Chapter

How Are Fractal Interpolation Functions Related to Several Contractions?

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Abstract

This chapter provides an overview of several types of fractal interpolation functions that are often studied by many researchers and includes some of the latest research made by the authors. Furthermore, it focuses on the connections between fractal interpolation functions resulting from Banach contractions as well as those resulting from Rakotch contractions. Our aim is to give theoretical and practical significance for the generation of fractal (graph of) functions in two and three dimensions for interpolation purposes that are not necessarily associated with Banach contractions.

Keywords: attractor, contraction, fixed point, iterated function system, fractal interpolation

1. Introduction

Interpolation is a method of constructing new data points within the range of a discrete set of known data points or the process of estimating the value of a function at a point from its values at nearby points. Although a large number of interpolation schemes are available in the mathematical field of numerical analysis, the majority of these conventional interpolation methods produce *interpolants*, i.e., functions used to generate interpolation, that are differentiable a number of times except possibly at a finite set of points. Taking into account that the *smoothness* of a function is a property measured by the number of continuous derivatives it has over some domain, the aforementioned interpolants are considered smooth.

On the other hand, many real-world and experimental signals are intricate and rarely show a sensation of smoothness in their traces. Consequently, to model these signals, we require interpolants that are nondifferentiable in dense sets of points in the domain. To address this issue, interpolation by fractal (graph of) functions is introduced in [1, 2], which is based on the theory of *iterated function system*. A *fractal interpolation function* can be considered as a continuous function whose graph is the *attractor*, a fractal set, of an appropriately chosen iterated function system. If this graph has a *Hausdorff-Besicovitch dimension* between 1 and 2, the resulting attractor is called *fractal interpolation curved line* or *fractal interpolation curve*. If this graph has a *Hausdorff-Besicovitch dimension* between 2 and 3, the resulting attractor is called *fractal interpolation surface*. Various types of fractal interpolation functions have been constructed, and some significant properties of them, including calculus, dimension, smoothness, stability, perturbation error, etc., have been widely studied [3–5].

Fractal interpolation is an advanced technique for analysis and synthesis of scientific and engineering data, whereas the approximation of natural curves and surfaces in these areas has emerged as an important research field. *Fractal functions* are currently being given considerable attention due to their applications in areas such as Metallurgy, Earth Sciences, Surface Physics, Chemistry and Medical Sciences. In the development of fractal interpolation theory, many researchers have generalised the notion in different ways [6–9]. Two key issues should be addressed in constructing fractal interpolation functions. They regard to ensuring continuity and the existence of the *contractivity*, or *vertical scaling*, *factors*; see [10, 11]. In [12], nonlinear fractal interpolation surfaces resulting from Rakotch or Geraghty contractions together with some continuity conditions were introduced as well as explicit illustrative examples were given.

The concept of iterated function system was originally introduced as a generalisation of the well-known Banach contraction principle. Since it has become a powerful tool for constructing and analysing fractal interpolation functions, one can use the well-known fixed point results obtained in the fixed point theory in order to construct them in a more general sense. A comparison of various definitions of contractive mappings as well as fixed point theorems that can be used to construct iterated function systems can be found in [13–15]. In [14], the authors proposed some iterated function systems by using various fixed point theorems, but unfortunately, one does not know whether fractal interpolation functions correspond to those may exist or not. As far as we know, the first significant generalisation of Banach's principle was obtained by Rakotch [16] in 1962. Recently, a method to generate nonlinear fractal interpolation functions by using the Rakotch or Geraghty fixed point theorem instead of Banach fixed point theorem was presented in [12, 17, 18].

The aim of our article is to provide the connections between several fractal interpolation functions and the contractions used to generate them; it is organised as follows. In Section 2, we recall the results obtained in construction of fractal interpolation curved lines and fractal interpolation surfaces by using Rakotch contractions (or Geraghty contractions) instead of Banach contractions. In Section 3, we only present the connection between fractal interpolation functions by using the Banach contractions and fractal interpolation functions by using the Rakotch contractions because in the case of Geraghty contractions, the existence of fractal interpolation curved lines and fractal interpolation surfaces is similar to the case of Rakotch contractions.

2. Preliminaries

Let (X, ρ) and (Y, σ) be metric spaces. A mapping $T : X \to Y$ is called a *Hölder mapping of exponent* or *order* a, if

$$\sigma(T(x), T(y)) \leq c\,[\rho(x, y)]^a$$

for $x, y \in X$, $a \geq 0$ and for some constant c. Note that, if $a > 1$, the functions are constants. Obviously, $c \geq 0$. The mapping T is called a *Lipschitz mapping*, if a may be taken to be equal to 1. If $c = 1$, T is said to be *nonexpansive*. A Lipschitz function is a *contraction* with *contractivity factor* c, if $c < 1$. We call T *contractive*, if for all $x, y \in X$ and $x \neq y$, we have $\sigma(T(x), T(y)) < \rho(x, y)$. Note that 'contraction $\Rightarrow$ contractive $\Rightarrow$ nonexpansive $\Rightarrow$ Lipschitz'.

An *iterated function system*, or *IFS* for short, is a collection of a complete metric space (X, ρ) together with a finite set of continuous mappings, $f_n : X \to X$, $n = 1, 2, \ldots, N$. It is often convenient to write an IFS formally as $\{X; f_1, f_2, \ldots, f_N\}$ or, somewhat more briefly, as $\{X; f_{1-N}\}$. The associated map of subsets $W : \mathcal{H}(X) \to \mathcal{H}(X)$ is given by:

$$W(E) = \bigcup_{n=1}^{N} f_n(E) \quad \text{for all} \quad E \in \mathcal{H}(X),$$

where $\mathcal{H}(X)$ is the metric space of all nonempty, compact subsets of X with respect to some metric, e.g., the Hausdorff metric. The map W is called the *Hutchinson operator* or the *collage map* to alert us to the fact that $W(E)$ is formed as a union or 'collage' of sets.

If w_n are contractions with corresponding contractivity factors s_n for $n = 1, 2, \ldots, N$, the IFS is termed *hyperbolic* and the map W itself is then a contraction with contractivity factor $s = \max\{s_1, s_2, \ldots, s_N\}$ ([2], Theorem 7.1, p. 81). In what follows, we abbreviate by f^k the k-fold composition $f \circ f \circ \cdots \circ f$.

