mathcal{N}$ as well, then we are done since $N/a_3$ will be weakly 4-connected.

Suppose that $cl(Y_4) \in \mathcal{N}$. Then $(cl(X_3),cl(Y_4))$ is a modular pairs and thus $cl(X_3) \cap cl(Y_4) = \{a_3, b_3, c_2\} \in \mathcal{N}$. In this case, we will show that $N/c_4$ or $N/a_1$ is weakly 4-connected. Suppose neither $N/c_4$ nor $N/a_1$ are weakly 4-connected. Therefore $cl(X_4') \in \mathcal{N}$ or $cl(Y_4') \in \mathcal{N}$. Similarly, $cl(X_1) \in \mathcal{N}$ or $cl(Y_1) \in \mathcal{N}$. If $cl(X_4') \in \mathcal{N}$ then $(cl(X_4'),\{a_3, b_3, c_2\})$ is a modular pair and thus their intersection, $\{a_3\} \in \mathcal{N}$. This contradicts the fact that $N$ is weakly 4-connected. Therefore $cl(Y_4')$ must be in $\mathcal{N}$. Also, $(cl(Y_4'),\{a_3, b_3, c_2\})$ is a modular pair so $\{b_3, c_2\} \in \mathcal{N}$. Similarly, if $cl(X_1) \in \mathcal{N}$ then we obtain contradiction as $(cl(X_1),\{a_3, b_3, c_2\})$ is a modular pair, which implies $\{c_2\} \in \mathcal{N}$. Therefore $cl(Y_1) \in \mathcal{N}$. Also,

($cl(Y_1)$,$\{a_3, b_3, c_2\}$) is a modular pair and thus, $\{a_3, b_2\} \in \mathcal{N}$. Now, because neither $N/a_1$ nor $N/c_4$ are weakly 4-connected, we showed $cl(Y_4')$ and $cl(Y_1) \in \mathcal{N}$ which implied $\{b_3, c_2\}$ and $\{a_3, b_2\} \in \mathcal{N}$. However, the two sets form a modular pair. Therefore $\{b_2\} \in \mathcal{N}$, a contradiction as $N$ is weakly 4-connected. Therefore we must have $cl(Y_4') \notin \mathcal{N}$ or $cl(X_1) \notin \mathcal{N}$ which implies $cl(X_4')$ and $cl(Y_4') \notin \mathcal{N}$ or $cl(X_1)$ and $cl(Y_1) \notin \mathcal{N}$. Then either $N/c_4$ is weakly 4-connected or $N/a_1$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{N}$ for all $i \in \{1, \cdots, n\}$, then there exists $j$, $k \in \{1, \cdots, n\}$ such that $N/a_j$ or $N/c_k$ is weakly 4-connected.

*Proof.* Since $cl(X_i) \notin \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. If there exists some $j \in \{1, \cdots, n\}$ such that $cl(Y_j) \notin \mathcal{N}$, then we are done as $M/a_j$ would be weakly 4-connected. Thus, suppose that $cl(Y_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. For $k \in \{4, \cdots, n\}$, the following statements hold:
(i) $\bigcap_{i=3}^{k} cl(Y_i) = E(P_n^*) - \{b_3, \cdots, b_{k+1}, c_2, \cdots, c_{i+1}\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i)) = r - 2 - (k-3) = r - k + 1$.
(ii) $\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1}) = E(P_n^*) - \{b_{k+1}, c_k, c_{k+3}\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r - 1$.
(iii) $\bigcap_{i=3}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{j=3}^{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cap cl(Y_{k+1})) = r - 2 - (k+1-3) = r - k$.
From (i), (ii), and (iii) we can deduce that ($\bigcap_{i=3}^{k} cl(Y_i)$,$cl(Y_{k+1})$) is a modular pair for each $k \in \{4, \cdots, n\}$, and thus, $\cap_{j=3}^{k+1} cl(Y_j) \in \mathcal{M}$. Therefore $\cap_{i=3}^{n} cl(Y_i) = \{a_1, \cdots, a_n, b_2, e\} \in \mathcal{M}$. Now, consider the set $\{a_1, \cdots, a_n, b_2, e\}$ and $cl(Y_2)$. Two sets form a modular pair and $\{a_1, \cdots, a_n, b_2, e\} \cap cl(Y_2) = \{a_1, \cdots, a_n, e\} \in \mathcal{N}$.

Now consider $cl(X_i')$. If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i') \notin \mathcal{N}$ but $cl(X_{i-1}') \in \mathcal{N}$, then by Claim 5, there exists $a \in E(L_n^*)$ such that $N/a$ is weakly 4-connected. If $cl(X_i') \notin \mathcal{N}$ for all $i$, then we look at $cl(Y_j')$. If there exists $j$ such that $cl(Y_j') \notin \mathcal{N}$ then $N/c_j$ is weakly 4-connected and we are done. If $cl(Y_i') \in \mathcal{N}$ for all $i$, then similar to $cl(Y_i)$, we have $\bigcap_{i=2}^{n} cl(Y_i') = \{c_1, \cdots, c_n, e\} \in \mathcal{N}$. However, $\{a_1, \cdots, a_n, e\} \in \mathcal{N}$. Then two sets form a modular pair and thus $\{e\} \in \mathcal{N}$, which is a contradiction as $N$ is weakly

4-connected. Therefore in all possible cases, there exists some $a \in E(L_n^*)$ such that $N/a$ is weakly 4-connected. □

In all possible cases, we have proven that there exists some $a \in E(L_n^*)$ such that $N/a$ is weakly 4-connected. Suppose $N/a$ is weakly 4-connected. Then we need to make sure $N/a$ and $(N/a)^*$ are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$. By comparing the rank and the number of elements we can confirm the previous statement. Also, as $N$ is a single element extension of $P_n^*$, it is not possible that $N \cong P_n^*$. Furthermore, $P_n^*$ is an internally 4-connected single element coextension of $L_n$ and $N$ is a weakly 4-connected single element extension of $P_n^*$. Therefore if $N/a$ is weakly 4-connected and $N/a$ has a $L_n$-minor, then $N/a$ has to be a weakly 4-connected single element extension of $L_n$. And by Lemma 7.2.2, there exists $b \in E(L_n)$ such that $N/a/b$ is weakly 4-connected. Also, by the same lemma, neither $N/a/b$ nor $(N/a/b)^*$ are isomorphic to $L_m$, $L^m$, $P_m$ or $P^m$ for $m \geq 4$. □

Note that the dual of a planar ladder is a graphic matroid. However, $(L^n)^*$ is not a graphic matroid. To compute the rank of sets in $(L^n)^*$, one can use the graph of a Möbius ladder. Also, we omit the proof of Claim 1, 2a, 2b, and 3 of the rest of the lemmas are same as the proof of the corresponding claims in Lemma 7.2.4, 7.2.5, and 7.2.6. We can do this as the proof only depends on the connectivity of the matroids $L^n$ and $P^m$ but not the structure.

**Lemma 7.2.7.** *If $M$ is a 3-connected single element extension of $(L^n)^*$, $n \geq 4$, then $M$ is weakly 4-connected. Moreover either,*
*(i) there exists an element $a \in E(L^n)$ such that $M \backslash a$ and $(M \backslash a)^*$ are weakly 4-connected and not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$, $m \geq 4$, or*
*(ii) $M \cong P^n$.*

*Proof.* Let $\mathcal{M}$ be a modular cut of $(L^n)^*$, $n \geq 4$ and let $M$ be a 3-connected extension of $(L^n)^*$ such that $M = (L^n)^* +_{\mathcal{M}} e$. Then $|E(M)| = |E((L^n)^*) + 1| = 3n+1$ and $r(M) = r((L^n)^*) =$

$n+1$. For this lemma, let $X_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subset E((L^n)^*)$, and $Y_i = E((L^n)^*) - (X_i \cup a_i)$ for $i \in \{3, \cdots, n\}$. Similarly, let $X_i' = \{b_i, b_{i+1}, a_{i-1}, a_i, a_{i+1}\} \subset E((L^n)^*)$ and $Y_i' = E(L_n) - (X_i' \cup c_i)$. For $i = 1$ and $2$, let $X_1 = \{c_1, a_2, c_n, b_1, b_2\}$ and $X_2 = \{a_1, c_2, c_3, b_2, b_3\}$. Define $Y_i$ and $Y_i'$ for $(L^n)^*$ in a same manner.

