Sparsity Optimization and RRSP-based Theory for 1-bit Compressive Sensing

by

Chunlei Xu

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ABSTRACT

Due to the fact that only a few significant components can capture the key information of the signal, acquiring a sparse representation of the signal can be interpreted as finding a sparsest solution to an underdetermined system of linear equations. Theoretical results obtained from studying the sparsest solution to a system of linear equations provide the foundation for many practical problems in signal and image processing, sample theory, statistical and machine learning, and error correction.

The first contribution of this thesis is the development of sufficient conditions for the uniqueness of solutions of the partial ℓ_0-minimization, where only a part of the solution is sparse. In particular, ℓ_0-minimization is a special case of the partial ℓ_0-minimization. To study and develop uniqueness conditions for the partial sparsest solution, some concepts, such as ℓ_p-induced quasi-norm, maximal scaled spark and maximal scaled mutual coherence, are introduced.

The main contribution of this thesis is the development of a framework for 1-bit compressive sensing and the restricted range space property based support recovery theories. The 1-bit compressive sensing is an extreme case of compressive sensing. We show that such a 1-bit framework can be reformulated equivalently as an ℓ_0-minimization with linear equality and inequality constraints. We establish a decoding method, so-called 1-bit basis pursuit, to possibly attack this 1-bit ℓ_0-minimization problem. The support recovery theories via 1-bit basis pursuit have been developed through the restricted range space

property of transposed sensing matrices.

In the last part of this thesis, we study the numerical performance of 1-bit basis pursuit. We present simulation results to demonstrate that 1-bit basis pursuit achieves support recovery, approximate sparse recovery and cardinality recovery with Gaussian matrices and Bernoulli matrices. It is not necessary to require that the sensing matrix be underdetermined due to the single-bit per measurement assumption. Furthermore, we introduce the truncated 1-bit measurements method and the reweighted 1-bit ℓ_1-minimization method to further enhance the numerical performance of 1-bit basis pursuit.

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Last but not least, my deepest gratitude goes to my parents, Yunhui Xu and Xiaoling Ren, for their endless love and uncompromising unconditional support throughout the

entire journey. To them I dedicate this thesis.

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CHAPTER 1

INTRODUCTION

In this information age, we are dealing with enormous flows of digital data everyday, such as images, video and audio signals. In order to cope with this large amount of data, we rely on compression to achieve a storage reduction, selectively sifting some significant features from data and discarding the rest. Such a compression method is called 'lossy compression', which may irretrievably cause loss of some parts of the true data so that the quality of the resulting data is poor. Instead of losing a large portion of the complete data, to promote the quality of the compression, a suitable *sparse* representation of the data can be useful to reduce the storage space. Such a sparse representation of the data is achievable due to the fact that in many situations only a few significant components can capture the key information required. This is the crucial idea behind compressive sensing, also known as compressed sensing or compressive sampling (see, [32, 35, 39, 45, 51, 59, 68]). Specifically, compressive sensing is a scheme by which the significant features of the signal can be reconstructed from only a limited number of linear measurements under the assumption that a signal has a sparse or compressible representation on a suitable basis. The reconstruction can be achieved by some efficient algorithms under certain conditions.

Plentiful works have been devoted to the study of compressive sensing over the past

decades. Such a promising development of compressive sensing has a great impact on many aspects of signal and image processing and stimulates a wide range of new applications. For instance, in biomedical image processing, (X-ray) computed tomography and magnetic resonance imaging (MRI) use the compressive sensing methodologies to significantly reduce scanning time and in the meantime preserve a high quality image [88, 91, 92]. Another well-known example is the single-pixel camera, implementing the compressive sensing methodologies on camera design, which is paid off when observations are beyond the visual spectrum [56, 117]. The compressive sensing framework has also inspired researches in sampling theory [37, 108, 109, 110], error correction [36, 79, 111], radar signal processing [61, 63, 77], statistical and machine learning [4, 97, 128], to name just a few. For more comprehensive introductions, applications and extensions of compressive sensing, we refer to books and articles on the subject [28, 32, 39, 50, 51, 58, 59, 68]. Numerous papers and information can be found at the compressive sensing resources webpage of the Rice University [124].

In this thesis, we focus on theoretical aspects of compressive sensing in the finite-dimensional real-valued setting. In the following sections, we start with a short review of important sparsity recovery models in compressive sensing and a brief introduction of some popular and efficient reconstruction algorithms.

1.1 Mathematical models for compressive sensing

A sparsest representation of a signal from a limited number of linear measurements can be achieved by solving the following problem:

$$\min\{\|x\|_0 : \ Ax = b\}, \tag{1.1.1}$$

where $\|x\|_0$, known as the ℓ_0-norm, denotes the number of nonzero components in x. $A \in R^{m \times n}$ with $m < n$ is called a sensing matrix and $b \in R^m$ is the measurements vector. The sensing matrix A is designed to map a signal from a higher dimension R^n to a lower dimension R^m so that the dimension of the measurements vector is lower than that of the signal to recover. Here, we refer to problem (1.1.1), finding a sparsest solution of a system of linear equations, as the standard compressive sensing problem, which is also called (standard) ℓ_0-minimization in the literature. Conventionally, we assume that A is a full-row-rank matrix with $m < n$. Thus, the linear system $Ax = b$ has infinitely many solutions and ℓ_0-minimization seeks a solution with the fewest nonzero components, namely, the sparsest solution. If the target signal is highly sparse, it can be recovered as the unique solution to ℓ_0-minimization under certain conditions [33, 52, 69, 120, 130]. In general, it is difficult to solve ℓ_0-minimization. For instance, given the measurements, even if we know a prior that the signal has k nonzero components, we still need to search through all possible combinations of any k columns of a matrix A in order to find the sparsest solution. Such an exhaustive intractable search indicates that ℓ_0-minimization is NP-hard [99].

Unfortunately, in practice, a signal is usually not exactly k-sparse and it may contain many insignificant small nonzero components below a small threshold; even if a signal is exactly sparse, measurements may often be inaccurate and contain a small amount of noise. This can be coped with by a more general problem by introducing a small deviation or tolerance to the linear system $Ax = b$. Typically, we have the following model:

$$\min\{\|x\|_0 : \ \|Ax - b\|_2 \leqslant \delta\} \tag{1.1.2}$$

for some $\delta > 0$. The choices of the parameter δ are often clear from applications. Note that problem (1.1.2) degenerates to ℓ_0-minimization when $\delta = 0$ and it is also NP-hard

[68, 99]. However, problem (1.1.2) seems more difficult to solve than (1.1.1) since the uniqueness property of solutions is lost. Any tiny appropriate perturbations of an optimal solution may yield an alternative optimal solution as long as it satisfies the constraint of the problem (1.1.2). Thus, instead of the exact recovery, only an approximate recovery is expected from (1.1.2). This means that a reconstructed signal is expected to be as close to the target sparse signal as possible [33, 43, 53, 130].

Additionally, we can consider the unconstrained version of (1.1.2),

$$\min \ \lambda \|x\|_0 + \|Ax - b\|_2^2, \tag{1.1.3}$$

which can be interpreted as the Lagrangian function of (1.1.2) with some $\lambda > 0$. But (1.1.3) seems more pessimistic than (1.1.2) due to the parameter λ. There is no tractable algorithm to solve the general ℓ_0-minimization, (1.1.2) or (1.1.3), directly. However, for some specially designed matrices A and the generated measurements $b = Ax^*$ by a highly sparse signal x^*, there is a variety of tractable and efficient algorithms to solve these problems and recover x^* from the measurements b. We now introduce some algorithms used for the sparse recovery in the next section.

1.2   Sparse recovery algorithms

In the past decades, various algorithms have been developed to possibly solve ℓ_0-minimization, (1.1.2) and (1.1.3) for some special matrices. We now give a brief overview of the algorithms and refer readers to [68, 45] and references therein for further information on sparse recovery algorithms.

1.2.1   Convex optimization algorithms

To attack either ℓ_0-minimization, (1.1.2) or (1.1.3), the main issue is the nonconvex objective function $\|x\|_0$. To convexify problems, we may replace $\|x\|_0$ by $\|x\|_1$, the convex

envelope of $\|x\|_0$ near the origin. As a result, we obtain the ℓ_1-minimizations as follows

$$\min\{\|x\|_1 : \ Ax = b\}, \tag{1.2.1}$$

$$\min\{\|x\|_1 : \ \|Ax - b\|_2 \leqslant \delta\} \text{ for some } \delta > 0, \tag{1.2.2}$$

and

$$\min \ \lambda\|x\|_1 + \|Ax - b\|_2^2, \tag{1.2.3}$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the ℓ_1-norm and $\lambda \geqslant 0$ is a parameter.

Problem (1.2.1) is known as (standard) ℓ_1-minimization. (1.2.1) and (1.2.2) are named as basis pursuit (BP) and basis pursuit denoising (BPDN), respectively [41]. Especially, BP can be cast as a linear program, thus, efficient methods are available to solve it. Remarkably, ℓ_1-norm objective can also promote the sparsity of solutions. We will give an example to explain and visualize the mechanism behind the ℓ_1-norm.

Example 1.2.1: Given the measurement $b \in R$, let $F \subseteq R^2$ be a feasible set of the linear system $Ax = b$, e.g., $F = \{x \in R^2 : \ Ax = b\}$, which is a line in R^2. As illustrated in Figure 1.1, expanding the ℓ_1-norm ball from the origin, it will intersect with F at the point $x^* \in F$, which is the sparsest solution with the least ℓ_1-norm. Squeezing edges of the ℓ_1-norm ball towards the origin as shown in Figure 1.2, the same sparsest solution $x^* \in F$ can be found in this case, which implies that the ℓ_p- (quasi-)norms with $0 < p < 1$, also promote sparsity of the solution. The ℓ_p-norm of a vector $x \in R^n$ is defined as $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$.

We now stretch the edges of the ℓ_1-norm ball until it becomes a circle centered at the origin, which is the ℓ_2-norm ball. The ℓ_2-norm of a vector $x \in R^n$ is defined as $\|x\|_2 :=$
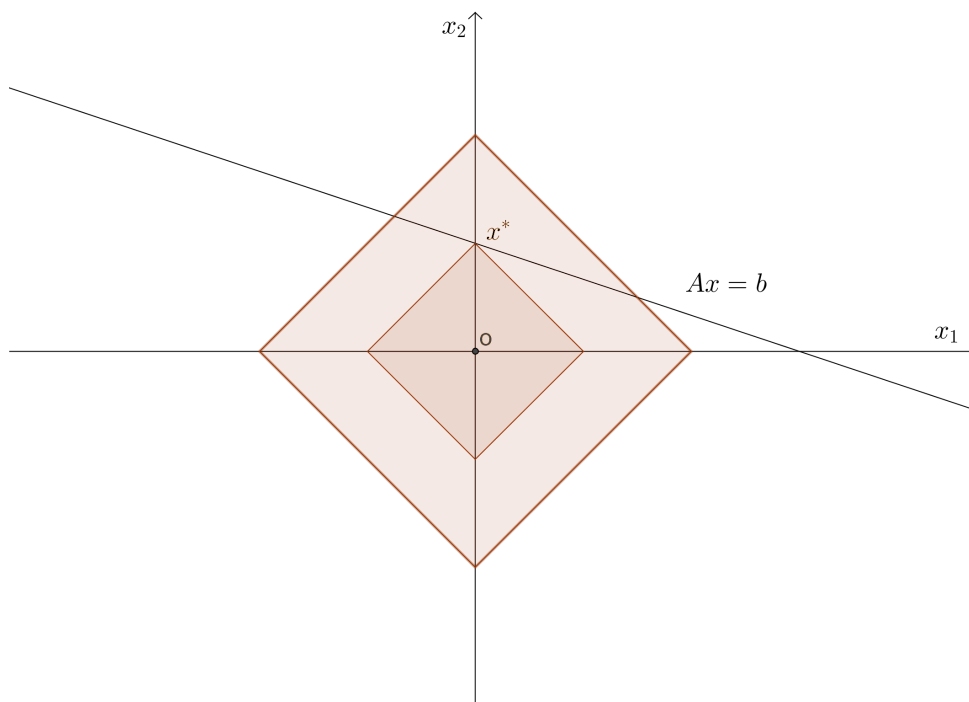
Figure 1.1: Geometry of ℓ_1-minimization in two dimensions. x^* is the sparsest solution of $Ax = b$ with the least ℓ_1-norm.
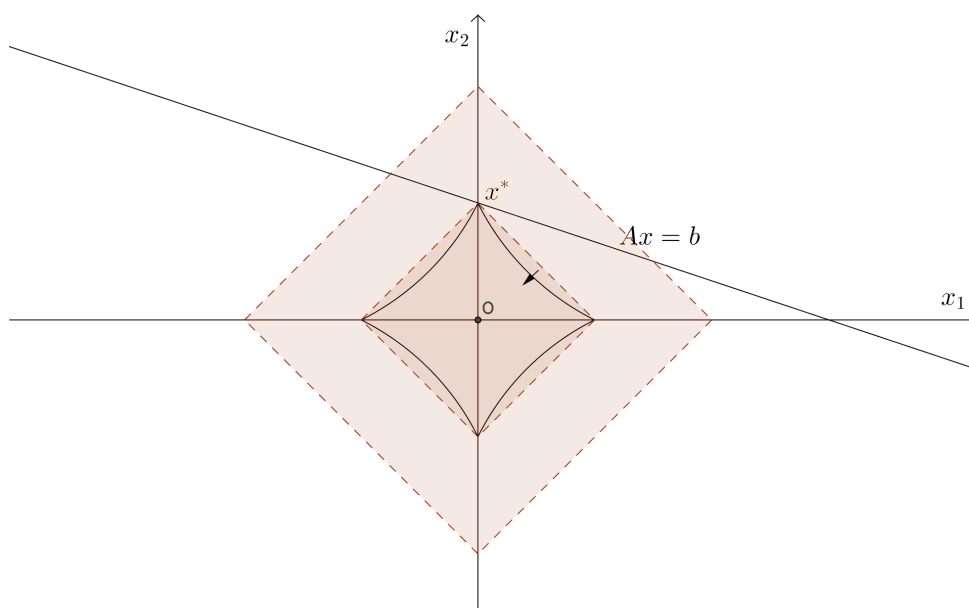
Figure 1.2: Geometry of ℓ_1-minimization and ℓ_p-minimization in two dimensions. x^* is the sparsest solution to ℓ_p-minimization: $\min\{\|x\|_p : Ax = b\}$ where $0 < p < 1$.

$(\sum_{i=1}^{n} |x_i|^2)^{1/2}$. As demonstrated in Figure 1.3, the ℓ_2-norm minimization problem, i.e., $\min\{\|x\|_2 : Ax = b\}$, has a unique solution $\hat{x}$ with the least ℓ_2-norm. If we then keep stretching edges of ℓ_2-norm ball until it becomes a ℓ_∞-norm square. Then the unique solution $\tilde{x}$ to the ℓ_∞-minimization problem, i.e., $\min\{\|x\|_\infty : Ax = b\}$ can be found, where ℓ_∞-norm of a vector $x \in R^n$ is defined as $\|x\|_\infty := \max_{i \in \{1,\cdots,n\}} |x_i|$. It is clear to see that both $\hat{x}$ and $\tilde{x}$ are not sparse in this case, unless the line is parallel to x-y axims. Therefore, ℓ_1-minimization and ℓ_p-minimization with $0 < p < 1$ can be used for achieving the sparsest solution of a system of linear equations.
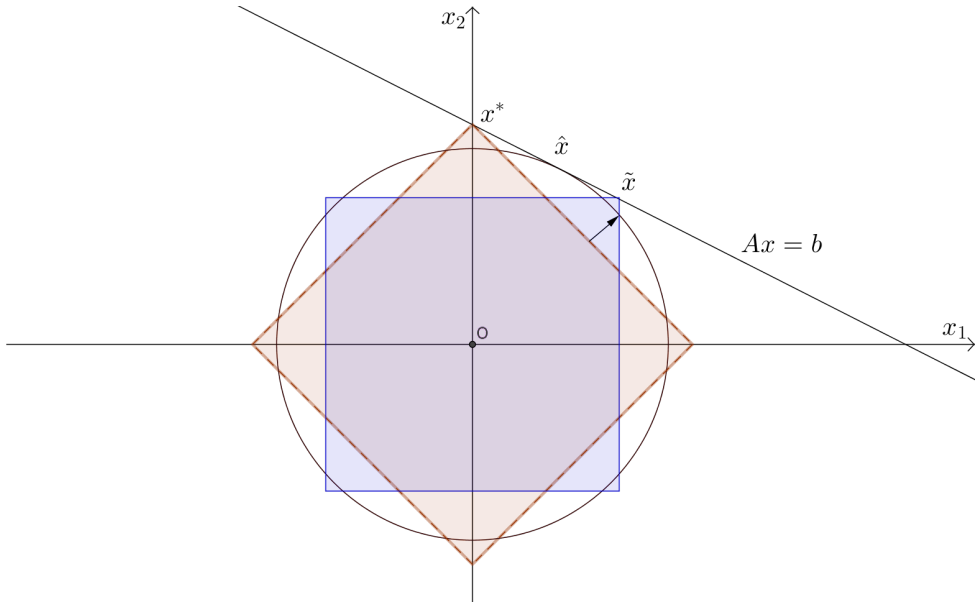
Figure 1.3: Geometry of ℓ_2-minimization and ℓ_∞-minimization in two dimensions. Dense solutions of $Ax = b$ are found by ℓ_2-minimization and ℓ_∞-minimization.

Additionally, Figure 1.2 and Figure 1.3 indicate that the ℓ_p-norm with $0 < p < 1$ can be used to approximate the ℓ_0-norm and is more efficient than the ℓ_p-norm with $p \geqslant 1$. But the ℓ_p-norm with $0 < p < 1$ is nonconvex and the corresponding minimization problem $\min\{\|x\|_p : Ax = b\}$ is NP-hard [70]. From Figure 1.3, we see that the sparsity of the solution may not be achieved for other ℓ_p-norm problems with $1 < p \leqslant \infty$. It is worth emphasizing that ℓ_p-norm with $p > 1$ does not favor sparsest solutions in general.

Between the ℓ_p-minimization with $0 < p < 1$ and the ℓ_p-minimization with $p > 1$, ℓ_1-minimization is a natural and well-suited approach to possibly attack ℓ_0-minimization. More theories concerning the efficiency of ℓ_1-minimization for solving compressive sensing problems will be introduced in the next chapter.

Problem (1.2.3) was firstly introduced to solve the BPDN problem in [41], where the parameter λ is used to balance between the ℓ_2-norm residual $\|b - Ax\|_2^2$ and the ℓ_1-term $\|x\|_1$. As λ tends to 0, (1.2.3) is minimizing $\|b - Ax\|_2^2$ which means that the solution to the linear system $Ax = b$, if it exists, will be found. As λ tends to ∞, (1.2.3) is focusing on minimizing the ℓ_1-term $\|x\|_1$ regardless of the values of $\|b - Ax\|_2^2$. For an appropriate choice of parameter $\lambda > 0$, the solution to (1.2.3) is the solution to (1.2.2), where the parameter λ is a function of A, b and δ [68]. Furthermore, it is worth mentioning that Fuchs [69] has proved that (1.2.3) can sufficiently recover a sparsest solution x^* to the linear system $Ax = b$ for nonzero λ. Due to the nonzero parameter λ, the recovery is defined in the sense that the unique solution to (1.2.3) which has at least the same sign of x^*. In this thesis, we restrict our attention on the noiseless case of sparse recovery, and more information about (1.2.2), (1.2.3) and their applications in statistics and signal processing, such as the least absolute shrinkage and selection operator (LASSO) and Dantzig selector, can be found in [118, 38, 68].

1.2.2 Greedy and thresholding-based algorithms

Besides convex approaches, there also exist various greedy methods that are easy to implement and relatively fast to solve large scaled sparse recovery problems. Greedy methods can be roughly categorized into two groups. One is the greedy pursuit algorithms, such as Matching Pursuit (MP) [94, 49], Orthogonal Matching Pursuit (OMP) [104, 93, 47, 121], Gradient Pursuit (GP) [18], Conjugate Gradient Pursuit (CGP) [18], Stagewise Orthogonal Matching Pursuit (StOMP) [55], and Regularized Orthogonal Matching Pursuit

(ROMP) [103, 102]. The other is the thresholding algorithms, such as Iterative Hard Thresholding (IHT) [19, 20], Compressive Sampling Matching Pursuit (CoSaMP) [101], and Subspace Pursuit (SP) [44]. Greedy pursuit algorithms build up an estimation of the sparse solution by iteratively adding new nonzero components until a stopping condition is met. Besides the estimation step in greedy pursuit algorithms, thresholding algorithms have one more step, setting all but a certain number of elements of the argument to zero. Performances of greedy algorithms can be guaranteed under certain conditions such as the restricted isometry property (RIP) based conditions, which can also theoretically ensure the performance of convex approaches. Moreover, compared to convex approaches, these greedy methods are easily applied to the union of subspaces models [60, 89] such as the structured sparse problems [9, 19, 57]. For instance, extensions of greedy pursuit algorithms can be adapted to the tree models [83]. Unfortunately, there is no strong theory to guarantee their performances. On the other hand, the thresholding algorithms are easily modified for the general union of subspaces models, such as the projected landweber algorithm introduced in [16], which is an extension of the IHT algorithm for the union of subspaces models [90, 19].

Here, we only consider the convex approaches for the sparse recovery and refer readers to [68, 17] and references therein for more information on greedy pursuit and thresholding-based algorithms.

1.3   Outline of the thesis and contributions

In this thesis, we focus our attention on the theoretical aspect of the noise-free sparse recovery in compressive sensing. Particularly, we focus on addressing the question of whether a linear system has a unique sparsest solution; and if the linear system is known to have a sparsest solution, how to exactly and efficiently recover such a solution of the underlying linear system from given measurements. We firstly introduce the uniqueness

and recovery conditions developed for the standard compressive sensing problem, and then extend these theoretical results to two other special applications, namely, the partial sparsity-seeking problem and the 1-bit compressive sensing problem. Our contributions are distributed in the following chapters.

First, Chapter 2 is a survey of current theoretical results of standard compressive sensing, using different techniques to establish uniqueness conditions for ℓ_0-minimization and recovery conditions via ℓ_1-minimization. In that chapter, some fundamental properties of matrix A are introduced, such as spark [54, 52], mutual coherence [54, 52], Babel function [120], exact recovery condition (ERC) [123, 122], null space property (NSP) [43, 130], restricted isometry property (RIP) [33], and range space property (RSP) [132, 133]. Based on these properties, uniqueness conditions for ℓ_0-minimization and recovery conditions via ℓ_1-minimization can be stated. All recovery results derived from those conditions are categorized into two groups, one ensuring to recover a specific sparsest vector through a linear system (namely, nonuniform recovery) while the other ensuring to recover every k-sparse vector x (i.e., $\|x\|_0 \leqslant k$) through a single sensing matrix A (namely, uniform recovery).

In practice, one may be interested in recovering a solution to a system of linear equations that only a part of it is sparse, known as the partial sparsity-seeking problem or partial ℓ_0-minimization. The standard ℓ_0-minimization (1.1.1) is a special case of partial ℓ_0-minimization [129]. Based on the well-founded uniqueness theories for ℓ_0-minimization, sufficient conditions for the uniqueness of solutions of partial ℓ_0-minimization will be developed in Chapter 3. These uniqueness criteria are established through ℓ_p-induced quasi-norm, the maximal scaled spark and the maximal scaled mutual coherence, which are the generalization of improved uniqueness conditions for ℓ_0-minimization based on coherence rank, submutual coherence and scaled mutual coherence introduced in [131].

As mentioned earlier, for the standard compressive sensing, to exactly recover the

sparsest solution to a system of linear equations from linear measurements, the sensing matrix needs to admit some properties, such as mutual coherence, NSP, RIP, RSP etc.. In Chapter 4, we consider an extreme case of the standard compressive sensing, i.e., 1-bit compressive sensing, which takes the sign information of linear measurements. Surprisingly, only signs of linear measurements might still provide adequate information for a certain level of reconstruction. We show that such a 1-bit model can be formulated equivalently as an ℓ_0-minimization problem with linear equality and inequality constraints, which can be seen as a partial ℓ_0-minimization problem. Like the basis pursuit method for standard compressive sensing, we develop a decoding method, 1-bit basis pursuit, for possibly attacking this 1-bit ℓ_0-minimization problem, for which recovery theories can be established by the RSP-based analysis through the restricted range space property (RRSP) of the transposed sensing matrix. Furthermore, the RRSP conditions ensure a certain level of nonuniform and uniform recoveries in this framework of 1-bit compressive sensing.

In Chapter 5, we study the numerical performance of 1-bit basis pursuit and verify some claims on recovery conditions developed in Chapter 4. In contrast to the standard compressive sensing, we carry out our experiments on both underdetermined and overdetermined Gaussian and Bernoulli matrices. Simulation results demonstrate that Gaussian matrices and Bernoulli matrices provide support recovery, approximate sparse recovery and cardinality recovery via 1-bit basis pursuit. Moreover, we introduce two approaches to further improve the numerical performance of 1-bit basis pursuit. One is truncating the 1-bit measurements by setting some relatively small linear measurements to zero components, and the other is using the reweighted 1-bit ℓ_1-minimization, the first-order method for solving concave approximation problems of the ℓ_0-minimization arising from 1-bit compressive sensing.

Finally, we conclude with Chapter 6. We will discuss some open questions on the

topics covered in this thesis and some possible extensions of compressive sensing for future research.

1.4 Preliminaries and notation

In this section, we introduce some notations and terminologies, and recall some results in linear algebra, linear programming and convex optimization, that will be used throughout the thesis.

1.4.1 Sets and Vectors

We denote R as the field of real numbers and R^n as the n-dimensional Euclidean space. Let R^n_+ be the set of nonnegative vectors in the Euclidean Space R^n, i.e., $R^n_+ := \{x \in R^n : x \geqslant 0\}$ and R^n_- be the set of nonpositive vectors in the Euclidean Space R^n, i.e., $R^n_- := \{x \in R^n : x \leqslant 0\}$. The empty set is denoted as $\emptyset$. For a set S, $|S|$ denotes the cardinality of S, the number of elements in the set S, while for a scalar $\alpha \in R$, $|\alpha|$ means the absolute value of α. Given a subset $S \subseteq V$ of a set V, the complement of S in V is denoted as $S^C = V \setminus S$.

For a vector x, the transpose of x is denoted as x^T. Any vector x in this thesis is a column vector (unless otherwise stated) and thus x^T denotes a row vector. Let x_i be the i-th component of the vector $x \in R^n$. The inner product of two vectors $x \in R^n$ and $y \in R^n$ is defined by $\langle x, y \rangle = x^T y = \sum_i^n x_i y_i$. If $x^T y = 0$, we say that x and y are orthogonal. Given a vector $x \in R^n$, the index sets of all the positive and negative components of x are denoted as S_+ and S_-, respectively, that is, $S_+ := \{i : x_i > 0\}$ and $S_- := \{i : x_i < 0\}$. Then $x_{S_+} \in R^{|S_+|}$ and $x_{S_-} \in R^{|S_-|}$ are the vectors obtained by deleting components indexed by $(S_+)^C$ and $(S_-)^C$, respectively. And the support set of the vector x is the index set for all nonzero components of x, denoted as $\text{Supp}(x) := \{i : x_i \neq 0\} = S_+ \bigcup S_-$. Let $\text{sign}(x)$ be the sign vector of $x \in R^n$ where $\text{sign}(x_i) = 1$ if $x_i > 0$, $\text{sign}(x_i) = -1$ if

$x_i < 0$, and $\text{sign}(x_i) = 0$ if $x_i = 0$. Let e_S denotes the vector $(1, \cdots, 1)^T \in R^{|S|}$.

1.4.2 Matrix and Linear Algebra

For a matrix A, the transpose of A is denoted as A^T. When A is a square matrix, A is symmetric if $A^T = A$. The identity matrix is denoted as I with a suitable size. A diagonal matrix $A \in R^{n \times n}$ with entries $a_1, \cdots, a_n$ along the diagonal is denoted as $A = \text{diag}(a_1, \cdots, a_n)$. Let a_{ij} be the element in row i and column j, and a_j be the j-th column of A or the j-th row of A, which will be clearly stated in the context. For a given matrix $A \in R^{m \times n}$, A_S (or $A_{m,S}$) denotes a submatrix of A by deleting columns indexed by S^C, and $A_{S,n}$ denotes a submatrix of A by deleting rows of A indexed by S^C. The range space (or column space) of $A^T \in R^{n \times m}$ is denoted by

$$\mathcal{R}(A^T) := \{z \in R^n : \ A^T y = z \text{ for some } y \in R^m\},$$

and the null space of A is denoted by

$$\mathcal{N}(A) := \{x \in R^n : \ Ax = 0\}.$$

Then the relation between $\mathcal{N}(A)$ and $\mathcal{R}(A)$ is

$$n = \text{dimension of } \mathcal{N}(A) + \text{dimension of } \mathcal{R}(A).$$

In addition, the range space of A^T is the orthogonal complement of the null space of A, namely,

$$\mathcal{R}(A^T) = \mathcal{N}(A)^\perp,$$

which can be written as $x^T z = 0$ for all $z \in \mathcal{R}(A^T)$ and $x \in \mathcal{N}(A)$. The rank of $A \in R^{m \times n}$ is defined as the maximal number of linearly independent rows, which is also

equal to the maximal number of linearly independent columns, i.e., $\text{rank}(A) = \text{rank}(A^T)$. A has full rank if and only if $\text{rank}(A) = \min\{m, n\}$. We say that A has column rank or row rank to emphasize that the rank of A is defined with columns or rows. Furthermore, if A is a square matrix with full rank, it is invertible and its inverse is A^{-1}. A general matrix inverse is the pseudo-inverse $A^\dagger$. If $\text{rank}(A) = m < n$, then $A^\dagger = A^T(AA^T)^{-1}$ (i.e., $AA^\dagger = I$) as AA^T is invertible; if $\text{rank}(A) = n < m$, then $A^\dagger = (A^TA)^{-1}A^T$ (i.e., $A^\dagger A = I$) as A^TA is invertible. We denote the singular values of $A \in R^{m \times n}$ by $\sigma_i(A)$ for $i \in \{1, \cdots, \min\{m, n\}\}$. If $m = n$, we denote the eigenvalues of $A \in R^{n \times n}$ by $\lambda_i(A)$ for $i \in \{1, \cdots, n\}$. Furthermore, the eigenvalues of a matrix is located in a bounded set that can be characterized by the Gersgorin discs [78].

Theorem 1.4.1 (Gersgorin discs theorem) :

Let $A = [a_{ij}] \in R^{n \times n}$, and let $R_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$, $1 \leqslant i \leqslant n$ denote the sum of all the off diagonal elements in the i-th row of A. Then all the eigenvalues of A are located in the union of n discs

$$\bigcup_{i=1}^{n}\{\lambda \in R : \ |\lambda - a_{ii}| \leqslant R_i(A)\}. \tag{1.4.1}$$

Taking two rows at a time, the geometrical region in (1.4.1) are not discs but sets known as ovals of Cassini [78, 27], then we have the following generalization of Gersgorin discs theorem.

Theorem 1.4.2 (Brauer theorem) :

Let $A = [a_{ij}] \in R^{n \times n}$. All the eigenvalues of A are located in the union of $n(n-1)/2$ ovals of Cassini

$$\bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} \{\lambda \in R : \ |\lambda - a_{ii}||\lambda - a_{jj}| \leqslant R_i(A)R_j(A)\}. \tag{1.4.2}$$

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A square matrix $A \in R^{n \times n}$ is positive definite if and only if $x^T A x > 0$ for all $x \in R^n \setminus \{0\}$ (or all eigenvalues of A are positive), and positive semi-definite if and only if $x^T A x \geqslant 0$ for all $x \in R^n$ (or all eigenvalues of A are nonnegative). The strictly diagonal dominant property provides a sufficient condition to verify the positive definite matrix [78].

Theorem 1.4.3 :

Let $A \in R^{n \times n}$ be symmetric and strictly diagonally dominant. If $a_{ii} > 0$ for all $i = 1, \cdots, n$, then A is positive definite.

The matrix $A \in R^{n \times n}$ is diagonally dominant if

$$|a_{ii}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for all } i = 1, \cdots, n,$$

and it is strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \text{ for all } i = 1, \cdots, n.$$

1.4.3   Functions, Vector and Matrix Norms

A set C is called convex set if $\alpha x + (1 - \alpha) y \in C$ for all $x, y \in C$ and for all $\alpha \in [-1, 1]$. Let D be a convex set. The function $f : D \to R$ is convex if $f(\alpha x + (1 - \alpha) y) \leqslant \alpha f(x) + (1 - \alpha) f(y)$ for all $x, y \in D$ and for all $\alpha \in [-1, 1]$. A function f is called concave if $-f$ is convex, i.e., $f(\alpha x + (1 - \alpha) y) \geqslant \alpha f(x) + (1 - \alpha) f(y)$. If the above inequalities are strict, then f are called strictly convex or strictly concave function, respectively. Particularly, linear functions are both convex and concave function.

Definition 1.4.4 : *Let $f : C \mapsto R$ be a function over a subset $C \subseteq R^n$. If $x^* \in C$ and there exists an open ball of radius $\varepsilon > 0$ at x^*, i.e., $B_\varepsilon(x^*) := \{x \in C : \|x - x^*\| < \varepsilon\}$, the point x^* is called*

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1. *a local minimizer of f if*

$$f(x^*) \leqslant f(x) \text{ for all } x \text{ in some } B_\varepsilon(x^*),$$

and a strict local minimizer of f if

$$f(x^*) < f(x) \text{ for all } x \text{ in some } B_\varepsilon(x^*);$$

2. *a global minimizer of f over C if*

$$f(x^*) \leqslant f(x) \text{ for all } x \in C,$$

and a strict global minimizer of f over C if

$$f(x^*) < f(x) \text{ for all } x \in C.$$

The (strictly) local maximizers and (strictly) global maximizers are defined by replacing all $\leqslant$ (or $<$) with $\geqslant$ (or $>$) in Definition 1.4.4. In particular, due to the convexity of the function f and the set C, all local minimizers of convex optimization are also global minimizers, as shown in the following theorem [12].

Theorem 1.4.5 *:*

Let $f : C \mapsto R$ be a convex function over a convex subset C of R^n. Then a local minimizer of f over C is also a global minimizer. If in addition f is strictly convex, there exists at most one global minimizer of f over C.

We say a function $\| \cdot \| : R^n \to R$ a vector norm if it has the following properties [12]:

1. $\|x\| \geqslant 0$ for all $x \in R^n$,

2. $\|cx\| = |c|\|x\|$ for all $c \in R$ and $x \in R^n$ (this is so-called homogeneous),

3. $\|x\| = 0$ if and only if $x = 0$,

4. $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in R^n$ (this is so-called triangle inequality).

Let ℓ_p-norm, $\|\cdot\|_p : R^n \to R$, be defined by $\|x\|_p = (\sum_i^n |x_i|^p)^{1/p}$. For $1 \leqslant p < \infty$, ℓ_p-norm is convex, in particular, if $p = 1$, we have

$$\ell_1\text{-norm}: \ \|x\|_1 = \sum_i^n |x_i| = \text{sign}(x)^T x;$$

if $p = 2$,

$$\ell_2\text{-norm (Euclidean norm)}: \ \|x\|_2 = (\sum_i^n |x_i|^2)^{1/2} = \sqrt{x^T x};$$

if $p = \infty$,

$$\ell_\infty\text{-norm}: \ \|x\|_\infty = \max_{i \in \{1, \cdots, n\}} |x_i|.$$

$\|\cdot\|_p$ with $0 < p < 1$ is a quasi-norm, which violates the triangle inequality property. While the ℓ_0-norm is the number of nonzero components of x, it is not a norm since it violates the homogeneous property that $\|\alpha x\|_0 \neq |\alpha|\|x\|_0$ for any $x \in R^n$ and $\alpha \notin \{0, 1, -1\}$. But we still call $\|\cdot\|_0$ the ℓ_0-norm. It is worth mentioning that

$$\lim_{p \to 0} \sum_i^n |x_i|^p = \|x\|_0,$$

which indicates that $\|x\|_0$ can be approximated by $\|x\|_p^p$ with sufficiently small $p \in (0, 1)$ for any $x \in R^n$.

We say a function $\|\cdot\| : R^{m \times n} \to R$ a matrix norm, which has the following properties [78]:

1. $\|A\| \geqslant 0$ for all $A \in R^{m \times n}$,

2. $\|A\| = 0$ if and only if $A = 0 \in R^{m \times n}$,

3. $\|cA\| = |c|\|A\|$ for all $c \in R$ and $A \in R^{m \times n}$ (this is so-called homogeneous),

4. $\|A + B\| \leqslant \|A\| + \|B\|$ for all $A, B \in R^{m \times n}$ (this is so-called triangle inequality),

5. $\|AB\| \leqslant \|A\|\|B\|$ for all $A, B \in R^{n \times n}$ (this is so-called submultiplicative).

We now introduce the operator norm [78]. For a matrix $A \in R^{m \times n}$, the matrix norm of A is induced by the vector norm as follows

$$\|A\|_{p,q} := \max_{x \neq 0} \frac{\|Ax\|_q}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_q.$$

For $p = q$, the induced operator norm $\| \cdot \|_{p(q)}$ is the submultiplicative matrix norm and $\|I\|_p = 1$. By the definition, we can find an upper bound for $\|Ax\|_q$,

$$\|Ax\|_q \leqslant \|A\|_{p,q} \|x\|_p.$$

In particular, some operator norms can be computed. For instance, the maximum column-sum matrix norm is defined by

$$\|A\|_1 = \max_{j \in \{1, \cdots, n\}} \sum_{i=1}^m |a_{ij}|,$$

the maximum row-sum matrix norm is

$$\|A\|_\infty = \max_{i \in \{1, \cdots, n\}} \sum_{j=1}^n |a_{ij}|,$$

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and the spectral norm is

$$\|A\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)},$$

where $\sigma_{\max}(A)$ ($\lambda_{\max}(A^T A)$) is the largest singular value (eigenvalue) of A ($A^T A$).

1.4.4 Linear Programming

Given a matrix $A \in R^{m \times n}$ and vectors $b \in R^m$ and $c \in R^n$, we define the standard form of linear programming as

$$\begin{aligned}
(LP) \quad \min \quad & c^T x \\
s.t. \quad & Ax = b, \\
& x \geqslant 0
\end{aligned} \tag{1.4.3}$$

and its dual problem as

$$\begin{aligned}
(DP) \quad \max \quad & b^T z \\
s.t. \quad & A^T z \leqslant c, \\
& z \in R^m.
\end{aligned} \tag{1.4.4}$$

By introducing a slack variable $s \in R^n$, the dual problem (1.4.4) is written as

$$\begin{aligned}
\max \quad & b^T z \\
s.t. \quad & A^T z + s = c, \\
& s \geqslant 0, z \in R^m.
\end{aligned} \tag{1.4.5}$$

The following theorem shows that the objective value of any primal problem is at least as large as the objective value of its dual problem [13].

Theorem 1.4.6 (Weak duality theorem) *:*

If x is a feasible solution to the primal problem (1.4.3) and z is a feasible solution to the dual problem (1.4.4), then $b^T z \leqslant c^T x$.

The weak duality theorem gives a flavor about the relation between primal and dual problems. The next corollary of the weak duality theorem shows a more deeper result.

Corollary 1.4.7 ([13]) :

Let x and z be feasible solutions to primal problem (1.4.3) and dual problem (1.4.4), respectively, and suppose that $b^T z = c^T x$. Then x and z are optimal solutions to the primal and dual problems, respectively.

This Corollary provides a sufficient condition for the optimality of primal and dual problems. The next theorem, which is the central result on linear programming, certifies that it is also a necessary optimality condition for primal and dual problems [13].

Theorem 1.4.8 (Strong duality theorem) :

If both the primal problem and dual problem have optimal solutions, for a primal optimal solution x and dual optimal solution z, we have $b^T z = c^T x$.

Another important relation between the primal problem (1.4.3) and its dual problem (1.4.5) is given by the complementary slackness condition [13], as shown in the next theorem.

Theorem 1.4.9 (Complementary slackness condition) :

Let x and (z, s) be feasible solutions to the primal and dual problems, respectively. Then x and (z, s) are optimal solutions if and only if the complementary slackness conditions holds, i.e., $x^T(c - A^T z) = 0 \Leftrightarrow x^T s = 0$.

Moreover, if a solution pair $(x, (z, s))$ satisfies that $x^T s = 0$ and $x + s > 0$, this solution pair is called the strictly complementary solution pair.

Theorem 1.4.10 (Strictly complementary slackness [114]) *:*

Let primal and dual problems both have feasible solutions. Then there exists a pair of strictly complementary solution $x \geqslant 0$ and $s \geqslant 0$ with $x^T s = 0$ and $x + s > 0$.

CHAPTER 2

THEORY FOR COMPRESSIVE SENSING

2.1 Introduction

In this chapter, we discuss two theoretical aspects of standard compressive sensing, uniqueness of solutions of ℓ_0-minimization and sparse signal recoverability of ℓ_1-minimization. When matrix $A \in R^{m \times n}$ with $m < n$ is full rank, the underdetermined linear equation system $Ax = b$ has infinitely many solutions. ℓ_0-minimization can locate either a unique sparsest solution or multiple sparsest solutions to the system $Ax = b$ from a given measurements vector b. Given the measurements b and a special designed matrix A, to achieve the exact recovery via ℓ_1-minimization, it is usually necessary to require the strong equivalence of ℓ_0- and ℓ_1-minimization [132], which implies the uniqueness of solutions to both ℓ_0-minimization and ℓ_1-minimization. On the other hand, in the case of multiple sparsest solutions, ℓ_1-minimization may find at most one sparsest solution of the system $Ax = b$ provided that ℓ_1-minimization admits a unique solution. This is referred to as the equivalence of ℓ_0- and ℓ_1-minimizations [132]. So far, many uniqueness conditions and recovery conditions have been developed for the uniqueness of solutions of ℓ_0-minimization and the strong equivalence or equivalence of ℓ_0- and ℓ_1-minimization. In this chapter, we briefly introduce ones in terms of spark, mutual coherence, Babel function, exact recovery con-

dition (ERC), restricted isometry property (RIP), null space property (NSP), and range space property (RSP). These conditions characterize certain properties of A such that ℓ_0-minimization can be solved by a tractable linear program, i.e., ℓ_1-minimization. All results obtained from the exact recovery can be categorized into two groups. One requires recovering a specific sparsest vector from a linear system (namely, nonuniform recovery), and the other requires recovering all K-sparse vectors (i.e., $\|x\|_0 \leqslant K$) through a single sensing matrix (namely, uniform recovery). The techniques and results established for the uniqueness of solutions of ℓ_0-minimization and recovery conditions via ℓ_1-minimization in this chapter will be extended to some special applications in compressive sensing, i.e., partial sparsity-seeking problem in Chapter 3 and 1-bit compressive sensing problem in Chapter 4.

This chapter is organized as follows. In section 2.2, we introduce sufficient conditions for the uniqueness of solutions of ℓ_0-minimization in terms of spark, mutual coherence and Babel function. In section 2.3, we introduce exact recovery coefficient, restricted isometry property, null space property and range space property-based analysis to ensure the nonuniform recovery and uniform recovery via ℓ_1-minimization.

2.2 Uniqueness conditions for ℓ_0-minimization

Firstly, we introduce two fundamental properties of a matrix, spark and mutual coherence, which are initially defined in [54, 52]. Based on these two properties, Donoho and Elad developed sufficient conditions for ℓ_0-minimization to admit a unique sparsest solution. In addition, mutual coherence provides a lower bound of spark [52], but such a lower bound is not tight and can be improved in some cases, e.g., the sensing matrix is a concatenation of orthonormal matrices [64, 73, 95]. Meanwhile, as mutual coherence characterizes the extreme relation of pairs of columns from a matrix, to reflect more correlations between a distinct column and a collection of a few columns, we introduce

Babel function proposed by Tropp [120], an alternative method to obtain some uniqueness conditions for ℓ_0-minimization.

2.2.1 Spark

We start with the *spark* of a matrix, which provides a sufficient condition for the uniqueness of solutions of ℓ_0-minimization.

Definition 2.2.1 (Spark(A)) *: For a given matrix A, the spark is the smallest number of linearly dependent columns from matrix A.*

By definition, spark is more difficult to evaluate as it requires to check all possible combinations of columns from a matrix. Also, note that the definition of spark resembles that of rank, the largest number of linearly independent columns from a matrix. By definition, the spark of any nonzero matrices should be greater than 1; for full rank matrices, such as rank(A)=m, the Spark(A) can rise up to $m+1$ with probability 1 if every entry of matrix A is from a random identical and independent distribution [68]. Additionally, we can get a tight upper bound for Spark(A) via the null space of matrix A, i.e., $\mathcal{N}(A) = \{h \in R^n | Ah = 0\}$. That is, Spark($A$) $\leqslant \|h\|_0$ for any vector $h \in \mathcal{N}(A) \setminus \{0\}$. Thus, the Spark($A$) can be evaluated as

$$\text{Spark}(A) = \min\{\|h\|_0 : \ Ah = 0, \ h \neq 0\}.$$

This gives the following sufficient condition for the uniqueness of solutions of ℓ_0-minimization.

Theorem 2.2.2 (Uniqueness via Spark(A), Corollary 4 in [52]) *:*
For a given linear system $Ax = b$, if there exists a solution x satisfying

$$\|x\|_0 < \frac{1}{2} Spark(A), \tag{2.2.1}$$

then x is necessarily the sparest solution to ℓ_0-minimization.

Proof. Assume the contrary that there exists another sparsest solution $\tilde{x} \neq x$, such that $\|\tilde{x}\|_0 \leqslant \|x\|_0$. Since both x and $\tilde{x}$ are solutions of $Ax = b$, we have $A(x - \tilde{x}) = 0$, *i.e.*, $x - \tilde{x} \in \mathcal{N}(A) \backslash \{0\}$. As the null space of A provides an upper bound for $\mathrm{Spark}(A)$, we have $\|x - \tilde{x}\|_0 \geqslant \mathrm{Spark}(A)$. By triangle inequality of ℓ_0-norm,

$$\|x\|_0 + \|\tilde{x}\|_0 \geqslant \|x - \tilde{x}\|_0 \geqslant \mathrm{Spark}(A). \tag{2.2.2}$$

If $\|x\|_0 < \frac{1}{2}\mathrm{Spark}(A)$, $\|\tilde{x}\|_0$ must be greater than $\frac{1}{2}\mathrm{Spark}(A)$ to satisfy inequality (2.2.2). This is a contradiction to the assumption $\|\tilde{x}\|_0 \leqslant \|x\|_0 < \frac{1}{2}\mathrm{Spark}(A)$. Hence, x is necessarily the sparsest solution of ℓ_0-minimization. $\qquad\square$

This following corollary can be achieved directly from the proof of Theorem 2.2.2.

