HARMONIC ANALYSIS USING METHODS OF NONSTANDARD ANALYSIS

by

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A thesis submitted to
The University of Birmingham
for the degree of
DOCTOR OF PHILOSOPHY

School of Mathematics
College of Engineering and Physical Science
The University of Birmingham
December 2014

Abstract

Throughout this research we use techniques of nonstandard analysis to derive and interpret results in classical harmonic analysis particularly in topological (metric) groups and theory of Fourier series.

We define monotonically definable subset $\mathscr{N}$ of a nonstandard *finite group F and prove some 'nice' properties of $\mathscr{N}$. Also we prove that $\mathscr{N}$ is monotonically definable if and only if $\mathscr{N}$ is the monad of the neutral element of F for some invariant *metric d on F. In addition, we show the nonstandard metrisation version of first-countable Hausdorff topological groups.

We introduce a variant of the notion of 'locally embeddable in finite groups' (LEF) for metric groups, as 'locally embeddable in finite metric groups' (LEFM). We show that every abelian group with an invariant metric is LEFM. We give a number of examples of classical metric groups represented by nonstandard *finite *metric groups using methods of nonstandard analysis. Generalising, we also define 'locally embeddable into (uniformly) discrete metric groups' (LEDM) and prove that every 2-sided invariant metric group is LEDM.

We present a nonstandard version of the main results of the classical space $L^1(\mathbb{T})$ (the space of Lebesgue integrable complex-valued functions defined on the topological circle group $\mathbb{T}$) such as: Fourier coefficients of piecewise continuous functions; some useful properties of Dirichlet and Fejér functions; the convolution properties of functions in $L^1(F)$; Also the relationship between the classical $L^1(\mathbb{T})$ and the nonstandard $L^1(F)$ via Loeb measure.

Furthermore, we introduce the proof of: the approximation of Lebesgue integrable functions by S-continuous functions in $L^1(F)$; the density of trigonometric polynomials with standard degree in $L^1(F)$; the 1-norm and pointwise convergence of the nth Cesàro mean; and if $f \in {}^*\mathbb{C}^F$ with

$\widehat{f}(m)=0$ for all $m \in F$, then $f=0$; and if $f \in L^1(F)$, then $\lim\limits_{|n| \in \mathbb{N}} \widehat{f}(n)=0$. In addition, we model functionals defined on the test space of exponential polynomial functions on $\mathbb{T}$ by functionals in NSA, using internal functions defined on nonstandard *finite sets.

We introduce the proof of the representation of continuous functions on the classical metric group $(G,+,d_G)$ by S-continuous functions on $(F,\cdot,d_F)$ whenever G is represented by $F/\mathscr{N}$ as a metric group, F_{fin} satisfies well-definedness conditions and $F_{\text{fin}}/\mathscr{N} \subseteq F/\mathscr{N}$ is an open set.

Acknowledgements

All praise and thanks be to God, my creator and the creator of all, for granting me the success to complete my thesis.

I would like to express my deep thanks, appreciation and sincere gratitude to my supervisor Dr. Richard Kaye for choosing this interesting topic, his scientific advice, patient guidance, insightful comments and his continuous support of my PhD research. His useful guidance helped me in all parts of my research and in the writing of this thesis.

I would like to express my special thanks to my co-supervisor Professor Chris Parker for his encouragement, support and guidance.

I am very grateful to the Iraqi Ministry of Higher Education and Scientific Research/ Scholarship and Cultural Relations Directorate for their financial support granted through a scholarship I received. I also thank Kurdistan Regional Government for their support.

I would like to thank the staff members of the School of Mathematics at the University of Birmingham who helped me.

I express my sincerely warm thanks to my parents, brothers and sisters for allowing me to be ambitious as I wanted.

My special thanks and appreciation to my beloved wife for all of the sacrifices that she has made on my behalf, her support, encouragement and help.

Finally, I thank my children for their patience and bearing during my absence from them due to my working on this project.

Contents

Chapter 1

Introduction to Material and Literature Review

Throughout this chapter we introduce a general introduction to materials as the first section and in the second section we attempt to present the basic definitions which we will use them implicitly in the following chapters.

1.1 General Introduction

Harmonic analysis is a diverse field concerned with the study of the notions of Fourier series and Fourier transforms with their generalisation, as well as the study of topological groups. It has many applications ranging across different areas of science, it has extensively used by the fields: signal processing; medical imaging; and quantum mechanics. Particularly, Fourier transform has applications in different fields of science. It is used in physics, astronomy, optics, communications, applied mathematics, engineering, geology, chemistry, etc. Therefore, the subject of Fourier transformation has an important role in major aspects of life [21].

Nonstandard analysis (the theory of infinitesimals) was introduced by Abraham Robinson in 1961 [6, 29], and provides a logical foundation to the idea of infinitesimals. This new theory and new number system obtained via model -theoretic methods was a satisfactory and complete solution of Leibniz's old problem (to give a rigorous meaning to infinitesimals and vanishing quantities). Newton and Leibniz originally formulated the calculus using the expressions 'vanishing quantity' and 'infinitesimal number' in their development of infinitesimal calculus, but were unable to make the idea of 'infinitesimal' precise in any consistent way.

In 1977 Nelson [24] axiomatised the Robinson's nonstandard analysis throughout a theory called *internal set theory*. This new approach to nonstandard analysis is based on the axioms of ZFC (Zermelo-Fraenkel set theory and the axiom of choice [17]) together with three other axioms which are: transfer principle (T); idealisation principle (I); standardisation principle (S).

Nonstandard analysis is a very reasonable way to study convergence, since NSA provides a uniform and simple approach to $\varepsilon - \delta$ analysis without excessive machinery associated with quantifiers. Our study of the convergence in Fourier series is an example in this aspect. Nonstandard analysis is better than the classical analysis for making examples and giving a uniform treatment of a number of different things. Because examples in NSA can be constructed by discrete means and shown to approximate continuous functions, for example. Very often nonstandard analysis simplifies the definitions and proofs.

Nonstandard analysis has been developed widely and considerably in various areas with great success. It has been applied in different subjects and directions such as: differential and integral calculus, differential equations, classical differential geometry, general topology, probability theory, theory of distributions, topological groups and Lie groups, boundary Layer theory, microeconomic theory and mathematical physics [5, 13, 14].

In 1970 Luxemburg [22] tried to link the harmonic analysis of *finite abelian groups and the abelian circle group $\mathbb{T}$. He studied the sequence of Fourier coefficients of a continuous function f on $L^p(\mathbb{T})$, for $p > 1$, as the standard part of the sequence of Fourier coefficients of the function $^*f$. Luxemburg used nonstandard analysis to find the discrete versions of Parseval's formula and Hausdorff-Young inequalities in Fourier analysis. He proved some theorems about representations of positive definite functions by using nonstandard analysis.

In 1991 Gordon [9, 10] considered an internal hyperfinite abelian group G and an internal *finite subgroup H of G. He showed how: the Haar measure can be approximated by the Loeb measure on H; a character of G can be approximated by an internal character of H; and the Fourier transform on G can be approximated by the *finite Fourier transform on H. Moreover, he introduced a new method based on nonstandard analysis for the construction of separable locally compact abelian (LCA) groups. He considered a *finite internal abelian group G and two subgroups H and K of infinitesimals and finite numbers of G respectively. Gordon also defined a topology on the quotient group K/H. The Loeb measure on G induces the Haar measure on K/H, if K/H is a separable locally abelian group. He investigated the approximation of Fourier transforms in LCA group by discrete Fourier transforms.

In 1997 Vershik and Gordon introduced the notion of locally embeddable into finite groups (LEF) as abstract groups. In 2012 this idea is also studied by others such as Pestov and Kwaitkowska [26], and corresponds to a group being embeddable in a nonstandard *finite group.

The aim of this research is to use methods of nonstandard analysis to study harmonic analysis in two aspects: the representation of classical topological (metric) groups by nonstandard metric groups; and the convergence of Fourier series in nonstandard universe. Nonstandard analysis has been applied to topological groups (for example, by Parikh [25], Gordon [9, 10]) but we know of no general techniques that allow topological groups to be studied by nonstandard methods applied to *finite groups. This possibility seemed particularly intriguing for abstract Fourier analysis, since Fourier series on the circle group and some other specific groups can be readily treated by nonstandard means [19].

In this work we shall set up our nonstandard mathematics (Chapter 2 and Chapter 4) that we need to provide an analogous nonstandard study of metric groups and nonstandard study of the convergence of Fourier series. For the classical theory, a number of texts could be followed. We shall follow Katznelson [16].

Chapter 2 introduces important nonstandard tools as a background, which form the core of nonstandard subjects used through next chapters. Section 2.1 explains the construction of an ultrapower of a first order structure M. The case when $M = (\mathbb{R}, \ldots)$ gives the construction of hyperreals ${}^*\mathbb{R}$. Section 2.2 defines first-order logic and first-order language and states Łoś's Theorem and the Transfer principle. Section 2.3 explains the construction of standard and nonstandard universe, defines internal sets with other important definitions. The definition of overspill and some interesting propositions are also given.

Sections 2.4 and 2.5 develop the nonstandard theory of L^1 functions on the circle $\mathbb{T}$. The elements of $L^1(\mathbb{T})$ are equivalence classes of complex-valued Lebesgue integrable functions such that $f \sim g$ if and only if $f = g$ almost everywhere on $\mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a subgroup of multiplicative abelian group $\mathbb{C} \setminus \{0\}$. Also we can identify $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$, then $\mathbb{T} = [0, 2\pi)$ or $\mathbb{T} = [-\pi, \pi)$ is an additive abelian group modulo 2π. Moreover, $\mathbb{T}$ has a topology as a subspace of the complex topological space.

