Chapter

Some Identities Involving 2-Variable Modified Degenerate Hermite Polynomials Arising from Differential Equations and Distribution of Their Zeros

Cheon Seoung Ryoo

Abstract

In this chapter, we introduce the 2-variable modified degenerate Hermite polynomials and obtain some new symmetric identities for 2-variable modified degenerate Hermite polynomials. In order to give explicit identities for 2-variable modified degenerate Hermite polynomials, differential equations arising from the generating functions of 2-variable modified degenerate Hermite polynomials are studied. Finally, we investigate the structure and symmetry of the zeros of the 2-variable modified degenerate Hermite equations.

Keywords: differential equations, symmetric identities, modified degenerate Hermite polynomials, complex zeros

2000 Mathematics Subject Classification: 05A19, 11B83, 34A30, 65L99

1. Introduction

The Hermite equation is defined as

$$u''(x) - 2xu'(x) + (\varepsilon - 1)u(x) = 0, x \in [-\infty, \infty], \tag{1}$$

where ε is unrestricted. Hermite equation is encountered in the study of quantum mechanical harmonic oscillator, where ε represent the energy of the oscillator. The ordinary Hermite numbers H_n and Hermite polynomials $H_n(x)$ are usually defined by the generating functions

$$e^{t(2x-t)} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \tag{2}$$

and

$$e^{-t^2} = \sum_{n=0}^{\infty} H_n \frac{t^n}{n!}. \tag{3}$$

Clearly, $H_n = H_n(0)$.

It is known that these numbers and polynomials play an important role in various fields of mathematics and physics, including number theory, combinations, special functions, and differential equations. Many interested properties about that have been studied (see [1–5]). The ordinary Hermite polynomials $H_n(x)$ satisfy the Hermite differential equation

$$\frac{d^2H(x)}{dx^2} - 2x\frac{dH(x)}{dx} + 2nH(x) = 0, n = 0, 1, 2, \dots . \tag{4}$$

Hence ordinary Hermite polynomials $H_n(x)$ satisfy the second-order ordinary differential equation

$$u'' - 2xu' + 2nu = 0. \tag{5}$$

We remind that the 2-variable Hermite polynomials $H_n(x, y)$ defined by the generating function (see [2])

$$\sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!} = e^{t(x+yt)} \tag{6}$$

are the solution of heat equation

$$\frac{\partial}{\partial y}H_n(x, y) = \frac{\partial^2}{\partial x^2}H_n(x, y), \quad H_n(x, 0) = x^n. \tag{7}$$

Observe that

$$H_n(2x, -1) = H_n(x). \tag{8}$$

Motivated by their importance and potential applications in certain problems of probability, combinatorics, number theory, differential equations, numerical analysis and other areas of mathematics and physics, several kinds of some special numbers and polynomials were recently studied by many authors (see [1–8]). Many mathematicians have studied in the area of the degenerate Stiling, degenerate Bernoulli polynomials, degenerate Euler polynomials, degenerate Genocchi polynomials, and degenerate tangent polynomials (see [6, 7, 9]).

Recently, Hwang and Ryoo [10] proposed the 2-variable degenerate Hermite polynomials $\mathcal{H}_n(x, y, \lambda)$ by means of the generating function

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x, y, \lambda)\frac{t^n}{n!} = (1 + \lambda)^{\frac{t(x+yt)}{\lambda}}. \tag{9}$$

Since $(1 + \mu)^{\frac{t}{\mu}} \to e^t$ as $\mu \to 0$, it is evident that (9) reduces to (6). The 2-variable degenerate Hermite polynomials $\mathcal{H}_n(x, y, \lambda)$ in generating function (9) are the solution of equation

$$\begin{aligned} &\frac{\partial}{\partial y}\mathcal{H}_n(x, y, \lambda) = \frac{\lambda}{\log(1+\lambda)}\frac{\partial^2}{\partial x^2}\mathcal{H}_n(x, y, \lambda), \\ &\mathcal{H}_n(x, 0, \lambda) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n x^n. \end{aligned} \tag{10}$$

Since $\frac{\log(1+\lambda)}{\lambda} \to 1$ as λ approaches to 0, it is apparent that (10) descends to (7).

Mathematicians have studied the differential equations arising from the generating functions of special numbers and polynomials (see [10–14]). Now, a new class of 2-variable modified degenerate Hermite polynomials are constructed based on the results so far. We can induce the differential equations generated from the generating function of 2-variable modified degenerate Hermite polynomials. By using the coefficients of this differential equation, we obtain explicit identities for the 2-variable modified degenerate Hermite polynomials. The rest of the paper is organized as follows. In Section 2, we construct the 2-variable modified degenerate Hermite polynomials and obtain basic properties of these polynomials. In Section 3, we give some symmetric identities for 2-variable modified degenerate Hermite polynomials. In Section 4, we derive the differential equations generated from the generating function of 2-variable modified degenerate Hermite polynomials. Using the coefficients of this differential equation, we have explicit identities for the 2-variable modified degenerate Hermite polynomials. In Section 5, we investigate the zeros of the 2-variable modified degenerate Hermite equations by using computer. Further, we observe the pattern of scattering phenomenon for the zeros of 2-variable modified degenerate Hermite equations. Our paper will finish with Section 6, where the conclusions and future directions of this work are showed.

2. Basic properties for the 2-variable modified degenerate Hermite polynomials

In this section, a new class of the 2-variable modified degenerate Hermite polynomials are considered. Furthermore, some properties of these polynomials are also obtained.

We define the 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x,y|\mu)$ by means of the generating function

$$\sum_{n=0}^{\infty} \mathrm{H}_n(x,y|\mu)\frac{t^n}{n!} = (1+\mu)^{\frac{xt}{\mu}} e^{yt^2}. \tag{11}$$

Since $(1+\mu)^{\frac{xt}{\mu}} \to e^{xt}$ as $\mu \to 0$, it is clear that (11) reduces to (6). Observe that degenerate Hermite polynomials $\mathcal{H}_n(x,y,\mu)$ and 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x,y|\mu)$ are totally different.