**Claim 1:** $M$ is weakly 4-connected.

**Claim 2a:** $M \backslash a$ is 3-connected.

**Claim 2b:** If $M \backslash a$ is 3-connected but not weakly 4-connected, then a 3-separation $(X \cup e, Y)$, $|X \cup e|$, $|Y| \geq 5$, in $M \backslash a$ induces a 3-separation $(X, Y)$ in $(L^n)^* \backslash a$.

**Claim 3:** If $cl_{(L^n)^*}(X_i)$, $cl_{(L^n)^*}(Y_i) \notin \mathcal{M}$ (resp. $cl_{(L^n)^*}(X_i')$, $cl_{(L^n)^*}(Y_i') \notin \mathcal{M}$), then $M \backslash a_i$ (resp. $M \backslash c_i$)) is weakly 4-connected.

**Claim 4:** There exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Then $cl(X_2)$, $cl(X_3)$ and $cl(X_4) \in \mathcal{M}$. Note that $r(cl(X_i)) = 3$ for all $i$. Also $r(cl(X_2) \cup cl(X_3)) = r(cl(\{a_1, b_2, b_3, b_4, c_2, c_3, c_4\})) = 4$. Lastly, $r(cl(X_2) \cap cl(X_3)) = r(\{b_3, c_2, c_3\}) = 2$. Therefore $(cl(X_2)$,$cl(X_3))$ is a modular pair and $cl(X_2) \cap cl(X_3) = \{b_3, c_2, c_3\} \in \mathcal{M}$. Also, $(\{b_3, c_2, c_3\}$,$cl(X_4))$ is a modular pair as well and $\{b_3, c_2, c_3\} \cap cl(X_4) = \{c_3\} \in \mathcal{M}$, a contradiction since $M$ is weakly 4-connected. Therefore there exist $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$. $\square$

**Claim 5:** If there exists $i$ such that $cl(X_i) \notin \mathcal{M}$ but $cl(X_{i-1}) \in \mathcal{M}$, then $M \backslash c_{i-1}$ is weakly 4-connected if $i \geq 4$ and $M \backslash c_1$, $M/c_2$, $M/c_3$, $M/c_n$ or $M/a_{n-1}$ is weakly 4-connected if $i = 1$, 2 or 3.

*Proof.* If $cl(X_i) \notin \mathcal{M}$ and $cl(Y_i) \notin \mathcal{M}$, then we are done as $M/c_i$ would be weakly 4-connected. Now, suppose there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{M}$ but $cl(X_{i-1})$, $cl(Y_i) \in \mathcal{M}$ for some $i \geq 4$. Then $(cl(X_{i-1})$,$cl(Y_i))$ is a modular pair and thus $cl(X_{i-1}) \cap$

$cl(Y_i) = \{b_{i-1}, c_{i-2}, c_{i-1}\} \in \mathcal{M}$. Suppose to the contrary that $M\backslash c_{i-1}$ is not weakly 4-connected. Then $cl(X'_{i-1}) \in \mathcal{M}$ or $cl(Y'_{i-1}) \in \mathcal{M}$. If $cl(X'_{i-1}) \in \mathcal{M}$, then $(cl(X'_{i-1}),\{b_{i-1}, c_{i-2}, c_{i-1}\}$ is a modular pair. Therefore $cl(X'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{b_{i-1}\} \in \mathcal{M}$, which implies $M$ is not weakly 4-connected. If $cl(Y'_{i-1}) \in \mathcal{M}$, then $(cl(Y'_{i-1}),\{b_{i-1}, c_{i-2}, c_{i-1}\}$ is a modular pair. Therefore $cl(Y'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{c_{i-1}\} \in \mathcal{M}$, which also creates a contradiction. Therefore $cl(X'_{i-1})$, $cl(Y'_{i-1}) \notin \mathcal{M}$ and $M\backslash c_{i-1}$ is weakly 4-connected.

We need to check the case when $i = 1$, 2 or 3. First, for $i = 1$, we suppose $cl(X_1) \notin \mathcal{M}$ but $cl(X_n) \in \mathcal{M}$. For $i = 2$, we suppose $cl(X_2) \notin \mathcal{M}$ but $cl(X_1) \in \mathcal{M}$ and lastly, for $i = 3$, we suppose $cl(X_3) \notin \mathcal{M}$ but $cl(X_2) \in \mathcal{M}$.

First, suppose $cl(X_1) \notin \mathcal{M}$ and $cl(X_n) \in \mathcal{M}$. If $cl(Y_1) \notin \mathcal{M}$ as well, then we are done and $M\backslash a_1$ is weakly 4-connected. Thus, suppose $cl(Y_1) \in \mathcal{M}$. Then $(cl(Y_1),cl(X_n))$ is a modular pair and $cl(Y_1) \cap cl(X_n) = \{b_{n-1}, b_n, c_n\} \in \mathcal{M}$. We will show that $M\backslash c_n$ is weakly 4-connected. Suppose not. Then $cl(X'_n) \in \mathcal{M}$ or $cl(Y'_n) \in \mathcal{M}$. If $cl(X'_n) \in \mathcal{M}$, then $(\{b_{n-1}, b_n, c_n\},cl(X'_n))$ is a modular pair and $\{b_n\} \in \mathcal{M}$, a contradiction. If $cl(Y'_n) \in \mathcal{M}$, then $(\{b_{n-1}, b_n, c_n\},cl(Y'_n))$ is a modular pair and $\{c_{n-1}\} \in \mathcal{M}$, a contradiction. Thus $cl(X'_n)$, $cl(Y'_n) \notin \mathcal{M}$ and $M\backslash c_n$ is weakly 4-connected.

Next, suppose $cl(X_1) \in \mathcal{M}$ and $cl(X_2) \notin \mathcal{M}$. If $cl(Y_2) \notin \mathcal{M}$ as well, then $M\backslash a_2$ is weakly 4-connected and we are done. Thus, suppose $cl(Y_2) \in \mathcal{M}$. Then $(cl(X_1),cl(Y_2))$ is a modular pair, and thus $\{b_1, c_1, c_n\} \in \mathcal{M}$. We will show that $M\backslash c_n$ is weakly 4-connected. Suppose to the contrary that $M\backslash c_n$ is not weakly 4-connected. Then $cl(X'_n) \in \mathcal{M}$ or $cl(Y'_n) \in \mathcal{M}$. If $cl(X'_n) \in \mathcal{M}$, then $(\{b_1, c_1, c_n\},cl(X'_n))$ is a modular pair and $\{b_1\} \in \mathcal{M}$, a contradiction. If $cl(Y'_n) \in \mathcal{M}$, then $(\{b_1, c_1, c_n\},cl(Y'_n))$ is a modular pair and $\{c_1\} \in \mathcal{M}$, a contradiction. Thus $cl(X'_n)$, $cl(Y'_n) \notin \mathcal{M}$ and $M\backslash c_n$ is weakly 4-connected.