We follow Cartier and Perrin [2] to progress our work and try to obtain a nonstandard version of L^1 by considering a *finite set F as a model of the circle $\mathbb{T}$. We define the distance function d on F by $d(r,s) = \frac{2\pi}{N}\min\{|r-s|, N-|r-s|\}$, then (F,d) is a precompact metric space. Moreover, these sections introduce the definitions of S-integrable, S-continuous, limitedness and L-integrable of functions on F in order to define $L^1(F)$, since the concept of integration plays a very important role in the theory of Fourier transform and Fourier series.

Sections 2.6 and 2.7 introduce the Loeb measure [20] of internal subsets of F in order to study the measurable functions in $L^1(F)$, and then to show the relation between the classical $L^1[-\pi, \pi]$ and the nonstandard $L^1(F)$. Finally, Section 2.8 presents some interesting applications of saturation in the nonstandard universe.

Chapter 3 introduces the basic material of the main branches of the classical theory of harmonic analysis. Section 3.1 focuses on the vector space V_G of complex-valued functions defined on a finite group G. The discrete Fourier transform (DFT) and its inverse (IDFT) are given via definition of the inner product $\langle \cdot, \cdot \rangle$ as defined on V_G. The convolution and its properties on V_G are given as well. Section 3.2 works on topological groups G, where G is abelian as a group and Hausdorff locally compact as a topology. The definition of Haar measure and L^1 on G are given in order to define Fourier coefficients and Fourier series. A number of interesting examples are given through the section. Section 3.3 restricts the definition of Fourier coefficients, Fourier series, partial sums of Fourier series and Cesàro mean on the L^1 space of the circle topological group $\mathbb{T}$.

Chapter 4 aims to represent classical metric groups by nonstandard metric groups through the notion of 'locally embeddable in finite metric groups' (LEFM) by a variant of the notion of ‘locally embeddable in finite groups’ (LEF) using methods of NSA which we develop through this chapter. When a group G has a metric structure, a number of interesting variations of this idea are possible, and one of the aims of this chapter is to explain these and give some interesting results. We work with standard metric groups G where the metric d is 2-sided, or invariant on both sides (Definition 1.2.9).

Section 4.1 introduces the definition of the standard part map of a *finite group F in the non-standard universe, the normal subgroup $\mathscr{N}$ of F and the definition of monotonically definable subset of F and its relation with $\mathscr{N}_d$ (the monad of the neutral element of F). Some nice properties of the monotonically definable subset are given. Theorem 4.1.9 shows the relationship between $\mathscr{N}$ and $\mathscr{N}_d$, that is, how to define a *metric d on F satisfied $\mathscr{N} = \mathscr{N}_d$. The definition of $^\circ d$ on $F/\mathscr{N}$ is given via the definition of d on F, in order to study some properties of the topological (metric) group $F/\mathscr{N}$. Section 4.2 presents the definition of counting measure on subsets of the nonstandard finite group F. The measure of subsets of the quotient group $F/\mathscr{N}$ is also defined. The section is ended by some interesting results related to the measure of the normal subgroup $\mathscr{N}$ when $\mathscr{N}$ is an external subset of F.

Section 4.3 starts with the definition of locally embeddable into finite metric groups (LEFM). Such a group G is (LEFM) if it is embeddable in $F/\mathscr{N}$ for some 2-sided *finite *metric group F, where $\mathscr{N}$ is the monad of the identity of F (Definition 4.3.1). In a sense this is not very far away from the idea of a sofic group as given by Pestov and Kwiatkowska [26].

A characterisation of such groups in terms of local embeddings appears in Theorem 4.3.2, which shows that how classical metric groups can be represented by nonstandard metric groups. We define a partial norm on the abelian group G, and then we can extend this norm on any subset A of G into an invariant metric d on the subgroup generated by A. As well as introducing the idea of LEFM groups, which we believe is very natural, and proving that every abelian group with invariant metric is LEFM (Theorem 4.3.7).

It is our belief that consideration of nonstandard finite groups can be used to streamline and simplify certain results from abstract harmonic analysis and locally compact groups in general. For example, Theorem 6.3.3 in Chapter 6 and Example 7.0.19 in Chapter 7.

We do not expect that all 2-sided metric groups will be LEFM. Since every Hausdorff first-countable topological group is metrisable with 2-sided metric, by a standard argument, then Theorem 4.1.12 shows a nonstandard version for this fact.

When an invariant metric group is not, or is not known to be, LEFM, a modified weaker notion may be helpful. Generalising the LEFM idea, we can also introduce the notion of groups locally embeddable into (uniformly) discrete metric groups (LEDM). This also has a natural nonstandard interpretation as groups embeddable in $F/\mathscr{N}$ where F is a discrete metric group in the nonstandard universe, and $\mathscr{N}$ is again the monad of the neutral element of F. This also gives some hope of studying metric groups as (subgroups of quotients of) discrete groups and Theorem 4.4.5 shows that all 2-sided metric groups are LEDM.

Chapter 5 introduces explicit descriptions of some familiar abelian and nonabelian groups as LEFM groups. Besides the obvious examples that arise directly from metric ultraproducts, it is of interest to show that other classical examples are LEFM groups. Section 5.1 gives interesting different examples of classical abelian metric groups as LEFM groups. What makes these examples even more interesting is that all these familiar abelian groups can be represented as a subgroup of the quotient $F/\mathscr{N}$ where F is a *cyclic group C_N of nonstandard order N. In particular, we wonder if all abelian groups with invariant metric occur as subgroups of such $C_N/\mathscr{N}$. Section 5.2 gives remarkable examples of classical nonabelian metric groups which are LEFM groups.

Chapter 6 presents a nonstandard approach to Fourier analysis on the topological circle group $\mathbb{T}$ by using methods of nonstandard analysis together with discrete Fourier transform. This chapter starts with the vector spaces of "smooth" functions over the field of complex numbers $\mathbb{C}$. The inner product of two functions is used to define the Fourier coefficients in $L^1(\mathbb{T})$ and DFT in $L^1(F)$. The Fourier coefficients of piecewise continuous functions in $L^1(\mathbb{T})$ can be written as the discrete Fourier transform in $L^1(F)$, whenever F is a *finite set, by using methods of nonstandard analysis.

The Dirichlet and Fejér functions are presented and used to discuss the convergence of Fourier series. Several important theorems on the Dirichlet and Fejér properties are introduced to facilitate the representation of functions in $L^1(F)$ as series. The convolution of internal functions in ${}^*\mathbb{C}^F$ is discussed. The relationship between the classical $L^1[-\pi,\pi]$ and the nonstandard $L^1(F)$ is explained. Also, this chapter presents the 1-norm and pointwise convergence of the nth Cesàro mean of functions in $L^1(F)$.

Chapter 7 introduces functionals in NSA as an analogy of functionals in classical analysis on the test space of the exponential polynomial functions in $\mathbb{C}^{\mathbb{T}}$. In our attempt to generalise functions, we notice that a Fourier series may not converge, but still gives useful information such as a "generalised function".

Chapter 8 aims to show how the nonstandard methods of Chapter 4 together with discrete harmonic analysis (Section 3.1) may be used to derive and interpret results in classical harmonic analysis (Sections 3.2 and 3.3). The relationship between functions ψ defined on F and functions ${}^\circ\psi$ defined on G are shown in Theorem 8.0.23 whenever $(G,+,d_G)$ embeds in $(F/\mathscr{N},+,d_F)$ as a metric group and $F_{\text{fin}}/\mathscr{N}$ is an open subset of $F/\mathscr{N}$ together with some further conditions.

1.2 Background of Basic Definitions

Throughout this section we introduce the basic definitions which are used in this research.

Definition 1.2.1. [23] Consider the set of all complex numbers $\mathbb{C}$. The set of *non zero complex numbers* is denoted by $\mathbb{C}^\times$. The (*multiplicative*) *circle subgroup* of $\mathbb{C}^\times$ is denoted by $\mathbb{T}_\mathbb{C}$ and defined by $\mathbb{T}_\mathbb{C} = \{z \in \mathbb{C}^\times : |z| = 1\}$. The metric function defined on $\mathbb{T}_\mathbb{C}$ is the metric induced from the usual metric on $\mathbb{C}$.

It will be convenient to have a notation for a group isomorphic to $\mathbb{T}_\mathbb{C}$ but written additively.

Definition 1.2.2. [16] The *additive circle group* (*modulo* 2π) is denoted by $\mathbb{T}$. This is the group $\mathbb{R}/2\pi\mathbb{Z}$ or equivalently $[0,2\pi)$ with addition modulo 2π.

Define the normalised metric $d_\mathbb{T}$ on $\mathbb{T} = [0,2\pi)$ by

$$d_\mathbb{T}(x,y) = \frac{1}{2\pi}\min\{|x-y|, 2\pi - |x-y|\}.$$

A group can act on a specific set and the action will be defined as follows.

Definition 1.2.3. [11] Let G be a group. A *group action* of G on a set X is a function from $G \times X$ to X (where the image of $(g,x) \in G \times X$ is written $g \cdot x$) satisfying:

(a) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g,h \in G$ and $x \in X$;

(b) $e \cdot x = x$ for all $x \in X$, where e is the identity element of G.

Following are the definitions of three different dihedral groups with suitable metrics defined on them.

