Chapter

Padé Approximation to Solve the Problems of Aerodynamics and Heat Transfer in the Boundary Layer

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Abstract

In this chapter, we describe the applications of asymptotic methods to the problems of mathematical physics and mechanics, primarily, to the solution of nonlinear singular perturbed problems. We also discuss the applications of Padé approximations for the transformation of asymptotic expansions to rational or quasi-fractional functions. The applications of the method of matching of internal and external asymptotics in the problem of boundary layer of viscous gas by means of Padé approximation are considered.

Keywords: asymptotic methods, Padé approximation, boundary-value problem of mathematical physics, boundary layer

1. Introduction

An important drawback of asymptotic methods is the local character of solutions obtained [1–4]. Since the constructed series are often asymptotic, a simple increase in the number of terms does not remove this drawback. Essence of the problem consists of divergence of obtained series. There exist a lot of approaches to these problems [5, 6]. The method of analytic continuation (e.g., the Euler transform or generalized Euler transform [7–12]) requires a priori information about the singularities of the searched function in the complex domain [4, 9]. These methods are useful if a large number of terms of the series are known. In this case, it is possible to use the Domb-Sykes plot [5, 8]. But usually only a few terms of asymptotic series are known, and to get information from them, the method of Padé approximations (PAs) is useful [1, 2, 5, 13–15]. PAs yield meromorphic continuations of functions defined by power series and can be used even in cases where analytic continuations are inapplicable. If a PAs converges to the given function, then roots of the denominator tend to points of singularities. One-point PAs give possibilities to improve convergence of series [16–20]. Two-point PAs (TPPAs) allow matching asymptotics in transition zones and are widely used in mechanics and physics [1, 2, 4, 14, 21–24]. Overcoming the mentioned limitations of asymptotic methods for practically important problem is the purpose of this chapter. We consider at the beginning (Section 2) the mathematical bases of asymptotic methods and the use of Padé approximants for the summations of the asymptotic series. Section 3 discusses the method of combining of internal and external asymptotics (matching method) by means of Padé approximants. In the Section 4, the methods of solving specific problems of mathematical physics and mechanics of fluid and gas are demonstrated. Section 5 presents a discussion of the obtained.

2. Mathematical background: summation of asymptotic series

2.1 Analysis of power series

We suppose that by the result of the asymptotic study, one obtains the following series:

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} C_n \varepsilon^n \text{ for } \qquad \varepsilon \to 0. \tag{1}$$

As is known, the radius of convergence ε_0 series (1) is determined by the distance to the nearest singularity of the function $f(\varepsilon)$ on the complex plane. To define ε_0, the Domb-Sykes plot may be useful [8, 10]. In many cases, one can effectively use the conformal mapping of the series, a fairly complete catalog of which is given in [9]. In particular, it sometimes turns out to be a successful Euler transformation [8, 10], based on the introduction of a new variable:

$$\tilde{\varepsilon} = \frac{\varepsilon}{1 - \varepsilon/\varepsilon_0}. \tag{2}$$

Recast the function f in terms of $\tilde{\varepsilon}, f \sim \sum_{n=0}^{\infty} d_n \tilde{\varepsilon}^n$, transfer the singularity at the point $\tilde{\varepsilon} = \infty$.

A natural generalization of Euler transformation looks as follows:

$$\tilde{\varepsilon} = \frac{\varepsilon}{(1 - \varepsilon/\varepsilon_0)^{\alpha}},$$

where α is the certain number.

2.2 Padé approximants

“The coefficients of the Taylor series in the aggregate have a lot more information about the values of features than its partial sums. It is only necessary to be able to retrieve it, and some of the ways to do this is to construct a Padé approximant” [11]. Padé approximants (PAs) allow us to transform of power series to a fractional-rational function. Let us define PAs, following Baker and Graves-Morris [25].

Suppose we are given the power series:

$$f(\varepsilon) = \sum_{i=1}^{\infty} c_i \varepsilon^i, \tag{3}$$

PAs can be written as the following expression:

$$f_{[n/m]}(\varepsilon) = \frac{a_0 + a_1 \varepsilon + \ldots + a_n \varepsilon^n}{1 + b_1 \varepsilon + \ldots + b_m \varepsilon^m}, \tag{4}$$

whose coefficients are determined from the condition

$$(1 + b_1\varepsilon + \dots + b_m\varepsilon^m)(c_0 + c_1\varepsilon + c_2\varepsilon^2 + \dots) = a_0 + a_1\varepsilon + \dots + a_n\varepsilon^n + O(\varepsilon^{n+m+1}) \tag{5}$$

Equating coefficients near the same powers ε, one obtains a system of linear algebraic equations. In the case where this system is solvable, one can obtain the Padé coefficients of the numerator and denominator of the PAs.

We note some properties of the PAs [5, 13, 19]. If the PAs at the chosen m and n exists, then it is unique.

1. If the PAs sequence converges to some function, the roots of its denominator tend to the poles of the function. This allows for a sufficiently large number of terms to determine the pole and then perform an analytical continuation.

2. PAs gives meromorphic continuation of a given power series.

3. PAs of the inverse function is treated as the PAs function inverse itself. This property is called duality and is more exactly formulated as follows. Let

$$q(\varepsilon) = f^{-1}(\varepsilon) \text{ and } f(0) \neq 0, \text{ then } q_{[n/m]}(\varepsilon) = f^{-1}_{[n/m]}(\varepsilon) \tag{6}$$

4. Diagonal PAs are invariant under fractional linear transformations of the argument. Suppose that the function is given by their expansion (3). Consider the linear fractional transformation that preserves the origin $W = a\varepsilon/(1 + b\varepsilon)$ and the function $q(W) = f(\varepsilon)$. Then $q_{[n/n]}(W) = f_{[n/n]}(\varepsilon)$, provided that one of these approximations exist. In particular, the diagonal PAs is invariant concerning Euler transformation (2).