Corollary 2.2.3 (Theorem 3 in [52]) *:*
If linear system $Ax = b$ has two distinct solutions, say x and $\widehat{x}$, the sum of the number of nonzeros of x and $\widehat{x}$ must be no less than $Spark(A)$.

2.2.2 Mutual Coherence

The sufficient condition for the sparest solution in Theorem 2.2.2 is intractable as the complexity of finding $\mathrm{Spark}(A)$ is at least the same as solving ℓ_0-minimization. We need an easier way to check whether the solution is unique. Now we introduce the *mutual coherence* which can provide a tractable uniqueness condition.

Definition 2.2.4 (mutual coherence) *: For a given matrix $A \in R^{m \times n}$, the mutual coherence denoted as $\mu(A)$ is the largest absolute value of inner products between any different normalized columns of A, i.e.,*

$$\mu(A) = \max_{1 \leqslant i \neq j \leqslant n} \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \cdot \|a_j\|_2},$$

where a_i and a_j $(i \neq j \in \{1, \cdots, n\})$ are the i-th column and the j-th column of matrix A, respectively.

The definition of mutual coherence states the dependency between any different columns of A. Thus, we can interpret mutual coherence as follows

$$\mu(A) = \max_{1 \leqslant i \neq j \leqslant n} |\cos \theta_{ij}|,$$

where θ_{ij} is the angle between column vectors a_i and a_j. If $\mu(A)$ is small, we say matrix A is *incoherent*. Compared to $\mathrm{Spark}(A)$, mutual coherence is easier to calculate and more direct to capture the relations between columns of matrix A. For instance, for unitary matrices, whose columns are pairwise orthogonal, its mutual coherence is zero. For the union of two orthonormal bases Φ and Ψ, i.e. $A = [\Phi, \Psi] \in R^{m \times 2m}$, $\frac{1}{\sqrt{m}} \leqslant \mu(A) \leqslant 1$ [54]. The lower bound is obtained from a well-known property in quantum physics [76][127], mutually unbiased bases (MUB), where two orthonormal bases $\Phi = [\phi_1, \cdots, \phi_m]$ and $\Psi = [\psi_1, \cdots, \psi_m]$ satisfy $|\langle \phi_i, \psi_j \rangle| = \frac{1}{\sqrt{m}}$, $i = 1 \cdots m$ and $j = 1 \cdots m$. For a general matrix $A \in R^{m \times n}$ with $m < n$, Theorem 2.3 in [116] shows that $\mu(A) \geqslant \sqrt{\frac{n-m}{m(n-1)}}$, and the equality is achieved by optimal Grassmnnian frames with $n \leqslant \frac{m(m+1)}{2}$.

By definitions of mutual coherence and spark, as both characterize matrix A in the context of dependency, the relationship between $\mathrm{Spark}(A)$ and $\mu(A)$ can be revealed by the following lemma.

Lemma 2.2.5 ([28, 52]) : *For a given matrix $A \in R^{m \times n}$, one has $\mathrm{Spark}(A) \geqslant 1 + \frac{1}{\mu(A)}$.*

Proof. Denote $\widehat{A}$ as the normalized matrix A, where each column of matrix $\widehat{A}$ is a unit ℓ_2-norm vector. Note that normalization will not change spark and mutual coherence. Let G be the Gram matrix of $\widehat{A}$, where

$$G_{ii} = 1, i = 1, \cdots, n, \qquad G_{ij} = \frac{\langle a_i, a_j \rangle}{\|a_i\|_2 \cdot \|a_j\|_2}, \quad i \neq j, 1 \leqslant i, j \leqslant n,$$

and a_i, a_j are the i-th and j-th columns of matrix A. Thus,

$$|G_{ij}| = \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \cdot \|a_j\|_2} \leqslant \mu(A), \quad \forall \, 1 \leqslant i \neq j \leqslant n.$$

Now we assume that $\mathrm{Spark}(A) = k$ and $\{a_1, \cdots, a_k\}$ are the smallest linearly dependent columns from matrix A. Let $G_{k \times k}$ be the submatrix of G generated by $\{a_1 \cdots, a_k\}$. $G_{k \times k}$ is not a positive definite matrix, so $G_{k \times k}$ is not a strictly diagonal dominant matrix. Thus, there exists a column a_i such that

$$|G_{ii}| = 1 \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{k} |G_{ij}|.$$

As $|G_{ij}| \leqslant \mu(A)$, we have

$$(k-1)\mu(A) \geqslant 1,$$

which implies that $k \geqslant 1 + \frac{1}{\mu(A)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In fact, the lower bound of $\mathrm{Spark}(A)$ given in Lemma 2.2.5 is quite conservative. We will give an example to show the difference between the mutual coherence uniqueness condition and the spark uniqueness condition.

Example 2.2.6 :

Let $A \in R^{m \times m+1}$ be a concatenation of the identity matrix $I_{m \times m}$ and a vector $e = [1, \cdots, 1]^\top \in R^m$, i.e.,

$$A = [I_{m \times m}; \ e]_{m \times m+1}.$$

Then, we have $\mu(A) = \frac{1}{\sqrt{m}}$. Obviously, A is full rank with $\mathrm{rank}(A) = m$. As the spark is the smallest number of linearly dependent columns, in this case, we have $\mathrm{Spark}(A) =$

$m + 1$. Thus,

$$\frac{1}{2}(1 + \frac{1}{\mu(A)}) = \frac{1}{2}(1 + \sqrt{m}) \quad \text{whereas} \quad \frac{1}{2}\text{Spark}(A) = \frac{1}{2}(m + 1).$$

When $m = 25$, the mutual coherence condition fails to verify any sparse solution with the sparsity between $\frac{1}{2}(1 + \sqrt{m}) = 3$ and $\frac{1}{2}(m + 1) = 13$. However, this lower bound of Spark(A) can be improved when matrix A is the concatenation of orthonormal matrices [64, 73, 95, 52] or by using new concepts, like sub-mutual coherence, (sub-)coherence rank established in [131].

From Theorem 2.2.2 and Lemma 2.2.5, we have the uniqueness and recovery condition for the strong equivalence of ℓ_0- and ℓ_1-minimization via mutual coherence.

Theorem 2.2.7 (Uniqueness and recovery via mutual coherence [52]) :
For a given linear system $Ax = b$, if there exists a solution x such that

$$\|x\|_0 < \frac{1}{2}(1 + \frac{1}{\mu(A)}), \tag{2.2.3}$$

then x is both the unique sparsest solution of ℓ_0-minimization and the unique solution of ℓ_1-minimization.

Theorem 2.2.7 states that the mutual coherence condition (2.2.3) can verify the unique sparsest solution to ℓ_0-minimization as well as provide a sufficient condition for the strong equivalence of ℓ_0- and ℓ_1-minimization.

2.2.3 Babel function

Mutual coherence reflects extreme relationships between columns from a matrix. If there is a normalized inner product of columns which is very large, the mutual coherence condition is too restrictive to verify the uniqueness of a sparsest solution. Instead of considering

such an extreme correlation between individual columns, Babel function measures the relation between a collection of columns and a fixed distinct column.

Definition 2.2.8 (Babel Function) : *For a given matrix $A \in R^{m \times n}$ with normalized columns, Babel function is defined as*

$$\mu_1(K) = \max_{|\Lambda|=K} \max_{j \notin \Lambda} \sum_{i \in \Lambda} |\langle a_j, a_i \rangle|,$$

where a_i and a_j are the i-th and j-th columns of A, and Λ is an index set with $\Lambda \subseteq \{1, \cdots, n\}$.

By the definition, Babel function is a maximum absolute sum of any $|\Lambda| = K$ inner products between a_i and a_j, and it is a nondecreasing function of K. Conventionally, we assume $\mu_1(0) = 0$ and notice that when $K = 1$, $\mu_1(1) = \mu(A)$.

Proposition 2.2.9 ([120]) : *For a given matrix $A \in R^{m \times n}$ with normalized columns,*

(1) if A is an orthonormal basis, then $\mu_1(K) = 0$ for any integer $K \geqslant 1$;

(2) $\mu_1(K) \leqslant K\mu(A)$ for any integer $K \geqslant 1$;

(3) if there is an index set Λ with $|\Lambda| = K$, for a submatrix A_Λ from A, the squared singular value of A_Λ satisfies $1 - \mu_1(K-1) \leqslant \sigma^2 \leqslant 1 + \mu_1(K-1)$.

Proof. (1) It is straightforward from the definition of Babel function since inner products between any distinct columns from a orthonormal matrix are zero;

(2) For an index set $\Lambda \subseteq \{1, \cdots, n\}$, as $|\langle a_j, a_i \rangle| \leqslant \mu(A)$ for any $i \neq j \in \{1, \cdots, n\}$,

$$\begin{aligned} \mu_1(K) &= \max_{|\Lambda|=K} \max_{j \notin \Lambda} \sum_{i \in \Lambda} |\langle a_j, a_i \rangle| \\ &\leqslant \max_{|\Lambda|=K} \sum_{i \in \Lambda} \mu(A) = K\mu(A). \end{aligned}$$

(3) Consider the Gram matrix $G_\Lambda = A_\Lambda^T A_\Lambda$, which is a positive semidefinite matrix. By Gersgorin Disc theorem, every eigenvalue λ of G_Λ lies in a Gersgorin disc centered at G_{ii},

namely,

$$|\lambda - G_{ii}| \leqslant \sum_{j \neq i} |G_{ij}|, \qquad \forall i \in \Lambda. \tag{2.2.4}$$

Since $G_{ii} = 1$ and

$$\begin{aligned}
\sum_{j \neq i} |G_{ij}| &= \sum_{\substack{j \neq i \\ j \in \Lambda/\{i\}}} |\langle a_i, a_j \rangle| \\
&\leqslant \max_i \sum_{j \in \Lambda' := \Lambda/\{i\}} |\langle a_i, a_j \rangle| \\
&\leqslant \max_{|\Lambda'| = K-1} \max_i \sum_{j \in \Lambda'} |\langle a_i, a_j \rangle| \\
&\leqslant \mu_1(K-1),
\end{aligned}$$

by the inequality (2.2.4), we get $1 - \mu_1(K-1) \leqslant \sigma^2 \leqslant 1 + \mu_1(K-1)$. $\qquad\square$

As discussed in section 2.2.2, the minimal mutual coherence can be achieved by Grassmannian frames with $n \leqslant \frac{m(m+1)}{2}$. Also, we can get a lower bound of Babel function by Grassannian frames [113]: if $K^2 < n - 1$, then

$$\mu_1(K) \geqslant K \sqrt{\frac{n-m}{m(n-1)}}.$$

This equality holds if and only if A is an equiangular unit norm tight frame [113], such as the orthonormal matrix. In addition, Babel function can be used to construct a lower bound on the spark, as shown in the next theorem.

Theorem 2.2.10 ([120]) :

For any normalized matrix $A \in R^{m \times n}$, Babel function gives a lower bound for Spark(A),

$$Spark(A) \geqslant \min\{K | \mu_1(K-1) \geqslant 1\}.$$

Proof. Assume that $\text{Spark}(A) = p$ and let Gram matrix of a submatrix A_Λ be $G_\Lambda = A_\Lambda^\top A_\Lambda$, where $\Lambda \subseteq \{1, \cdots, n\}$ is an index set with $|\Lambda| = p$. Since matrix A is normalized, we have

$$G_{ii} = 1 \text{ and } G_{ij} = \frac{\langle a_i, a_j \rangle}{\|a_i\|_2 \|a_j\|_2}, \quad i \neq j \in \Lambda.$$

As A_Λ has linearly dependent columns, there is an $i \in \Lambda$ such that the corresponding eigenvalue λ_i of G_Λ is zero. By Gersgorin Disc theorem, we have

$$
\begin{aligned}
|\lambda_i - G_{ii}| = 1 \leqslant \sum_{j \neq i} \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \|a_j\|_2} \quad &\leqslant \quad \max_i \sum_{j \in \Lambda/\{i\}} \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \|a_j\|_2} \\
&\leqslant \quad \max_{|\Lambda/\{i\}|=p-1} \max_i \sum_{j \in \Lambda/\{i\}} \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \|a_j\|_2} \\
&= \quad \mu_1(p-1),
\end{aligned}
$$

which implies that

$$p \in \{K | \ \mu_1(K-1) \geqslant 1\}.$$

Thus, as $\mu_1(K-1)$ is a nondecreasing function of K, p should at least be the minimum number of K such that $\mu_1(K-1) \geqslant 1$. Hence, we have $\text{Spark}(A) \geqslant \min\{K | \ \mu_1(K-1) \geqslant 1\}$. $\qquad\square$

Combining Theorem 2.2.10 and Theorem 2.2.2, we obtain another uniqueness condition for ℓ_0-minimization in terms of Babel function.

Theorem 2.2.11 (Uniqueness by Babel function) :

For a given linear system $Ax = b$, if there exists a solution x satisfying

$$\|x\|_0 < \frac{1}{2} \min\{K | \ \mu_1(K-1) \geqslant 1\}, \tag{2.2.5}$$

then x is the sparsest solution to ℓ_0-minimization.

2.3 Recovery conditions for ℓ_1-minimization

We have seen that if a system of linear equations has a unique sparsest solution, conditions in Theorems 2.2.2, 2.2.7 and 2.2.11 can verify such a solution. To exactly recover the sparsest solution of a system of linear equations, more conditions on the sensing matrix are established to ensure the strong equivalence of ℓ_0- and ℓ_1-minimization. Results from such a recovery will be discussed by nonuniform recovery, recovering a specific sparsest vector from the measurements, and uniform recovery, recovering all K-sparse vectors (i.e., $\{x \in R^n : \|x\|_0 \leqslant K\}$) through a single sensing matrix.

2.3.1 Exact Recovery Coefficient

We begin with the Exact Recovery Coefficient (ERC) recovery condition, which gives an idea that what property of the sensing matrix can ensure the exact recovery.

Definition 2.3.1 (ERC) : *Given a normalized matrix $A \in R^{m \times n}$, let Λ be an index set of a full column-rank submatrix from A, then define Exact Recovery Coefficient, $ERC(\Lambda; A)$ as*

$$ERC(\Lambda; A) := 1 - \max_{j \notin \Lambda} \|A_\Lambda^\dagger a_j\|_1,$$

where $A_\Lambda^\dagger$ is the pseudoinverse of A_Λ. That is, $A_\Lambda^\dagger = (A_\Lambda^T A_\Lambda)^{-1} A_\Lambda^T$.

We omit the matrix symbol from the notation of ERC. Note that $ERC(\Lambda)$ is developed from Babel function, measuring the difference between any full column-rank submatrix A_Λ and a column a_j that is not participated in A_Λ from matrix A. Tropp gave a brief discussion about the necessity of ERC in [123] and proved that $ERC(\Lambda)$ provides an sufficient condition for the nonuniform recovery.

Theorem 2.3.2 (Theorem 3.3 in [123]) :

For a given linear system $Ax = b$, where matrix A is normalized, let Λ be the support set of x^* such that $b = A_\Lambda x_\Lambda^*$ and x^* be the sparsest solution to ℓ_0-minimization. If $ERC(\Lambda) > 0$, then ℓ_1-minimization can recover the sparsest solution x^*.

Proof. Assume the contrary that there exists a solution $\widehat{x} \neq x^*$ to ℓ_1-minimization such that $b = \widehat{A}\widehat{x}_{\mathrm{Supp}(\widehat{x})}$ and $\|\widehat{x}\|_1 \leqslant \|x^*\|_1$, where $\widehat{A}$ is a full column-rank submatrix of A. By assumption, as x^* is the sparsest solution, A_Λ has full column-rank. Thus, we have $\|\widehat{x}\|_1 \leqslant \|x^*\|_1 = \|A_\Lambda^\dagger \widehat{A}\widehat{x}_{\mathrm{Supp}(\widehat{x})}\|_1 \leqslant \max_\Lambda \|A_\Lambda^\dagger \widehat{A}\widehat{x}_{\mathrm{Supp}(\widehat{x})}\|_1$ and $\max_\Lambda \|A_\Lambda^\dagger \widehat{A}\widehat{x}_{\mathrm{Supp}(\widehat{x})}\|_1$ means maximizing $\|A_\Lambda^\dagger \widehat{A}\widehat{x}_{\mathrm{Supp}(\widehat{x})}\|_1$ over any index set $\Lambda \subseteq \{1, \cdots, n\}$ with $|\Lambda| = \mathrm{Supp}(x^*)$, where the corresponding submatrix A_Λ is full column rank.

Let a_i's be row vectors from $A_\Lambda^\dagger$ and $\widehat{a}_j$'s be column vectors from $\widehat{A}$, then we get

$$
\begin{aligned}
\max_\Lambda \|A_\Lambda^\dagger \widehat{A}\widehat{x}_{\mathrm{Supp}(\widehat{x})}\|_1 &= \max_\Lambda \sum_{i \in \Lambda} \left| \sum_j a_i \widehat{a}_j \widehat{x}_j \right| \\
&\leqslant \max_\Lambda \sum_{i \in \Lambda} \sum_j |a_i \widehat{a}_j \widehat{x}_j| \\
&\leqslant \max_\Lambda \sum_{i \in \Lambda} \sum_j |a_i \widehat{a}_j| \, |\widehat{x}_j| \qquad (2.3.1) \\
&= \max_\Lambda \sum_j \sum_{i \in \Lambda} |a_i \widehat{a}_j| \, |\widehat{x}_j| .
\end{aligned}
$$

As $b = \widehat{A}\widehat{x}$ is a different representation from $b = A_\Lambda x^*$, there exists at least one column $\widehat{a}_j$ belonging to $\widehat{A}$ but not in A_Λ satisfying $ERC(\Lambda) > 0$. For such a column $\widehat{a}_j$, as $ERC(\Lambda) > 0$, $1 - \|A_\Lambda^\dagger \widehat{a}_j\|_1 \geqslant 1 - \max_{j \notin \Lambda} \|A_\Lambda^\dagger \widehat{a}_j\|_1 > 0$, which is equivalent to $\max_{j \notin \Lambda} \sum_{i \in \Lambda} |a_i, \widehat{a}_j| < 1$. Relaxing the last inequality in equation (2.3.1), we have

$$
\begin{aligned}
\max_\Lambda \|A_\Lambda^\dagger \widehat{A}\widehat{x}\|_1 &\leqslant \max_\Lambda \sum_j \left(\sum_{i \in \Lambda} |a_i \widehat{a}_j| \right) |\widehat{x}_j| \\
&< \max_\Lambda \sum_j |\widehat{x}_j| = \|\widehat{x}\|_1 .
\end{aligned}
$$

This contradicts the assumption that $\|\widehat{x}\|_1 \leqslant \|x^*\|_1 < \|\widehat{x}\|_1$. Hence, ℓ_1-minimization can recover the sparsest solution under the ERC condition. $\qquad\square$

Note that the sparsity level of the sparsest solution is implicitly shown in the ERC condition, which is the cardinality of the support set Λ. Compared to the mutual coherence condition (2.2.3), ERC condition is more difficult to check, unless we know the support set Λ. Even if the sparsity level $|\Lambda|$ is known, it is still costly to test all the possible $\binom{n}{|\Lambda|}$ combinations of columns of matrix A. But we can use Babel function and mutual coherence to verify the ERC condition, i.e., $ERC(\Lambda) > 0$.

Proposition 2.3.3 (Proposition 3.7 in [122]) :

Suppose that Λ is an index set of a full column-rank submatrix A_Λ from a normalized matrix $A \in R^{m \times n}$ with $|\Lambda| \leqslant K$, a lower bound of Exact Recovery Coefficient is

$$ERC(\Lambda) \geqslant \frac{1 - \mu_1(K-1) - \mu_1(K)}{1 - \mu_1(K-1)}. \tag{2.3.2}$$

If $\mu_1(K-1) + \mu_1(K) < 1$ holds, then we have $ERC(\Lambda) > 0$.

Proof. As $ERC(\Lambda) = 1 - \max_{j \notin \Lambda} \|A_\Lambda^\dagger a_j\|_1$, we need to relax $\max_{j \notin \Lambda} \|A_\Lambda^\dagger a_j\|_1$ to find a lower bound of $ERC(\Lambda)$.

Substituting $A_\Lambda^\dagger = (A_\Lambda^T A_\Lambda)^{-1} A_\Lambda^T$ into the definition of $ERC(\Lambda)$, as $A_\Lambda^T A_\Lambda$ is invertible, we have

$$\begin{aligned} \max_{j \notin \Lambda} \|A_\Lambda^\dagger a_j\|_1 &= \max_{j \notin \Lambda} \|(A_\Lambda^T A_\Lambda)^{-1} A_\Lambda^T a_j\|_1 \\ &\leqslant \|(A_\Lambda^T A_\Lambda)^{-1}\|_{1,1} \max_{j \notin \Lambda} \|A_\Lambda^T a_j\|_1. \end{aligned} \tag{2.3.3}$$

Now, we relax the first term on the right hand side of inequality (2.3.3).

By Proposition 2.2.9 (3), the squared singular value of a submatrix A_Λ is the eigenvalue

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of a Gram matrix $G = A_\Lambda^T A_\Lambda$, then we have,

$$1 - \mu_1(K-1) \leqslant \lambda(G) \leqslant 1 + \mu_1(K-1).$$

Since G is invertible, we get $\lambda(G^{-1}) = \frac{1}{\lambda(G)}$. Thus,

$$\frac{1}{1 + \mu_1(K-1)} \leqslant \lambda(G^{-1}) \leqslant \frac{1}{1 - \mu_1(K-1)}.$$

By definition of operator norm,

$$
\begin{aligned}
\|(A_\Lambda^T A_\Lambda)^{-1}\|_{1,1} = \max_{\|x\|_1=1} \|(A_\Lambda^T A_\Lambda)^{-1}x\|_1 &= \max_{\|x\|_1=1} \|\lambda(G^{-1})x\|_1 \\
&\leqslant \frac{1}{1 - \mu_1(K-1)}\|x\|_1 = \frac{1}{1 - \mu_1(K-1)}.
\end{aligned}
$$

Then, relax the second term $\max_{j\notin\Lambda} \|A_\Lambda^T a_j\|_1$ of inequality (2.3.3) and we have

$$
\begin{aligned}
\max_{j\notin\Lambda} \|A_\Lambda^T a_j\|_1 &\leqslant \max_{|\Lambda|=K} \max_{j\notin\Lambda} \|A_\Lambda^T a_j\|_1 \\
&= \max_{|\Lambda|=K} \max_{j\notin\Lambda} \sum_{i\in\Lambda} |\langle a_i, a_j\rangle| \stackrel{def}{=} \mu_1(K).
\end{aligned}
$$

Hence,

$$\max_{j\notin\Lambda} \|A_\Lambda^\dagger a_j\|_1 \leqslant \frac{1}{1 - \mu_1(K-1)}\mu_1(K),$$

and the inequality (2.3.2) holds. To achieve $ERC(\Lambda) > 0$, the lower bound should be greater than 0. Hence, we have $\mu_1(K-1) + \mu_1(K) < 1$ holds to guarantee it. $\qquad\square$

By Proposition 2.2.9 (2), the mutual coherence also can build a lower bound of ERC.

Proposition 2.3.4 (Theorem 4.7 in [58]) :

Suppose that Λ is an index set of a full column-rank submatrix A_Λ, which is from a normalized matrix $A \in R^{m \times n}$ with $|\Lambda| \leqslant K$. Then, a lower bound of Exact Recovery

Coefficient is

$$ERC(\Lambda) \geqslant \frac{1 - 2K\mu(A) + \mu(A)}{1 - (K-1)\mu(A)}. \tag{2.3.4}$$

If $K < \frac{1}{2}(1 + \frac{1}{\mu(A)})$ holds, then we have $ERC(\Lambda) > 0$.

Proof. We follow the proof similar to that of Proposition 2.3.3. From the definition of $ERC(\Lambda)$, we have

$$
\begin{aligned}
\max_{j \notin \Lambda} \|A_\Lambda^\dagger a_j\|_1 &= \max_{j \notin \Lambda} \|(A_\Lambda^T A_\Lambda)^{-1} A_\Lambda^T a_j\|_1 \\
&\leqslant \|(A_\Lambda^T A_\Lambda)^{-1}\|_{1,1} \max_{j \notin \Lambda} \|A_\Lambda^T a_j\|_1.
\end{aligned} \tag{2.3.5}
$$

This inequality holds by the definition of operator norms. Now, we consider the first term on the right hand side of inequality (2.3.5), $\|(A_\Lambda^T A_\Lambda)^{-1}\|_{1,1}$. As A_Λ is a full column-rank submarix, the Gram matrix $G = A_\Lambda^T A_\Lambda$ is nonsingular. By Gersgorin Disc theorem, every eigenvalue λ of G lies in a Gersgorin disc that is

$$|\lambda - 1| \leqslant \sum_{j \neq i} |G_{ij}| \leqslant (K-1)\mu(A) \ \forall i \in \Lambda,$$

which equals to

$$1 - (K-1)\mu(A) \leqslant \lambda(G) \leqslant 1 + (K-1)\mu.$$

Thus, the eigenvalue of G^{-1} satisfies

$$\frac{1}{1 + (K-1)\mu} \leqslant \lambda(G^{-1}) \leqslant \frac{1}{1 - (K-1)\mu(A)}.$$

Hence,

$$\|(A_\Lambda^T A_\Lambda)^{-1}\|_{1,1} = \max_{\|x\|_1=1} \|(A_\Lambda^T A_\Lambda)^{-1}x\|_1 = \max_{\|x\|_1=1} \|\lambda(G^{-1})x\|_1$$

$$\leqslant \frac{1}{1-(K-1)\mu(A)}\|x\|_1 = \frac{1}{1-(K-1)\mu(A)}.$$

Next, since $\max_{j\notin\Lambda}\|A_\Lambda^T a_j\|_1$ can be bounded as

$$\max_{j\notin\Lambda}\|A_\Lambda^T a_j\|_1 = \max_{j\notin\Lambda}\sum_{i\in\Lambda}|\langle a_i, a_j\rangle| \leqslant K\mu(A),$$

we have

$$\max_{j\notin\Lambda}\|A_\Lambda^\dagger a_j\|_1 \leqslant \frac{K\mu(A)}{1-(K-1)\mu(A)}.$$

Then,

$$ERC(\Lambda) = 1 - \max_{j\notin\Lambda}\|A_\Lambda^\dagger a_j\|_1 \geqslant \frac{1-2K\mu(A)+\mu(A)}{1-(K-1)\mu(A)},$$

and hence, ERC(Λ)> 0 can be provided by $K < \frac{1}{2}(1+\frac{1}{\mu(A)})$. $\qquad\square$

Due to the mutual coherence sufficient condition in Theorem 2.2.7, this proposition explains why $ERC(\Lambda) > 0$ is also a sufficient condition for the exact recovery via ℓ_1-minimization. We have shown that mutual coherence condition (2.2.3) and ERC condition can guarantee the nonuniform recovery. To recover all K-sparse vectors through a single matrix A, we will further introduce some restrictive conditions on matrix A. Candès and Tao [32, 33, 36], and Donoho [51] have shown that under a certain Restricted Isometry Property (RIP) condition, some recovery methods including ℓ_1-minimization can ensure the uniform recovery.

2.3.2 Restricted Isometry Property

Let $\sum_K$ be the set of all K-sparse vectors, i.e.

$$\sum_K := \{x \in R^n | \|x\|_0 \leqslant K\}.$$

Definition 2.3.5 (Restricted Isometry Property) :

For any given matrix A, if there exists a smallest isometry constant number $\delta_K \in (0,1)$, such that

$$(1 - \delta_K)\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1 + \delta_K)\|x\|_2^2 \tag{2.3.6}$$

holds for all $x \in \sum_K$, then we say that matrix A satisfies the Restricted Isometry Property (RIP) of order K.

This property shows that submatrices consisted of any K columns of A have similar behaviors with small isometry constant δ. Note that, for any $t < K$, if matrix A satisfies the RIP of order K with δ_K, A must satisfy the RIP of order t with δ_t where $\delta_t < \delta_K$. Additionally, if matrix A has arbitrary bounds, such as

$$a\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant b\|x\|_2^2,$$

where $0 < a \leqslant b < \infty$, we can scale A to satisfy (2.3.6). The uniqueness of a K-sparse solution can be obtained from the RIP, as indicated in the following result.

Theorem 2.3.6 (Uniqueness via RIP [33]) :

Suppose that matrix A satisfies the RIP of order $2K$ with $\delta_{2K} < 1$, then ℓ_0-minimization has a unique K-sparse solution if such a solution exists.

Proof. Assume the contrary that there exist two distinct solutions of ℓ_0-minimization x and y, such that $x \in \Sigma_K$ and $y \in \Sigma_K$. Then we have $h = x - y \in \mathcal{N}(A) \setminus \{0\}$ and $h \in \Sigma_{2K}$. By the RIP assumption, we have

$$(1 - \delta_{2K})\|h\|_2^2 \leqslant \|Ah\|_2^2 \leqslant (1 + \delta_{2K})\|h\|_2^2.$$

This yields a contradiction since $Ah = 0$ whereas $1 - \delta_{2K} > 0$. Therefore, ℓ_0-minimization has a unique K-sparse solution. $\qquad\square$

Moreover, for a certain restricted isometry constant δ_K, RIP can also provide a sufficient condition for the uniform recovery via ℓ_1-minimization.

Theorem 2.3.7 (Theorem 1.2 in [33]) :

 Suppose that matrix A satisfies the RIP of order $2K$ with $\delta_{2K} < \sqrt{2} - 1$, then the solution x^* of ℓ_1-minimization satisfies

$$\|x^* - x\|_2 \leqslant C \frac{\|x - x_K\|_1}{\sqrt{K}}, \tag{2.3.7}$$

where $C = 2\frac{1 - (1-\sqrt{2})\delta_{2K}}{1 - (1+\sqrt{2})\delta_{2K}}$, x is the unique sparsest solution to ℓ_0-minimization and x_K is a sub-vector of the largest K components of absolute value of x. Particularly, if x is K-sparse, the ℓ_1-recovery is exact.

Theorem 2.3.7 gives a nice result for the uniform recovery via ℓ_1-minimization under certain RIP condition. It is worth stressing that the δ_{2K} condition, i.e., $\delta_{2K} < \sqrt{2} - 1$, has been improved by several researches in [29, 30, 31, 15, 48]. Also, the uniform recovery via other greedy algorithms, such as orthogonal matching pursuit (OMP) [47, 120] and iterative hard thresholding (IHT) [19, 17], can be ensured by certain RIP conditions. Now, problems are what matrices satisfy the RIP condition and how to evaluate the RIP constant δ_K such that δ_K obeys the condition in Theorem 2.3.7 or other improved δ_K

conditions in [29, 30, 31, 65, 66, 68, 96]. In fact, computing the constant δ_K for a given matrix and any number K is NP-hard [119, 6]. Fortunately, there do exist some families of random matrices that admit a certain RIP condition with high probability, for instance, Fourier matrix [37], Gaussian matrix [37, 10], Bernoulli matrix [10], randomly restricted Hadmard matrix [112] and Toeplitz matrix [5]. It is worth mentioning that Gaussian matrix and Bernoulli matrix are often used in experiments to show performances of any recovery algorithms in compressive sensing. We will also test the performance of 1-bit basis pursuit, which is a basis pursuit method for 1-bit compressive sensing problem, by using Gaussian and Bernoulli matrices in Chapter 5.

The RIP property is one of the widely used analyzing tools in compressed sensing. However, the RIP property has its limitations. For example, the linear system is determined by (WA, Wb), where (WA, Wb) and (A, b) have the same solution structure since W is an invertible weight matrix. But the restricted isometry constants of WA and A may be massively different. Zhang gave an example for this in [130], showing that ℓ_1-minimization can recover the sparsest solution of $Ax = b$ whereas it cannot recover the sparsest solution of $WAx = Wb$, as the restricted isometry constant $\delta(WA)$ of WA violates the requirement $\delta(WA) < \sqrt{2} - 1$. Due to the limitation of RIP, non-RIP analysis brings a new perspective for the recovery of compressive sensing via ℓ_1-minimization. Among those analysis, null space property (NSP) and range space property (RSP) are the popular tools, which will be introduced in the following sections.

2.3.3 Null Space Property

In this section, the recoverability of ℓ_1-minimization will be considered by another property of matrix A, the Null Space Property (NSP). Firstly, we give a general definition of the Null Space Property on mixed norm spaces [43]. Recall that we denote the null space of a matrix A by $\mathcal{N}(A) := \{x \in R^n | Ax = 0\}$. For a given norm $\| \cdot \|_X$, the best K-term

approximation error is [43]

$$\sigma_K(x)_X := \min_{\widehat{x} \in \Sigma_K} \|x - \widehat{x}\|_X.$$

Definition 2.3.8 (Null Space Property (NSP)) :

For a matrix $A \in R^{m \times n}$, if there exists a constant $C > 0$ such that

$$\|\eta\|_X \leqslant CK^{-s}\|\eta_{\Lambda^c}\|_Y \quad for \ all \ \eta \in \mathcal{N}(A), \tag{2.3.8}$$

where $\|\cdot\|_X$ denotes the ℓ_p-norm and $\|\cdot\|_Y$ denotes the ℓ_q-norm, $1 \leqslant q \leqslant p \leqslant 2, s = \frac{1}{q} - \frac{1}{p}$, holding for all $\Lambda \subseteq \{1, \cdots, n\}$ with $|\Lambda| \leqslant K$ and Λ^c is the complement set of Λ in $\subseteq \{1, \cdots, n\}$, then we say that the matrix A satisfies the Null Space Property in (X, Y) of order K.

Remark 2.3.9 :

1. *The NSP condition (2.3.8) can be reformulated as*

$$\|\eta\|_X \leqslant CK^{-s}\sigma_K(\eta)_Y.$$

   As $\|\eta_{\Lambda^c}\|_Y = \|\eta - \eta_\Lambda\| \geqslant \sigma_K(\eta)_Y$ for any Λ in $\subseteq \{1, \cdots, n\}$ with $|\Lambda| \leqslant K$, any vector $\eta \in \mathcal{N}(A)$ can be bounded by its best K-term approximation error. If the vector η is a vector in $\sum_K$ and there exists an index set Λ with $|\Lambda| = K$, $\|\eta_{\Lambda^c}\|_Y = \sigma_K(\eta)_Y = 0$ for any ℓ_q-norm, which implies $\eta \equiv 0$.

2. *Let $\triangle : R^m \longrightarrow R^n$ be any recovery methods such that*

$$z = arg \ \triangle(Ax) \ and \ z \in R^n.$$

   $\triangle$ is called a decoder in the literature as it is expected to extract x by decoding from

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b = Ax. For different choices of p and q, the corresponding stability condition [43]

is

$$\|x - \triangle(Ax)\|_X \leqslant C_0 K^{-s} \sigma_K(\eta)_Y \text{ for any } x \in R^n,$$

where $C_0 > 0$ is a constant.

Throughout this dissertation, we only consider the recoverability of ℓ_1-minimization and the case when $X = \ell_2$ and $Y = \ell_1$. Thus, the definition of NSP yields to that there exists a constant $C > 0$ such that

$$\|\eta\|_2 \leqslant CK^{-1/2}\|\eta_{\Lambda^c}\|_1 \tag{2.3.9}$$

holds for all $\eta \in \mathcal{N}(A)$ and $\Lambda \subseteq \{1, \cdots, n\}$ with $|\Lambda| \leqslant K$.

Lemma 2.3.10 : *For any $h \in \Sigma_K$, we have the following inequalities between different norms,*

$$\frac{\|h\|_1}{\sqrt{K}} \leqslant \|h\|_2 \leqslant \sqrt{K}\|h\|_\infty.$$

The first inequality can be obtained by applying Cauchy-Schwarz inequality as $\|h\|_1 = |\langle \text{sign}(h), h \rangle|$. The second inequality is more straightforward.

By the Lemma 2.3.10, the definition of NSP (2.3.9) can be refined into the same norm, namely,

$$\|\eta\|_1 \leqslant C\|\eta_{\Lambda^c}\|_1. \tag{2.3.10}$$

For all other cases of mixed norm spaces and recovery methods, we refer readers to the paper by Cohen, Dahmen and DeVore [43]. To reveal the recovery condition via NSP,

we start with characterizing the uniqueness condition for ℓ_1-minimization, and then build uniform recovery conditions via ℓ_1-minimization.

Theorem 2.3.11 (Uniqueness condition for ℓ_1-minimization via NSP [130]) :

For a given linear system $Ax = b$ and any integer $K \geqslant 1$, $x^* = \arg\min\{\|x\|_1 | Ax = Ax^*\}$ holds for all $x^* \in R^n$ such that $\|x^*\|_0 \leqslant K$ if and only if matrix A satisfies the NSP of order K for some constant $0 < C < 1$ with all $\eta \in \mathcal{N}(A) \setminus \{0\}$.

Proof. Let $y \neq x^*$ be a solution of ℓ_1-minimization. Let $\eta = y - x^* \in \mathcal{N} \setminus \{0\}$ and Λ be the support set of x^* with $|\Lambda| = K$. Denote Λ^c as the complement set of Λ in $\{1, \cdots, n\}$, then we have the following relation

$$\|y\|_1 = \|\eta + x^*\|_1 = \|\eta_\Lambda + x^*_\Lambda\|_1 + \|\eta_{\Lambda^c}\|_1. \tag{2.3.11}$$

Due to the triangle inequality, we have

$$\|\eta_\Lambda + x^*_\Lambda\|_1 \geqslant \|x^*_\Lambda\|_1 - \|\eta_\Lambda\|_1,$$

thus, equation (2.3.11) becomes

$$\|y\|_1 = \|\eta + x^*\|_1 \geqslant \|x^*_\Lambda\|_1 - \|\eta_\Lambda\|_1 + \|\eta_{\Lambda^c}\|_1.$$

Hence, x^* is the unique solution of ℓ_1-minimization (namely, $\|y\|_1 > \|x^*\|_1$) if and only if $\|\eta_{\Lambda^c}\|_1 - \|\eta_\Lambda\|_1 > 0$ holds. Matrix A satisfies the NSP of order K, for the index set Λ and vector η defined above, by definition (2.3.10), then we have

$$\|\eta\|_1 \leqslant C\|\eta_{\Lambda^c}\|_1 \text{ for some constant } C > 0,$$

which implies

$$\|\eta_\Lambda\|_1 \leqslant \|\eta\|_1 \leqslant C\|\eta_{\Lambda^c}\|_1 < \|\eta_{\Lambda^c}\|_1.$$

Hence, matrix A satisfying the NSP of order K, by definition (2.3.10) for any $0 < C < 1$, x^* is the unique solution to ℓ_1-minimization. $\qquad\square$

Remark 2.3.12 : *It is worth mentioning that there is an alternative to define the null space property [68], which says that if $\|\eta_\Lambda\|_1 < \|\eta_{\Lambda^c}\|_1$ for all $\eta \in \mathcal{N}(A)\backslash\{0\}$ holds for any sets $\Lambda \subseteq \{1,\cdots,n\}$ with $|\Lambda| \leqslant K$, matrix A satisfies the NSP of order K. Based on this definition, it is more straightforward to see that NSP can fully characterize the uniqueness of ℓ_1-minimization.*

The following theorem will show the NSP of order K is also a sufficient condition for finding a unique K-sparse solution of ℓ_0-minimization.

Theorem 2.3.13 (Uniqueness via NSP [130]) :

For a given linear system $Ax = b$ and any integer $K \geqslant 1$, if matrix A satisfies the NSP of order K, then ℓ_0-minimization has a unique K-sparse solution if such a solution exists.

Proof. Let $x,y \in R^n$ be two distinct solutions of $Ax = b$ and Λ be the support set of x with $|\Lambda| = K$. Let $\eta = y - x \in \mathcal{N}(A) \setminus \{0\}$. By Theorem 2.3.11, A satisfies the NSP of order K, then x is the unique minimizer of ℓ_1-minimization with $\|x\|_1 < \|y\|_1$ and then by the proof of the same theorem $\|\eta_\Lambda\|_1 < \|\eta_{\Lambda^c}\|_1$. By the Lemma 2.3.10, if $\sqrt{\|x\|_0}\|\eta_\Lambda\|_2 < \|\eta_{\Lambda^c}\|_1$ holds, we have

$$\|\eta_\Lambda\|_1 \leqslant \sqrt{\|x\|_0}\|\eta_\Lambda\|_2 < \|\eta_{\Lambda^c}\|_1,$$

which implies $\sqrt{\|x\|_0} < \frac{\|\eta_{\Lambda^c}\|_1}{\|\eta_\Lambda\|_2}$ or $\sqrt{\|x\|_0} \leqslant C\frac{\|\eta_{\Lambda^c}\|_1}{\|\eta_\Lambda\|_2}$ for some constant $0 < C < 1$. Necessarily, we have $\sqrt{\|y\|_0} \geqslant \frac{\|\eta_{\Lambda^c}\|_1}{\|\eta_\Lambda\|_2}$. Otherwise, if $\|y\|_0 \leqslant \|x\|_0$, y is also a K-sparse

vector, and then it should be a minimizer of ℓ_1-minimization, which contradicts with the uniqueness of solutions of ℓ_1-minimization. Hence, $\|y\|_0 > \|x\|_0$, so x is a unique K-sparse solution of ℓ_0-minimization. $\qquad\square$

Combining Theorem 2.3.11 and Theorem 2.3.13, it indicates that NSP of order K ensures the uniform recovery via ℓ_1-minimization provided that ℓ_0-minimization has a unique sparsest solution. Furthermore, Gribonval and Nielson [73, 74] have shown a more precise statement on the NSP condition for the uniform recovery via ℓ_p-minimization with $0 \leqslant p \leqslant 1$.

2.3.4 Range Space Property

So far, all the recovery conditions stated in this chapter concern about the strong equivalence of ℓ_0- and ℓ_1-minimization, which cannot completely demonstrate the capability of ℓ_1-minimization in terms of finding a sparsest solution of a linear system. To deterministically characterize the solvability of ℓ_1-minimization, Zhao [132] introduced range space property (RSP) of matrix A^T to guarantee the equivalence of ℓ_0- and ℓ_1-minimization and further elaborated the uniform recovery from the perspective of RSP. From a mathematical point of view, due to $\mathcal{N}(A)^\perp = \mathcal{R}(A^T)$, as the NSP-based recovery theory can demonstrate the recoverability of ℓ_1-minimization, it is natural to consider the range space of A^T to construct a recovery theory.

Definition 2.3.14 (Range Space Property of $\boldsymbol{A^T}$ at $\boldsymbol{x}$) : *For a given matrix $A \in R^{m \times n}$, let $x \in R^n$ be a given vector. If there exists a vector $\eta \in \mathcal{R}(A^T)$ obeying that*

$$\begin{cases} \eta_i = 1 & \text{for all } x_i > 0, \\ \eta_i = -1 & \text{for all } x_i < 0, \\ |\eta_i| < 1 & \text{for all } x_i = 0, \end{cases} \tag{2.3.12}$$

then we say that matrix A^T satisfies the range space property (RSP) at x.

By this definition, RSP of A^T at x indicates that there exists a vector $\eta \in \mathcal{R}(A^T)$ such that it reflects sign values of nonzero components of x. It is easy to verify the existence of such a vector η by solving the following linear program [132],

$$
\begin{aligned}
\min \quad & \varrho \\
s.t. \quad & A_{S_+}^T \omega = e_{S_+}, \\
& A_{S_-}^T \omega = e_{S_-}, \\
& |A_{S_0}^T \omega| \leqslant \varrho e_{S_0},
\end{aligned}
$$

where $S_0 := \{i : x_i = 0\}$. The optimal value ϱ^* of the linear program above is strictly less than 1 if and only if there exists a vector $\eta \in \mathcal{R}(A)$ satisfying (2.3.12) [132].

As ℓ_1-minimization is a linear program, necessary and sufficient conditions for the uniqueness of ℓ_1-minimization can be derived from the strong duality theorem [132].

Theorem 2.3.15 (Uniqueness condition for ℓ_1-minimization via RSP [107, 132]) :

For a given matrix $A \in R^{m \times n}$, x is the unique least ℓ_1-norm solution to the linear system $Ax = b$ if and only if (i) the matrix $H = \begin{pmatrix} A_{S_+} & A_{S_-} \\ -e_{S_+} & e_{S_-} \end{pmatrix}$ has full column rank, and (ii) the RSP of A^T (2.3.12) holds at x.

Theorem 2.3.15 states that full-rank property (FRP) (condition (i)) and RSP of A^T can accurately capture the uniqueness of ℓ_1-minimization. In experiments, without the requirement of uniqueness of sparsest solutions of a linear system, ℓ_1-minimization can still solve ℓ_0-minimization in the sense that it finds a sparsest solution, which goes beyond the scope of ERC, RIP and NSP-based recovery theories. The RSP-based analysis provides a new perspective to explain such an extraordinary performance of ℓ_1-minimization.

Theorem 2.3.16 (Equivalence condition via RSP [132]) :

Let $x \in R^n$ be a sparsest solution to the system $Ax = b$. Then x is the unique ℓ_1-norm

solution to this system if and only if the range space property of A^T defined by (2.3.12) holds at x.

It is worth mentioning that the 'if' part of Theorem 2.3.16 is derived by Fuchs in [69]. To our knowledge, the RSP-based condition in Theorem 2.3.16 is the first equivalence condition directly derived from the strong duality theorem and strictly complementary theorem in linear programming [114] for finding a sparsest solution of ℓ_0-minimization. Moreover, the RSP-based analysis can be adapted to different linear systems, for instance, a system of linear equations with nonnegative variables [133] and a linear system with mixed inequality and equality constraints [137]. Particularly, for the linear system with mixed inequality and equality constraints in [137], it has multiple sparsest solutions. Thus, it is hard to derive the null space property or the restricted isometry property based conditions for such a linear system. However, it is still possible to construct RSP-based recovery theories for finding a sparsest solution. In Chapter 4, we will give more details about how the RSP-based analysis can be used to develop recovery conditions for a certain level of nonuniform and uniform recoveries for 1-bit compressive sensing. We now introduce the RSP-based uniform recovery condition defined in [132].

Definition 2.3.17 (RSP of order K) *For a given matrix $A \in R^{m \times n}$ with $m < n$, the matrix A^T is said to satisfy the range space property of order K if for any disjoint subsets S_1, S_2 of $\{1, \cdots, n\}$ with $|S_1| + |S_2| \leqslant K$, the range space $\mathcal{R}(A^T)$ contains a vector η such that $\eta_i = 1$ for all $i \in S_1$, $\eta_i = -1$ for all $i \in S_2$ and $|\eta_i| < 1$ for all other components.*

The next theorem states that the uniform recovery via ℓ_1-minimization can be completely characterized by the RSP of order K.

Theorem 2.3.18 (Uniform recovery via RSP of order K [132]) *:*
Let $A \in R^{m \times n}$ be a given matrix with $m < n$, if A^T has the RSP of order K, then any

$x \in R^n$ with $\|x\|_0 \leqslant K$ is both the unique least ℓ_1-norm solution and the unique sparsest solution to the system $Ax = b = A\overline{x}$.