Definition 1.2.4. [30] Let C_N denotes the cyclic group of order N. The *dihedral group* $D_{2N} = \{g^j h^k : 0 \leqslant j < N, 0 \leqslant k < 2, g^N = 1 = h^2, gh = hg^{-1}\}$ of order $2N$ is defined to be the semidirect product of the cyclic groups C_N and C_2. Also is written as $D_{2N} = C_N \rtimes C_2$. We define an action of C_2 on C_N by $g^h = h^{-1}gh \in C_N$ such that $g^1 = g$, $(g^h)^k = g^{(hk)}$ and $(g^{-1})^h = (g^h)^{-1}$ for all $g \in C_N$ and $h,k \in C_2$. The multiplication on the set $D_{2N} = \{(g,h) \in C_N \rtimes C_2 : g \in C_N, h \in C_2\}$ is defined by $(g,h)(g',h') = (g(g')^{h^{-1}}, hh')$.

The metric d defined on D_{2N} by

$$d((g,h),(g',h')) = d_N(g,g') + d_2(h,h')$$

where d_N is the usual metric on C_N defined by $d_N(g,g') = \frac{1}{N}\min\{|g-g'|, N-|g-g'|\}$ and d_2 is the discrete metric on C_2.

Definition 1.2.5. We define the *dihedral circle group* as the semidirect product of the circle group $\mathbb{T}$ and the cyclic group C_2. We denoted it by $D_{2\mathbb{T}}$, that is, $D_{2\mathbb{T}} := \mathbb{T} \rtimes C_2$. The non identity element of C_2 acts on $\mathbb{T}$ by inverting elements. The multiplication on $D_{2\mathbb{T}}$ is defined by $(g,h)(g',h') = (g(g')^{h^{-1}}, hh')$. The metric d defined on $D_{2\mathbb{T}} = \{(g,h) \in \mathbb{T} \rtimes C_2 : g \in \mathbb{T}, h \in C_2\}$ by

$$d((g,h),(g',h')) = d_{\mathbb{T}}(g,g') + d_2(h,h')$$

where $d_{\mathbb{T}}$ and d_2 are metrics on the circle group $\mathbb{T}$ and the cyclic group C_2 respectively.

Definition 1.2.6. [30] The *infinite dihedral group* D_∞ is defined to be the semidirect product of the infinite cyclic group of integers $\mathbb{Z}$ by the cyclic group C_2. That is $D_\infty \cong \mathbb{Z} \rtimes C_2$. The group C_N acts on $\mathbb{Z}$ by inverting the elements. The set D_∞ is the Cartesian product $\mathbb{Z} \times C_2$. The multiplication on D_∞ is defined by $(g,h)(g',h') = (g(g')^{h^{-1}}, hh')$. The metric d defined on D_∞ is the discrete metric $d(x,y) = 1$ if $x \neq y$.

The groups p-adic integers and profinite completion of $\mathbb{Z}$ with well-defined metric functions are two interesting metric groups constructed via the inverse limit of quotients $\mathbb{Z}/(p^n\mathbb{Z})$ and $\mathbb{Z}/(n\mathbb{Z})$ respectively as given in the following two definitions.

Definition 1.2.7. [31] Let p be a prime number. The *inverse limit* of cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$ with reduction modulo p^n by the natural maps $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$, where $n \geqslant m$ is called the group of *p-adic integers* and denoted by $\mathbb{Z}_p$. That is,

$$\mathbb{Z}_p := \lim_{\leftarrow}(\mathbb{Z}/p^n\mathbb{Z}).$$

Alternatively, we can define $\mathbb{Z}_p$ as the set of sequences (with componentwise addition) as

$$\mathbb{Z}_p := \{(x_n) \in \prod_{n\in\mathbb{N}}(\mathbb{Z}/p^n\mathbb{Z}) : \text{ for all } n \in \mathbb{N},\ x_{n+1} \equiv x_n \bmod p^n\}.$$

Also define the metric d_p on $\mathbb{Z}_p$ by

$$d_p(x,y) = \begin{cases} 2^{-k} & \text{if } k \text{ is the least such that } x_k \neq y_k \text{ and } k \in \mathbb{N}, \\ 0 & \text{if } x_k = y_k \text{ for all } k \in \mathbb{N}. \end{cases}$$

Definition 1.2.8. [31] The *inverse limit* of cyclic groups $\mathbb{Z}/n\mathbb{Z}$ with reduction modulo n by the natural maps $\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is called the *profinite completion of* $\mathbb{Z}$ and denoted by $\widehat{\mathbb{Z}}$. That is,

$$\widehat{\mathbb{Z}} := \lim_{\leftarrow}(\mathbb{Z}/n\mathbb{Z}).$$

Alternatively we can define $\widehat{\mathbb{Z}}$ as the set of sequences (with componentwise addition) as follows.

$$\widehat{\mathbb{Z}} := \{(x_n) \in \prod_{n\geqslant 1}(\mathbb{Z}/n\mathbb{Z}) : \text{ for all } n|m,\ x_m \equiv x_n \bmod n\}.$$

That is, $(x_0, x_1, x_2, \ldots)$ is an element of $\widehat{\mathbb{Z}}$ if and only if for all functions $\phi_{mn} \colon \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$

such that $\phi_{mn}(x_{mn}) = x_n$ for all $m,n \in \mathbb{N}$. Also the metric $\widehat{d}$ is defined on $\widehat{\mathbb{Z}}$ by

$$\widehat{d}(x,y) = \begin{cases} 2^{-k} & \text{if } k \text{ is the least such that } x_k \neq y_k \text{ and } k \in \mathbb{N}, \\ 0 & \text{if } x_k = y_k \text{ for all } k \in \mathbb{N}, \end{cases}$$

where $x = (x_0, x_1, x_2, \ldots)$ and $y = (y_0, y_1, y_2, \ldots)$.

The invariance of metrics defined on groups is given as follows.

Definition 1.2.9. [26] Let G be a group and d be a metric defined on G. Then d is said to be *left invariant* if $d(xy,xz) = d(y,z)$ for all $x,y,z \in G$, *right invariant* if $d(xz,yz) = d(x,y)$ for all $x,y,z \in G$, and *invariant* if d is both left and right invariant.

The direct sum of any two groups is given as follows.

Definition 1.2.10. [30] The *direct sum* of two groups G_1 and G_2 is denoted by $G_1 \oplus G_2$. Its elements are the elements of the Cartesian product $G_1 \times G_2$ and the group operation is defined componentwise.

For finitely generated abelian groups we recall the following theorem.

Theorem 1.2.11. Let A be a finitely generated abelian group. Then there are $n,m \in \mathbb{N}$ and $a_i \in \mathbb{N}$ for $i = 0,1,2,\ldots,m-1$, such that

$$A \cong \mathbb{Z}^n \oplus \bigoplus_{i=0}^{m-1} C_{a_i}.$$

Proof. For a proof see Rotman [30, Chapter 9]. □

A precompact metric space is defined as follows.

Definition 1.2.12. [2] A metric space (X,d) is called *precompact*, if for all real $\varepsilon > 0$ there is a finite family of open subsets of diameter ε whose union coincides with X. That is, for a given real $\varepsilon > 0$, there is a finite family $A_1, A_2, \ldots, A_n$ of open subsets of X such that $X \subseteq \bigcup_{i=1}^{n} A_i$ and $\mathrm{dia}(A_i) < \varepsilon$ for all $i = 1,2,\ldots,n$.

The Hausdorff property of topological spaces is given as follows.

Definition 1.2.13. [23] A topological space T is said to be *Hausdorff* or T_2 if for every two distinct points x and y in T, there are two disjoint open sets U_x and U_y in T containing x and y respectively.

Locally compact of topological groups has an essential role in the structure of topological groups in harmonic analysis. Its definition given as follows.

Definition 1.2.14. [23] A topological space T is said to be *locally compact* if every point of T has a compact neighbourhood, that is a neighbourhood contained in a compact set.

In order to define Borel measurable function, we start with regular Borel measure as follows.

Definition 1.2.15. [16] Let G be a locally compact Hausdorff space. The *σ-algebra of Borel sets* $\mathscr{B}(G)$ is the smallest σ-algebra on G containing open sets of G. A measure μ on $\mathscr{B}(G)$ is called a *Borel measure*. A Borel measure μ is *inner regular* on $A \in \mathscr{B}(G)$ if $\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ compact}\}$, *outer regular* on A if $\mu(A) = \inf\{\mu(U) : U \supseteq A, U \text{ open}\}$, and μ is called *regular* if it is both inner regular and outer regular on $\mathscr{B}(G)$.

Definition 1.2.16. [32] A function $f\colon G \to \mathbb{C}$ is said to be *Borel measurable* if for all open subset $U \subseteq \mathbb{C}$, $f^{-1}(U)$ is a Borel set in G.

Let G be a multiplicative group. The definition of the topological group G will be as follows.

Definition 1.2.17. [23] A *topological group* G is a group that is also a topological space such that the multiplication map $(x,y) \mapsto xy$ of $G \times G$ into G and the inversion map $x \mapsto x^{-1}$ of G into G are continuous, where $G \times G$ carries the product topology.

The inner product on the complex vector space is defined as follows.