Now, we recall that the μ-analogue of the falling factorial sequences as follows:

$$(x|\mu)_0 = 1, (x|\mu)_n = x(x-\mu)(x-2\mu)\cdots(x-(n-1)\mu), (n \geq 1). \tag{12}$$

Note that $\lim_{\mu\to 1}(x|\mu)_n = x(x-1)(x-2)\cdots(x-(n-1)) = (x)_n, (n \geq 1)$. We also need the binomial theorem: for a variable y,

$$\begin{aligned}(1+\mu)^{yt/\mu} &= \sum_{m=0}^{\infty}\binom{\frac{ty}{\mu}}{m}\frac{\mu^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m} S_1(m,l)\left(\frac{ty}{\mu}\right)^l \frac{\mu^m}{m!}\right)\\ &= \sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty} S_1(m,l) y^l \mu^{m-l}\frac{l!}{m!}\right)\frac{t^l}{l!}.\end{aligned} \tag{13}$$

We remember that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations (see [6–13])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{14}$$

respectively. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{15}$$

As another application of the differential equation for $\mathrm{H}_n(x,y|\mu)$ is as follows: Note that

$$G(t,x,y,\mu) = (1+\mu)^{\frac{xt}{\mu}} e^{yt^2} \tag{16}$$

satisfies

$$\frac{\partial G(t,x,y,\mu)}{\partial y} - \left(\frac{\log(1+\mu)}{\mu}\right)^2 \frac{\partial^2 G(t,x,y,\mu)}{\partial x^2} = 0. \tag{17}$$

Substitute the series in (11) for $G(t,x,y,\mu)$ to get

$$\frac{\partial}{\partial y}\mathrm{H}_n(x,y|\mu) = \left(\frac{\mu}{\log(1+\mu)}\right)^2 \frac{\partial^2}{\partial x^2}\mathrm{H}_n(x,y|\mu).$$

Thus the 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x,y|\mu)$ in generating function (11) are the solution of equation

$$\begin{aligned} &\left(\frac{\log(1+\mu)}{\mu}\right)^2 \frac{\partial}{\partial y}\mathrm{H}_n(x,y|\mu) - \frac{\partial^2}{\partial x^2}\mathrm{H}_n(x,y|\mu) = 0, \\ &\mathrm{H}_n(x,0|\mu) = \left(\frac{\log(1+\mu)}{\mu}\right)^n x^n. \end{aligned} \tag{18}$$

The generating function (11) is useful for deriving several properties of the 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x,y|\mu)$. For example, we have the following expression for these polynomials:

Theorem 1. For any positive integer n, we have

$$\mathrm{H}_n(x,y|\mu) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \left(\frac{\log(1+\mu)}{\mu}\right)^{n-2k} x^{n-2k} y^k \frac{n!}{k!(n-2k)!}, \tag{19}$$

where [ ] denotes taking the integer part.

Proof. By (11) and (13), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{H}_n(x,y|\mu)\frac{t^n}{n!} &= (1+\mu)^{\frac{xt}{\mu}} e^{yt^2} \\ &= \sum_{k=0}^{\infty} y^k \frac{t^{2k}}{k!} \sum_{l=0}^{\infty} \left(\frac{\log(1+\mu)}{\mu}\right)^l x^l \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{n}{2}\right]} y^k \left(\frac{\log(1+\mu)}{\mu}\right)^{n-2k} x^{n-2k} \frac{n!}{k!(n-2k)!} \right) \frac{t^n}{n!}. \end{aligned} \tag{20}$$

By comparing the coefficients of $\frac{t^n}{n!}$, the expected result of Theorem 1 is achieved. □

Since $\lim_{\mu\to 0}\frac{\log(1+\mu)}{\mu}=1$, we get

$$H_n(x,y)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{y^k x^{n-2k}}{k!(n-2k)!}. \tag{21}$$

The following basic properties of the 2-variable degenerate Hermite polynomials $\mathrm{H}_n(x,y|\mu)$ are induced form (11). Therefore, it is enough to delete involved detail explanation.

Theorem 2. For any positive integer n, we have

$$\begin{aligned}
&1.\ \mathrm{H}_n(x,y|\mu)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\sum_{m=n-2k}^{\infty} y^k S_1(m,n-2k)x^{n-2k}\mu^{m-(n-2k)}\frac{n!}{m!k!}.\\
&2.\ \mathrm{H}_n(x_1+x_2,y|\mu)=\sum_{l=0}^{n}\binom{n}{l}\left(\frac{\log(1+\mu)}{\mu}\right)^l x_2^l\mathrm{H}_{n-l}(x_1,y|\mu).\\
&3.\ \mathrm{H}_n(x_1+x_2,y|\mu)=\sum_{l=0}^{n}\binom{n}{l}\mathrm{H}_{n-l}(x_1,y|\mu)\sum_{m=l}^{\infty}S_1(m,l)x_2^l\mu^{m-l}\frac{l!}{m!}.\\
&4.\ \mathrm{H}_n(x,y_1+y_2|\mu)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\mathrm{H}_{n-2k}(x,y_1|\mu)\frac{y_2^k n!}{k!(n-2k)!}.\\
&5.\ \mathrm{H}_n(x_1+x_2,y_1+y_2|\mu)=\sum_{l=0}^{n}\binom{n}{l}\mathrm{H}_l(x_1,y_1|\mu)\mathrm{H}_{n-l}(x_2,y_2|\mu).
\end{aligned} \tag{22}$$

3. Symmetric identities for 2-variable modified degenerate Hermite polynomials

In this section, we give some new symmetric identities for 2-variable modified degenerate Hermite polynomials. We also get some explicit formulas and properties for 2-variable modified degenerate Hermite polynomials.

Theorem 3. Let $a,b>0$ $(a\neq b)$. The following identity holds true:

$$b^m\mathrm{H}_m(ax,a^2y|\mu)=a^m\mathrm{H}_m(bx,b^2y|\mu). \tag{23}$$

Proof. Let $a,b>0$ $(a\neq b)$. We start with

$$\mathcal{G}(t,\mu)=(1+\mu)^{\frac{abxt}{\mu}}e^{a^2b^2yt^2}. \tag{24}$$

Then the expression for $\mathcal{G}(t,\mu)$ is symmetric in a and b

$$\mathcal{G}(t,\mu)=\sum_{m=0}^{\infty}\mathrm{H}_m(ax,a^2y|\mu)\frac{(bt)^m}{m!}=\sum_{m=0}^{\infty}b^m\mathrm{H}_m(ax,a^2y|\mu)\frac{t^m}{m!}. \tag{25}$$

By the similar way, we get that

$$\mathcal{G}(t,\mu)=\sum_{m=0}^{\infty}\mathrm{H}_m(bx,b^2y|\mu)\frac{(at)^m}{m!}=\sum_{m=0}^{\infty}a^m\mathrm{H}_m(bx,b^2y|\mu)\frac{t^m}{m!}. \tag{26}$$

By comparing the coefficients of $\frac{t^m}{m!}$ in last two equations, the expected result of Theorem 3 is achieved. □

Again, we now use

$$\mathcal{F}(t,\mu)=\frac{abt(1+\mu)^{\frac{abxt}{\mu}}e^{a^2b^2yt^2}\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}. \tag{27}$$