5. Diagonal PAs are invariant under fractional linear transformations of functions. Let us analyze a function (3). Let

$$q(\varepsilon) = \frac{a + bf(\varepsilon)}{c + df(\varepsilon)}.$$

If $c + df(0) \neq 0$, then

$$q_{[n/n]}(\varepsilon) = \frac{a + bf_{[n/n]}(\varepsilon)}{c + df_{[n/n]}(\varepsilon)}$$

provided that there is $f_{[n/n]}(\varepsilon)$.

6. PAs can get the upper and lower bounds for $f_{[n/n]}(\varepsilon)$. For the diagonal PAs, one has the following estimate:

$$f_{[n/n-1]}(\varepsilon) \le f_{[n/n]}(\varepsilon) \le f_{[n/n+1]}(\varepsilon). \tag{7}$$

Typically, this estimate is valid for the function itself, that is, $f_{[n/n]}(\varepsilon)$ in Eq. (7) can be replaced by $f(\varepsilon)$.

7. Diagonal and close to them a sequence of PAs often possesses the property of autocorrection [17, 18]. It consists of the following. To determine the coefficients of the numerator and denominator of PAs, we have to solve

systems of linear algebraic equations. This is an ill-posed procedure, so the coefficients of PAs can be determined with large errors. However, these errors in a certain sense are of self-consistent, so the accuracy of PAs is high. This is the radical difference the PAs from the Taylor series, the calculation error of which only increases with increasing number of terms.

Autocorrection property is verified for a number of special functions. At the same time, even for elliptic functions, the so-called Froissart doublets phenomenon arises [26]. Thus, in general, having no information about the location of the poles of the PAs, but relying solely on the very PAs (computed exactly as you wish), we cannot say that you have found a good approximated function. Now consider the question: In what sense the available mathematical results on the convergence of the PAs can facilitate the solution of practical problems? Gonchar's theorem [16] states: If none of the diagonal PAs $f_{[n/n]}(\varepsilon)$ has poles in the circle of radius R, then the sequence $f_{[n/n]}(\varepsilon)$ is uniformly convergent in the circle to the original function $f(\varepsilon)$. Moreover, the absence of poles of the sequence of the $f_{[n/n]}(\varepsilon)$ in a circle of radius R confirms convergence of the Taylor series in the circle. Since the diagonal PAs is invariant under fractional linear maps $\varepsilon \to \varepsilon/(a\varepsilon + b)$, the theorem is true for any open circle containing the point of decomposition, and for any area, which is the union of these circles. A significant drawback in practice is the need to check all diagonal PAs. The fact is that if a circle of radius R has no poles only for a subsequence of the sequence of diagonal PAs, then the uniform convergence to its original holomorphic in the disk is guaranteed only with $r < r_0$, where $0.583 < r_0 < 0.584$ [27]. How can we use these results? Suppose that there are a few terms of the perturbation series and one wants to estimate its radius of convergence R. Consider the interval $[0,\varepsilon_0]$, where the truncated perturbation series and the diagonal PAs of the maximal possible order differ by no more than 5% (adopted in the engineering accuracy of the calculations). If none of the previous diagonal PAs does not have in a circle of radius ε_0 poles, then it is a high level of confidence to assert that $R \geq \varepsilon_0$.

3. Matching of limiting asymptotic expansions

3.1 Method of asymptotically equivalent functions

This method was originally proposed by Slepyan and Yakovlev for the inversion of the integral transformations. Here is a description of this method, following [26].

Suppose that the Laplace transform of a function of a real variable $f(t)$ is

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st} ds.$$

To obtain an approximate expression for the inverse transform, it is necessary to clarify the behavior of the transform to the vicinity of the points $s = 0$ and $s = \infty$ and to determine whether the nature and location of its singular points are on the exact boundary of the regularity or near it. Then the transform $F(s)$ is replaced by the function $F_0(s)$, approximated the exact inversion and satisfying the following conditions:

1. Functions $F_0(s)$ and $F(s)$ are asymptotically equivalent at $s \to \infty$ and $s \to 0$, that is,

$$F_0(s) \sim F(s) \text{ at } s \to 0 \text{ and } s \to \infty.$$

2. Singular points of functions $F_0(s)$ and $F(s)$, located on the exact boundary of the regularity, coincide.

The free parameters of the function $F_0(s)$ are chosen so as to satisfy the conditions of the good approximation of $F(s)$ in the sense of minimum relative error for all real values $s \geq 0$:

$$\max \left| \frac{F_0(s, \alpha_1, \alpha_2, \dots, \alpha_k)}{F(s)} - 1 \right| \to \min \tag{8}$$

Condition (8) is achieved by variation of free parameters α_k. Often the implementation of equalities

$$\int_0^\infty F_0(s)ds = \int_0^\infty F(s)ds$$

or $F_0'(s) \sim F'(s)$ at $s \to 0$ leads to a rather precise fulfillment of the requirements (8).

Constructed in such a way function $F_0(s)$ is called asymptotically equivalent function for $F(s)$ (AEF). Let's dwell on the terminology. In the following sections, we will use the symbols of ordinal relations. We will give strict definitions of these concepts.

Let's consider the function $f(x)$. To describe the ordinal relationships with respect to another function $\varphi(x)$, enter the following definitions:

Definition 1. Let us say that $f(x)$ is a value of order $\varphi(x)$ at $x \to x_0$, that is,

$$f(x) = O(\varphi(x))$$

if $\forall \delta > 0 \exists A : |x - x_0| < \delta \Rightarrow |f(x)| \leq A|\varphi(x)|$.