Compared to the nonuniform recovery condition given in Theorem 2.3.16, the uniform recovery ensures the uniqueness of any K-sparse solution since the RSP of order K is independent of any individual solution. Even though the RSP of order K is a strong condition, fortunately, it is possible to find some matrices satisfying the RSP of certain order. Zhao has proven in Lemma 4.4 [132] that any sufficient conditions to guarantee the uniform recovery via ℓ_1-minimization must imply the RSP of certain order. Hence, any matrices has RIP, NSP or mutual coherence properties, then their transposed matrices must have the RSP property.

Chapter 3

Uniqueness Conditions for Partial ℓ_0-minimization

3.1 Introduction

In compressive sensing, one might be interested in recovering a solution to an underdetermined linear system, for which only a part of the solution is sparse [1, 7, 80, 125]. In other words, it may be known in advance that the solution to an underdetermined linear system consists of two parts, where one is sparse and the other is possibly dense. To locate such a partially sparse solution, in this chapter[1], we consider the following model for the partially sparse representation of the measurements $b \in R^m$:

$$\min \left\{ \|x\|_0 : \ M \begin{pmatrix} x \\ y \end{pmatrix} = b, \ y \in C \right\}, \tag{3.1.1}$$

where $M = [A_1, A_2] \in R^{m \times (n_1 + n_2)}$ with $m < n_1$ is a concatenation of $A_1 \in R^{m \times n_1}$ and $A_2 \in R^{m \times n_2}$, and C is a convex set in R^{n_2} which can be interpreted as certain constraints on the variable $y \in R^{n_2}$. Throughout the chapter, we assume that A_1 has full-row-rank.

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[1]Part of the work in this chapter was carried out with Yun-Bin Zhao [137].

The solution to the system

$$M \begin{pmatrix} x \\ y \end{pmatrix} = b, \ y \in C, \tag{3.1.2}$$

includes two parts: $x \in R^{n_1}$ and $y \in R^{n_2}$. The ℓ_0-minimization problem (3.1.1) is to seek a solution $z = (x, y)$ to the system (3.1.2) such that the x-part is the sparsest one, but there is no requirement on the sparsity of the y-part of the solution. Such a sparsest solution x can be called the *sparsest x-part solution* to the system (3.1.2). The ℓ_0-minimization problem (3.1.1) can be called a partial ℓ_0-minimization problem, or partial sparsity-seeking problem. Clearly, when $A_2 = 0$ and matrix $A_1 \in R^{m \times n_1}$ has a full row rank, the problem (3.1.1) is reduced to the standard ℓ_0-minimization

$$\min\{\|x\|_0 : A_1 x = b\}. \tag{3.1.3}$$

Thus, the problem (3.1.1) is NP-hard (see Natarajan [99]) since the standard ℓ_0-minimization (3.1.3) is a special case of it.

To convexify the problem (3.1.1), we may replace $\|x\|_0$ by $\|x\|_1$, and then the partially sparsest x-part solution to the system (3.1.2) may be found by the following convex optimization

$$\min \left\{ \|x\|_1 : \ M \begin{pmatrix} x \\ y \end{pmatrix} = b, \ y \in C \right\}, \tag{3.1.4}$$

which is referred as the partially sparse recovery model [7, 8]. Many practical problems can be formulated as a partially sparse recovery problem, such as the image reconstruction in [125, 2] and the sparse Hessian recovery in [7]. Analogues to NSP and RIP conditions for uniform recovery in compressive sensing, partial null space property (partial NSP) and partial restricted isometry property (partial RIP) [7, 8] are developed for partial x-part

uniform recovery, namely, recovering any solution (x, y) with $\|x\|_0 \leqslant K$ by a single sensing matrix.

Moreover, a nonconvex approach is also proposed for the partially sparse reconstruction by replacing the ℓ_0-norm objective with the ℓ_p quasi-norm with $0 < p \leqslant 1$ in (3.1.1), which is named as partially ℓ_p-norm minimization [14] given as

$$\min \left\{ \|x\|_p^p : \ M \begin{pmatrix} x \\ y \end{pmatrix} = b, \ y \in C \right\}. \tag{3.1.5}$$

Analogues of the partial RIP condition, partially p-RIC (restricted isometry constant) condition is developed to recover any partially sparse solution (x, y) with $\|x\|_0 \leqslant K$ via the partially ℓ_p-norm minimization [14]. Note that partial NSP, partial RIP and partially p-RIC conditions are based on the assumption that matrix A_2 is full column rank. However, for a more general case that A_2 is not full column rank, it remains a question that under partial RIP and partial NSP conditions whether the unique solution of partially sparse recovery with the least ℓ_1-norm x-part is also necessarily the sparsest x-part solution of partial ℓ_0-minimization; and whether other conditions can be established for the partial x-part uniform recovery via partially sparse recovery or partially ℓ_p-norm minimization. In the next chapter, we will further study a special model of partial ℓ_0-minimization, namely, 1-bit ℓ_0-minimization, and develop criteria for a certain level of recovery of 1-bit ℓ_0-minimization via its associated ℓ_1-norm minimization problem.

So far, the uniqueness of the sparsest x-part solution has not been well developed for partial ℓ_0-minimization. The main contribution of this chapter is to study such the uniqueness and to establish some criteria under which the partial ℓ_0-minimization has a unique sparsest x-part solution. These results will be established through some new concepts such as the l_p-induced quasi-norm, the (maximal) scaled spark, coherence, and coherence rank associated with a pair of matrices $(A_1 \in R^{m \times n_1}, A_2 \in R^{m \times n_2})$. These

concepts can be seen as a generalization of concepts of coherence rank, submutual coherence and scaled mutual coherence in [131]. It is worth stressing that the uniqueness criteria based on the scaled spark and coherence can be reduced to the spark and mutual coherence uniqueness conditions for the standard ℓ_0-minimization.

This chapter is organized as follows. In Section 3.2, we develop sufficient conditions for the uniqueness of x-part solution to partial ℓ_0-minimization in terms of l_p-induced quasi-norm and such concepts as maximal scaled spark, and minimal or maximal scaled mutual coherence. A further improvement of these conditions is provided in Section 3.3.

3.2 Uniqueness criteria for the partial ℓ_0-minimization

The uniqueness of the sparsest x-part solution to the system (3.1.2) can be developed through different concepts and properties of matrices. One of such important concepts is *spark* together with its variants, which provides a connection between the null space of a matrix and the sparsest solution to linear equations. In this section, we borrow the method used for developing uniqueness conditions for the standard ℓ_0-minimization in Chapter 2 to establish similar uniqueness claims to the system (3.1.2), while the extra variable y in the system $M \begin{pmatrix} x \\ y \end{pmatrix} = b$ increases the complexity of the partial ℓ_0-minimization (3.1.1). Our first sufficient uniqueness condition for the sparsest x-part solution to (3.1.1) can be developed by using the so-called l_p-induced quasi-norm, as shown in the following subsection.

3.2.1 An l_p-induced quasi-norm-based uniqueness condition

For any $0 < p < \infty$ and a vector $x \in R^n$, let $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$. When $p \in (0,1)$, $\|x\|_p$ is called the l_p quasi-norm of x. We now introduce the l_p-induced quasi-norm of a matrix.

Definition 3.2.1 (ℓ_p-induced quasi-norm) *: For any given matrix $A \in R^{m \times n}$, when*

$0 < p < 1$, *the l_p-induced quasi-norm of A, denoted by $\psi_p(A)$, is defined by*

$$\psi_p(A) = \sup_{z \in R^n \setminus \{0\}} \frac{\|Az\|_p^p}{\|z\|_p^p} = \sup_{\substack{\|z\|_p^p \leqslant 1 \\ z \in R^n \setminus \{0\}}} \|Az\|_p^p = \sup_{\|z\|_p^p = 1} \|Az\|_p^p = \|A\|_p^p. \qquad (3.2.1)$$

Clearly, for a fixed $p \in (0,1)$, $\psi_p(A)$ satisfies the following properties: $\psi_p(A) \geqslant 0$, $\psi_p(A) > 0$ for any $A \neq 0$, and $\psi_p(A+B) \leqslant \psi_p(A) + \psi_p(B)$ for any matrices A, B with same dimensions. It is worth mentioning that the triangle inequality above follows from the property: $\|x+y\|_p^p \leqslant \|x\|_p^p + \|y\|_p^p$ (see, e.g., [67]). We see that for any $\alpha > 0$, $\psi_p(\alpha A) \neq \alpha \psi_p(A)$ in general, but $\psi_p(A)$ is referred as a quasi-norm of A. Note that, for every entry z_i, as p tends to zero, $|z_i|^p$ approaches to 1 for $z_i \neq 0$ and 0 for $z_i = 0$. Thus for any given $z \in R^n$, we have

$$\lim_{p \to 0^+} \|z\|_p^p = \lim_{p \to 0^+} \sum_{i=1}^{n} |z_i|^p = \|z\|_0, \qquad (3.2.2)$$

which indicates that the 'ℓ_0-norm' $\|z\|_0$ can be approximated by $\|z\|_p^p$ with sufficiently small $p \in (0,1)$. Note that for a given matrix A, $\psi_p(A)$ is continuous with respect to $p \in (0,1)$. Thus there might exist a positive number η such that $\eta = \lim_{p \to 0^+} \psi_p(A)$. We assume that the following property holds for the matrix $M = [A_1, A_2]$ when p tends to 0.

Assumption 3.2.2 *Assume that matrices A_1, A_2 satisfy the following properties: (i) $A_2^T A_2$ is a nonsingular matrix, and (ii) there exists a positive constant, denoted by $\psi_0(A_2^\dagger A_1)$, such that*

$$\psi_0(A_2^\dagger A_1) = \lim_{p \to 0^+} \psi_p(A_2^\dagger A_1),$$

where $A_2^\dagger = (A_2^T A_2)^{-1} A_2^T$ is the pseudo-inverse of A_2.

Under Assumption 3.2.2 and by (3.2.1) and (3.2.2), we immediately have the following inequality:

$$\|(A_2^\dagger A_1)z\|_0 = \lim_{p \to 0^+} \|(A_2^\dagger A_1)z\|_p^p \leqslant \lim_{p \to 0^+} \psi_p(A_2^\dagger A_1)\|z\|_p^p = \psi_0(A_2^\dagger A_1)\|z\|_0 \qquad (3.2.3)$$

for any $z \in R^n \setminus \{0\}$. We now state a uniqueness condition for problem (3.1.1) under Assumption 3.2.2.

Theorem 3.2.3 :

Consider the system (3.1.2) with $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$ and $m < n_1$. Let Assumption 3.2.2 be satisfied. Then if there exists a solution (x, y) to the system (3.1.2) satisfying that

$$\|x\|_0 < \frac{1}{2} \frac{Spark(M)}{(1 + \psi_0(A_2^\dagger A_1))}, \qquad (3.2.4)$$

x must be the unique sparsest x-part solution to the system (3.1.2).

Proof. Assume the contrary that there is another solution $(x^{(1)}, y^{(1)})$ to the system (3.1.2) such that $x^{(1)}$ is the sparsest x-part and $x^{(1)} \neq x$ and $\|x^{(1)}\|_0 \leqslant \|x\|_0 < \frac{1}{2} \frac{Spark(M)}{(1+\psi_0(A_2^\dagger A_1))}$. Since both (x, y) and $(x^{(1)}, y^{(1)})$ are solutions to the linear system $M \begin{pmatrix} x \\ y \end{pmatrix} = b$, we have

$$A_1(x - x^{(1)}) + A_2(y - y^{(1)}) = 0. \qquad (3.2.5)$$

Since $A_2^T A_2$ is nonsingular, $y - y^{(1)}$ can be uniquely determined by $x - x^{(1)}$, i.e.,

$$y^{(1)} - y = A_2^\dagger A_1(x - x^{(1)}), \qquad (3.2.6)$$

where $A_2^\dagger$ is the pseudo-inverse of A_2 given by $A_2^\dagger = (A_2^T A_2)^{-1} A_2^T$. From (3.2.5), we know

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that $\begin{pmatrix} x - x^{(1)} \\ y - y^{(1)} \end{pmatrix}$ is in the null space of the matrix $M = [A_1, A_2]$. This implies that the

Spark(M) is a lower bound for $\left\| \begin{pmatrix} x - x^{(1)} \\ y - y^{(1)} \end{pmatrix} \right\|_0$, i.e.,

$$\|x - x^{(1)}\|_0 + \|y - y^{(1)}\|_0 = \left\| \begin{pmatrix} x - x^{(1)} \\ y - y^{(1)} \end{pmatrix} \right\|_0 \geqslant \text{Spark}(M). \qquad (3.2.7)$$

Substituting (3.2.6) into (3.2.7) leads to

$$\|x - x^{(1)}\|_0 + \|A_2^\dagger A_1 (x - x^{(1)})\|_0 \geqslant \text{Spark}(M). \qquad (3.2.8)$$

Under Assumption 3.2.2 and by the inequality (3.2.3), one has

$$\|A_2^\dagger A_1 (x^{(1)} - x)\|_0 \leqslant \psi_0(A_2^\dagger A_1) \cdot \|x - x^{(1)}\|_0.$$

Merging (3.2.8) and the inequality above leads to

$$(1 + \psi_0(A_2^\dagger A_1))\|x - x^{(1)}\|_0 \geqslant \text{Spark}(M).$$

Therefore,
$$2\|x\|_0 \geqslant \|x^{(1)}\|_0 + \|x\|_0 \geqslant \|x - x^{(1)}\|_0 \geqslant \frac{\text{Spark}(M)}{1 + \psi_0(A_2^\dagger A_1)}.$$

Thus $\|x\|_0 \geqslant \frac{1}{2} \frac{\text{Spark}(M)}{(1+\psi_0(A_2^\dagger A_1))}$, contradicting with (3.2.4). Therefore x must be the unique sparsest x-part solution to system (3.1.2). $\qquad \square$

Remark 3.2.4 : *The spark condition for partial ℓ_0-minimization in Theorem 3.2.3 is stronger than the spark condition for ℓ_0-minimization (2.2.1).*

Considering the linear system $M\binom{x}{y} = b$, the spark condition of the whole system is

$$\|\binom{x}{y}\|_0 < \frac{1}{2}Spark(M),$$

which is equivalent to

$$\|x\|_0 + \|y\|_0 < \frac{1}{2}Spark(M).$$

Based on Assumption 3.2.2, as $\psi_0(A_2^\dagger A_1) + 1 > 1$, we have

$$\|x\|_0 < \frac{1}{2}\frac{Spark(M)}{(1 + \psi_0(A_2^\dagger A_1))} < \frac{1}{2}Spark(M).$$

Therefore, the sparsity of the sparsest x-part solution to the linear system $M\binom{x}{y} = b$ is bounded by the spark condition for ℓ_0-minimization.

The above result provides a new uniqueness criteria for the partial ℓ_0-minimization by using l_p-induced quasi-norm. However, the above analysis relies on the nonsingularity of $A_2^T A_2$ which might not be satisfied in more general situations. Thus we develop more general uniqueness criteria for the partial ℓ_0-minimization from other perspectives.

3.2.2 Uniqueness based on scaled spark and scaled mutual coherence

In this section, we discuss the case that the null space of A_2^T is nonzero, namely, $\mathcal{N}(A_2^T) \neq \{0\}$. We develop uniqueness conditions for problem (3.1.1) by using the so-called scaled spark and scaled mutual coherence.

Lemma 3.2.5 ([28]) : *For any matrix M and any scaling matrix W, one has*

$$Spark(WM) \geqslant 1 + \frac{1}{\mu(WM)}.$$

Recall that $\mathcal{N}(A)$ denotes the null space of matrix A. Our first uniqueness criterion based on the scaled spark is given as follows.

Theorem 3.2.6 :

Consider the system (3.1.2) where $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$ and $m < n_1$. If there exists a solution (x, y) to the system (3.1.2) satisfying

$$\|x\|_0 < \frac{1}{2} Spark(B^T A_1), \tag{3.2.9}$$

where B is a basis of $\mathcal{N}(A_2^T)$, then x is the unique sparsest x-part solution to the system (3.1.2).

Proof. Assume the contrary that $(x^{(1)}, y^{(1)}) \neq (x, y)$ is a solution to the system (3.1.2) satisfying that $x^{(1)} \neq x$ and $\|x^{(1)}\|_0 \leqslant \|x\|_0 < \frac{1}{2} \text{Spark}(B^T A_1)$, where B is a basis of $\mathcal{N}(A_2^T)$. Note that $\begin{pmatrix} x - x^{(1)} \\ y - y^{(1)} \end{pmatrix}$ is in the null space of $M = [A_1, A_2]$, so

$$A_1(x - x^{(1)}) = -A_2(y - y^{(1)}). \tag{3.2.10}$$

Note that the range space of A_2 is orthogonal to the null space of A_2^T, namely, $\mathcal{R}(A_2) = \mathcal{N}(A_2^T)^\perp$. Let B be an arbitrary basis of $\mathcal{N}(A_2^T)$. Since the right-hand side of (3.2.10) is in $\mathcal{R}(A_2)$, multiplying both sides of the equation (3.2.10) by B^T, we get

$$B^T A_1(x - x^{(1)}) = 0,$$

which implies that

$$\|x - x^{(1)}\|_0 \geqslant \text{Spark}(B^T A_1). \tag{3.2.11}$$

Therefore,

$$2\|x\|_0 \geqslant \|x^{(1)}\|_0 + \|x\|_0 \geqslant \mathrm{Spark}(B^T A_1).$$

i.e., $\|x\|_0 \geqslant \frac{1}{2}\mathrm{Spark}(B^T A_1)$, leading to a contradiction. Therefore, the system (3.1.2) has a unique sparsest x-part solution. $\qquad\square$

Let F be a set of all bases of $\mathcal{N}(A_2^T)$, namely,

$$F = \{B \in R^{m \times q} : \ B \text{ is a basis of } \ \mathcal{N}(A_2^T)\},$$

where q is the dimension of $\mathcal{N}(A_2^T)$. As assumed before $\mathcal{N}(A_2^T) \neq \{0\}$, the set F is nonempty.

From the definition of the spark, we know that $\mathrm{Spark}(B^T A_1)$ is bounded. Hence, there exists the supremum of $\mathrm{Spark}(B^T A_1)$ over the set F, defined as follows.

Definition 3.2.7 : *For any matrix $A_1 \in R^{m \times n_1}$ with $m < n_1$, let*

$$Spark^*_{A_2}(A_1) = \sup_{B \in F} Spark(B^T A_1). \qquad\qquad (3.2.12)$$

$Spark^*_{A_2}(A_1)$ *is called the maximal scaled spark of A_1 over F (the set of bases of $\mathcal{N}(A_2^T)$).*

The inequality (3.2.11) in the proof of Theorem 3.2.6 holds for all bases B of $\mathcal{N}(A_2^T)$. Therefore, the spark condition (3.2.9) can be further enhanced by using $\mathrm{Spark}^*_{A_2}(A_1)$.

Theorem 3.2.8 :

Consider the system (3.1.2) where $A_1 \in R^{m \times n_1}$ and $A_2 \in R^{m \times n_2}$ and $m < n_1$. If there exists a solution (x, y) to the system (3.1.2) satisfying

$$\|x\|_0 < \frac{1}{2}Spark^*_{A_2}(A_1), \qquad\qquad (3.2.13)$$

where $Spark^*_{A_2}(A_1)$ is given by (3.2.12), then x is the unique sparsest x-part solution to the system (3.1.2).

From Lemma 3.2.5, the scaled mutual coherence may provide a lower bound for the scaled spark. An immediate consequence of Theorem 3.2.6 is the corollary below.

Corollary 3.2.9 :

Consider the system (3.1.2) where $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$ and $m < n_1$. If there exists a solution (x, y) to the system (3.1.2) satisfying

$$\|x\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu(B^T A_1)}\right), \tag{3.2.14}$$

where B is a basis of $\mathcal{N}(A_2^T)$, then x is the unique sparsest x-part solution to the system (3.1.2).

Note that Corollary 3.2.9 holds for any basis B of $\mathcal{N}(A_2^T)$. So it makes sense to further enhance the bound (3.2.14) by introducing the following definition.

Definition 3.2.10 : For any matrix $A_1 \in R^{m \times n_1}$ $(m < n_1)$ and $A_2 \in R^{m \times n_2}$, let

$$\mu^*_{A_2}(A_1) = \inf_{B \in F} \mu(B^T A_1), \quad \mu^{**}_{A_2}(A_1) = \sup_{B \in F} \mu(B^T A_1). \tag{3.2.15}$$

$\mu^*_{A_2}(A_1)$ is called the minimal scaled coherence of A_1 over F, and $\mu^{**}_{A_2}(A_1)$ is called the maximal scaled coherence of A_1 over F.

Based on Lemma 3.2.5 and the above definition, we have the following result.

Lemma 3.2.11 : For any basis B of $\mathcal{N}(A_2^T)$, we have

$$1 + \frac{1}{\mu(B^T A_1)} \leqslant 1 + \frac{1}{\mu^*_{A_2}(A_1)} \leqslant Spark^*_{A_2}(A_1), \tag{3.2.16}$$

Proof. The first inequality holds by the definition of $\mu^*_{A_2}(A_1)$. From Lemma 3.2.5, we have

$$1 + \frac{1}{\mu(B^T A_1)} \leqslant \mathrm{Spark}(B^T A_1) \quad \text{for every basis } B \text{ of } \mathcal{N}(A_2^T).$$

By (3.2.12), we see that $\mathrm{Spark}(B^T A_1) \leqslant \mathrm{Spark}^*_{A_2}(A_1)$, thus

$$1 + \frac{1}{\mu(B^T A_1)} \leqslant \mathrm{Spark}^*_{A_2}(A_1) \quad \text{for all } B \in F. \qquad \square$$

Since the right-hand side of the above inequality is fixed, which is an upper bound for the left-hand side for any $B \in F$, we conclude that

$$\mathrm{Spark}^*_{A_2}(A_1) \geqslant \sup_{B \in F} \left\{ 1 + \frac{1}{\mu(B^T A_1)} \right\} = 1 + \frac{1}{\inf_{B \in F}\{\mu(B^T A_1)\}} = 1 + \frac{1}{\mu^*_{A_2}(A_1)}.$$

By Theorem 3.2.8 and Lemma 3.2.11, we have the next enhanced uniqueness claim of (3.2.14).

Theorem 3.2.12 *:*

Consider the system (3.1.2) with $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$ and $m < n_1$. If there exists a solution $(x, y)^T$ satisfying

$$\|x\|_0 < \frac{1}{2} \left(1 + \frac{1}{\mu^*_{A_2}(A_1)} \right), \tag{3.2.17}$$

where $\mu^*_{A_2}(A_1)$ is the minimal scaled coherence of A_1 over F, then x is the unique sparsest x-part solution to the system (3.1.2).

Remark 3.2.13 *: The uniqueness criteria established in this section can be seen as certain generalization of that of sparsest solutions to systems of linear equations. For instance, when $A_2 = 0$, the null space of A_2^T is the whole space R^m. Hence, by letting*

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$B = I$, the corresponding scaled mutual coherence and scaled spark become

$$\mu(B^T A_1) = \mu(A_1) \quad and \quad Spark(B^T A_1) = Spark(A_1).$$

Then results in this section are reduced to the existing ones in [28, 45, 52]. It is worth noting that the spark type uniqueness conditions are derived from the property of null spaces. It is worth mentioning that the null space based analysis is not the unique way to derive uniqueness criteria for sparsest solutions. Some other approaches such as the so-called range space property (see, e.g., [131, 132, 133]) and orthogonal projection from $R^{n_1+n_2}$ to $\mathcal{N}(A_2^T)$ [8] can be also used to develop uniqueness criteria.

3.3 Further Improvement of some uniqueness conditions

Since spark conditions are difficult to verify, mutual coherence conditions play an important role in the uniqueness theory for the partial ℓ_0-minimization problem (3.1.1). As shown in Lemma 3.2.11, $1 + \frac{1}{\mu^*_{A_2}(A_1)}$ is a good lower bound for $\text{Spark}^*_{A_2}(A_1)$ which is an improved version of the bound (3.2.14). In this section, we aim to further enhance the uniqueness claim (3.2.17) by further improving the lower bound of $\text{Spark}^*_{A_2}(A_1)$ under some situations. Following the discussions in [131], we introduce the so-called scaled coherence rank, scaled sub-coherence and scaled sub-coherence rank to achieve certain improvement on uniqueness conditions developed in Section 3.2.

3.3.1   Maximal scaled (sub) coherence and rank

Let us first recall several concepts which were introduced by Zhao [131]. For a given matrix $A \in R^{m \times n}$ with columns $a_i, i = 1, ..., n$, consider the index set

$$S_i(A) := \left\{ j : \ j \neq i, \ \frac{|a_i^T a_j|}{\|a_i\|_2 \cdot \|a_j\|_2} = \mu(A) \right\}, \ i = 1, ..., n.$$

Let $\alpha_i(A)$ be the cardinality of $S_i(A)$, and $\alpha(A)$ be the largest one among $\alpha_i(A)$'s, i.e.,

$$\alpha(A) = \max_{1 \leqslant i \leqslant n} \alpha_i(A) = \max_{1 \leqslant i \leqslant n} |S_i(A)|.$$

$\alpha(A)$ is called the coherence rank of A.

Let i_0 be an index such that $\alpha(A) = \alpha_{i_0}(A) = |S_{i_0}(A)|$. Define

$$\beta(A) = \max_{1 \leqslant i \leqslant n, \ i \neq i_0} \alpha_i(A) = \max_{1 \leqslant i \leqslant n, \ i \neq i_0} |S_i(A)|,$$

which is called the sub-coherence rank of A.

Also we define by

$$\mu^{(2)}(A) = \max_{i \neq j} \left\{ \frac{|a_i^T a_j|}{\|a_i\|_2 \cdot \|a_j\|_2} : \ \frac{|a_i^T a_j|}{\|a_i\|_2 \cdot \|a_j\|_2} < \mu(A) \right\},$$

the second largest absolute value of the inner product between two normalized columns of A. $\mu^{(2)}(A)$ is called the sub-mutual coherence of A.

Consider the sub-mutual coherence $\mu^{(2)}(B^T A_1)$ with a scaling matrix $B \in F$. We introduce the following new concepts.

Definition 3.3.1 : *Let $A_1 \in R^{m \times n_1}$ ($m < n_1$) and $A_2 \in R^{m \times n_2}$ be two matrices, and F is the nonempty set of bases of $\mathcal{N}(A_2^T)$.*

(i) The maximal scaled sub-mutual coherence of A_1 on F, denoted by $\mu_{A_2}^{**(2)}(A_1)$, is

defined as

$$\mu_{A_2}^{**(2)}(A_1) = \sup_{B \in F} \mu^{(2)}(B^T A_1). \tag{3.3.1}$$

(ii) The maximal scaled coherence rank of A_1 on F, denoted by $\alpha_{A_2}^*(A_1)$, is defined as

$$\alpha_{A_2}^*(A_1) = \sup_{B \in F} \{\alpha(B^T A_1)\}. \tag{3.3.2}$$

(iii) The maximal scaled sub-coherence rank of A_1 on F, denoted by $\beta_{A_2}^*(A_1)$, is defined as

$$\beta_{A_2}^*(A_1) = \sup_{B \in F} \{\beta(B^T A_1)\}. \tag{3.3.3}$$

It is easy to see the following relationship between $\alpha(B^T A_1)$, $\beta(B^T A_1)$, $\alpha_{A_2}^*(A_1)$ and $\beta_{A_2}^*(A_1)$: for every basis B of $\mathcal{N}(A_2^T)$, we have

$$1 \leqslant \beta(B^T A_1) \leqslant \alpha(B^T A_1) \leqslant \alpha_{A_2}^*(A_1) \text{ and } 1 \leqslant \beta(B^T A_1) \leqslant \beta_{A_2}^*(A_1) \leqslant \alpha_{A_2}^*(A_1). \tag{3.3.4}$$

3.3.2 Improved lower bounds of $Spark_{A_2}^*(A_1)$

Following the method used to improve the lower bound of Spark(A) in [131], we can find an enhanced lower bound of $\text{Spark}_{A_2}^*(A_1)$ via the concepts introduced in Section 3.3.1 based on the following two lemmas.

Lemma 3.3.2 (Brauer theorem [27]) : *For any matrix $A \in R^{n \times n}$ with $n \geqslant 2$, if λ is an eigenvalue of A, there is a pair (i, j) of positive integers with $i \neq j$ ($1 \leqslant i, j \leqslant n$) such that*

$$|\lambda - a_{ii}| \cdot |\lambda - a_{jj}| \leqslant \Delta_i \Delta_j,$$

where $\Delta_i := \sum_{j=1, j \neq i}^n |a_{ij}|$ for $1 \leqslant i \leqslant n$.

Merging Theorem 2.5 and Proposition 2.6 in [131] yields the following result.

Lemma 3.3.3 ([131]) : *Let $A \in R^{m \times n}$, and let $\alpha(A)$ and $\beta(A)$ be the coherence rank and subcoherence rank of A, respectively. Suppose that one of the following conditions holds: (i) $\alpha(A) < \frac{1}{\mu(A)}$; (ii) $\alpha(A) \leqslant \frac{1}{\mu(A)}$ and $\beta(A) < \alpha(A)$. Then $\mu^{(2)}(A) > 0$ and*

$$
\begin{aligned}
Spark(A) &\geqslant 1 + \frac{2[1 - \alpha(A)\beta(A)\overline{\mu}(A)^2]}{\mu^{(2)}(A)\{\overline{\mu}(A)(\alpha(A) + \beta(A)) + \sqrt{\overline{\mu}(A)^2(\alpha(A) - \beta(A))^2 + 4}\}} \\
&> 1 + \frac{1}{\mu(A)},
\end{aligned}
$$

where $\overline{\mu}(A) = \mu(A) - \mu^{(2)}(A)$ and $\mu^{(2)}(A)$ is the subcoherence of A.

Based on Lemma 3.3.3, we can construct an enhanced lower bound of $Spark^*_{A_2}(A_1)$ under some conditions, in terms of the scaled coherence rank and scaled sub-coherence rank.

Theorem 3.3.4 :

Consider the system (3.1.2) where $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$ and $m < n_1$. Suppose that one of the following conditions holds: (i) $\alpha(B^T A_1) < \frac{1}{\mu(B^T A_1)}$ for all $B \in F$; (ii) $\alpha(B^T A_1) \leqslant \frac{1}{\mu(B^T A_1)}$ and $\beta(B^T A_1) < \alpha(B^T A_1)$ for all $B \in F$. Then for any $B \in F$, we have that $\mu^{(2)}(B^T A_1) > 0$ and

$$
\begin{aligned}
Spark^*_{A_2}(A_1) &\geqslant \sup_{B \in F} \left\{ 1 + \frac{2[1 - \alpha(B^T A_1)\beta(B^T A_1)\overline{\mu}(B^T A_1)^2]}{\mu^{(2)}(B^T A_1)\{\overline{\mu}(B^T A_1)(\alpha(B^T A_1) + \beta(B^T A_1)) + \sqrt{\Delta}\}} \right\} \\
&\geqslant 1 + \frac{1}{\mu^*_{A_2}(A_1)},
\end{aligned}
$$

where $\overline{\mu}(B^T A_1) = \mu(B^T A_1) - \mu^{(2)}(B^T A_1)$ and $\Delta = [\overline{\mu}(B^T A_1)]^2(\alpha(B^T A_1) - \beta(B^T A_1))^2 + 4$.

Proof. Under conditions (i) and (ii), by Lemma 3.3.3, for any $B \in F$ we have that

$\mu^{(2)}(B^T A_1) > 0$ and

$$\begin{aligned}
\text{Spark}(B^T A_1) \quad &\geqslant \quad \varphi(B^T A_1) \\
&=: \quad 1 + \frac{2[1 - \alpha(B^T A_1)\beta(B^T A_1)\overline{\mu}(B^T A_1)^2]}{\mu^{(2)}(B^T A_1)\{\overline{\mu}(B^T A_1)(\alpha(B^T A_1) + \beta(B^T A_1)) + \sqrt{\Delta}\}} \quad (3.3.5)
\end{aligned}$$

where $\overline{\mu}(B^T A_1) = \mu(B^T A_1) - \mu^{(2)}(B^T A_1)$ and $\Delta = [\overline{\mu}(B^T A_1)]^2(\alpha(B^T A_1) - \beta(B^T A_1))^2 + 4$.

The above inequality holds for any basis $B \in F$. By the definition of $\text{Spark}_{A_2}^*(A_1)$, we have

$$\text{Spark}_{A_2}^*(A_1) \geqslant \text{Spark}(B^T A_1) \text{ for any } B \in F.$$

Thus it follows from (3.3.5) that

$$\text{Spark}_{A_2}^*(A_1) \geqslant \varphi(B^T A_1) \quad \text{for all } B \in F. \tag{3.3.6}$$

Inequality (3.3.6) implies that the value of $\varphi(B^T A_1)$ is bounded by the constant $\text{Spark}_{A_2}^*(A_1)$. Hence, the supremum of $\varphi(B^T A_1)$ over F should be bounded by $\text{Spark}_{A_2}^*(A_1)$, namely,

$$\text{Spark}_{A_2}^*(A_1) \geqslant \sup_{B \in F} \varphi(B^T A_1).$$

By Lemma 3.3.3 again, under conditions (i) and (ii), we see that $\varphi(B^T A_1) > 1 + \frac{1}{\mu(B^T A_1)}$. Therefore, the superimum of $\varphi(B^T A_1)$ should be greater than the value of $1 + \frac{1}{\mu(B^T A_1)}$ for any basis $B \in F$, i.e.,

$$\sup_{B \in F} \varphi(B^T A_1) > 1 + \frac{1}{\mu(B^T A_1)} \text{ for any } B \in F.$$

This in turn implies that

$$\sup_{B \in F}\{\varphi(B^T A_1)\} \geqslant \sup_{B \in F}\left\{1 + \frac{1}{\mu(B^T A_1)}\right\} = 1 + \frac{1}{\mu^*_{A_2}(A_1)},$$

where the last equality follows from the definition of $\mu^*_{A_2}(A_1)$. Therefore, under conditions (i) and (ii), we conclude that

$$\text{Spark}^*_{A_2}(A_1) \geqslant \sup_{B \in F}\{\varphi(B^T A_1)\} \geqslant 1 + \frac{1}{\mu^*_{A_2}(A_1)},$$

as claimed. $\hspace{1cm} \square$

Conditions (i) and (ii) in Theorem 3.3.4 rely on $B \in F$. A similar condition without relying on B can be also established as shown by the next result.

Theorem 3.3.5 :

Consider the system (3.1.2) with $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$ and $m < n_1$. Let $\mu^{**}_{A_2}(A_1)$, and $\mu^{**(2)}_{A_2}(A_1), \alpha^*_{A_2}(A_1), \beta^*_{A_2}(A_1)$ be four constants defined by (3.2.15), (3.3.1)-(3.3.3), respectively. Suppose that one of the following conditions holds: (i) $\alpha^*_{A_2}(A_1) < \frac{1}{\mu^{**}_{A_2}(A_1)}$; (ii) $\alpha^*_{A_2}(A_1) \leqslant \frac{1}{\mu^{**}_{A_2}(A_1)}$ and $\beta^*_{A_2}(A_1) < \alpha^*_{A_2}(A_1)$. Then $\mu^{**(2)}_{A_2}(A_1) > 0$ and

$$Spark^*_{A_2}(A_1) \geqslant \varphi^* = 1 + \frac{\sqrt{\rho} - (\alpha^*_{A_2}(A_1) + \beta^*_{A_2}(A_1))\overline{\mu}^*}{2\mu^{**(2)}_{A_2}(A_1)},$$

where $\overline{\mu}^* = \mu^{**}_{A_2}(A_1) - \mu^{**(2)}_{A_2}(A_1)$ and $\rho = \left(\alpha^*_{A_2}(A_1) - \beta^*_{A_2}(A_1)\right)^2 (\overline{\mu}^*)^2 + 4$.

Proof. Note that $\alpha(B^T A_1) \in \{1, ..., n_1 - 1\}$ for any $B \in F$. By the definition of $\alpha^*_{A_2}(A_1)$ which is the maximum value of $\alpha(B^T A_1)$ over F, this maximum is attainable, that is, there exists a $\widehat{B} \in F$ such that

$$\alpha^*_{A_2}(A_1) = \alpha(\widehat{B}^T A_1).$$

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For such a basis $\widehat{B} \in F$, without loss of generality, we assume that all columns of $\widehat{B}^T A_1$ are normalized in the sense that the l_2-norm of every column of $\widehat{B}^T A_1$ is 1. Note also that the spark, mutual coherence, sub-coherence, coherence rank, and sub-coherence rank are invariant under normalization.

Let $p = \text{Spark}(\widehat{B}^T A_1)$ and $\{c_1, \cdots, c_p\}$ be the set of p columns from $\widehat{B}^T A_1$ that are linearly dependent. Denote C_p the submatrix consisting of these p columns. Then the Gram matrix of C_p, $G_{pp} = C_p^T C_p \in R^{p \times p}$, is singular. Since all diagonal entries of G_{pp} are 1's, and the absolute values of off-diagonal entries are less than or equal to $\mu(\widehat{B}^T A_1)$, under either condition (i) or (ii) of the theorem, we have

$$\alpha_{A_2}^*(A_1) \leqslant \frac{1}{\mu_{A_2}^{**}(A_1)} \leqslant \frac{1}{\mu(B^T A_1)} \text{ for any } B \in F.$$

In particular, we have

$$\alpha_{A_2}^*(A_1) \leqslant \frac{1}{\mu(\widehat{B}^T A_1)} \leqslant \text{Spark}(\widehat{B}^T A_1) - 1 = p - 1. \tag{3.3.7}$$

Since G_{pp} is a $p \times p$ matrix, in each row of G_{pp}, there are at most $\alpha_{A_2}^*(A_1) = \alpha(\widehat{B}^T A_1)$ entries whose absolute values are equal to $\mu(\widehat{B}^T A_1)$, and the absolute values of the remaining $(p - 1 - \alpha_{A_2}^*(A_1))$ entries are less than or equal to $\mu^{(2)}(\widehat{B}^T A_1)$. By the singularity of G_{pp}, we know that $\lambda = 0$ is an eigenvalue of G_{pp}. By Lemma 3.3.2, there exist two rows of G_{pp}, say, the ith row and the jth row $(i \neq j)$, satisfying that

$$|0 - G_{ii}| \cdot |0 - G_{jj}| \leqslant \Delta_i \cdot \Delta_j = \sum_{t=1, t \neq i}^{p} |c_i^T c_t| \cdot \sum_{t=1, t \neq j}^{p} |c_j^T c_t|. \tag{3.3.8}$$

By the definitions of coherence rank and sub-coherence rank, if there are $\alpha_{A_2}^*(A_1)(= \alpha(\widehat{B}^T A_1))$ entries whose absolute values are $\mu(\widehat{B}^T A_1)$ in the ith row, then for the jth row, there are at most $\beta(\widehat{B}^T A_1)$ entries whose absolute values are $\mu(\widehat{B}^T A_1)$. And the

absolute values of the remaining entries in either row are less than or equal to $\mu^{(2)}(\widehat{B}^T A_1)$. Therefore, from (3.3.8), we have that

$$
\begin{aligned}
1 \;\leqslant\; & [\alpha^*_{A_2}(A_1)\mu(\widehat{B}^T A_1) + (p - 1 - \alpha^*_{A_2}(A_1))\mu^{(2)}(\widehat{B}^T A_1)] \cdot \\
& [\beta(\widehat{B}^T A_1)\mu(\widehat{B}^T A_1) + (p - 1 - \beta(\widehat{B}^T A_1))\mu^{(2)}(\widehat{B}^T A_1)].
\end{aligned}
\tag{3.3.9}
$$

Let $p^* = \mathrm{Spark}^*_{A_2}(A_1)$. Since $\mathrm{Spark}^*_{A_2}(A_1)$ is the supremum of $\mathrm{Spark}(B^T A_1)$ over F, we have $p \leqslant p^*$. Thus it follows from (3.3.9) that

$$
\begin{aligned}
1 \;\leqslant\; & [\alpha^*_{A_2}(A_1)\mu(\widehat{B}^T A_1) + (p^* - 1 - \alpha^*_{A_2}(A_1))\mu^{(2)}(\widehat{B}^T A_1)] \cdot \\
& [\beta(\widehat{B}^T A_1)\mu(\widehat{B}^T A_1) + (p^* - 1 - \beta(\widehat{B}^T A_1))\mu^{(2)}(\widehat{B}^T A_1)].
\end{aligned}
\tag{3.3.10}
$$

By the definition of $\beta^*_{A_2}(A_1)$, we have $\beta(\widehat{B}^T A_1) \leqslant \beta^*_{A_2}(A_1)$. This, together with $\mu(\widehat{B}^T A) \geqslant \mu^{(2)}(\widehat{B}^T A_1)$, implies that

$$
\begin{aligned}
& \beta(\widehat{B}^T A_1)\mu(\widehat{B}^T A_1) + (p^* - 1 - \beta(\widehat{B}^T A_1))\mu^{(2)}(\widehat{B}^T A_1) \\
\leqslant\; & \beta^*_{A_2}(A_1)\mu(\widehat{B}^T A_1) + (p^* - 1 - \beta^*_{A_2}(A_1))\mu^{(2)}(\widehat{B}^T A_1).
\end{aligned}
$$

Combining (3.3.10) with the inequality above yields

$$
\begin{aligned}
1 \;\leqslant\; & [\alpha^*_{A_2}(A_1)\mu(\widehat{B}^T A_1) + (p^* - 1 - \alpha^*_{A_2}(A_1))\mu^{(2)}(\widehat{B}^T A_1)] \cdot \\
& [\beta^*_{A_2}(A_1)\mu(\widehat{B}^T A_1) + (p^* - 1 - \beta^*_{A_2}(A_1))\mu^{(2)}(\widehat{B}^T A_1)].
\end{aligned}
\tag{3.3.11}
$$

Note that

$$
\beta^*_{A_2}(A_1) \leqslant \alpha^*_{A_2}(A_1) \leqslant p - 1 \leqslant p^* - 1, \;\; \mu(\widehat{B}^T A_1) \leqslant \mu^{**}_{A_2}(A_1), \;\; \mu^{(2)}(\widehat{B}^T A_1) \leqslant \mu^{**(2)}_{A_2}(A_1).
$$

So from (3.3.11), we obtain

$$1 \leqslant [\alpha^*_{A_2}(A_1)\mu^{**}_{A_2}(A_1) + (p^* - 1 - \alpha^*_{A_2}(A_1))\mu^{**(2)}_{A_2}(A_1)] \cdot$$
$$[\beta^*_{A_2}(A_1)\mu^{**}_{A_2}(A_1) + (p^* - 1 - \beta^*_{A_2}(A_1))\mu^{**(2)}_{A_2}(A_1)].$$

Denote by $\overline{\mu}^* := \mu^{**}_{A_2}(A_1) - \mu^{**(2)}_{A_2}(A_1)$. The above inequality can be written as

$$\left[(p^* - 1)\mu^{**(2)}_{A_2}(A_1)\right]^2 + (p^* - 1)(\alpha^*_{A_2}(A_1) + \beta^*_{A_2}(A_1))\overline{\mu}^*\mu^{**(2)}_{A_2}(A_1) \quad (3.3.12)$$
$$+\alpha^*_{A_2}(A_1)\beta^*_{A_2}(A_1)(\overline{\mu}^*)^2 \geqslant 1.$$

By the definition of $\mu^{**(2)}_{A_2}(A_1)$, we know that $\mu^{**(2)}_{A_2}(A_1) \geqslant 0$. We now prove that $\mu^{**(2)}_{A_2}(A_1) > 0$. In fact, if $\mu^{**(2)}_{A_2}(A_1) = 0$, then the quadratic inequality (3.3.12) becomes

$$\alpha^*_{A_2}(A_1)\beta^*_{A_2}(A_1)\left(\mu^{**}_{A_2}(A_1)\right)^2 \geqslant 1,$$

which contradicts to either condition (i) or condition (ii) of the theorem. Thus $\mu^{**(2)}_{A_2}(A_1)$ must be positive. Consider the following quadratic equation in variable t:

$$h(t) := t^2 + t(\alpha^*_{A_2}(A_1) + \beta^*_{A_2}(A_1))\overline{\mu}^* + \alpha^*_{A_2}(A_1)\beta^*_{A_2}(A_1)(\overline{\mu}^*)^2 - 1 = 0$$

which has only one positive root under conditions (i) and (ii). This positive root is given by

$$t^* = \frac{-(\alpha^*_{A_2}(A_1) + \beta^*_{A_2}(A_1))\overline{\mu}^* + \sqrt{\rho}}{2},$$

where $\rho = (\alpha^*_{A_2}(A_1) - \beta^*_{A_2}(A_1))^2(\overline{\mu}^*)^2 + 4$. Let $\gamma = (p^* - 1)\mu^{**(2)}_{A_2}(A_1)$. The inequality

(3.3.12) shows that $h(\gamma) \geqslant 0$. Thus $\gamma \geqslant t^*$, that is,

$$(p^* - 1)\mu_{A_2}^{**(2)}(A_1) \geqslant \frac{-(\alpha_{A_2}^*(A_1) + \beta_{A_2}^*(A_1))\overline{\mu}^* + \sqrt{\rho}}{2}.$$

Therefore,

$$\text{Spark}_{A_2}^*(A_1) = p^* \geqslant 1 + \frac{\sqrt{\rho} - (\alpha_{A_2}^*(A_1) + \beta_{A_2}^*(A_1))\overline{\mu}^*}{2\mu_{A_2}^{**(2)}(A_1)},$$

as desired. $\square$

By Theorem 3.2.8 and Theorem 3.3.5, we immediately have the next uniqueness condition.

Corollary 3.3.6 *:*
Consider the system (3.1.2) where $A_1 \in R^{m \times n_1}$, $A_2 \in R^{m \times n_2}$, and $m < n_1$. Under the same condition of Theorem 3.3.5, if there exists a solution (x, y) to the system (3.1.2) satisfying that

$$\|x\|_0 < \frac{1}{2}\varphi^* =: \frac{1}{2}\left(1 + \frac{\sqrt{\rho} - (\alpha_{A_2}^*(A_1) + \beta_{A_2}^*(A_1))\overline{\mu}^*}{2\mu_{A_2}^{**(2)}(A_1)}\right),$$

then x is the unique sparsest x-part solution to the system (3.1.2).

The above corollary may also provide a tighter lower bound of $\text{Spark}_{A_2}^*(A_1)$ than Theorems 3.2.12 under some conditions, as indicated by the following proposition.