Last, suppose $cl(X_3) \notin \mathcal{M}$ but $cl(X_2) \in \mathcal{M}$. If $cl(Y_3) \notin \mathcal{M}$, then $M\backslash a_3$ is weakly 4-connected and we are done. Suppose $cl(Y_3) \in \mathcal{M}$. Then $(cl(Y_2),cl(X_3))$ is a modular pair and $\{a_1, b_2, c_2\} \in \mathcal{M}$. Then we will show that $M\backslash c_2$ is weakly 4-connected. Suppose not.

Then $cl(X_2') \in \mathcal{M}$ or $cl(Y_2') \in \mathcal{M}$. If $cl(X_2') \in \mathcal{M}$, then $(\{a_1, b_2, c_2\}, cl(X_3'))$ is a modular pair and $\{b_2\} \in \mathcal{M}$, a contradiction. If $cl(Y_2') \in \mathcal{M}$, then $(\{a_1, b_2, c_2\}, cl(Y_2'))$ is a modular pair and $\{a_1\} \in \mathcal{M}$, a contradiction. Thus $cl(X_2')$, $cl(Y_2') \notin \mathcal{M}$ and $M \backslash c_2$ is weakly 4-connected. Therefore in all possible cases such that $cl(X_i) \in \mathcal{M}$ and $cl(X_{i-1}) \notin \mathcal{M}$, there exists $a \in E((L^n)^*)$ such that $M \backslash a$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ then either there exists $j$ such that $cl(Y_j) \notin \mathcal{M}$ or $M \cong P^n$ where $P^n \backslash a$ is not weakly 4-connected for all $a \in E((L^n)^*)$.

*Proof.* Suppose $cl(X_i) \notin \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. If there exists $cl(Y_j) \notin \mathcal{M}$ for some $j \in \{1, \cdots, n\}$, then we are done as $cl(X_j)\ cl(Y_j) \notin \mathcal{M}$ implies $M \backslash a_j$ is weakly 4-connected. Note that $cl(Y_i) = cl(Y_i') = E((L^n)^*) - \{a_i, b_i, b_{i+1}, c_i\}$ for all $i \in \{1, \cdots, n\}$. This implies that if $cl(Y_i) \in \mathcal{M}$ then $cl(Y_i') \in \mathcal{M}$. Also, the same holds if $i = 1$, 2, or 3.

Suppose that $\mathcal{M}$ is a modular cut of $(L^n)^*$ such that $cl(Y_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$. Then the following statements hold for $k \geq 2$.

(i) $\bigcap_{i=3}^{k} cl(Y_i) = E((L^n)^*) - \{a_2, \cdots, a_k, b_2, \cdots, b_{k+1}, c_2, \cdots, c_k\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i)) = r - k + 1$.

(ii) $\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1}) = E((L^n)^*) - c_{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r$.

(iii) $\bigcap_{i=3}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r - k$.

From (i), (ii), and (iii) we can deduce that for each $k \in \{3, \cdots, n\}$, the sets $\bigcap_{i=3}^{k} cl(Y_j)$ and $cl(Y_{k+1})$ form a modular pair and thus, $(\bigcap_{i=2}^{k} cl(Y_i)) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1} cl(Y_i) \in \mathcal{M}$. Therefore $\bigcap_{i=2}^{n} cl(Y_j) = \{a_1, c_1\} \in \mathcal{M}$. This applies for any intersection of $n-1$ number of $cl(Y_i)$. Thus $\{a_i, c_i\} \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$.

Then for all $i \in \{1, \cdots, n\}$, $M \backslash a_i$ and $M \backslash c_i$ are not weakly 4-connected. It is also not too difficult to check that $M \backslash b_i$ is also not weakly 4-connected. Therefore if $cl(X_i) \notin \mathcal{M}$ for all $i$, then either there exists $j$ such that $M / a_j$ is weakly 4-connected or $M \cong P^n$. □

Let $P_n$ be the internally 4-connected single element extension of $L_n^*$ such that $P^n\backslash a$ is not weakly 4-connected for all $a \in E(L_n)$. And let $X_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E(L_n^*) \subseteq E(P^n)$, for all $i \in \{1, \cdots, n\}$. And let $Y_i = E(P^n) - (X_i \cup a_i)$. Also, let $X_i' = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E(L_n) \subseteq E()$ and $Y_i' = E(L_n) - (X_i' \cup a_i)$. Let $X_i$, $X_i'$ be defined in a same manner for $E(P^n)$ and the dual except for $i = 1$ and 2. Let $X_1 = \{c_1, a_2, c_n, b_1, b_2\}$ and $X_2 = \{a_1, c_2, c_3, b_2, b_3\}$. Let $X_1' = \{a_2, c_2, a_n, b_1, b_2\}$ and $X_2' = \{c_1, a_2, a_3, b_2, b_3\}$. Define $Y_i$ and $Y_i'$ for $P^n$ and the dual in a same manner for all $i \in \{1, \cdots, n\}$.

**Claim 7:** $P^n$ is the only weakly 4-connected single element extension of $(L^n)^*$ such that $P^n\backslash a$ is not weakly 4-connected for all $a \in E((L^n)^*)$.

*Proof.* Suppose that $\mathcal{M}$ is a modular cut of $(L^n)^*$ such that $cl(Y_i) \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ and $M \cong (L^n)^* +_{\mathcal{M}} e$. Then by Claim 6, $\{a_i, c_i\} \in \mathcal{M}$ for all $i \in \{1, \cdots, n\}$ and $M \cong P^n$. We will prove that there does not exist any other modular cut of $(L^n)^*$, say $\mathcal{M}_1$, such that $\{a_i, c_i\}$ for all $i \in \{1, \cdots, n\}$ and $\mathcal{M}_1 \neq \mathcal{M}$. Let $M_1 = (L^n)^* +_{\mathcal{M}_1} e$. We will show that $M_1$ is not 3-connected and thus, Claim 7 holds.

Let $F$ be a flat of $(L^n)^*$ such that $F \in \mathcal{M}_1$ but $F \notin \mathcal{M}$. Then $\{a_i, c_i\} \nsubseteq F$ for all $i \in \{1, \cdots, n\}$. If $\{a_i, c_i\} \subseteq F$ for some $i \in \{1, \cdots, n\}$, then $F \in \mathcal{M}$ because $\mathcal{M}$ is a modular cut. Therefore $|\{a_i, c_i\} \cap F| = 1$ or 0 for all $i$.

Suppose there exists $i \in \{1, \cdots, n\}$ such that $|\{a_i, c_i\} \cap F| = 1$. Assume without loss of generality that $a_i \in F$ and $c_i \notin F$. Then $r(F \cup \{a_i, c_i\}) = r(F)$ or $r(F) + 1$. If $r(F \cup \{a_i, c_i\}) = r(F) + 1$, then $(F, \{a_i, c_i\})$ is a modular pair, and $F \cap \{a_i, c_i\} = \{a_i\} \in \mathcal{M}_1$. This implies $M_1$ is not 3-connected. If $r(F \cup \{a_i, c_i\}) = r(F)$, then $\{a_i, c_i\} \subseteq F$. This contradicts our assumption that $\{a_i, c_i\} \nsubseteq F$. Therefore there does not exist $i$ such that $|\{a_i, c_i\} \cap F| = 1$. Thus $\{a_i, c_i\} \cap F = \emptyset$ for all $i$. This implies that $F \subseteq \{b_1, \cdots, b_n\}$. Since $F$ is a flat, $r(F \cup a_i) = r(F) + 1$ and $r(F \cup c_i) = r(F) + 1$ for all $i$. If $r(F \cup \{a_i, c_i\}) = r(F) + 1$, then there exists a circuit $C \subseteq F \cup \{a_i, c_i\}$ such that $\{a_i, c_i\} \subseteq C$. However, there does not exist a circuit in $(L^n)^*$ where the circuit only contains $\{a_i, c_i\}$ and some elements from the set $\{b_1, \cdots, b_n\}$. Therefore $r(F \cup \{a_i, c_i\}) = 2$. Then $(F,\{a_i, c_i\})$ is a modular pair. This implies $F \cap \{a_i, c_i\} = \emptyset \in \mathcal{M}_1$, and $M_1$ is not 3-connected. Therefore Claim 7 is true. □