Definition 2. Let us say that $f(x)$ is a value of order less than $\varphi(x)$ at $x \to x_0$, that is,

$$f(x) = o(\varphi(x))$$

if $\forall \delta > 0 \exists \varepsilon : |x - x_0| < \delta \Rightarrow |f(x)| \leq \varepsilon|\varphi(x)|$.

Here A is a finite number, and ε, δ are infinitely small.

Definition 3. Let us say that $f(x)$ is asymptotically equal to $\varphi(x)$ at $x \to x_0$, that is,

$$f(x) \sim \varphi(x) \text{ if } \frac{f(x)}{\varphi(x)} \to 1.$$

Here we use the term "asymptotically equivalent function." Other terms ("reduced method of matched asymptotic expansions" [28], "quasi-fractional approximants" (QAs) [29], and "mimic function" [30]) are also used.

3.2 Two-point Padé approximants

The analysis of numerous examples confirms "complementarity principle": if for $\varepsilon \to 0$, one can construct a physically meaningful asymptotics, there is a nontrivial asymptotics and for $\varepsilon \to \infty$. The most difficult in the asymptotic approach is the intermediate case of $\varepsilon \sim 1$. In this domain, typically numerical methods work well; however, if the task is to investigate the solution depending on the parameter ε, then it is inconvenient to use different solutions in different areas. Construction of a unified solution on the basis of limiting asymptotics is not a trivial task, and for this purpose, one can use a two-point Padé approximants (TPPAs). We give the definition following [25]. Let

$$F(\varepsilon) = \sum_{i=0}^{\infty} c_i \varepsilon^i \text{ at } \varepsilon \to 0, \tag{9}$$

$$F(\varepsilon) = \sum_{i=0}^{\infty} d_i \varepsilon^{-i} \text{ at } \varepsilon \to \infty \tag{10}$$

TPPA is a rational function of the form:

$$f_{[n/m]}(\varepsilon) = \frac{a_0 + a_1\varepsilon + \dots + a_n\varepsilon^n}{1 + b_1\varepsilon + \dots + b_m\varepsilon^m}, \tag{11}$$

k coefficients which are determined from the condition

$$(1 + b_1\varepsilon + \dots + b_m\varepsilon^m)(c_0 + c_1\varepsilon + c_2\varepsilon^2 + \dots) = a_0 + a_1\varepsilon + \dots + a_n\varepsilon^n + O(\varepsilon^{n+m+1}) \tag{12}$$

and the remaining coefficients from a similar condition for ε^{-1}.

4. Application of Padé approximants

4.1 Using of TPPAs in boundary-value problems

For boundary-value problems, we assume that there exist two asymptotics for limit values of the parameter. In this case, the method of matching of asymptotic expansions is usually used [4]. However, for correct application of the matching method, it is necessary to know the matching point or, at least, the domain of overlapping of asymptotics. An exact description of the transition layer $0 < \varepsilon < \infty$ exists only in the cases where solutions with different behaviors on opposite sides of the layer can be matched by a special function (e.g., the Airy function).

For the matching of nonoverlapping asymptotics, a method based on TPPAs has recently been developed. In [15, 21, 23], this method was applied for the construction of thermal profiles in a boundary layer of gas. In [2, 6], this method allowed one to examine the heat exchange in hypersonic boundary layers.

Two-point Padé approximations (TPPAs) are defined in Section 3.2 [see formulas (2)–(4)]. As an example of application of TPPAs, we consider the Airy boundary-value problem [4, 10, 31]:

$$y'' - \lambda^2 x y = g(x) y \text{ as } \lambda \to \infty \tag{13}$$

with boundary conditions

$$y(0) = 1, \;\; y(\infty) = 0 \tag{14}$$

This boundary-value problem has the form in terms of Airy function $U(s)$:

$$U'' - sU = 0, \quad U(0) = 1, \quad U(\infty) = 0 \tag{15}$$

The asymptotic solution for problems (13) and (14) has the form:

$$y(\mathrm{x}) = U(\mathrm{s})\left[1 + O\left(-\lambda^{-1}\right)\right] \text{ as } s = x\lambda^{2/3}. \tag{16}$$

The interior asymptotic ($s \to 0$) has the form of a power function:

$$U^i = 1 - as + \frac{1}{6}s^3 + O\left(s^4\right) \tag{17}$$

The exterior asymptotic has the form of an exponential function:

$$U^e = bs^{-1/4} \exp\left(-\frac{2}{3}s^{1/2}\right)\left[1 - \frac{5}{48}s^{-3/2} + O\left(s^{-3}\right)\right] \tag{18}$$

as $a \cong 0.7290, \ b \cong 0.7946$.

The transition layer is defined by the domain, where $x = O\left(\lambda^{-2/3}\right)$

Airy function approaches with TPPA:

$$U_a = \frac{1 - as + \frac{2}{3}s^{3/2} - \frac{2}{3}as^{5/2} + \frac{32}{5}as^4}{1 + \frac{32}{5}\frac{a}{b}s^{12/4}} \exp\left(-\frac{2}{3}s^{3/2}\right) \tag{19}$$

The TPPA (19) preserves three terms of the asymptotics at both ends and provides accuracy with relative error:

$$\Delta = \frac{|U - U_a|}{U} \sim 1.5\%$$

Parameters a and b are obtained from the integral equations (relations). The relations (20) and (21) can be obtained by multiplying Eq. (18) by 1, s, s^2, … and then by integrating from 0 to ∞.