Definition 1.2.18. [32] A complex vector space V is called an *inner product space* if for each $(x,y) \in V \times V$ there is associated complex number $\langle x,y \rangle$ the so-called *inner product of x and y* such that:

(a) $\langle x,y \rangle = \overline{\langle y,x \rangle}$ for all $x,y \in V$ (the bar denotes the complex conjugation),

(b) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$ for all $x,y,z \in V$,

(c) $\langle x,\lambda y \rangle = \lambda \langle x,y \rangle$ for all $x,y \in V$ and all scalar $\lambda \in \mathbb{C}$,

(d) $\langle x,x \rangle \geqslant 0$ for all $x \in V$,

(e) $\langle x,x \rangle = 0$ if and only if $x = 0$ for $x \in V$.

Definition 1.2.19. [8] A *subdivision* of $[-\pi,\pi)$ is a finite subset $\{x_0,x_1,\ldots,x_n\}$ of $[-\pi,\pi]$ such that $-\pi = x_0 < x_1 < \ldots < x_n = \pi$.

Definition 1.2.20. [1] A function $f\colon \mathbb{T} \to \mathbb{C}$ is said to be *piecewise continuous*, if $\mathbb{T} = [-\pi,\pi)$ can be partitioned by a finite number of points $-\pi = x_0 < x_1 < \ldots < x_n = \pi$ such that:

(a) f is continuous on each subinterval (x_{i-1},x_i), for all $i = 1,2,\ldots,n$;

(b) f has finite limits as x approaches the end points, in the subintervals (x_{i-1},x_i).

Definition 1.2.21. [8] A function $f\colon \mathbb{T} \to \mathbb{C}$ is said to be *bounded*, if there is $K \in \mathbb{N}$ such that $|f(t)| < K$, for all $t \in \mathbb{T}$.

Chapter 2

Introduction to Nonstandard Analysis

Our aim in this chapter is to present all concepts and tools that we need in the nonstandard methods that we use in this research. Therefore, we start with ultrafilters and ultraproducts in order to construct the set of hyperfinite ${}^*M$ of an infinite set M in general. The concepts of first-order logic and first-order language can be used and interpreted in many classes of mathematical structures, particularly in our work area. Throughout construction of standard and nonstandard universe, we define some nonstandard concepts such as internal sets, standard elements and finite elements.

To understand precisely the space of Lebesgue integrable functions L^1 on nonstandard finite sets F, we explain what are the S-integrable and S-continuous functions on F. Dealing with measure on subsets of F requires to know what is Loeb measure and then the relation of measurable functions and integrals as well. Saturation is another part of this chapter in which we attempt to show its use in our work. Throughout this chapter, several interesting examples are given in order to illustrate the concepts introduced.

2.1 Ultrafilters and Ultraproducts

In order to build the set of hyper numbers ${}^*M$ of an infinite set of numbers M we will present the following definitions and theorems with some interesting examples. All theorems here are easy and proofs can be found in the literature, for example, see Keisler [18].

Definition 2.1.1. [13] Let $\mathscr{P} = \mathscr{P}(\mathbb{N}) = \{X : X \subseteq \mathbb{N}\}$. A *nontrivial filter* on $\mathscr{P}$ (or $\mathbb{N}$) is a set

$D \subseteq \mathscr{P}$, such that:

(F1) D is non-empty and $\emptyset \notin D$;

(F2) if $X \supseteq Y$, and $Y \in D$, then $X \in D$;

(F3) if $X, Y \in D$, then $X \cap Y \in D$.

Example 2.1.2. Let $n \in \mathbb{N}$, $D_n = \{X \subseteq \mathbb{N} : n \in X\}$, then D_n is obviously a filter.

Theorem 2.1.3. $D_{\text{cofin}} = \{X \subseteq \mathbb{N} : \mathbb{N} \setminus X \text{ is finite}\}$ is a filter.

Proof. For proof see Keisler [18]. □

Definition 2.1.4. [13] A filter E on $\mathbb{N}$ is called an *ultrafilter* if, for all $X \subseteq \mathbb{N}$, either $X \in E$ or $\mathbb{N} \setminus X \in E$.

Definition 2.1.5. [18] An ultrafilter D is a *nonprincipal ultrafilter* if for all $n \in \mathbb{N}$, $\{n\} \notin D$.

Theorem 2.1.6. An ultrafilter D is nonprincipal if and only if D does not contain finite sets.

Proof. For proof see Keisler [18]. □

Following are two well-known theorems about the existence of filter extensions.

Theorem 2.1.7. If D is a filter and $X \subseteq \mathbb{N}$, such that neither X, nor $\mathbb{N} \setminus X$ is in D, then there is a filter $E \supseteq D$, such that $X \in E$.

Theorem 2.1.8. If D is a filter, then there is a filter $E \supseteq D$, such that, for every $X \subseteq \mathbb{N}$, either $X \in E$, or $\mathbb{N} \setminus X \in E$, i.e., E is an ultrafilter extending D.

Definition 2.1.9. Let M be an infinite set, then $M^{\mathbb{N}} = \{(m_0, m_1, \ldots) : m_i \in M, i \in \mathbb{N}\}$.

Definition 2.1.10. Let D be a nonprincipal ultrafilter on $\mathbb{N}$. Define $\sim_D$ on $M^{\mathbb{N}}$ as follows:

$$(m_0, m_1, \ldots) \sim_D (n_0, n_1, \ldots) \Leftrightarrow \{i : m_i = n_i\} \in D.$$

Following is an easy theorem about the equivalence relation $\sim_D$ on the set $M^{\mathbb{N}}$.

Theorem 2.1.11. The relation $\sim_D$ is an equivalence relation on $M^{\mathbb{N}}$.

By Theorem 2.1.3, Theorem 2.1.6 and Theorem 2.1.8 there is a nonprincipal ultrafilter on $\mathbb{N}$. From now on, we fix a nonprincipal ultrafilter D on $\mathbb{N}$.

Definition 2.1.12. [18] The set of all equivalence classes $(m_0, m_1, \ldots)/_D$ with respect to the equivalence relation $\sim_D$ on $M^{\mathbb{N}}$ is called the *ultrapower of M modulo D* or is called the *set of hyper numbers of M* and denoted by $^*M$ or $M^{\mathbb{N}}/_D$ or $\Pi_D M$. That is, $^*M = \{(m_0, m_1, \ldots)/_D : (m_0, m_1, \ldots) \in M^{\mathbb{N}}\}$.

Definition 2.1.13. [13] For each $m \in M$, the hyper number $^*m = (m, m, m, \ldots)/_D \in {}^*M$ called a *standard number*.

Embedding of M into $^*M$ will be satisfied via the function $m \mapsto {}^*m = (m, m, m \cdots)/_D$, which is 1-1. So M can be identified with the image of this embedding, but for this chapter at least we continue to use the $^*m$ notation.

Definition 2.1.14. The set of all equivalence classes $(r_0, r_1, \ldots)/_D$ with respect to the equivalence relation $\sim_D$ on $\mathbb{R}^{\mathbb{N}}$ is called the *set of hyperreal numbers* and denoted by $^*\mathbb{R}$ or $\mathbb{R}^{\mathbb{N}}/_D$ or $\Pi_D\mathbb{R}$. That is $^*\mathbb{R} = \{(r_0, r_1, \ldots)/_D : (r_0, r_1, \ldots) \in \mathbb{R}^{\mathbb{N}}\}$.

Remark 2.1.15. These notions depend on some fixed choice of nonprincipal ultrafilter D.

Definition 2.1.16. We denote the set of hypernatural numbers, the set of hyperinteger numbers, the set of hyperrational numbers and the set of hypercomplex numbers by $^*\mathbb{N}$, $^*\mathbb{Z}$, $^*\mathbb{Q}$ and $^*\mathbb{C}$, respectively.

2.2 First-Order Logic and First-Order Language

Our work is within the framework of first-order logic, as mentioned in Kaye's book [17]. A *first-order logic* or *predicate logic* is a formal system for proofs, extending the propositional logic (logic of statements that can be true or false) or some values in Boolean algebra with mathematical objects from a domain or more such as the set of natural numbers and the set of complex numbers.

Here an important point we should know is that the first-order logic is not only a system for writing and checking proofs, but also it has a wide use in the theory of algebraic structures. That is, it has applications in a massive area of mathematical subjects such as theory of definability.

First-order logic covers predicates and quantification. It is distinguished from propositional logic by its use of quantified variables and equality.

Definition 2.2.1. [17] A *first-order language* consists of the following symbols:

(a) the logical symbols $\wedge, \vee, \neg, \top, \bot$, for propositional logic;

(b) a countably infinite set of variables $x, y, z, \ldots$;

(c) the equality and quantifier symbols '$=$','$\forall$','$\exists$';

(d) a (possibly empty) set of constant symbols such as 0,1;

(e) a (possibly empty) set of function symbols such as $+, \times, -$;

(f) a (possibly empty) set of relation symbols such as $<$;

(g) the punctuation symbols such as '(',')' and ','.

If M is a structure for a first-order language say $\mathscr{L}$ then ${}^*M$ (Definition 2.1.12) can be regarded as an $\mathscr{L}$ structure in a natural way. For example, if M has a binary relation $<$ we can define $<$ on ${}^*M$ as

$$(m_0, m_1, m_2, \ldots)/_D < (n_0, n_1, n_2, \cdots)/_D \Leftrightarrow \{i : m_i < n_i\} \in D.$$

Also if M has a binary function $+$, then $+$ is defined on ${}^*M$ by

$$(m_0, m_1, m_2, \ldots)/_D + (n_0, n_1, n_2, \cdots)/_D = (m_0 + n_0, m_1 + n_1, m_2 + n_2, \cdots)/_D.$$

Similarly, the functions multiplication and subtraction can be defined on ${}^*M$ as well.