Definition 2.1. *Let X be a set. A* self-map on *X or a* transformation *is a mapping from X to itself.*

i. *A self-map f on a metric space (X, ρ) is called a φ-*contraction*, if there exists a function $\varphi : (0, +\infty) \to (0, +\infty)$ with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$ such that for all $x, y \in X$, $\rho(f(x), f(y)) \le \varphi(\rho(x, y))$.*

ii. *We say that f is a* Rakotch contraction, *if f is a φ-contraction such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0, +\infty) \ni t \to \frac{\varphi(t)}{t}$ is nonincreasing.*

iii. *If f is a φ-contraction for some function $\varphi : (0, +\infty) \to (0, +\infty)$ such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0, +\infty) \ni t \to \frac{\varphi(t)}{t}$ is nonincreasing (or nondecreasing, or continuous), then we call such a function a* Geraghty contraction.

From [14], we have the following.

Theorem 2.1. *Let X be a complete metric space and $\{X; f_{1-N}\}$ be an IFS consisting of Rakotch or Geraghty contractions. Then there is a unique nonempty compact set $K \in \mathcal{H}(X)$ such that*

$$K = \bigcup_{n=1}^{N} f_n(K).$$

2.1 Fractal interpolation in $\mathbb{R}$

Let N be a positive integer greater than 1 and $I = [x_0, x_N] \subset \mathbb{R}$. Let a set of *interpolation points* $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \ldots, N\}$ be given, where $x_0 < x_1 < \cdots < x_N$ and $y_0, y_1, \ldots, y_N \in \mathbb{R}$. Set $I_n = [x_{n-1}, x_n] \subset I$ and define, for all $n = 1, 2, \ldots, N$, contractive homeomorphisms $L_n : I \to I_n$ by

$$L_n(x) := a_n x + b_n,$$

where the real numbers a_n, b_n are chosen to ensure that $L_n(I) = I_n$.

Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a nondecreasing continuous function such that for any $t > 0$, $\alpha(t) := \frac{\varphi(t)}{t} < 1$ and the function $(0, +\infty) \ni t \to \frac{\varphi(t)}{t}$ is nonincreasing. Let $d_n : I \to \mathbb{R}$ be a continuously differentiable function such that

$$\max_{x \in I} |d_n(x)| \leq 1.$$

Now, consider an IFS of the form $\{I \times \mathbb{R}; w_n, n = 1, 2, \ldots, N\}$ in which the maps are nonlinear transformations of the special structure

$$w_n\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_n(x) \\ F_n(x,y) \end{pmatrix} = \begin{pmatrix} a_n x + b_n \\ c_n x + d_n(x)s_n(y) + e_n \end{pmatrix},$$

where the transformations are constrained by the data according to

$$w_n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \quad w_n\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

for $n = 1, 2, \ldots, N$, and s_n are some Rakotch or Geraghty contractions.

Let us denote by $C(D)$ the linear space of all real-valued continuous functions defined on D, i.e., $C(D) = \{ f : D \to \mathbb{R} \mid f \text{ continuous}\}$. Let $C^*(I) \subset C(I)$ denote the set of continuous functions $f : I \to \mathbb{R}$ such that $f(x_0) = y_0$ and $f(x_N) = y_N$, that is,

$$C^*(I) := \{ f \in C(I) : f(x_0) = y_0, f(x_N) = y_N \}.$$

Let $C^{**}(I) \subset C^*(I) \subset C(I)$ be the set of continuous functions that pass through the given data points $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \ldots, N\}$, that is,

$$C^{**}(I) := \{ f \in C^*(I) : f(x_i) = y_i, i = 0, 1, \ldots, N\}.$$

Define a metric $d_{C(I)}$ on the space $C(I)$ by

$$d_{C(I)}(g,h) := \max_{x \in [x_0, x_N]} |g(x) - h(x)|$$

for all $g, h \in C(I)$. Define a mapping $T : C^*(I) \to C(I)$ for all $f \in C^*(I)$ by

$$\begin{aligned} Tf(x) &:= F_n\left(L_n^{-1}(x), f\left(L_n^{-1}(x)\right)\right) \\ &= c_n L_n^{-1}(x) + d_n\left(L_n^{-1}(x)\right)s_n\left(f\left(L_n^{-1}(x)\right)\right) + e_n \end{aligned}$$

for $x \in [x_{n-1}, x_n]$ and $n = 1, 2, \ldots, N$. From [17], we have the following.

Theorem 2.2. *Let $\{I \times \mathbb{R}; w_n, n = 1, 2, \ldots, N\}$ denote the IFS defined above. Let each s_n be a bounded Rakotch or Geraghty contraction. Then,*

i. *there is a unique continuous function $f : I \to \mathbb{R}$ which is a fixed point of T;*

ii. *$f(x_i) = y_i$ for all $i = 0, 1, \ldots, N$;*

iii. *if $G \subset I \times \mathbb{R}$ is the graph of f, then*

$$G = \bigcup_{n=1}^{N} w_n(G).$$

An extremely explicit simple example is the following; cf. [12].

Example 1. *Let $\varphi(t) := \frac{t}{1+t}$ for $t \in (0, +\infty)$. Let a set of data $\{(x_i, y_i) : i = 0, 1, \ldots, N\}$ be given, where $0 = x_0 < x_1 < \ldots < x_N = 1$ and $y_i \in [0,1]$ for all $i = 0, 1, \ldots, N$. Let for all $n = 1, 2, \ldots, N$, $d_n(x) := x^n$. Let for $y \in [0, +\infty)$ and*

$n = 1, 2, \dots, N$, $s_n(y) := \frac{y}{1+ny}$. *That is, each* s_n *is a Rakotch contraction (with the same function* φ*) that is not a Banach contraction on* $[0, +\infty)$. *Let for all* $n = 1, 2, \dots, N$,

$$w_n(x, y) := (a_n x + b_n, c_n x + d_n(x) s_n(y) + e_n),$$

where

$$a_n = x_n - x_{n-1}, \quad b_n = x_{n-1},$$
$$c_n = y_n - y_{n-1}, \quad e_n = y_{n-1}.$$

Then, there exists a continuous function $f : [0, 1] \to \mathbb{R}$ *that interpolates the given points* $\{(x_i, y_i) : i = 0, 1, \dots, N\}$. *Moreover, the graph* G *of* f *is invariant with respect to* $\{[0, 1] \times \mathbb{R}; w_1, w_2, \dots, w_N\}$, *i.e.,*

$$G = \bigcup_{n=1}^{N} w_n(G).$$

2.2 Fractal interpolation in $\mathbb{R}^2$

Let M, N be two positive integers greater than 1. Let us represent the given set of *interpolation points* as $\left\{\left(x_i, y_j, z_{i,j}\right) \in K : i = 0, 1, \dots, M; j = 0, 1, \dots, N\right\}$, where $x_0 < x_1 < \dots < x_M$, $y_0 < y_1 < \dots < y_N$ and $z_{i,j} \in [a, b]$ for all $i = 0, 1, \dots, M$ and $j = 0, 1, \dots, N$. Set $I = [x_0, x_M] \subset \mathbb{R}$ and $J = [y_0, y_N] \subset \mathbb{R}$. Throughout this section, we will work in the complete metric space $K = D \times \mathbb{R}$, where $D = I \times J$, with respect to the Euclidean, or to some other equivalent, metric.