Proposition 3.3.7 *: Let φ^* be a lower bound of $Spark_{A_2}^*(A_1)$ given in Theorem 3.3.5. Assume that $\alpha_{A_2}^*(A_1) = 1$ and $\alpha_{A_2}^*(A_1) < \frac{1}{\mu_{A_2}^{**}(A_1)}$. If $\mu_{A_2}^{**(2)}(A_1) < \mu_{A_2}^*(A_1)(1 - \overline{\mu}^*)$ where $\overline{\mu}^* = \mu_{A_2}^{**}(A_1) - \mu_{A_2}^{**(2)}(A_1)$, we have $\varphi^* > 1 + \frac{1}{\mu_{A_2}^*(A_1)}$.*

Proof. Under condition $\alpha_{A_2}^*(A_1) < \frac{1}{\mu_{A_2}^{**}(A_1)}$, by Theorem 3.3.5 we get the following lower

bond of $\mathrm{Spark}^*_{A_2}(A_1)$:

$$\varphi^* = 1 + \frac{\sqrt{\rho} - (\alpha^*_{A_2}(A_1) + \beta^*_{A_2}(A_1))\overline{\mu}^*}{2\mu^{**(2)}_{A_2}(A_1)}. \tag{3.3.13}$$

By (3.3.4), we see that $\alpha^*_{A_2}(A_1) = 1$ implies that $\beta^*_{A_2}(A_1) = 1$. Thus (3.3.13) is reduced to $\varphi^* - 1 = \frac{1-\overline{\mu}^*}{\mu^{**(2)}_{A_2}(A_1)}$. Note that

$$\begin{aligned}
\frac{1-\overline{\mu}^*}{\mu^{**(2)}_{A_2}(A_1)} &= \frac{1}{\mu^*_{A_2}(A_1)} + \left(\frac{1-\overline{\mu}^*}{\mu^{**(2)}_{A_2}(A_1)} - \frac{1}{\mu^*_{A_2}(A_1)}\right) \\
&= \frac{1}{\mu^*_{A_2}(A_1)} + \frac{\mu^*_{A_2}(A_1)(1-\overline{\mu}^*) - \mu^{**(2)}_{A_2}(A_1)}{\mu^{**(2)}_{A_2}(A_1)\mu^*_{A_2}(A_1)}.
\end{aligned}$$

Thus if $\mu^{**(2)}_{A_2}(A_1) < \mu^*_{A_2}(A_1)(1-\overline{\mu}^*)$, we must have $\varphi^* > 1 + \frac{1}{\mu^*_{A_2}(A_1)}$. □

The discussion in this section demonstrates that the concepts, such as maximal scaled coherence rank and sub-coherence rank, and minimal/maximal scaled mutual coherence, are useful in the development of uniqueness criteria for partial ℓ_0-minimization. Now, to further study the partial ℓ_0-minimization, the main issue is how to efficiently reconstruct the partially sparest x-part solution to the linear system (3.1.2). As introduced in section 3.1, partial NSP and partial RIP [7, 8] conditions are sufficient for the partial x-part uniform recovery via partially sparse recovery if the matrix A_2 is full column rank. However, few work has been done on whether the unique solution of partially sparse recovery with the least ℓ_1-norm x-part is also necessarily the unique sparsest x-part solution of partial ℓ_0-minimization without the assumption on matrix A_2, and whether other conditions can be developed to ensure the partial x-part uniform recovery via partially sparse recovery in a more general situation. In the next chapter, we will study a special model of partial ℓ_0-minimization, namely, 1-bit ℓ_0-minimization, and use the range space property (RSP)

based analysis to develop criteria for a certain level of recovery for such a problem. Hence, the RSP-based analysis can be possibly applied to establish the partial x-part recovery for partial ℓ_0-minimization.

CHAPTER 4

RRSP-BASED THEORY FOR 1-BIT COMPRESSIVE SENSING

4.1 Introduction

As demonstrated in Chapter 2, to achieve a sparse representation of a signal from a reduced number of nonadaptive measurements, sensing matrix should admit certain properties (see, e.g., [52, 120, 36, 35, 34, 51, 43, 130, 132, 68, 133]). In some situations, however, only a limited information of measurements can be acquired, and thus achieving a certain level of sparse recovery in these situations faces some mathematical challenges.

In this chapter [1], we study an extreme case of compressive sensing, to recover a sparse signal within certain factors when every measurement is quantized into a single bit, i.e., the sign information of the measurement. This is called 1-bit compressive sensing (see, e.g., [22, 24, 75, 85, 87, 86, 105, 21]). In practice, 1-bit compressive sensing is widely used in many applications, such as the imaging system [25, 26] and the matrix completion [46]. Moreover, the 1-bit techniques can be applied to the nonlinear monotonic distorted measurements to recover the signal with high accuracy [23]. Surprisingly, only signs of

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[1]Part of the work in this chapter was carried out with Yun-Bin Zhao [137].

measurements may contain enough information for a certain level of sparse reconstruction. Empirical evidences in [22, 24] showed that a sparse signal up to a certain factor can be reconstructed from signs of measurements via some 1-bit compressive sensing models.

Unfortunately, in contrast to the standard compressive sensing, the theoretical analysis for the performances of 1-bit compressive sensing and 1-bit compressive sensing recovery algorithms is far from complete. As the amplitude information of sparse signals in 1-bit measurement is lost, all the analysis techniques for standard compressive sensing cannot apply to 1-bit compressive sensing directly. Hence, how to characterize behaviors of 1-bit compressive sensing and its recovery algorithms is a key issue to be addressed. In this chapter, we focus on theoretical analysis for the noiseless 1-bit compressive sensing and work towards establishing a certain nonuniform and uniform recovery theory for 1-bit compressive sensing from a new perspective of range space properties. In compressive sensing scenario, it has been shown in [132, 133] that the K-sparse signals can be exactly recovered by the standard basis pursuit (i.e., ℓ_1-minimization) if and only if the transposed sensing matrix admits the so-called range space property (RSP) of order K. This property arises naturally from the optimality conditions of linear programs. It captures an intrinsic feature of linear basis-pursuit-type decoding methods. The RSP-based analysis makes it possible to develop an analogous recovery guarantee for 1-bit compressive sensing algorithms. To build the RSP-based recovery theories, we start with proposing a general 1-bit compressive sensing model which can cope with the situations where the measurements might include zero components. We show that such a 1-bit model can be formulated equivalently as an ℓ_0-minimization problem with linear equality and inequality constraints. To possibly attack this ℓ_0-minimization problem, it is naturally to consider the so-called 1-bit basis pursuit (as a decoding method). Like the standard compressive sensing, the uniqueness of solutions to a decoding method plays a fundamental role in sparse signal reconstruction. We then develop the uniqueness condition for the solution

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of the 1-bit basis pursuit via the linear programming theory. This development will naturally leads to the important concept of restricted range space property (RRSP) which eventually gives rise to the desired conditions for a certain level of nonuniform and uniform recovery for 1-bit compressive sensing.

This chapter is organized as follows. In section 4.2, we introduce and summarize some existing frameworks of 1-bit compressive sensing. In section 4.3, we provide a general 1-bit compressive sensing model and its equivalent reformulation via noiseless measurements. In section 4.4, we establish uniqueness conditions for the solution of 1-bit basis pursuit. The concept of restricted range space property (RRSP) is introduced in section 4.5, and is used to develop criteria for nonuniform and uniform support recoveries for 1-bit compressive sensing.

4.2 Existing frameworks of 1-bit compressive sensing

Due to the sign-process, the amplitude of the signal is lost. It is impossible to exactly recover the sparse signal. Thus, to achieve a certain level of sparse recovery for 1-bit compressive sensing, the following questions should be addressed:

1. What properties should the sensing matrix (Φ) admit for 1-bit compressive sensing?

2. How efficient are reconstruction algorithms? In other words, how can a reconstruction algorithm ensure a certain level of sparse recovery?

In this section, we introduce some existing frameworks of 1-bit compressive sensing, which have partially answered the above questions. Let $x^* \in R^n$ be a sparse signal, 1-bit measurements vector y is acquired by taking signs of measurements from a linear system $\Phi x^* = b$, e.g., $y = \text{sign}(\Phi x^*)$, where $\Phi \in R^{m \times n}$ is a sensing matrix. We start with the first 1-bit compressive sensing framework proposed in [24], which can be stated as the

following ℓ_0-minimization problem:

$$\min\{\|x\|_0 : \operatorname{sign}(\Phi x) = y\}, \qquad (4.2.1)$$

where $y = \operatorname{sign}(\Phi x^*) \in R^m$ is the vector of sign measurements in the binary space $\{-1, 1\}^m$. The sign function is applied element-wise, where $y_i = \operatorname{sign}[(\Phi x^*)_i] = 1$ if $(\Phi x^*)_i \geqslant 0$ and $y_i = \operatorname{sign}[(\Phi x^*)_i] = -1$ if $(\Phi x^*)_i < 0$ for any $i \in \{1, \cdots, m\}$ [24].

Note that if x^* is a solution to the problem (4.2.1), then αx^* remains a solution to this problem for any scalar $\alpha > 0$ [24]. Also, the 1-bit measurements y are robust to any sparsest solution Px^* for some weight $P = \operatorname{diag}(p)$ satisfying $\|Px^*\|_0 = \|x^*\|_0$ (i.e., $\operatorname{Supp}(Px^*) = \operatorname{Supp}(x^*)$ and $\operatorname{sign}(Px^*) = \operatorname{sign}(x^*)$, where p is defined as $|p_i| > 0$ for $i \in \operatorname{Supp}(x^*)$ and $p_i = 0$ for $i \notin \operatorname{Supp}(x^*)$. More generally, the 1-bit measurements y are robust to any small change of sparsest solutions $x^* + \Delta x$, where the perturbation Δx satisfies $\|x^* + \Delta x\|_0 = \|x^*\|_0$ and $\operatorname{sign}(\Phi(x^* + \Delta x)) = \operatorname{sign}(\Phi x^*)$. Thus, problem (4.2.1) has infinitely many sparsest solutions. If any sparsest solution of (4.2.1) is of the form αx^* for some $\alpha > 0$, the sign information of measurements might be enough to reconstruct the signal x^* within a positive scalar factor [24]. Due to the minimization-based objective, problem (4.2.1) may find the trivial solution (zero solution). To exclude such a solution, an artificial constraint $\|x\|_2 = 1$ is enforced to (4.2.1) in [24] so that all solutions are restricted on the unit ℓ_2-sphere. Since $\operatorname{sign}(\Phi x) = y$ is a discrete constraint, a linear inequality relaxation of $\operatorname{sign}(\Phi x)$ is introduced in [24], i.e., $Y\Phi x \geqslant 0$, where $Y = \operatorname{diag}(y)$ is a diagonal matrix with zero elements and y along the diagonal. Thus, a relaxation of (4.2.1) is written as

$$\min\{\|x\|_0 : Y\Phi x \geqslant 0, \ \|x\|_2 = 1\}. \qquad (4.2.2)$$

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Let $S_{n,K}$ be a set of all K-sparse signals restricted on the unit ℓ_2-sphere, i.e., $S_{n,K} = \{x \in R^n : \|x\|_0 \leqslant K, \|x\|_2 = 1\}$. And for given 1-bit measurements y, $\widehat{P}$ is a polyhedral set of all vectors satisfying $Y\Phi x \geqslant 0$, namely, $\widehat{P} = \{x \in R^n : Y\Phi x \geqslant 0\}$. Hence, any sparse solution of (4.2.2) is in the set $S_{n,K} \bigcap \widehat{P}$.

Due to the computational intractability caused by the ℓ_0-norm, an ℓ_1-norm recovery model of (4.2.2) is then introduced in [24]

$$\min\{\|x\|_1 : Y\Phi x \geqslant 0, \|x\|_2 = 1\}. \tag{4.2.3}$$

As the unit ℓ_2-sphere constraint is nonconvex, how to theoretically analyze the recoverability of nonconvex problem (4.2.3) remains a challenge.

Then, a convex 1-bit recovery model is proposed in [105], namely,

$$\min\{\|x\|_1 : Y\Phi x \geqslant 0, \|\Phi x\|_1 = p \text{ for any } p > 0\},$$

which can be written as

$$
\begin{aligned}
\min_{x,u} \quad & e^T u \\
\text{s.t.} \quad & -u \leqslant x \leqslant u, \\
& Y\Phi x \geqslant 0, \\
& \tfrac{1}{p}y^T\Phi x \geqslant 1, \\
& u \geqslant 0,
\end{aligned}
\tag{4.2.4}
$$

where $Y = \operatorname{diag}(y)$ is the diagonal matrix and $e = (1, \cdots, 1)^T$ is a unit vector in R^n.

An alternative linear program is introduced in [85], i.e.,

$$\min\{\|x\|_1 : Y\Phi x \geqslant 0, \varpi^T x = 1\}, \tag{4.2.5}$$

where ϖ is the centroid of the hyperplanes defined by row vectors of Φ, i.e., $\varpi = \sum_i^m \phi_i$ [85].

Instead of the unit ℓ_2-sphere constraint in (4.2.3), 1-bit recovery models (4.2.4) and (4.2.5) use linear constraints to avoid the trivial solution.

Furthermore, for a more general case that the measurements Φx has noises, an optimization problem is introduced in [106]

$$\max\{\langle y, \Phi x \rangle :\ \|x\|_2 \leqslant 1,\ \|x\|_1 \leqslant \sqrt{K}\}. \tag{4.2.6}$$

The problem (4.2.6) is derived from (4.2.3) but requires the sparsity level in the constraint. Note that the feasible set of (4.2.6) is convex, denoted as $L_{n,k}$, i.e.,

$$L_{n,K} = \{x \in R^n :\ \|x\|_2 \leqslant 1,\ \|x\|_1 \leqslant \sqrt{K}\}.$$

Thus, any K-sparse signal can be efficiently estimated via the convex problem (4.2.6) [106], where $L_{n,K}$ is almost exactly the convex hull of $S_{n,K}$ as shown in [105].

Remark 4.2.1 : *The noisy measurements y for 1-bit compressive sensing are usually defined as the 1-bit measurements with some random sign-flips caused by the noise-corruptions in measurements, namely, $y = sign(\Phi x + v)$, where the noise $v \in R^n$ is drawn from the standard Gaussian distribution. The convex program (4.2.6) pursues the maximal consistency between the observed noisy 1-bit measurements y and signs of measurements Φx. Instead of maximizing the consistency, some 1-bit recovery algorithms aim to minimize the inconsistency (against sign flips) via one-sided ℓ_2-norm or one-sided ℓ_1-norm penalty minimization problems [22, 24, 80, 82, 87, 98, 115]. More 1-bit recovery algorithms will be reviewed and explored in the next chapter.*

If the recovery model (4.2.6) returns a K-sparse solution or an approximately K-sparse solution, one may ask what properties that the sensing matrix should admit to ensure such a recovery. It has been proved in [80] that when the mapping T from R^n to $\{-1,1\}^m$, i.e., $T(x) = \text{sign}(\Phi x)$, admits the binary ε-stable embedding (BεSE) property [80, 85], an estimate of a sparse signal in the set $S_{n,K}$ can be found by any reconstruction algorithm. Fortunately, some random matrices can provide the BεSE property for T, for instance, if every element of the sensing matrix Φ follows the standard Gaussian distribution, then the mapping T has the BεSE property, which is proved by Lemma 2 and Lemma 4 in [80]. Note that the BεSE property for 1-bit compressive sensing is analogous to the restricted isometry property (RIP) for standard compressive sensing. Furthermore, if every element of the sensing matrix Φ follows sub-gaussian distributions, such as Bernoulli distribution, it is also possible to recover an approximate sparse solution via (4.2.6) as shown in [3]. However, the recovery conditions in terms of the property of Φ and the 1-bit measurements y are still under development. As discussed in Chapter 2, for the standard compressive sensing, it is well known that when the sensing matrix has some properties such as the mutual coherence [52, 28], null space property (NSP) [43, 130], restricted isometry property (RIP) [36] and the transposed sensing matrix has the range space property (RSP) [132, 133], the sparse signal can be exactly recovered by basis pursuit methods (i.e., ℓ_1-minimization) and other decoding algorithms. This motivates us to further investigate whether such recovery criteria can be established for 1-bit compressive sensing.

Also, notice that existing 1-bit models do not distinguish between zero and positive measurements of Φx^* in the sense that $\text{sign}[(\Phi x^*)_i] = 1$ if $(\Phi x^*)_i = 0$ and $\text{sign}[(\Phi x^*)_i] = 1$ if $(\Phi x^*)_i > 0$, while zero and positive measurements usually stand for different information in practice. Thus, in the next section, we define the 1-bit measurements in a way that zero and positive components of Φx^* are treated differently, for instance, $\text{sign}[(\Phi x^*)_i] = 0$

if $(\Phi x^*)_i = 0$ and $\text{sign}[(\Phi x^*)_i] = 1$ if $(\Phi x^*)_i > 0$. Furthermore, we propose a general 1-bit compressive sensing model which can cope with the situations where the 1-bit measurements might include zero components and then develop certain restricted range space property on the transposed matrix of Φ to ensure a certain level of sparse recovery for 1-bit compressive sensing.

4.3   Reformulation of 1-bit compressive sensing

In existing 1-bit compressive sensing models, the measurement vector y is often confined to the binary space $\{-1, 1\}^m$, where the sign value of any positive or zero component $(\Phi x)_i$ is assigned as $y_i = 1$, and any negative component $(\Phi x)_i$ as $y_i = -1$. This means that the positive components and zero components of Φx are represented by the same 1-bit measurements. From the mathematical point of view, all positive, negative and zero components of Φx should be more clearly differentiated by 1-bit measurements. This motivates us to consider the measurements $y \in \{-1, 1, 0\}^m$ instead of $y \in \{-1, 1\}^m$. More specifically, for each $i = 1, ..., m$, the 1-bit measurements take the values

$$
\begin{cases}
y_i = \text{sign}[(\Phi x)_i] = 1 & \text{for } (\Phi x)_i > 0, \\
y_i = \text{sign}[(\Phi x)_i] = 0 & \text{for } (\Phi x)_i = 0, \\
y_i = \text{sign}[(\Phi x)_i] = -1 & \text{for } (\Phi x)_i < 0.
\end{cases}
\tag{4.3.1}
$$

As introduced in the last section, when the measurement vector $y \in \{-1, 1, 0\}^m$ is given as (4.3.1), the sign constraint in (4.2.1) is no longer equivalent to the condition $Y\Phi x \geqslant 0$ for which $y_i = 0$ does not necessarily correspond to the case $(\Phi x)_i = 0$. Thus existing optimization models based on the formulation $Y\Phi x \geqslant 0$ is no longer suitable for the 1-bit compressive sensing with measurements $y \in \{-1, 1, 0\}^m$. We now propose a more general 1-bit compressive sensing model and its equivalent reformulation, which are different from existing ones in that our model and reformulation can cope with the situations where some

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components of the measurement vector vanish, and in the mean time the trivial solutions are naturally avoided in our model without imposing any extra constraints to the problem.

For given 1-bit measurements $y \in \{-1, 1, 0\}^m$, let J_+, J_- and J_0 denote, throughout the chapter, the indices of positive, negative and zero components of y, respectively, i.e.,

$$J_+ = \{i : y_i = 1\}, \ J_- = \{i : y_i = -1\}, \ J_0 = \{i : y_i = 0\}, \tag{4.3.2}$$

by which the system (4.3.1) can be written as

$$\text{sign}(\Phi_{J_+,n}x) = e_{J_+}, \ \text{sign}(\Phi_{J_-,n}x) = -e_{J_-}, \ \Phi_{J_0,n}x = 0, \tag{4.3.3}$$

where $\Phi_{J_+,n}$, $\Phi_{J_-,n}$ and $\Phi_{J_0,n}$ denote the submatrices of $\Phi \in R^{m \times n}$ with row indices in J_+, J_- and J_0, respectively. Thus we consider the 1-bit compressive sensing model with measurements $y \in \{-1, 1, 0\}^m$ which can be stated as

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & \text{sign}(\Phi_{J_+,n}x) = e_{J_+}, \ \text{sign}(\Phi_{J_-,n}x) = -e_{J_-}, \ \Phi_{J_0,n}x = 0. \end{aligned} \tag{4.3.4}$$

Let $\varepsilon > 0$ be a fixed positive number throughout the chapter (ε can be fixed as any positive number, for instance, $\varepsilon = 1$). Consider the following system in $u \in R^n$

$$\Phi_{J_+,n}u \geqslant \varepsilon e_{J_+}, \ \Phi_{J_-,n}u \leqslant -\varepsilon e_{J_-}, \ \Phi_{J_0,n}u = 0. \tag{4.3.5}$$

Clearly, if x satisfies (4.3.3), then there exists a positive scalar $\alpha > 0$ such that $u = \alpha x$ satisfies the system (4.3.5); Conversely, if u satisfies the system (4.3.5), then $x = u$ satisfies the system (4.3.3). Thus the 1-bit model (4.3.4) can be reformulated as the

ℓ_0-minimization problem with linear constraints

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & \Phi_{J_+,n}x \geqslant \varepsilon e_{J_+}, \\ & \Phi_{J_-,n}x \leqslant -\varepsilon e_{J_-}, \\ & \Phi_{J_0,n}x = 0. \end{aligned} \qquad (4.3.6)$$

From the relation of (4.3.3) and (4.3.5), we immediately have the following observation.

Proposition 4.3.1 : *If x^* is a solution to the 1-bit model (4.3.4), then there exists a positive scalar $\alpha > 0$ such that αx^* is a solution to the ℓ_0-problem (4.3.6); Conversely, if x^* is a solution to the ℓ_0-problem (4.3.6), then x^* must be a solution to the 1-bit model (4.3.4).*

Remark 4.3.2 :

(i) *In contrast to the underdetermined linear system in standard compressive sensing, there is no restriction on the dimensions of the sensing matrix in 1-bit compressive sensing. The underdetermined requirement provides that the linear system $\Phi x = b$ has infinitely many solutions since the null space of Φ is nonempty. Otherwise, if the matrix Φ is square or overdetermined, the linear system $\Phi x = b$ has at most one solution. However, it is not necessary to require the matrix Φ to be underdetermined in 1-bit compressive sensing. By Proposition 4.3.1, if x^* is a solution to either the system (4.3.3) or the system (4.3.5), any positive scalar of x^* is also a solution to each system independent of the dimensions of matrix Φ. Therefore, the 1-bit model (4.3.4) and ℓ_0-minimization problem (4.3.6) have infinitely many sparsest solutions independent of dimensions of matrix Φ.*

(ii) *Due to the sign-process and inequality constraints, there exist perturbation-type of solutions to both systems (4.3.3) and (4.3.5). For instance, if x^* is a sparsest solu-*

tion to the system (4.3.5), satisfying $(\Phi x^*)_i > 0 \ \forall i \in J_+$, $(\Phi x^*)_i < 0 \ \forall i \in J_-$ and $(\Phi x^*)_i = 0 \ \forall i \in J_0$. It is easy to find another sparsest solution $\overline{x} = x^* + \lambda \Delta x$ such that $Supp(\overline{x}) = Supp(x^*)$, satisfying

$$\Phi_{J_+,n}(x^* + \lambda \Delta x) > 0,$$
$$\Phi_{J_-,n}(x^* + \lambda \Delta x) < 0,$$
$$\Phi_{J_0,n}\Delta x = 0,$$

for some sufficient small $\lambda \neq 0$. Similarly, for the system (4.3.3), it is possible to find another sparsest solution $\overline{x} = x^* + \lambda \Delta x$ such that $Supp(\overline{x}) = Supp(x^*)$, satisfying $sign(\Phi_{J_+,n}\overline{x}) = sign(\Phi_{J_+,n}x^*)$, $sign(\Phi_{J_-,n}\overline{x}) = sign(\Phi_{J_-,n}x^*)$, and $\Phi_{J_0,n}\Delta x = 0$, for some sufficient small $\lambda \neq 0$. It is worths mentioning that $Supp(\overline{x}) = Supp(x^*)$ implies that there exists a factor p such that $\overline{x} = Px^*$, satisfying $|p_i| > 0$ for $i \in Supp(x^*)$ and $p_i = 0$ for $i \notin Supp(x^*)$, where $P = diag(p)$.

As problems (4.3.4) and (4.3.6) have multiple sparsest solutions, we focus on the support recovery for 1-bit compressive sensing throughout this thesis, where the support recovery is defined as finding a solution x to any reconstruction algorithm satisfying $Supp(x) = Supp(x^*)$.

As a result, to study the 1-bit model (4.3.4), it is sufficient to investigate the ℓ_0-minimization problem (4.3.6). Indeed, the equivalence of (4.3.3) and (4.3.5) makes it possible to use the methodology for the standard compressive sensing to study the 1-bit model (4.3.6). Motivated by (4.3.6), we may consider the ℓ_1-minimization

$$
\begin{aligned}
\min \quad & \|x\|_1 \\
\text{s.t.} \quad & \Phi_{J_+,n}x \geqslant \varepsilon e_{J_+}, \\
& \Phi_{J_-,n}x \leqslant -\varepsilon e_{J_-}, \\
& \Phi_{J_0,n}x = 0,
\end{aligned}
\tag{4.3.7}
$$

which can be seen as a natural decoding method for the 1-bit ℓ_0-minimization (4.3.6). In this thesis, the problem (4.3.7) is referred to as 1-bit basis pursuit. It is worth stressing that unlike existing 1-bit models, our reformulation of 1-bit compressive sensing only admits linear constraints and it automatically excludes the trivial solution without imposing any extra constraint. More importantly, later analysis indicates that our model and reformulation make it possible to develop a recovery theory for sparse signals from the new perspective of the range space property (RSP) of Φ^T.

For the convenience of analysis, we define the index sets $\mathcal{A}(\cdot)$, $\widetilde{\mathcal{A}}_+(\cdot)$ and $\widetilde{\mathcal{A}}_-(\cdot)$ which will be used frequently in this chapter. Let x^* be a signal satisfying the constraints of (4.3.7). At x^*, let $\mathcal{A}(x^*)$ be the index set of active constraints among the inequality constraints of (4.3.7), i.e.,

$$\mathcal{A}(x^*) = \{i : \ (\Phi x^*)_i = \varepsilon\} \cup \{i : \ (\Phi x^*)_i = -\varepsilon\}, \tag{4.3.8}$$

and let

$$\tilde{\mathcal{A}}_+(x^*) = J_+ \setminus \mathcal{A}(x^*), \ \ \tilde{\mathcal{A}}_-(x^*) = J_- \setminus \mathcal{A}(x^*). \tag{4.3.9}$$

Clearly, $\tilde{\mathcal{A}}_+(x^*)$ is the index set of inactive constraints in the first group of inequalities of (4.3.7) (i.e., $\Phi_{J_+,n}x^* \geqslant \varepsilon e_{J_+}$), and $\tilde{\mathcal{A}}_-(x^*)$ is the index set of inactive constraints in the second group of inequalities of (4.3.7) (i.e., $\Phi_{J_-,n}x^* \leqslant -\varepsilon e_{J_-}$). Thus we see that

$$(\Phi x^*)_i = \varepsilon \text{ for } i \in \mathcal{A}(x^*) \cap J_+, \quad (\Phi x^*)_i > \varepsilon \text{ for } i \in \tilde{\mathcal{A}}_+(x^*),$$
$$(\Phi x^*)_i = -\varepsilon \text{ for } i \in \mathcal{A}(x^*) \cap J_-, \quad (\Phi x^*)_i < -\varepsilon \text{ for } i \in \tilde{\mathcal{A}}_-(x^*).$$

We also need the symbols $\pi(\cdot)$ and $\varrho(\cdot)$ defined as follows. Let $p = |J_+|$ and denote the elements in J_+ by $i_k \in \{1, ..., m\}, k = 1, ..., p$, i.e., $J_+ = \{i_1, i_2, ..., i_p\}$. Without loss of generality, we let the elements be sorted in ascending order $i_1 < i_2 < \cdots < i_p$. Then we

define the bijective mapping $\pi : J_+ \to \{1, ..., p\}$ as

$$\pi(i_k) = k \text{ for all } k = 1, ..., p. \qquad (4.3.10)$$

Similarly, let $J_- = \{j_1, j_2, ..., j_q\}$ where $q = |J_-|$ and $j_1 < j_2 < \cdots < j_q$. We define the bijective mapping $\varrho : J_- \to \{1, ..., q\}$ as

$$\varrho(j_k) = k \text{ for all } k = 1, ..., q. \qquad (4.3.11)$$

To develop some conditions for a certain level of sparse recovery from 1-bit measurements, we first characterize the uniqueness of solutions to the 1-bit basis pursuit in the next section.

4.4 Uniqueness characterization for 1-bit basis pursuits

The uniqueness of the solution to a decoding method plays a fundamental role in its guaranteed performance in sparse recovery. As indicated in [69, 68, 132, 133], the uniqueness conditions often lead to certain criteria for nonuniform and uniform recovery of sparse vectors. In this section, we establish a necessary and sufficient condition for the uniqueness of solutions of the 1-bit basis pursuit through the strict complementarity theory of linear programs, which was used in [132] for the first time to develop the RSP-based recovery criteria for the standard compressive sensing. First, let us develop some necessary conditions.

4.4.1 Necessary condition (I): Range space property

By introducing variables $\alpha \in R_+^{|J_+|}$ and $\beta \in R_+^{|J_-|}$, the problem (4.3.7) becomes

$$
\begin{aligned}
\min \quad & \|x\|_1, \\
\text{s.t.} \quad & \Phi_{J_+,n} x - \alpha = \varepsilon e_{J_+}, \\
& \Phi_{J_-,n} x + \beta = -\varepsilon e_{J_-}, \\
& \Phi_{J_0,n} x = 0, \\
& \alpha \geqslant 0, \ \beta \geqslant 0.
\end{aligned}
\tag{4.4.1}
$$

Note that for any solution (x^*, α^*, β^*) of (4.4.1), we have $\alpha^* = \Phi_{J_+,n} x^* - \varepsilon e_{J_+}$ and $\beta^* = -\varepsilon e_{J_-} - \Phi_{J_-,n} x^*$. Using (4.3.8), (4.3.9), (4.3.10) and (4.3.11), we immediately have the following observation.

Lemma 4.4.1 : (i) For any solution (x^*, α^*, β^*) to the problem (4.4.1), we have

$$
\begin{cases}
\alpha^*_{\pi(i)} = 0 & \text{for } i \in \mathcal{A}(x^*) \cap J_+, \\
\alpha^*_{\pi(i)} = (\Phi x^*)_i - \varepsilon > 0 & \text{for } i \in \tilde{\mathcal{A}}_+(x^*), \\
\beta^*_{\varrho(i)} = 0 & \text{for } i \in \mathcal{A}(x^*) \cap J_-, \\
\beta^*_{\varrho(i)} = -\varepsilon - (\Phi x^*)_i > 0 & \text{for } i \in \tilde{\mathcal{A}}_-(x^*).
\end{cases}
\tag{4.4.2}
$$

(ii) x^* is the unique solution to the 1-bit basis pursuit (4.3.7) if and only if (x^*, α^*, β^*) is the unique solution to the problem (4.4.1), where (α^*, β^*) is given by (4.4.2).

Proof. (i) Let (x^*, α^*, β^*) be a solution to the problem (4.4.1), satisfying the linear system

$$
\begin{aligned}
\Phi_{J_+,n} x - \alpha &= \varepsilon e_{J_+}, \\
\Phi_{J_-,n} x + \beta &= -\varepsilon e_{J_-}, \\
\Phi_{J_0,n} x &= 0,
\end{aligned}
$$

then we have $\alpha^* = \Phi_{J_+,n}x^* - \varepsilon e_{J_+}$ and $\beta^* = -\varepsilon e_{J_-} - \Phi_{J_-,n}x^*$.

Let π be the bijective mapping $\pi : J_+ \to \{1, \cdots, p\}$ and ϱ be the bijective mapping

$\varrho : J_- \to \{1, \cdots, q\}$ defined as (4.3.10) and (4.3.11)

At the point x^*, one has

$$(\Phi x^*)_{i_k} = \varepsilon \text{ and } (\Phi x^*)_{j_k} = -\varepsilon,$$

for any active constraints $i_k \in \mathcal{A}(x^*) \cap J_+$ and $j_k \in \mathcal{A}(x^*) \cap J_-$ of (4.3.7).

This indicates that

$$\alpha_k^* = \alpha_{\pi(i_k)}^* = (\Phi x^*)_{i_k} - \varepsilon = 0$$

and

$$\beta_k^* = \beta_{\varrho(j_k)}^* = -\varepsilon - (\Phi x^*)_{j_k} = 0.$$

Also, one has

$$(\Phi x^*)_{i_k} > \varepsilon \text{ and } (\Phi x^*)_{j_k} < -\varepsilon,$$

for any inactive constraints $i_k \in \tilde{\mathcal{A}}_+(x^*)$ and $j_k \in \tilde{\mathcal{A}}_-(x^*)$ of (4.3.7).

This indicates that

$$\alpha_k^* = \alpha_{\pi(i_k)}^* = (\Phi x^*)_{i_k} - \varepsilon > 0$$

and

$$\beta_k^* = \beta_{\varrho(j_k)}^* = -\varepsilon - (\Phi x^*)_{j_k} > 0.$$

Hence, for any solution (x^*, α^*, β^*) to the problem (4.4.1), (α^*, β^*) is given as (4.4.2).

(ii) First, we prove that if x^* is the unique solution to problem (4.3.7), then (x^*, α^*, β^*)

is the unique solution to the problem (4.4.1), where (α^*, β^*) is given by (4.4.2).

Let (x, α, β) be any feasible solution to (4.4.1), by (i) and constraints of (4.4.1), then

we have

$$\alpha_{\pi(i)} = 0, \quad \text{if } (\Phi x)_i - \varepsilon = 0 \quad \text{for } i \in \mathcal{A}(x) \cap J_+,$$

$$\alpha_{\pi(i)} > 0, \quad \text{if } (\Phi x)_i - \varepsilon > 0 \quad \text{for } i \in \tilde{\mathcal{A}}_+(x),$$

$$\beta_{\varrho(i)} = 0, \quad \text{if } -\varepsilon - (\Phi x)_i = 0 \quad \text{for } i \in \mathcal{A}(x) \cap J_-,$$

$$\beta_{\varrho(i)} > 0, \quad \text{if } -\varepsilon - (\Phi x)_i > 0 \quad \text{for } i \in \tilde{\mathcal{A}}_-(x).$$

This implies that x is a feasible solution to (4.3.7). Since x^* is the unique solution to the 1-bit basis pursuit, we have $\|x^*\|_1 < \|x\|_1$. Based on the unique solution x^*, we construct $\alpha^* \in R^{|J_+|}$ and $\beta^* \in R^{|J_-|}$ by (4.4.2), then (x^*, α^*, β^*) is a feasible solution to (4.4.1). Together with $\|x^*\|_1 < \|x\|_1$, (x^*, α^*, β^*) is the unique solution to (4.4.1).

Now, we prove that if (x^*, α^*, β^*) is the unique solution to (4.4.1), where (α^*, β^*) is given by (4.4.2), then x^* is the unique solution to (4.3.7). Let x be any feasible solution to (4.3.7). Then we construct a feasible solution (x, α, β) to the problem (4.4.1), where

$$\alpha_{\pi(i)} = 0 \qquad\qquad \text{for } i \in \mathcal{A}(x) \cap J_+,$$

$$\alpha_{\pi(i)} = (\Phi x)_i - \varepsilon > 0 \quad \text{for } i \in \tilde{\mathcal{A}}_+(x),$$

$$\beta_{\varrho(i)} = 0 \qquad\qquad \text{for } i \in \mathcal{A}(x) \cap J_-,$$

$$\beta_{\varrho(i)} = -\varepsilon - (\Phi x)_i > 0 \quad \text{for } i \in \tilde{\mathcal{A}}_-(x).$$

Note that α and β defined above are nonnegative variables. This together with the uniqueness of (x^*, α^*, β^*) implies that $\|x^*\|_1 < \|x\|_1$. As $\alpha^* \geqslant 0$ and $\beta^* \geqslant 0$, the first two equality constraints of (4.4.1) can be reformulated as

$$\Phi_{J_+,n} x^* \geqslant \varepsilon e_{J_+},$$

$$\Phi_{J_-,n} x^* \leqslant -\varepsilon e_{J_-},$$

combining with the constraint $\Phi_{J_0,n} x^* = 0$, and thus x^* is a feasible solution to (4.3.7) with $\|x^*\|_1 < \|x\|_1$ for any feasible solution x. Hence, x^* is the unique solution to (4.3.7). $\square$

By introducing a nonnegative variable $t \in R^n$ such that $|x_i| \leqslant t_i$ for $i = 1, \cdots, n$, the problem (4.4.1) becomes

$$
\begin{aligned}
\min \quad & e^T t, \\
s.t. \quad & x \leqslant t, \\
& -x \leqslant t, \\
& \Phi_{J_+,n} x - \alpha = \varepsilon e_{J_+}, \\
& \Phi_{J_-,n} x + \beta = -\varepsilon e_{J_-}, \\
& \Phi_{J_0,n} x = 0, \\
& (t, \alpha, \beta) \geqslant 0.
\end{aligned}
\tag{4.4.3}
$$

Note that for any solution $(x^*, t^*, \alpha^*, \beta^*)$ of (4.4.3), we have $t^* = |x^*|$, $\alpha^* = \Phi_{J_+,n} x^* - \varepsilon e_{J_+}$ and $\beta^* = -\varepsilon e_{J_-} - \Phi_{J_-,n} x^*$ defined as (4.4.2). We immediately have the following observation.

Lemma 4.4.2 : (x^*, α^*, β^*) *is the unique solution to (4.4.1) if and only if* $(x^*, t^*, \alpha^*, \beta^*) = (x^*, |x^*|, \alpha^*, \beta^*)$ *is the unique solution to (4.4.3) where* (α^*, β^*) *is given by (4.4.2).*

Proof. Firstly, we prove that if (x^*, α^*, β^*) is the unique solution to (4.4.1), then

$$
(x^*, t^*, \alpha^*, \beta^*) = (x^*, |x^*|, \alpha^*, \beta^*)
$$

is the unique solution to (4.4.3) where (α^*, β^*) is given by (4.4.2). Let (x, t, α, β) be any feasible solution to (4.4.3), thus, $|x| \leqslant t$, $\alpha = \Phi_{J_+,n} x - \varepsilon e_{J_+}$ and $\beta = -\varepsilon e_{J_-} - \Phi_{J_-,n} x$. It follows from the constraint of (4.4.3) that (x, α, β) is a feasible solution to (4.4.1). Since (x^*, α^*, β^*) is the unique solution to (4.4.1), we have $\|x^*\|_1 < \|x\|_1 \leqslant e^T t$. Based

on (x^*, α^*, β^*), we construct a feasible solution $(x^*, t^*, \alpha^*, \beta^*)$ to (4.4.3) where t^* is a nonnegative variable such that $t^* = |x^*|$. This together with the fact that $e^T t^* = \|x^*\|_1 < \|x\|_1 \leqslant e^T t$ for any solution (x, t, α, β) implies that $(x^*, t^* = |x^*|, \alpha^*, \beta^*)$ is the unique solution to (4.4.3).

Now, we prove that if $(x^*, t^* = |x^*|, \alpha^*, \beta^*)$ is the unique solution to (4.4.3), then (x^*, α^*, β^*) is the unique solution to (4.4.1). Let (x, α, β) be any solution to (4.4.1). We construct a feasible solution (x, t, α, β) to (4.4.3) where t is a nonnegative variable such that $t = |x|$. This together with the uniqueness of $(x^*, t^*, \alpha^*, \beta^*)$ implies that $e^T t^* = \|x^*\|_1 < e^T t = \|x\|_1$. Also, (x^*, α^*, β^*) satisfies the linear system of (4.4.1), namely,

$$\Phi_{J_+,n} x^* - \alpha^* = \varepsilon e_{J_+},$$

$$\Phi_{J_-,n} x^* + \beta^* = -\varepsilon e_{J_-},$$

$$\Phi_{J_0,n} x^* = 0,$$

$$\alpha^* \geqslant 0, \ \beta^* \geqslant 0,$$

thus, (x^*, α^*, β^*) is a feasible solution to (4.4.1) with $\|x^*\|_1 < \|x\|_1$ for any solution (x, α, β). Hence, (x^*, α^*, β^*) is the unique solution to (4.4.1). $\qquad \square$

Furthermore, introducing slack variables $u \geqslant 0$, $v \geqslant 0$, the problem (4.4.3) can be further written as the linear program

$$
\begin{aligned}
\min \quad & e^T t \\
\text{s.t.} \quad & x + u = t, \\
& -x + v = t, \\
& \Phi_{J_+,n} x - \alpha = \varepsilon e_{J_+}, \\
& \Phi_{J_-,n} x + \beta = -\varepsilon e_{J_-}, \\
& \Phi_{J_0,n} x = 0, \\
& (t, \ u, \ v, \ \alpha, \ \beta) \geqslant 0.
\end{aligned}
\tag{4.4.4}
$$

Note that for any solution $(x^*, t^*, u^*, v^*, \alpha^*, \beta^*)$ of the problem (4.4.4), we must have $t^* = |x^*|$, $u^* = |x^*| - x^*$ and $v^* = |x^*| + x^*$, and (α^*, β^*) is given by (4.4.2). By the same reason as Lemma 4.4.1, we have the following statement.

Lemma 4.4.3 : $(x^*, t^*, \alpha^*, \beta^*) = (x^*, |x^*|, \alpha^*, \beta^*)$ *is the unique solution to (4.4.3) if and only if* $(x^*, t^*, u^*, v^*, \alpha^*, \beta^*)$ *is the unique solution to (4.4.4) where* (α^*, β^*) *is given by (4.4.2) and* $u^* = |x^*| - x^*$, $v^* = |x^*| + x^*$.

From Lemma 4.4.1, Lemma 4.4.2 and Lemma 4.4.3, we can claim that the following lemma holds.

Lemma 4.4.4 x^* *is the unique solution to the 1-bit basis pursuit (4.3.7) if and only if* $(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*)$ *is the unique solution to the linear program (4.4.4), where* (α^*, β^*) *is given by (4.4.2) and* $u^* = |x^*| - x^*$, $v^* = |x^*| + x^*$.

In matrix form, the problem (4.4.4) can be stated as

$$\min \quad e^T t$$

$$\text{s.t.} \quad \begin{bmatrix} I & -I & I & 0 & 0 & 0 \\ -I & -I & 0 & I & 0 & 0 \\ \Phi_{J_+,n} & 0 & 0 & 0 & -I & 0 \\ \Phi_{J_-,n} & 0 & 0 & 0 & 0 & I \\ \Phi_{J_0,n} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ t \\ u \\ v \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \varepsilon e_{J_+} \\ -\varepsilon e_{J_-} \\ 0 \end{bmatrix},$$

$$x \in R^n, \ (t, u, v, \alpha, \beta) \geqslant 0,$$

through which it is very easy to verify that the dual problem of (4.4.4) is given as

$$(\text{DLP}) \qquad \max \quad \varepsilon e_{J_+}^T h_3 - \varepsilon e_{J_-}^T h_4$$

$$\text{s.t.} \quad h_1 - h_2 + (\Phi_{J_+,n})^T h_3 + (\Phi_{J_-,n})^T h_4 + (\Phi_{J_0,n})^T h_5 = 0,$$

$$-h_1 - h_2 \leqslant e, \tag{4.4.5}$$

$$h_1 \leqslant 0, \tag{4.4.6}$$

$$h_2 \leqslant 0, \tag{4.4.7}$$

$$-h_3 \leqslant 0, \tag{4.4.8}$$

$$h_4 \leqslant 0. \tag{4.4.9}$$

The (DLP) is always feasible in the sense that there exists a point, for instance, $(h_1, ..., h_5) = (0, ..., 0)$, satisfies all constraints. Furthermore, let $s^{(1)}, ..., s^{(5)}$ be the nonnegative slack variables associated with the constraints (4.4.5) through (4.4.9), respectively. Then the

(DLP) can be also written as

$$\max \quad \varepsilon e_{J_+}^T h_3 - \varepsilon e_{J_-}^T h_4$$

$$\text{s.t.} \quad h_1 - h_2 + (\Phi_{J_+,n})^T h_3 + (\Phi_{J_-,n})^T h_4 + (\Phi_{J_0,n})^T h_5 = 0, \qquad (4.4.10)$$

$$s^{(1)} - h_1 - h_2 = e, \qquad (4.4.11)$$

$$s^{(2)} + h_1 = 0, \qquad (4.4.12)$$

$$s^{(3)} + h_2 = 0, \qquad (4.4.13)$$

$$s^{(4)} - h_3 = 0, \qquad (4.4.14)$$

$$s^{(5)} + h_4 = 0, \qquad (4.4.15)$$

$$s^{(1)}, ..., s^{(5)} \geqslant 0.$$

We now prove that when x^* is the unique solution to the problem (4.3.7), the range space of Φ^T, denoted by $\mathcal{R}(\Phi^T)$, must satisfy certain properties.

