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# **A Numerical Method for the Computation of Faber Polynomials for Starlike Domains**

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**Abstract** We describe a simple numerical process (based on the Theodorsen method for conformal mapping ) for computing approximations to Faber polynomials for starlike domains.

**Keywords** : Faber polynomials, conformal mapping, starlike domains, Theodorsen method.

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## 1 Introduction

Faber polynomials have well-known classical applications as basis sets for polynomial and ational approximations in the complex plane. In addition, their study has received considerable attention recently, in connection with the construction of efficient iterative methods for solving systems of linear algebraic equations (see eg. [4]). Thus, the problem of developing numerical methods for computing approximations to the Faber polynomials for a region in the complex plane is of considerable current interest. This paper is concerned with the study of one such numerical method.

Let  $\Gamma$  be a Jordan curve in the complex  $z$ -plane, and let  $\psi$  denote the conformal map of

$D_E := \{ w : |w| > 1 \}$  onto  $\Omega_E := \text{Ext } \Gamma$  normalized by the conditions

$$\psi(\infty) = \infty \quad \text{and} \quad \psi'(\infty) > 0.$$

Also, let  $\Phi$  denote the inverse conformal map  $\Phi := \psi^{-1}$ , from  $\Omega_E$  onto  $D_E$ , and observe that the Laurent series expansions of  $\psi$  and  $\Phi$  at infinity are of the form

$$\psi(w) = cw + c_0 + c_1/w + c_2/w^2 + \dots, \quad (1.1)$$

and

$$\Phi(z) = bz + b_0 + b_1/z + b_2/z^2 + \dots \quad \text{with} \quad b = 1/c, \quad (1.2)$$

where  $c = \psi'(\infty)$  is the capacity of the curve  $\Gamma$ , ie.  $c := \text{cap } \Gamma$ . Then, the  $n$ th Faber polynomial  $p_n$  ( $n = 0, 1, 2, \dots$ ) for the domain  $\Omega_1 := \text{Int } \Gamma$  is the polynomial part of the Laurent series expansion at infinity of the function  $\Phi^n$ .

Full details of the theory of Faber polynomials and their approximating properties can be found in the two books by Gaier [9, Chap.1, §6] and Henrici [12, §18.1-18.2] and the other important references cited there. Of particular interest to us here is the relation

$$p_{n+1}(z) = \frac{1}{c} \left\{ z p_n(z) - \sum_{k=0}^n c_k p_{n-k}(z) - n c_n \right\}, \quad n = 0, 1, \dots, \quad (1.3)$$

which can be used to determine recursively the Faber polynomials  $p_n$ . However, because of the presence of the generally unknown mapping coefficients  $c$  and  $c_i$ , the use of (1.3) has received very little attention from the computational point of view (see e.g. the remark made by Gaier in [9, p.44]).

As far as we are aware the most recent references on the computation of Faber polynomials are the papers by Coleman and Smith [2] and Ellacott [5]. Of these, [2] concerns the computation of Faber polynomials in circular sectors, and involves the derivation of an explicit formula for the corresponding mapping function  $\psi$ . The paper also contains a recurrence formula for determining the mapping coefficients  $c_i$ , but Coleman and Smith do not make use of (1.3) in their computations. They choose, instead, to compute the coefficients of the Faber polynomials directly from a standard integral representation that makes use of their formula for  $\psi$ . (We note, however, that their recurrence formula was subsequently used by Ellacott and Saff [6], for computing Faber polynomials in circular sectors, by means of (1.3).) In [5], Ellacott suggests computing the coefficients of Faber polynomials, from their integral representation in terms of the mapping function  $\Phi$ , by using the fast Fourier

transform (FFT). He also considers the recursive computation of the polynomials by means of (1.3), but only for cases where the boundary curve  $\Gamma$  is a polygon. For this, he proposes using the Schwartz-Christoffel formula for determining the expansion of  $\psi^1$  and hence, by termwise integration, that of  $\psi$ .

Regarding numerical experiments, all the examples of Faber polynomials that we have come across in the literature, including those of [2] and [5], are for regions for which the mapping function  $\Phi$ , or its inverse  $\psi$ , is known exactly. The purpose of the present paper is to consider a fully numerical technique, based on using the well-known Theodorsen integral equation method for computing approximations to the coefficients  $c$  and  $c_j$  of the mapping function  $\psi$ , and then determining recursively the corresponding approximate Faber polynomials by means of (1.3). Our main objective is to show, by means of numerical examples, that this technique is very well-suited for computing accurate approximations to the Faber polynomials corresponding to a wide class of starlike curves  $\Gamma$ .

## 2 The numerical method

Let  $\alpha_j^{(n)}$  denote the coefficients of the Faber polynomial  $p_n$ , ie.

$$p_n(z) = \sum_{j=0}^n \alpha_j^{(n)} z^j. \quad (2.1)$$

Then, by substituting (2.1) in the recurrence relation (1.3) and comparing coefficients of like powers of  $z$ , we obtain the following relations for  $n \geq 1$ :

$$\alpha_0^{(n+1)} = -\frac{1}{c} \left\{ \sum_{k=0}^{n-1} c_k \alpha_0^{(n-k)} + (n+1) c_n \right\}, \quad (2.2)$$

$$\alpha_j^{(n+1)} = \frac{1}{c} \left\{ \alpha_{j-1}^{(n)} - \sum_{k=0}^{n-j} c_k \alpha_j^{(n-k)} \right\}, \quad j=1, 2, \dots, n, \quad (2.3)$$

$$\alpha_{n+1}^{(n+1)} = \frac{1}{c} \alpha_n^{(n)}. \quad (2.4)$$

Since

$$\alpha_0^{(1)} = -c_0/c \quad \text{and} \quad \alpha_1^{(1)} = 1/c, \quad (2.5)$$

the relations (2.2)-(2.5) can be used to determine recursively the coefficients of the Faber polynomials  $p_n$ ,  $n=2,3,\dots$ , in terms of the Laurent coefficients  $c$  and  $c_j$ ,  $j=1,2,\dots$ , of the mapping function  $\psi$ .