Set $I_m = [x_{m-1}, x_m]$, $J_n = [y_{n-1}, y_n]$, $D_{m,n} = I_m \times J_n$ and let $u_m : I \to I_m$, $v_n : J \to J_n$, $L_{m,n} : D \to D_{m,n}$ be defined for $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$, by

$$L_{m,n}(x, y) = (u_m(x), v_n(y)) = (a_m x + b_m, c_n y + d_n).$$

Thus, for $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$,

$$a_m = \frac{x_m - x_{m-1}}{x_M - x_0}, \quad b_m = x_{m-1} - \frac{x_m - x_{m-1}}{x_M - x_0} x_0,$$

$$c_n = \frac{y_n - y_{n-1}}{y_N - y_0}, \quad d_n = y_{n-1} - \frac{y_n - y_{n-1}}{y_N - y_0} y_0.$$

Furthermore, for $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$, let mappings $F_{m,n} : K \to \mathbb{R}$ be continuous with respect to each variable. We consider an IFS of the form $\{K; w_{m,n}, m = 1, 2, \dots, M; n = 1, 2, \dots, N\}$ in which maps $w_{m,n} : D \times \mathbb{R} \to D_{m,n} \times \mathbb{R}$ are transformations of the special structure

$$w_{m,n}(x, y, z) := (L_{m,n}(x, y), F_{m,n}(x, y, z)),$$

where the transformations are constrained by the data according to

$$w_{m,n}\begin{pmatrix} x_0 \\ y_0 \\ z_{0,0} \end{pmatrix} = \begin{pmatrix} x_{m-1} \\ y_{n-1} \\ z_{m-1,n-1} \end{pmatrix}, \quad w_{m,n}\begin{pmatrix} x_0 \\ y_N \\ z_{0,N} \end{pmatrix} = \begin{pmatrix} x_{m-1} \\ y_n \\ z_{m-1,n} \end{pmatrix},$$

$$w_{m,n}\begin{pmatrix} x_M \\ y_0 \\ z_{M,0} \end{pmatrix} = \begin{pmatrix} x_m \\ y_{n-1} \\ z_{m,n-1} \end{pmatrix}, \quad w_{m,n}\begin{pmatrix} x_M \\ y_N \\ z_{M,N} \end{pmatrix} = \begin{pmatrix} x_m \\ y_n \\ z_{m,n} \end{pmatrix}$$

for $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$.

Let $B(D)$ denote the set of bounded functions $f : D \to \mathbb{R}$ and

$$B^*(D) = \{f \in B(D) : f(x_0, y_0) = z_{0,0},\ f(x_0, y_N) = z_{0,N},$$
$$f(x_M, y_0) = z_{M,0}, f(x_M, y_N) = z_{M,N}\}.$$

Let $B^{**}(D) \subset B^*(D)$ be the set of bounded functions that pass through the given interpolation points $\left\{\left(x_i, y_j, z_{i,j}\right) \in K = D \times [a,b] : i = 0, 1, \dots, M; j = 0, 1, \dots, N\right\}$, that is,

$$B^{**}(D) = \left\{ f \in B^*(D) : f\left(x_i, y_j\right) = z_{i,j}, i = 0, 1, \dots, M; j = 0, 1, \dots, N\right\}.$$

Define an operator $T : B^*(D) \to B(D)$ for all $f \in B^*(D)$ by

$$Tf(x,y) = F_{m,n}\left(u_m^{-1}(x), v_n^{-1}(y), f\left(u_m^{-1}(x), v_n^{-1}(y)\right)\right)$$

for $(x,y) \in D_{m,n}$, $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$. In [18], we see the following.

Theorem 2.3. *Let $\{D \times \mathbb{R}; w_{m,n}, m = 1, 2, \dots, M; n = 1, 2, \dots, N\}$ denote the IFS defined above. Assume that the maps $F_{m,n}$ are Rakotch or Geraghty contractions with respect to the third variable, and uniformly Lipschitz with respect to the first and second variable. Then,*

1. *there is a unique bounded function $f : D \to \mathbb{R}$ which is a fixed point of T;*

2. *$f\left(x_i, y_j\right) = z_{i,j}$ for $i = 0, 1, \dots, M$ and $j = 0, 1, \dots, N$;*

3. *if $G \subset D \times \mathbb{R}$ is the graph of f, then*

$$G = \bigcup_{m=1}^{M} \bigcup_{n=1}^{N} w_{m,n}(G).$$

Let for all $i = 0, 1, \dots, M$ and $j = 0, 1, \dots, N$, $z_{0,j} = z_{i,0} = z_{M,j} = z_{i,N}$ and define

$$F_{m,n}(x,y,z) = e_{m,n}x + f_{m,n}y + g_{m,n}xy + s_{m,n}(z) + h_{m,n},$$

where $s_{m,n}$ are Rakotch or Geraghty contractions. Let

$$C^*(D) = \{f \in C(D) : f(x_0, y_0) = z_{0,0}, f(x_0, y_N) = z_{0,N},$$
$$f(x_M, y_0) = z_{M,0}, f(x_M, y_N) = z_{M,N}\}$$

and

$$C^{**}(D) = \left\{ f \in C^*(D) : f\left(x_i, y_j\right) = z_{i,j}, i = 0, 1, \dots, M; j = 0, 1, \dots, N\right\}.$$