Theorem 2.2.2. (***Łoś's Theorem***) For every first-order formula $\theta(x_1, \ldots, x_n)$ in the first-order language $\mathscr{L}$ for M, ${}^*M \vDash \theta(a_1, \ldots, a_n) \Leftrightarrow \{i \in \mathbb{N} : M \vDash \theta(a_{1i}, \ldots, a_{ni})\} \in D$, where ${}^*M$ is the set of all hyper numbers of an infinite set M and $a_j = (a_{j0}, a_{j1}, \ldots)/_D$, for all $j = 1, 2, \cdots, n$.

Corollary 2.2.3. For a sentence σ, ${}^*M \vDash \sigma \Leftrightarrow M \vDash \sigma$.

Corollary 2.2.4. (***Transfer principle***)[5] If ϕ is a first-order statement with parameters from M, then ϕ holds in M if and only if ${}^*\phi$ holds in ${}^*M$, where ${}^*M$ is the set of all hyper numbers of an infinite set M, ${}^*\phi(m_1, m_2, \ldots, m_n) = \phi({}^*m_1, {}^*m_2, \ldots, {}^*m_n)$ and ${}^*m_i = (m_i, m_i, \ldots)/_D$ for all $i = 1, 2, 3, \ldots, n$.

Remark 2.2.5. For the rest of this section we look at the set of real numbers $\mathbb{R}$, for the sake of illustration.

Example 2.2.6. Since the field of real numbers is the ordered field $(\mathbb{R},+,\cdot,0,1,<)$ and all axioms of ordered fields are first-order, then by Łoś's Theorem all first-order properties are preserved, which implies that the hyperreal structure $^*(\mathbb{R},+,\cdot,0,1,<)$ is an ordered field. On the other hand, when D is a nonprincipal ultrafilter $^*(\mathbb{R},+,\cdot,0,1,<)$ is non Archimedean [35], showing the "Archimedean property" is not first-order.

Definition 2.2.7. $^*\mathbb{R}_{\text{fin}} = \{x \in {}^*\mathbb{R} : \exists r,s \in \mathbb{R}, {}^*r < x < {}^*s\}$.

Notice that the ordered field $(\mathbb{R},+,\cdot,0,1,<)$ is embedded within the subfield $(^*\mathbb{R}_{\text{fin}},+,\cdot,0,1,<)$ of the field $(^*\mathbb{R},+,\cdot,0,1,<)$.

Definition 2.2.8. [35] A real number $r \in {}^*\mathbb{R}$ is called *infinitesimal*, if $|r| < {}^*(1/n)$, for all $n \in \mathbb{N}$.

Example 2.2.9. 0 is the only standard number which is infinitesimal.

Definition 2.2.10. [34] Let $r,s \in {}^*\mathbb{R}$. We say that r is *infinitely close* to s, if $r-s$ is infinitesimal. This is denoted by $r \approx s$.

Definition 2.2.11. [34] If $r \in {}^*\mathbb{R}$, the *monad* of r is the set of all $s \in {}^*\mathbb{R}$ such that $s \approx r$. It is denoted by $\text{monad}(r)$.

Theorem 2.2.12. (***Standard part principle***) A finite $x \in {}^*\mathbb{R}$ is infinitely close to a unique real number $r \in \mathbb{R}$. That is, the monad of each finite hyperreal number contains a unique real number.

Proof. For proof see Keisler [18]. □

Definition 2.2.13. [18] Let $x \in {}^*\mathbb{R}_{\text{fin}}$. Then the *standard part* of x is denoted by $^\circ x$ or $\text{st}(x)$ and defined to be the real number r such that $^*r \approx x$. That is, $\text{st}(x) = \sup\{t \in \mathbb{R} : {}^*t < x\}$.

2.3 Standard and Nonstandard Universe

There is a standard way of extending a first-order structure $(M, f, \ldots, R, \ldots)$ into an object that behaves in many respects like a higher-order structure.

Let $S_0 = M$ and for each $n \in \mathbb{N}$, $S_{n+1} = S_n \dot\cup \mathscr{P}(S_n)$.

For each $n \in \mathbb{N}$ let $\text{rank}_n(x)$ denote $x \in S_n \setminus S_{n-1}$, and let $u \in v$ be the membership relation relating an object u of rank n to an object v of rank $n+1$, for some n [20]. Interpret the original functions and relations $f,\ldots,R,\ldots$ of M on objects of rank 0 in the usual way. This gives the structure

$$V(M) = (\cup_{n\in\mathbb{N}} S_n, \ldots, \mathrm{rank}_n, \ldots, \in, f, \ldots, R).$$

For example, for $\mathbb{R} = (\mathbb{R}, +, \cdot, 0, 1, <)$, $V(\mathbb{R})$ contains an object for the natural numbers, $\mathbb{N}$, of rank 1. Thus the theory of $V(\mathbb{R})$ is much more expressive than that of $\mathbb{R}$ itself.

$S = V(M)$ is called the *superstructure over M* [20], $V(M)$ is also called the *standard universe* and elements of $V(M)$ are called *standard* [13].

Fix a nonprincipal ultrafilter D. We are going to apply the ultrapower construction of Sections 2.1 and 2.2 to M, obtaining ${}^*M$, and also to $V(M)$, obtaining ${}^*V(M)$.

It is clear that the objects of rank 0 in ${}^*V(M)$ are precisely the elements of ${}^*M$. More generally, an object of rank $n+1$ in ${}^*V(M)$ can be regarded (via the $\in$ relation in ${}^*V(M)$) as a set of objects of rank at most n in ${}^*V(M)$. It follows that ${}^*M \subseteq {}^*V(M)$ but it is not quite true that ${}^*V(M) \subseteq V({}^*M)$.

An object $A = (A_0, A_1, \ldots)/_D$ in ${}^*V(M)$ has *bounded rank* if $\mathrm{rank}_n(A)$ holds for some $n \in \mathbb{N}$. That is, if

$$\{i : A_i \text{ has rank } n\} \in D. \tag{1}$$

Conversely, suppose that $A_0, A_1, \ldots, A_n, \ldots \in V(M)$ are given, and $n \in \mathbb{N}$ be such that

$$U = \{i : A_i \text{ has rank at most } n\} \in D \tag{2}$$

(in particular, this holds when every A_i has rank at most n).

Then $U = U_0 \dot\cup U_1 \dot\cup \ldots \dot\cup U_n$ where $U_k = \{i : A_i \text{ has rank } k\}$. So $U_k \in D$ for some k, hence $(A_0, A_1, \ldots, A_n, \ldots)/_D = (B_0, B_1, \ldots, B_n, \ldots)/_D$, where

$$B_i = \begin{cases} A_i & \text{if } A_i \text{ has rank } k \\ C & \text{otherwise ,} \end{cases}$$

where C is some arbitrary fixed element of $V(M)$ of rank k.

Thus, if (2) holds then $A = (A_0, A_1, \ldots, A_n, \ldots)/_D$ is equal to the equivalence class of a sequence of elements B_i of some fixed rank $k \leqslant n$.

Definition 2.3.1. [20] We say that $A \in {}^*V(M)$ has *bounded rank* if (1) or (2) holds for some $n \in \mathbb{N}$.

${}^*V(M)_{\mathrm{bdd}}$ is the substructure of ${}^*V(M)$ of objects of bounded rank, and by observations already made ${}^*V(M)_{\mathrm{bdd}} \subseteq V({}^*M)$.

In our work, given an infinite first-order structure M, we will usually work with the superstructure $V(M)$ over M, using usual set theoretic definitions in $V(M)$ to define higher-order concepts in the first-order language of $V(M)$.

The ultrapower ${}^*V(M)$ will be considered, and all work we do will take place in the bounded part ${}^*V(M)_{\mathrm{bdd}}$ of it.

We will relate ${}^*V(M)_{\mathrm{bdd}}$ to $V({}^*M)$. In particular, we will need to pay attention to which elements of $V({}^*M)$ are *internal*, that is, are elements of ${}^*V(M)_{\mathrm{bdd}}$ [20].

For example, with $M = (\mathbb{R}, +, \cdot, 0, 1, <)$, the set $\mathbb{N}$ is in $V(M)$ and embeds into ${}^*\mathbb{N} = (\mathbb{N}, \mathbb{N}, \ldots)/_D$ in ${}^*V(\mathbb{R})_{\mathrm{bdd}}$. Thus ${}^*\mathbb{N}$ is internal. However, its subset, the set of standard natural numbers, $\mathbb{N} = \{{}^*n : n \in \mathbb{N}\}$, is an element of $V({}^*\mathbb{R})$ (since $\mathbb{N} \subseteq {}^*\mathbb{N} \subseteq {}^*\mathbb{R}$) but is not internal as shown by the following.

Proposition 2.3.2. [20](***Overspill principle***) The set of standard natural numbers ($\mathbb{N}$) is not an internal subset of ${}^*\mathbb{N}$.

Equivalently, an internal subset A of ${}^*\mathbb{N}$ containing arbitrarily large finite numbers, must A contain an infinite element. Since if otherwise $\mathbb{N} = \{n \in {}^*\mathbb{N} : \exists a \in A, n < a\}$ shows $\mathbb{N}$ is internal, contradicting overspill. Conversely, $\mathbb{N}$ contains arbitrary large finite numbers but no infinite element. Hence, is not internal.

Notice also that any set definable (in the first-order logic of ${}^*V(M)$) from an internal set is in ${}^*V(M)_{\mathrm{bdd}}$. It follows that if $A \in V({}^*M)$ and $\mathbb{N}$ is definable from A using first-order logic then A is not internal.