Consider now the "auxiliary" function

$$\Psi(w) := \log \frac{\psi(w)}{w}, \quad (2.6)$$

and observe that it has a Laurent expansion of the form

$$\Psi(w) = \gamma_0 + \gamma_1/w + \gamma_2/w^2 + \dots, \quad (2.7)$$

where  $\gamma_0 = \log c$ . Hence

$$\psi^1(w) - \frac{1}{w} \psi(w) = \psi(w) \left\{ -\gamma_1/w^2 - 2\gamma_2/w^3 - \dots \right\},$$

which, by substituting the expansions of  $\psi$  and  $\psi^1$  and comparing coefficients of like powers of  $w$ , yields the following relations:

$$c_0 = c \gamma_1, \quad (2.8)$$

$$c_n = c \gamma_{n+1} + \frac{1}{n+1} \sum_{j=0}^{n-1} (n-j) c_j \gamma_{n-j}, \quad n \geq 1. \quad (2.9)$$

Since

$$c = \exp \gamma_0, \quad (2.10)$$

the relations (2.8)-(2.10) can be used to determine recursively the mapping coefficients  $c$  and  $c_j$  from the coefficients  $\gamma_j$  in the expansion (2.7) of the auxiliary function  $\Psi$ .

The numerical method considered in this section is based on the use of the recurrence relations (2.2)-(2.5) and (2.8)-(2.10), and is motivated by the following observation concerning the computation of the coefficients  $\gamma_j$  by means of the Theodorsen method.

Assume that the boundary curve  $\Gamma$  is starlike with respect to the origin and is given in polar coordinates by

$$\Gamma := \{ z : z = \rho(\theta) e^{i\theta}, 0 \leq \theta \leq 2\pi \}, \quad (2.11)$$

where  $\rho$  is piecewise differentiable in  $[0, 2\pi]$ . Also let  $\Theta$  denote the boundary correspondence function associated with the conformal map  $\psi$ , ie.

$$\psi(e^{i\tau}) = \rho(\Theta(\tau)) e^{i\Theta(\tau)}.$$

Then,  $\Theta$  satisfies the Theodorsen integral equation

$$\Theta(\tau) = \tau - k[\log \rho(\Theta(\tau))], \quad (2.12)$$

where  $K$  denotes the well-known operator for conjugation on the unit circle ( see e.g. Gaier [8, Kap. II] and Henrici [12, §16.8]).

In the basic Theodorsen method equation (2.12) is solved iteratively for  $\Theta$ , by using a Jacobi type iteration of the form

$$\Theta^0(\tau) = \tau, \quad (2.13a)$$

$$\Theta^{k+1}(\tau) = \tau - k[\log \rho(\Theta^k(\tau))], \quad k = 0, 1, \dots, \quad (2.13b)$$

In practice, this iteration is performed in discretized form where, at the  $k$  th step, the function  $\log \rho(\Theta^k(\tau))$  is replaced by its trigonometric interpolating polynomial

$$T_N(\tau) := \frac{1}{2} a_0^{(k)} + \sum_{j=1}^{N-1} (a_j^{(k)} \cos j\tau + b_j^{(k)} \sin j\tau) + \frac{1}{2} a_N^{(k)} \cos N\tau, \quad (2.14)$$

of degree  $N$  and nodes

$$\tau_r := r\pi / N, \quad r = 0, 1, \dots, 2N-1. \quad (2.15)$$

Then, because of the properties of the operator  $K$ , the  $(k+1)$ th approximation to the boundary

correspondence function is given, from (2.13), by

$$\Theta_N^{(k+1)}(\tau) = \tau + \sum_{j=1}^{N-1} (b_j^{(k)} \cos j\tau - a_j^{(k)} \sin j\tau) - \frac{1}{2} a_N^{(k)} \sin N\tau. \quad (2.16)$$

The resulting discrete iteration can be stated as follows:

- Set  $\Theta^0(\tau) = \tau$ .
- Do steps (i) and (ii) with  $k = 0, 1, 2, \dots$ , until convergence:
  - (i) Compute the coefficients  $a_j^{(k)}$  and  $b_j^{(k)}$  of the trigonometric polynomial (2.14).
  - (ii) Use (2.16) to compute the values  $\Theta_N^{(k+1)}(\tau_r)$ ,  $r = 0, 1, \dots, 2N-1$ .

The coefficients  $a_j^{(k)}$ ,  $b_j^{(k)}$  in step (i) of the iteration can be computed efficiently in  $O(N \log N)$  operations by the use of the FFT. Similarly, in step (ii), the computations of the values  $\Theta_N^{(k+1)}(\tau_r)$  can be performed by the use of the FFT. That is, the basic Theodorsen method involves the application of two FFTs in each iterative step.