For $\mu\in\mathbb{C}$, we introduce the modified degenerate Bernoulli polynomials given by the generating function

$$\sum_{n=0}^{\infty}\beta_n(x|\mu)\frac{t^n}{n!}=\frac{t}{(1+\mu)^{\frac{t}{\mu}}-1}(1+\mu)^{\frac{xt}{\mu}},\ (\text{see}[6,7]). \tag{28}$$

When $x=0$ and $\beta_n(\mu)=\beta_n(0|\mu)$ are called the modified degenerate Bernoulli numbers. Note that

$$\lim_{\mu\to 0}\beta_n(\mu)=B_n, \tag{29}$$

where B_n are called the Bernoulli numbers. The first few of them are

$$\begin{aligned}
\beta_0(x|\mu)&=\frac{\mu}{\log(1+\mu)},\\
\beta_1(x|\mu)&=-\frac{1}{2}+x,\\
\beta_2(x|\mu)&=\frac{\log(1+\mu)}{6\mu}-\frac{x\log(1+\mu)}{\mu}+\frac{x^2\log(1+\mu)}{\mu},\\
\beta_3(x|\mu)&=\frac{x\log(1+\mu)^2}{2\mu^2}-\frac{3x^2\log(1+\mu)^2}{2\mu^2}+\frac{x^3\log(1+\mu)^2}{\mu^2},\\
\beta_4(x|\mu)&=-\frac{\log(1+\mu)^3}{30\mu^3}+\frac{x^2\log(1+\mu)^3}{\mu^3}-\frac{2x^3\log(1+\mu)^3}{\mu^3}+\frac{x^4\log(1+\mu)^3}{\mu^3}.
\end{aligned} \tag{30}$$

For each integer $k\geq 0$, $S_k(n)=0^k+1^k+2^k+\cdots+(n-1)^k$ is called sum of integers. A modified generalized falling factorial sum $\sigma_k(n,\mu)$ can be defined by the generating function

$$\sum_{k=0}^{\infty}\sigma_k(n|\mu)\frac{t^k}{k!}=\frac{(1+\mu)^{\frac{(n+1)t}{\mu}}-1}{(1+\mu)^{\frac{t}{\mu}}-1}. \tag{31}$$

Note that $\lim_{\mu\to 0}\sigma_k(n|\mu)=S_k(n)$. From $\mathcal{F}(t,\mu)$, we get the following result:

$$\begin{aligned}
\mathcal{F}(t,\mu)&=\frac{abt(1+\mu)^{\frac{abxt}{\mu}}e^{a^2b^2yt^2}\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}\\
&=\frac{abt}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)}(1+\mu)^{\frac{abxt}{\mu}}e^{a^2b^2yt^2}\frac{\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}\\
&=b\sum_{n=0}^{\infty}\beta_n(\mu)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\mathrm{H}_n(bx,b^2y|\mu)\frac{(at)^n}{n!}\sum_{n=0}^{\infty}\sigma_k(a-1|\mu)\frac{(bt)^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}a^ib^{n+1-i}\beta_m(\mu)\mathrm{H}_{i-m}(bx,b^2y|\mu)\sigma_{n-i}(a-1|\mu)\right)\frac{t^n}{n!}.
\end{aligned} \tag{32}$$

In a similar fashion we have

$$\begin{aligned}\mathcal{F}(t,\mu)&=\frac{abt}{\left((1+\mu)^{\frac{bt}{\mu}}-1\right)}(1+\mu)^{\frac{abxt}{\mu}}e^{a^2b^2yt^2}\frac{\left((1+\mu)^{\frac{abt}{\mu}}-1\right)}{\left((1+\mu)^{\frac{at}{\mu}}-1\right)}\\&=a\sum_{n=0}^{\infty}\beta_n(\mu)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\mathrm{H}_n\left(ax,a^2y|\mu\right)\frac{(bt)^n}{n!}\sum_{n=0}^{\infty}\sigma_k(b-1|\mu)\frac{(at)^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}b^ia^{n+1-i}\beta_m(\mu)\mathrm{H}_{i-m}\left(ax,a^2y|\mu\right)\sigma_{n-i}(b-1|\mu)\right)\frac{t^n}{n!}.\end{aligned}\tag{33}$$

By comparing the coefficients of $\frac{t^m}{m!}$ on the right hand sides of the last two equations, we have the below theorem.

Theorem 4. Let $a,b>0(a\neq b)$. The the following identity holds true:

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}a^ib^{n+1-i}\beta_m(\mu)\mathrm{H}_{i-m}\left(bx,b^2y|\mu\right)\sigma_{n-i}(a-1|\mu)\\&=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}b^ia^{n+1-i}\beta_m(\mu)\mathrm{H}_{i-m}\left(ax,a^2y|\mu\right)\sigma_{n-i}(b-1|\mu).\end{aligned}\tag{34}$$

By taking the limit as $\mu\to0$, we have the following corollary.

Corollary 5. Let $a,b>0$ $(a\neq b)$. The the following identity holds true:

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}a^ib^{n+1-i}\mathrm{S}_{n-i}(a-1)B_mH_{i-m}\left(bx,b^2y\right)\\&=\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n}{i}\binom{i}{m}b^ia^{n+1-i}\mathrm{S}_{n-i}(b-1)B_mH_{i-m}\left(ax,a^2y\right).\end{aligned}\tag{35}$$

4. Differential equations associated with 2-variable modified degenerate Hermite polynomials

In this section, we construct the differential equations with coefficients $a_i(N,x,y,\mu)$ arising from the generating functions of the 2-variable modified degenerate Hermite polynomials:

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\right)^NG(t,x,y,\mu)-a_0(N,x,y,\mu)G(t,x,y,\mu)-\cdots\\&-a_{2N}(N,x,y,\mu)t^NG(t,x,y,\mu)=0.\end{aligned}\tag{36}$$

By using the coefficients of this differential equation, we can get explicit identities for the 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x,y,\mu)$. Recall that

$$\begin{aligned}G&=G(t,x,y,\mu)\\&=(1+\mu)^{\frac{xt}{\mu}}e^{yt^2}\\&=\sum_{n=0}^{\infty}\mathrm{H}_n(x,y|\mu)\frac{t^n}{n!},\quad\mu,x,t\in\mathbb{C}.\end{aligned}\tag{37}$$