Theorem 4.4.5 : *If x^* is the unique solution to the problem (4.3.7), then there exist vectors $h_1, h_2 \in R^n$ and $w \in R^m$ satisfying*

$$\begin{cases}
h_2 - h_1 = \Phi^T w \in \mathcal{R}(\Phi^T), \\
(h_1)_i = -1, \ (h_2)_i = 0 & \text{for } x_i^* > 0, \\
(h_1)_i = 0, \ (h_2)_i = -1 & \text{for } x_i^* < 0, \\
(h_1)_i, (h_2)_i < 0, (h_1 + h_2)_i > -1 & \text{for } x_i^* = 0, \\
w_i > 0 & \text{for } i \in \mathcal{A}(x^*) \cap J_+, \\
w_i < 0 & \text{for } i \in \mathcal{A}(x^*) \cap J_-, \\
w_i = 0 & \text{for } i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*).
\end{cases} \qquad (4.4.16)$$

Proof. Assume that x^* is the unique solution to the problem (4.3.7). By Lemma 4.4.4,

$$(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*) \qquad (4.4.17)$$

is the unique solution to the problem (4.4.4), where (α^*, β^*) is given by (4.4.2). Since (4.4.4) has a finite solution and (DLP) is always feasible, by the strict complementarity theory of linear programs, there exists a solution $(h_1, ..., h_5)$ of (DLP) such that the associated slack vectors $s^{(1)}, ..., s^{(5)}$ determined by (4.4.11)–(4.4.15) and the vectors (t, u, v, α, β) given by (4.4.17) are strictly complementary, i.e., these vectors satisfy the following conditions:

$$t^T s^{(1)} = u^T s^{(2)} = v^T s^{(3)} = \alpha^T s^{(4)} = \beta^T s^{(5)} = 0 \qquad (4.4.18)$$

and

$$t + s^{(1)} > 0, \ u + s^{(2)} > 0, \ v + s^{(3)} > 0, \alpha + s^{(4)} > 0, \ \beta + s^{(5)} > 0. \qquad (4.4.19)$$

For the above-mentioned solution $(h_1, ..., h_5)$ of (DLP), let $w \in R^m$ be the vector defined by $w_{J_+} = h_3, w_{J_-} = h_4$ and $w_{J_0} = h_5$. Then it follows from (4.4.10) that

$$h_2 - h_1 = (\Phi_{J_+,n})^T h_3 + (\Phi_{J_-,n})^T h_4 + (\Phi_{J_0,n})^T h_5 = \Phi^T w. \qquad (4.4.20)$$

From (4.4.17), we see that the solution of (4.4.4) satisfies the following properties:

$$
\begin{aligned}
&t_i = x_i^* > 0, &u_i &= 0, &v_i &= 2x_i^* > 0 &&\text{for } x_i^* > 0, \\
&t_i = |x_i^*| > 0, &u_i &= 2|x_i^*| > 0, &v_i &= 0 &&\text{for } x_i^* < 0, \\
&t_i = 0, &u_i &= 0, &v_i &= 0 &&\text{for } x_i^* = 0.
\end{aligned}
$$

Thus from (4.4.18) and (4.4.19), it follows that

$$s_i^{(1)} = 0, \quad s_i^{(2)} > 0, \quad s_i^{(3)} = 0 \quad \text{for } x_i^* > 0,$$

$$s_i^{(1)} = 0, \quad s_i^{(2)} = 0, \quad s_i^{(3)} > 0 \quad \text{for } x_i^* < 0,$$

$$s_i^{(1)} > 0, \quad s_i^{(2)} > 0, \quad s_i^{(3)} > 0 \quad \text{for } x_i^* = 0.$$

From (4.4.11), (4.4.12) and (4.4.13), the above relations imply that

$$(h_1 + h_2)_i = -1, \quad (h_1)_i < 0, \quad (h_2)_i = 0 \quad \text{for } x_i^* > 0,$$

$$(h_1 + h_2)_i = -1, \quad (h_1)_i = 0, \quad (h_2)_i < 0 \quad \text{for } x_i^* < 0,$$

$$(h_1 + h_2)_i > -1, \quad (h_1)_i < 0, \quad (h_2)_i < 0 \quad \text{for } x_i^* = 0.$$

From (4.4.14) and (4.4.15), we see that $s^{(4)} = h_3 \geqslant 0$ and $s^{(5)} = -h_4 \geqslant 0$. Let $\pi(\cdot)$ and $\varrho(\cdot)$ be defined as (4.3.10) and (4.3.11), respectively. It follows from (4.4.2), (4.4.18) and (4.4.19) that

$$(h_3)_{\pi(i)} = s_{\pi(i)}^{(4)} > 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_+, \ (h_3)_{\pi(i)} = s_{\pi(i)}^{(4)} = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(x^*),$$

$$(-h_4)_{\varrho(i)} = s_{\varrho(i)}^{(5)} > 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_-, \ (-h_4)_{\varrho(i)} = s_{\varrho(i)}^{(5)} = 0 \text{ for } i \in \tilde{\mathcal{A}}_-(x^*).$$

By the definition of w (i.e., $w_{J_+} = h_3$, $w_{J_-} = h_4$ and $w_{J_0} = h_5$), the above conditions imply that

$$w_i = (h_3)_{\pi(i)} > 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_+, \ w_i = (h_3)_{\pi(i)} = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(x^*),$$

$$w_i = (h_4)_{\varrho(i)} < 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_-, \ w_i = (h_4)_{\varrho(i)} = 0 \text{ for } i \in \tilde{\mathcal{A}}_-(x^*).$$

Thus, the vector (h_1, h_2, w) satisfies (4.4.20) and the following properties:

$$
\begin{aligned}
(h_1)_i = -1, \ (h_2)_i = 0 && \text{for } x_i^* > 0, \\
(h_1)_i = 0, \ (h_2)_i = -1 && \text{for } x_i^* < 0, \\
(h_1)_i, (h_2)_i < 0, \ (h_1 + h_2)_i > -1 && \text{for } x_i^* = 0, \\
w_i > 0 && \text{for } i \in \mathcal{A}(x^*) \cap J_+, \\
w_i = 0 && \text{for } i \in \tilde{\mathcal{A}}_+(x^*), \\
w_i < 0 && \text{for } i \in \mathcal{A}(x^*) \cap J_-, \\
w_i = 0 && \text{for } i \in \tilde{\mathcal{A}}_-(x^*).
\end{aligned}
$$

Therefore, the condition (4.4.16) is a necessary condition for x^* to be the unique solution of the problem (4.3.7). $\qquad\square$

We now present an equivalent statement for (4.4.16), based on which we will introduce the concept of restricted range space property (RRSP).

Lemma 4.4.6 : *Let $x^* \in R^n$ be a given vector satisfying the constraints of (4.3.7). There exist vectors h_1, h_2 and w satisfying (4.4.16) if and only if there exists a vector $\eta \in \mathcal{R}(\Phi^T)$ satisfying the following two conditions:*

(i) $\eta_i = 1$ for $x_i^* > 0$, $\eta_i = -1$ for $x_i^* < 0$, and $|\eta_i| < 1$ for $x_i^* = 0$;

(ii) $\eta = \Phi^T w$ for some $w \in \mathcal{F}(x^*)$ which is a set defined as

$$
\begin{aligned}
\mathcal{F}(x^*) = \{ w \in R^m : \quad & w_i > 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_+, \ w_i < 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_-, \\
& w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*) \}. \qquad (4.4.21)
\end{aligned}
$$

Proof. Assume that (h_1, h_2, w) satisfies (4.4.16). Setting $\eta = h_2 - h_1$, from (4.4.16), it is easy to see that $\eta \in \mathcal{R}(\Phi^T)$, $\eta_i = 1$ for $x_i^* > 0$, and $\eta_i = -1$ for $x_i^* < 0$. Note that for

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$x_i^* = 0$, we have

$$(h_2 - h_1)_i > (h_2 + h_1)_i > -1, \ (h_2 - h_1)_i < -(h_2 + h_1)_i < 1,$$

which implies that $|\eta_i| = |(h_2 - h_1)_i| < 1$ for $x_i^* = 0$. Therefore the condition (i) holds. Also, the condition (ii) follows from (4.4.16) immediately.

Conversely, for a given x^*, we assume that the conditions (i) and (ii) hold, i.e., there exist vectors $w \in \mathcal{F}(x^*)$ and $\eta \in R^n$ satisfying that $\eta = \Phi^T w$, $\eta_i = 1$ for $x_i^* > 0$, $\eta_i = -1$ for $x_i^* < 0$, and $|\eta_i| < 1$ for $x_i^* = 0$. We now construct vectors h_1, h_2 so that (h_1, h_2, w) satisfies (4.4.16). First, set $(h_1)_i = -\eta_i = -1$ and $(h_2)_i = 0$ for $x_i^* > 0$, and set $(h_1)_i = 0$ and $(h_2)_i = \eta_i = -1$ for $x_i^* < 0$. For those components corresponding to $x_i^* = 0$, since $|\eta_i| < 1$, we have only two cases:

Case 1: $-1 < \eta_i < 0$. In this case, $\frac{\eta_i - 1}{2} < \eta_i < 0$. Let μ be a fixed number in the interval $\left(\frac{\eta_i - 1}{2}, \eta_i\right)$.

Case 2: $0 \leqslant \eta_i < 1$. In this case, $\frac{\eta_i - 1}{2} < 0$. Let μ be a fixed number in the interval $\left(\frac{\eta_i - 1}{2}, 0\right)$.

For each of the above cases, we set $(h_2)_i = \mu$ and $(h_1)_i = \mu - \eta_i$. Then we see that $(h_1)_i < 0, (h_2)_i < 0, (h_2 + h_1)_i = 2\mu - \eta_i > -1$ and $(h_2 - h_1)_i = \eta_i$. Clearly, the constructed vector (h_1, h_2) satisfies that $\eta = h_2 - h_1$. This construction, together with $\eta = \Phi^T w$ where $w \in \mathcal{F}(x^*)$, implies that the vectors h_1, h_2 and w satisfy (4.4.16). $\qquad \square$

For the standard basis pursuit $\min\{\|z\|_1 : Az = b\}$ where the linear system is under-determined, Zhao [132] has shown that *if x is the unique solution to the standard basis pursuit then there exists a vector η satisfying the following conditions: (i) $\eta = A^T w$ for some $w \in R^m$; (ii) $\eta_i = 1$ for $x_i > 0$, $\eta_i = -1$ for $x_i < 0$, and $|\eta_i| < 1$ for $x_i = 0$.* For this situation, there is no restriction on $w \in R^n$, even when this result has been generalized

to the nonnegative sparse signals satisfying $Ax = b$ and $x \geqslant 0$ (see [133]). However, from the above analysis, the 1-bit basis pursuit (4.3.7) with mixed (equality and inequality) constraints is more complicated than the standard basis pursuit. The necessary uniqueness condition for the solution x^* of the 1-bit basis pursuit has a restricted choice of w, which is confined to the set (4.4.21). Motivated by the above analysis, we introduce the following concept.

Definition 4.4.7 (RRSP of Φ^T at x^*) : *Given the partition (J_+, J_-, J_0) of the set $\{1, ..., m\}$ as (4.3.2) and a point $x^* \in R^n$ satisfying the constraints of (4.3.7), we say that Φ^T satisfies the restricted range space property (RRSP) at x^* if there exist vectors $\eta \in \mathcal{R}(\Phi^T)$ and $w \in \mathcal{F}(x^*)$, defined by (4.4.21), such that $\eta = \Phi^T w$ and*

$$\begin{cases} \eta_i = 1 & \text{for } x_i^* > 0, \\ \eta_i = -1 & \text{for } x_i^* < 0, \\ |\eta_i| < 1 & \text{for } x_i^* = 0. \end{cases}$$

4.4.2   Necessary condition (II): Full column rank

The RRSP at x^* is a necessary condition for x^* to be unique solution of (4.3.7), but it is not sufficient to ensure the uniqueness of x^*. (This has been pointed out in [132] for the standard basis pursuit.) We need to develop another necessary condition (called the full-column-rank property) which, combined with the RRSP at x^*, turns out to be sufficient for the uniqueness of x^*, as shown in the next subsection. We now develop such a necessary condition. Still, we assume that x^* is the unique solution to the 1-bit basis pursuit (4.3.7). Denote the index sets of positive and negative components of x^* by

$$S_+ = \{i : x_i^* > 0\}, \text{ and } S_- = \{i : x_i^* < 0\}.$$

First, the following lemma is obvious.

Lemma 4.4.8 : *Consider the matrix*

$$
G(x^*) = \begin{bmatrix}
\Phi_{\mathcal{A}(x^*)\cap J_+,S_+} & \Phi_{\mathcal{A}(x^*)\cap J_+,S_-} & 0 & 0 \\
\Phi_{\mathcal{A}(x^*)\cap J_-,S_+} & \Phi_{\mathcal{A}(x^*)\cap J_-,S_-} & 0 & 0 \\
\Phi_{J_0,S_+} & \Phi_{J_0,S_-} & 0 & 0 \\
\Phi_{\widetilde{\mathcal{A}}_+(x^*),S_+} & \Phi_{\widetilde{\mathcal{A}}_+(x^*),S_-} & -I^{(1)} & 0 \\
\Phi_{\widetilde{\mathcal{A}}_-(x^*),S_+} & \Phi_{\widetilde{\mathcal{A}}_-(x^*),S_-} & 0 & I^{(2)}
\end{bmatrix}
\tag{4.4.22}
$$

where $I^{(1)}$ and $I^{(2)}$ are $|\widetilde{\mathcal{A}}_+(x^*)| \times |\widetilde{\mathcal{A}}_+(x^*)|$ and $|\widetilde{\mathcal{A}}_-(x^*)| \times |\widetilde{\mathcal{A}}_-(x^*)|$ identity matrices, respectively. Then $G(x^*)$ has a full-column rank if and only if the matrix

$$
H(x^*) = \begin{bmatrix}
\Phi_{\mathcal{A}(x^*)\cap J_+,S_+} & \Phi_{\mathcal{A}(x^*)\cap J_+,S_-} \\
\Phi_{\mathcal{A}(x^*)\cap J_-,S_+} & \Phi_{\mathcal{A}(x^*)\cap J_-,S_-} \\
\Phi_{J_0,S_+} & \Phi_{J_0,S_-}
\end{bmatrix}
\tag{4.4.23}
$$

has a full-column rank.

We now prove that $H(x^*)$ having a full-column rank is a desired necessary condition for x^* to be unique.

Theorem 4.4.9 : *If x^* is the unique solution to the problem (4.3.7), then the matrix $H(x^*)$, defined by (4.4.23), has a full-column rank.*

Proof. Assume the contrary that $H(x^*)$ has linearly dependent columns. By Lemma 4.4.8, the matrix $G(x^*)$ defined by (4.4.22) also has linearly dependent columns. Thus there exists a nonzero vector $d = (d_1, d_2, d_3, d_4) \neq 0$ such that

$$
G(x^*)d = 0.
$$

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By the structure of $G(x^*)$, it is easy to see that $(d_1, d_2) \neq 0$, since otherwise d must be zero. Since x^* is the unique solution to the problem (4.3.7), there exist nonnegative variables α^* and β^*, determined by (4.4.2), such that (x^*, α^*, β^*) is the unique solution to the problem (4.4.1) with the least objective value $\|x^*\|_1$. The vector (x^*, α^*, β^*) satisfies that

$$\Phi_{J_+, n} x^* - \alpha^* = \varepsilon e_{J_+}, \quad \Phi_{J_-, n} x^* + \beta^* = -\varepsilon e_{J_-}, \quad \Phi_{J_0, n} x^* = 0. \tag{4.4.24}$$

From (4.4.2), we have

$$\alpha^*_{\pi(i)} = 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_+, \quad \beta^*_{\varrho(i)} = 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_-. \tag{4.4.25}$$

From (4.3.8) and (4.3.9), we see that

$$J_+ = \widetilde{\mathcal{A}}_+(x^*) \cup (\mathcal{A}(x^*) \cap J_+), \quad J_- = \widetilde{\mathcal{A}}_-(x^*) \cup (\mathcal{A}(x^*) \cap J_-). \tag{4.4.26}$$

Thus, by (4.4.25) and (4.4.26), eliminating the zero components of x^*, α^* and β^* from the system (4.4.24) leads to

$$
\begin{aligned}
\Phi_{\mathcal{A}(x^*) \cap J_+, S_+} x^*_{S_+} + \Phi_{\mathcal{A}(x^*) \cap J_+, S_-} x^*_{S_-} &= \varepsilon e_{\mathcal{A}(x^*) \cap J_+}, \\
\Phi_{\mathcal{A}(x^*) \cap J_-, S_+} x^*_{S_+} + \Phi_{\mathcal{A}(x^*) \cap J_-, S_-} x^*_{S_-} &= -\varepsilon e_{\mathcal{A}(x^*) \cap J_-}, \\
\Phi_{J_0, S_+} x^*_{S_+} + \Phi_{J_0, S_-} x^*_{S_-} &= 0, \\
\Phi_{\widetilde{\mathcal{A}}_+(x^*), S_+} x^*_{S_+} + \Phi_{\widetilde{\mathcal{A}}_+(x^*), S_-} x^*_{S_-} - \alpha^*_{\pi(\widetilde{\mathcal{A}}_+(x^*))} &= \varepsilon e_{\widetilde{\mathcal{A}}_+(x^*)}, \\
\Phi_{\widetilde{\mathcal{A}}_-(x^*), S_+} x^*_{S_+} + \Phi_{\widetilde{\mathcal{A}}_-(x^*), S_-} x^*_{S_-} + \beta^*_{\varrho(\widetilde{\mathcal{A}}_-(x^*))} &= -\varepsilon e_{\widetilde{\mathcal{A}}_-(x^*)},
\end{aligned}
\tag{4.4.27}
$$

where $\alpha^*_{\pi(\widetilde{\mathcal{A}}_+(x^*))}$ denotes the subvector of α^* obtained by deleting the components $\alpha^*_{\pi(i)}$ with $i \in J_+ \backslash \widetilde{\mathcal{A}}_+(x^*)$, and $\beta^*_{\varrho(\widetilde{\mathcal{A}}_-(x^*))}$ is the subvector of β^* formed by deleting the components $\beta^*_{\varrho(i)}$ with $i \in J_- \backslash \widetilde{\mathcal{A}}_-(x^*)$. Thus the vector $Z^* = (z_1^*, z_2^*, z_3^*, z_4^*)$ with $z_1^* = x^*_{S_+} > 0$, $z_2^* = x^*_{S_-} < 0$, $z_3^* = \alpha^*_{\pi(\widetilde{\mathcal{A}}_+(x^*))} > 0$ and $z_4^* = \beta^*_{\varrho(\widetilde{\mathcal{A}}_-(x^*))} > 0$ is a solution to the following

system:

$$
G(x^*)
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{pmatrix}
=
\begin{bmatrix}
\varepsilon e_{\mathcal{A}(x^*)\cap J_+} \\
-\varepsilon e_{\mathcal{A}(x^*)\cap J_-} \\
0 \\
\varepsilon e_{\tilde{\mathcal{A}}_+(x^*)} \\
-\varepsilon e_{\tilde{\mathcal{A}}_-(x^*)}
\end{bmatrix}.
\tag{4.4.28}
$$

Based on the vectors Z^* and d, we now construct another solution to the problem (4.4.1). In fact, note that $z_1^* > 0$, $z_2^* < 0$, $z_3^* > 0$, $z_4^* > 0$. There exists a small number $\delta > 0$ such that for any $\lambda \neq 0$ with absolute value $|\lambda| \in (0, \delta)$, the vector $Z^* + \lambda d$ satisfies

$$
\widetilde{z}_1 = z_1^* + \lambda d_1 > 0, \ \widetilde{z}_2 = z_2^* + \lambda d_2 < 0,
$$

$$
\widetilde{z}_3 = z_3^* + \lambda d_3 > 0, \ \widetilde{z}_4 = z_4^* + \lambda d_4 > 0.
$$

In particular, let $\lambda^* \neq 0$ satisfy $|\lambda^*| \in (0, \delta)$ and the following condition:

$$
\lambda^*(e_{S_+}^T d_1 - e_{S_-}^T d_2) \leqslant 0.
\tag{4.4.29}
$$

Denote by $\widetilde{Z} = (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3, \widetilde{z}_4) = Z^* + \lambda^* d$. Since $G(x^*)d = 0$, the vector $\widetilde{Z}$ is also a solution to the system (4.4.28). Obviously, $\widetilde{Z} \neq Z^*$ as $\lambda^* \neq 0$ and $d \neq 0$. Let $(\tilde{x}, \tilde{\alpha}, \tilde{\beta}) \in R^n \times R_+^{|J_+|} \times R_+^{|J_-|}$ be defined as

$$
\tilde{x}_{S_+} = \widetilde{z}_1, \ \tilde{x}_{S_-} = \widetilde{z}_2, \ \tilde{\alpha}_{\pi(\tilde{\mathcal{A}}_+(x^*))} = \widetilde{z}_3, \ \tilde{\beta}_{\varrho(\tilde{\mathcal{A}}_-(x^*))} = \widetilde{z}_4
$$

and let all remaining components of $\tilde{x}, \tilde{\alpha}$ and $\tilde{\beta}$ be zeros. Then $(\tilde{x}, \tilde{\alpha}, \tilde{\beta})$ satisfies all constraints of the problem (4.4.1). By the construction, we see that $\tilde{x} \neq x^*$ since $\lambda^* \neq 0$

and $(d_1, d_2) \neq 0$. Moreover, we have

$$
\begin{aligned}
\|\tilde{x}\|_1 &= e_{S_+}^T (x_{S_+}^* + \lambda d_1) - e_{S_-}^T (x_{S_-}^* + \lambda d_2), \\
&= e_{S_+}^T x_{S_+}^* - e_{S_-}^T x_{S_-}^* + \lambda e_{S_+}^T d_1 - \lambda e_{S_-}^T d_2, \\
&= \|x^*\|_1 + \lambda (e_{S_+}^T d_1 - e_{S_-}^T d_2), \\
&\leqslant \|x^*\|_1,
\end{aligned}
$$

where the inequality follows from (4.4.29). As $\|x^*\|_1$ is the least objective value of the problem (4.3.7), the above relation implies that $\tilde{x}$ is also a solution to this problem, contradicting to the uniqueness of x^*. Hence, the matrix $H(x^*)$ must have a full-column rank. $\qquad\square$

Combining the aforementioned two necessary conditions yields the next theorem.

Theorem 4.4.10 : *If x^* is the unique solution to the problem (4.3.7), then*

(i) $H(x^*) = \begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+, S_+} & \Phi_{\mathcal{A}(x^*) \cap J_+, S_-} \\ \Phi_{\mathcal{A}(x^*) \cap J_-, S_+} & \Phi_{\mathcal{A}(x^*) \cap J_-, S_-} \\ \Phi_{J_0, S_+} & \Phi_{J_0, S_-} \end{bmatrix}$ *has a full-column rank, and*

(ii) the RRSP of Φ^T holds at x^*.

4.4.3 Sufficient conditions

In this section, we show that the converse of Theorem 4.4.10 is also valid, i.e., the RRSP of Φ^T at x^* combined with the full-column-rank property of $H(x^*)$ specified in Theorem 4.4.10 is also a sufficient condition for the uniqueness of x^*. We start with a property of (DLP).

Lemma 4.4.11 : *Let x^* be a feasible solution to the problem (4.3.7). If the vector $(h_1, h_2, w) \in R^n \times R^n \times R^m$ satisfies that*

$$\begin{cases} (h_1)_i = -1, \ (h_2)_i = 0 & \text{for } x_i^* > 0, \\ (h_1)_i = 0, \ (h_2)_i = -1 & \text{for } x_i^* < 0, \\ (h_1)_i < 0, \ (h_2)_i < 0, \ (h_1 + h_2)_i > -1 & \text{for } x_i^* = 0, \\ h_2 - h_1 = \Phi^T w, \\ w_{J_+} \geqslant 0, \\ w_{J_-} \leqslant 0, \\ w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*), \end{cases} \quad (4.4.30)$$

then the vector $(h_1, h_2, h_3, h_4, h_5)$, with $h_3 = w_{J_+}, h_4 = w_{J_-}$ and $h_5 = w_{J_0}$, is an optimal solution to the problem (DLP). Moreover, x^* must be an optimal solution to the problem (4.3.7).

Proof. Let $(h_1, ..., h_5)$ be the vector satisfying the condition (4.4.30). It is evident that $(h_1, ..., h_5)$ satisfies the constraints of (DLP). We now further prove that $(h_1, ..., h_5)$ is a solution of (DLP). Since $\Phi^T w = h_2 - h_1$, we have

$$(\Phi^T w)^T x^* = (h_2 - h_1)^T x^* = \|x^*\|_1,$$

where the second equality follows from the choices of h_1 and h_2. Thus,

$$\begin{aligned} \|x^*\|_1 = w^T \Phi x^* &= w_{J_+}^T \Phi_{J_+,n} x^* + w_{J_-}^T \Phi_{J_-,n} x^* + w_{J_0}^T \Phi_{J_0,n} x^* \\ &= h_3^T \Phi_{J_+,n} x^* + h_4^T \Phi_{J_-,n} x^* + h_5^T \Phi_{J_0,n} x^*. \end{aligned}$$

Since $\Phi_{J_0,n} x^* = 0$, we have

$$h_3{}^T \Phi_{J_+,n} x^* + h_4{}^T \Phi_{J_-,n} x^* = \|x^*\|_1. \tag{4.4.31}$$

Note that

$$(\Phi x^*)_i = \varepsilon \text{ for } i \in \mathcal{A}(x^*) \cap J_+, \ (\Phi x^*)_i = -\varepsilon \text{ for } i \in \mathcal{A}(x^*) \cap J_-. \tag{4.4.32}$$

We also note that $h_3 = w_{J_+}$, $h_4 = w_{J_-}$, and $w_i = 0$ for $i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*)$. This implies that

$$(h_3)_{\pi(i)} = w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(x^*), \ (h_4)_{\varrho(i)} = w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_-(x^*). \tag{4.4.33}$$

By (4.4.26), (4.4.32) and (4.4.33), the equality (4.4.31) is reduced to

$$\sum_{i \in \mathcal{A}(x^*) \cap J_+} (h_3)_{\pi(i)} (\Phi x^*)_i + \sum_{i \in \mathcal{A}(x^*) \cap J_-} (h_4)_{\varrho(i)} (\Phi x^*)_i = \sum_{i \in \mathcal{A}(x^*) \cap J_+} \varepsilon (h_3)_{\pi(i)} - \sum_{i \in \mathcal{A}(x^*) \cap J_-} \varepsilon (h_4)_{\varrho(i)}$$
$$= \|x^*\|_1, \tag{4.4.34}$$

which, together with (4.4.33) again, implies that

$$\varepsilon e_{J_+}^T h_3 - \varepsilon e_{J_-}^T h_4 = \|x^*\|_1.$$

Thus the objective value of (DLP) at $(h_1, ..., h_5)$ coincides with that of its primal problem (4.3.7) at x^*. By strong duality of linear programs, $(h_1, ..., h_5)$ must be an optimal solution to the problem (DLP), and x^* must be an optimal solution to the problem (4.3.7) as well.$\square$

We now prove the desired sufficient condition for the uniqueness of solutions to the 1-bit basis pursuit.

Theorem 4.4.12 : *Let x^* be a feasible solution to the problem (4.3.7). Assume that the following conditions hold: (i) The RRSP of Φ^T holds at x^*; (ii) The matrix $H(x^*)$, defined by (4.4.23), has a full-column rank. Then x^* is the unique solution to the problem (4.3.7).*

Proof. By the assumption of the theorem, the RRSP of Φ^T holds at x^*. Then by Lemma 4.4.6, there exists a vector $(h_1, h_2, w) \in R^n \times R^n \times R^m$ satisfying (4.4.16), which implies that the condition (4.4.30) holds. As x^* is a feasible solution to the problem (4.3.7), by Lemma 4.4.11, $(h_1, h_2, h_3, h_4, h_5)$ with $h_3 = w_{J_+}, h_4 = w_{J_-}$ and $h_5 = w_{J_0}$ is an optimal solution to the problem (DLP). At this solution, let the slack vectors $s^{(1)}, ..., s^{(5)}$ be given as (4.4.11)–(4.4.15). Also, from Lemma 4.4.11, x^* is an optimal solution to the problem (4.3.7). Based on x^*, we can construct an optimal solution

$$(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*),$$

where (α^*, β^*) is given by (4.4.2), to the problem (4.4.4). We now further show that x^* is the unique solution to the problem (4.3.7).

Since the vector (x^*, α^*, β^*) satisfies (4.4.24), as shown in the proof of Theorem 4.4.9, the system (4.4.24) can be written as (4.4.27), i.e.,

$$G(x^*) \begin{bmatrix} x^*_{S_+} \\ x^*_{S_-} \\ \alpha^*_{\pi(\tilde{A}_+(x^*))} \\ \beta^*_{\varrho(\tilde{A}_-(x^*))} \end{bmatrix} = \begin{bmatrix} \varepsilon e_{\mathcal{A}(x^*) \cap J_+} \\ -\varepsilon e_{\mathcal{A}(x^*) \cap J_-} \\ 0 \\ \varepsilon e_{\tilde{A}_+(x^*)} \\ -\varepsilon e_{\tilde{A}_-(x^*)} \end{bmatrix} \tag{4.4.35}$$

where the coefficient matrix $G(x^*)$ is given by (4.4.22). Let $(\widetilde{x}, \widetilde{t}, \widetilde{u}, \widetilde{v}, \widetilde{\alpha}, \widetilde{\beta})$ be an arbitrary solution to the problem (4.4.4). Then, we have $\widetilde{t} = |\widetilde{x}|, \widetilde{u} = |\widetilde{x}| - \widetilde{x}$ and $\widetilde{v} = |\widetilde{x}| + \widetilde{x}$. By com-

plementary slackness property of linear programs, the nonnegative vectors $(\widetilde{t}, \widetilde{u}, \widetilde{v}, \widetilde{\alpha}, \widetilde{\beta})$ and $(s^{(1)}, ..., s^{(5)})$ are complementary, i.e.,

$$\widetilde{t}^T s^{(1)} = \widetilde{u}^T s^{(2)} = \widetilde{v}^T s^{(3)} = \widetilde{\alpha}^T s^{(4)} = \widetilde{\beta}^T s^{(5)} = 0. \tag{4.4.36}$$

As (h_1, h_2, w) satisfies (4.4.16), the vector (h_1, h_2) satisfies that $(h_1)_i = -1 < 0$ for all $x_i^* > 0$, $(h_2)_i = -1 < 0$ for all $x_i^* < 0$ and that $(h_1 + h_2)_i > -1$, $(h_1)_i < 0$ and $(h_2)_i < 0$ for all $x_i^* = 0$. By the choice of (h_1, h_2) and $(s^{(1)}, ..., s^{(5)})$, we see that the following components of slack variables are positive:

$$s_i^{(1)} = 1 + (h_1 + h_2)_i > 0 \qquad \text{for } x_i^* = 0,$$
$$s_{\pi(i)}^{(4)} = (h_3)_{\pi(i)} = w_i > 0 \qquad \text{for } i \in \mathcal{A}(x^*) \cap J_+,$$
$$s_{\varrho(i)}^{(5)} = -(h_4)_{\varrho(i)} = -w_i > 0 \quad \text{for } i \in \mathcal{A}(x^*) \cap J_-.$$

The positiveness of these components, together with (4.4.36), implies that

$$\begin{cases} \widetilde{t}_i = 0 & \text{for } x_i^* = 0, \\ \widetilde{\alpha}_{\pi(i)} = 0 & \text{for } i \in \mathcal{A}(x^*) \cap J_+, \\ \widetilde{\beta}_{\varrho(i)} = 0 & \text{for } i \in \mathcal{A}(x^*) \cap J_-. \end{cases} \tag{4.4.37}$$

Still, we denote by $S_+ = \{i : x_i^* > 0\}$ and $S_- = \{i : x_i^* < 0\}$. Since $\widetilde{t} = |\widetilde{x}|$, the first relation in (4.4.37) implies that $\widetilde{x}_i = 0$ for all $i \notin S_+ \cup S_-$. Note that

$$\Phi_{J_+, n} \widetilde{x} - \widetilde{\alpha} = \varepsilon e_{J_+}, \ \ \Phi_{J_-, n} \widetilde{x} + \widetilde{\beta} = -\varepsilon e_{J_-}, \ \ \Phi_{J_0, n} \widetilde{x} = 0.$$

Since $\widetilde{x}_i = 0$ for all $i \notin S_+ \cup S_-$, the above system is reduced to

$$
\begin{cases}
\Phi_{J_+,S_+}\widetilde{x}_{S_+} + \Phi_{J_+,S_-}\widetilde{x}_{S_-} - \widetilde{\alpha} = \varepsilon e_{J_+}, \\
\Phi_{J_-,S_+}\widetilde{x}_{S_+} + \Phi_{J_-,S_-}\widetilde{x}_{S_-} + \widetilde{\beta} = -\varepsilon e_{J_-}, \\
\Phi_{J_0,S_+}\widetilde{x}_{S_+} + \Phi_{J_0,S_-}\widetilde{x}_{S_-} = 0.
\end{cases}
$$

By (4.4.26) and (4.4.37), splitting up the first two equalities of the above system into two, respectively, the above system is equivalent to

$$
\Phi_{\mathcal{A}(x^*)\cap J_+,S_+}\widetilde{x}_{S_+} + \Phi_{\mathcal{A}(x^*)\cap J_+,S_-}\widetilde{x}_{S_-} = \varepsilon e_{\mathcal{A}(x^*)\cap J_+},
$$
$$
\Phi_{\tilde{\mathcal{A}}_+(x^*),S_+}\widetilde{x}_{S_+} + \Phi_{\tilde{\mathcal{A}}_+(x^*),S_-}\widetilde{x}_{S_-} - \widetilde{\alpha}_{\pi(\tilde{\mathcal{A}}_+(x^*))} = \varepsilon e_{\tilde{\mathcal{A}}_+(x^*)},
$$
$$
\Phi_{\mathcal{A}(x^*)\cap J_-,S_+}\widetilde{x}_{S_+} + \Phi_{\mathcal{A}(x^*)\cap J_-,S_-}\widetilde{x}_{S_-} = -\varepsilon e_{\mathcal{A}(x^*)\cap J_-},
$$
$$
\Phi_{\tilde{\mathcal{A}}_-(x^*),S_+}\widetilde{x}_{S_+} + \Phi_{\tilde{\mathcal{A}}_-(x^*),S_-}\widetilde{x}_{S_-} + \widetilde{\beta}_{\varrho(\tilde{\mathcal{A}}_-(x^*))} = -\varepsilon e_{\tilde{\mathcal{A}}_-(x^*)},
$$
$$
\Phi_{J_0,S_+}\widetilde{x}_{S_+} + \Phi_{J_0,S_-}\widetilde{x}_{S_-} = 0,
$$

which can be written as

$$
G(x^*)
\begin{bmatrix}
\widetilde{x}_{S_+} \\
\widetilde{x}_{S_-} \\
\widetilde{\alpha}_{\pi(\tilde{\mathcal{A}}_+(x^*))} \\
\widetilde{\beta}_{\varrho(\tilde{\mathcal{A}}_-(x^*))}
\end{bmatrix}
=
\begin{bmatrix}
\varepsilon e_{\mathcal{A}(x^*)\cap J_+} \\
-\varepsilon e_{\mathcal{A}(x^*)\cap J_-} \\
0 \\
\varepsilon e_{\tilde{\mathcal{A}}_+(x^*)} \\
-\varepsilon e_{\tilde{\mathcal{A}}_-(x^*)}
\end{bmatrix},
\tag{4.4.38}
$$

where $G(x^*)$ is given by (4.4.22). By the assumption of the theorem, the

$$
H(x^*) =
\begin{bmatrix}
\Phi_{\mathcal{A}(x^*)\cap J_+,S_+} & \Phi_{\mathcal{A}(x^*)\cap J_+,S_-} \\
\Phi_{\mathcal{A}(x^*)\cap J_-,S_+} & \Phi_{\mathcal{A}(x^*)\cap J_-,S_-} \\
\Phi_{J_0,S_+} & \Phi_{J_0,S_-}
\end{bmatrix}
$$

has a full-column rank. By Lemma 4.4.8, the matrix $G(x^*)$ has a full-column rank. Thus it follows from (4.4.35) and (4.4.38) that $\widetilde{x}_{S_+} = x^*_{S_+}$ and $\widetilde{x}_{S_-} = x^*_{S_-}$ which, together with the fact $\widetilde{x}_i = 0$ for all $i \notin S_+ \cup S_-$, implies that $\widetilde{x} = x^*$. By assumption, $(\widetilde{x}, \widetilde{t}, \widetilde{u}, \widetilde{v}, \widetilde{\alpha}, \widetilde{\beta})$ is an arbitrary solution to (4.4.4). Thus $(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*)$ is the unique solution to the problem (4.4.4), and hence (by Lemma 4.4.4) x^* is the unique solution of the problem (4.3.7). $\qquad\square$

4.4.4   A necessary and sufficient condition

Based on the results developed in sections 4.2–4.3, we summarize the necessary and sufficient conditions for the uniqueness of the solution to the problem (4.3.7) as follows.

Theorem 4.4.13 (Necessary and sufficient condition) *x^* is the unique solution to the 1-bit basis pursuit (4.3.7) if and only if the RRSP of Φ^T holds at x^* and the matrix $H(x^*)$, defined as (4.4.23), has a full-column rank.*

The uniqueness of solutions to a decoding method like (4.3.7) is an important property needed in signal reconstruction. Theorem 4.4.13, together with the new concept of the RRSP of order K that will be introduced in the next section, makes it possible to develop a recovery theory for sparse signals from 1-bit measurements (see section 4.5.2 for details).

Remark 4.4.14 *:*

(i) *The importance of the restricted range space property (RRSP) goes beyond characterizing the uniqueness of solutions of 1-bit basis pursuit. The RRSP-based analysis provides an insight into a certain level of sparse recovery from a linear system with inequality constraints or with mixed equality and inequality constraints, in which case the null space property has lost the ability to identify a sparsest solution to the linear system.*

(ii) *1-bit basis pursuit is a special case of partial sparse recovery. Specifically, introducing variables $\alpha \in R_+^{|J_+|}$ and $\beta \in R_+^{|J_-|}$ to the problem (4.3.7), one has the problem (4.4.1), i.e.,*

$$
\begin{aligned}
\min \quad & \|x\|_1, \\
\text{s.t.} \quad & \Phi_{J_+,n} x - \alpha = \varepsilon e_{J_+}, \\
& \Phi_{J_-,n} x + \beta = -\varepsilon e_{J_-}, \\
& \Phi_{J_0,n} x = 0, \\
& \alpha \geqslant 0, \ \beta \geqslant 0.
\end{aligned}
$$

in matrix form, which can be stated as

$$
\begin{aligned}
\min \quad & \|x\|_1, \\
\text{s.t.} \quad & \Phi x + \begin{bmatrix} -I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \varepsilon e_{J_+} \\ -\varepsilon e_{J_-} \\ 0 \end{bmatrix}, \\
& \alpha \geqslant 0, \ \beta \geqslant 0.
\end{aligned}
\tag{4.4.39}
$$

Let $z = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in R_+^{|J_+|+|J_-|}$, $B = \begin{bmatrix} -I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}$ *and* $b = \begin{bmatrix} \varepsilon e_{J_+} \\ -\varepsilon e_{J_-} \\ 0 \end{bmatrix}$, *then the problem (4.4.39) becomes*

$$
\begin{aligned}
\min \quad & \|x\|_1, \\
\text{s.t.} \quad & \Phi x + B z = b, \\
& z \geqslant 0,
\end{aligned}
$$

where the null space of B^T is nonzero (This satisfies the assumption for the partial ℓ_0-minimization). Hence, the RRSP and Full column rank based analysis of uniqueness conditions for solutions of 1-bit basis pursuit can be modified to establish uniqueness conditions for solutions of partial sparse recovery.

4.5 Nonuniform and uniform support recovery via 1-bit basis pursuit

4.5.1 Nonuniform support recovery

The main purpose of this section is to develop the nonuniform support recovery conditions for 1-bit compressive sensing. Let us begin with a certain full-rank property of the submatrix of Φ associated with a (sparsest) solution of the ℓ_0-problem (4.3.6).

Lemma 4.5.1 *: Let x^* be a sparsest solution of ℓ_0-problem (4.3.6) and let $S_+ = \{i : x_i^* > 0\}$ and $S_- = \{i : x_i^* < 0\}$. Then the matrix*

$$\widetilde{H}(x^*) = \begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+, S_+} & \Phi_{\mathcal{A}(x^*) \cap J_+, S_-} \\[6pt] \Phi_{\mathcal{A}(x^*) \cap J_-, S_+} & \Phi_{\mathcal{A}(x^*) \cap J_-, S_-} \\[6pt] \Phi_{J_0, S_+} & \Phi_{J_0, S_-} \\[6pt] \Phi_{\tilde{\mathcal{A}}_+(x^*), S_+} & \Phi_{\tilde{\mathcal{A}}_+(x^*), S_-} \\[6pt] \Phi_{\tilde{\mathcal{A}}_-(x^*), S_+} & \Phi_{\tilde{\mathcal{A}}_-(x^*), S_-} \end{bmatrix}$$

has a full-column rank. Moreover, if any sparsest solution of (4.3.6) is of the form Px^*, for some factor p satisfying $|p_i| > 0$ for $i \in Supp(x^*)$ and $p_i = 0$ for $i \notin Supp(x^*)$, where

$P = diag(p) \in R^{n \times n}$, then the matrix

$$
H' = \begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+, Supp(x^*)} \\ \Phi_{\mathcal{A}(x^*) \cap J_-, Supp(x^*)} \\ \Phi_{J_0, Supp(x^*)} \end{bmatrix}
$$

has a full-column rank, which implies that $H(x^*)$ defined by (4.4.23) has a full-column rank.

Proof. By assumption, x^* is a sparsest solution to the following system

$$
\Phi_{J_+,n} x^* \geqslant \varepsilon e_{J_+}, \ \ \Phi_{J_-,n} x^* \leqslant -\varepsilon e_{J_-}, \ \ \Phi_{J_0,n} x^* = 0. \tag{4.5.1}
$$

Without loss of generality, we assume that $\mathcal{A}(x^*) \neq \emptyset$ which implies that either $\mathcal{A}(x^*) \cap J_+ \neq \emptyset$ or $\mathcal{A}(x^*) \cap J_- \neq \emptyset$. This can be always guaranteed by taking a scalar multiplication operation of the vector x^*, if necessary. Including α^* and β^*, given by (4.4.2), into the system (4.5.1) leads to

$$
\Phi_{J_+,n} x^* - \alpha^* = \varepsilon e_{J_+}, \ \ \Phi_{J_-,n} x^* + \beta^* = -\varepsilon e_{J_-}, \ \ \Phi_{J_0,n} x^* = 0. \tag{4.5.2}
$$

By eliminating the zero components of x^*, the system (4.5.2) is equivalent to

$$
\begin{aligned}
\Phi_{J_+,S_+} x^*_{S_+} + \Phi_{J_+,S_-} x^*_{S_-} - \alpha^* &= \varepsilon e_{J_+}, \\
\Phi_{J_-,S_+} x^*_{S_+} + \Phi_{J_-,S_-} x^*_{S_-} + \beta^* &= -\varepsilon e_{J_-}, \\
\Phi_{J_0,S_+} x_{S_+} + \Phi_{J_0,S_-} x^*_{S_-} &= 0.
\end{aligned} \tag{4.5.3}
$$

Since x^* is a sparsest solution of (4.3.6), it is not very difficult to see that the coefficient

matrix

$$\widehat{H} = \begin{bmatrix} \Phi_{J_+,S_+} & \Phi_{J_+,S_-} \\ \Phi_{J_-,S_+} & \Phi_{J_-,S_-} \\ \Phi_{J_0,S_+} & \Phi_{J_0,S_-} \end{bmatrix}$$

has a full-column rank, since otherwise at least one column of $\widehat{H}$ can be represented by its other columns, and hence the system (4.5.3), which is equivalent to (4.5.1), has a solution sparser than x^*. By (4.4.26), performing row permutations on $\widehat{H}$ if necessary, we obtain the following matrix

$$\widetilde{H}(x^*) = \begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+, S_+} & \Phi_{\mathcal{A}(x^*) \cap J_+, S_-} \\ \Phi_{\mathcal{A}(x^*) \cap J_-, S_+} & \Phi_{\mathcal{A}(x^*) \cap J_-, S_-} \\ \Phi_{J_0, S_+} & \Phi_{J_0, S_-} \\ \Phi_{\tilde{\mathcal{A}}_+(x^*), S_+} & \Phi_{\tilde{\mathcal{A}}_+(x^*), S_-} \\ \Phi_{\tilde{\mathcal{A}}_-(x^*), S_+} & \Phi_{\tilde{\mathcal{A}}_-(x^*), S_-} \end{bmatrix}.$$

Since row permutations will not change the column rank of $\widehat{H}$, $\widetilde{H}(x^*)$ has a full-column rank.

Moreover, if there exists a factor p satisfying $|p_i| > 0$ for $i \in \mathrm{Supp}(x^*)$ and $p_i = 0$ for $i \notin \mathrm{Supp}(x^*)$ such that Px^* is another sparsest solution to (4.3.6), where $P = \mathrm{diag}(p)$, we can further prove that H' given as

$$\begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+, S} \\ \Phi_{\mathcal{A}(x^*) \cap J_-, S} \\ \Phi_{J_0, S} \end{bmatrix}$$

has a full-column rank, where $S = \mathrm{Supp}(x^*) = S_+ \cup S_-$.

We prove this by contradiction. Assume that the columns of H' are linearly dependent. Then, there exists a nonzero vector $u \in R^{|S|}$ satisfying $H'u = 0$. Since $u \neq 0$ and $\widetilde{H}(x^*)$

has a full-column rank, we have that the matrix

$$
\widetilde{H}' = \begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+, S} \\ \Phi_{\mathcal{A}(x^*) \cap J_-, S} \\ \Phi_{J_0, S} \\ \Phi_{\tilde{\mathcal{A}}_+(x^*), S} \\ \Phi_{\tilde{\mathcal{A}}_-(x^*), S} \end{bmatrix}
$$

has a full-column rank and

$$
\begin{bmatrix} \Phi_{\tilde{\mathcal{A}}_+(x^*), S} \\ \Phi_{\tilde{\mathcal{A}}_-(x^*), S} \end{bmatrix} u \neq 0.
$$

Let $\widehat{x}$ be the vector with components

$$
\widehat{x}_S = x_S^* + \lambda u, \ \widehat{x}_i = 0 \text{ for all } i \notin S,
$$

where $\lambda \neq 0$ is chosen such that $|\widehat{x}_i| > 0$ for $i \in S$ (this can be always guaranteed for some sufficient small $|\lambda|$ since $|x_i^*| > 0$ for $i \in S$). According to the active constraints and inactive constraints given by (4.4.2) and (4.4.26), and removing zero components of x^*, the system (4.5.3) is equivalent to

$$
\Phi_{\mathcal{A}(x^*) \cap J_+, S} x_S^* = \varepsilon e_{\mathcal{A}(x^*) \cap J_+},
$$

$$
\Phi_{\mathcal{A}(x^*) \cap J_-, S} x_S^* = -\varepsilon e_{\mathcal{A}(x^*) \cap J_-},
$$

$$
\Phi_{J_0, S} x_S^* = 0,
$$

$$
\Phi_{\tilde{\mathcal{A}}_+(x^*), S} x_S^* > \varepsilon e_{\tilde{\mathcal{A}}_+(x^*)},
$$

$$
\Phi_{\tilde{\mathcal{A}}_-(x^*), S} x_S^* < -\varepsilon e_{\tilde{\mathcal{A}}_-(x^*)}.
$$

From the above system and the construction of $\widehat{x}$, we see that for any sufficient small

$|\lambda| \neq 0$, the vector $\widehat{x}_S$ satisfies the system

$$H'\widehat{x}_S = \begin{bmatrix} \varepsilon e_{\mathcal{A}(x^*) \cap J_+} \\ -\varepsilon e_{\mathcal{A}(x^*) \cap J_-} \\ 0 \end{bmatrix},$$

$$\Phi_{\tilde{\mathcal{A}}_+(x^*),S}\widehat{x}_S > \varepsilon e_{\tilde{\mathcal{A}}_+(x^*)},$$

$$\Phi_{\tilde{\mathcal{A}}_-(x^*),S}\widehat{x}_S < -\varepsilon e_{\tilde{\mathcal{A}}_-(x^*)}.$$

Then, it is not difficult to see that the vector $\widehat{x}_S$ together with certain nonnegative vector $\widehat{a} \in R^{|\pi(\tilde{\mathcal{A}}_+(x^*))|}$ and $\widehat{\beta} \in R^{|\varrho(\tilde{\mathcal{A}}_-(x^*))|}$, satisfies the system (4.4.27). This implies that $\widehat{x}$ is feasible to (4.3.6). By the construction of $\widehat{x}$, we see that $\|\widehat{x}\|_0 = \|x^*\|_0$ and $\widehat{x} \neq x^*$ (since $\lambda \neq 0$ and $u \neq 0$). Thus $\widehat{x}$ is also a sparsest solution to the problem (4.3.6). By our assumption, there exists a weight $P = \text{diag}(p)$ such that $\widehat{x} = Px^*$, where the factor p satisfies $|p_i| > 0$ for $i \in S$ and $p_i = 0$ for $i \notin S$.