As the above remarks indicate, the Theodorsen method is based essentially on approximating the coefficients of the trigonometric polynomial

$$T_N(\tau) = \frac{1}{2} a_0 + \sum_{j=1}^{N-1} (a_j \cos j\tau + b_j \sin j\tau) + \frac{1}{2} a_N \cos N\tau, \quad (2.17)$$

which interpolates, the function  $\log \rho\Theta(\tau)$  at the nodes (2.15). The relevance of this observation with our work here is that once the approximations  $\bar{a}_j$  and  $\bar{b}_j$  to  $a_j$  and  $b_j$  are found, then the corresponding approximations to the Laurent coefficients  $\gamma_j, j=0, 1, \dots, N$ , of the auxiliary function  $\Psi$  are given quite simply by

$$\bar{\gamma}_0 = \frac{1}{2} \bar{a}_0 \quad \text{and} \quad \bar{\gamma}_j = \bar{a}_j + i \bar{b}_j, \quad j = 1, 2, \dots, N, \quad (2.18)$$

with  $b_N = 0$  ( see e.g. [8, p.87] and [12, p.407] ). Thus, the details of the proposed algorithm are as follows :

STEP 1 : Use the Theodorsen method to compute approximations  $\bar{a}_j, \bar{b}_j$  to the coefficients  $a_j, b_j$  of the trigonometric polynomial (2.17). Hence, use (2.18) to compute the approximations  $\bar{\gamma}_j, j = 0, 1, \dots, N$ , to the Laurent coefficients of the auxiliary function  $\Psi$ .

STEP 2 : Use the relations (2.8)-(2.10) to compute recursively the approximations to the mapping coefficients  $c$  and  $c_j, j=0, 1, \dots, N$ .

STEP 3 : Use the relations (2.2)-(2.5) to compute recursively approximations to the coefficients  $\alpha_j^{(n)}$  of the Faber polynomials  $p_n, n = 1, 2, \dots, N$ .



We end this section by making the following remarks concerning the convergence of the Theodorsen method and our implementation of Step 1 of the algorithm outlined above.

It is well-known that convergence of the iteration (2.13), and of the corresponding discrete iteration defined by (2.16), can be guaranteed only if the curve (2.11) satisfies a so-called "ε-condition", which requires that

$$\varepsilon := \sup_{0 \leq \tau \leq 2\pi} |\rho'(\tau) / \rho(\tau)| < 1. \quad (2.19)$$

It is also well-known that if (2.19) is violated, then convergence can often be recovered by using underrelaxation with appropriate relaxation factor. This approach of using the Jacobi iteration (2.16), with underrelaxation if necessary, is sufficient for the purpose of the present paper, ie. for illustrating the application of the algorithm outlined above. We note, however, that it might be more efficient to perform Step 1 of the algorithm by solving the discrete Theodorsen integral equation using one of the several alternative iterative procedures which have been proposed in recent years. Full details of these recent developments can be found in the survey paper by Gutknecht [11] and the other references cited there.

### 3 Numerical examples

In this section we present five examples illustrating the use of the numerical algorithm of Section 2, for computing approximations to Faber polynomials associated with starlike curves. For comparison purposes, in each of our examples we choose a curve  $\Gamma$  for which the mapping function  $\psi$  is known exactly. Then, by using if necessary the symbolic manipulation package MAPLE, we determine in rational form the mapping coefficients  $c, c_j$  and ( whenever  $\phi := \psi^{-1}$  is also available ) the corresponding exact Faber polynomials.

All our numerical results were computed on a Pyramid 9825 computer, using programs written in double precision Fortran. ( Double length working on the Pyramid is between 15 and 16 significant figures. ) In presenting these results we give in each case the following information, regarding the use of the Theodorsen method in Step 1 of the algorithm:

- The degree  $N$  of the trigonometric interpolating polynomial used.
- The relaxation parameter  $\omega$  used in the iteration.
- The number  $K$  of iterations needed for convergence.

**3.1 Ellipse** Let  $z = x + iy$  and let  $\Gamma$  be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0.$$

Then,

$$\psi(w) = \frac{1}{2} \{ (a+b)w + (a-b)w^{-1} \},$$

and hence

$$\phi(z) = \frac{1}{a+b} \left\{ z + z \left( 1 - \frac{a^2 - b^2}{z^2} \right)^{1/2} \right\}.$$

In particular, if

$$a = 1+d \quad \text{and} \quad b = 1-d, \quad 0 < d < 1$$

, then

$$\psi(w) = w + d/w,$$

and

$$\phi(z) = \frac{1}{2} \left\{ z + z \left( 1 - \frac{4d}{z^2} \right)^{1/2} \right\},$$

see e.g. [3, p.225] and [12, p.412]). Thus, in this case,

$$c = 1, \quad c_0 = 0, \quad c_1 = d, \quad c_2 = c_3 = \dots = 0,$$

and, for  $n \geq 2$ , the recurrence formula (1.3) simplifies to

$$p_{n+1}(z) = z p_n(z) - d p_{n-1}(z).$$

Hence

$$\begin{aligned} P_1(z) &= z, & p_2(z) &= z^2 - 2d, & p_3(z) &= z^3 - 3dz, \\ P_4(z) &= z^4 - 4dz^2 + 2d^2, & p_5(z) &= z^5 - 5dz^3 + 5d^2z, \end{aligned}$$

e.t.c. More generally, it follows easily from the solution of the difference equation

$$p_{n+1} - z p_n + d p_{n-1} = 0$$

that

$$p_j(z) = \frac{1}{2^{j-1}} \sum_{k=0}^{[j/2]} \begin{bmatrix} j \\ 2k \end{bmatrix} z^{j-2k} (z^{j-2k} - 4d)^k, \quad j \geq 1, \quad (3.1)$$

where  $[J]$  denotes the largest integer  $\leq J$  ( See [3, p.226] and note that due to a printing error the factor  $2^{j+1}$  in the left hand side of (3.1) is given incorrectly in [3] as 2.)