$$U'' = sU \Rightarrow \int_0^\infty U''ds = \int_0^\infty sUds \Rightarrow \int_0^\infty (U')'ds = \int_0^\infty sUds \Rightarrow U'|_o^\infty = \int_0^\infty sUds$$

This is the first integral relation.

$$\int_0^\infty sUds = a \tag{20}$$

$$sU'' = s^2U \Rightarrow \int_0^\infty sU''ds = \int_0^\infty s^2Uds \Rightarrow \begin{vmatrix} s = t, & dt = ds \\ U''ds = dV, & V = U' \end{vmatrix} \Rightarrow U'|_0^\infty - \int_0^\infty U'ds$$
$$= \int_0^\infty s^2Uds \Rightarrow \int_0^\infty s^2Uds = 1$$

This is the next integral relation.

$$\int_0^\infty s^2 U ds = 1 \tag{21}$$

Substituting in Eqs. (20) and (21) instead of U (4) interpolation U_a (7), calculate using quadrature integration formulas a = 0.7287 and b = 0.7922.

In the same manner, integral relations with weights U, U' can be obtained by part integration. Multiplying Eq. (18) by $U,\ \ U',\ \ U''\ \ldots$, we get after integration from 0 to ∞,

$$UU'' = sU^2 \Rightarrow \int_0^\infty UU'' ds = \int_0^\infty sU^2 ds \Rightarrow \left| \begin{matrix} U = t, & dt = dU \\ U'' ds = dV, & V = U' \end{matrix} \right| \Rightarrow UU'^2\Big|_0^\infty - \int_0^\infty U'^2 ds$$
$$= \int_0^\infty sU^2 ds \Rightarrow a - \int_0^\infty U'^2 ds = \int_0^\infty sU^2 ds \Rightarrow \int_0^\infty \left(U'^2 + sU^2\right) ds = a$$

This is the first integral relation for the second method of producing it:

$$\int_0^\infty \left(U'^2 + sU^2\right) ds = a \tag{22}$$

$$U'U'' = sU'U \Rightarrow \int_0^\infty U'U'' ds = \int_0^\infty sU'U ds \Rightarrow \left| \begin{matrix} U' = t, & dt = U'' ds \\ U'' ds = dV, & V = U' \end{matrix} \right|$$
$$\Rightarrow U'^2\Big|_0^\infty - \int_0^\infty U'U'' ds = \int_0^\infty sU'U ds \Rightarrow -a^2 = 2\int_0^\infty sU'U ds$$
$$\Rightarrow \left| \begin{matrix} s = t, & dt = ds \\ U'U ds = dV, & V = \dfrac{U^2}{2} \end{matrix} \right| \Rightarrow -a^2 = 2\left[\frac{sU^2}{2}\Big|_0^\infty - \int_0^\infty \frac{U^2}{2} ds \right] \Rightarrow \int_0^\infty sU^2 ds$$
$$\Rightarrow \int_0^\infty (U^2) ds = a^2$$

And this is the next integral relation for the second method of producing it:

$$\int_0^\infty U^2 ds = a^2 \tag{23}$$

Using Eq. (19), from Eqs. (22) and (23), we calculate a = 0.7277, and b = 0.7966.

From the given example, it follows that the features of the asymptotic connection method are the ambiguity of the algorithm, the freedom to choose both the form of TPPAs, integral relations, and methods for calculating the parameters of the TPPAs. The question of choosing integral relations is, in fact, a question of controlling the asymptotic approximation using weights selected to obtain integral relations. Choosing the weight allows you to achieve acceptable accuracy in a particular area of the boundary layer: a weight equal to 1 means that the uniform influence of the entire layer is taken into account; a weight equal to $1,\ \ s,\ \ s^2,\ \ldots$ increases the influence of the outer region of the layer; and if the desired solution U, U', U'' is chosen as the weight, then its inhomogeneity increases the influence of the local region where the inhomogeneity is concentrated.

4.2 Quasi-fractional Padé approximants (modification of TPPA)

In the illustrated example (5), Eq. (18) TPPA represents a modified (quasi-fractional) two-point Padé approximant (10) by an exponential weight function, the choice of which is dictated by a kind of exterior asymptotics. Evidently, the TPPAs are not panacea. For example, one of the "bottlenecks" of the TPPAs method is related to the presence of logarithmic components in numerous asymptotic expansions. This problem is the most essential for the TPPAs, because, as a rule, one of the limits $\varepsilon \to 0 \;\; or \;\; \varepsilon \to \infty$ for a real mechanical problem gives expansions with logarithmic terms or other complicated functions. It is worth noting that in some cases these obstacles may be overcome by using an approximate method of TPPAs' construction by tacking as limit points not $\varepsilon = 0 \;\; and \;\; \varepsilon = \infty$, but some small and large values. On the other hand, Martin and Baker [32] proposed the so-called quasi-fractional approximants (QAs). Let us suppose that we have a perturbation approach in powers of ε for $\varepsilon \to 0$ and asymptotic expansions $F(\varepsilon)$ containing, for example, logarithm for $\varepsilon \to \infty$. By definition, QA is a ratio R with unknown coefficients a_i, b_i, containing both powers of ε and $F(\varepsilon)$. We give this modification of TPPA [2, 14, 15]. Let the series give for Eq. (5). Then the modification of TPPA is represented by the irrational function:

$$F(\varepsilon) = \frac{\sum_{k=0}^{m} a_k \varepsilon^k}{\sum_{k=0}^{n} b_k \varepsilon^k} \exp\left(-\sum_{k=0}^{l} c_k \varepsilon^k\right), \tag{24}$$

where $k+1$ coefficients $c_k, (k = 0, 1, 2, \ldots)$ are determined by means of $l+1$ integral equations for function from Eqs. (20) and (21). We notice that exponential terms [multiplier in expressions (17) and (18)] give for $\varepsilon = 0 \;\; and \;\; \varepsilon = \infty$ coincidence with TPPA (19). When considering the computational aspects of the connection method, it should first be assumed that the system of equations for determining the TPPA parameters is substantially nonlinear. To solve it, we developed a modification of the method of solving nonlinear algebraic systems [4, 23, 24].