Let $C_0^*(D) \subset C^*(D)$ be the set of continuous functions $f : D \to \mathbb{R}$ such that

$$\begin{aligned} f\big(x_0, (1-\lambda)y_0 + \lambda y_N\big) &= z_{*,*}, \\ f\big(x_M, (1-\lambda)y_0 + \lambda y_N\big) &= z_{*,*}, \\ f\big((1-\lambda)x_0 + \lambda x_M, y_0\big) &= z_{*,*}, \\ f\big((1-\lambda)x_0 + \lambda x_M, y_N\big) &= z_{*,*} \end{aligned}$$

for all $\lambda \in [0, 1]$, where for all $i = 0, 1, \dots, M$ and $j = 0, 1, \dots, N$,

$$z_{**} := z_{0,j} = z_{i,0} = z_{M,j} = z_{i,N}.$$

Let $C_0^{**}(D) := \left\{ f \in C_0^*(D) : f\left(x_i, y_j\right) = z_{i,j}, i = 0, 1, \dots, M; j = 0, 1, \dots, N\right\} \subset C^{**}(D)$. For $f \in C_0^*(D)$, we define $T : C_0^*(D) \to B(D)$ by

$$\begin{aligned} Tf(x,y) &= F_{m,n}\big(u_m^{-1}(x), v_n^{-1}(y), f\big(u_m^{-1}(x), v_n^{-1}(y)\big)\big) \\ &= e_{m,n}u_m^{-1}(x) + f_{m,n}v_n^{-1}(y) + g_{m,n}u_m^{-1}(x)v_n^{-1}(y) \\ &\quad + s_{m,n}\big(f\big(u_m^{-1}(x), v_n^{-1}(y)\big)\big) + h_{m,n} \end{aligned}$$

for $(x, y) \in D_{m,n}$, $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, N$.

Corollary 2.1 *(see* [18]*) Let* $\{D \times \mathbb{R}; w_{m,n}, m = 1, 2, \dots, M; n = 1, 2, \dots, N\}$ *denote the IFS defined above. Then,*

1. *there is a unique continuous function* $f : D \to \mathbb{R}$ *which is a fixed point of* T;

2. $f\left(x_i, y_j\right) = z_{i,j}$ *for all* $i = 0, 1, \dots, M$ *and* $j = 0, 1, \dots, N$;

3. *if* $G \subset D \times \mathbb{R}$ *is the graph of* f*, then*

$$G = \bigcup_{m=1}^{M} \bigcup_{n=1}^{N} w_{m,n}(G).$$

The most simple example is the following; cf. [12].

Example 2. *Let* $\varphi(t) := \frac{t}{1+t}$ *for* $t \in (0, +\infty)$*. Let a set of data* $\left\{\left(x_i, y_j, z_{i,j}\right) : i = 0, 1, 2; j = 0, 1, 2\right\}$ *be given, where* $0 = x_0 < x_1 < x_2 = 1$, $0 = y_0 < y_1 < y_2 = 1$ *and* $z_{i,j} \in [0, 1]$ *for all* $i = 0, 1, 2; j = 0, 1, 2$*. Let for all* $i = 0, 1, 2$ *and* $j = 0, 1, 2$,

$$z_{0,j} = z_{i,0} = z_{2,j} = z_{i,2} = 0.$$

Let for $z \in [0, +\infty)$,

$$\begin{aligned} s_{1,1}(z) &:= \frac{z}{1+z}, \quad s_{1,2}(z) := \frac{z}{1+2z}, \\ s_{2,1}(z) &:= \frac{z}{1+3z}, \quad s_{2,2}(z) := \frac{z}{1+4z}. \end{aligned}$$

Then, $s_{1,1}$, $s_{1,2}$, $s_{2,1}$, $s_{2,2}$ *are Rakotch contractions (with the same function* φ*) that are not Banach contractions on* $[0, +\infty)$*. So, there exists a continuous function* $f : [0, 1] \times [0, 1] \to \mathbb{R}$ *that interpolates the given data* $\left\{\left(x_i, y_j, z_{i,j}\right) : i = 0, 1, 2; j = 0, 1, 2\right\}$.

Let $d_{m,n} : D \to \mathbb{R}$ be a function such that $\max_{(x,y) \in D}|d_{m,n}(x, y)| \le 1$,

$$d_{m,n}(x_0,y) = d_{m,n}(x_M,y) = d_{m,n}(x,y_0) = d_{m,n}(x,y_N) = 0$$

and for some $L_1, L_2 > 0$,

$$|d_{m,n}(x,y) - d_{m,n}(x',y')| \le L_1|x-x'| + L_2|y-y'|.$$

Let

$$F_{m,n}(x,y,z) = e_{m,n}x + f_{m,n}y + g_{m,n}xy + d_{m,n}(x,y)s_{m,n}(z) + h_{m,n},$$

where $s_{m,n}$ is a Rakotch or Geraghty contraction. For $f \in C^*(D)$, we define $T : C^*(D) \to B(D)$ by

$$\begin{aligned} Tf(x,y) &= F_{m,n}\left(u_m^{-1}(x), v_n^{-1}(y), f\left(u_m^{-1}(x), v_n^{-1}(y)\right)\right) \\ &= e_{m,n}u_m^{-1}(x) + f_{m,n}v_n^{-1}(y) + g_{m,n}u_m^{-1}(x)v_n^{-1}(y) \\ &\quad + d_{m,n}\left(u_m^{-1}(x), v_n^{-1}(y)\right)s_{m,n}\left(f\left(u_m^{-1}(x), v_n^{-1}(y)\right)\right) + h_{m,n} \end{aligned}$$

for $(x,y) \in D_{m,n}$, $m = 1, 2, \ldots, M$ and $n = 1, 2, \ldots, N$.

For the next, see [12] for details.

Corollary 2.2. *Let $\{D \times \mathbb{R}; w_{m,n}, m = 1, 2, \ldots, M; n = 1, 2, \ldots, N\}$ denote the IFS defined above. If each $s_{m,n}$ be a bounded function, then.*

1. *there is a unique continuous function $f : D \to \mathbb{R}$ which is a fixed point of T;*
2. *$f\left(x_i, y_j\right) = z_{i,j}$ for all $i = 0, 1, \ldots, M$ and $j = 0, 1, \ldots, N$;*
3. *if $G \subset D \times \mathbb{R}$ is a graph of f, then*

$$G = \bigcup_{m=1}^{M} \bigcup_{n=1}^{N} w_{m,n}(G).$$

An especially simple example is the following; see [12].