In the case  $d = 0.4$ , the Theodorsen method with  $N = 32$  and  $\omega = 0.5$  gives the following results, after  $K = 54$  iterations:

- The approximations to  $c = 1$  and  $c_l = 0.4$  are respectively

$$1.000\ 000\ 000\ 002\ 4 \quad \text{and} \quad 0.400\ 000\ 000\ 0010.$$

- The maximum error in the approximations to the coefficients

$$c_0 = c_2 = c_3 = \dots = c_{64} = 0$$

is less than  $7.9 \times 10^{-11}$ .

- The corresponding approximations to the nonzero coefficients

$$\alpha_0^{(10)} = -0.020\ 48, \quad \alpha_2^{(10)} = 0.64, \quad \alpha_4^{(10)} = -3.2,$$

and

$$\alpha_6^{(10)} = 56, \quad \alpha_8^{(10)} = -4, \quad \alpha_{10}^{(10)} = 1,$$

of the Faber polynomial  $p_{10}$  are respectively

$$-0.020479\ 999\ 97, \quad -0.639\ 999\ 999\ 98, \quad -3.200\ 000\ 000\ 12,$$

and

$$5.600\ 000\ 000\ 01, \quad -3.999\ 999\ 999\ 92, \quad 0.999\ 999\ 999\ 98.$$

- The maximum error in the approximations to the coefficients

$$\alpha_1^{(10)} = \alpha_3^{(10)} = \alpha_5^{(10)} = \alpha_7^{(10)} = \alpha_9^{(10)} = 0,$$

is less than  $2.9 \times 10^{-15}$ .

**3.2 Two intersecting circles** Let  $\Omega_1$  be the domain bounded by the two intersecting circles

$$|z - 1.6| = 2.0 \quad \text{and} \quad |z - 0.9| = 1.5.$$

That is  $\Omega_1$ , is bounded by the curve

$$\Gamma := \{ z : z = \rho(\theta)e^{i\theta}, \quad 0 \leq \theta \leq 2\pi \},$$

where

$$\rho(\theta) = \begin{cases} 0.9 \cos \theta + \{2.25 - 0.81 \sin^2 \theta\}^{1/2}, & 0 \leq \theta \leq \pi/2, \\ -1.6 \cos \theta + \{4.00 - 2.56 \sin^2 \theta\}^{1/2}, & \pi/2 < \theta \leq \pi \end{cases}$$

With

$$\rho(\theta) = \rho(2\pi - \theta), \quad \pi < \theta \leq 2\pi$$

Then

$$\phi(z) = \frac{2}{5z} \left( z - \frac{9}{10} \right) \left( z + \frac{8}{5} \right), \quad (3.2)$$

and hence

$$\psi(w) = \frac{5}{4}w - \frac{7}{20} + \frac{5}{4}w \left\{ 1 - \frac{14}{25} \frac{1}{w} + \frac{1}{w^2} \right\}^{1/2}, \quad (3.3)$$

(see e.g. [13, p.48]). Thus, from (3.3),

$$c = \frac{5}{2}, c_0 = -\frac{7}{10}, c_1 = \frac{72}{125}, \dots \text{e.t.c.},$$

and, from (3.2), the first four Faber polynomials are :

$$p_1(z) = \frac{2}{5}z + \frac{7}{25}, \quad p_2(z) = \frac{4}{25}z^2 + \frac{28}{125}z - \frac{239}{625},$$

$$p_3(z) = \frac{8}{125}z^3 + \frac{84}{625}z^2 - \frac{114}{625}z - \frac{1,141}{3,125},$$

$$p_4(z) = \frac{16}{625}z^4 + \frac{224}{3,125}z^3 - \frac{1,128}{15,625}z^2 - \frac{21,448}{78,125}z + \frac{8,429}{78,125}.$$

The Theodorsen method with  $N=1,024$  and  $\omega = 0.4$  gives, after  $K=86$  iterations, the approximation

$$2.49999999957$$

to the capacity  $c = 2.5$  of the curve  $\Gamma$ . The corresponding approximations to the coefficients  $c_j, j = 0, 1, \dots, 20$  of the mapping function  $\psi$  and the coefficients  $\alpha_k^{\{20\}}, k = 0, 1, \dots, 20$  of the Faber

polynomial  $p_{20}(z)$  are listed in Tables 3.1 and 3.2 respectively. In each of the tables we also include the corresponding exact values which were obtained by rounding, if necessary, the rational MAPLE values to 10 significant figures.

### 3.3 Oval of Cassini

Let  $\Gamma$  be the oval of Cassini

$$\Gamma := \{ z : z = \rho(\theta)e^{i\theta}, 0 \leq \theta \leq 2\pi \},$$

where

$$\rho(\theta) := \{ \cos 2\theta + [\cos^2 2\theta + a^4 - 1]^{1/2} \}^{1/2}, \quad 0 \leq \theta \leq \pi, \quad a \geq 1,$$

and

$$\rho(\theta) = \rho(2\pi - \theta), \quad \pi < \theta \leq 2\pi$$

Then,

$$\psi(w) = aw \left\{ 1 + \frac{1}{a^2 w^2} \right\}^{1/2},$$

$$\phi(z) = \frac{z}{a} \left\{ 1 - \frac{1}{z^2} \right\}^{1/2},$$

and the exact value of  $c := \text{cap } \Gamma$  is  $c = a$ .