Definition 2.3.3. [20] A set $A \in V(^*M)$ is said to be *external* if it is not internal.

Proposition 2.3.4. If $A \subseteq \mathbb{Z}$ is internal and *finite, then A is finite.

Proof. Suppose $n \in {}^*\mathbb{N}$ is the nonstandard natural number such that card $A = n$.

If $n > \mathbb{N}$ then for all $k \in \mathbb{N}$, there exists $a \in A$ and $|a| > k$ since the set $\{-k, \ldots, 0, \ldots, k-1, k\}$ is finite of size $2k+1 < n$. So by overspill there is $\nu > \mathbb{N}$ such that $|a| > \nu$ for some $a \in A$. In other words $A \nsubseteq \mathbb{Z}$. $\square$

Proposition 2.3.5. If $N > \mathbb{N}$, there exists $K \in {}^*\mathbb{N}$ such that $n < K$ and $nK < N$ for all $n \in \mathbb{N}$.

Proof. Suppose $N > \mathbb{N}$. Then for all standard $k \in \mathbb{N}$, $k^2 < N$. By overspill there exists $K > \mathbb{N}$ such that $K^2 < N$. So $n < K$ and $nK < N$ for all $n \in \mathbb{N}$. $\square$

2.4 S-Integrable Functions (SL^1)

Throughout this section we will introduce a type of integral on *finite sets called the *S-integral*. This requires the definition of the measure m of individual elements and internal subsets of nonstandard finite sets, as given in Cartier and Perrin [2]. Therefore, to understand precisely what are S- integrable functions exactly, we have to give the following definitions and examples.

Definition 2.4.1. [20] If $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, then the set $\{0, 1, 2, \ldots, N\} = \{k \in {}^*\mathbb{N} : k \leqslant N\}$ is called **finite* or *hyperfinite*.

Example 2.4.2. Let $F = \left\{ \lfloor -\frac{N}{2} \rfloor + 1, \lfloor -\frac{N}{2} \rfloor + 2, \ldots, 0, \ldots, \lfloor \frac{N}{2} \rfloor \right\}$, where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then F is a *finite set. If N is even, then $F = \left\{ -\frac{N}{2} + 1, -\frac{N}{2} + 2, \ldots, 0, \ldots, \frac{N}{2} \right\}$ and if N is odd, then $F = \left\{ -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, 0, \ldots, \frac{N-1}{2} \right\}$.

Definition 2.4.3. [2] Let F be a *finite set. An internal function $m \colon F \to {}^*\mathbb{R}$ such that $m(t) \geqslant 0$ for all $t \in F$ is said to be a *measure* on F.

Example 2.4.4. If F is a *finite set, define $m(t) = \frac{1}{|F|}$, for each $t \in F$. This is the usual (or uniform) measure on F normalised to 1.

Definition 2.4.5. [2] Given a measure m on a *finite set F. The *measure of an internal set* $A \subseteq F$ is

$$m(A) = \sum_{t \in A} m(t).$$

For the usual measure, $m(A) = 0$ if and only if A is the empty subset of F.

In practice, we usually work with the uniform measure $m(t) = \frac{1}{|F|}$, for each $t \in F$.

Definition 2.4.6. [20] A real number $r \in {}^*\mathbb{R}$ is called *finite* or *limited*, if there is a positive $a \in \mathbb{R}$ such that $|r| < a$. An element $s \in {}^*\mathbb{R}$ is called *infinite* or *unlimited*, if s is not finite (or not limited).

Definition 2.4.7. [27] A real number $r \in {}^*\mathbb{R}$ is called *appreciable*, if it is limited and not infinitesimal.

Example 2.4.8. 0 is not an appreciable number, since 0 is an infinitesimal.

Definition 2.4.9. [2] An internal or external subset A of a *finite set F, is called *rare*, if for every appreciable number $r > 0$, there exists an internal set E of F, such that $A \subseteq E$ and $m(E) < r$.

Example 2.4.10. Let $A = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$. Then A is an internal subset of F of Example 2.4.2 and $m(A) = n(\frac{1}{N}) \approx 0$. So A is a rare set.

Definition 2.4.11. Let F be a *finite set with measure m and $f\colon F \to {}^*\mathbb{C}$ be a function. Then the *integral* of f on F is denoted by $\int_F f dm$, and defined by $\int_F f dm = \sum\limits_{t \in F} f(t)m(t)$.

Definition 2.4.12. [2] Let F be a *finite set with measure m. An internal function $f\colon F \to {}^*\mathbb{C}$ is said to be *S-integrable*, if the integral $\int_F |f| dm$ is limited, and $\int_A f dm$ is infinitesimal for all internal rare subsets A of F.

Definition 2.4.13. [2] The internal sequence $(f_n)_{0 \leqslant n \leqslant \nu}$ of functions $f_n\colon F \to {}^*\mathbb{C}$, converges to f *almost everywhere*, if there is a rare subset E of the *finite set F, such that for all $t \in F \setminus E$ and all unlimited $n \leqslant \nu$, $f_n(t) \approx f(t)$.

Definition 2.4.14. [2] Let $\mathscr{P}$ be an internal partition of F. Then each element A in $\mathscr{P}$ is called an *atom* of $\mathscr{P}$.

Definition 2.4.15. [2] Let $\mathscr{P}$ be an internal partition of a *finite set F and $f\colon F \to {}^*\mathbb{C}$ be a function. The function $E^{\mathscr{P}}[f]\colon F \to {}^*\mathbb{C}$ defined by $E^{\mathscr{P}}[f](t) = \frac{1}{m(A)} \int_A f dm$, where $A \in \mathscr{P}$ is the unique atom containing t, is called the *average of f relative to* $\mathscr{P}$. The function $E^{\mathscr{P}}[f]$ is constant on each atom $A \in \mathscr{P}$ and defined by $E^{\mathscr{P}}[f](t) = E^{\mathscr{P}}[f](A)$.

Proposition 2.4.16. Let F be a *finite set. If $f\colon F \to {}^*\mathbb{C}$ is S-integrable, then $E^{\mathscr{P}}[f]\colon F \to {}^*\mathbb{C}$ is S-integrable, for all internal partitions $\mathscr{P}$ of F.

Proof. For a proof see Cartier and Perrin [2]. □

Definition 2.4.17. [2] Let F be a *finite subset of $^*\mathbb{C}$. A function $d\colon F \times F \to {}^*\mathbb{R}$ is said to be a **metric* on F if it satisfies the following axioms:

(M1) $d(r,r) = 0$ for all $r \in F$;

(M2) $d(r,s) = d(s,r) > 0$ for all $r \neq s$, where $r,s \in F$;

(M3) $d(r,t) \leqslant d(r,s) + d(s,t)$ for all $r,s,t \in F$.

The pair (F,d) is said to be a **metric space* if d is a *metric function on F.

Definition 2.4.18. [2] Let A be a set in a *finite metric space (F,d). The *boundary* of a set A is denoted by $\partial_F(A)$ and defined by

$$\partial_F(A) = \{t \in F : \mathrm{monad}(t) \cap A \neq \emptyset \wedge \mathrm{monad}(t) \cap (F \setminus A) \neq \emptyset\}.$$

Definition 2.4.19. [2] A set A in a *finite metric space (F,d) is called *quadrable*, if it is internal and $\partial_F(A)$ is rare.

2.5 S-Continuity and L^1-Integrability on *Finite Sets

Definition 2.5.1. [2] Given a *finite set F, equipped with a *metric d, an internal function $f\colon F \to {}^*\mathbb{C}$ is said to be *S-continuous* on a subset A of F, if for all $r,s \in A$, $d(r,s) \approx 0$ implies $f(r) \approx f(s)$.

Definition 2.5.2. [2] Let F be a *finite set. An internal function $f\colon F \to {}^*\mathbb{C}$ is said to be *almost S-continuous*, if there exists a rare subset E of F such that f is S-continuous on $F \setminus E$.

Definition 2.5.3. [2] Given a *finite set F with *metric d and uniform measure m, an internal function $f\colon F \to {}^*\mathbb{C}$ is said to be *Lebesgue integrable* or *L-integrable* if it is S-integrable and almost S-continuous on F.

Therefore, for the given *finite set F with *metric d, we may define the space $L^1(F)$ as the space of "*L-integrable*" functions $f\colon F \to {}^*\mathbb{C}$.

Definition 2.5.4. Given a *finite set F, $L^1(F)$ is the set of all L-integrable internal functions

$f\colon F \to {}^*\mathbb{C}$. This is endowed with extra structure as follows. $L^1(F)$ is a vector space over the field of complex numbers $\mathbb{C}$, with componentwise vector addition and scalar multiplication defined as

$$(f+g)(\omega) = f(\omega)+g(\omega), \forall f,g \in L^1(F) \text{ and } (\lambda f)(\omega) = \lambda(f(\omega)), \forall f \in L^1(F), \forall \lambda \in \mathbb{C}.$$

We see that the vector space axioms are easily verified. Also, if f and g are L-integrable functions, then $f+g$, and λf are L-integrable provided that $\lambda \in \mathbb{C}$ is limited.

Note 2.5.5. Notice that, $L^1(F)$ is not a vector space over the field of hypercomplex numbers ${}^*\mathbb{C}$. For example, if $f(t) = 1$, for all $t \in F$ and $\lambda = N \in {}^*\mathbb{C} \setminus \mathbb{C}$, then $\lambda f(t) = N$ is not L-integrable.