Let $P_S = \text{diag}(p_S) \neq I \in R^{|S| \times |S|}$ (due to $\widehat{x} \neq x^*$). Thus, one has

$$\widehat{x}_S = x_S^* + \lambda u = P_S x_S^*$$

which implies that $u = \frac{1}{\lambda}(P_S - I)x_S^*$. Note that as $\mathcal{A}(x^*) \neq \emptyset$, one has either $\mathcal{A}(x^*) \cap J_+ \neq \emptyset$ or $\mathcal{A}(x^*) \cap J_- \neq \emptyset$. As $P_S \neq I$, one has

$$\begin{aligned} 0 &= H'u = \frac{1}{\lambda}H'(P_S - I)x_S^* \\ &= \frac{1}{\lambda}\begin{bmatrix} \Phi_{\mathcal{A}(x^*) \cap J_+,S}(P_S - I)x_S^* \\ \Phi_{\mathcal{A}(x^*) \cap J_-,S}(P_S - I)x_S^* \\ \Phi_{J_0,S}(P_S - I)x_S^* \end{bmatrix} \neq 0, \end{aligned}$$

which leads to a contradiction. Since column permutations on H' does not change the

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linear independency between columns of H', $H(x^*)$ defined by (4.4.23) has a full-column rank. □

From Lemma 4.5.1, we immediately have the following result, providing a connection between 1-bit compressive sensing and 1-bit basis pursuit.

Theorem 4.5.2 *:*

(i) Suppose that x^* is a sparsest solution to the ℓ_0-problem (4.3.6) and any solution of this problem is of the form Px^*, for some factors p satisfying $|p_i| > 0$ for $i \in Supp(x^*)$ and $p_i = 0$ for $i \notin Supp(x^*)$, where $P = diag(p)$. Then x^* is the unique solution of 1-bit basis pursuit (4.3.7) if and only if the RRSP of Φ^T holds at x^*.

(ii) Suppose that x^* is a sparsest solution to the 1-bit model (4.3.4) and any solution of this problem is of the form Px^*, for some factor p satisfying $|p_i| > 0$ for $i \in Supp(x^*)$ and $p_i = 0$ for $i \notin Supp(x^*)$, where $P = diag(p)$. Then the support set of x^* coincides with the support set of the unique solution of 1-bit basis pursuit (4.3.7) if and only if there exists a factor p^* satisfying $|p_i^*| > 0$ for $i \in Supp(x^*)$ and $p_i^* = 0$ for $i \notin Supp(x^*)$, such that P^*x^*, where $P^* = diag(p^*)$, is feasible to 1-bit basis pursuit (4.3.7), and the RRSP of Φ^T holds at P^*x^* and $H(P^*x^*)$ has a full-column rank.

Proof. (i) By Theorem 4.4.13, if x^* is the unique solution to the problem (4.3.7), the RRSP of Φ^T holds at x^*. Conversely, if x^* is a sparsest solution to the problem (4.3.6) and all its solution can be represented as Px^*, for some factor p satisfying $|p_i| > 0$ for $i \in Supp(x^*)$ and $p_i = 0$ for $i \notin Supp(x^*)$, then by Lemma 4.5.1, the matrix $H(x^*)$ has a full-column rank. Thus by Theorem 4.4.13, the RRSP of Φ^T at x^* implies that x^* is the unique solution to the problem (4.3.7).

(ii) If the support set of x^* coincides with the support set of the unique solution $\overline{x}$ of (4.3.7), $\overline{x}$ can be written as $\overline{x} = P^*x^*$ for a certain weight satisfying $|p_i^*| > 0$ for $i \in Supp(x^*)$ and $p_i^* = 0$ for $i \notin Supp(x^*)$, where $P^* = diag(p^*)$. By Theorem 4.4.13,

the RRSP of Φ^T holds at P^*x^* and $H(P^*x^*)$ has a full-column rank. Conversely, if there exists a factor p^* satisfying $|p_i^*| > 0$ for $i \in \text{Supp}(x^*)$ and $p_i^* = 0$ for $i \notin \text{Supp}(x^*)$, where $P^* = \text{diag}(p^*)$ such that P^*x^* is feasible to (4.3.7), and RRSP of Φ^T holds at P^*x^* and $H(P^*x^*)$ has a full-column rank. By Theorem 4.4.13, P^*x^* is the unique solution of 1-bit basis pursuit (4.3.7). By the definition of P^*, we have $\text{Supp}(P^*x^*) = \text{Supp}(x^*)$. $\qquad \square$

The above results provide some insight into the nonuniform recovery of the support set of a signal, i.e., the support recovery of an individual sparse vector via 1-bit basis pursuit. This result indicates that the key to the support recovery of a sparse vector x is the RRSP of Φ^T at x. It is worths stressing that the support recovery given in Theorem 4.5.2 (ii) includes recovering solutions with certain perturbations and solutions within a positive scalar. However, this property is defined at x which is not known in advance. Thus we need to further strengthen this concept in order to develop certain support recovery conditions. Note that in 1-bit models, the amplitude of signal is not available. To develop support recovery conditions, the reconstruction of a sparse representation is achieved largely within a certain factor, and in the mean time the matrix $\Phi \in R^{m \times n}$ should admit certain restrictive properties independent of individual vectors. In what follows, we develop the concept of *RRSP of order K with respect to 1-bit measurements* and a stronger one called *RRSP of order K*.

For a given 1-bit measurements $y \in \{-1, 0, 1\}^m$, let $S(y)$ be the support set of signals consistent with y, namely,

$$S(y) = \{\text{Supp}(x) : y = \text{sign}(\Phi x)\}.$$

Definition 4.5.3 (RRSP of order K with respect to y) : *Let y be the given 1-bit measurements and (J_+, J_-, J_0) be given as (4.3.2). The matrix Φ^T is said to satisfy the restricted range space property (RRSP) of order K with respect to y if for any disjoint*

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subsets S_+ and S_- of $\{1, \cdots, n\}$ satisfying $|S_+| + |S_-| \leqslant K$, where $S = S_+ \cup S_- \in S(y)$,

and for a pair of subsets (T_1, T_2) with $T_1 \subseteq J_+$, $T_2 \subseteq J_-$, and $|T_1| + |T_2| \leqslant |J_+| + |J_-| - 1$,

there exists a vector $\eta \in \mathcal{R}(\Phi^T)$ satisfying the following properties:

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^T w$ for some $w \in \mathcal{F}(T_1, T_2)$, defined as

$$\mathcal{F}(T_1, T_2) = \{w \in R^m : w_{J_+ \backslash T_1} > 0, \ w_{J_- \backslash T_2} < 0, \ w_{T_1 \cup T_2} = 0\}. \qquad (4.5.4)$$

When the matrix Φ^T has the RRSP of order K with respect to 1-bit measurements y, its column vectors satisfy some properties, as shown by the next lemma.

Lemma 4.5.4 : *Let Φ^T satisfy the RRSP of order K with respect to y. Then any K columns of Φ are linearly independent. Also, any K columns of the matrix $Q = $*

$$\begin{bmatrix} \Phi_{J_+ \backslash T_1, n} \\ \Phi_{J_- \backslash T_2, n} \\ \Phi_{J_0, n} \end{bmatrix}$$

are linearly independent for a pair of subsets (T_1, T_2) with $T_1 \subseteq J_+$ and $T_2 \subseteq J_-$ satisfying $|T_1| + |T_2| \leqslant |J_+| + |J_-| - 1$.

Proof. If the matrix Φ^T has the RRSP of order K with respect to y, from the Definition 4.5.3, we see that for any disjoint subsets S_+ and S_- of $\{1, \cdots, n\}$ with $|S_+| + |S_-| \leqslant K$, there exists a vector η satisfying that $\eta = \Phi^T w$, where $w \in \mathcal{F}(T_1, T_2)$ is defined by (4.5.4), and that $\eta_i = 1$ for all $i \in S_+$, $\eta_i = -1$ for all $i \in S_-$, and $|\eta_i| < 1$ otherwise. This implies that the matrix Φ^T has the standard RSP of order K introduced in [132] (see Definition 4.1 therein). It follows directly from Theorem 4.2 in [132] that any K columns of the matrix Φ are linearly independent.

Let $T_1 \subseteq J_+$ and $T_2 \subseteq J_-$ be any given sets satisfying $|T_1| + |T_2| \leqslant |J_+| + |J_-| - 1$. By Definition 4.5.3, for any disjoint subsets S_+ and S_- of $\{1, \cdots, n\}$ with $|S_+| + |S_-| \leqslant K$,

there exists a vector $\bar{\eta}$ such that $\bar{\eta} = \Phi^T \overline{w}$ for some $\overline{w} \in \mathcal{F}(T_1, T_2)$, and the vector $\bar{\eta}$ satisfies that $\bar{\eta}_i = 1$ for $i \in S_+$, $\bar{\eta}_i = -1$ for $i \in S_-$, and $|\bar{\eta}_i| < 1$ otherwise. Note that $\overline{w}_{T_1} = 0$ and $\overline{w}_{T_2} = 0$.

The vector $\bar{\eta}$ can be written as

$$
\begin{aligned}
\bar{\eta} = \Phi^T \overline{w} &= \left[(\Phi_{J_+,n})^T, (\Phi_{J_-,n})^T, (\Phi_{J_0,n})^T \right] \begin{bmatrix} \overline{w}_{J_+} \\ \overline{w}_{J_-} \\ \overline{w}_{J_0} \end{bmatrix} \\
&= \left[(\Phi_{J_+ \backslash T_1, n})^T, (\Phi_{J_- \backslash T_2, n})^T, (\Phi_{J_0, n})^T \right] \begin{bmatrix} \overline{w}_{J_+ \backslash T_1} \\ \overline{w}_{J_- \backslash T_2} \\ \overline{w}_{J_0} \end{bmatrix} \\
&= Q^T \begin{bmatrix} \overline{w}_{J_+ \backslash T_1} \\ \overline{w}_{J_- \backslash T_2} \\ \overline{w}_{J_0} \end{bmatrix},
\end{aligned}
\tag{4.5.5}
$$

where $\overline{w}_{J_+ \backslash T_1}$ and $\overline{w}_{J_- \backslash T_2}$ are the subvectors of $\overline{w}_{J_+}$ and $\overline{w}_{J_-}$, obtained by deleting those components indexed by T_1 and T_2, respectively.

Therefore, $Q^T = \left[(\Phi_{J_+ \backslash T_1, n})^T, (\Phi_{J_- \backslash T_2, n})^T, (\Phi_{J_0, n})^T \right]$ has the standard RSP of order K. By Theorem 4.2 in [132] again, we conclude that any K columns of the matrix Q are linearly independent. $\qquad \square$

We now prove the main result concerning the nonuniform support recovery for a fixed 1-bit measurements, which claims that the support set of a sparse signal can be exactly reconstructed by 1-bit basis pursuit (4.3.7) if matrix Φ^T has the RRSP of order K with respect to the 1-bit measurements.

Theorem 4.5.5 : *Given the 1-bit measurements $y \in \{-1, 1, 0\}^m$, suppose that Φ^T has the RRSP of order K with respect to y. Then, 1-bit basis pursuit (4.3.7) admits a solution*

$\tilde{x}$ *satisfying* $Supp(\tilde{x}) \subseteq Supp(x^*)$ *for any K-sparse signal x^* (i.e., $\|x^*\|_0 \leqslant K$) consistent with the measurements y in the sense that $y = sign(\Phi x^*)$. Furthermore, if x^* is a sparsest signal consistent with y, then $Supp(\tilde{x}) = Supp(x^*)$.*

Proof. Let x^* be any given K-sparse vector consistent with the 1-bit measurements y, i.e., $\text{sign}(\Phi x^*) = y$. Let (J_+, J_-, J_0) be given as (4.3.2) and let $S_+ = \{i : x_i^* > 0\}$ and $S_- = \{i : x_i^* < 0\}$, where $S = S_+ \cup S_- \in S(y)$. The above consistency implies that $(\Phi x^*)_i > 0$ for all $i \in J_+, (\Phi x^*)_i < 0$ for all $i \in J_-$ and $(\Phi x^*)_i = 0$ for all $i \in J_0$. This implies that there exists a scalar $\alpha > 0$ such that $\alpha(\Phi x^*)_i \geqslant \varepsilon$ for all $i \in J_+$ and $\alpha(\Phi x^*)_i \leqslant -\varepsilon$ for all $i \in J_-$. Thus αx^* satisfies the constraints of 1-bit basis pursuit (4.3.7), i.e.,

$$\Phi_{J_+,n}(\alpha x^*) \geqslant \varepsilon e_{J_+}, \tag{4.5.6}$$

$$\Phi_{J_-,n}(\alpha x^*) \leqslant -\varepsilon e_{J_-}, \tag{4.5.7}$$

$$\Phi_{J_0,n}(\alpha x^*) = 0. \tag{4.5.8}$$

Hence, αx^* is a feasible solution to (4.3.7). Note that for any scalar $\alpha > 0$, $\|\alpha x^*\|_0 = \|x^*\|_0 \leqslant K$ and $\text{sign}(\alpha x^*) = \text{sign}(x^*)$. Thus x^* and αx^* share the same index sets S_+ and S_-. From (4.5.6) and (4.5.7), we see that

$$\alpha \geqslant \frac{\varepsilon}{(\Phi x^*)_i} \text{ for } i \in J_+, \ \alpha \geqslant \frac{\varepsilon}{-(\Phi x^*)_i} \text{ for } i \in J_-.$$

Let $\alpha^* > 0$ be the smallest α satisfying the above inequalities, i.e.

$$\alpha^* = \max \left\{ \max_{i \in J_+} \frac{\varepsilon}{(\Phi x^*)_i}, \ \max_{i \in J_-} \frac{\varepsilon}{-(\Phi x^*)_i} \right\} = \max_{i \in J_+ \cup J_-} \frac{\varepsilon}{|(\Phi x^*)_i|}.$$

Define the index sets

$$T_0' = \left\{ i \in J_+ \cup J_- : \ \frac{\varepsilon}{|(\Phi x^*)_i|} = \alpha^* \right\},$$

$$T_1' = \left\{ i \in J_+ : \ \frac{\varepsilon}{(\Phi x^*)_i} < \alpha^* \right\}, \ T_2' = \left\{ i \in J_- : \ \frac{\varepsilon}{-(\Phi x^*)_i} < \alpha^* \right\}. \tag{4.5.9}$$

Clearly, T_0' represents the active constraints in (4.5.6) and (4.5.7) at the point $\alpha^* x^*$, and $T_1' \subseteq J_+$ and $T_2' \subseteq J_-$ represent the inactive constraints at $\alpha^* x^*$. Clearly, $T_1' \cup T_2' = (J_+ \cup J_-) \backslash T_0'$. By the definition of α^*, we see that $T_0' \neq \emptyset$ and hence

$$|T_1'| + |T_2'| \leqslant |J_+| + |J_-| - 1.$$

We now prove that the unique solution $\tilde{x}$ of the problem (4.3.7) satisfying $\mathrm{Supp}(\tilde{x}) \subseteq \mathrm{Supp}(x^*)$. To prove the uniqueness of $\tilde{x}$, by Theorem 4.4.13, it is sufficient to prove that Φ^T has the RRSP at $\tilde{x}$ and the matrix $H(\tilde{x})$ has a full-column rank.

Let $S = \mathrm{Supp}(x^*)$. If $\mathcal{N}\left(\begin{bmatrix} \Phi_{J_+ \backslash T_1', S} \\ \Phi_{J_- \backslash T_2', S} \\ \Phi_{J_0, S} \end{bmatrix} \right) \neq \{0\}$, let d be a nonzero vector in

$\mathcal{N}\left(\begin{bmatrix} \Phi_{J_+ \backslash T_1', S} \\ \Phi_{J_- \backslash T_2', S} \\ \Phi_{J_0, S} \end{bmatrix} \right)$ and consider the vector $x(\lambda)$ such that $x_S(\lambda) = \alpha^* x_S^* + \lambda d$ and

$x(\lambda)_i = 0$ for $i \notin S$. By Lemma 4.5.4, as $\Phi_{m,S}$ has a full-column rank, one has

$$\begin{bmatrix} \Phi_{T_1', S} \\ \Phi_{T_2', S} \end{bmatrix} d \neq 0.$$

For such a construction of $x(\lambda)$, we have $\mathrm{Supp}(x(\lambda)) \subseteq \mathrm{Supp}(x^*)$ for any $\lambda \in R$. It is easy to see that $x(\lambda)$ is feasible to the problem (4.3.7) provided by some sufficient small λ so

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that $\alpha^* x^*$ and $x(\lambda)$ have the same active and inactive constraints, namely,

$$\Phi_{J_+ \setminus T_1', S} x_S(\lambda) = \varepsilon e_{J_+ \setminus T_1'},$$

$$\Phi_{J_- \setminus T_2', S} x_S(\lambda) = -\varepsilon e_{J_- \setminus T_2'},$$

$$\Phi_{J_0, S} x_S(\lambda) = 0,$$

$$\Phi_{T_1', S} x_S(\lambda) > \varepsilon e_{T_1'},$$

$$\Phi_{T_2', S} x_S(\lambda) < -\varepsilon e_{T_2'}.$$

Now, for such a nonzero vector d, vary $|\lambda|$ continuously from zero to a positive number $|\lambda_1| \neq 0$ such that at least one inactive constraint in $T_1' \cup T_2'$ becomes active at $x(\lambda_1)$ and $x(\lambda_1)$ is still feasible to the problem (4.3.7). Thus, the active constraints at $x(\lambda_1)$ is augmented.

Let $\widehat{x} = x(\lambda_1)$, $T_1'' = \{i \in J_+ : (\Phi \widehat{x})_i > \varepsilon\}$, $T_2'' = \{i \in J_- : (\Phi \widehat{x})_i < -\varepsilon\}$, and T_0'' be the index set of active constraints at $\widehat{x}$ satisfying $T_0' \subseteq T_0''$. Now, replacing the role of $\alpha^* x^*$ by

$\widehat{x}$, if $\mathcal{N}\left(\begin{bmatrix} \Phi_{J_+ \setminus T_1'', S} \\ \Phi_{J_- \setminus T_2'', S} \\ \Phi_{J_0, S} \end{bmatrix} \right) \neq \{0\}$, let d be a nonzero vector in $\mathcal{N}\left(\begin{bmatrix} \Phi_{J_+ \setminus T_1'', S} \\ \Phi_{J_- \setminus T_2'', S} \\ \Phi_{J_0, S} \end{bmatrix} \right)$ and con-

sider the vector $x'(\lambda)$ such that $x'_S(\lambda) = \widehat{x}_S + \lambda d$ and $x'(\lambda)_i = 0$ for $i \notin S$. Thus, $x'(\lambda)$ is feasible to the problem (4.3.7), and $\widehat{x}$ and $x'(\lambda)$ have the same active and inactive constraints for some sufficient small λ. Then, continuing to update active and inactive constraints,

we repeat the above process until find a point $\tilde{x}$ at which $\mathcal{N}\left(\begin{bmatrix} \Phi_{J_+ \setminus T_1, S} \\ \Phi_{J_- \setminus T_2, S} \\ \Phi_{J_0, S} \end{bmatrix} \right) = \{0\}$,

where $T_1 = \{i \in J_+ : (\Phi \tilde{x})_i > \varepsilon\}$ and $T_2 = \{i \in J_- : (\Phi \tilde{x})_i < -\varepsilon\}$. Note that

$$\mathcal{N}\left(\begin{bmatrix} \Phi_{J_+\setminus T_1,S} \\ \Phi_{J_-\setminus T_2,S} \\ \Phi_{J_0,S} \end{bmatrix}\right) = \{0\} \text{ implies that the matrix}$$

$$\begin{bmatrix} \Phi_{J_+\setminus T_1,S} \\ \Phi_{J_-\setminus T_2,S} \\ \Phi_{J_0,S} \end{bmatrix} \tag{4.5.10}$$

has a full-column rank.

Due to such a construction of $\tilde{x}$, $\tilde{x}$ is still feasible to (4.3.7) and $\mathcal{A}(\tilde{x}) \neq \emptyset$. Notice that T_1 and T_2 are inactive constraints at point $\tilde{x}$, namely, $T_1 = \tilde{\mathcal{A}}_+(\tilde{x})$ and $T_2 = \tilde{\mathcal{A}}_-(\tilde{x})$. Thus, $J_+\setminus T_1 = \mathcal{A}(\tilde{x}) \cap J_+$ and $J_-\setminus T_2 = \mathcal{A}(\tilde{x}) \cap J_-$. Note that as $\mathcal{A}(\tilde{x}) \neq \emptyset$, one has either $\mathcal{A}(\tilde{x}) \cap J_+ \neq \emptyset$ or $\mathcal{A}(\tilde{x}) \cap J_- \neq \emptyset$. Hence, (4.5.10) is equivalent to

$$\begin{bmatrix} \Phi_{\mathcal{A}(\tilde{x})\cap J_+,S} \\ \Phi_{\mathcal{A}(\tilde{x})\cap J_-,S} \\ \Phi_{J_0,S} \end{bmatrix}, \tag{4.5.11}$$

which has a full-column rank.

Now, we prove that $\tilde{x}$ is the unique solution of (4.3.7). Let $S' = \text{Supp}(\tilde{x})$. For T_1 and T_2 defined above, as $\tilde{x}$ is consistent with 1-bit measurements y and satisfies $\text{Supp}(\tilde{x}) \subseteq \text{Supp}(x^*)$, i.e., $S' \subseteq S$, we have $S' \in S(y)$ and $\begin{bmatrix} \Phi_{\mathcal{A}(\tilde{x})\cap J_+,S'} \\ \Phi_{\mathcal{A}(\tilde{x})\cap J_-,S'} \\ \Phi_{J_0,S'} \end{bmatrix}$ has a full-column rank since the matrix (4.5.11) has a full-column rank.

Let $S'_+ = \{i : \tilde{x}_i > 0\}$ and $S'_- = \{i : \tilde{x}_i < 0\}$, then $|S'_+| + |S'_-| \leqslant K$. By the assumption, Φ^T has the RRSP of order K with respect to y. Thus there exists a vector $\eta \in \mathcal{R}(\Phi^T)$ and $w \in \mathcal{F}(T_1, T_2)$ satisfying that $\eta = \Phi^T w$ and $\eta_i = 1$ for $i \in S'_+$, $\eta_i = -1$

for all $i \in S'_-$, and $|\eta_i| < 1$ otherwise, where the set $\mathcal{F}(T_1, T_2)$ is defined as (4.5.4). As $T_1 = \widetilde{\mathcal{A}}_+(\tilde{x})$ and $T_2 = \widetilde{\mathcal{A}}_-(\tilde{x})$, the conditions $w_{T_1} = 0$ and $w_{T_2} = 0$ in the set $\mathcal{F}(T_1, T_2)$ coincide with the condition

$$w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(\tilde{x}) \cup \tilde{\mathcal{A}}_-(\tilde{x}).$$

As a result, the set $\mathcal{F}(T_1, T_2)$ coincides with $\mathcal{F}(\tilde{x})$ defined by (4.4.21). Thus the RRSP of Φ^T holds at the point $\tilde{x}$ (see Definition 4.4.7). Hence, $\tilde{x}$ is the unique solution of the problem (4.3.7).

Furthermore, if x^* is a sparsest signal consistent with y, as $\text{Supp}(\tilde{x}) \subseteq \text{Supp}(x^*)$, we immediately have $\text{Supp}(\tilde{x}) = \text{Supp}(x^*)$. $\qquad \square$

Theorem 4.5.6 : *Given the 1-bit measurements $y \in \{-1, 1, 0\}^m$, let x^* be an unknown K-sparse signal, if the 1-bit basis pursuit (4.3.7) has a unique solution $\tilde{x}$ satisfying $\text{Supp}(\tilde{x}) = \text{Supp}(x^*)$, then Φ^T has the RRSP of order K with respect to y.*

Proof. For a given 1-bit measurements y and a K-sparse unknown signal x^*, suppose that the problem (4.3.7) has a unique solution $\tilde{x}$ satisfying $\text{Supp}(\tilde{x}) = \text{Supp}(x^*)$ and $\text{sign}(\Phi\tilde{x}) = y$. Let $\tilde{S}_+ = \{i : \tilde{x}_i > 0\}$ and $\tilde{S}_- = \{i : \tilde{x}_i < 0\}$. Thus, we have $|\tilde{S}_+| + |\tilde{S}_-| = |\text{Supp}(x^*)| \leqslant K$ for any disjoint subsets $\tilde{S}_+$ and $\tilde{S}_-$ with $\tilde{S}_+ \cup \tilde{S}_- \in S(y)$. By Theorem 4.4.13, the uniqueness of $\tilde{x}$ indicates that the RRSP of Φ^T at $\tilde{x}$ holds. Let $T_1 = \widetilde{\mathcal{A}}_+(\tilde{x})$ and $T_2 = \widetilde{\mathcal{A}}_-(\tilde{x})$, thus, $J_+ \backslash T_1 = \mathcal{A}(\tilde{x}) \cap J_+$ and $J_- \backslash T_2 = \mathcal{A}(\tilde{x}) \cap J_-$. Due to the optimality of solution $\tilde{x}$, there exists at least one active constraint at $\tilde{x}$, namely, $\mathcal{A}(\tilde{x}) \neq \emptyset$. Thus, $T_1 \cup T_2 \neq J_+ \cup J_-$. The RRSP of Φ^T at $\tilde{x}$ implies that properties (i) and (ii) in Definition 4.5.3 are held with $S_+ = \tilde{S}_+$ and $S_- = \tilde{S}_-$ satisfying $|\tilde{S}_+| + |\tilde{S}_-| \leqslant K$, and T_1, T_2 defined above. Hence, Φ^T has the RRSP of order K with respect to y is a necessary condition for the uniqueness of solutions of (4.3.7). $\qquad \square$

4.5.2 Uniform support recovery

Theorem 4.5.5 and Theorem 4.5.6 provide conditions for the nonuniform support recovery that any K-sparse vectors are consistent with the given 1-bit measurements y. Clearly, for a given vector $y \in \{-1, 1, 0\}^m$, not every K-sparse vector is always consistent with y. To develop a condition for the uniform support recovery of all K-sparse signals, we need to further strengthen the Definition 4.5.3 and introduce the following concepts.

Let Y_k be a subspace of all the possible values of 1-bit measurements y for any K-sparse vectors, i.e., $Y_k = \{y : y = \mathrm{sign}(\Phi x), \|x\|_0 \leqslant K\}$. For a given 1-bit measurements $y \in Y_k$, let $J_+(y)$, $J_-(y)$ and $J_0(y)$ be index sets of components of y with values $1, -1$ and 0, respectively.

Definition 4.5.7 (S-RRSP of order K) : *The matrix Φ^T is said to satisfy the sufficient restricted range space property (S-RRSP) of order K if any disjoint subsets S_+ and S_- of $\{1, \cdots, n\}$ satisfying $|S_+| + |S_-| \leqslant K$, and for any $y \in Y_k$, $S = S_+ \cup S_- \in S(y)$ and there exist subsets $T_1 \subseteq J_+(y)$ and $T_2 \subseteq J_-(y)$ satisfying $|T_1| + |T_2| \leqslant |J_+(y)| + |J_-(y)| - 1$, and for any pair of (T_1, T_2), there exists a vector $\eta \in \mathcal{R}(\Phi^T)$ satisfying the following properties:*

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^T w$ for some $w \in \mathcal{F}(T_1, T_2)$, defined as

$$\mathcal{F}(T_1, T_2) = \{w \in R^m : w_{J_+ \backslash T_1} > 0, \ w_{J_- \backslash T_2} < 0, \ w_{T_1 \cup T_2} = 0\}.$$

The above concept is stronger than Definition 4.5.3. Clearly, if the matrix has the S-RRSP of order K, it must have the RRSP of order K with respect to any given vector $y \in Y_k$. This gives a sufficient condition for the uniform support recovery shown as follows.

Theorem 4.5.8 : *Let $\Phi \in R^{m \times n}$ be a given matrix and suppose that Φ^T has the S-RRSP of order K, for any K-sparse signal x^*, then 1-bit basis pursuit (4.3.7) has a unique solution $\tilde{x}$ satisfying $Supp(\tilde{x}) \subseteq Supp(x^*)$ with (J_+, J_-, J_0) being determined by the acquired 1-bit measurements $y = sign(\Phi x^*) \in Y_k$, i.e., $J_+ = \{i : sign[(\Phi x^*)_i] = 1\}$, $J_- = \{i : sign[(\Phi x^*)_i] = -1\}$, $J_0 = \{i : sign[(\Phi x^*)_i] = 0\}$. Furthermore, if x^* is a sparsest signal satisfying the sign constraints in (4.3.3), then $Supp(\tilde{x}) = Supp(x^*)$.*

Proof. : Let x^* be an arbitrary K-sparse signal, and let the 1-bit measurements $y = sign(\Phi x^*)$ be taken, which determines a partition (J_+, J_-, J_0) of $\{1, ..., m\}$ as (4.3.2). Since Φ^T has the S-RRSP of order K (Definition 4.5.7), this implies that Φ^T has the RRSP of order K with respect to this particular vector y. By Theorem 4.5.5, the problem (4.3.7) has a unique solution $\tilde{x}$ satisfying $Supp(\tilde{x}) \subseteq Supp(x^*)$. Furthermore, as $sign(\Phi x^*) = sign(\Phi \tilde{x})$, if x^* is a sparsest signal satisfying the sign constraints in (4.3.3), we immediately have $Supp(\tilde{x}) = Supp(x^*)$. $\square$

Definition 4.5.9 (N-RRSP of order K) : *The matrix Φ^T is said to satisfy the necessary restricted range space property (N-RRSP) of order K if for any $y \in Y_k$ and any disjoint subsets S_+ and S_- of $\{1, \cdots, n\}$ satisfying $|S_+| + |S_-| \leqslant K$, where $S = S_+ \cup S_-$, and there exist $y \in Y_k$ and a pair of subsets (T_1, T_2) such that $S \in S(y)$, $T_1 \subseteq J_+(y)$ and $T_2 \subseteq J_-(y)$ satisfying $|T_1| + |T_2| \leqslant |J_+(y)| + |J_-(y)| - 1$, and there exists a vector $\eta \in \mathcal{R}(\Phi^T)$ satisfying the following properties:*

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^T w$ for some $w \in \mathcal{F}(T_1, T_2)$, defined as

$$\mathcal{F}(T_1, T_2) = \{w \in R^m : w_{J_+ \setminus T_1} > 0, \ w_{J_- \setminus T_2} < 0, \ w_{T_1 \cup T_2} = 0\}.$$

Theorem 4.5.10 : *Let $\Phi \in R^{m \times n}$ be a given matrix. For any K-sparse signal x^* (i.e., $\|x^*\|_0 \leqslant k$), 1-bit basis pursuit (4.3.7) has a unique solution $\tilde{x}$ satisfying $Supp(\tilde{x}) = Supp(x^*)$ with $J_+ = \{i : sign[(\Phi x^*)_i] = 1\}$, $J_- = \{i : sign[(\Phi x^*)_i] = -1\}$, $J_0 = \{i : sign[(\Phi x^*)_i] = 0\}$, then Φ^T has the N-RRSP of order K.*

Proof. : Let x^* be an arbitrary K-sparse signal with $S = \text{Supp}(x^*)$. Let $y = \text{sign}(\Phi x^*)$ be the acquired measurements. Assume that $\tilde{x}$ is the unique solution of (4.3.7) satisfying $\text{Supp}(\tilde{x}) = S$. Then, we have $y \in Y_k$, $S \in S(y)$ and $|\text{Supp}(\tilde{x})| = |S| \leqslant K$. By Theorem 4.4.13, the uniqueness of $\tilde{x}$ implies that the matrix $Q = \begin{bmatrix} \Phi_{\mathcal{A}(\tilde{x}) \cap J_+, S} \\ \Phi_{\mathcal{A}(\tilde{x}) \cap J_-, S} \\ \Phi_{J_0, S} \end{bmatrix}$ has a full-column rank and there exists a vector $\eta \in \mathcal{R}(\Phi^T)$ satisfying

(a) $\eta_i = 1$ for $i \in S_+(\tilde{x})$, $\eta_i = -1$ for $i \in S_-(\tilde{x})$, and $|\eta_i| < 1$ for $i \notin S$;

(b) $\eta = \Phi^T w$ for some $w \in \mathcal{F}(\tilde{x})$ defined as

$$\mathcal{F}(\tilde{x}) = \{w \in R^m : \quad w_i > 0 \text{ for } i \in \mathcal{A}(\tilde{x}) \cap J_+(y), \ w_i < 0 \text{ for } i \in \mathcal{A}(\tilde{x}) \cap J_-(y),$$
$$w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(\tilde{x}) \cup \tilde{\mathcal{A}}_-(\tilde{x})\},$$

where $S_+(\tilde{x}) = \{i : \tilde{x}_i > 0\}$ and $S_-(\tilde{x}) = \{i : \tilde{x}_i < 0\}$.

Note that $(S_+(\tilde{x}), S_-(\tilde{x}))$ is a partition of $S \in S(y)$ satisfying $S_+(\tilde{x}) \cup S_-(\tilde{x}) = S$. Let $T_1 = \tilde{\mathcal{A}}_+(\tilde{x}) \subseteq J_+(y)$ and $T_2 = \tilde{\mathcal{A}}_-(\tilde{x}) \subseteq J_-(y)$. Due to the optimality of solution $\tilde{x}$, we have $\mathcal{A}(\tilde{x}) \neq \emptyset$. Thus, one has $|T_1| + |T_2| \leqslant |J_+(y)| + |J_-(y)| - 1$, $\mathcal{A}(\tilde{x}) \cap J_+ = J_+ \backslash T_1$, $\mathcal{A}(\tilde{x}) \cap J_- = J_- \backslash T_2$. Hence, the above properties (a) and (b) coincide with the properties (i) and (ii) in Definition 4.5.9. Considering all K-sparse signal x^* and its corresponding unique solution $\tilde{x}$ of (4.3.7), for any possible $y \in Y_K$ and disjoint subsets $S_+(\tilde{x})$, $S_-(\tilde{x})$ of $\{1, \cdots, n\}$ with $S = S_+(\tilde{x}) \cup S_-(\tilde{x})$ satisfying $|S| \leqslant K$, there exist a 1-bit measurements

$y \in Y_k$ and a pair of (T_1, T_2) defined above such that $S \in S(y)$. Hence, Φ^T has the N-RRSP of order K. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Due to the sign-process, 1-bit measurements is robust to signals with certain perturbations. Thus, it is impossible to exactly recover sparse signals. To achieve a certain level of sparse recovery, one of the contributions of this chapter is a new reformulation for the 1-bit compressive sensing model, based on which we develop a decoding method, 1-bit basis pursuit, for 1-bit compressive sensing. And we then establish the restricted range space property (RRSP) to provide a connection between the sensing matrix and the support recovery of sparse signals from 1-bit measurements. In particular, we have shown that the transposed sensing matrix satisfying RRSP is necessary and sufficient for recovering the support set of an individual sparse signal via 1-bit basis pursuit; furthermore, the support set of any K-sparse signals can be exactly recovered via 1-bit basis pursuit if the transposed sensing matrix satisfies the S-RRSP of order K.

CHAPTER 5

EMPIRICAL VERIFICATION: 1-BIT BASIS

PURSUIT

5.1   Introduction

Having studied the theoretical aspect of 1-bit ℓ_0-minimization and 1-bit basis pursuit, and established the RRSP-based conditions for the nonuniform and uniform support recovery for 1-bit compressive sensing, we now study the numerical performance of 1-bit basis pursuit. In our experiments, in contrast to the standard basis pursuit (ℓ_1-minimization) for compressive sensing, we investigate the performance of 1-bit basis pursuit from the following perspectives:

1. the constant $\varepsilon = 1$ is fixed in the linear system (4.3.5) for 1-bit basis pursuit, namely,

$$
\begin{aligned}
\Phi_{J_+,n} x &\geqslant e_{J_+}, \\
\Phi_{J_-,n} x &\leqslant -e_{J_-}, \qquad\qquad (5.1.1)\\
\Phi_{J_0,n} x &= 0,
\end{aligned}
$$

for given index sets $J_+, \ J_-, \ J_0 \subseteq \{1, \cdots, m\}$, where $J_+ = \{i : \ y_i = 1\}$, $J_- = \{i : \ y_i = -1\}$, and $J_0 = \{i : \ y_i = 0\}$;

2. the matrix Φ is overdetermined rather than underdetermined in the linear system (5.1.1);

3. we aim to find the support set of a sparse signal x^* satisfying the linear system 5.1.1 or a sparser solution within the support set of the sparse signal x^*;

4. we aim to find a sparsest solution to the linear system (5.1.1).

We carry out experiments on Gaussian matrices and Bernoulli matrices to empirically reveal certain features of 1-bit basis pursuit and 1-bit compressive sensing. To further enhance the sparsity and improve the performance of 1-bit basis pursuit, we propose two approaches: truncating the 1-bit measurements and adding weights to the ℓ_1-norm objective of 1-bit basis pursuit, namely, the reweighted 1-bit ℓ_1-minimization, where weights are defined by some merit functions demonstrated in [40, 135, 66, 42, 84, 134].

This chapter is organized as follows. We begin with introducing some existing 1-bit recovery models in section 5.2. Then, we carry out numerical simulations to show the solvability of 1-bit basis pursuit and to reveal certain features of 1-bit compressive sensing in section 5.3. To further improve the performance of 1-bit basis pursuit, we carry out experiments on the truncated 1-bit measurements method and the reweighted 1-bit ℓ_1-minimization method in section 5.4.

5.2 Existing 1-bit recovery models

Due to the sign constraint (4.3.3), the 1-bit measurements are robust to solutions within a certain factor or solutions with certain perturbations. This implies that it is generally impossible to exactly recover a sparse signal via the problem (4.3.4). Thus, in general, the majority 1-bit recovery models focus on the following two types of recovery:

1. support recovery: recovering the support set of a sparse signal x^*, namely, $\mathrm{Supp}(x^*) = \mathrm{Supp}(x)$, where x is a solution to any recovery algorithms;

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2. approximate sparse recovery: recovering an approximately sparse solution x which is close to a sparse signal x^* up to a constant $\varepsilon > 0$, namely,

$$\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2 < \varepsilon, \tag{5.2.1}$$

which is referred to as the normalized ℓ_2-norm residual in this chapter.

Remark 5.2.1 : *Note that both support recovery and approximate sparse recovery include the positive proportional recovery that algorithms recover the signal x^* within a positive scalar. In fact, if a solution x to an algorithm has the same support set as the signal x^* and satisfies the condition (5.2.1) for a sufficiently small ε tending to 0, we say that the algorithm successfully recovers a solution positive proportional to the signal x^*.*

So far, a few algorithms are developed for the support recovery. Among those algorithms, Gupta et al. [75] demonstrated the first measurement bonds for the support recovery via passive algorithms and adaptive algorithms, respectively. And Gopi et al. [71] proposed two combinatorial algorithms based on so-called union free families of sets [62] and expanders [11, 81] to achieve the uniform support recovery. They further showed that an approximately sparse vector can be obtained by firstly recovering the support set via the expanders algorithms, and then solving a linear program subject to this support set with an adequate number of measurements. Some greedy algorithms, such as matching sign pursuit (MSP) introduced by Boufounos and Baraniuk in [24] and binary iterative hard thresholding (BIHT) introduced by Jacques et al. in [80], are proposed to estimate the largest k-nonzero components of a sparse signal. In particular, the BIHT algorithm performs better than MSP as it minimizes the inconsistency via the one-sided ℓ_1-norm minimization problem, as shown in Figure 2 and Figure 3 in [80]. But there is no theory for guarantee performances of both algorithms. For the approximate sparse recovery, Boufounos and Baraniuk introduced the first algorithm, namely, the renormal-

ized fixed point iteration (RFPI), to tackle the 1-bit compressive sensing. Based on the BIHT and RFPI algorithms, other methods such as adaptive outlier pursuit (AOP) and noise-adaptive renormalized fixed point iteration (NARFPI) are designed for the noisy 1-bit compressive sensing without theoretical guarantees. And Laska et al. [87], and Plan and Vershynin [105, 106] provided provable recovery methods for the approximate sparse recovery. In particular, Ai et al. [3] have proved that the proposed convex approach in [106] allows an approximately sparse reconstruction from 1-bit measurements obtained by the sub-gaussian matrices, such as the Bernoulli matrix.

However, when we investigate the performance of 1-bit basis pursuit, we notice that, besides the support recovery and the approximate sparse recovery defined above, solutions of 1-bit basis pursuit may have the following two settings:

1. 1-bit basis pursuit found a sparsest solution x within the support set of a sparse signal x^*, namely, $\text{Supp}(x) \subset \text{Supp}(x^*)$;

2. 1-bit basis pursuit found a sparsest solution x to the linear system (5.1.1) satisfying $\|x\|_0 \leqslant \|x^*\|_0$, which includes the case that the nonzero locations of x and x^* are different.

We give the following two examples to demonstrate that 1-bit basis pursuit is possible to obtain a sparse solution to the linear system (5.1.1) either within the support set of the sparse signal or having a different support pattern from the sparse signal.

Example 5.2.2 *:*

For a given matrix $\Phi \in R^{3 \times 6}$ and the signal $x^* \in R^6$ with

$$\Phi = \begin{bmatrix} -1.3455 & -2.3419 & -0.2322 & 0.3843 & 1.6182 & -0.0759 \\ 0.0007 & 1.2481 & 0.2870 & -0.3798 & -0.8472 & -0.1325 \\ 0.0535 & 2.8092 & -0.4646 & -0.1018 & -0.5759 & 1.4393 \end{bmatrix}, \quad x^* = \begin{bmatrix} 0 \\ 1.6359 \\ 0.5590 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where every entry in Φ and the nonzero entries of x^* are drawn from the standard normal distribution. Compute the 1-bit measurements $y = \text{sign}(\Phi x^*) = [\ -1, 1, 1\ ]^T$. According to 1-bit measurements y, index sets J_+, J_- and J_0 are $J_+ = \{2, 3\}$, $J_- = \{1\}$ and $J_0 = \emptyset$. For the given matrix Φ, the linear system (5.1.1) is written as

$$\begin{bmatrix} 0.0007 & 1.2481 & 0.2870 & -0.3798 & -0.8472 & -0.1325 \\ 0.0535 & 2.8092 & -0.4646 & -0.1018 & -0.5759 & 1.4393 \end{bmatrix} x \geqslant \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} -1.3455 & -2.3419 & -0.2322 & 0.3843 & 1.6182 & -0.0759 \end{bmatrix} x \leqslant -1.$$

Hence, 1-bit basis pursuit obtains a sparser solution $x = [\ 0,\ 0.8012,\ 0,\ 0,\ 0,\ 0\ ]$ within the support set of signal x^* and the normalized ℓ_2-norm residual is $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2 = 0.3278$.

Example 5.2.3 :

For a given matrix $\Phi \in R^{3\times 6}$ and the signal $x^* \in R^6$ with

$$\Phi = \begin{bmatrix} -1.6118 & 1.0205 & -0.0708 & -2.1924 & -0.9485 & 0.8577 \\ -0.0245 & 0.8617 & -2.4863 & -2.3193 & 0.4115 & -0.6912 \\ -1.9488 & 0.0012 & 0.5812 & 0.0799 & 0.6770 & 0.4494 \end{bmatrix}, \; x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.0078 \\ -2.1237 \end{bmatrix},$$

where every entry in Φ and the nonzero entries of x^* are drawn from the standard normal distribution. Compute the 1-bit measurements $y = \text{sign}(\Phi x^*) = [\; -1, 1, -1\;]^T$. According to the 1-bit measurements y, index sets J_+, J_- and J_0 are $J_+ = \{2\}$, $J_- = \{1, 3\}$ and $J_0 = \emptyset$. For the given matrix Φ, the linear system (5.1.1) is written as

$$\begin{bmatrix} -0.0245 & 0.8617 & -2.4863 & -2.3193 & 0.4115 & -0.6912 \end{bmatrix} x \geqslant 1,$$

$$\begin{bmatrix} -1.6118 & 1.0205 & -0.0708 & -2.1924 & -0.9485 & 0.8577 \\ -1.9488 & 0.0012 & 0.5812 & 0.0799 & 0.6770 & 0.4494 \end{bmatrix} x \leqslant \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

For this example, 1-bit basis pursuit obtains a solution $x = [\; 0.6384, \quad 0, \quad -0.4085, \quad 0, \quad 0, \quad 0\;]$, which has a different support pattern from the signal x^* and the normalized ℓ_2-norm residual is $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2 = 1.4142$.

Therefore, in this chapter, we present the performance of 1-bit basis pursuit in three ways:

(i) support recovery: finding a solution x within the support set of a sparse signal x^*, e.g., $\text{Supp}(x) \subseteq \text{Supp}(x^*)$, including exactly recovering the support set of the sparse signal x^*, namely, $\text{Supp}(x) = \text{Supp}(x^*)$;

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(ii) approximate sparse recovery: finding an approximately sparse solution x close to the sparse signal x^*, which is examined by the cosine stopping condition

$$\cos(\theta(x^*, x)) \geqslant \tau,$$

where τ is a cosine-stopping threshold and $\cos(\theta(x^*, x))$ is the angle between x^* and x;

(iii) cardinality recovery: finding a sparsest solution to the linear system (5.1.1), which shows the ability of 1-bit basis pursuit to solve the ℓ_0-problem of 1-bit compressive sensing, namely,

$$\begin{aligned}
\min \quad & \|x\|_0 \\
\text{s.t.} \quad & \Phi_{J_+, n} x \geqslant e_{J_+}, \\
& \Phi_{J_-, n} x \leqslant -e_{J_-}, \\
& \Phi_{J_0, n} x = 0,
\end{aligned} \qquad (5.2.2)$$

where index sets $J_+,\ J_-,\ J_0 \subseteq \{1, \cdots, m\}$ are given by the 1-bit measurements.

5.3 Performances of 1-bit basis pursuit

In our implementation, all examples are solved by the CVX, a Matlab software for convex programs [72]. And experiments are set up as follows. Firstly, we generate the sensing matrix $\Phi \in R^{m \times n}$, where each entry of the matrix follows either i.i.d standard normal distribution, $\phi_{ij} \backsim \mathcal{N}(0, 1)$, or i.i.d symmetric Bernoulli ± 1 (the distribution will be specified in every experiment). We then compute a length-n sparse vector x^* with k-nonzero entries drawn from the standard normal distribution. Lastly, we calculate the 1-bit measurements y through the sign function of measurements, namely, $y_i = 1$ if $\text{sign}[(\Phi x^*)_i] > 0$, $y_i = -1$ if $\text{sign}[(\Phi x^*)_i] < 0$, and $y_i = 0$ if $\text{sign}[(\Phi x^*)_i] = 0$.