For the case  $a = 2$ , the Theodorsen method with  $N = 64$  and  $\omega = 1$  gives, after  $K = 18$  iterations, the capacity  $c = 2$  correct to 14 significant figures. The corresponding approximations to the non-zero Laurent coefficients  $c_{2j-1}, j = 1, 2, \dots, 10$  of the mapping function  $\psi$  and the non-zero coefficients  $\alpha_{2k}^{\{20\}}, k = 0, 1, \dots, 10$ , of the Faber polynomial  $p_{20}$  are listed in Tables 3.3 and 3.4. The tables also include the corresponding exact values which were obtained by rounding, if necessary, the rational MAPLE values to 12 significant figures.

Similarly, in the case  $a = 1.2$ , the Theodorsen method with  $N = 64$  and  $\omega = 0.6$  gives, after  $K = 62$  iterations, the capacity  $c = 1.2$  correct to 13 significant figures. The corresponding approximations to the non-zero coefficients  $c_{2j-1}, j = 1, 2, \dots, 10$  and  $\alpha_{2k}^{\{20\}}, k = 0, 1, \dots, 10$  are listed together with the exact values in Tables 3.5 and 3.6.

<i>J</i>	<i>Exact c<sub>j</sub></i>	<i>Approximate c<sub>j</sub></i>	<i>J</i>	<i>Exact c<sub>j</sub></i>	<i>Approximate c<sub>j</sub></i>
0	-7.0(-1)	-7.000 000 003 (-1)	11	2.496 089 277 (-2)	2.496 089 397 (-2)
1	5.76 (-1)	5.760 000 008 (-1)	12	8.569 622 489 (-3)	8.569 623 452 (-3)
2	1.612 8 (-1)	1.612 800 008(-1)	13	-1.532 731 879 (-2)	-1.532 731 942 (-2)
3	-8.755 2 (-2)	-8.755 200 040 (-2)	14	-1.458 066 666 (-2)	-1.458 066 803 (-2)
4	-9.883 238 4 (-2)	-9.883 238 511 (-2)	15	5.053 758 185 (-3)	5.053 757 968 (-3)
5	2.266 398 72 (-3)	2.266 398 481 (-3)	16	1.458 799 731 (-2)	1.458 799 856 (-2)
6	5.747 286 344 (-2)	5.747 286 449 (-2)	17	3.277 040 132 (-3)	3.277 041 137 (-3)
7	2.473 365 366 (-2)	2.473 365 457 (-2)	18	-1.059 436 651 (-2)	-1.059 436 713 (-2)
8	-2.677 287 058 (-2)	-2.677 287 114 (-2)	19	-8.273 365 964 (-3)	-8.273 367 367 (-3)
9	-3.005 744 396 (-2)	-3.005 744 525 (-2)	20	4.778 735 278 (-3)	4.778 735 005 (-3)
10	4.934 305 707 (-3)	4.934 305 491 (-3)			

**Table 3.1** (Laurent coefs for 2 intersecting circles.)

<i>K</i>	<i>Exact <math>\alpha_k^{\{20\}}</math></i>	<i>Approximate <math>\alpha_k^{\{20\}}</math></i>	<i>K</i>	<i>Exact <math>\alpha_k^{\{20\}}</math></i>	<i>Approximate <math>\alpha_k^{\{20\}}</math></i>
0	1.608 445 417 (-1)	1.608 445 650 (-1)	11	-1.165 648 426 (-3)	-1.165 648 442 (-3)
1	6.637 133 211 (-2)	6.637 132 432 (-2)	12	-1.339 408 792 (-5)	-1.339 408 918 (-5)
2	-1.000 767 335 (-1)	-1.000 767 435 (-1)	13	2.183 991 056 (-4)	2.183 991 076 (-4)
3	-4.230 563 949 (-2)	-4.230 563 472 (-2)	14	4.278 154 752 (-5)	4.278 154 797 (-5)
4	4.955 265 047 (-2)	4.955 264 757 (-2)	15	-2.201 457 486 (-5)	-2.201 457 499 (-5)
5	2.383 175 181 (-2)	2.383 175 556 (-2)	16	-9.410 538 603 (-6)	-9.410 538 664 (-6)
6	-1.894 848 188 (-2)	-1.894 848 299 (-2)	17	8.774 102 790 (-8)	8.774 102 583 (-8)
7	-1.124 125 419 (-2)	-1.124 125 409 (-2)	18	7.069 859 767 (-7)	7.069 859 795 (-7)
8	5.213 209 722 (-3)	5.213 209 672 (-3)	19	1.539 316 279 (-7)	1.539 316 285 (-7)
9	4.198 521 885 (-3)	4.198 521 940 (-3)	20	1.099 511 628 (-8)	1.099 511 632 (-8)
10	-8.358 293 364 (-4)	-8.358 293 397 (-4)			