Example 3. *Let $\varphi(t) \coloneqq \frac{t}{1+t}$ for $t \in (0, +\infty)$. Let a set of data $\left\{\left(x_i, y_j, z_{i,j}\right) : i = 0, 1, 2; j = 0, 1, 2\right\}$ be given, where $0 = x_0 < x_1 < x_2 = 1$, $0 = y_0 < y_1 < y_2 = 1$ and $z_{i,j} \in [0, 1]$ for all $i = 0, 1, 2; j = 0, 1, 2$. Here, a set of data points is not necessarily the case that $z_{0,j} = z_{i,0} = z_{2,j} = z_{i,2}$ for all $i = 0, 1, 2; j = 0, 1, 2$. Let for all $i = 1, 2; j = 1, 2$ and $(x, y) \in [0, 1] \times [0, 1]$,*

$$d_{m,n}(x,y) \coloneqq 2^{2(m+n)}x^m(1-x)^m y^n(1-y)^n.$$

Let for $z \in [0, +\infty)$,

$$s_{1,1}(z) \coloneqq \frac{1}{1+z}, s_{1,2}(z) \coloneqq \frac{z}{1+z},$$

$$s_{2,1}(z) \coloneqq \frac{z}{1+2z}, s_{2,2}(z) \coloneqq \frac{z}{1+3z}.$$

Then, $s_{1,1}$, $s_{1,2}$, $s_{2,1}$, $s_{2,2}$ are Rakotch contractions (with the same function φ) that are not Banach contractions on $[0,+\infty)$. So, there exists a continuous function $f: [0,1]\times[0,1]\to\mathbb{R}$ that interpolates the given data $\left\{\left(x_i,y_j,z_{i,j}\right): i=0,1,2; j=0,1,2\right\}$.

3. Interconnections between FIFs and contractions

In this section, we only present the interconnections between FIFs resulting from Banach contractions and FIFs resulting from Rakotch contractions because in the case of Geraghty contractions, the existence of FICs and FISs is derived similarly to the case of Rakotch contractions.

Connection 1

1. Each Banach contraction is a Rakotch contraction, since a self-map is a Banach contraction if and only if it is a φ-contraction for a function $\varphi(t)=\alpha t$, for some $0\le\alpha<1$. There exist examples of Rakotch contraction maps that are not Banach contraction maps on $X\subset\mathbb{R}$ with respect to the Euclidean metric (see [13]).

2. The Rakotch's functional condition for convergence of a contractive iteration in a complete metric space can be replaced by an equivalent (or another) functional condition; for instance, a map is a Rakotch contraction if and only if it is a φ-contraction for some nondecreasing function $\varphi:(0,+\infty)\to(0,+\infty)$ such that additionally $\varphi(t)<t$ for $t>0$ and the map $t\to\frac{\varphi(t)}{t}$ is nonincreasing (see [19]).

Connection 2

1. $\left(C(I),d_{C(I)}\right)$, $\left(C^*(I),d_{C(I)}\right)$ and $\left(C^{**}(I),d_{C(I)}\right)$ are complete metric spaces, where

$$d_{C(I)}(f,g)\coloneqq\max_{x\in I}|f(x)-g(x)|$$

for all $f,g\in C(I)$ (see [2]).

2. $\left(B(D),d_{B(D)}\right)$, $\left(B^*(D),d_{B(D)}\right)$ and $\left(B^{**}(D),d_{B(D)}\right)$ are complete metric spaces, where

$$d_{B(D)}(f,g)\coloneqq\sup_{(x,y)\in D}|f(x,y)-g(x,y)|$$

for all $f,g\in B(D)$ [10].

3. $C_0^{**}(D)$, $C_0^*(D)$, $C^{**}(D)$, $C^*(D)$ and $C(D)$ are closed subspaces of $B(D)$ with $C_0^{**}(D)\subset C_0^*(D)\subset C^*(D)\subset C(D)\subset B(D)$ and $C_0^{**}(D)\subset C^{**}(D)\subset C^*(D)\subset C(D)\subset B(D)$, and so they are complete metric spaces.

Connection 3

Let $d_n : I \to \mathbb{R}$ be a continuously differentiable function such that

$$\max_{x\in I} |d_n(x)| \leq 1.$$

Then, by the Differential Mean Value Theorem and the extreme value theorem, we can see that for some $L_{d_n} > 0$,

$$|d_n(x') - d_n(x'')| \leq L_{d_n} |x' - x''|,$$

where $x', x'' \in I$. Hence, d_n is Lipschitz continuous function defined on I satisfying $\max_{x\in I}|d_n(x)| \leq 1$, but the converse is not true in general.

Connection 4

1. The function $d_n(x)s_n(y)$ is a generalisation of the bivariable function $d_n(x)y$ with vertical scaling factors as (continuous) 'contraction functions'. In fact, in the case when $0 < \max_{x\in I}|d_n(x)| < 1$ (see [20], p. 3), obviously,

$$d_n(x)y = \frac{d_n(x)}{\max_{x\in I}|d_n(x)|} \max_{x\in I} |d_n(x)|y.$$

Let $s_n(y) = \max_{x\in I}|d_n(x)|y$ and $d_n^*(x) = \frac{d_n(x)}{\max_{x\in I}|d_n(x)|}$. Then $d_n(x)y = d_n^*(x)s_n(y)$, $\max_{x\in I}|d_n^*(x)| = 1$ and s_n is a Banach (or Rakotch) contraction.

2. The functional condition $\max_{x\in I}|d_n(x)| \leq 1$ is essential in order to show the difference between Banach contractibility of $F_n(\cdot, y)$ and Rakotch contractibility of $F_n(\cdot, y)$; compare with [20]. In fact, since $\varphi(t) < t$ for any $t > 0$,

$$\begin{aligned}
|F_n(x, y') - F_n(x, y'')| &= |d_n(x)||s_n(y') - s_n(y'')| \\
&\leq \max_{x\in I} |d_n(x)||s_n(y') - s_n(y'')| \\
&\leq \max_{x\in I} |d_n(x)|\varphi(|y' - y''|) \\
&\leq \max_{x\in I} |d_n(x)||y' - y''|,
\end{aligned}$$

where $(x, y'), (x, y'') \in \mathbb{R}^2$. Hence, if $\max_{x\in I}|d_n(x)| < 1$, as can be seen, notwithstanding each s_n is a Rakotch contraction that is not a Banach contraction, each F_n is Banach contraction with respect to the second variable because

$$|F_n(x, y') - F_n(x, y'')| \leq \max_{x\in I} |d_n(x)||y' - y''|.$$

On the other hand, if $\max_{x\in I}|d_n(x)| = 1$, then we conclude that each F_n is Rakotch contraction with respect to the second variable whenever each s_n is a Rakotch contraction because

$$|F_n(x, y') - F_n(x, y'')| \leq \max_{x\in I} |d_n(x)|\varphi(|y' - y''|).$$

3. In Theorem 2.2, for all $(x, y'), (x, y'') \in I \times \mathbb{R}$,

$$|F_n(x, y') - F_n(x, y'')| = |d_n(x)||s_n(y') - s_n(y'')| \leq |s_n(y') - s_n(y'')| \leq \varphi(|y' - y''|).$$

That is, each $w_n(x, y)$ is chosen so that function $F_n(x, y)$ is Rakotch contraction with respect to the second variable.