Then, by (37), we have

$$\begin{aligned} G^{(1)} &= \frac{\partial}{\partial t} G(t,x,y,\mu) \\ &= \frac{\partial}{\partial t}\left((1+\mu)^{\frac{xt}{\mu}} e^{yt^2}\right) \\ &= \left(\frac{\log(1+\mu)}{\mu}x + 2yt\right)\left((1+\mu)^{\frac{xt}{\mu}} e^{yt^2}\right) \\ &= \left(\frac{\log(1+\mu)}{\mu}x + 2yt\right) G(t,x,y,\mu), \end{aligned} \tag{38}$$

$$\begin{aligned} G^{(2)} &= \frac{\partial}{\partial t} G^{(1)}(t,x,y,\mu) \\ &= 2yG(t,x,y,\mu) + \left(\frac{\log(1+\mu)}{\mu}x + 2yt\right) G^{(1)}(t,x,y,\mu) \\ &= \left(2y + \left(\frac{\log(1+\mu)}{\mu}\right)^2 x^2\right) G(t,x,y,\mu) \\ &+ \left(\frac{\log(1+\mu)}{\mu} 4xy\right) tG(t,x,y,\mu) \\ &+ (2y)^2 t^2 G(t,x,y,\mu). \end{aligned} \tag{39}$$

By continuing this process as shown in (39), we can get easily that

$$\begin{aligned} G^{(N)} &= \left(\frac{\partial}{\partial t}\right)^N G(t,x,y,\mu) \\ &= \sum_{i=0}^{N} a_i(N,x,y,\mu) t^i G(t,x,y,\mu), (N = 0,1,2, \dots). \end{aligned} \tag{40}$$

By differentiating (40) with respect to t, we have

$$\begin{aligned} G^{(N+1)} = \frac{\partial G^{(N)}}{\partial t} &= \sum_{i=0}^{N} a_i(N,x,y,\mu)(i) t^{i-1} G(t,x,y,\mu) \\ &+ \sum_{i=0}^{N} a_i(N,x,y,\mu) t^i G^{(1)}(t,x,y,\mu) \\ = \sum_{i=0}^{N} (i) a_i(N,x,y,\mu) t^{i-1} G(t,x,y,\mu) & \\ &+ \sum_{i=0}^{N} \left(\frac{\log(1+\mu)}{\mu}x\right) a_i(N,x,y,\mu) t^i G(t,x,y,\mu) \\ &+ \sum_{i=0}^{N} (2y) a_i(N,x,y,\mu) t^{i+1} G(t,x,y,\mu) \\ = \sum_{i=0}^{N-1} (i+1) a_{i+1}(N,x,y,\mu) t^i G(t,x,y,\mu) & \\ &+ \sum_{i=0}^{N} \left(\frac{\log(1+\mu)}{\mu}x\right) a_i(N,x,y,\mu) t^i G(t,x,y,\mu) \\ &+ \sum_{i=1}^{N+1} (2y) a_{i-1}(N,x,y,\mu) t^i G(t,x,y,\mu). \end{aligned} \tag{41}$$

Now we replace N by $N+1$ in (40). We find

$$G^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,y,\mu)t^i G(t,x,y,\mu). \tag{42}$$

By comparing the coefficients on both sides of (41) and (42), we get

$$\begin{aligned} a_0(N+1,x,y,\mu) &= a_1(N,x,y,\mu) + \left(\frac{\log(1+\mu)}{\mu}x\right)a_0(N,x,y,\mu),\\ a_N(N+1,x,y,\mu) &= \left(\frac{\log(1+\mu)}{\mu}x\right)a_N(N,x,y,\mu)\\ &\quad +2ya_{N-1}(N,x,y,\mu),\\ a_{N+1}(N+1,x,y,\mu) &= (2y)a_N(N,x,y,\mu), \end{aligned} \tag{43}$$

and

$$\begin{aligned} a_i(N+1,x,y,\mu) &= (i+1)a_{i+1}(N,x,y,\mu)\\ &\quad +\left(\frac{\log(1+\mu)}{\mu}x\right)a_i(N,x,y,\mu)\\ &\quad +(2y)a_{i-1}(N,x,y,\mu), (1 \leq i \leq N-1). \end{aligned} \tag{44}$$

In addition, by (37), we have

$$G(t,x,y,\mu) = G^{(0)}(t,x,y,\mu) = a_0(0,x,y,\mu)G(t,x,y,\mu). \tag{45}$$

By (45), we get

$$a_0(0,x,y,\mu) = 1. \tag{46}$$

It is not difficult to show that

$$\begin{aligned} &\frac{x\log(1+\mu)}{\mu}G(t,x,y,\mu) + 2ytG(t,x,y,\mu)\\ &= G^{(1)}(t,x,y,\mu)\\ &= \sum_{i=0}^{1} a_i(1,x,y,\mu)t^i G(t,x,y,\mu)\\ &= a_0(1,x,y,\mu)G(t,x,y,\mu) + a_1(1,x,y,\mu)tG(t,x,y,\mu). \end{aligned} \tag{47}$$

Thus, by (38) and (47), we also get

$$a_0(1,x,y,\mu) = \frac{x\log(1+\mu)}{\mu}, \quad a_1(1,x,y,\mu) = 2y. \tag{48}$$

From (43) and (44), we note that

$$\begin{aligned} a_0(N+1,x,y,\mu) &= a_1(N,x,y,\mu) + \frac{x\log(1+\mu)}{\mu}a_0(N,x,y,\mu),\\ a_0(N,x,y,\mu) &= a_1(N-1,x,y,\mu) + \frac{x\log(1+\mu)}{\mu}a_0(N-1,x,y,\mu),\\ &\cdots\\ a_0(N+1,x,y,\mu) &= \sum_{i=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^i a_1(N-i,x,y,\mu)\\ &\quad +\left(\frac{\log(1+\mu)}{\mu}\right)^{N+1}x^{N+1}, \end{aligned} \tag{49}$$

$$\begin{aligned}
a_N(N+1,x,y,\mu) &= \frac{x\log(1+\mu)}{\mu}a_N(N,x,y,\mu)\\
&\quad +(2y)a_{N-1}(N,x,y,\mu),\\
a_{N-1}(N,x,y,\mu) &= \frac{x\log(1+\mu)}{\mu}a_{N-1}(N-1,x,y,\mu)\\
&\quad +(2y)a_{N-2}(N-1,x,y,\mu),\ \dots\\
a_N(N+1,x,y,\mu) &= (N+1)x(2y)^N\left(\frac{\log(1+\mu)}{\mu}\right),
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
a_{N+1}(N+1,x,y,\mu) &= (2y)a_N(N,x,y,\mu),\\
a_N(N,x,y,\mu) &= (2y)a_{N-1}(N-1,x,y,\mu),\ \dots\\
a_{N+1}(N+1,x,y,\mu) &= (2y)^{N+1}.
\end{aligned} \tag{51}$$

For $i=1$ in (44), we have

$$\begin{aligned}
a_1(N+1,x,y,\mu) &= 2\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_2(N-k,x,y,\mu)\\
&\quad +2y\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_0(N-k,x,y,\mu),
\end{aligned} \tag{52}$$