In standard compressive sensing, for every pair of (Φ, x^*), we set the measurements b as $b = \Phi x^*$, where $\Phi \in R^{m \times n}$ with $m < n$ is a full row rank matrix. As discussed in chapter 2, if x^* is sufficiently sparse, for example, $\|x^*\|_0 < \frac{1}{2}\mathrm{Spark}(\Phi)$, x^* is the unique sparsest solution to the linear system $\Phi x^* = b$. Theoretically, under the mutual coherence condition, ℓ_1-minimization can exactly recover the sparsest signal x^*. Moreover, under the RIP, NSP and RSP-based conditions, the sparsest signal x^* can be exactly recovered by some algorithms, such as greedy algorithms [104, 93, 47, 121, 55, 19, 20, 101] and reweighted ℓ_1-minimization [134, 135, 40, 66, 42, 84, 126]. On the other hand, to numerically measure the performance of a recovery algorithm as mentioned in chapter 2, some families of random matrices admit a certain RIP condition with high probability. Thus, performances of recovery algorithms are usually tested by Gaussian matrices and Bernoulli matrices in experiments. And, if x is the solution to a recovery algorithm, one may use either ℓ_2-norm of residual $\|x - x^*\|_2$, ℓ_∞-norm of the difference $\|x - x^*\|_\infty$, or relative error (RE) $\frac{\|x^* - x\|_2}{\|x^*\|_2}$ as a stopping condition. For instance, if the solution x satisfies the stopping condition

$$\mathrm{RE} = \frac{\|x^* - x\|_2}{\|x^*\|_2} < \xi,$$

where $\xi \in (0, 1)$ is often chosen as a sufficient small number, such as 10^{-6}, we say that both values and support set of x^* are exactly recovered by the algorithm.

In 1-bit compressive sensing, since the amplitude of any signal is lost, in experiment, we expect 1-bit basis pursuit to find either the support set of a sparse signal x^* or a vector approximately sparse to x^* or another k-sparse vector x such that $\|x\|_0 \leqslant \|x^*\|_0$. One may ask that what sensing matrices and stopping conditions can ensure such a performance of 1-bit basis pursuit. Plan and Vershynin [105] proved that the approximate sparse recovery can be achieved via the linear program (4.2.4) when the sensing matrix Φ has independent standard normal entries. As 1-bit basis pursuit can be reformulated as a linear program, it can be shown that solution sets of 1-bit basis pursuit and linear program (4.2.4) are

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equal under certain conditions on the matrix Φ. Thus, it motivates us to use Gaussian matrices to test the approximate sparse recovery for 1-bit basis pursuit. Since Gopi et al. constructed a Bernoulli-type of sensing matrix using the union free family of sets [62] and expander graphs [81, 11] for the support recovery in [71], it encourages us to test the support recovery for 1-bit basis pursuit via Bernoulli matrices. We will explain more why choose Bernoulli matrix for 1-bit compressive sensing in later sections.

For the approximate sparse recovery, most recovery algorithms in 1-bit compressive sensing use the normalized ℓ_2-norm residual defined by (5.2.1) as a criterion to verify any approximately sparse reconstruction. As normalized ℓ_2-norm residual only demonstrates the differences between components of signal x^* and approximately sparse reconstruction x, in order to capture a geometry-relationship between x^* and x, we introduce another stopping criterion based on the cosine value of the angle between x^* and x to directly visualize the geometric distance.

Let $\theta(t, z)$ be the angle between two distinct vectors t and z in R^n evaluated as follows

$$\theta(t, z) = \arccos(\frac{\langle t, z \rangle}{\|t\|_2 \cdot \|z\|_2}).$$

Note that the angle reveals a natural distance between any two vectors with different magnitudes and the cosine value of it maps the distance to a real value in $[-1, 1]$. Thus, in our experiments, we use the cosine value of an angle as an alternative criterion. And we say that 1-bit basis pursuit successfully recovers an approximately sparse vector x close to the sparse signal x^* if the solution x satisfies the cosine-stopping condition

$$\cos(\theta(x^*, x)) \geqslant \tau, \tag{5.3.1}$$

where the cosine-stopping threshold τ is a large number close to 1, such as 0.99 or 0.995. For instance, when $\tau = 0.99$, the angle between x^* and x is less than 8 degree, which

indicates that x is geometrically close to x^*.

5.3.1 Regime shifting: From underdetermined matrix to overdetermined matrix

Following the framework of standard compressive sensing that aims to recover the sparse signal from limited number of measurements, our first experiment is to show the performance of 1-bit basis pursuit when the matrix $\Phi \in R^{m \times n}$ with $m < n$ has the full row rank. In this experiment, each entry of the matrix $\Phi \in R^{m \times n}$ is randomly drawn from the standard normal distribution with fixed $m = 200$ and $n = 1000$, and the sparsity level k of vector x^* varies from $k = 1$ to $k = 200$ ($k = m$). Each reconstruction is repeated for 50 trials with generated random matrices Φ at each sparsity level. And the cosine-stopping threshold τ is set as 0.99. The results of this experiment are shown in Figure 5.1.

Let x be the solution of 1-bit basis pursuit algorithm. Figure 5.1 demonstrates the performance of 1-bit basis pursuit with underdetermined Gaussian matrices. We display the performance of 1-bit basis pursuit in three ways: (i) approximate sparse recovery: Figure 5.1(a) and 5.1(b) show the trends of two stopping criteria, i.e., the normalized ℓ_2-norm residual and the cosine value of the angle, and Figure 5.1(d) shows the success rate of approximate sparse recovery based on the cosine-stopping criterion; (ii) support recovery: the success rate of finding a solution x satisfying $\text{Supp}(x) \subseteq \text{Supp}(x^*)$ is shown in Figure 5.1(c); (iii) cardinality recovery: the ability of 1-bit basis pursuit to solve the problem (5.2.2) is plotted in Figure 5.1(e) . Figure 5.1(a) shows that 1-bit basis pursuit obtains large normalized ℓ_2-norm residuals even at high sparsity level $k \in [1, 25]$. Thus, the normalized ℓ_2-norm residual criterion is not enough for identifying an approximate sparse solution. On the other hand, it is easy to numerically visualize the distance between x^* and x by the cosine-stopping criterion as shown in Figure 5.1(b). Based on the cosine-stopping condition, it also shows in Figure 5.1(d) that 1-bit basis pursuit cannot achieve
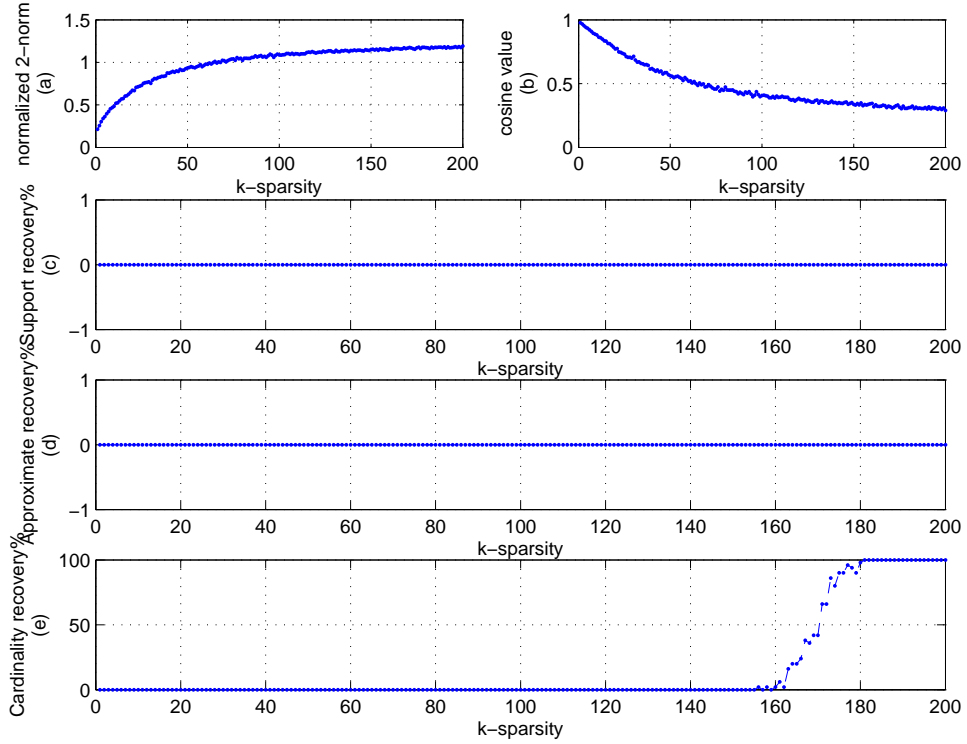
Figure 5.1: Performance of 1-bit basis pursuit with underdetermined Gaussian matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x and x^*, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate sparse recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system $(5.1.1)$ at each sparsity level.

the approximate sparse recovery. In addition, Figure 5.1(c) demonstrates that the support recovery is hard to be achieved via 1-bit basis pursuit in this experiment. As the sparsity level increases, 1-bit basis pursuit is more likely to find a sparse solution x to the linear system $(5.1.1)$, where x has different support patterns from the signal x^* with $\|x\|_0 \leqslant \|x^*\|_0$. Conventionally, the cardinality recovery for 1-bit basis pursuit can always be achieved provided by any matrix $\Phi \in R^{m \times n}$ when the sparsity level k is large.

To observe more clearly, we zoom in Figure 5.1 in the sense that we show performances of 1-bit basis pursuit for a random matrix $\Phi \in R^{200 \times 1000}$ with independent standard normal entries when $k = 1$ and $k = 2$.

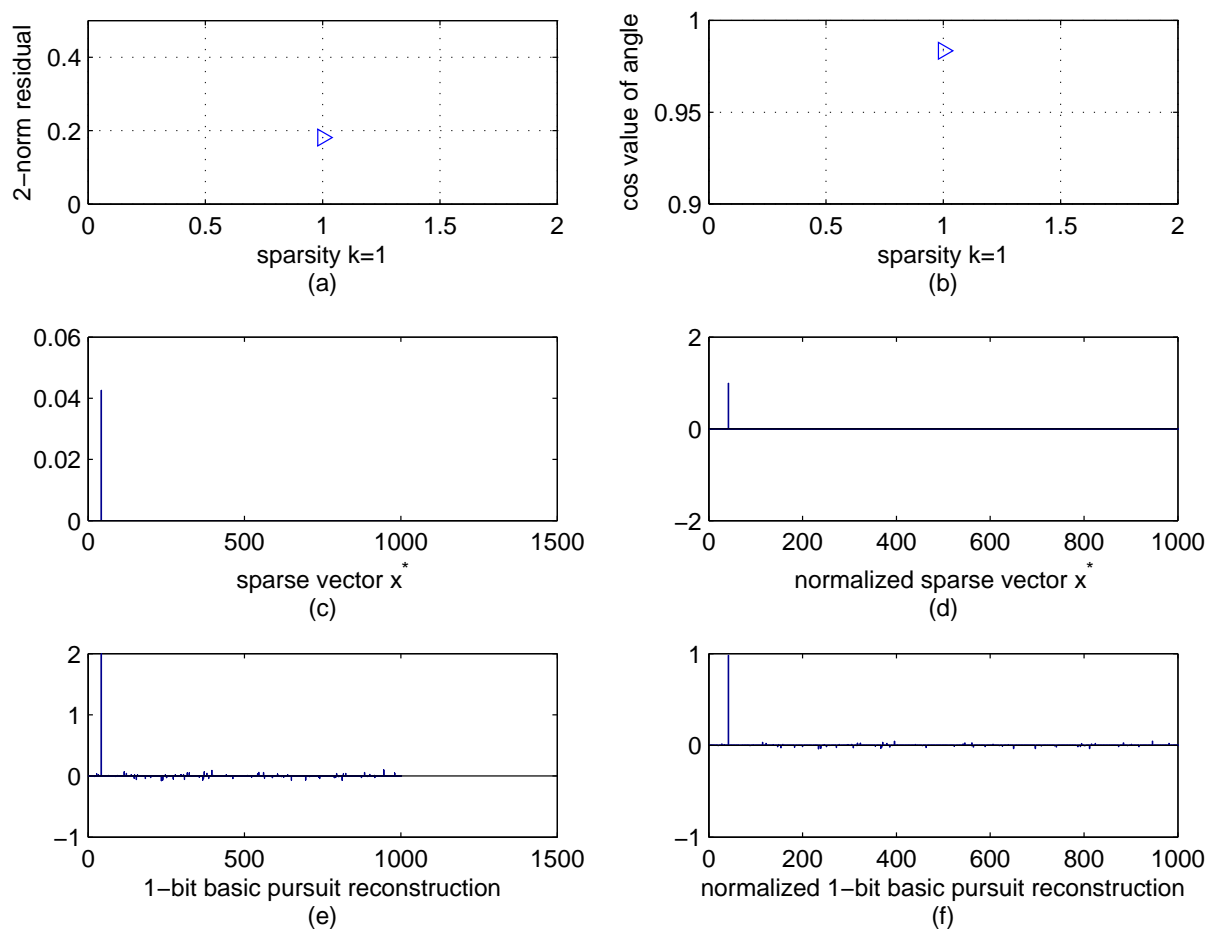We see from Figure 5.2 and Figure 5.3 that 1-bit basis pursuit can detect locations

Figure 5.2: Performance of 1-bit basis pursuit via a Gaussian matrix $\Phi \in R^{200 \times 1000}$ at the fixed sparsity level $k = 1$. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Sparse vector x^* and its normalized value (d), and (e) Reconstruction of 1-bit basis pursuit and its normalized value (f).

of nonzero components of sparse signal x^*. Unfortunately, the solution x of 1-bit basis pursuit has some coefficients that are relatively small but cannot be counted as zeros. More specifically, due to such a perturbation in the solution, when $k = 1$, the normalized ℓ_2-norm residual is nearly 0.2 and the cosine value of $\theta(x^*, x)$ is around 0.98, while, when $k = 2$, the normalized ℓ_2-norm residual is more than 0.35 and the cosine value of $\theta(x^*, x)$ is around 0.94, respectively. This gives a hint why 1-bit basis pursuit may not achieve support recovery and approximate sparse recovery in the experiment demonstrated in
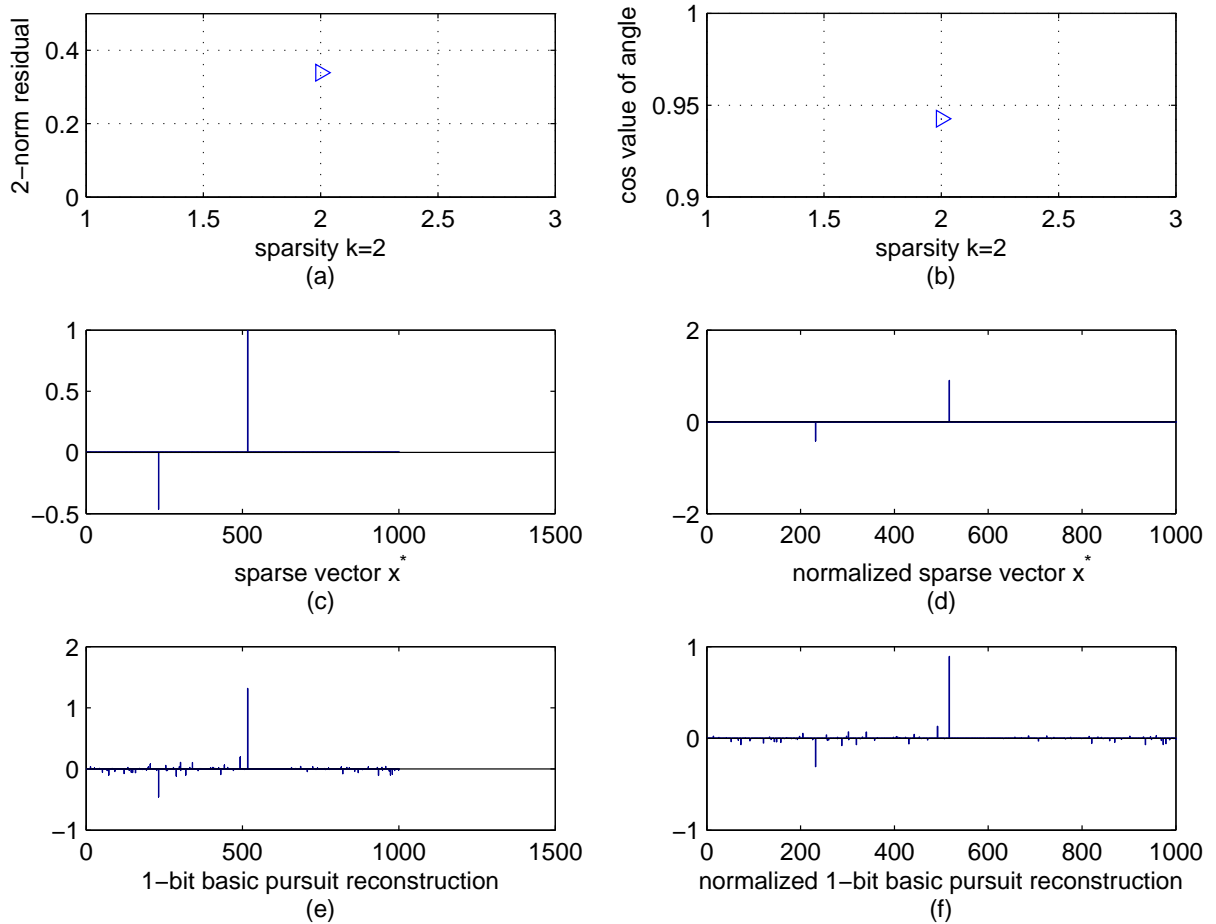
Figure 5.3: Performance of 1-bit basis pursuit via a Gaussian matrix $\Phi \in R^{200 \times 1000}$ at the fixed sparsity level $k = 2$. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Sparse vector x^* and its normalized value (d), and (e) Reconstruction of 1-bit basis pursuit and its normalized value (f).

Figure 5.1.

The main issue that limits the performance of 1-bit basis pursuit is perturbations in the solution. Because of the sign-mapping from a continuous space to a discrete space, it causes perturbations in solutions to the linear system (5.1.1). In principle, ℓ_1-norm favors sparse solutions, but 1-bit basis pursuit may pick a solution with a few small nonzero components but having the least ℓ_1-norm.

If we dig deeper, one of sources causes perturbation-type of solutions of 1-bit basis

pursuit is the limited number of 1-bit measurements. When the matrix $\Phi \in R^{m \times n}$ with $m < n$ is full row rank, the null space of Φ is nonempty and contributes to the perturbations in the solution. To avoid such a situation, one approach is to refine the null space of Φ such that it only contains zero element. This can be achieved when the matrix $\Phi \in R^{m \times n}$ has a full column rank, which implies that Φ may be a square or overdetermined matrix. In general, having more number of 1-bit measurements forces the feasible region of 1-bit basis pursuit narrowing towards the location of the sparse signal.

We perform the next experiment to reveal the relationship between the number of measurements and capability of 1-bit basis pursuit. In this experiment, the sparsity level k of vector x^* is fixed at $k = 5$, and the sensing matrix $\Phi \in R^{m \times n}$ is randomly drawn from the standard normal distribution with $n = 400$ and varied number of measurements from $m = 50$ to $m = 2000$ ($m = 5n$). At each measurements level, the reconstruction is repeated for 50 trials with generated random matrix Φ and the cosine-stopping threshold τ is set as 0.99. The results are shown in Figure 5.4

Figure 5.4 demonstrates that the performance of 1-bit basis pursuit on the approximate sparse recovery has improved by having more number of 1-bit measurements. This indicates that the matrix Φ changing from underdetermined matrix to overdetermined matrix promotes the approximate sparse recovery provided by Gaussian matrices. Specifically, as the sparsity level is fixed at $k = 5$, in the underdetermined case like $m/n = \frac{1}{4}$, 1-bit basis pursuit cannot obtain any approximately sparse solutions to the linear system (5.1.1). When the number of measurements increases to the overdetermined case like $m/n = 2.5$ ($m = 1000$), 1-bit basis pursuit starts stably returning approximately sparse reconstructions satisfying the cosine-stopping condition, as shown in Figure 5.4(d).

But one may argue that having a large number of measurements goes beyond the goal of compressive sensing, to recover a sparse signal from limited measurements. Note that the regime of compressive sensing concerns about two factors: the number of mea-
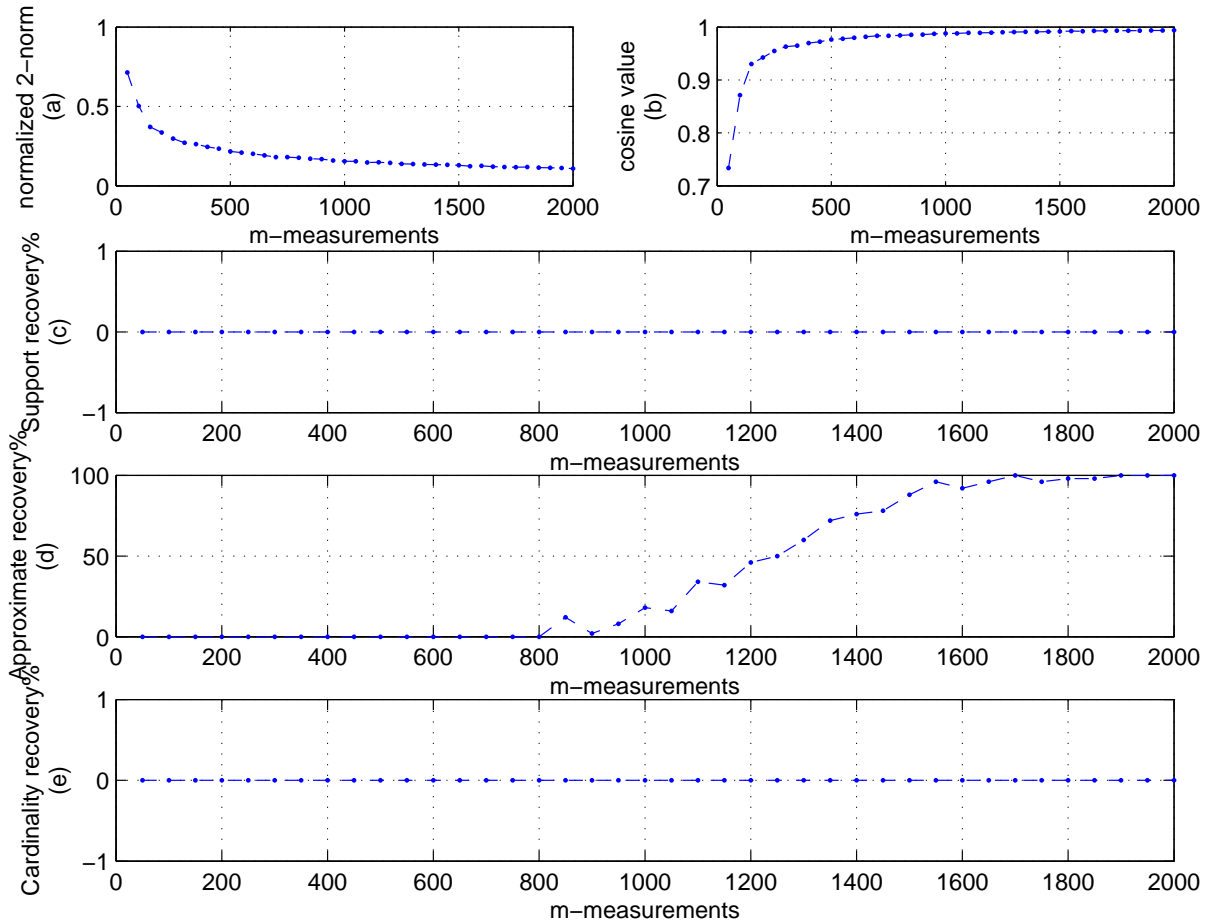
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Figure 5.4: Performance of 1-bit basis pursuit with Gaussian matrices and varied measurements at the fixed sparsity level $k = 5$. (*a*) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (*b*) Cosine value of the angle between x^* and x, (*c*) Success rate of support recovery at each measurements level, (*d*) Success rate of approximate sparse recovery at each measurements level, and (*e*) Success rate of finding a sparsest solution to the linear system (5.1.1) at each measurements level.

surements and the number of bit precision. The focus of compressive sensing is on the least number of measurements but infinite bit precision, in other words, having the least number of measurements but keeping the number of bit precision as accurate as possible. Thus, since the focus of 1-bit compressive sensing is on the least number of bit precision, namely, only one bit per measurement, ideally, the number of 1-bit measurements should not be bounded. Hence, the regime of 1-bit compressive sensing including the case that

the matrix $\Phi \in R^{m \times n}$ is overdetermined is not against the regime of compressive sensing. On the other hand, in practice, when the cost of having precise measurements is more than the cost of increasing the number of measurements, it is practical and reasonable to study the 1-bit compressive sensing models from a large number of coarsely quantized measurements [24].

5.3.2   Bernoulli sensing matrices for 1-bit compressive sensing

Based on the above empirical results, another source may cause the perturbation-type of solutions of 1-bit basis pursuit is the random Gaussian matrix. When the 1-bit measurements y are coarsely quantized from measurements Φx^*, as it only stores the sign information of the product between the sparse signal x^* and the Gaussian matrix Φ, limited information of the sparse signal x^* can be expressed from 1-bit measurements. To transmit more information of the sparse signal x^* to the 1-bit measurements and to reduce the influence of each entry of Φ on the 1-bit measurements, it encourages us to construct a Bernoulli-type of sensing matrix by keeping the sign of every entry of the Gaussian matrix for improving the performance of 1-bit basis pursuit. In addition, Gopi et al. [71] also proved that Bernoulli-type of sensing matrix can be used for the uniform support recovery via some combinatorial algorithms. Thus, we conduct the next experiment to show the performance of 1-bit basis pursuit with Bernoulli matrices.

In this experiment, we compare success rates of support recovery, approximate sparse recovery and cardinality recovery via 1-bit basis pursuit with both Gaussian and Bernoulli matrices, respectively. Based on the result shown in Figure 5.4(d), we set the dimension of Φ as $m = 1000$ and $n = 400$. Let each entry of Φ follows the standard normal distribution and the symmetric Bernoulli distribution $\{-1, 1\}$, respectively, and let the sparsity level k of x^* vary from $k = 1$ to $k = 50$. At each sparsity level, the reconstruction is repeated for 50 trials with generated random matrices Φ, and the cosine-stopping threshold τ is

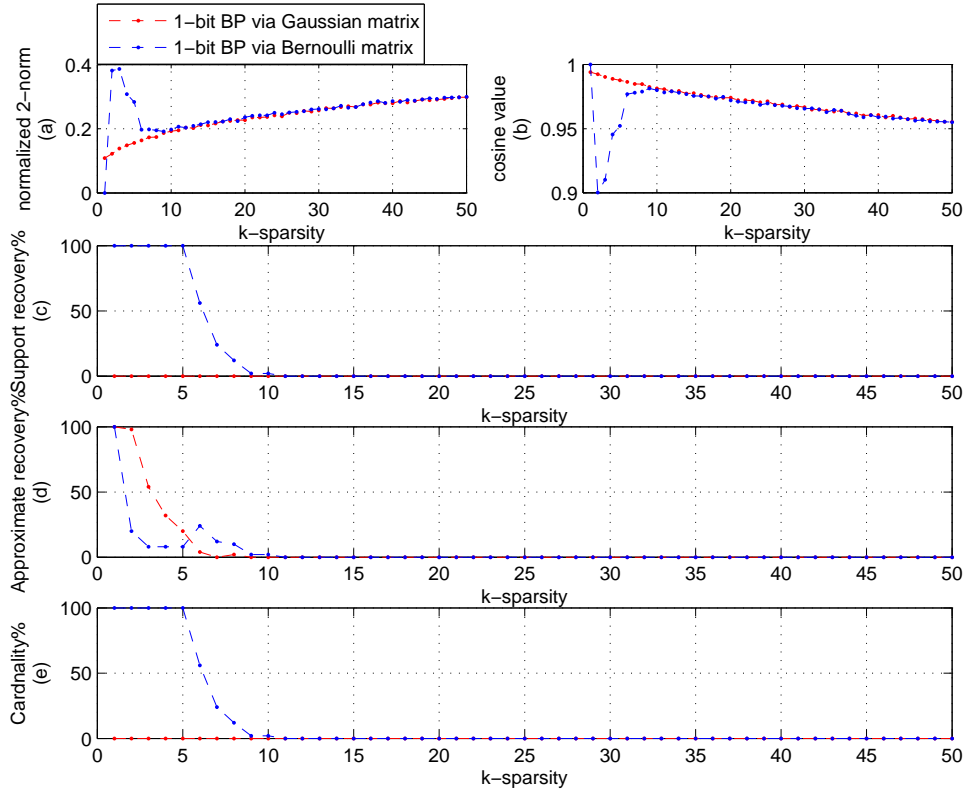set as 0.99. The results are depicted in Figure 5.5.

Figure 5.5: Comparison of performances of 1-bit basis pursuit with overdetermined Gaussian matrices and Bernoulli matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate sparse recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

Figure 5.5 demonstrates that 1-bit basis pursuit with Bernoulli matrices achieves support recovery, approximate sparse recovery and cardinality recovery when $k \in [1, 10]$, while 1-bit basis pursuit with Gaussian matrices only has approximate sparse recovery. As shown in Figure 5.5(d), Gaussian matrices provide a more stable and better performance of 1-bit basis pursuit on the approximate sparse recovery than Bernoulli matrices when $k \in [2, 5]$, while the performance of Bernoulli matrices can exceed that of Gaussian matrices on the approximate sparse recovery when $k \in [6, 8]$. Remarkably, as shown in

Figure 5.5(c) and 5.5(e), Bernoulli matrices outperform Gaussian matrices on both support recovery and cardinality recovery at each sparsity level. From Figure 5.5(c) and 5.5(d), we note that, in this experiment, 1-bit basis pursuit with Bernoulli matrices can recover the sparse signal within a positive scalar when $k = 1$ and may find a sparser solution within the support set of the sparse signal when $k = 2$.

To observe more clearly, we zoom in Figure 5.5 in the sensing that we show the solvability of 1-bit basis pursuit for a random Bernoulli matrix $\Phi \in R^{1000 \times 400}$ with independent symmetric Bernoulli ± 1 entries when $k = 1$ and $k = 2$, respectively, as illustrated in Figure 5.6 and Figure 5.6.

As demonstrated in Figure 5.6, 1-bit basis pursuit can successfully find a sparse reconstruction that is positive proportional to the sparse signal. Such a result is due to the linear independency of columns from an overdetermined Bernoulli matrix Φ. Specifically, when $k = 1$, the 1-bit measurements is actually obtained from the column of Φ associated with the nonzero coefficient of the sparse signal. Since all columns from Φ are linearly independent, 1-bit basis pursuit can accurately detect the support set of the sparse signal and recover the sparse signal within a positive scalar factor.

Results shown in Figure 5.7 reveal an interesting phenomenon that 1-bit basis pursuit can find a sparser reconstruction x within the support set of the sparse signal x^* such that $\text{Supp}(x) \subseteq \text{Supp}(x^*)$ and $\text{sign}(\Phi x) = \text{sign}(\Phi x^*)$ when $k = 2$. This phenomenon has also been discussed in [3, 105] through the following example. Suppose that all entries of Φ are independent ± 1 valued symmetric random variables. Then, for the vectors $x^* = (1, \frac{1}{2}, 0, \cdots, 0)^T$ and $\widehat{x} = (1, 0, 0, \cdots, 0)^T$, one can easily have $\text{sign}(\Phi x^*) = \text{sign}(\Phi \widehat{x})$. Even if it is hard to distinguish signals x^* and $\widehat{x}$ from 1-bit measurements, it is still possible to achieve the support recovery via 1-bit basis pursuit.

Therefore, as the 1-bit measurements from the Bernoulli matrix contains more information of the sparse signal, 1-bit basis pursuit is more likely to achieve support recovery
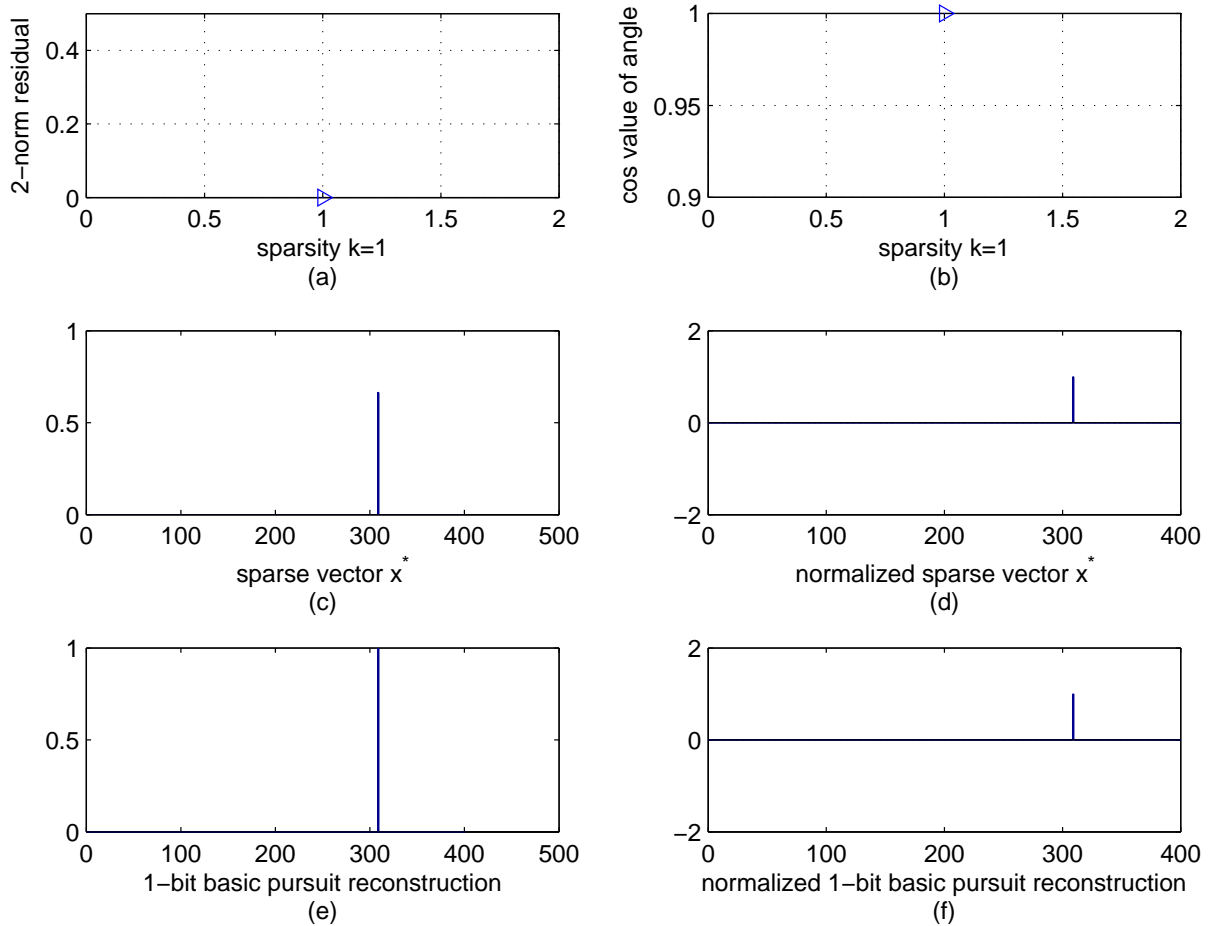
Figure 5.6: Performance of 1-bit basis pursuit via a Bernoulli matrix $\Phi \in R^{1000 \times 400}$ at the fixed sparsity level $k = 1$. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Sparse vector x^* and its normalized value (d), and (e) Reconstruction of 1-bit basic pursuit and its normalized value (f).

and cardinality recovery when the sparsity level k is small.

5.4 Enhancing the performance of 1-bit basis pursuit

In standard compressive sensing, it is well known that the Gaussian matrix is one of 'good' matrices, which satisfies certain RIP properties with a high probability, a variety of algorithms can find a sparse solution via Gaussian matrices in experiments. However, from
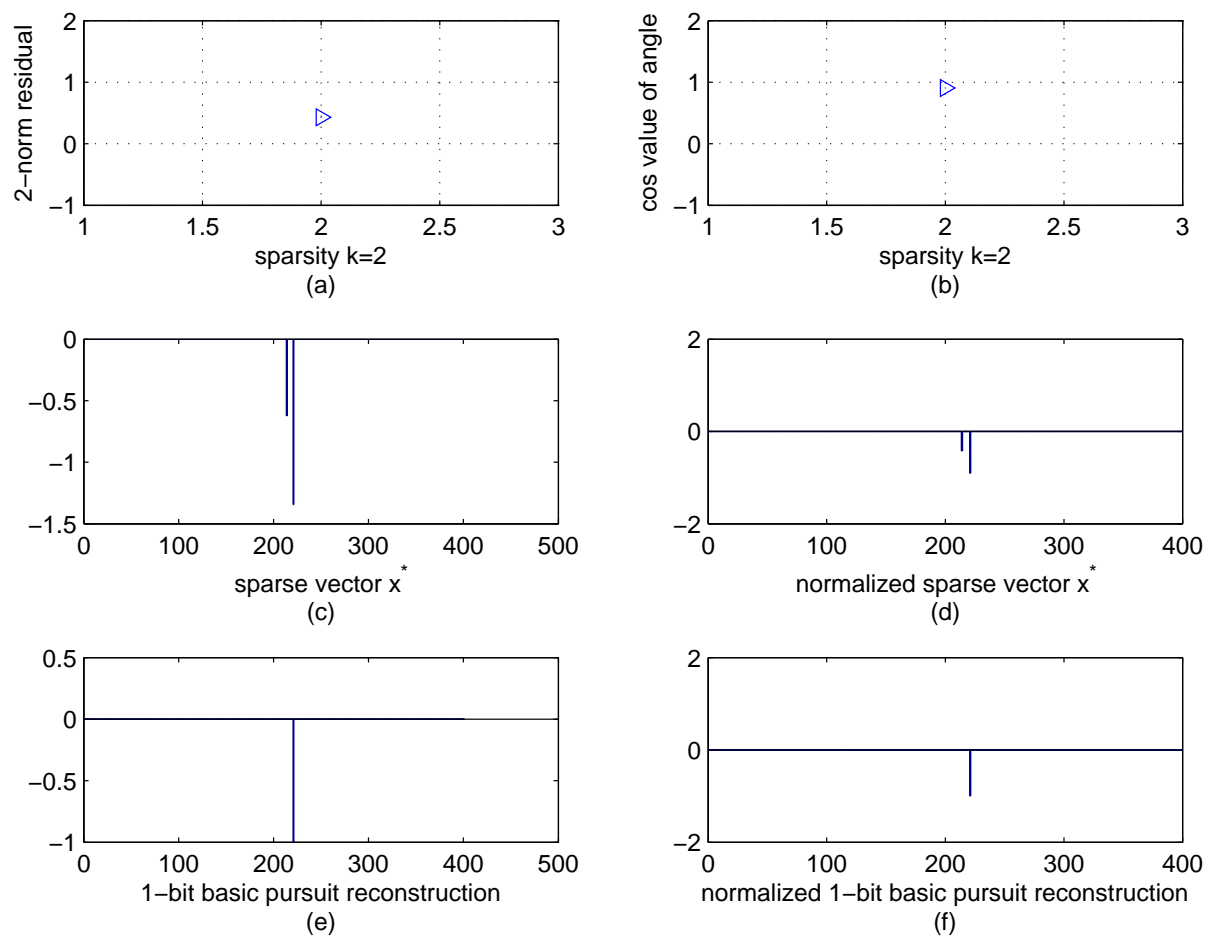
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Figure 5.7: Performance of 1-bit basis pursuit via a Bernoulli matrix $\Phi \in R^{1000 \times 400}$ at the fixed sparsity level $k = 2$. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Sparse vector x^* and its normalized value (d), and (e) Reconstruction of 1-bit basic pursuit and its normalized value (f).

all the results shown above, 1-bit basis pursuit with Gaussian matrices performs poorly for support recovery and approximate sparse recovery even if the matrix is overdetermined. Such the results motivate us to consider how to enhance the behavior of 1-bit basis pursuit with both Gaussian matrices and Bernoulli matrices. Here, we introduce two approaches to improve the performance of 1-bit basis pursuit: the truncated 1-bit measurements method and the reweighted 1-bit ℓ_1-minimization method inspired by the reweighted ℓ_1-minimization for the standard compressive sensing [40, 42, 100, 66, 135, 134, 136].

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5.4.1 Truncated 1-bit measurements

In general, it is hard to have zero components in the 1-bit measurements generated from Gaussian and Bernoulli matrices in experiments. Thus, some information of measurements may not be accurately transmitted to the corresponding 1-bit measurements without zero components. We will use the following examples to illustrate the necessity of having zero components in the 1-bit measurements in experiments.

Example 5.4.1 :

For a given matrix $\Phi \in R^{5\times 10}$ and the signal $x^* \in R^{10}$ with $\Phi =$

$$
\begin{bmatrix}
1.4434 & 1.7641 & -0.7808 & 0.9520 & 2.2803 & 0.1464 & 0.1896 & -1.5235 & 0.5265 & -0.0665 \\
-0.9239 & 0.4175 & 1.7765 & 0.2216 & 0.2388 & 0.4551 & 0.7829 & -0.0186 & 1.8219 & -0.3533 \\
-1.3759 & -0.6684 & 1.6528 & -0.6419 & -1.0060 & 0.8174 & 1.4438 & -0.0821 & -0.2555 & 0.1124 \\
0.7818 & -0.9291 & 1.3999 & 1.9165 & -1.1382 & 0.0747 & -0.0374 & -1.2145 & 0.4200 & -0.8387 \\
-0.2621 & 0.4465 & -0.3989 & -0.6691 & -0.0936 & 0.2502 & -0.4323 & -1.7792 & 0.5645 & -0.1667
\end{bmatrix},
$$

and $x^* = [\ -0.9028,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0.1129,\ 0,\ 0\ ]$, where every entry in the matrix Φ and nonzero entries of x^* are drawn from the standard normal distribution, compute the measurements b and 1-bit measurements y, i.e.,

$$
b = \Phi x^* = \begin{bmatrix} -1.4751 \\ 0.8319 \\ 1.2329 \\ -0.8429 \\ 0.0357 \end{bmatrix}, y = \mathrm{sign}(\Phi x^*) = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.
$$

Compared with the measurements b, some information cannot be observed from 1-bit measurements. For instance, 1-bit measurements for both 0.0357 and 1.2329 are 1, while 0.0357 has the least absolute value among all measurements and 1.2329 is the largest positive component in measurements. This indicates that 1-bit measurements $y \in \{-1, 1\}^m$ ignores the diversity among measurements and unifies all positive and negative measurements into 1 and -1, respectively. To accurately transport messages of measurements, the 1-bit representations of 0.0375 and 1.2329 should be different. Hence, we artificially include zero components in 1-bit measurements to represent some relatively small measurements in experiments, for instance, the 1-bit measurement of 0.0375 in this example can be considered as 0. And we name such an approach as the truncated sign-mapping, defined as follows.

Definition 5.4.2 : *For a matrix $\Phi \in R^{m \times n}$ and a signal $x \in R^n$, the truncated sign-mapping $\widetilde{sign} \colon R^m \mapsto \{-1, 0, 1\}^m$ is defined as*

$$
\begin{aligned}
\widetilde{sign}[(\Phi x)_i] &= 1 && if \ (\Phi x)_i > g_t && for \ any \ i \in \{1, \cdots, m\}, \\
\widetilde{sign}[(\Phi x)_i] &= 0 && if \ |(\Phi x)_i| \leqslant g_t && for \ any \ i \in \{1, \cdots, m\}, \\
\widetilde{sign}[(\Phi x)_i] &= -1 && if \ (\Phi x)_i < -g_t && for \ any \ i \in \{1, \cdots, m\},
\end{aligned}
\tag{5.4.1}
$$

where the vector $g \in R^m$ consists of absolute values of all elements in Φx in an ascending order and g_t denotes the t-th component of g.

Remark 5.4.3 : *g_t is a threshold dividing all the measurements into three groups: absolute positive measurements, absolute negative measurements and relatively small measurements close to zero, such that the truncated 1-bit measurements $\tilde{y} = \widetilde{sign}(\Phi x)$ defined by (5.4.1) has zero components defined from the relatively small measurements. To evaluate the threshold g_t, in our experiments, we set the index t as the number of relatively small measurements, namely, $t = h \times m$, where m is the number of measurements and h is the*

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assumed percentage of relatively small measurements in total. Furthermore, the Definition 5.4.2 can be applied to any matrices with a suitable threshold.

Continuing to Example 5.4.1, we compute the truncated 1-bit measurements and compare solutions of 1-bit basis pursuit from the 1-bit measurement and from the truncated 1-bit measurements.

Example 5.4.4 :

For the Gaussian matrix Φ and the signal x^* given in Example 5.4.1, let the percentage of relatively small measurements h be 0.2, then compute the vector g, in an ascending order of absolute values of Φz, which is

$$g = [\ 0.0357, \quad 0.8319, \quad 0.8429, \quad 1.2329, \quad 1.4751\ ]^T.$$

As $t = h \times m = 1$, the threshold g_1 is 0.0357. Thus, by Definition 5.4.2, the truncated 1-bit measurements becomes $\tilde{y} = \widetilde{\text{sign}}(\Phi z) = [\ -1, 1, 1, -1, 0\ ]^T$. According to the 1-bit measurements y, index sets J_+, J_- and J_0 are $J_+ = \{2, 3, 5\}$, $J_- = \{1, 4\}$ and $J_0 = \emptyset$. For the given matrix Φ, the linear system (5.1.1) is written as

$$\begin{bmatrix} -0.9239 & 0.4175 & 1.7765 & 0.2216 & 0.2388 & 0.4551 & 0.7829 & -0.0186 & 1.8219 & -0.3533 \\ -1.3759 & -0.6684 & 1.6528 & -0.6419 & -1.0060 & 0.8174 & 1.4438 & -0.0821 & -0.2555 & 0.1124 \\ -0.2621 & 0.4465 & -0.3989 & -0.6691 & -0.0936 & 0.2502 & -0.4323 & -1.7792 & 0.5645 & -0.1667 \end{bmatrix}$$

$x \geqslant [\ 1, 1, 1\ ]^T,$

$$\begin{bmatrix} 1.4434 & 1.7641 & -0.7808 & 0.9520 & 2.2803 & 0.1464 & 0.1896 & -1.5235 & 0.5265 & -0.0665 \\ 0.7818 & -0.9291 & 1.3999 & 1.9165 & -1.1382 & 0.0747 & -0.0374 & -1.2145 & 0.4200 & -0.8387 \end{bmatrix}$$

$x \leqslant [\ -1, -1\ ]^T.$

Hence, 1-bit basis pursuit obtains a solution

$$x = [\ -0.7422,\ \ 0,\ \ 0,\ \ -0.4118,\ \ 0,\ \ 0,\ \ 0,\ \ -0.2280,\ \ 0.2203,\ \ 0\ ].$$

According to the truncated 1-bit measurements $\tilde{y}$, index sets J_+, J_- and J_0 are $J_+ = \{2,3\}$, $J_- = \{1,4\}$ and $J_0 = \{5\}$.