**Claim 8:** $P^n$ is internally 4-connected.

*Proof.* Suppose to the contrary that $P_n$ is not internally 4-connected. Since $P_n$ is still weakly 4-connected, there exists 3-separation $(X \cup e, Y)$ such that $|X \cup e|$, $|Y| \geq 4$. If $e \notin cl_{P^n}(X)$, then $cl(X) \notin \mathcal{M}_1$. Then $r_{P^n}(X \cup e) = r_{P^n}(X) + 1$ and $r_{(L^n)^*}(X) + 1 + r_{(L^n)^*}(Y) - r((L^n)^*) = 2$, a contradiction as $(L^n)^*$ is internally 4-connected. Therefore $e \in cl_{P^n}(X)$ and $cl(X) \in \mathcal{M}$. This implies that $r_{(L^n)^*}(X) + r(Y)_{(L^n)^*}(Y) - r((L^n)^*) = 2$. Because $(L^n)^*$ is internally 4-connected, $X$ must be one of the triangle, $\{a_i, a_{i+1}, b_{i+1}\}$ for some $i$ and $e \in cl_{P^n}(X)$. Therefore $\{a_i, c_i\}, \{a_i, a_{i+1}, b_{i+1}\} \in \mathcal{M}_1$ and because two sets form a modular pair, this implies $\{a_i\} \in \mathcal{M}$. This is a contradiction as $P^n$ is weakly 4-connected. Therefore $P^n$ is also internally 4-connected as well. □

Suppose $M$ is a weakly 4-connected single element extension of $(L^n)^*$ such that $M \ncong P^n$. Then there exists element $a \in E((L^n)^*)$ such that $M \backslash a$ is weakly 4-connected. If $M \backslash a$ or $(M \backslash a)^*$ is isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for some $n \geq 4$, it must be the case that $M \backslash a \cong L_n^*$ or $(L^n)^*$. If $m \neq n$, then the rank and the number of element does not match. Suppose there exists $a_i \in E((L^n)^*)$ such that $M \backslash a_i$ is weakly 4-connected and $M \backslash a_i \cong (L^n)^*$. Since $\{a_{i-1}, a_i, b_i\}$ and $\{a_i, a_{i+1}, b_{i+1}\}$ is a triangle in $(L^n)^*$, if $M \backslash a_i \cong (L^n)^*$, then $\{a_{i-1}, e, b_i\}$ and $\{e, a_{i+1}, b_{i+1}\}$ is a triangle in $M \backslash a_i$ and $M$. Then $cl(\{a_{i-1}, b_i\}) = \{a_{i-1}, a_i, b_i\}$ and $cl(\{a_{i+1}, b_{i+1}\}) = \{a_i, a_{i+1}, b_i\} \in \mathcal{M}$. However, two sets form a modular pair. Therefore $\{a_i\} \in \mathcal{M}$, a contradiction as $M$ is weakly 4-connected. This holds if $i = 2$ where the triangles have the form $\{c_1, b_1, a_2\}$. Therefore if $M \backslash a_i$ is weakly 4-connected, $M \backslash a_i$ is not isomorphic to $(L^n)^*$. For the same reason, $M \backslash a_i$ is not isomorphic to $L_n^*$ as well. □

Now, we prove results similar to Lemma 7.2.3 and 7.2.7 for $P^n$. As for the coextension of $P^n$, we look at the extension of $(P^n)^*$ instead. Let $X_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E((L^n)^*) \subseteq E(P^n)$, for all $i \in \{3, \cdots, n\}$. For $i = 1$ and 2, let $X_1 = \{c_1, a_2, c_n, b_1, b_2\}$ and $X_2 = \{a_1, c_2, c_3, b_2, b_3\}$. And let $Y_i = E(P^n) - (X_i \cup a_i)$. Also, let $X'_i = \{b_i, b_{i+1}, c_{i-1}, c_i, c_{i+1}\} \subseteq E((L^n)^*) \subseteq E(P^n)$ and $Y'_i = E(P^n) - (X'_i \cup a_i)$. Define $X'_i$ and $Y'_i$ in a similar manner.

**Lemma 7.2.8.** *If $N$ is a 3-connected extension of $P^n$, then $N$ is weakly 4-connected and there exists an element $a \in E((L^n)^*)$ such that $N \backslash a$ and $(N \backslash a)^*$ are weakly 4-connected and are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$.*

*Proof.* Let $\mathcal{N}$ be a modular cut of $P^n$, $n \geq 4$ and let $N$ be a 3-connected extension of $P^n$ such that $N = P^n +_{\mathcal{N}} f$. Let $r$, denote the rank of $P^n$. Then $r(N) = r() = r(L_n^*) = n + 1$.

**Claim 1:** $N$ is weakly 4-connected.

**Claim 2a:** If $N \backslash a$ is 3-connected.

**Claim 2b:** If $N \backslash a$ is 3-connected but not weakly 4-connected, then a 3-separation $(X \cup e, Y)$ in $N \backslash a$ where $|X \cup f|$, $|Y| \geq 5$ induces a 3-separation $(X - e, Y - e)$ in $(L^n)^* \backslash a$.

**Claim 3:** If $cl(X_i)$, $cl(Y_i) \notin \mathcal{N}$ (resp. $cl(X'_i)$, $cl(Y'_i) \notin \mathcal{N}$), then $N \backslash a_i$ (resp. $N \backslash c_i$) is weakly 4-connected.

In the rest of the proof of Lemma 7.2.8, when we write $cl(X)$, we mean $cl_{P^n}(X)$. When we take closure of a set in a different matroid, say $N$, we will denote it properly as $cl_N(X)$ in the proof.

**Claim 4:** There exists $i$ such that $cl(X_i) \notin \mathcal{N}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Then $cl(X_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Therefore $cl(X_2)$, $cl(X_3)$ and $cl(X_4) \in \mathcal{M}$. Note that $f \notin cl(X_i)$ for all $i$ since $cl(X_i)$ does not contain $a_i$ and $\{a_i, c_i, f\}$ is a triangle for all $i \in \{1, \cdots, n\}$ in $P^n$. Therefore $cl(X_i) = cl_{(L^n)^*}(X_i)$ and we can apply the proof of the Claim 4 of Lemma 7.2.7 to prove this claim. □

**Claim 5:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1}) \in \mathcal{N}$, then $N\backslash c_i$ is weakly 4-connected for $i \geq 4$ and $N\backslash c_1$, $N/c_2$, $N/c_3$, $N/c_n$ or $N/a_{n-1}$ is weakly 4-connected if $i = 1$, 2 or 3.