4. Even though $s_n : \mathbb{R} \to \mathbb{R}$ are Rakotch contractions, $w_n : I \times \mathbb{R}$ are not in general Rakotch contractions on the metric space $(I \times \mathbb{R}, d_0)$, and thus, the IFSs defined above are not IFSs of [14] (cf. second and third line in p. 215 of [2]).

Connection 5

In the case where the vertical scaling factors are constants, in [1], the existence of affine FIFs by using the Banach fixed point theorem was investigated, whereas in [20], a generalisation of affine FIFs by using vertical scaling factors as (continuous) 'contraction functions' and Banach's fixed point theorem was introduced. Theorem 2.2 gives the existence of fractal interpolation curves by using the Rakotch fixed point theorem and vertical scaling factors as (continuous) 'contraction functions'.

Connection 6

The boundedness of s_n is the essential condition to establish a unique invariant set of an iterated function system. In the fractal interpolation curve with vertical scaling factors as 'contraction function', $0 < \max_{x \in I}|d_n(x)| < 1$ (see [20]). Let $M := \max_{x \in I}|c_n x + f_n|$ and $h \geq \frac{M}{1 - \max_{x \in I}|d_n(x)|}$. Then for all $y \in [-h, h]$,

$$|F_n(x, y)| = |c_n x + d_n(x)y + f_n| \leq M + \max_{x \in I}|d_n(x)||y| \leq M + \max_{x \in I}|d_n(x)|h \leq h.$$

So, for all $(x, y) \in I \times [-h, h]$, we can see that $F_n(x, y) \in [-h, h]$. That is, an IFS of the form $\{I \times [-h, h]; w_{1-N}\}$ has been constructed (cf. [21], p. 1897). Thus $D(s_n) = [-h, h]$ and $s_n(y) := \max_{x \in I}|d_n(x)|y$ is bounded in $D(s_n)$. Hence the boundedness of s_n in $D(s_n)$ is the essential condition to establish a unique invariant set of an IFS (cf. [21], p. 1897).

Connection 7

In view of a φ-contraction, the connections between the coefficients of y variable are obtained as follows:

1. In the affine FIF (cf. [1], p. 308, Example 1), for all $t > 0$,

$$\varphi(t) := \max_{n=1,2,\ldots,N}|d_n|t,$$

where $|d_n| < 1$ for all $i = 1, 2, \ldots, N$.

2. In the FIF with vertical scaling factors as (continuous) 'contraction functions' (cf. [20], p. 3), for all $t > 0$,

$$\varphi(t) := \max_{i=1,2,\ldots,N} \max_{x \in I}|d_n(x)|t,$$

where $d_n(x)$ is Lipschitz function defined on I satisfying $\sup_{x \in I}|d_n(x)| < 1$ for all $n = 1, 2, \ldots, N$.

Connection 8

We refer to f of Theorem 2.2 as a nonlinear FIF. The reason is that the functions F_n take the form

$$F_n(x,y) = c_n x + d_n(x) s_n(y) + e_n,$$

where $\max_{x\in I}|d_n(x)| \le 1$ and each s_n is Rakotch contraction. That is, each F_n, in general, is nonlinear with respect to the second variable (cf. [17]). In fact, in [2] or [20], since $0 < |d_n(x)| \equiv |d_n| < 1$ or $0 < \max_{x\in I}|d_n(x)| < 1$ and

$$d_n(x)y = \frac{d_n(x)}{\max\limits_{x\in I}|d_n(x)|}\max_{x\in I}|d_n(x)|y,$$

we can see that

$$F_n(x,y) = c_n x + d_n(x)y + e_n = c_n x + d_n^*(x) s_n(y) + e_n,$$

where $d_n^*(x) \coloneqq \frac{d_n(x)}{\max_{x\in I}|d_n(x)|}$ and $s_n(y) \coloneqq \max_{x\in I}|d_n(x)|y$, and thus, each s_n is a special Banach contraction and linear with respect to the second variable. Obviously, we can say that nonlinear FIFs may have more flexibility and applicability.

Connection 9

1. The well-known FIS in theory and applications is generated by an IFS of the form $\{K, w_{m,n} : m = 1, 2, \dots, M; n = 1, 2, \dots, N\}$ under some conditions, where the maps are transformations of the special structure

$$w_{m,n}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_m(x) \\ v_n(y) \\ F_{m,n}(x,y,z) \end{pmatrix} = \begin{pmatrix} a_m x + b_m \\ c_n y + d_n \\ e_{m,n}x + f_{m,n}y + g_{m,n}xy + d_{m,n}(x,y)z + h_{m,n} \end{pmatrix},$$

where $|d_{m,n}(x,y)| < 1$ for all $(x,y) \in D \subset \mathbb{R}^2$. Then for all $(x,y,z), (x,y,z') \in K$,

$$\begin{aligned} |F_{m,n}(x,y,z) - F_{m,n}(x,y,z')| &= |d_{m,n}(x,y)z - d_{m,n}(x,y)z'| \\ &\le \max_{(x,y)\in D}|d_{m,n}(x,y)||z - z'|. \end{aligned}$$

That is, each $w_{m,n}(x,y,z)$ is chosen so that function $F_{m,n}(x,y,z)$ is a Banach contraction with respect to the third variable. So, the existence of bivariable FIFs follows from Banach's fixed point theorem. In fact, in [22], since for all $(x,y) \in D \subset \mathbb{R}^2$, $d_{m,n}(x,y) \equiv s_{m,n}$ and $0 \le |s_{m,n}| < 1$, we can see that each $w_{m,n}(x,y,z)$ is chosen so that function $F_{m,n}(x,y,z)$ is Banach contraction with respect to the third variable. Also in [21], since

$$d_{m,n}(x,y) = \lambda_{m,n}(x - x_0)(x_M - x)(y - y_0)(y_N - y)$$

and

$$|\lambda_{m,n}| < \frac{16}{(x_M - x_0)^2 (y_N - y_0)^2},$$

we can see that $\max_{(x,y)\in D}|d_{m,n}(x,y)|<1$, and so each $w_{m,n}(x,y,z)$ is chosen so that function $F_{m,n}(x,y,z)$ is a Banach contraction with respect to the third variable.