For the given matrix Φ, the linear system (5.1.1) is written as

$$\begin{bmatrix} -0.9239 & 0.4175 & 1.7765 & 0.2216 & 0.2388 & 0.4551 & 0.7829 & -0.0186 & 1.8219 & -0.3533 \\ -1.3759 & -0.6684 & 1.6528 & -0.6419 & -1.0060 & 0.8174 & 1.4438 & -0.0821 & -0.2555 & 0.1124 \end{bmatrix}$$

$\widehat{x} \geqslant [\ 1,1,1\ ]^T,$

$$\begin{bmatrix} 1.4434 & 1.7641 & -0.7808 & 0.9520 & 2.2803 & 0.1464 & 0.1896 & -1.5235 & 0.5265 & -0.0665 \\ 0.7818 & -0.9291 & 1.3999 & 1.9165 & -1.1382 & 0.0747 & -0.0374 & -1.2145 & 0.4200 & -0.8387 \end{bmatrix}$$

$\widehat{x} \leqslant [\ -1,-1\ ]^T,$

$$\begin{bmatrix} -0.2621 & 0.4465 & -0.3989 & -0.6691 & -0.0936 & 0.2502 & -0.4323 & -1.7792 & 0.5645 & -0.1667 \end{bmatrix}$$

$\widehat{x} = 0.$

Hence, 1-bit basis pursuit obtains a solution

$$\widehat{x} = [\ -1.0417,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0.1605,\ \ 0.0223,\ \ 0\ ],$$

which is sparser than the solution x solved from the 1-bit measurements y. And the normalized ℓ_2-norm residual is reduced from $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2 = 0.6609$ to $\left\| \frac{x^*}{\|x^*\|_2} - \frac{\widehat{x}}{\|\widehat{x}\|_2} \right\|_2 = 0.0355$.

Hence, we carry out the next experiment to compare performances of 1-bit basis pursuit from the 1-bit measurements and from the truncated 1-bit measurements with Bernoulli and Gaussian matrices, respectively. Firstly, we perform the experiment on the Gaussian matrix. In this experiment, we choose the dimension of Gaussian matrices as

$m = 1000$ and $n = 400$ since 1-bit basis pursuit stably returns a reconstruction $x \in R^{400}$ with 1000 1-bit measurements in Figure 5.4(d). In each trial, we draw a Gaussian matrix $\Phi \in R^{1000 \times 400}$ with independent standard normal entries and set the sparsity level k of signal x^* to be varied from $k = 1$ to $k = 50$. The reconstruction at each sparsity level is repeated for 50 trials with the cosine-stopping threshold $\tau = 0.99$, and the assumed percentage of relatively small measurements is $h = 5\%$.
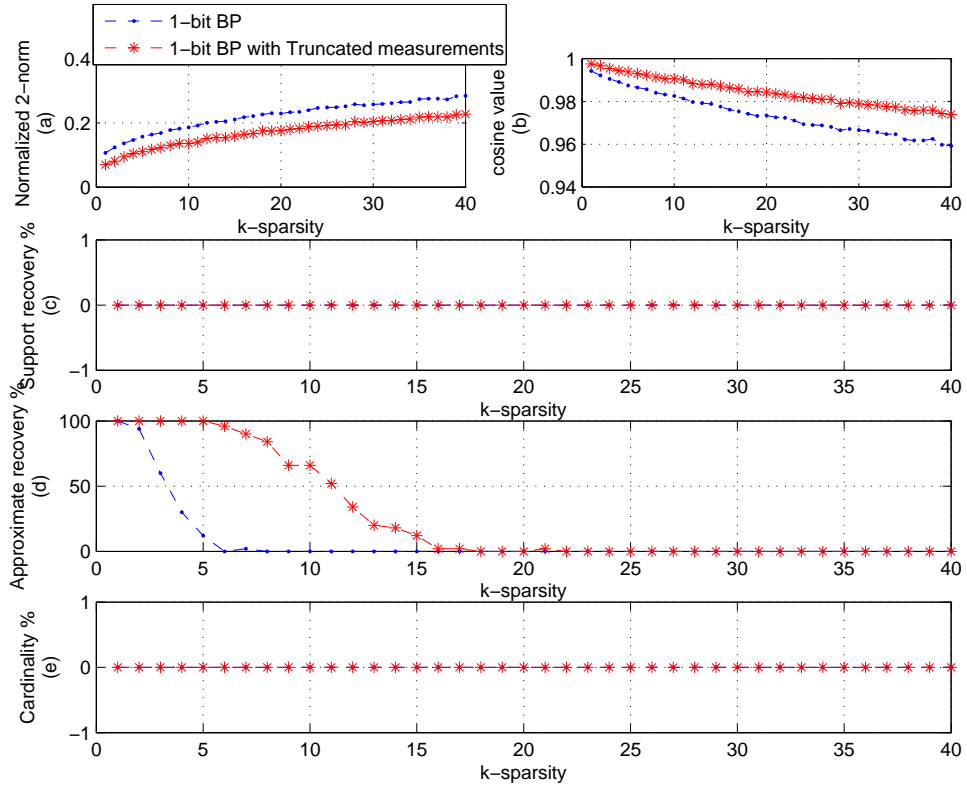

Figure 5.8: Performance of 1-bit basis pursuit from the truncated 1-bit measurements with overdetermined Gaussian matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

The results of this experiment are plotted in Figure 5.8. Specifically, in Figure 5.8(d), we see that the performance of 1-bit basis pursuit on the approximate sparse reconstruc-

tion has significantly been improved by the truncated 1-bit measurements. When $k = 5$, the success rate of 1-bit basis pursuit from 1-bit measurements has dropped to around 20%, while an approximately sparse vector to the sparse signal can be 100% recovered from the truncated 1-bit measurements. Unfortunately, for Gaussian matrices, it still seems impossible to have support recovery and cardinality recovery by the truncated 1-bit measurements method. Next, we perform the same experiment on the Bernoulli
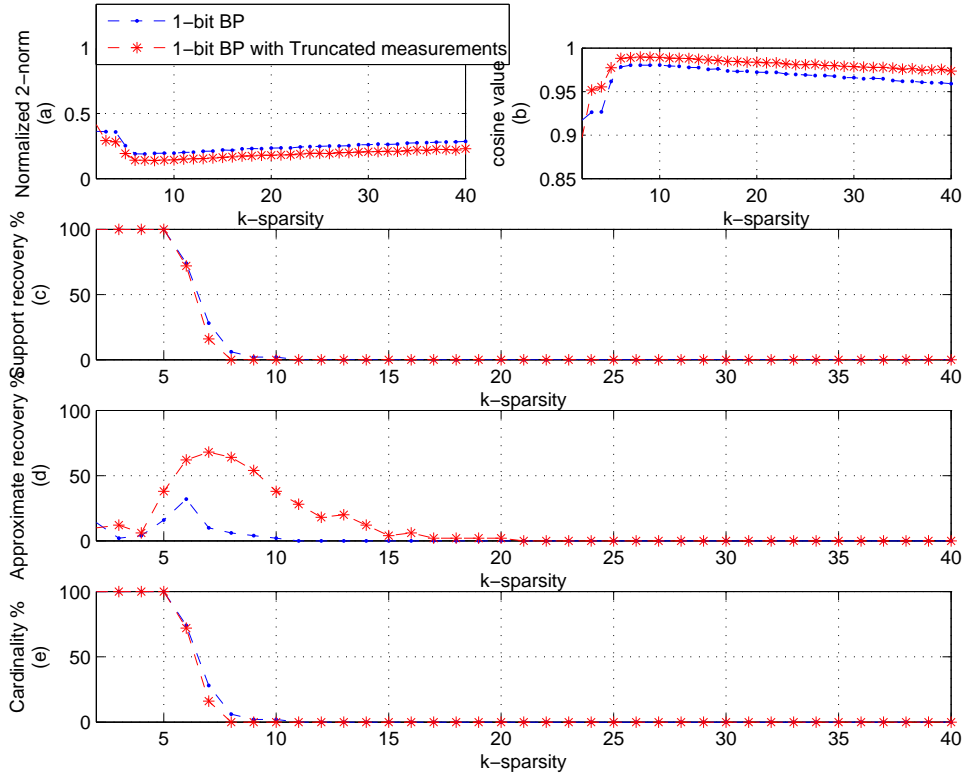

Figure 5.9: Performance of 1-bit basis pursuit from the truncated 1-bit measurements with overdetermined Bernoulli matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

matrix. In each trial, we draw a Bernoulli matrix $\Phi \in R^{1000 \times 400}$ with independent symmetric Bernoulli ± 1 entries and set the sparsity level k of signal x^* varied from $k = 2$ to $k = 50$ (1-bit basis pursuit successfully recovers the sparse signal within a positive

scalar by Bernoulli matrix when $k = 1$, hence, truncating the 1-bit measurements is not necessary in this case). The reconstruction at each sparsity level is repeated for 50 trials with the cosine-stopping threshold $\tau = 0.99$, and the assumed percentage of relatively small measurements is $h = 5\%$. Surprisingly, the results of this experiment shown in Figure 5.9(c) and 5.9(e) demonstrate that truncating 1-bit measurements generated from Bernoulli matrix does not promote support recovery and cardinality recovery, while the approximate sparse recovery has been significantly improved, as shown in Figure 5.9(d).

Based on results illustrated in Figure 5.8 and Figure 5.9, truncating 1-bit measurements is not enough to promote sparsity for 1-bit basis pursuit with Gaussian and Bernoulli matrices respectively. To further reduce perturbations and obtain sparser solutions, we next introduce reweighted 1-bit ℓ_1-minimization, analogous to reweighted ℓ_1-minimization for standard compressive sensing.

5.4.2   Reweighted 1-bit ℓ_1-minimization

In standard compressive sensing, to enhance the sparsity, an alternative to find a sparsest solution to the linear system is reweighted ℓ_1-minimization, namely, the first order method for solving concave approximation problems of ℓ_0-minimization [135]. Numerous experiments have demonstrated that reweighted ℓ_1-minimization outperforms the standard ℓ_1-minimization in many situations [134, 135, 40, 66, 42, 84, 126]. Unlike the convex approximation to $\|x\|_0$, some concave functions can approximate $\|x\|_0$ to any level of accuracy, such as $\|x\|_p^p$ for $0 < p < 1$ [66, 42, 84] and $n - \frac{\sum_i^n \log(|x_i| + \varepsilon)}{log(\varepsilon)}$ for some sufficiently small constants $\varepsilon > 0$ [40, 135, 126]. The idea of reweighted ℓ_1-minimization is to promote the sparsity iteratively by penalizing heavily on small components and keeping large components of the current iterate. Specifically, reweighted ℓ_1-minimization attempts to find a local minimum for the following problem at each iteration [135]:

$$x^{k+1} = \arg\min\{\|\text{diag}(\nabla F_\varepsilon(|x^k|))x\|_1 : \ Ax = b\}, \tag{5.4.2}$$

where $F_\varepsilon(x)$ is a separable merit function defined as $F_\varepsilon(x) = \sum_i^n \phi_i(|x_i| + \varepsilon)$ for any kernel functions ϕ_i and the weight vector $\omega^k = \nabla F_\varepsilon(|x^k|) \in R_{++}^n$ is defined on the current iterate x^k.

The problem (5.4.2) is a general framework of the separable reweighted ℓ_1-minimization. Zhao and Li [135] proved that various reweighted ℓ_1-minimization algorithms can be constructed by merit functions satisfying Assumption 2.1 in [135] and the convergence of the iterative scheme (5.4.2) can be guaranteed under a certain range space condition on A^T. In particular, the iterative reweighted ℓ_1-minimization proposed by Candès, Wakin and Boyd [40] is a special case of (5.4.2) with the weight vector defined as

$$\omega_i^k = \frac{1}{|x_i^k| + \varepsilon}, \ i = 1, \cdots, n, \ \text{for a small } \varepsilon > 0, \tag{5.4.3}$$

where the weight vector is the gradient of the merit function $F_\varepsilon(x) = \sum_i^n \log(|x_i| + \varepsilon)$. In this chapter, we refer the weight vector ω^k defined in (5.4.3) as CWB weight. Needell [100] further analyzed and proved that, under a certain restricted isometry property condition, the error bound for the noisy reconstruction via the reweighted ℓ_1-minimization with the CWB weight is tighter than that given by Candès [33] via the ℓ_1-minimization from noisy measurements.

Also, the ℓ_p-norm based reweighted ℓ_1-minimization studied by Foucart and Lai [66] is a special case of (5.4.2) with the following weight vector, e.g.,

$$\omega_i^k = \frac{1}{(|x_i^k| + \varepsilon)^{1-p}}, \ i = 1, \cdots, n, \ \text{for any } 0 < p < 1, \tag{5.4.4}$$

where the weight vector is the gradient of the merit function $F_\varepsilon(x) = \frac{1}{p} \sum_i^n (|x_i| + \varepsilon)^p$. In this chapter, we refer the weight vector ω^k defined in (5.4.4) as Wlp weight. Chen and Zhou proved that any sequence generated by such a reweighted ℓ_1-minimization converges to a stationary point of an ℓ_p-norm minimization problem that is an approximation of the

ℓ_0-minimization [42]. Besides separable reweighted ℓ_1-minimization, Wipf and Nagarajan proposed the nonseparable iterative reweighted algorithms [126]. It worths mentioning that, instead of iteratively defining the weight vector, Zhao and Kočvara [134] and Zhao and Luo [136] developed a new class of weighted ℓ_1-minimization algorithms, where the weight vector is computed from a dual space via a certain convex optimization.

In 1-bit compressive sensing, the main obstacle to enhance the sparsity of solutions of 1-bit basis pursuit is perturbations in solutions. Motivated by reweighted ℓ_1-minimization, to reduce the perturbations and then further enhance the sparsity, one approach is to construct weights via concave merit functions to force small and nonzero perturbations tending to zero. In this chapter, we only consider the separable concave merit functions. Based on the framework of reweighted ℓ_1-minimization (5.4.2), we introduce the so-called reweighted 1-bit ℓ_1-minimization to find a local minimum of a concave approximation problem that resembles the ℓ_0-problem of 1-bit compressive sensing (5.2.2) at each iteration for a given 1-bit measurements y, namely,

$$x^{k+1} = \arg\min\{\|\mathrm{diag}(\nabla F_\varepsilon(|x^k|))x\|_1 : \Phi_{J_+,n}x \geqslant e_{J_+}, \Phi_{J_-,n}x \leqslant -e_{J_-}, \Phi_{J_0,n}x = 0\},$$

(5.4.5)

where $F_\varepsilon(x)$ is a separable merit function and the weight vector $\omega^k = \nabla F_\varepsilon(|x^k|) \in R_{++}^n$ is defined on the current iterate x^k.

The reweighted 1-bit ℓ_1-minimization algorithm is formally described as follows.

Algorithm 1. (Reweighted 1-bit ℓ_1-minimization)

- **Task**: For a given 1-bit measurements y, find x that approximately solves reweighted 1-bit ℓ_1-minimization: $\min\left\{\|\mathrm{diag}(\omega)x\|_1 : \Phi_{J_+,n}x \geqslant e_{J_+}, \Phi_{J_-,n}x \leqslant -e_{J_-}, \Phi_{J_0,n}x = 0\right\}$, where J_+, J_- and J_0 are index sets defined as $J_+ = \{i : y_i = 1\}$, $J_- = \{i : y_i = -1\}$ and $J_0 = \{i : y_i = 0\}$.

- **Step 1**: Let $\omega^0 = (1, \cdots, 1)^T \in R^n$ and $x^0 = \arg\min\left\{\|\text{diag}(\omega^0)x\|_1 : \; \Phi_{J_+,n}x \geqslant e_{J_+}, \; \Phi_{J_-,n}x \leqslant -e_{J_-}, \; \Phi_{J_0,n}x = 0\right\}$ be initial points.

- **Step 2**: At the current iterate x^k with the fixed $\varepsilon > 0$, compute

$$x^{k+1} = \arg\min\left\{\|\text{diag}(\omega^k)x\|_1 : \; \Phi_{J_+,n}x \geqslant e_{J_+}, \; \Phi_{J_-,n}x \leqslant -e_{J_-}, \; \Phi_{J_0,n}x = 0\right\},$$

  where $F_\varepsilon(x)$ is a separable concave merit function and $\omega^k = \nabla F_\varepsilon(|x^k|)$.

- **Step 3**: Update the weight vector ω^{k+1} by x^{k+1}, e.g., $\omega^{k+1} = \nabla F_\varepsilon(|x^{k+1}|)$, and repeat Step 2.

Note that the initialization step in Algorithm 1 implies that the initial point x^0 is actually the solution of 1-bit basis pursuit. Using the current iterate x^0 to construct the first weight vector ω^1 by giving large weight to small nonzero coefficients (small nonzero coefficients are the perturbations in x^0), this tends to eliminate perturbations and helps identify the true nonzero coefficient locations. Following this argument, compared with 1-bit basis pursuit, Algorithm 1 may achieve a better estimation of nonzero locations of the sparse signal. In this section, to show the improvement provided by the reweighted ℓ_1-minimization method, we compare performances of 1-bit basis pursuit and reweighted 1-bit ℓ_1-minimization methods with the following weight vectors, e.g.,

- Candès-Wakin-Boyd (CWB) reweighted 1-bit ℓ_1-minimization method [40]

$$x^{k+1} = \arg\min\left\{\omega^{k^T}|x| : \; \Phi_{J_+,n}x \geqslant e_{J_+}, \; \Phi_{J_-,n}x \leqslant -e_{J_+}, \; \Phi_{J_0,n}x = 0\right\},$$

  where $\omega_i^k = \frac{1}{|x_i^k|+\varepsilon}$ for $i = 1, \cdots, n$;
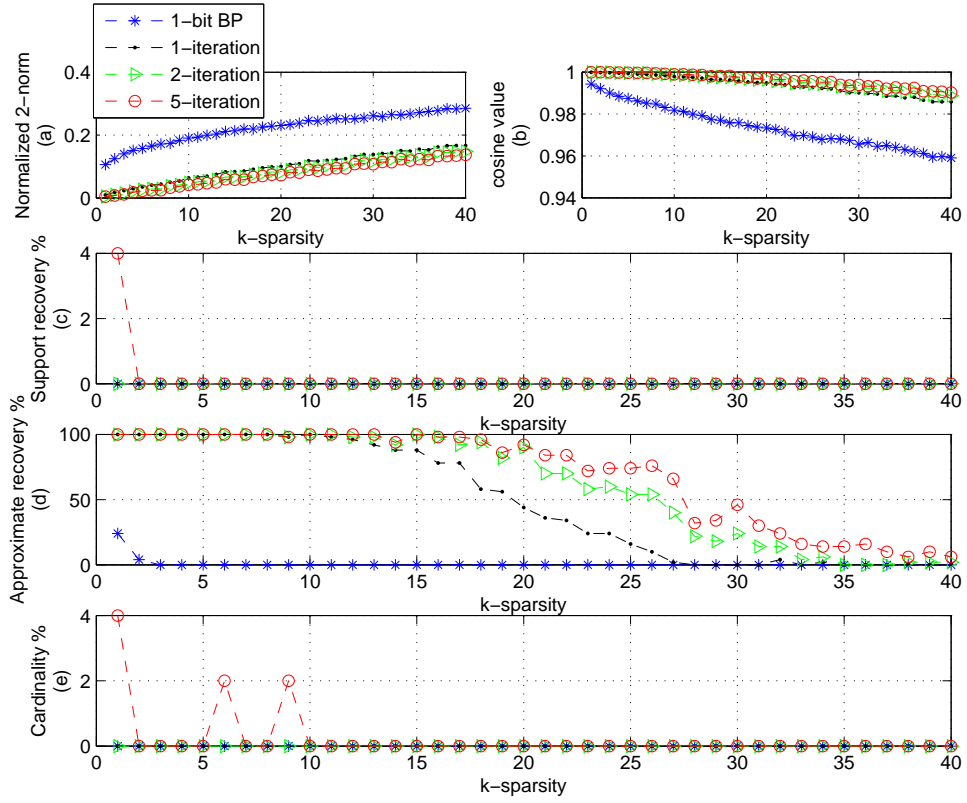
Figure 5.10: Performance of CWB reweighted 1-bit ℓ_1-minimization with overdetermined Gaussian matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\|\frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2}\right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

- Wlp reweighted 1-bit ℓ_1-minimization method [66]

$$x^{k+1} = \arg\min\left\{\omega^{k^T}|x| : \ \Phi_{J_+,n}x \geqslant e_{J_+}, \ \Phi_{J_-,n}x \leqslant -e_{J_+}, \ \Phi_{J_0,n}x = 0\right\},$$

where $\omega_i^k = \frac{1}{(|x_i^k|+\varepsilon)^{1-p}}$ for $i = 1, \cdots, n$ and $0 < p < 1$;

- NW2 reweighted 1-bit ℓ_1-minimization method [135]

$$x^{k+1} = \arg\min\left\{\omega^{k^T}|x| : \ \Phi_{J_+,n}x \geqslant e_{J_+}, \ \Phi_{J_-,n}x \leqslant -e_{J_+}, \ \Phi_{J_0,n}x = 0\right\},$$

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where $\omega_i^k = \frac{q+(|x_i^k|+\varepsilon)^{1-q}}{(|x_i^k|+\varepsilon)^{1-p}[|x_i^k|+\varepsilon+(|x_i^k|+\varepsilon)^q]^{1-p}}$ for $i = 1, \cdots, n$ and $0 < p, q < 1$.

Remark 5.4.5 : *The NW2 reweighted ℓ_1-minimization proposed in [135] performs well in finding a sparsest solution to the linear system (5.1.1), which is quite comparable to the Wlp reweighted ℓ_1-minimization in certain situations. Hence, we also consider the NW2 reweighted 1-bit ℓ_1-minimization in this section (For more examples of weight vectors, see [135, 134] and other reweighted ℓ_1-minimization methods in the literature).*

Additionally, for all the experiments in this section, we set the cosine-stopping threshold $\tau = 0.995$ and $\varepsilon = 10^{-3}$ in all weight vectors. We perform a total of 5 reweighting iterations for every reweighted 1-bit ℓ_1-minimization method. Firstly, we test the CWB reweighted 1-bit ℓ_1-minimization method on both Gaussian and Bernoulli matrices. To show the improvement, we compare performances of CWB reweighted 1-bit ℓ_1-minimization and 1-bit basis pursuit. In this experiment, the dimension of matrix Φ is set as $m = 1000$ and $n = 400$ in each trial, and each entry of Φ is randomly drawn from either the standard normal distribution or the symmetric Bernoulli distribution $\{-1, 1\}$, and the sparsity level k of x^* varies from $k = 1$ to $k = 40$. At each sparsity level, the reconstruction is run by 5 CWB reweighting iterations and repeated for 50 trials with generated random matrices Φ. The results are shown in Figure 5.10 and Figure 5.11 respectively.

As demonstrated in Figure 5.10 and Figure 5.11, success rates of approximate sparse recovery on both Gaussian matrix and Bernoulli matrix have been significantly improved by CWB reweighted 1-bit ℓ_1-minimization method at each iteration. Note that most of the improvement comes from the first and second reweighting iterations. In particular, in Figure 5.11(c) and 5.11(e), the sparsity level of the sparse reconstruction given by Bernoulli matrices has been enhanced at each iteration. Note that, in Figure 5.10, success rate of support recovery is 4% when $k = 1$, and success rates of cardinality recovery are 4% when $k = 1$ and 2% when $k = 6, 9$. But such a random result is not convincible to
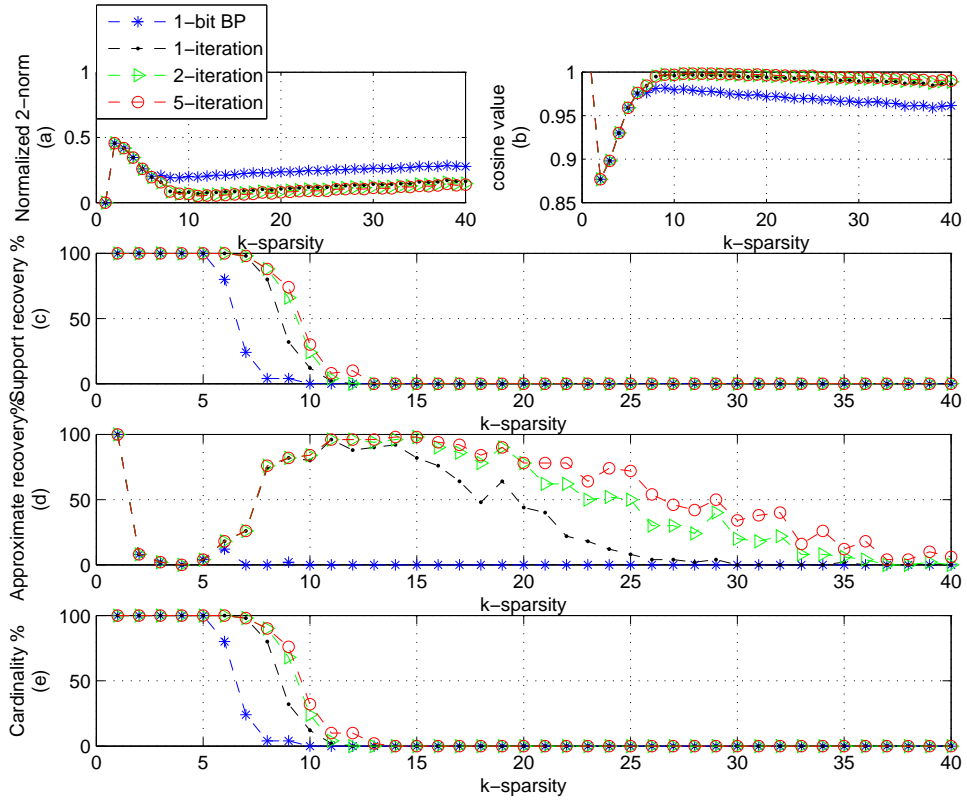
Figure 5.11: Performance of CWB reweighted 1-bit ℓ_1-minimization with overdetermined Bernoulli matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

claim that the CWB weight can improve the performance of 1-bit basis pursuit on support and cardinality recovery provided by Gaussian matrices.

Using Wlp and NW2 reweighted 1-bit ℓ_1-minimization methods, we perform the same experiment as for CWB reweighted 1-bit ℓ_1-minimization method detailed above. Due to the fixed ε in all experiments, performances of Wlp and NW2 reweighted 1-bit ℓ_1-minimization methods are based on parameters p and q. In numerical experiments, we notice that Wlp and NW2 reweighted 1-bit ℓ_1-minimization methods perform well when p is small and q is large. And also notice that performances of Wlp and NW2 reweighted
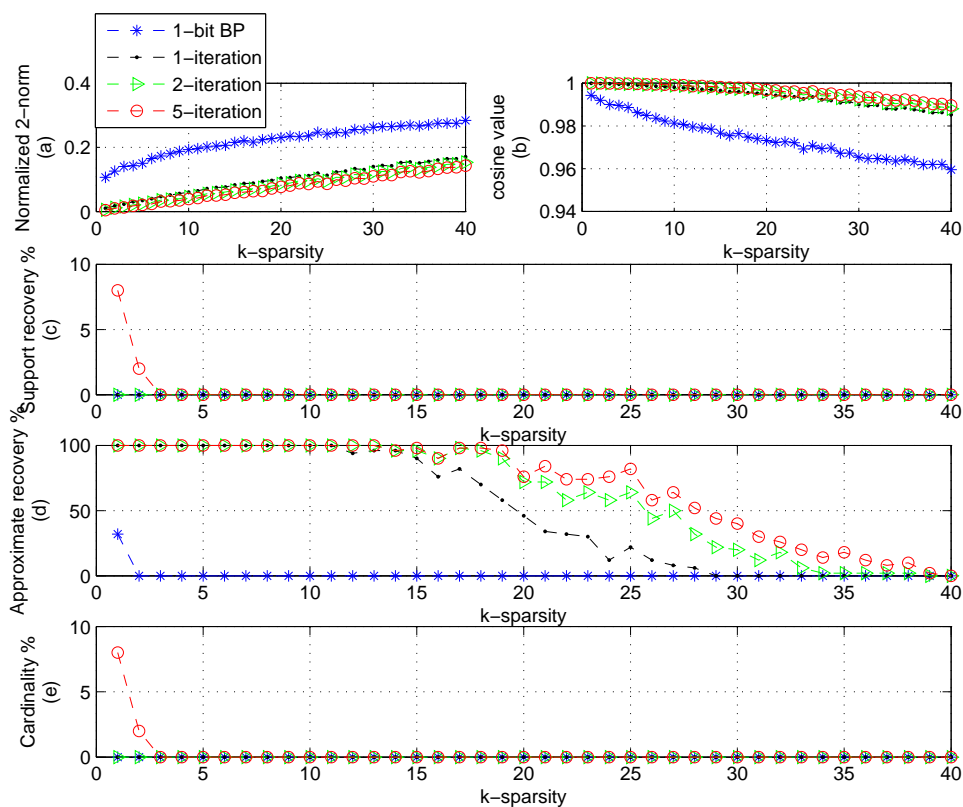
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Figure 5.12: Performance of Wlp reweighted 1-bit ℓ_1-minimization with overdetermined Gaussian matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\|\frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2}\right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

1-bit ℓ_1-minimization methods resemble the performance of CWB reweighted 1-bit ℓ_1-minimization method for small p and large q. In our experiments, we choose $p = 0.01$ and $q = 1 - p = 0.99$ for Wlp and NW2 reweighted 1-bit ℓ_1-minimization methods.

Figure 5.12 and Figure 5.13 depict performances of Wlp reweighted 1-bit ℓ_1-minimization with $p = 0.01$ from Gaussian and Bernoulli matrices. And Figure 5.14 and Figure 5.15 depict performances of NW2 reweighted 1-bit ℓ_1-minimization with $p = 0.01$ and $q = 0.99$ from Gaussian and Bernoulli matrices. Specifically, the Wlp weight and the NW2 weight show the ability to enhance the sparsity on support recovery and cardinality recovery,
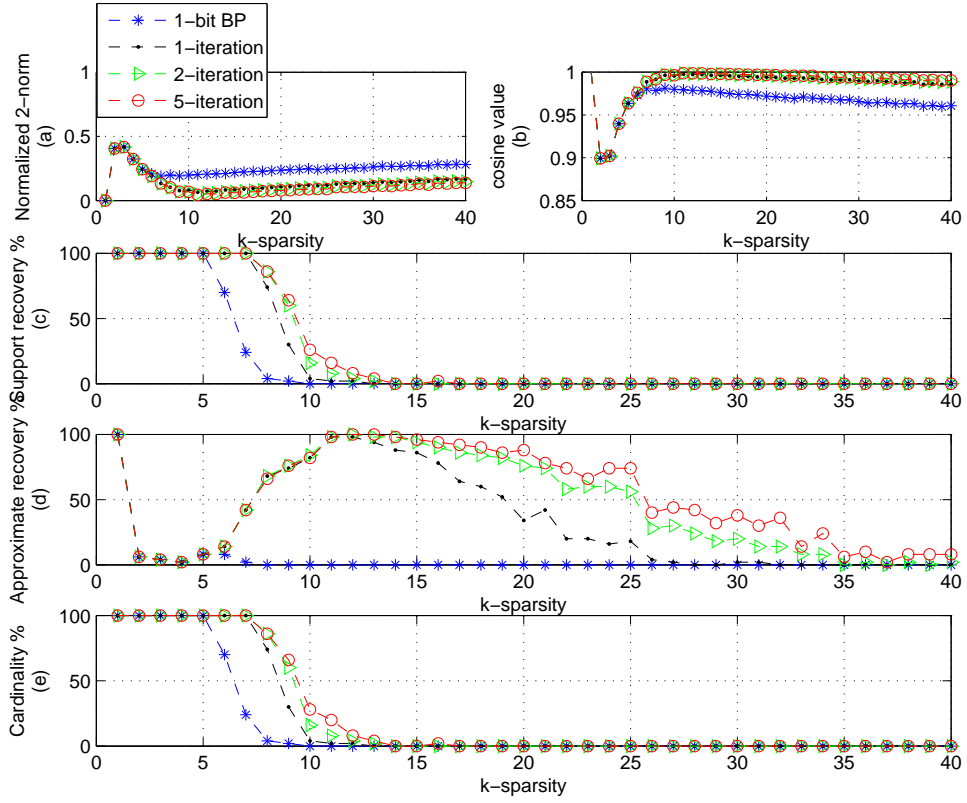
Figure 5.13: Performance of Wlp reweighted 1-bit ℓ_1-minimization with overdetermined Bernoulli matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

and to boost the approximate sparse recovery at each iteration provided by Bernoulli matrices in Figure 5.13 and Figure 5.15. In experiments, results shown in Figure 5.12(c) and 5.12(e), and Figure 5.14(c) and 5.14(e) demonstrate that Wlp and NW2 reweighted ℓ_1-minimization methods can also detect some random successes of support recovery and cardinality recovery at extremely high sparsity level. To further investigate such a phenomenon, more theoretical analysis and experiments can be studied.

To sum up, in this chapter, we have revealed the ability of 1-bit basis pursuit in three ways: support recovery, approximate sparse recovery and cardinality recovery. Also, we
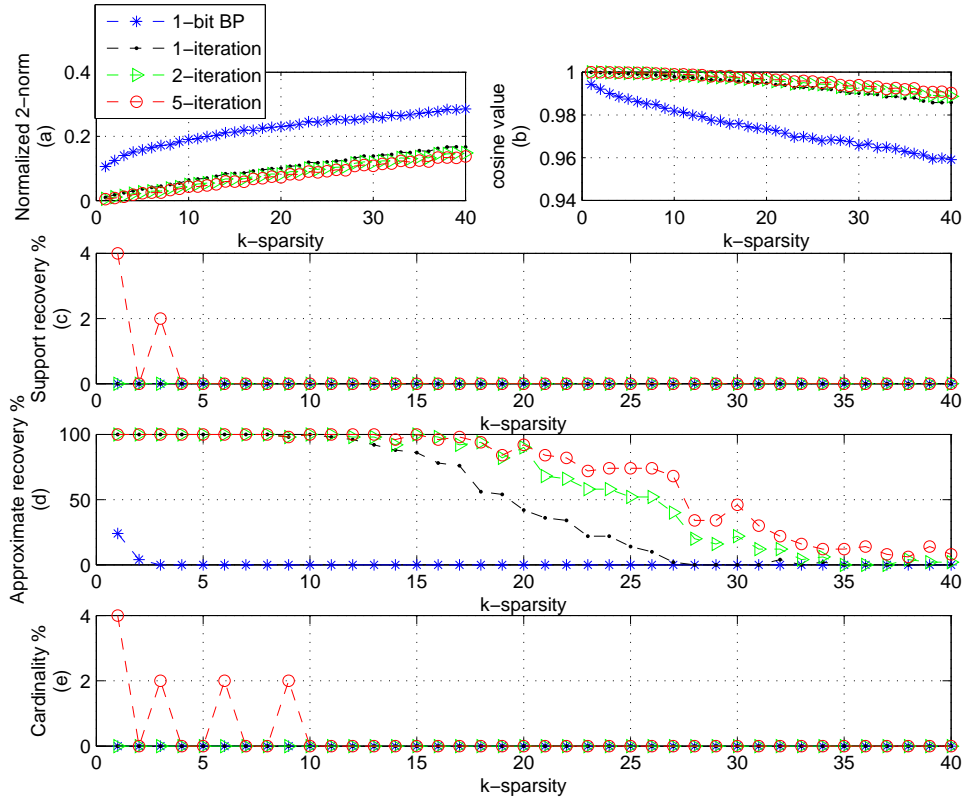
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Figure 5.14: Performance of NW2 reweighted 1-bit ℓ_1-minimization with overdetermined Gaussian matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

have numerically demonstrated that Gaussian matrix and Bernoulli matrix can be used for 1-bit compressive sensing. Especially, the Bernoulli type of sensing matrix performs well on support recovery and cardinality recovery while Gaussian matrix works well on approximate sparse recovery. Additionally, we have numerically demonstrated that the regime of 1-bit compressive sensing can include the overdetermined linear system. Last but not least, we have introduced truncated 1-bit measurements method and reweighted 1-bit ℓ_1-minimization methods to improve the performance of 1-bit basis pursuit. Particularly, we have illustrated that the sparsity level of the solution of 1-bit basis pursuit
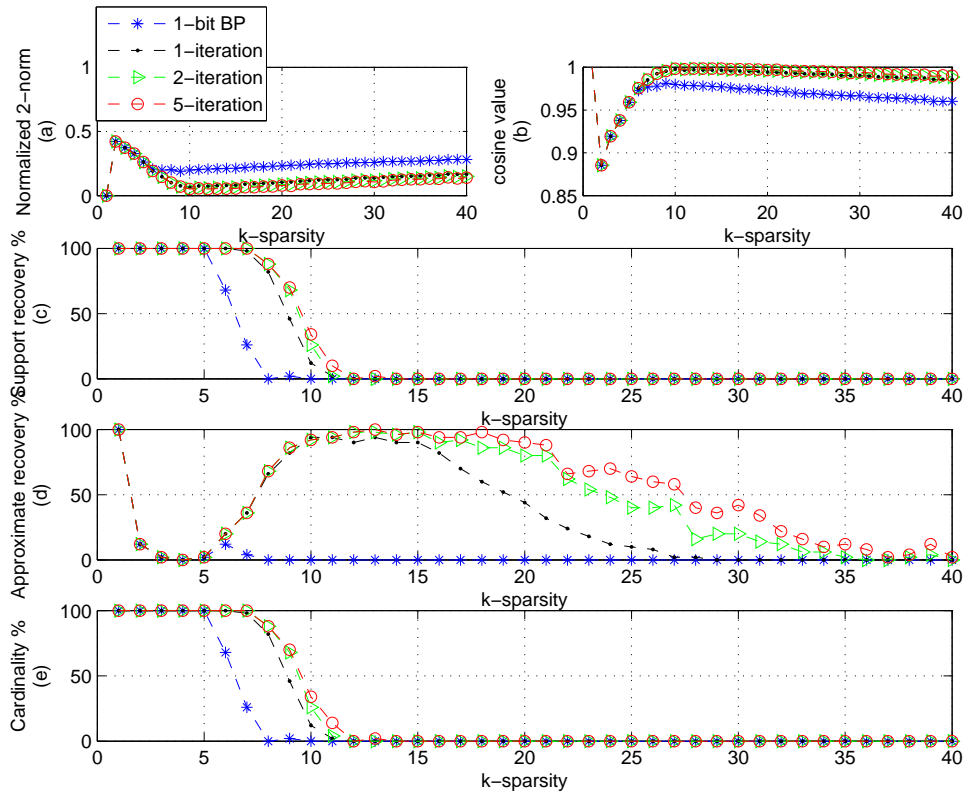
Figure 5.15: Performance of NW2 reweighted 1-bit ℓ_1-minimization with overdetermined Bernoulli matrices at varied sparsity levels. (a) Normalized ℓ_2-norm residual $\left\|\frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2}\right\|_2$, ($b$) Cosine value of the angle between x^* and x, (c) Success rate of support recovery at each sparsity level, (d) Success rate of approximate recovery at each sparsity level, and (e) Success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

can be enhanced by reweighted 1-bit ℓ_1-minimization, analogous to the reweighted ℓ_1-minimization for standard compressive sensing.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

Sparse recovery lies at the heart of compressive sensing, acquiring a sparse representation of a signal from a limited number of measurements or noisy measurements. In particular, various recovery algorithms are developed to recover the exact or approximate sparse signal under certain conditions. Based on the theoretical analysis for the standard compressive sensing and basis pursuit methods (e.g., ℓ_1-minimization), we study two special applications in compressive sensing: the partial ℓ_0-minimization problem and the 1-bit compressive sensing problem.

Partial sparsity-seeking. In chapter 3, we have developed sufficient conditions for the uniqueness of solutions of partial ℓ_0-minimization, where the ℓ_0-minimization is a special case of partial ℓ_0-minimization. Based on the well-founded uniqueness properties for ℓ_0-minimization, we have shown that the sufficient conditions can be developed through the ℓ_p-induced quasi-norm, the maximal scaled spark and the maximal scaled mutual coherence. We notice that the study of a certain sparse recovery for the partial ℓ_0-minimization is still incomplete. This leads to a number of future works. First, a certain level of uniform recovery via partial recovery models introduced in chapter 3 are ensured under the so-called partial RIP, partial NSP and partially p-RIC conditions. It would be interesting to study and derive the uniqueness conditions for solutions of these partial

recovery models and the corresponding recovery conditions from a new perspective, such as the range space property. And some experiments could be carried out to analyze the performance of these partial recovery models. Second, it is worth discussing whether it is possible to derive recovery conditions for a certain level of sparse recovery for partial ℓ_0-minimization without the full-column rank assumption on matrix A_2.

1-bit compressive sensing. The main contribution of this thesis is the development of a framework for 1-bit compressive sensing and the support recovery conditions based on the restricted range space property (RRSP). In this framework, 1-bit compressive sensing can be interpreted as solving an ℓ_0-minimization problem that are subject to a sign constraint. Specifically, we have shown that such a 1-bit framework can be formulated equivalently as an ℓ_0-minimization problem with linear equality and inequality constraints. And a decoding method, so-called 1-bit basis pursuit, is developed for possibly attacking this 1-bit ℓ_0-minimization problem. The recovery theories for 1-bit basis pursuit are established through the restricted range space property (RRSP) of transposed sensing matrices. Also, the RRSP-based conditions ensure the nonuniform and uniform support recoveries for 1-bit compressive sensing. It is worth stressing that the 1-bit ℓ_0-minimization and the RRSP-based theories have broadened the horizon of the investigation of 1-bit compressive sensing. This stimulates a few works to study in the future. First, showing the existence of RRSP matrices, we plan to prove that some random matrices may have certain RRSP properties with a high probability. Second, we plan to construct and study other reformulations of 1-bit compressive sensing. Moreover, we will develop a framework for the noisy 1-bit compressive sensing and extend the RRSP-based analysis to achieve a certain level of sparse recovery for the noisy 1-bit compressive sensing.

Our numerical experiments in chapter 5 have shown that 1-bit basis pursuit obtains support recovery from Bernoulli matrices and approximate sparse recovery from Gaussian matrices. Now, the main problem with the current simulations is that most experiments
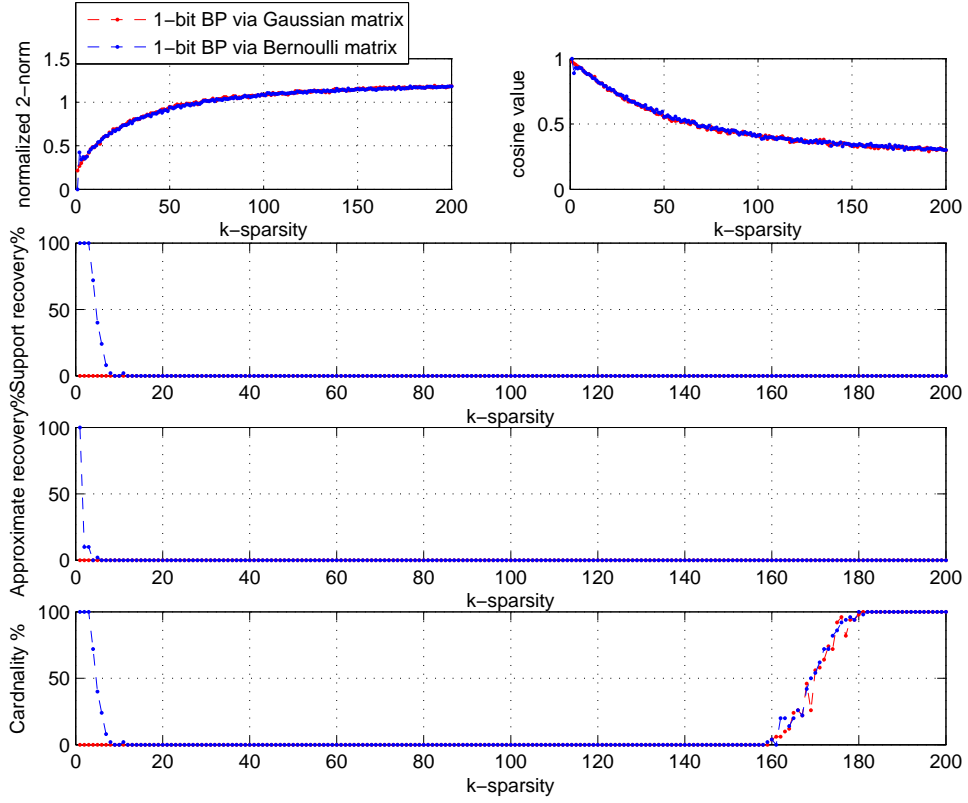
Figure 6.1: Comparison of performances of 1-bit basis pursuit with underdetermined matrices $\Phi \in R^{200 \times 1000}$ at varied sparsity level, where every entry of Φ follows the standard normal distribution and the symmetric Bernoulli distribution $\{-1, 1\}$, respectively. (a) normalized ℓ_2-norm residual $\left\| \frac{x^*}{\|x^*\|_2} - \frac{x}{\|x\|_2} \right\|_2$, (b) cosine value of the angle between x^* and x, (c) success rate of support recovery at each sparsity level, (d) success rate of approximate sparse recovery at each sparsity level, and (e) success rate of finding a sparsest solution to the linear system (5.1.1) at each sparsity level.

performed in this thesis are concerned with overdetermined matrices rather than underdetermined matrices due to the single-bit measurements. We notice that 1-bit basis pursuit can achieve both support recovery and approximate sparse recovery with underdetermined Bernoulli matrices when the signal is extremely sparse, while 1-bit basis pursuit can barely achieve any type of recoveries with underdetermined Gaussian matrices in this case, as shown in Figure 6.1. It motives us to study the measurement bounds for 1-bit compressive sensing with respect to Gaussian matrices and Bernoulli matrices on support recovery and approximate sparse recovery separately in the future. In addition, we have considered

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two approaches to improve the performance of 1-bit basis pursuit: the truncated 1-bit measurements method and the reweighted 1-bit ℓ_1-minimization method. The former is designed to cope with the inaccurate 1-bit representations in experiments and the latter is designed to enhance the sparsity level of solutions. Based on our numerical results, remarkably, reweighted 1-bit ℓ_1-minimization methods have significantly promoted success rates of support recovery and approximate sparse recovery with Bernoulli matrices. To theoretically analyze such results, we need to further study and characterize the convergence of reweighted 1-bit ℓ_1-minimization problems. Also, it would be interesting to develop other reweighted 1-bit ℓ_1-minimization problems (for instance, the weight vector is computed from a dual space via a certain convex optimization [134]), or some greedy recovery algorithms to achieve either support recovery or approximate sparse recovery from 1-bit compressive measurements.

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