*Proof.* Suppose there exists an $i$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1})$, $cl(Y_i) \in \mathcal{N}$ for some $i \geq 4$. Then $(cl(X_{i-1}), cl(Y_i))$ is a modular pair and thus $cl(X_{i-1}) \cap cl(Y_i) = \{b_{i-1}, c_{i-2}, c_{i-1}\} \in \mathcal{N}$. Now, suppose to the contrary that $M\backslash c_{i-1}$ is not weakly 4-connected. Then $cl(X'_{i-1}) \in \mathcal{N}$ or $cl(Y'_{i-1}) \in \mathcal{N}$. If $cl(X'_{i-1}) \in \mathcal{N}$, then $(cl(X'_{i-1}),\{b_{i-1}, c_{i-2}, c_{i-1}\})$ is a modular pair. Therefore $cl(X'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{b_{i-1}\} \in \mathcal{N}$. This is a contradiction since $N$ is weakly 4-connected. If $cl(Y'_{i-1}) \in \mathcal{N}$, then $(cl(Y'_{i-1}),\{b_{i-1}, c_{i-2}, c_{i-1}\})$ is a modular pair. Therefore $cl(Y'_{i-1}) \cap \{b_{i-1}, c_{i-2}, c_{i-1}\} = \{c_{i-1}\} \in \mathcal{N}$. This is a contradiction since $N$ is weakly 4-connected. Therefore $cl(X'_{i-1})$, $cl(Y'_{i-1}) \notin \mathcal{N}$ and $M\backslash c_{i-1}$ is weakly 4-connected.

We still need to check the case where $i = 1$, 2 and 3. For $i = 1$, we suppose $cl(X_1) \notin \mathcal{M}$ but $cl(X_n) \in \mathcal{M}$. For $i = 2$, we suppose $cl(X_2) \notin \mathcal{N}$ but $cl(X_1) \in \mathcal{N}$. Lastly, for $i = 3$, we suppose $cl(X_3) \notin \mathcal{N}$ but $cl(X_2) \in \mathcal{N}$.

First, suppose $cl(X_1) \notin \mathcal{N}$ and $cl(X_n) \in \mathcal{N}$. If $cl(Y_1) \notin \mathcal{N}$ as well, then we are done and $M\backslash a_1$ is weakly 4-connected. Suppose $cl(Y_1) \in \mathcal{N}$. Then $(cl(Y_1),cl(X_n))$ is a modular pair and $\{b_{n-1}, b_n, c_n\} \in \mathcal{N}$. Then we will show that $M\backslash c_n$ is weakly 4-connected. Suppose $M\backslash c_n$ is not weakly 4-connected. Then $cl(X'_n) \in \mathcal{N}$ or $cl(Y'_n) \in \mathcal{N}$. If $cl(X'_n) \in \mathcal{N}$, then $(\{b_{n-1}, b_n, c_n\},cl(X'_n))$ is a modular pair and $\{b_n\} \in \mathcal{N}$, a contradiction. If $cl(Y'_n) \in \mathcal{N}$, then $(\{b_{n-1}, b_n, c_n\},cl(Y'_n))$ is a modular pair and $\{c_{n-1}\} \in \mathcal{N}$, a contradiction. Thus $cl(X'_n)$, $cl(Y'_n) \notin \mathcal{N}$ and $N\backslash c_n$ is weakly 4-connected.

Next, suppose $cl(X_1) \in \mathcal{N}$ and $cl(X_2) \notin \mathcal{N}$. If $cl(Y_2) \notin \mathcal{N}$ as well, then $M\backslash a_2$ is weakly 4-connected and we are done. So, suppose $cl(Y_2) \in \mathcal{N}$. Then $(cl(X_1),cl(Y_2))$ is a modular pair, and thus $\{b_1, c_1, c_n\} \in \mathcal{N}$. We will show that $N\backslash c_n$ is weakly 4-connected. Suppose to the contrary that $M\backslash c_n$ is not weakly 4-connected. Then $cl(X'_n) \in \mathcal{N}$ or $cl(Y'_n) \in$

$\mathcal{N}$. If $cl(X'_n) \in \mathcal{N}$, then $(\{b_1, c_1, c_n\}$,$cl(X'_n))$ is a modular pair and $\{b_1\} \in \mathcal{N}$. Which is a contradiction. If $cl(Y'_n) \in \mathcal{N}$, then $(\{b_1, c_1, c_n\}$,$cl(Y'_n))$ is a modular pair and $\{c_1\} \in \mathcal{N}$, a contradiction. Thus $cl(X'_n)$, $cl(Y'_n) \notin \mathcal{N}$ and $N \backslash c_n$ is weakly 4-connected.

Last, suppose $cl(X_3) \notin \mathcal{N}$ but $cl(X_2) \in \mathcal{N}$. If $cl(Y_3) \notin \mathcal{N}$, then $N \backslash a_3$ is weakly 4-connected and we are done. So, suppose $cl(Y_3) \in \mathcal{N}$. Then $(cl(Y_2)$,$cl(X_3))$ is a modular pair and $\{a_1, b_2, c_2\} \in \mathcal{N}$. Then we will show that $N \backslash c_2$ is weakly 4-connected. Suppose to the contrary $M \backslash c_2$ is not weakly 4-connected. Then $cl(X'_2) \in \mathcal{N}$ or $cl(Y'_2) \in \mathcal{N}$. If $cl(X'_2) \in \mathcal{N}$, then $(\{a_1, b_2, c_2\}$,$cl(X'_3))$ is a modular pair and $\{b_2\} \in \mathcal{N}$, a contradiction. If $cl(Y'_2) \in \mathcal{N}$, then $(\{a_1, b_2, c_2\}$,$cl(Y'_2))$ is a modular pair and $\{a_1\} \in \mathcal{N}$, a contradiction. Thus $cl(X'_2)$, $cl(Y'_2) \notin \mathcal{N}$ and $N \backslash c_2$ is weakly 4-connected. Therefore in all possible cases such that $cl(X_i) \in \mathcal{N}$ and $cl(X_{i-1}) \notin \mathcal{N}$, there exists $a \in E((L^n)^*)$ such that $N \backslash a$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{N}$ for all $i \in \{1, \cdots, n\}$, then there exists $j$ such that $cl(Y_j) \in \mathcal{N}$ and $N \backslash a_j$ is weakly 4-connected.

*Proof.* Suppose to the contrary that $cl(Y_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Then, $r(Y_i) = r - 1$ and the following statements holds:

(i) $\bigcap_{i=3}^{k} cl(Y_j) = E(P^n) - \{a_2, \cdots, a_k, b_2, \cdots, b_{k+1}, c_2, \cdots, c_k\}$ and $r(\bigcap_{i=3}^{k} cl(Y_i)) = r - k + 1$.

(ii) $\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1}) = E(P^n) - c_{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r$.

(iii) $\bigcap_{i=3}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1}$ and $r(\bigcap_{i=3}^{k} cl(Y_i) \cup cl(Y_{k+1})) = r - k$.

From (i), (ii), and (iii) we can deduce that for each $k \in \{3, \cdots, n\}$, $\bigcap_{i=3}^{k} cl(Y_i)$ and $cl(Y_{k+1})$ form a modular pair and thus, $\bigcap_{i=2}^{k} cl(Y_i) \cap cl(Y_{k+1}) = \bigcap_{i=3}^{k+1} cl(Y_i) \in \mathcal{N}$. Therefore $\bigcap_{i=2}^{n} cl(Y_j) = \{a_1, c_1, e\} \in \mathcal{N}$. Consider the set $\{a_1, c_1, e\}$ and $cl(Y_1)$. Two sets form a modular pair and their intersection $\{e\} \in \mathcal{N}$, a contradiction as $N$ is weakly 4-connected. Therefore there exists some $j$ such that $cl(Y_j) \notin \mathcal{N}$ and $N \backslash a_j$ is weakly 4-connected. □