2. In Theorem 2.3, for all $(x,y,z), (x,y,z')\in K\subset\mathbb{R}^3$,

$$\begin{aligned}|F_{m,n}(x,y,z)-F_{m,n}(x,y,z')| &= |d_{m,n}(x,y)||s_{m,n}(z)-s_{m,n}(z')|\\ &\le |s_{m,n}(z)-s_{m,n}(z')|\le\varphi(|z-z'|).\end{aligned}$$

That is, each $F_{m,n}(x,y,z)$ is Rakotch contraction with respect to the third variable. So, each $w_{m,n}(x,y,z)$ is chosen so that the function $F_{m,n}(x,y,z)$ is a Rakotch contraction with respect to the third variable.

Connection 10

In view of a φ-contraction, the connections between the coefficients of variable z are obtained as follows:

1. In the affine FIS (cf. [22]), for all $t>0$,

$$\varphi(t) \coloneqq \max_{m=1,2,\dots,M}\ \max_{n=1,2,\dots,N}|d_{m,n}|t,$$

where $|d_{m,n}|<1$ for all $m=1,2,\dots,M$ and $n=1,2,\dots,N$.

2. In the FIS with vertical scaling factors as function (cf. [21]), for all $t>0$,

$$\varphi(t) \coloneqq \max_{m=1,2,\dots,M}\ \max_{n=1,2,\dots,N}\ \max_{x\in I}|d_{m,n}(x)|t,$$

where $d_{m,n}(x)$ is Lipschitz function defined on I satisfying $\sup_{x\in I}|d_{m,n}(x)|<1$ for all $m=1,2,\dots,M$ and $n=1,2,\dots,N$.

Connection 11

The continuity of bivariable FIFs differ from the continuity of univariable FIFs.

1. The graphs of linear univariable FIFs are always continuous curves.

2. There are bivariable discontinuous functions that interpolate the given data; (see for instance [23], p. 630, 631).

3. Theorem 2.3 ensures that attractors of constructed IFSs are graphs of some bounded functions which interpolate the given data, but these graphs (i.e., the graphs of bivariable FIFs) are not always continuous surfaces. Some continuity conditions of bivariable FIFs are given explicitly by Corollary 2.1 and Corollary 2.2.

Connection 12

The key difficulty in constructing fractal interpolation surfaces (or volumes) involves ensuring continuity. Another important element necessary in modelling complicated surfaces of this type is the existence of the contractivity, or vertical scaling, factors.

1. In order to ensure continuity of a fractal interpolation surface, in [22], the interpolation points on the boundary was assumed collinear, whereas in [21], vertical scaling factors as (continuous) 'contraction functions' were used.

2. A new bivariable fractal interpolation function by using the Matkowski fixed point theorem and the Rakotch contraction is presented in [18]. In order to ensure the continuity of nonlinear FIS, the coplanarity of all the interpolation points on the boundaries instead of collinearity of interpolation points on the boundary was assumed in [18], whereas in [12], vertical scaling factors as (continuous) 'contraction functions' were used.

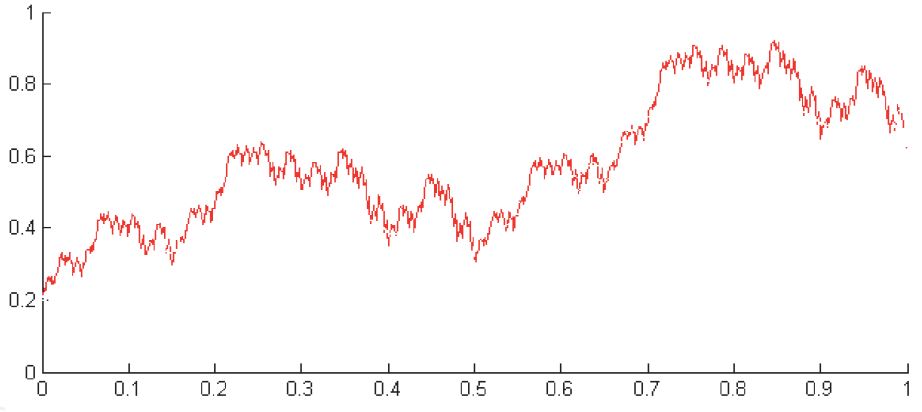
Connection 13

1. In Theorem 2.2, we can see that

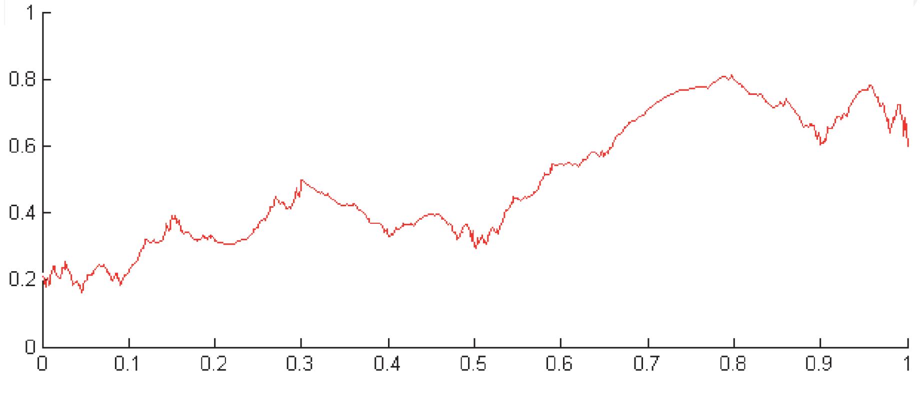
$$a_n = \frac{x_n - x_{n-1}}{x_N - x_0}, \quad b_n = \frac{x_N x_{n-1} - x_0 x_n}{x_N - x_0}$$

$$c_n = \frac{y_n - y_{n-1}}{x_N - x_0} - \frac{d_n(x_N)s_n(y_N) - d_n(x_0)s_n(y_0)}{x_N - x_0},$$

$$f_n = \frac{x_N y_{n-1} - x_0 y_n}{x_N - x_0} - \frac{x_N d_n(x_0)s_n(y_0) - x_0 d_n(x_N)s_n(y_N)}{x_N - x_0}.$$



(a)