**Table 3.2** (Faber poly. for  $2_{20}$  intersecting circles)

$J$	<i>Exact <math>c_j</math></i>	<i>Approximate <math>c_j</math></i>
1	2.5 (-1)	2.500 000 000 00 (-1)
3	-1.562 5 (-2)	-1.562 500 000 00 (-2)
5	1.953 125 (-3)	1.953 125 000 04 (-3)
7	-3.051 757 812 5 (-4)	-3.051 757 811 94 (-4)
9	5.340 576 171 88 (-5)	5.340 576 165 58 (-5)
11	-1.001 358 032 23 (-5)	-1. 001 358 034 97 (-5)
13	1.966 953 277 59 (-6)	1.966 953 338 67 (-6)
15	-3.995 373 845 10 (-7)	-3.995 373 893 14 (-7)
17	8.323 695 510 63 (-8)	8.323 691 995 13 (-8)
19	-1.768 785 296 01 (-8)	-1.768 783 710 48 (-8)

**Table 3.3** (Laurent coefs for oval of Cassini

a=2.0)

$k$	<i>Exact <math>\alpha_k^{\{20\}}</math></i>	<i>Approximate <math>\alpha_k^{\{20\}}</math></i>
0	9.536 743 164 (-7)	9.536 741 181 35 (-7)
2	-9.536 743 164 (-6)	-9.536 743 099 12(-6)
4	4.291 534 423 8 (-5)	4.291 534 426 49 (-5)
6	-1.144 409 179 68 (-4)	-1.144 409 179 76 (-4)
8	2.002 716 064 44 (-4)	2.002 716 064 43 (-4)
10	-2.403 259 277 33 (-4)	-2.403 259 277 33 (-4)
12	2. 002 716 064 44 (-4)	2. 002 716 064 45 (-4)
14	-1.144 409 179 68 (-4)	-1.144 409 179 69 (-4)
16	4.291 534 423 8 (-5)	4.291 534 423 83 (-5)
18	-9. 536 743 164 06(-6)	-9. 536 743 164 06 (-6)
20	9. 536 743 164 06(-7)	9. 536 743 164 06 (-7)

**Table 3.4** (Faber poly.  $p_{20}$  for oval of Cassini

a=2.0)

$j$	<i>Exact <math>c_j</math></i>	<i>Approximate <math>c_j</math></i>
1	4.166 666 666 67 (-1)	4.166 666 666 66 (-1)
3	-7.233 796 296 29 (-2)	-7. 233 796 296 32 (-2)
5	2.511 734 825 10 (-2)	2.511 734 825 19 (-2)
7	-1.090 162 684 51 (-2)	-1.090 162 684 58 (-2)
9	5.299 401 938 57 (-3)	5.299 401 938 41 (-3)
11	-2.760 105 176 34 (-3)	-2.760 105 175 09 (-3)
13	1.506 009 768 84 (-3)	1.506 009 767 08 (-3)
15	-8.497 450 952 63 (-4)	-8.497 450 952 34 (-4)
17	4.917 506 338 33 (-4)	4.917 506 335 46 (-4)
19	-2.902 694 713 60 (-4)	-2.902 694 719 28 (-4)

**Table 3.5** (Laurent coefs for oval of Cassini a=1 .2 )

$K$	<i>Exact <math>\alpha_k^{\{20\}}</math></i>	<i>Approximate <math>\alpha_k^{\{20\}}</math></i>
0	2.608 405 331 (-2)	2.608 405 330 44 (-2)
2	-2.608 405 331 (-1)	-2.608 405 330 03 (-1)
4	1. 173 782 398 95	1. 173 782 398 67 (-1)
6	-3.130 086 397 2	-3.130 086 396 56
8	5.477 651 195 1	5.477 651 193 98
10	-6.573 181 434 12	-6.573 181 432 75
12	5.477 651 195 10	5.477 651 193 95
14	-3.130 086 397 20	-3.130 086 396 54
16	1.173 782 398 95	1.173 782 398 70
18	-2.608 405 331 (-1)	-2.608 405 330 45 (-1)
20	2.608 405 331 (-2)	2.608 405 330 45 (-2)

**Table 3.6** (Faber poly.  $p_{20}$  for oval of Cassini a=1.2)

3.4 **Circular sectors** Let  $\Omega_1$  be a circular sector of the form

$$\{\zeta : |\zeta| < 1, \quad |\arg \zeta| < \alpha\}, \quad 0 < \alpha < \pi/2$$

For the application of the Theodorsen method we translate  $\Omega_1$  along the real axis ( so that the vertex at  $\zeta = 0$  goes to the point  $z = 0.5$  ) and express the boundary  $\Gamma$  in polar form:

$$\Gamma := \{ z = 0.5 + \rho(\theta)e^{i\theta}, \quad 0 \leq \theta \leq 2\pi \},$$

where

$$\rho(\theta) := \begin{cases} 0.5 \left\{ -\cos \theta + [\cos^2 \theta + 3]^{1/2} \right\}, & 0 < \theta \leq \pi - \beta, \\ 0.5 \sin \alpha / \sin (\theta - \alpha), & \pi - \beta < \theta \leq \pi, \end{cases}$$

with

$$\rho(\theta) = \rho(2\pi - \theta), \quad \pi < \theta \leq 2\pi$$

And

$$\beta := \arctan \left\{ \frac{2 \sin \alpha}{1 - 2 \cos \alpha} \right\}.$$

A formula for the conformal mapping  $\psi$  and a recurrence relation for the coefficients of its Laurent expansion have been derived recently by Coleman and Smith in [2]. In particular, it is shown in [2] that the capacity of  $F$  is given by

$$c = \frac{\hat{\alpha}^2}{(2\hat{\alpha} - 1)^{2-1/\hat{\alpha}}}, \quad \text{Where } \hat{\alpha} := \pi/\alpha.$$