Suppose $N\backslash a_i$ is weakly 4-connected. Then we need to make sure $N\backslash a_i$ and $(N\backslash a_i)^*$ are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$. By comparing the rank and the number of elements, we only need to check if $N\backslash a_i$ is isomorphic to $P_n$ or $P^n$. Note that $\{a_i, a_{i+1}, b_{i+1}\}$ is a triangle for all $i \in \{1, \cdots, n\}$ in $P^n$ if $i \neq 2$. Also $a_i$ is in two such triangles as $\{a_{i-1}, a_i, b_i\}$ is also a triangle if $i \neq 2$. However, $a_2$ is also in two triangles of $P^n$ as well. Therefore if $N\backslash a_i \cong$, then $\{f, a_{i+1}, b_{i+1}\}$ and $\{a_{i-1}, f, b_i\}$ must be two triangles in $N\backslash a_i$. Since $N$ is an extension of $P^n$, the two set $\{f, a_{i+1}, b_{i+1}\}$ and $\{a_{i-1}, f, b_i\}$ must be also a triangle in $N$ as well. If $\{a_i, f\}$ is not in parallel, by applying the CEA to $\{a_i, a_{i+1}, b_{i+1}\}$ and $\{f, a_{i+1}, b_{i+1}\}$ we see that $\{a_{i+1}, f, b_{i+1}\}$ must be a triangle in $N$ as well. This implies that $\{a_{i+1}, b_{i+1}\} \in \mathcal{N}$ and by the construction of $P^n$, we also have $\{a_{i+1}, c_{i+1}\} \in \mathcal{N}$ as well. Two sets form a modular pair and this forces $\{a_{i+1}\} \in \mathcal{N}$, a contradiction as $N$ is weakly 4-connected matroid. Therefore if $N/a_i \cong P^n$, then $\{a_i, f\}$ is in parallel, which again, is a contradiction. Thus, $N\backslash a_i \ncong P^n$. For the same reason, $N\backslash a_i \ncong P_n$ as well. Furthermore, if $N\backslash a_i$ has a $(L^n)^*$-minor, then $N\backslash a_i$ must be one of weakly 4-connected extension of $(L^n)^*$ which is not isomorphic to $P^n$ and in this case, by Lemma 7.2.7, there exists an element $b \in E((L^n)^*)$ such that $N\backslash a_i\backslash b$ is weakly 4-connected. This complete the proof of Lemma 7.2.8.

□

**Lemma 7.2.9.** *If $N$ is a 3-connected extension of $(P^n)^*$, then $N$ is weakly 4-connected and there exists an element $a \in E((L^n)^*)$ such that $N/a$ and $(N/a)^*$ are weakly 4-connected and are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$.*

*Proof.* Let $\mathcal{N}$ be a modular cut of $(P^n)^*$, $n \geq 4$ and let $N$ be a 3-connected extension of $(P^n)^*$ such that $N = (P^n)^* +_{\mathcal{N}} e$. Let $r$, denote the rank of $(P^n)^*$.

**Claim 1:** $N$ is weakly 4-connected.

**Claim 2a:** If $N/a$ is 2-connected but not 3-connected, then there exists $x \in E((L^n)^*) - a$ such that $\{f, a, x\}$ is a triangle in $(P^n)^*$.

**Claim 2b:** If $N/a$ is 3-connected but not weakly 4-connected for $a = a_i \in E(L^n)$, then a 3-separation $(X \cup f, Y)$ where $|X \cup f|$, $|Y| \geq 5$, in $N/a$ induces a 3-separation $(X - e, Y - e)$ in $L^n/a$.

**Claim 3:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i)$, $cl(Y_i) \notin \mathcal{N}$ (resp. $cl(X_i')$, $cl(Y_i') \notin \mathcal{N}$), then $N/a_i$ (resp. $N/c_i$) is weakly 4-connected.

In the rest of the proof of Lemma 7.2.9, when we write $cl(X)$, we mean $cl_{(P^n)^*}(X)$. When we take closure of a set in a different matroid, say $N$, we will denote it properly as $cl_N(X)$ in the proof.

**Claim 4:** There exists $i$ such that $cl(X_i) \notin \mathcal{N}$.

*Proof.* Suppose to the contrary that $cl(X_i) \in \mathcal{N}$ for all $i \in \{1, \cdots, n\}$. Therefore $cl(X_2)$, $cl(X_3)$, $cl(X_4) \in \mathcal{N}$. Note that $f \notin cl(X_i)$ for all $i$ since $f \in cl^*(a_i, c_i)$ not $cl(a_i, c_i)$. Therefore the proof of this part is exactly same as the proof of the Claim 4 of Lemma 7.2.7 as $(P^n)^*$ is equivalent to the coextension of $(L^n)^*$. □

**Claim 5:** If there exists $i \in \{1, \cdots, n\}$ such that $cl(X_i) \notin \mathcal{N}$ but $cl(X_{i-1}) \in \mathcal{N}$, then $N/a_i$, $N/c_i$ or $N/a_{i-3}$ is weakly 4-connected for $i \geq 4$ and $N/c_1$, $N/c_2$, $N/c_3$, $N/c_n$ or $N/a_{n-1}$ is weakly 4-connected if $i = 1$, 2 or 3.

*Proof.* Suppose there exists some $i$ such that $cl(X_i) \notin \mathcal{N}$ and $cl(X_{i-1}) \in \mathcal{N}$. For $i \geq 4$, without loss of generality, we suppose that $i = 4$ and prove the claim. Suppose $cl(X_4) \notin \mathcal{N}$ but $cl(X_3) \in \mathcal{N}$. If $cl(Y_4) \notin \mathcal{N}$, then we are done as $N/a_4$ is weakly 4-connected. Thus, suppose $cl(Y_4) \in \mathcal{N}$. Then ($cl(X_3)$,$cl(Y_4)$) is a modular pair and $\{c_2, b_3, a_3\} \in \mathcal{N}$. Now, we show that $N/c_3$ or $N/c_4$ is weakly 4-connected. Suppose both matroids are not weakly 4-connected. Then $cl(X_3') \in \mathcal{N}$ or $cl(Y_3') \in \mathcal{N}$. Similarly, $cl(X'4) \in \mathcal{N}$ or $cl(Y_4') \in \mathcal{N}$. If $cl(X_3') \in \mathcal{N}$, then ($\{c_2, b_3, a_3\}$,$cl(X_3')$) is a modular pair and $\{c_2\} \in \mathcal{M}$. If $cl(X_4') \in \mathcal{N}$, then

$(\{c_2, b_3, a_3\}$,$cl(X_4'))$ is a modular pair and $\{a_3\} \in \mathcal{M}$. Thus $cl(Y_3')$, $cl(Y_4') \in \mathcal{N}$. However,both $(cl(Y_3')$,$\{c_2, b_3, a_3\})$ and $(cl(Y_4')$,$\{c_2, b_3, a_3\})$ are modular pairs. Therefore $\{b_3, a_3\}$, $\{c_2, b_3\} \in \mathcal{N}$. But, the two sets also form a modular pair which forces $\{b_3\} \in \mathcal{N}$, a contradiction. Thus either $N/c_3$ or $N/c_4$ is weakly 4-connected.

Now suppose $cl(X_1) \in \mathcal{N}$ and $cl(X_2) \notin \mathcal{N}$. If $cl(Y_2) \notin \mathcal{N}$ as well, then we are done and $N/a_2$ is weakly 4-connected. Thus, suppose $cl(Y_2) \in \mathcal{N}$. Then $(cl(X_1)$,$cl(Y_2))$ is a modular pair, and $\{a_2, b_1, c_1, c_n\} \in \mathcal{N}$. Let $\{a_2, b_1, c_1, c_n\} = Z$. We will show that either $N/c_1$, $N/c_2$ or $N/c_3$ is weakly 4-connected. Suppose neither $N/c_1$, $N/c_2$ nor $N/c_3$ are weakly 4-connected. For each case, $cl(X_i') \in \mathcal{N}$ or $cl(Y_i') \in \mathcal{N}$. In all cases, if $cl(X_i') \in \mathcal{N}$ or $cl(Y_i') \in \mathcal{N}$ then they form a modular pair with $Z$. Thus the following statements hold:

If $cl(X_1')$ and $Z \in \mathcal{N}$ then $\{b_1, c_1\} \in \mathcal{N}$.