Definition 2.5.6. Given a *finite set F. Let $f \in L^1(F)$. Then the L^1 *norm* of f is defined by $\|f\|_1 = \int_F |f| dm$.

Note 2.5.7. Given a *finite set F, notice that, in $L^1(F)$, we get

$$\|f\|_1 = \int_F |f| dm = \sum_{t\in F} |f(t)| m(t) = \frac{1}{|F|} \sum_{t\in F} |f(t)|.$$

Definition 2.5.8. [2] The internal sequence $(f_n)_{0 \leqslant n \leqslant \nu}$ L^1*-converges to* f if $\|f_n - f\|_1 \approx 0$ for all unlimited natural numbers $n \leqslant \nu$.

Definition 2.5.9. [2] Let (F,d) be a *metric space. The *diameter* of an internal subset A of F is $\mathrm{dia}(A) = \max\{d(x,y) : x,y \in A\}$.

Definition 2.5.10. [2] An internal partition $\mathscr{P}$ of a *finite set F is called *infinitely fine* if each of its atoms has an infinitesimal diameter.

Definition 2.5.11. [2] A family of internal partitions $(\mathscr{P}_n)_{0 \leqslant n \leqslant \nu}, (\nu > \mathbb{N})$ of a *finite set F is called a *dissection* of F if satisfies the following properties:

(a) $n \leqslant m \leqslant \nu \Rightarrow \mathscr{P}_m$ is finer than $\mathscr{P}_n$;

(b) n is limited $\Rightarrow \mathscr{P}_n$ is composed of a limited number of quadrable sets;

(c) n is unlimited $\Rightarrow \mathscr{P}_n$ is infinitely fine.

Proposition 2.5.12. A dissection $(\mathscr{P}_n)_{0 \leqslant n \leqslant \nu}, (\nu > \mathbb{N})$ of a *finite set F exists.

Proof. For a proof see Cartier and Perrin [2]. □

Theorem 2.5.13. Given a *finite set F and $f\colon F \to {}^*\mathbb{C}$ is an S-integrable function, then the following statements are equivalent:

(a) f is L-integrable;

(b) for all infinitely fine partitions $\mathscr{P}$, the function $f - E^{\mathscr{P}}[f] \approx 0$ almost everywhere;

(c) for all infinitely fine partitions $\mathscr{P}$, $\|f - E^{\mathscr{P}}[f]\|_1 \approx 0$.

Proof. For a proof see Cartier and Perrin [2]. □

2.6 Loeb Measure

The Loeb measure is a type of measure introduced by Loeb in 1973 [5] using nonstandard analysis. We follow the approach given by Lindstrøm [20]. Consider a *finite internal set F with the uniform normalised measure (and a *metric d). To understand the Loeb measure and its properties we have to consider the following.

Definition 2.6.1. Let $\mathscr{A}$ be the set of all internal subsets of F. Then the *internal finitely additive measure* $\mu\colon \mathscr{A} \to {}^*\mathbb{R}$ is defined by $\mu(A) = \frac{|A|}{|F|}$, for all internal subsets A of F.

Example 2.6.2. $\mu(F) = 1$. Thus, $\mu(A)$ is finite, for every internal subset A of F.

We can turn the internal finitely additive measure $\mu\colon \mathscr{A} \to {}^*\mathbb{R}$ into a finitely additive, real-valued measure ${}^\circ\mu\colon \mathscr{A} \to \mathbb{R}$ by taking the standard part ${}^\circ\mu(A) = \mathrm{st}(\mu(A))$ for all internal subset $A \in \mathscr{A}$.

Definition 2.6.3. [20] A subset B of F is μ-*approximable* if for each real $\varepsilon > 0$, there are sets $A, C \in \mathscr{A}$ such that $A \subseteq B \subseteq C$ and $\mu(C) - \mu(A) < \varepsilon$.

Definition 2.6.4. [20] The *Loeb algebra* $L(\mathscr{A})$ is defined to be the set of all subsets B of F such that B is μ-approximable.

Definition 2.6.5. [20] The *Loeb measure* of μ is the map $L(\mu)\colon L(\mathscr{A}) \to \mathbb{R}$ defined by

$$L(\mu)(B) = \inf\{{}^\circ\mu(C) : C \in \mathscr{A}, C \supseteq B\}.$$

Notice that the Loeb measure $L(\mu)$ is an extension of ${}^\circ\mu$ to a σ-additive measure.

Lemma 2.6.6. $L(\mathscr{A})$ is a σ-algebra extending $\mathscr{A}$.

Proof. For a proof see Lindstrøm [20]. $\square$

Definition 2.6.7. [20] A measure μ on a σ-algebra $\mathscr{A}$ is called *complete*, if for all $B \in \mathscr{A}$ whenever $A \subset B$ and $\mu(B) = 0$, then $A \in \mathscr{A}$ and $\mu(A) = 0$.

Lemma 2.6.8. $L(\mu)$ is a complete measure on $L(\mathscr{A})$.

Proof. For a proof see Lindstrøm [20]. $\square$

2.7 Measurable Functions and Integrals

Our aim in this section is to discuss the Lebesgue measure on subsets of $\mathbb{T} = [-\pi, \pi)$ and the Loeb measure on subsets of a *finite set F, and then measurable and integrable functions on these sets, in order to study the relationship between functions in the standard space $L^1(\mathbb{T})$ and functions in the nonstandard space $L^1(F)$ (as given later in Theorem 6.5.1). Throughout this section, fix a *finite set $F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$ of order $N > \mathbb{N}$ and $\mathbb{T} = [-\pi, \pi)$. Additional material here is based on Lindstrøm [20].

Definition 2.7.1. The function $\mathrm{st}_N\colon F \to [-\pi, \pi]$ is defined by $\mathrm{st}_N(t) = \mathrm{st}(\frac{2\pi t}{N})$ is said to be the *"normalised" standard part map.*

Definition 2.7.2. Let $\mathscr{A}$ be the set of all internal subsets of F. The *normalised internal finitely additive measure* $\mu\colon \mathscr{A} \to {}^*\mathbb{R}$ is defined by $\mu(A) = \frac{2\pi|A|}{N}$, for all internal subsets A of F.

Definition 2.7.3. [7] Let $E \subseteq \mathbb{R}$. The *Lebesgue outer measure* $\lambda^*(E)$ is defined by

$$\lambda^*(E) = \inf\{\sum_{k=0}^{\infty} |I_k| : (I_k)_{k\in\mathbb{N}} \text{ is a sequence of half-open intervals with } E \subseteq \bigcup_{k\in\mathbb{N}} I_k\}.$$

The *Lebesgue measure* of E is defined by its Lebesgue outer measure. That is, $\lambda(E) = \lambda^*(E)$, if for every $A \subseteq \mathbb{R}$, $\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$, where $E^c = \mathbb{R} \setminus E$.

Note 2.7.4. A subset $Y \subseteq \mathbb{R}$ is Lebesgue measurable if and only if $\mathrm{st}_N^{-1} Y \in L(\mathscr{A})$. In addition, if $\mathrm{st}_N^{-1} Y \in L(\mathscr{A})$, then $L(\mu)(\mathrm{st}_N^{-1} Y) = \lambda(Y)$. See Lindstrøm [20, page33].

Definition 2.7.5. [20] A function $h\colon F \to {}^*\mathbb{R}$ is $\mathscr{A}$*-measurable* if for every closed set $[u,v] \subset {}^*\mathbb{R}$,

$$h^{-1}([u,v]) = \{\omega \in F : h(\omega) \in [u,v]\} \in \mathscr{A}.$$

Note 2.7.6. Every internal function $h\colon F \to {}^*\mathbb{R}$ is $\mathscr{A}$*-measurable.*

Definition 2.7.7. A function $h\colon F \to {}^*\mathbb{C}$ is $\mathscr{A}$*-measurable* if and only if Re h and Im h are $\mathscr{A}$ -measurable as real-valued functions, where Re $h(\omega) = \mathrm{Re}(h(\omega))$ and Im $h(\omega) = \mathrm{Im}(h(\omega))$.

Definition 2.7.8. If $h\colon F \to {}^*\mathbb{C}$ is an internal function, then ${}^\circ h\colon F \to \mathbb{C}$ is defined by ${}^\circ h(\omega) = \mathrm{st}(h(\omega))$ provided that $\mathrm{Re}(h(\omega))$ and $\mathrm{Im}(h(\omega))$ are finite in ${}^*\mathbb{R}$, where $\mathrm{st}(a+ib) = \mathrm{st}(a) + i\mathrm{st}(b)$ for $a, b \in {}^*\mathbb{R}_{\mathrm{fin}}$.

Note 2.7.9. If $h\colon F \to {}^*\mathbb{C}$ is $\mathscr{A}$-measurable, then ${}^\circ h\colon F \to \mathbb{C}$ is $L(\mathscr{A})$-measurable.

Definition 2.7.10. For $h\colon F \to {}^*\mathbb{C}$ internal, given $A \subseteq F$, we define $\int_A h d\mu = \frac{1}{|F|} \sum\limits_{\omega \in A} h(\omega)$.

Lemma 2.7.11. For internal $A \subseteq F$, if $h\colon F \to {}^*\mathbb{C}$ is internal and $h(\omega) \approx 0$ for all $\omega \in A$, then $\int_A |h| d\mu \approx 0$.

Proof. For a proof see Lak [19]. □

Definition 2.7.12. [20] An internal function $h\colon F \to {}^*\mathbb{C}$ is called *finite* if $\mu(\{t : h(t) \neq 0\})$ and $\max\limits_{t \in F} |h(t)|$ are both limited.