Continuing this process, we can deduce that, for $1\le i\le N-1$,

$$\begin{aligned}
a_i(N+1,x,y,\mu) &= (i+1)\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_{i+1}(N-k,x,y,\mu)\\
&\quad +2y\sum_{k=0}^{N}\left(\frac{x\log(1+\mu)}{\mu}\right)^k a_{i-1}(N-k,x,y,\mu).
\end{aligned} \tag{53}$$

Note that, from (37)–(53), here the matrix $a_i(j,x,y,\mu)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix}
1 & \frac{x\log(1+\mu)}{\mu} & 2y+\left(\frac{\log(1+\mu)}{\mu}\right)^2x^2 & \cdots & \cdot\\
0 & 2y & \left(\frac{\log(1+\mu)}{\mu}\right)4xy & \cdots & \cdot\\
0 & 0 & (2y)^2 & \cdots & \cdot\\
\vdots & \vdots & \vdots & \ddots & \cdot\\
0 & 0 & 0 & \cdots & (2y)^{N+1}
\end{pmatrix} \tag{54}$$

Therefore, from (37)–(53), we obtain the following theorem.
Theorem 5. For $N=0,1,2,\dots,$ the differential equation

$$\left(\frac{\partial}{\partial t}\right)^N G(t,x,y,\mu)-\left(\sum_{i=0}^{N}a_i(N,x,y,\mu)t^i\right)G(t,x,y,\mu)=0 \tag{55}$$

has a solution

$$G = G(t,x,y,\mu) = (1+\mu)^{\frac{xt}{\mu}} e^{yt^2}, \tag{56}$$

where

$$\begin{aligned} a_0(N+1,x,y,\mu) &= \sum_{i=0}^{N} \left(\frac{x \log(1+\mu)}{\mu}\right)^i a_1(N-i,x,y,\mu) \\ &\quad + \left(\frac{\log(1+\mu)}{\mu}\right)^{N+1} x^{N+1}, \\ a_N(N+1,x,y,\mu) &= (N+1)(2y)^N \left(\frac{x \log(1+\mu)}{\mu}\right), \\ a_{N+1}(N+1,x,y,\mu) &= (2y)^{N+1}, \\ a_i(N+1,x,y,\mu) &= (i+1) \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu}\right)^k a_{i+1}(N-k,x,y,\mu) \\ &\quad + 2y \sum_{k=0}^{N} \left(\frac{x \log(1+\mu)}{\mu}\right)^k a_{i-1}(N-k,x,y,\mu), (1 \le i \le N-1). \end{aligned} \tag{57}$$

Here is a plot of the surface for this solution. In the left picture of **Figure 1**, we choose $-2 \le x \le 2,\ -1 \le t \le 1, \mu = 1/10,$ and $y = 0.1$. In the right picture of **Figure 1**, we choose $-2 \le y \le 2,\ -1 \le t \le 1, \mu = 1/10,$ and $x = 0.1$.

Making N-times derivative for (10) with respect to t, we have

$$\left(\frac{\partial}{\partial t}\right)^N G(t,x,y,\mu) = \sum_{m=0}^{\infty} \mathrm{H}_{m+N}(x,y,\mu) \frac{t^m}{m!}. \tag{58}$$

By (58) and Theorem 5, we have

$$\begin{aligned} &a_0(N,x,y,\mu)G(t,x,y,\mu) \\ &+ a_1(N,x,y,\mu)tG(t,x,y,\mu) \\ &+ \cdots \\ &+ a_N(N,x,y,\mu)t^N G(t,x,y,\mu) \\ &= \sum_{m=0}^{\infty} \mathrm{H}_{m+N}(x,y,\mu) \frac{t^m}{m!}. \end{aligned} \tag{59}$$

Hence we have the following theorem.

Theorem 6. For $N = 0, 1, 2, \ldots,$ we get

$$\mathrm{H}_{m+N}(x,y,\mu) = \sum_{i=0}^{m} \frac{\mathrm{H}_{m-i}(x) a_i(N,x,y,\mu) m!}{(m-i)!}. \tag{60}$$

If we take $m = 0$ in (60), then we have the below corollary.

Corollary 7. For $N = 0, 1, 2, \ldots,$ we have

$$\mathrm{H}_N(x,y,\mu) = a_0(N,x,y,\mu), \tag{61}$$

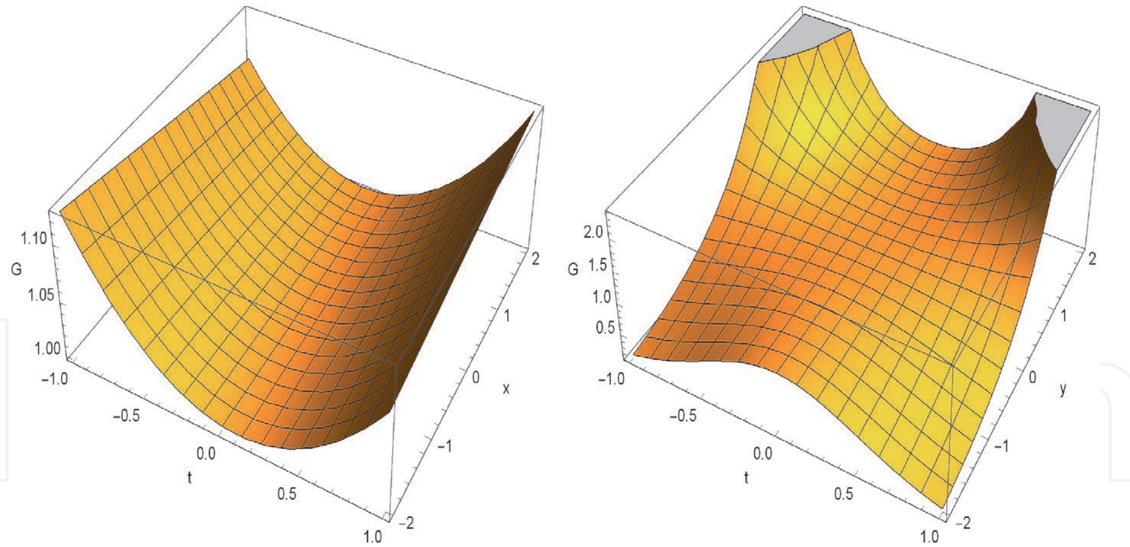

Figure 1.
The surface for the solution $G(t,x,y,\mu)$.

where

$$\begin{aligned} a_0(0,x,y,\mu) &= 1, \\ a_0(N+1,x,y,\mu) &= \sum_{i=0}^{N} \left(\frac{x\log(1+\mu)}{\mu}\right)^i a_1(N-i,x,y,\mu) \\ &\quad + \left(\frac{\log(1+\mu)}{\mu}\right)^{N+1} x^{N+1}. \end{aligned} \tag{62}$$