If $cl(Y_1')$ and $Z \in \mathcal{N}$ then $\{c_1, c_n, a_2\} \in \mathcal{N}$.

If $cl(X_2')$ and $Z \in \mathcal{N}$ then $\{b_1, c_n\} \in \mathcal{N}$.

If $cl(Y_2')$ and $Z \in \mathcal{N}$ then $\{a_2, c_1\} \in \mathcal{N}$.

If $cl(X_3')$ and $Z \in \mathcal{N}$ then $\{a_2\} \in \mathcal{N}$.

If $cl(Y_3')$ and $Z \in \mathcal{N}$ then $\{b_1, c_1, c_n\} \in \mathcal{N}$.

Therefore if $cl(X_1')$ and $cl(Y_2') \in \mathcal{N}$ then $(\{b_1, c_1\}$,$\{b_1, c_n\})$ is a modular pair as well and $\{b_1\} \in \mathcal{N}$. Following this idea, the following statements hold as well:

If $cl(X_1')$ and $cl(Y_2') \in \mathcal{N}$, then $\{b_1\} \in \mathcal{N}$.

If $cl(X_1')$ and $cl(X_2') \in \mathcal{N}$, then $\{c_1\} \in \mathcal{N}$.

If $cl(Y_1')$ and $cl(Y_2') \in \mathcal{N}$, then $\{c_n\} \in \mathcal{N}$.

If $cl(Y_1')$ and $cl(X_2') \in \mathcal{N}$, then $\{a_2, c_1\} \in \mathcal{N}$.

Because neither $N/c_1$ nor $N/c_2$ are weakly 4-connected, we have $cl(X_1') \in \mathcal{N}$ or $cl(Y_1') \in \mathcal{N}$ and same holds for $cl(X_2')$ and $cl(Y_2')$. However, only case in which this does not induce a contradiction is when $cl(Y_1')$, $cl(X_2')$, and $\{a_2, c_1\} \in \mathcal{N}$. If $\{a_2, c_1\}$ and $cl(X_3') \in \mathcal{N}$ then $(\{a_2, c_1\}$,$cl(X_3'))$ is a modular pairs and $\{a_2\} \in \mathcal{N}$, a contradiction. If $\{a_2, c_1\}$ and $cl(Y_3') \in \mathcal{N}$

then $(\{a_2, c_1\}, cl(Y_3'))$ is a modular pair and $\{c_1\} \in \mathcal{N}$, a contradiction. Therefore neither $N/c_1$, $N/c_2$ nor $N/c_3$ are weakly 4-connected, then we obtain a contradiction. Thus, at least one of $N/c_1$, $N/c_2$ or $N/c_3$ is weakly 4-connected.

Now, suppose $cl(X_2)$ and $cl(Y_3) \in \mathcal{N}$ and $cl(X_3) \notin \mathcal{N}$. In this case, $(cl(Y_3), cl(X_2))$ is a modular pair and thus, $\{a_1, a_2, b_2\} \in \mathcal{N}$. Let $\{a_1, a_2, b_2\} = Z$. Now, we will show that either $N/c_2$ or $N/c_3$ is weakly 4-connected. Suppose to the contrary that neither $N/c_2$ nor $N/c_3$ are weakly 4-connected. If $cl(X_2') \in \mathcal{N}$ then $(Z, cl(Y_2'))$ is a modular pair and $\{c_2\} \in \mathcal{N}$, a contradiction. If $cl(X_3') \in \mathcal{N}$ then $(Z, cl(X_3'))$ is a modular pair and $\{a_2\} \in \mathcal{N}$, a contradiction. Thus it must be $cl(X_2')$ and $cl(Y_3') \in \mathcal{N}$. Therefore $N/c_2$ and $N/c_3$ are not weakly 4-connected. However, $cl(X_2')$ and $Cl(Y_3')$ form a modular pair with $Z$ and thus $\{b_2, a_2\}$ and $\{a_1, b_2\} \in \mathcal{N}$, respectively. Then again, $(\{b_2, a_2\}, \{a_1, b_2\})$ is a modular pair and thus $\{b_2\} \in \mathcal{N}$, a contradiction. Therefore $N/c_2$ or $N/c_3$ must be weakly 4-connected.

Lastly, suppose $cl(X_n)$ and $cl(Y_1) \in \mathcal{N}$ and $cl(X_1) \notin \mathcal{N}$. Then $(cl(Y_1), cl(X_1))$ is a modular pair and $\{a_n, b_n, c_{n-1}\} \in \mathcal{N}$. Using same reasoning as previous proofs in this claim, either $N/c_n$ or $N/a_{n-1}$ is weakly 4-connected. In all possible cases noted in the claim, there exists some $a \in E((L^n)^*)$ such that $N/a$ is weakly 4-connected. □

**Claim 6:** If $cl(X_i) \notin \mathcal{N}$ for all $i \in \{1, \cdots, n\}$, then there exists $j$, $k \in \{1, \cdots, n\}$ such that $N/a_j$ or $N/c_k$ is weakly 4-connected.

*Proof.* Since $cl(X_i) \notin \mathcal{N}$ for all $i \in \{1, \cdots, n\}$, if there exists $j \in \{1, \cdots, n\}$ such that $cl(Y_j) \notin \mathcal{N}$, then we are done. Now, suppose that $cl(Y_j) \in \mathcal{N}$ for all $j \in \{1, \cdots, n\}$. Then $\bigcap_{j=1}^{n} cl(Y_j) = \{e, a_n, a_{n-1}, \cdots, a_3\} \in \mathcal{N}$. If $cl(Y_j') \in \mathcal{M}$ for all $j$ as well, then $\bigcap_{j=1}^{n} cl(Y_j') = \{e, c_n, c_{n-1}, \cdots, c_3\} \in \mathcal{N}$ as well. However, $\{e, a_n, a_{n-1}, \cdots, a_3\}$ and $\{e, c_n, c_{n-1}, \cdots, c_3\}$ form a modular pair, which forces $\{e\}$ to be in $\mathcal{N}$, a contradiction. Therefore there exists $k \in \{1, \cdots, n\}$ such that $cl(Y_k') \notin \mathcal{N}$ and thus $N/c_k$ is weakly 4-connected.

Now suppose there exists $i$ such that $cl(X'_i) \notin \mathcal{N}$ but $cl(X_{i-2}) \in \mathcal{N}$. If $cl(Y'_i) \notin \mathcal{N}$, then we are done. If $cl(Y'_i) \in \mathcal{N}$ then by Claim 5, there exists some $a \in E(L^n)$ such that $N/a$ is weakly 4-connected. $\square$

In all possible cases, we have proven that there exists some $a \in E((L^n)^*)$ such that $N/a$ is weakly 4-connected. Suppose $N/a$ is weakly 4-connected. Then we need to make sure $N/a$ and $(N/a)^*$ are not isomorphic to $L_m$, $L^m$, $P_m$, or $P^m$ for $m \geq 4$. By comparing the rank and the number of elements we can confirm the previous statement. Also, as $N$ is a single element extension of $(P^n)^*$, it is not possible that $N \cong (P^n)^*$. Furthermore, $(P^n)^*$ is an internally 4-connected single element coextension of $L^n$ and $N$ is a weakly 4-connected single element extension of $(P^n)^*$. Therefore if $N/a$ is weakly 4-connected and $N/a$ has a $L^n$-minor, then $N/a$ has to be a weakly 4-connected single element extension of $L^n$. And by Lemma 7.2.3, there exists $b \in E(L^n)$ such that $N/a/b$ is weakly 4-connected. Also, by the same lemma, neither $N/a/b$ nor $(N/a/b)^*$ are isomorphic to $L^m$, $L_m$, $P^m$ or $P_m$ for $m \geq 4$. $\square$