Our numerical results are for the three cases  $\alpha = \pi/2, \pi/4$  and  $\pi/12$ . The values of the degree  $N$  of the interpolating polynomial, the relaxation parameter  $\omega$  and the number of iterations  $K$  used, in each case, are given below together with the value of  $c := \text{cap } \Gamma$  and the computed approximation  $\tilde{c}$ . (Here, for each  $\alpha$ , the relaxation parameter  $\omega$  was determined as indicated by Gutknecht [10, p.412], by estimating the value of  $e$  in (2.19) and taking  $\omega = 1/(1 + \varepsilon^2)$ .) The corresponding approximations to the Laurent coefficients  $c_j, j = 0, 1, \dots, 10$ , of the mapping function  $\psi$ , and the coefficients  $\alpha_j^{(10)}, j = 0, 1, \dots, 10$ , of the Faber polynomial  $p_{10}$ , are listed in Tables 3.7 - 3.12.

(i)  $\alpha = \pi/2$

$$N=2,048, \quad \omega = 0.2, \quad K=114,$$

$$c=0.769\ 800\ 359, \quad \tilde{c} = 0.769\ 800\ 296.$$

(ii)  $\alpha = \pi/4$

$$N=2,048, \quad \omega = 0.23,$$

$$K=100, \quad c=0.531127\ 041, \quad \tilde{c} = 0.531127\ 014.$$



$$(iv) \alpha = \pi/12$$

$$N = 2,048, \quad \omega = 0.05895, \quad K = 288,$$

$$c = 0.353495704, \quad \tilde{c} = 0.353495718.$$

For comparison purposes, in each of the Tables 3.7, 3.9, and 3.11 we include the exact values of the Laurent coefficients  $c_j, j=0,1,\dots, 10$ , rounded to eight decimal places. These were obtained by means of MAPLE from the formula for  $\psi$  given in [2, p.232]. Similarly, in each of the Tables 3.8,

3.10, and 3.12 we give the exact values of  $\alpha_j^{\{10\}}, j=0,1,\dots,10$ . These were obtained by rounding to

eight significant figures the appropriately normalized values given in the supplement of [2]. (For the case  $\alpha = n/2$  the exact  $\alpha_j^{\{10\}}$  can also be determined, by means of MAPLE, from the exact formula for the mapping function  $\phi$  which is given in [7, p.138].)

### 3.5 Square Let $\Omega_1$ be the square

$$\Omega_1 := \{z = x + iy : |x| < 1, |y| < 1\}.$$

Then,

$$\begin{aligned} \psi(w) &= c \int \left\{ 1 + \frac{1}{w^4} \right\}^{1/2} dw \\ &= c \left\{ w - \frac{1}{6} \frac{1}{w^3} + \frac{1}{56} \frac{1}{w^7} - \frac{1}{176} \frac{1}{w^{11}} + \frac{1}{384} \frac{1}{w^{15}} - \frac{7}{4,864} \frac{1}{w^{19}} \right. \\ &\quad \left. + \frac{21}{23,552} \frac{1}{w^{23}} - \frac{33}{55,296} \frac{1}{w^{27}} \dots \right\}, \end{aligned} \tag{3.4a}$$

where to 11 decimal places the exact value of  $\text{cap } \Gamma$  is

$$c = 1.18034059902, \tag{3.4b}$$

(see e.g. [1], [5, p.582] and [7, p.114]).

In this case the Theodorsen method with  $N = 2,048$  and  $\omega = 0.5$  gives, after  $K = 46$  iterations, the approximation

$$1.18034071$$

to the capacity  $c = 1.18034060$ . The corresponding approximations to the non-zero coefficients  $c_j; j=3(4)23$  of  $\psi$  are listed in Table 3.13 and are compared with the exact values obtained from (3.4). In Table 3.14 we list the approximations to the non-zero coefficients  $\alpha_k^{\{18\}}, k=2(4)18$  of the Faber polynomial  $p$ . These are compared with the corresponding exact values, which were obtained by means of MAPLE from the exact formula for  $\phi$  given by Elliot in [7, p.142].

$j$	<i>Exact</i> $c_j$	<i>Appromimate</i> $c_i$	$j$	<i>Exact</i> $c_j$	<i>Approximate</i> $c_j$
0	0.384 900 18	0.384 900 24	6	-0.006 577 88	-0.006 577 85
1	-0.240 562 61	-0.240 562 57	7	-0.000 716 52	-0.000 716 51
2	0.120 281 31	0.120 281 22	8	0.003 729 41	0.003 729 38
3	-0.039 091 42	-0.039 091 37	9	-0.002 342 02	-0.002 341 87
4	-0.001 503 52	-0.001 503 50	10	-0.000 375 55	-0.000 373 67
5	0.011 652 25	0.011 652 20			

**Table 3.7 ( Laurent coefs for sector;  $\alpha = \pi/2$  )**

$K$	<i>Extrat</i> $\alpha_k^{\{10\}}$	<i>Approximate</i> $\alpha_k^{\{10\}}$	$K$	<i>Extrat</i> $\alpha_k^{\{10\}}$	<i>Approximate</i> $\alpha_k^{\{10\}}$
0	1.022 800 4	1.022 800 1	6	2.133 343 6 (+2)	2.133 345 6 (+2)
1	-7.701 740 04	-7.701 741 5	7	-1.814 204 4(+2)	-1.814 206 1 (+2)
2	2.960 669 5 (+1)	2.960 670 5 (+1)	8	1.165 689 8(+2)	1.165 690 9 (+2)
3	-7.497 972 8 (+1)	-7.497 976 4 (+1)	9	-5.267 044 9 (+1)	-5.267 050 1 (+1)
4	1.382 948 7 (+2)	1.382 949 6 (+2)	10	1.368 418 (+1)	1.368 419 5 (+1)
5	-1.947 289 4 (+2)	-1.947 290 9 (+2)			