The first few of them are

$$\begin{aligned} &\mathrm{H}_0(x,y,\mu) = 1, \\ &\mathrm{H}_1(x,y,\mu) = \frac{\log(1+\mu)}{\mu}x, \\ &\mathrm{H}_2(x,y,\mu) = 2y + \frac{(\log(1+\mu))^2}{\mu^2}x^2, \\ &\mathrm{H}_3(x,y,\mu) = 6xy\frac{\log(1+\mu)}{\mu} + \frac{(\log(1+\mu))^3}{\mu^3}x^3, \\ &\mathrm{H}_4(x,y,\mu) = 12y^2 + 12x^2y\frac{(\log(1+\mu))^2}{\mu^2} + \frac{(\log(1+\mu))^4}{\mu^4}x^4, \\ &\mathrm{H}_5(x,y,\mu) = 60xy^2\frac{\log(1+\mu)}{\mu} + 20x^3y\frac{(\log(1+\mu))^3}{\mu^3} + \frac{(\log(1+\mu))^5}{\mu^5}x^5. \end{aligned} \tag{63}$$

5. Zeros of the 2-variable modified degenerate Hermite polynomials

This section shows the benefits of supporting theoretical prediction through numerical experiments and finding new interesting pattern of the zeros of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x,y|\mu) = 0$. By using computer, the 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x,y|\mu)$ can be determined explicitly. We investigate the zeros of the 2-variable modified degenerate

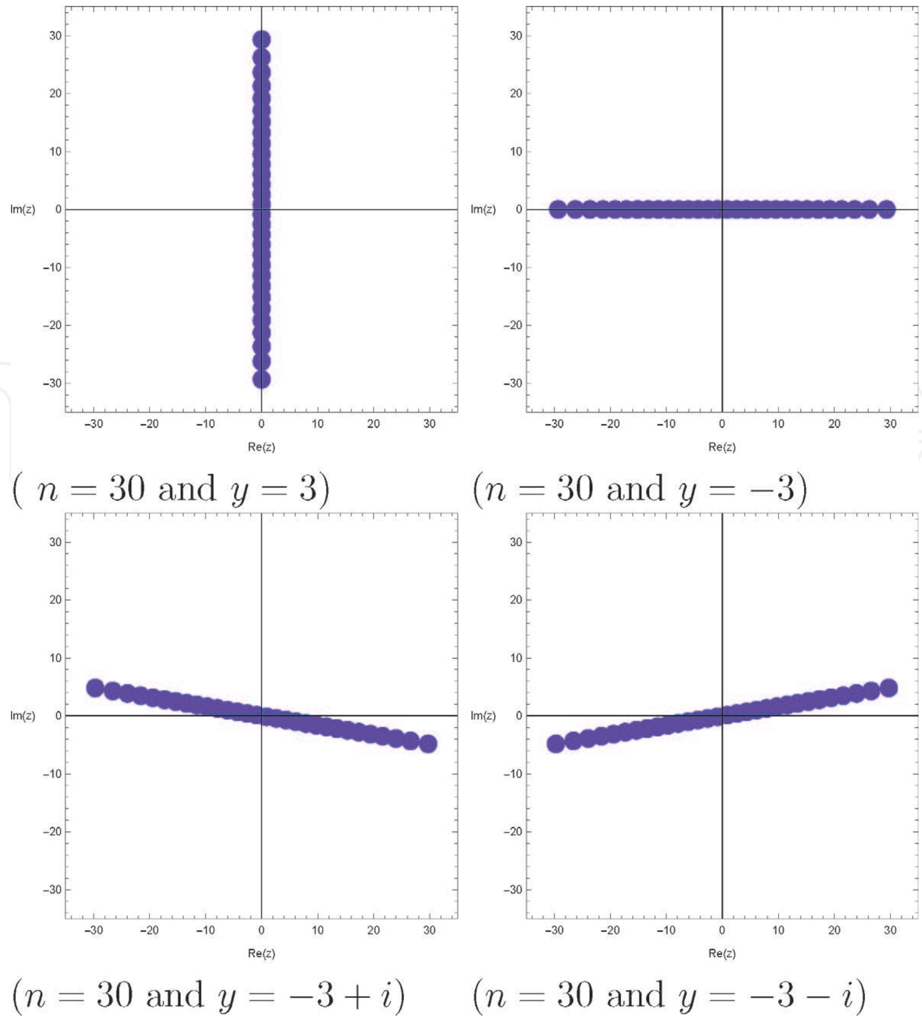

Figure 2.
Zeros of $\mathrm{H}_n(x, y|\mu) = 0$.

Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$. The zeros of the $\mathrm{H}_n(x, y|\mu) = 0$ for $n = 30, y = 3, -3, 3 + i, -3 - i, \mu = 1/2,$ and $x \in \mathbb{C}$ are displayed in **Figure 2**. In the top-left picture of **Figure 2**, we choose $n = 30$ and $y = 3$. In the top-right picture of **Figure 2**, we choose $n = 30$ and $y = -3$. In the bottom-left picture of **Figure 2**, we choose $n = 30$ and $y = -3 + i$. In the bottom-right picture of **Figure 2**, we choose $n = 30$ and $y = -3 - i$.

Stacks of zeros of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ for $1 \leq n \leq 50, \mu = 1/2$ from a 3-D structure are presented in **Figure 3**. In the top-left picture of **Figure 3**, we choose $y = 3$. In the top-right picture of **Figure 3**, we choose $y = -3$. In the bottom-left picture of **Figure 3**, we choose $y = -3 + i$. In the bottom-right picture of **Figure 3**, we choose $y = -3 - i$.

Our numerical results for approximate solutions of real zeros of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ are displayed (**Tables 1** and **2**).

We observed a remarkable regular structure of the complex roots of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ and also hope to verify same kind of regular structure of the complex roots of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ (**Table 1**).