4.3 Application of TPPAs in problems of incompressible liquid and gas mechanics

Consider the Blasius equation (45), which describes laminar boundary layers on a flat plate:

$$\begin{aligned} &\varphi''' + \varphi\varphi'' = 0; \\ &\varphi(0) = \varphi'(0) = 0; \;\; \varphi'(\infty) = 2 \end{aligned} \tag{25}$$

where $\varphi(\zeta) = \psi/\sqrt{x}$, $\psi(y)$ is the stream function, $\zeta = \frac{y}{2}\sqrt{\frac{\mathrm{Re}}{x}}$ is the automodel variable, and x and y are the Cartesian coordinates such that the axis x is directed along the flow. The interior asymptotic ($\zeta \to 0$) has the form:

$$\varphi = a_2\zeta^2 - \frac{a_2^2}{30}\zeta^5 + O(\zeta^8) \tag{26}$$

The procedure for obtaining external asymptotics is nontrivial due to the presence of logarithmic components in the main elements. We describe in detail the mechanism for obtaining and evaluating both primary and secondary members of asymptotic. From Eq. (25) follows:

$$\frac{\varphi'''}{\varphi''} = \varphi \tag{27}$$

After integration of Eq. (27) by the coordinate ζ follows:

$$\ln\left[\varphi''(\zeta)\right]\Big|_0^\zeta = -\int_0^\zeta \varphi d\zeta \Rightarrow \ln\left[\varphi''(\zeta)\right] - \ln\left(2a_2\right) = -\int_0^\zeta \varphi d\zeta \Rightarrow \varphi''(\zeta) = 2a_2 \exp\left(-\int_0^\zeta \varphi d\zeta\right) \tag{28}$$

After reintegration of Eq. (28) by the coordinate

$$\int_0^\zeta \varphi''(\zeta_1)d\zeta_1 = \int_0^\zeta 2a_2 \exp\left(-\int_0^{\zeta_4} \varphi d\zeta_2\right)d\zeta_1$$

follows:

$$\varphi'(\zeta_1)\Big|_0^\zeta = \int_0^\zeta 2a_2 \exp\left(-\int_0^{\zeta_4} \varphi d\zeta_2\right)d\zeta_1$$

subject to boundary conditions

$$\varphi'(\zeta_1) = \int_0^\zeta 2a_2 \exp\left(-\int_0^{\zeta_1} \varphi d\zeta_2\right)d\zeta_1 \Rightarrow \varphi'(\zeta_1) = \int_0^\zeta 2a_2 \frac{\varphi(\zeta_1)\exp\left(-\int_0^{\zeta_1}\varphi(\zeta_2)d\zeta_2\right)}{\varphi(\zeta_1)}d\zeta_1$$
$$\Rightarrow \varphi'(\zeta_1) = 2a_2 \int_0^\zeta \frac{1}{\varphi(\zeta_1)} d\left(\exp\left(-\int_0^{\zeta_1}\varphi(\zeta_2)d\zeta_2\right)\right)$$

Let us make a limit transition $\zeta \to \infty$ in the last equation and represent the integration interval as
$[0, \infty) = [0, \zeta] \cup [\zeta, \infty)$ follows:

$$\varphi'(\zeta) = 2 + 2a_2 \int_\zeta^\infty \frac{1}{\varphi(\zeta_1)} d\left(\exp\left(-\int_0^{\zeta_1}\varphi(\zeta_2)d\zeta_2\right)\right)$$

We use the mean theorem in the last equation

$$\varphi'(\zeta) = 2 + 2a_2 \frac{1}{\varphi(\zeta)} \exp\left(-\int_0^{\zeta_1}\varphi(\zeta_2)d\zeta_2\right) \tag{29}$$

In the resulting equation, the first compound is the principal member of the external asymptotics. To obtain the following members of the asymptotic, we will present the function as

$$\varphi = 2\zeta - c + z$$

where $z \to 0$, if $\zeta \to \infty$. Given the last expression of the function φ, Eq. (29) is obtained as follows:

$$\varphi'(\zeta) = 2 + \frac{2a_2}{2\zeta - c + z} \exp\left(-\int_0^{\zeta} (2\zeta_1 - c + z) d\zeta_1\right)$$

If $z = \varphi - 2\zeta + c$, then

$$\varphi'(\zeta) = 2 + \frac{2a_2}{2\zeta - c + z} \exp\left(-\zeta^2 + \mathrm{c}\zeta\right) \exp\left(-\int_0^{\zeta} (\varphi - 2\zeta_1 + c) d\zeta_1\right)$$

In the external domain, where $\zeta \to \infty$ and $z \to 0$, let us receive an exterior asymptotic:

$$\varphi'(\zeta) = 2 + \frac{2a_2 D}{2\zeta - c} \exp\left(-\zeta^2 + \mathrm{c}\zeta\right) + o\left(\frac{1}{\zeta^2}\right) \tag{30}$$

where $D = \exp\left(-\int_0^{\infty} (\varphi - 2\zeta + c) d\zeta\right)$.