Before proving Theorem 7.1.1, we provide an example how Lemma 7.2.2 is applied. Suppose $M$ is an internally 4-connected matroid such that $|E(M)| \geq 6$ and $M \backslash e \cong L_5$ for some $e \in E(M)$. Since $M$ is internally 4-connected, it has no 4-element 3-separating set $A$. Therefore we can apply Theorem 5.1.9 (1) to obtain a smaller weakly 4-connected matroid. However, we can not tell if this smaller matroid is isomorphic to $L_5$ or not. Since $M \backslash e \cong L_5$, by applying Theorem 5.1.9, we can obtain $L_5$ as a result. If we obtain $L_5$, we can not apply Theorem 5.1.9 to obtain a smaller weakly 4-connected matroid and the chain which started from $M$ stops at $L_5$. However, by Lemma 7.2.2, $M$ has an element $f$ such that $M/f$ is weakly 4-connected neither $M/f$ nor $(M/f)^*$ is isomorphic to $L_n$, $L^n$, $P_n$, and $P_n$ for all $n \geq 4$. Then we can apply Theorem 5.1.9 to $M/f$ to obtain a smaller matroid and continue the chain of weakly 4-connected matroids.

As we will see in the proof, by applying Lemmas 7.2.2, $\cdots$, 7.2.9 and Splitter Theorem to Theorem 5.1.9, we obtain Theorem 7.1.1. Now, we state the Theorem 7.1.1 once more before proving the theorem.

## 7.3 Proof of the Main Result

**Theorem.** *Let $M$ be a weakly 4-connected matroid with $|E(M)| \geq 7$ such that $M$ or $M^*$ is not isomorphic to the cycle matroid of a ladder($L_n$ and $L^n$), $P_n$, $P^n$, or a trident. Then there exists a weakly 4-connected $W'M$-chain where $W' = M_0 \in \{W_3, W^3, W^2, trident\}$, and $M = M_n$ and for each $i \in \{0, \cdots, n\}$,*
*1. there exists $e \in E(M_i)$ such that $M_i \backslash e \cong M_{i-1}$ or $M_i/e \cong M_{i-1}$ is weakly 4-connected, or*
*2. $M_i$ has a 4-element 3-separating set $A$ with elements $c$, $d \in A$ such that $M_i \backslash d/c \cong M_{i-1}$ is weakly 4-connected.*

*Proof.* (of Theorem 7.1.1) Let $M$ be a weakly 4-connected matroid with $|E(M)| \geq 7$ and suppose $M$ is not a planar ladder, Möbius ladder, their duals or a trident. By Theorem 5.1.9, there exists a matroid $M_1$ such that either $M\backslash a$, $M/a$ or $M\backslash d/c \cong M_1$ and $M_1$ is weakly 4-connected. Apply Theorem 5.1.9 to the matroid $M_1$ to obtain a smaller weakly 4-connected $M_2$. Suppose $M_i$ is isomorphic to a *trident*. Then we can not apply Theorem 5.1.9 to $M_i$. Then the chain of matroids starting from $M_i$, a *trident*, to $M_1 = M$ is the weakly 4-connected $W'M$-chain described by the above theorem.

Now suppose $M_i \cong L$, where $L$ is a planar ladder or a Möbius ladder for some $i \geq 2$. If $M_{i+1}$ has a 4-element 3-separating set $A$ with elements $c$, $d \in A$ such that $M_{i+1}\backslash d/c = M_i$, then $|E(M_{i+1}) - E(M_i)| = 2$. $M$ has a 4-element 3-separating set $A$ with elements $c$, $d \in A$ such that $M\backslash d/c$ is weakly 4-connected. By the Splitter Theorem, there exist a 3-connected matroid $M'$ and element $e \in E(M_{i+1})$ such that $M_{i+1}\backslash e$ or $M_{i+1}/e = M'$. Furthermore, there exists an element $f \in E(M')$ such that $M' \backslash f$ or $M'/f \cong M_i$. Then $M'$ is either a 3-connected single element extension or coextension of $M_i$. Then $M'$ is a 3-connected single element extension or coextension of $M_i$, a planar ladder or a Möbius ladder, and all such matroids are also weakly 4-connected. Therefore $M'$ is weakly 4-connected. Therefore we can always relabel the matroids in the chain such that if $M_i \cong L$ then $M_{i+1}$ is a single element extension or coextension of $M_i$.

If $M_{i+1}$ is an extension of a planar ladder or Möbius ladder, then by Lemmas 7.2.2 and 7.2.3, there exists $a \in E(L)$ such that $M_{i+1}/a$ is weakly 4-connected and is not isomorphic to $L$ or $L^*$. Then we relabel the matroid $M_i$ such that $M_{i+1}/a = M_i$ and apply Theorem 5.1.9. If $L$ is a coextension of a planar ladder or a Möbius ladder, then by the dual of Lemmas 7.2.4 and 7.2.7, except for one case, there exists an element $a \in E(L)$ such that $M_{i+1}/a$ is weakly 4-connected and is not isomorphic to $L$ or $L^*$. If $M_{i+1}/a$ is not weakly 4-connected for all $a \in E(L)$, then by Lemmas 7.2.4 and 7.2.7, $M_{i+1}$ is internally 4-connected and like the ladder, we can relabel the matroids in the chain such that $M_{i+2}$ is weakly 4-connected and $|E(M_{i+2} - E(M_{i+1})| = 1$. Then $M_{i+2}$ is either an extension or coextension of $M_{i+1}$ and by Lemmas 7.2.5, 7.2.6, 7.2.8, and 7.2.9 there exists an element $a \in E(M_i)$ such that $M_{i+2}/a$ is weakly 4-connected and $M_{i+2}/a \ncong M_{i+1}$. In this case, $M_{i+1}$ is the only matroid such that $M_{i+1}/a$ is not weakly 4-connected for all $a \in E(L)$ by Claim 7 of Lemmas 7.2.4 and 7.2.7. Then by the same lemmas, there exists $b \in E(L)$ such that $M_{i+2}/a/b$ is weakly 4-connected and is not isomorphic to $L$ or $L^*$. Then relabel the matroid such that $M_{i+2}/a = M_{i+1}$ and $M_{i+2}/a/b = M_i$ and apply Theorem 5.1.9. If $M_i$ is a dual of the ladder or a Möbius ladder, then we can apply the dual of Lemma 7.2.2, 7.2.3 and Lemmas 7.2.4-7.2.9. Therefore in all cases, whenever $M_i$ is a planar ladder, Möbius ladder, or their duals, we can go around the chain and obtain a chain of matroid which does not contain a planar ladder, Möbius ladder.

Because Theorem 5.1.9 can be applied to any weakly 4-connected matroid with 7-elements or more and any 3-connected matroid with elements 7, 8, or 9 are trivially weakly

4-connected, by repetitively applying Theorem 5.1.9 and Lemmas 7.2.2 - 7.2.9, we can obtain a matroid $M_j$ such that $|E(M_j)| = 5$ or 6 or $M_j$ is a trident. If $M_j$ is a 3-connected binary matroid with 6-elements, then $M_j \cong W_3$ are we are done. If $M_j$ is a non-binary matroid, then either $M_j \cong W^3$ or $M_j$ has $W^2$-minor. If $M_j$ has $W^2$-minor, then we can delete or contract an element from $M_j$ and obtain $W^2$. In all cases, the chain stops at the *trident*, $W_3$, $W^3$ or $W^2$ as noted in the Theorem 7.1.1. □

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