**Table 3.8 ( Faber poly.  $p_{10}$  for sector;  $\alpha = \pi/2$  )**

$j$	<i>Exact</i> $c_j$	<i>Appromimate</i> $c_i$	$j$	<i>Exact</i> $c_j$	<i>Approximate</i> $c_j$
0	0.597 517 92	0.597 517 88	6	-0.007 179 60	-0.007 179 59
1	-0.093 362 18	-0.093 362 08	7	0.005 285 46	0.005 285 42
2	-0.081 691 90	-0.081 691 91	8	0.000 585 69	0.000 210 39
3	-0.055 652 61	-0.055 652 58	9	0.000 211 06	0.000 210 39
4	-0.004 531 35	-0.004 531 38	10	-0.002 514 03	-0.002 525 44
5	-0.002 421 24	-0.002 421 20			

**Table3.9 (Laurent coefs sector ;  $\alpha = \pi/4$ )**

$k$	<i>Extra</i> $\alpha_k^{\{10\}}$	<i>Approximate</i> $\alpha_k^{\{10\}}$	$K$	<i>Extra</i> $\alpha_k^{\{10\}}$	<i>Approximate</i> $\alpha_k^{\{10\}}$
0	1.515 342 7	1.515 335 1	6	1.722 120 8 (+4)	1.722 121 0 (+4)
1	-4.924 891 1(+1)	-4.924 881 9(+1)	7	-1.552 798 6 (+4)	-1.552 799 0(+4)
2	4.734 517 4(+2)	-4.734 512 5(+2)	8	9.271 109 6(+3)	9.271 112 8 (+4)
3	-2.296 669 7(+3)	-2.296 668 3(+3)	9	-3.344 774 2(+3)	-3.344 775 7 (+3)
4	6.750 601 4 (+3)	6.750 599 3 (+3)	10	5.597 780 6 (+3)	5.597 783 4 (+4)
5	-1.305 799 6(+4)	-1.305 799 (+4)			

**Table 3.10** (Faber poly.  $P_{10}$  for sector ;  $\alpha = \pi 4$  )

$j$	<i>Extras</i> $c_j$	<i>Approximate</i> $c_j$	$j$	<i>Extras</i> $c_j$	<i>Approximate</i> $c_j$
0	0.594 069 17	0.594 069 17	6	0.003 435 77	0.003 435 76
1	0.154 705 51	0.154 705 52	7	0.002 796 11	0.002 795 96
2	-0.091 15211	0.091 152 13	8	-0.000 001 69	-0.000 002 15
3	-0.015 24769	-0.015 247 70	9	-0.000 724 77	-0.000 728 16
4	-0.04 957 97	-0.004 957 97	10	-0.001 339 83	-0.001 332 63
5	0.004 765 05	0.004 765 06			

**Table 3.11** (Laurent coefs for sector ;  $\alpha = \pi /12$  )

$k$	<i>Extra</i> $\alpha_k^{\{10\}}$	<i>Approximate</i> $\alpha_k^{\{10\}}$	$k$	<i>Extra</i> $\alpha_k^{\{10\}}$	<i>Approximate</i> $\alpha_k^{\{10\}}$
0	1.844 379 0	1.844 373 5	6	6.692 1180	6.692 115 3 (+5)
1	-1.832 133 4 (+2)	-1.832 131 (+2)	7	-7.367 107 1(+5)	-7.367 104 2 (+5)
2	3.668 461 5 (+3)	3.668 458 9 (+3)	8	5.032 995 5 (+5)	5.032 993 5 (+5)
3	-3.108 707 4 (+4)	-3.108 705 7 (+4)	9	-1.949 824 9(+5)	-1.949 824 1 (+5)
4	1.417 392 8 (+5)	1.417 392 1 (+5)	10	3.282 151 3 (+4)	3.282 150 (+4)
5	-3.877 779 (+5)	-3.877 777 9 (+5)			

**Table 3.12** (Faber poly.  $p_{10}$  for sector;  $a = \pi /12$ )

$j$	<i>Exact</i> $c_j$	<i>Approximate</i> $c_j$	$j$	<i>Exact</i> $c_j$	<i>Approximate</i> $c_j$
3	-0.196 723 43	-0.196 723 61	15	0.003 073 80	0.003 073 87
7	0.021 077 51	0.021 077 61	19	-0.001 698 68	-0.001 698 74
11	-0.006 706 48	-0.006 706 56	23	0.001 052 44	0.001 052 50

**Table 3.13** (Laurent coefs for square )

$K$	<i>Exact</i> $\alpha_k^{\{18\}}$	<i>Approximate</i> $\alpha_k^{\{18\}}$	$k$	<i>Exact</i> $\alpha_k^{\{18\}}$	<i>Approximate</i> $\alpha_k^{\{18\}}$
2	-0.004 290 94	-0.004 291 32	14	0.294 454 65	0.294 454 48
6	0.186 762 96	0.186 763 21	18	0.050 567 07	0.050 566 98
10	0.462 676 49	0.462 676 57			

**Table 3.14** (Faber poly.  $p_{18}$  for square )

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