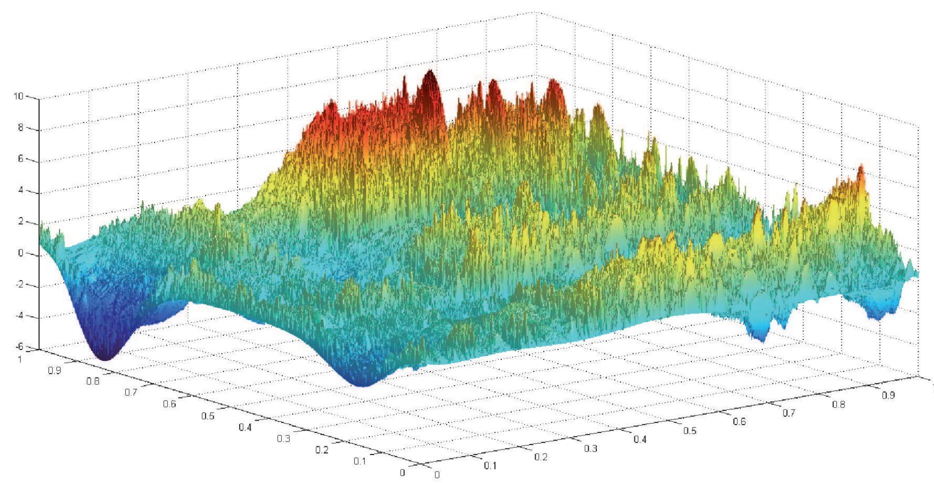


(b)

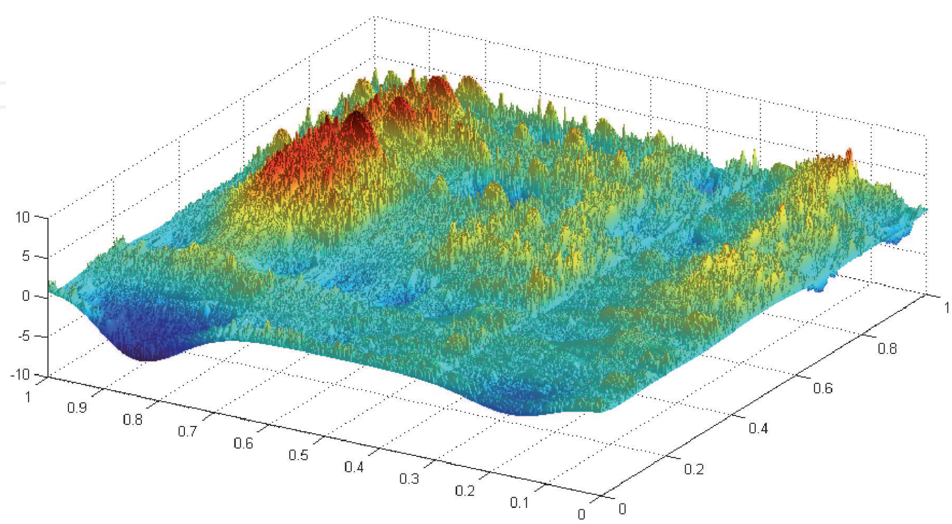
Figure 1.
The graph of a fractal interpolation function (a) that is associated with Banach contractions, (b) that is not necessarily associated with Banach contractions.

2. In Corollary 2.1, we can see that

$$
\begin{aligned}
a_m &= \frac{x_m - x_{m-1}}{x_M - x_0}, \quad b_m = \frac{x_M x_{m-1} - x_0 x_m}{x_M - x_0},\\
c_n &= \frac{y_n - y_{n-1}}{y_N - y_0}, \quad d_n = \frac{y_N y_{n-1} - y_0 y_n}{y_N - y_0},\\
g_{m,n} &= \frac{(z_{m,n} - z_{m-1,n}) - (z_{m,n-1} - z_{m-1,n-1})}{(x_M - x_0)(y_N - y_0)},\\
e_{m,n} &= \frac{y_N(z_{m,n-1} - z_{m-1,n-1}) - y_0(z_{m,n} - z_{m-1,n})}{(x_M - x_0)(y_N - y_0)},\\
f_{m,n} &= \frac{x_M(z_{m-1,n} - z_{m-1,n-1}) - x_0(z_{m,n} - z_{m,n-1})}{(x_M - x_0)(y_N - y_0)},\\
h_{m,n} &= \frac{x_0 y_0 z_{m,n} - x_0 y_N z_{m,n-1} - x_M y_0 z_{m-1,n} + x_M y_N z_{m-1,n-1}}{(x_M - x_0)(y_N - y_0)} - s_{m,n}(z_{M,N}).
\end{aligned}
$$



(a)



(b)

Figure 2.
A fractal interpolation surface (a) that is associated with Banach contractions, (b) that is not necessarily associated with Banach contractions.

3. In Corollary 2.2, we can see that (compare with above coefficients)

$$g_{m,n} = \frac{(z_{m,n} - z_{m-1,n}) - (z_{m,n-1} - z_{m-1,n-1})}{(x_M - x_0)(y_N - y_0)},$$

$$e_{m,n} = \frac{y_N(z_{m,n-1} - z_{m-1,n-1}) - y_0(z_{m,n} - z_{m-1,n})}{(x_M - x_0)(y_N - y_0)},$$

$$f_{m,n} = \frac{x_M(z_{m-1,n} - z_{m-1,n-1}) - x_0(z_{m,n} - z_{m,n-1})}{(x_M - x_0)(y_N - y_0)},$$

$$h_{m,n} = \frac{x_0 y_0 z_{m,n} - x_0 y_N z_{m,n-1} - x_M y_0 z_{m-1,n} + x_M y_N z_{m-1,n-1}}{(x_M - x_0)(y_N - y_0)}.$$

Figures 1(a) and **2(a)** are associated with Banach contractions, whereas **Figures 1(b)** and **2(b)** are not necessarily associated with Banach contractions.

4. Conclusions and further work

We reviewed nonlinear fractal interpolation functions by using the Geraghty fixed point theorem instead of the Banach fixed point theorem (or the Rakotch fixed point theorem) since Banach contraction (or Rakotch contraction) is a special case of Geraghty contraction. Theorems 2.1, 2.2 and 2.3 ensure that attractors of constructed nonlinear iterated function systems are graphs of some continuous functions which interpolate the given data. In particular, Examples 1, 2 and 3 show that our results remain still true under essentially weaker conditions on the maps of iterated function systems. The methods presented here can be directly extended to piecewise fractal interpolation functions that are based on recurrent IFS. A premise for future work is to extend these methods to hidden-variable fractal interpolation surfaces as well as to identify the parameters of such surfaces.

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