Plot of real zeros of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ for $1 \leq n \leq 50, \mu = 1/2$ structure are presented in **Figure 4**. In the top-left picture of **Figure 4**, we choose $y = 3$. In the top-right picture of **Figure 4**, we

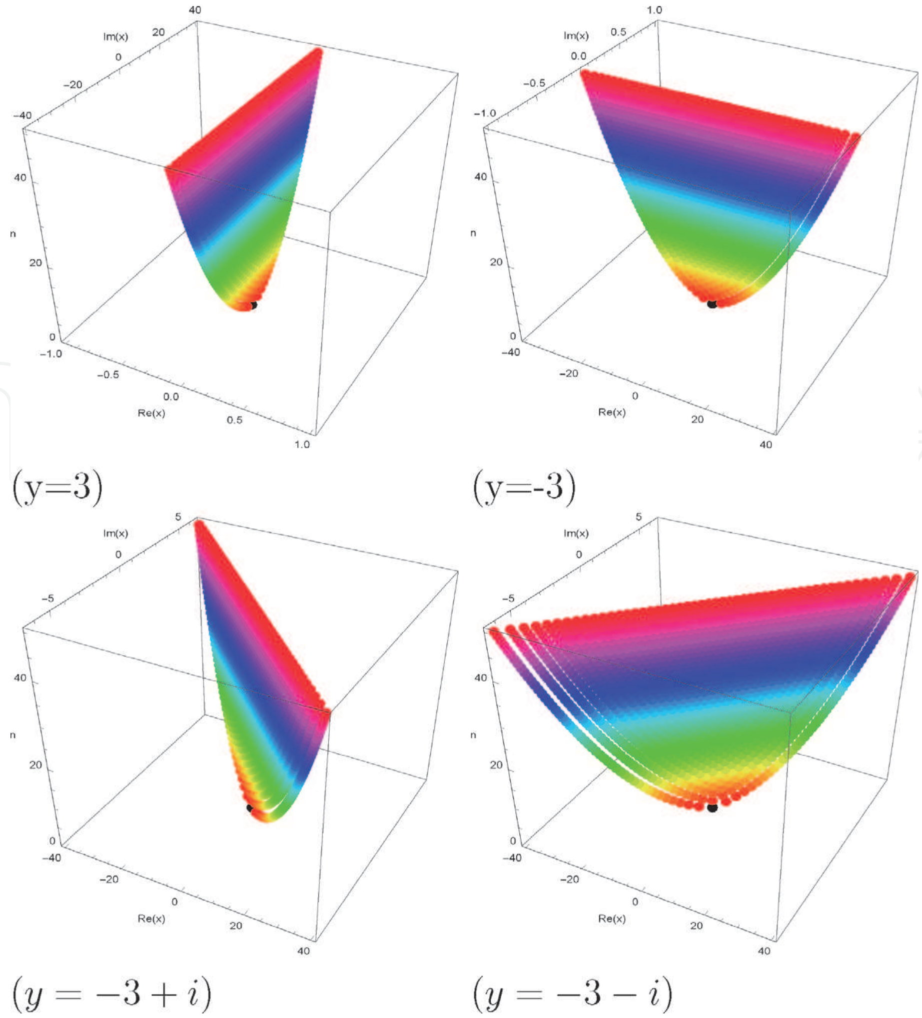

Figure 3.
Stacks of zeros of $\mathrm{H}_n(x, y|\mu) = 0$, $1 \leq n \leq 50$.

	$y = 3, \mu = 1/2$		$y = -3, \mu = 1/2$	
Degree n	**Real zeros**	**Complex zeros**	**Real zeros**	**Complex zeros**
1	1	0	1	0
2	0	2	2	0
3	1	2	3	0
4	0	4	4	0
5	1	4	5	0
6	0	6	6	0
7	1	6	7	0
8	0	8	8	0
9	1	8	9	0
10	0	10	10	0

Table 1.
Numbers of real and complex zeros of $\mathrm{H}_n(x, y|\mu) = 0$.

Degree n	x
1	0
2	$-3.0206, \quad 3.0206$
3	$-5.2318, \quad 0, \quad 5.2318$
4	$-7.0513, \quad -2.2412, \quad 2.2412, \quad 7.0513$
5	$-8.6297, \quad -4.0948, \quad 0, \quad 4.0948, \quad 8.6297$
6	$-10.041, \quad -5.7064, \quad -1.8628, \quad 1.8628, \quad 5.7064, \quad 10.041$
7	$-11.329, \quad -7.1490, \quad -3.4870, \quad 0, \quad 3.4870, \quad 7.1490, \quad 11.329$
8	$-12.519, \quad -8.4652, \quad -4.9433, \quad -1.6283, \quad 1.6283, \quad 4.9433, \quad 8.4652, \quad 12.519$

Table 2.
Approximate solutions of $\mathrm{H}_n(x, y|\mu) = 0, x \in \mathbb{R}$.

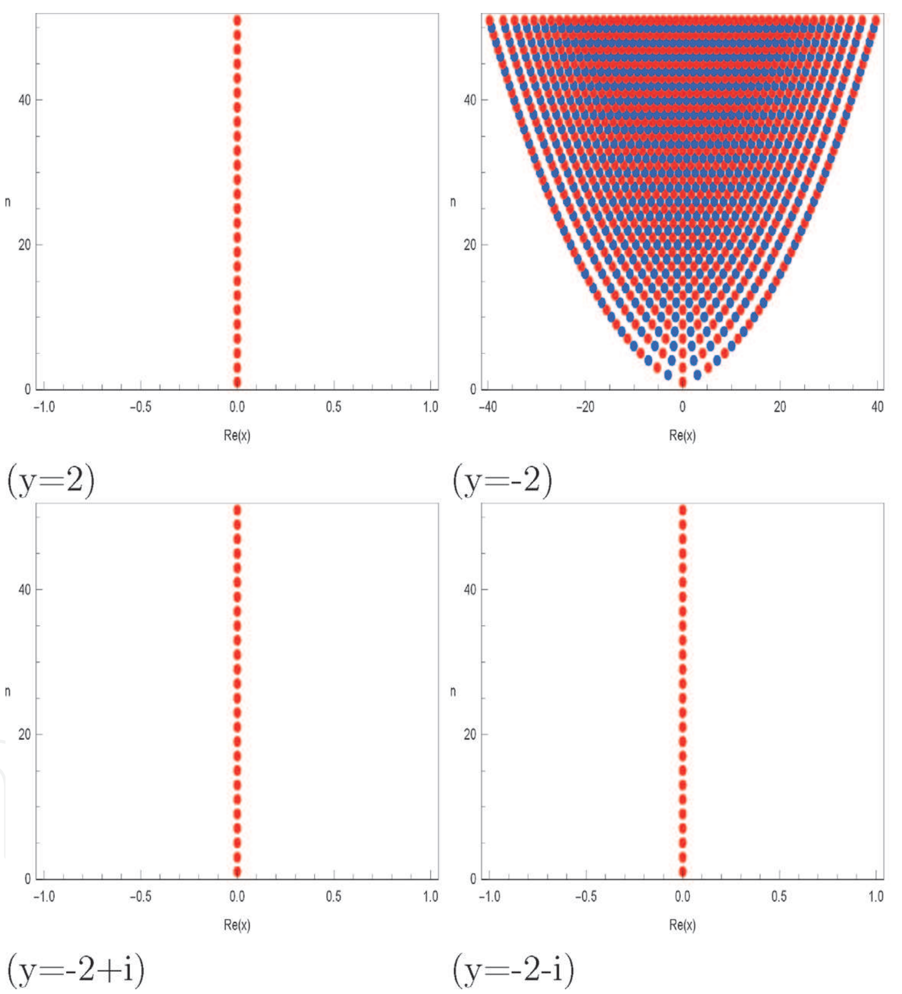

Figure 4.
Real zeros of $\mathrm{H}_n(x, y|\mu) = 0$ for $1 \le n \le 50$, $\mu = \frac{1}{2}$.

choose $y = -3$. In the bottom-left picture of **Figure 4**, we choose $y = -3 + i$. In the bottom-right picture of **Figure 4**, we choose $y = -3 - i$.