To calculate parameter a_2, use the procedure of Section 4.1 [see formula (20)], and using weight equal to 1:

$$a_2 = \frac{1}{2} \int_0^{\infty} \varphi_a' (2 - \varphi') d\zeta \tag{31}$$

At that, in external domain, $\zeta \to \infty$

$$\varphi = 2\zeta - c, \quad (z \to 0)$$

Therefore,

$$c = \int_0^{\infty} (2 - \varphi') d\zeta \tag{32}$$

and

$$D = \exp\left(-\int_0^{\infty} (\varphi - 2\zeta + c) d\zeta\right) \tag{33}$$

Type of generalized and normalized TPPA of order (4,4):

$$\varphi_a'(\zeta) = 2\left[1 - \frac{\left(1 + \alpha_1\zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3 + \alpha_4\zeta^4\right) \exp\left(-\zeta^2 + c\zeta\right)}{1 + \beta_1\zeta + \beta_2\zeta^2 + \beta_3\zeta^3 + \beta_4\zeta^4}\right] \tag{34}$$

Species of TPPA taking into account four nontrivial parameters:

$$\alpha_3,\ \beta_1,\ \beta_2,\ \beta_4$$

Therefore,

$$\varphi_a'(\zeta) = 2\left[1 - \frac{\left(1+\alpha_3\zeta^3\right)\exp\left(-\zeta^2 + c\zeta\right)}{1+\beta_1\zeta+\beta_2\zeta^2+\beta_4\zeta^4}\right] \tag{35}$$

Parameter values are determined using local asymptotic and TPPA in the respective domain. Taking into account the decomposition of the exponent in the internal domain, we will write down the local equality:

$$2a_2\zeta - \frac{a_2^2}{6}\zeta^4 = 2\left[1 - \frac{\left(1+\alpha_3\zeta^3\right)\left(1+c\zeta-\zeta^2+\frac{\left(c\zeta-\zeta^2\right)^2}{2}+\dots\right)}{1+\beta_1\zeta+\beta_2\zeta^2+\beta_4\zeta^4}\right] \tag{36}$$

Taking into account Eq. (33) in the external domain, we will write down the second local equality:

$$2 - \frac{2a_2D}{2\zeta - c}\exp\left(-\zeta^2 + c\zeta\right) = 2\left[1 - \frac{\left(1+\alpha_3\zeta^3\right)\exp\left(-\zeta^2+c\zeta\right)}{1+\beta_1\zeta+\beta_2\zeta^2+\beta_4\zeta^4}\right]$$
$$\Rightarrow a_2D\left(1+\beta_1\zeta+\beta_2\zeta^2+\beta_4\zeta^4\right) = (2\zeta - c)\left(1+\alpha_3\zeta^3\right) \tag{37}$$

Equalizing the coefficients in Eqs. (36) and (37) at the same degrees ζ, we get

$$\alpha_3 = a_2D,\ \ \beta_1 = a_2 + c,\ \ \beta_2 = -\frac{c^2}{2} + a_2(a_2+c),\ \ \beta_4 = 2.$$

Therefore, the TPPA has the form:

$$\varphi_a'(\zeta) = 2\left[1 - \frac{\left(1+a_2D\zeta^3\right)\exp\left(-\zeta^2+c\zeta\right)}{1+(a_2+c)\zeta+\left(a_2^2+a_2c+\frac{c^2}{2}-1\right)\zeta^2+2\zeta^4}\right] \tag{38}$$

After systems (31)–(33) are solved, we will obtain

$$\begin{aligned} a_2 &= 0.6641, \\ c &= 1.7308, \\ D &= 0.3357 \end{aligned} \tag{39}$$

By substituting (39) in (38), we get an explicit expression for the TPPA.

4.4 Combining method of interior and exterior asymptotics for boundary layer of supersonic flow in compressed viscous gas by TPPA

We consider the boundary layer in hypersonic flow of viscous gas and solve a model problem which reduces to ordinary differential equations with appropriate boundary conditions. The TPPAs parameters are calculated and relevant questions are discussed. The equations of laminar boundary layer near a semi-infinite plate in the supersonic flow of viscous perfect gas, as it is known [2, 7], can be reduced to the form:

$$\left(\varphi'' \frac{\mu}{T}\right)' + \varphi\varphi'' = 0, \tag{40}$$

$$\left(\mu \frac{T'}{T}\right)' + \sigma\varphi T' + a\sigma \frac{\mu}{T}\varphi''^2 = 0 \tag{41}$$

where

$$\varphi = \frac{\psi}{\sqrt{x}} = \varphi(\zeta), \;\; T = T(\zeta), \;\; \zeta = \frac{\eta}{2\sqrt{x}}, \;\; \eta = \int_0^y \frac{dy}{T}, \;\; a = \frac{1}{4}M^2(\kappa - 1)$$

M is the Mach number, σ is the Prandtl number, κ is the adiabatic index, ψ is the stream function, T is the temperature, μ is the viscosity coefficient, and x and y are the Cartesian coordinates.

The boundary conditions at the wall are

$$\varphi(0) = \varphi'(0) = 0, \;\; T(0) = T_s \tag{42}$$

At external boundary of layer is

$$\varphi'(\infty) = 2, \;\; T(\infty) = 1. \tag{43}$$

Interior asymptotic expansions are for $\mu = T^n$

$$\begin{aligned} \varphi' &= 2a_2\zeta - (n-1)a_2 \frac{T_1}{T_s}\zeta^2 + O\left(\zeta^3\right), \\ T &= T_s + T_1\zeta - \left(2a\sigma a_2^2 + \frac{(n-1)}{2}\frac{T_1^2}{T_s}\right)\zeta^2 + O\left(\zeta^3\right) \end{aligned} \tag{44}$$

where two constants a_2 and T_1 remain undefined.