Next, we calculated an approximate solution satisfying $\mathrm{H}_n(x, y|\mu) = 0, x \in \mathbb{C}$. The results are given in **Table 2**. In **Table 2**, we choose $y = -3$ and $\mu = 1/2$.

6. Conclusions

In this chapter, we constructed the 2-variable modified degenerate Hermite polynomials and got some new symmetric identities for 2-variable modified degenerate Hermite polynomials. We constructed differential equations arising from the generating function of the 2-variable modified degenerate Hermite polynomials $\mathrm{H}_n(x, y|\mu)$. We also investigated the symmetry of the zeros of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ for various variables x and y. As a result, we found that the distribution of the zeros of 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y|\mu) = 0$ is very regular pattern. So, we make the following series of conjectures with numerical experiments:

Let us use the following notations. $R_{\mathrm{H}_n(x,y|\mu)}$ denotes the number of real zeros of $\mathrm{H}_n(x, y|\mu) = 0$ lying on the real plane $Im(x) = 0$ and $C_{\mathrm{H}_n(x,y|\mu)}$ denotes the number of complex zeros of $\mathrm{H}_n(x, y|\mu) = 0$. Since n is the degree of the polynomial $\mathrm{H}_n(x, y|\mu)$, we have $R_{\mathrm{H}_n(x,y|\mu)} = n - C_{\mathrm{H}_n(x,y|\mu)}$.

We can see a good regular pattern of the complex roots of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y, \mu) = 0$ for y and μ. Therefore, the following conjecture is possible.

Conjecture 1. Let n be odd positive integer. For $a > 0$ or $a \in \mathbb{C} \backslash \{a \mid a < 0\}$, prove or disprove that

$$R_{\mathrm{H}_n(x,a,\mu)} = 1, \quad C_{\mathrm{H}_n(x,a,\mu)} = 2\left[\frac{n}{2}\right], \tag{64}$$

where $\mathbb{C}$ is the set of complex numbers.

Conjecture 2. Let n be odd positive integer and $a \in \mathbb{C}$. Prove or disprove that

$$\mathrm{H}_n(0, a, \mu) = 0. \tag{65}$$

As a result of investigating more y and μ variables, it is still unknown whether the conjecture 1 and conjecture 2 is true or false for all variables y and μ.

We observe that solutions of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y, \mu) = 0$ has not $Re(x) = b$ reflection symmetry for $b \in \mathbb{R}$. It is expected that solutions of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y, \mu) = 0$, has not $Re(x) = b$ reflection symmetry (see **Figures 2–4**).

Conjecture 3. Prove that the zeros of $\mathrm{H}_n(x, a, \mu) = 0, a \in \mathbb{R}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. Prove that the zeros of $\mathrm{H}_n(x, a, \mu) = 0, a < 0, a \in \mathbb{C} \backslash \mathbb{R}$, has not $Im(x) = 0$ reflection symmetry analytic complex functions.

Finally, we consider the more general problems. How many zeros does $\mathrm{H}_n(x, y, \mu)$ have? We are not able to decide if $\mathrm{H}_n(x, y, \mu) = 0$ has n distinct solutions. We would like to know the number of complex zeros $C_{\mathrm{H}_n(x,y,\mu)}$ of $\mathrm{H}_n(x, y, \mu) = 0$.

Conjecture 4. For $a \in \mathbb{C}$, prove or disprove that $\mathrm{H}_n(x, a, \mu) = 0$ has n distinct solutions.

As a result of investigating more n variables, it is still unknown whether the conjecture is true or false for all variables n (see **Tables 1** and **2**).

We expect that research in these directions will make a new approach using the numerical method related to the research of the 2-variable modified degenerate Hermite equations $\mathrm{H}_n(x, y, \mu) = 0$ which appear in applied mathematics and mathematical physics.

Author details

Cheon Seoung Ryoo
Department of Mathematics, Hannam University, Daejeon, Republic of Korea

*Address all correspondence to: ryoocs@hnu.kr

References

[1] Andrews LC. Special Functions for Engineers and Mathematicians. New York, NY, USA: Macmillan. Co.; 1985

[2] Appell P, Hermitt Kampéde Fériet J. Fonctions Hypergéométriques et Hypersphériques: Polynomes d Hermite. Paris, France: Gauthier-Villars; 1926

[3] Erdelyi A, Magnus W, Oberhettinger F, Tricomi FG. Higher Transcendental Functions. Vol. 3. New York, NY, USA: Krieger; 1981

[4] Andrews GE, Askey R, Roy R. Special Functions. Cambridge, England: Cambridge University Press; 1999

[5] Arfken G. Mathematical Methods for Physicists. 3rd ed. Orlando, FL: Academic Press; 1985

[6] Carlitz L. Degenerate Stirling, Bernoulli and Eulerian numbers. Utilitas Mathematica. 1979;**15**:51-88

[7] Young PT. Degenerate Bernoulli polynomials, generalized factorial sums, and their applications. Journal of Number Theory. 2008;**128**:738-758

[8] Ryoo CS. A numerical investigation on the structure of the zeros of the degenerate Euler-tangent mixed-type polynomials. Journal of Nonlinear Sciences and Applications. 2017;**10**: 4474-4484

[9] Ryoo CS. Notes on degenerate tangent polynomials. The Global Journal of Pure and Applied Mathematics. 2015; **11**:3631-3637

[10] Hwang KW, Ryoo CS. Differential equations associated with two variable degenerate Hermite polynomials. Mathematics. 2020;**8**:1-18. DOI: 10.3390/math8020228

[11] Hwang KW, Ryoo CS, Jung NS. Differential equations arising from the generating function of the (r,β)-Bell polynomials and distribution of zeros of equations. Mathematics. 2019;**7**:1-11. DOI: 10.3390/math7080736

[12] Ryoo CS. Differential equations associated with tangent numbers. Journal of Applied Mathematics and Informatics. 2016;**34**:487-494

[13] Ryoo CS. Some identities involving Hermitt Kampé de Fériet polynomials arising from differential equations and location of their zeros. Mathematics. 2019;**7**:1-11. DOI: 10.3390/math7010023

[14] Ryoo CS, Agarwal RP, Kang JY. Differential equations associated with Bell-Carlitz polynomials and their zeros. Neural, Parallel & Scientific Computations. 2016;**24**:453-462