Exterior asymptotics for $\varsigma \to \infty$

$$\begin{aligned} &\ln \varphi'' = c^2 + c\zeta + \ln A + o(1), \\ &\ln(-T') = -\sigma\zeta^2 + \sigma c\zeta + \ln B + o(1) \end{aligned} \tag{45}$$

where three constants are unknown: c, A, and B.

We solve boundary problems (40) and (41) approximately by connecting asymptotics (44) and (45) TPPA

$$\varphi_a'(\zeta) = 2\left[1 - \frac{\left(1 + A\zeta^3\right)\exp\left(-\zeta^2 + c\zeta\right)}{1 + \alpha_1\zeta + \alpha_2\zeta^2 + \alpha_4\zeta^4}\right] \tag{46}$$

$$T_a'(\zeta) = \frac{\zeta_m - \zeta}{\beta_0 + \beta_1\zeta}\exp\left(\sigma\left(-\zeta^2 + c\zeta\right)\right) \tag{47}$$

Boundary conditions (45) and (46) are satisfied if to put

$$\varphi_a(\zeta) = \int_0^\infty \varphi_a'(\zeta)d\zeta, \quad T_a(\zeta) = T_s + \int_0^\infty T_a'(\zeta)d\zeta \tag{48}$$

We complement the last equalities (50) and (51) with a normalizing condition:

$$1 = T_s + \int_0^\infty T_a'(\zeta)d\zeta \tag{49}$$

Following the procedure of the previous section, we will calculate the coefficients at ζ and ζ^2 in asymptotic expansions (44) and, equating them with the corresponding expressions from Eqs. (46) and (47), we will obtain equalities, from which values $\alpha_1, \alpha_2, \alpha_4, \beta_0, \beta_1$ are expressed through a_1, c, T_1, ζ_m:

$$\begin{aligned} &\alpha_1 = a_2 + c, \\ &\alpha_2 = -\frac{1}{2}(n-1)a_2\frac{T_1}{T_s} - 1 + \frac{c^2}{2} + a_2(a_2+c), \quad \alpha_4 = 4, \\ &\beta_0 = \frac{\zeta_m}{T_1}, \quad \beta_1 = \sigma c\frac{\zeta_m}{T_1} - \frac{1}{T_1} + (n-1)\frac{\zeta_m}{{T_1}^2} \end{aligned} \tag{50}$$

Three parameters in asymptotics (44) are defined in the outer region if the following condition is met:

$$\beta_1 = -1/B \tag{51}$$

A priori at large M numbers, it is known that the temperature profile is non-monotonic and has a maximum within the layer at point ς_m at which, as can be seen from the second equation of the systems (40) and (41), the following condition is used:

$$T''(\zeta_m) = -a\sigma\varphi''^2(\zeta_m) \tag{52}$$

From the convexity condition of the temperature profile in the vicinity of the point ς_m, the following equality is used:

$$(\beta_0 + \beta_1\zeta_m)a\sigma\varphi''^2_a(\zeta_m) = \exp\left(-\sigma\left(\zeta_m^2 - \zeta_m\right)\right) \tag{53}$$

Let us add the received equations with the integrated ratios received on the basis of coincidence of TPPAs (46) and (47); in this case, three members in asymptotic decompositions (50) and (51), the initial system of Eqs. (40) and (41), with boundary conditions (42) and (43), by using the technique stated in the previous sections.

$$\begin{aligned} a_2 &= \frac{1}{2}\int_0^\infty \varphi_a'(2-\varphi')d\zeta \\ c &= \int_0^\infty (2-\varphi')d\zeta \end{aligned} \tag{54}$$

The integral relation for parameter A is obtained by multiplying Eq. (40) by

$$\exp\left(\zeta^2 - c\zeta\right)$$

and integrating from 0 to ∞ taking into account Eq. (48):

$$A = 2a_2\frac{\mu(T_s)}{T_s} - \int_0^\infty \left(\varphi - \frac{\mu(T)}{T}(2\zeta - c)\right)\varphi'' \exp\left(\zeta^2 - c\zeta\right)d\zeta \tag{55}$$

Similarly, from Eq. (41), we get

$$B = \sigma\int_0^\infty \left(T'\left(\varphi - \frac{\mu(T)}{T}(2\zeta - c)\right) + a_2\frac{\mu(T)}{T}\varphi''^2\right) \exp\left(\sigma\left(\zeta^2 - c\zeta\right)\right)d\zeta \tag{56}$$

Thus, the integral relations (52) and (55)–(47) form a nonlinear system of equations for determining the following parameters:

$$T_1,\ a_2,\ c,\ A,\ B.$$

Integrals of the systems (37) and (42)–(44) solution were approximated using Simpson quadrature formulas. The behavior of magnitude B proved to be highly dependent on the behavior of the exponent at large, so the integral relation had to be replaced by the local condition (52), besides controlling the behavior of the TPPA near the maximum is more important than the weight of the exponent away from the wall. Thus, instead of the value of B, we include the value among the parameters sought, and the value of B is expressed from Eqs. (50) and (51).

5. Results

As an example of TPPA (see Section 3.2) used for matching of limiting asymptotics, consider the paper by Grasman et al. [33]. They dealt with Lyapunov exponents which characterize the dynamics of a system near its attractor. For the Van der Pol oscillator:

$$\ddot{x} + \mu\dot{x}\left(x^2 - 1\right) + x = 0 \tag{57}$$

Similar to the asymptotic approximation of amplitude and period, expressions are derived for the nonzero Lyapunov exponent λ_2 for both small and large parameter μ values:

$$\lambda_2 = -\mu - \frac{1}{16}\mu^3 + \frac{263}{18432}\mu^5 + \ldots, \mu \to 0, \tag{58}$$

$$\lambda_2 = -\frac{3 + 4\ln 2}{2(3 - 2\ln 2)}\mu + \ldots, \mu \to \infty. \tag{59}$$

The overlap of these series does not take place. The authors of [33] remark: "Such an overlap comes within reach if in the regular expansion a large number of terms is included." This is not correct, because the obtained series is asymptotic; so, with increasing of number of terms, the results will be worst. So, one needs a summation procedure. Some authors [34] proposed to use PAs, but in this case one needs hundreds of perturbation series terms. That is why we use TPPA. Using two terms from expansion (58) and one term from expansion (59), one obtains

$$L \approx -\frac{\lambda_2}{\mu} = \frac{1 + 0.14\mu^2}{1 + 0.079\mu^2} \tag{60}$$

Expression (60) has a pole at $\mu = -12.66$. Below, one can see some numerical results.

In **Table 1**, the second column is made by calculation results by formula (4), the third column is made by paper data [33]. One can see that TPPA gives good result for any value of used parameter.

μ	L (4)	L (NR)
1	1.057	1.0648
5	1.513	1.4724
10	1.685	1.6358
25	1.759	1.7398
50	1.768	1.7691

Table 1.
Comparison for L of numerical results (NR) from paper by [33] with TPPA formulate (60).

In Section 4.4, the problem was solved for several variants of the Mach number and the heating temperature: $M = 5; 10; 15$, $T_s = 3; 5; 7$ of the streamlined flat plate, with constant Prandtl number values $\sigma = 0.76$, adiabatic index $\kappa = 1.4$, and two values of dynamical viscosity index $\mu = T^n : n = 1;\ \ 0.76$. When the first equation of the systems (43) and (44) is solved, it becomes independent of the second equation and can be compared with the known Blasius solution (see Section 3), which was used as a test when compared to our method [35–40]. Thus, the value of

M	T_s	ς_m	T_1
5	3	0.426	1.340
10	3	0.744	9.127
10	5	0.637	7.929
10	7	0.531	6.676
15	3	0.806	22.00
15	5	0.756	20.82
15	7	0.712	19.75

Table 2.
TPPAs parameters for different Mach numbers M*, temperature* T_S*, and* n = 1 *values.*

M	5	10	10
T_s	3	3	5
a_2	0.17	0.74	0.56
ς_m	0.729	0.714	0.798
c	1.45	1.42	1.38
T_1	0.56	8.69	8.16

Table 3.
TPPAs parameters for different Mach numbers M*, temperature* T_S*, and* n = 0.76 *values.*

the parameters according to the exact solution is equal: $a_2 = 0.664;\ c = 1.72$. Our decision gives $a_2 = 0.6641;\ \ c = 1.7308$. Of course, such a good match is due to the fact that these parameters are largely determined by local internal asymptotics, more precisely, derived from the function on the wall. But also within the transition area, the deviation from the exact solution does not exceed 1÷2% (for φ' and T, respectively). Design values of parameters for determining approximations (37) and (38) for n = 1 are given in **Table 2**.

If $n = 0.76$, this value corresponds to the physical characteristics of the air, and the constant calculation results for the approximation formulas (49) and (50) are shown in **Table 3**.

6. Conclusion

The procedure of constructing the PA is much less labor-intensive than the construction of higher approximations of perturbation theory. PA can be applied to power series but also to the series of orthogonal polynomials. PA is locally the best rational approximation of a given power series. They are constructed directly and allow for efficient analytic continuation of the series outside its circle of convergence, and their poles in a certain sense localize the singular points (including the poles and their multiplicities) of the function at the corresponding region of convergence and on its boundary. PA is fundamentally different from rational approximations with (fully or partially) fixed poles, including the polynomial approximation, when all the poles are fixed in infinity. That is the above property of PA—effectively solving the problem of analytic continuation of power series—lies at the basis of their many successful applications in the analysis and the study of applied problems. Currently, the PA method is one of the most promising nonlinear methods of summation of power series and the localization of its singular points. Including the reason why the theory of the PA turned into a completely independent section of approximation theory, and these approximations have found a variety of applications both directly in the theory of rational approximations, and in perturbation theory.

Thus, the main advantages of PA compared with the Taylor series are as follows:

1. Typically, the rate of convergence of rational approximations greatly exceeds the rate of convergence of polynomial approximation. For example, the function e^{ε} in the circle of convergence approximated by rational polynomials $P_n(\varepsilon)/Q_n(\varepsilon)$ in 4^n times better than an algebraic polynomial of degree $2n$. More tangible, it is property for functions of limited smoothness. Thus, the function $|\varepsilon|$ on the interval $[-1,1]$ cannot be approximated by algebraic polynomials so that the order of approximation was better than $1/n$, where n is the degree of polynomial. PA gives the rate of convergence $\sim \exp\left(-\sqrt{2n}\right)$.

2. Typically, the radius of convergence of rational approximation is large compared with the power series. Thus, for the function $\arctan(x)$, Taylor polynomials converge only if $|\varepsilon| \leq 1$, and PA is everywhere in $C\backslash((-\,i\infty,\,-\,i] \cup [i,\,i\infty))$.

3. PA can establish the position of singularities of the function.

TPPA allows to overcome the locality of asymptotic expansions, using only a few terms of asymptotics. Unfortunately, the situations when both asymptotic limits have the form of power expansions are rarely encountered in practice, so we have to resort to other methods of AEFs construction, for example, the method quasirational approximation which is described in [23]. The method of combination (combining method) of asymptotics by using TPPA is alternative to the well-known matching method [6]; it is useful in local domains of transition layers where asymptotics are not uniform. This method was tested on well-known problems of mathematical physics, in particular, problems of fluid dynamics. The main advantage of the method is that it has an analytic form.

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