Lemma 2.7.13. If $h\colon F \to {}^*\mathbb{C}$ is finite, then $\mathrm{st}(\int_A h d\mu) = \int_A {}^\circ h dL(\mu)$, for all internal $A \subseteq F$.

Proof. For a proof see Lindstrøm [20]. □

Theorem 2.7.14. If $h\colon F \to {}^*\mathbb{C}$ is S-integrable, then $\mathrm{st}(\int_A h d\mu) = \int_A {}^\circ h dL(\mu)$, for all internal subset A in F.

Proof. For a proof see Lindstrøm [20]. □

Definition 2.7.15. [20] Let $h\colon F \to \mathbb{C}$ be an $L(\mathscr{A})$-measurable function. Then a *lifting* of h is an internal, $\mathscr{A}$-measurable function $H\colon F \to {}^*\mathbb{C}$ such that ${}^\circ H(\omega) = h(\omega)$ L(μ)-almost everywhere.

Theorem 2.7.16. If $\mu(F)$ is finite, then all $L(\mathscr{A})$-measurable functions have lifting.

Proof. For a proof see Lindstrøm [20]. □

Notice that Theorem 2.7.16 can be extended to give the following theorem.

Theorem 2.7.17. If $h\colon F \to \mathbb{C}$ is $L(\mathscr{A})$-measurable and has 1-norm $\|\cdot\|_1$, then h has a lifting $H\colon F \to {}^*\mathbb{C}$, which is S-integrable.

Proof. For a proof see Lak [19]. □

2.8 Some Applications of Saturation

Theorem 2.8.1. ($\aleph_1$***-saturation***) If $(A_i)_{i\in\mathbb{N}}$ is a sequence of internal sets such that $\cap_{i<I}A_i \neq \emptyset$ for all $I \in \mathbb{N}$ then $\cap_{i\in\mathbb{N}}A_i \neq \emptyset$.

Proof. For a proof see Lindstrøm [20]. □

Following are some application examples of saturation.

Example 2.8.2. Let $A_0, A_1, \ldots, A_i, \ldots$ $(i \in \mathbb{N})$ be a sequence of internal sets and $\phi_i(x, \overline{A})$ be a sequence of formulas of the language of $V(M)$ with parameters $\overline{A}$ from $\{A_0, A_1, \ldots, A_i, \ldots\}$. Suppose

$$\exists x \bigwedge_{i<I} \phi_i(x, \overline{A}) \tag{3}$$

for each $I \in \mathbb{N}$. Then there is an internal X satisfying $\phi_i(X, \overline{A})$ for all $i \in \mathbb{N}$. To explain this, let

$$B_I = \{x : \bigwedge_{i<I} \phi_i(x, \overline{A})\}.$$

Then B_I is internal for each I and nonempty (by 3). So by $\aleph_1$-saturation there is some internal $X \in \cap_{I\in\mathbb{N}}B_I$.

Definition 2.8.3. [20] M is κ-*saturated* if whenever $|A| < \kappa$ and $\overline{\Psi}$ is a family of formulas $\psi(x, \overline{a})$, $\overline{a} \in A$, that is, finitely satisfied, then $\overline{\Psi}$ is satisfied.

Example 2.8.4. ${}^*V(M)$ is $\aleph_1$-saturated.

Remark 2.8.5. For each cardinality κ, by replacing the indexing set $\mathbb{N}$ with some other set I and using an appropriate ultrafilter D on I we can arrange that ${}^*V(M)$ is κ-saturated.

Example 2.8.6. The function $\mathrm{st}\colon {}^*\mathbb{Q}_{\mathrm{fin}} \to \mathbb{R}$ defined by

$$\mathrm{st}(x) = {}^\circ x$$

is surjective, where ${}^*\mathbb{Q}_{\mathrm{fin}}$ is the set of nonstandard finite rational numbers.

Given $\alpha \in \mathbb{R}$, there exists a rational sequence (a_n) such that (a_n) converges to α and without loss of generality we may assume $|a_n - \alpha| < 1/n$ for all $n \in \mathbb{N}$.

So $\{a_n : n \in \mathbb{N}\} \subseteq {}^*\mathbb{Q}_{\mathrm{fin}}$. We have to show that there exists a sequence $b = (b_0, b_1, \ldots, b_\nu)$ in ${}^*\mathbb{Q}_{\mathrm{fin}}$, where $\nu > \mathbb{N}$ such that $b_0 = a_0, b_1 = a_1, \ldots, b_i = a_i$ for all $i \in \mathbb{N}$.

Let $\phi_i(b, a_0, a_1, \ldots, a_i)$ be the statement "b is a sequence $(b_0, b_1, \ldots, b_\nu)$ where $\nu \geqslant i$ and $b_j = a_j$ for all $j \leqslant i$". Then it is clear that this set of statements is finitely satisfied. So by saturation there is some $b = (b_0, b_1, \ldots, b_\nu)$ satisfying every ϕ_i.

Then for all $n \in \mathbb{N}$, $b_n \in {}^*\mathbb{Q}_{\mathrm{fin}}$ and $|b_n - \alpha| < 1/n$. So by overspill there is $m > \mathbb{N}$ in ${}^*\mathbb{N}$ with $b_m \in {}^*\mathbb{Q}_{\mathrm{fin}}$ and $|b_m - \alpha| < 1/m$ and $\alpha = \mathrm{st}(b_m)$ as required.

Example 2.8.7. The function $\phi : C_{p^N} \to \mathbb{Z}_p$ defined by

$$\phi(x_0 + x_1 p + x_2 p^2 + \ldots + x_{N-1} p^{N-1}) = x_0 + x_1 p + x_2 p^2 + \ldots.$$

is surjective, where $C_{p^N} = \{x_0 + x_1 p + \ldots + x_{N-1} p^{N-1} : 0 \leqslant x_i < p \text{ for all } i = 1, 2, \ldots, N-1\}$, p is standard prime, $N > \mathbb{N}$ and $\mathbb{Z}_p$ is the set of p-adic integers.

Given $z = \sum_{k=0}^{\infty} z_k p^k$ in $\mathbb{Z}_p$, we have to show that there is $w = \sum_{k=0}^{N-1} w_k p^k$ in C_{p^N} such that $\phi(w) = z$.

Let $\psi_i(w, w_0, w_1, \ldots, w_i)$ be the statement "$w \equiv w_0 + w_1 p + \ldots + w_i p^i \bmod p^{i+1}$". Then for all $I \in \mathbb{N}$ we have

$$\exists w \bigwedge_{i<I} \psi_i(w, w_0, w_1, \ldots, w_i).$$

Therefore, we can take $w = w_0 + w_1 p + \ldots + w_I p^I$. So by saturation there is w with

$$w \equiv w_0 + w_1 p + \ldots + w_i p^i \bmod p^{i+1}, \text{ for all } i.$$

Example 2.8.8. Assume $(a_n)_{n\in\mathbb{N}}$ is a standard Cauchy sequence of rational numbers.

By saturation there is $a\colon {}^*\mathbb{N} \to {}^*\mathbb{Q}$ such that $a(n) = a_n$ for all $n \in \mathbb{N}$.

Since $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence,

$$\forall k \exists N_k \forall i, j \in \mathbb{N}(i, j \geqslant N_k \Rightarrow |a_i - a_j| < 1/k).$$

By saturation there is $N\colon {}^*\mathbb{N} \to {}^*\mathbb{N}$ such that $N(k) = N_k$ for all $k \in \mathbb{N}$.

For all $n \in \mathbb{N}$, we have the single sentence

$$\forall k < n \forall i, j < n(i, j \geqslant N(k) \Rightarrow |a(i) - a(j)| < 1/k).$$

So there is $\nu > \mathbb{N}$ such that

$$\forall k < \nu \forall i, j < \nu(i, j \geqslant N(k) \Rightarrow |a(i) - a(j)| < 1/k).$$

Let $k = 2$, $i = N_k = N(k)$ and $b = a_{\nu-1}$ then $|a_{\nu-1} - a_i| < 1/2$. So, $a_i - 1/2 < b < a_i + 1/2$. Therefore, $b \in {}^*\mathbb{Q}_{\text{fin}}$.

Now assume $\ell = \text{st}(b)$. Given k, let $N = N_k \in \mathbb{N}$. So, for $i \geqslant N$,

$$|a_i - \ell| \approx |a_i - b| < 1/k.$$

So $|a_i - \ell| \leqslant 1/k$. Therefore, $a_i \to \ell$ as $i \to \infty$.

Chapter 3

Abstract Harmonic Analysis

The subject of harmonic analysis describes complex-valued functions whose domain is an abelian group. Such functions are described by their Fourier series.

Our purpose in this chapter is to give the basic definitions of the classical theory in the cases of a finite group and a locally compact group. The theory for finite groups will be used in the nonstandard context later.

We divide this chapter into three main sections: abstract harmonic analysis on finite groups; abstract harmonic analysis on topological groups; and Fourier coefficients and Fourier series.

3.1 Abstract Harmonic Analysis on Finite Groups

Let G be a finite abelian group, written additively. Let $N \in \mathbb{N}$ be the order of G. Such G has an obvious counting measure, $\mu\colon \mathscr{P}(G) \to \mathbb{N}$ defined by $\mu(A) = \operatorname{card} A$ for every $A \subseteq G$, where $\mathscr{P}(G)$ is the power set of G, which is a σ-algebra. So all subsets of G are measurable and of course μ is additive, that is, for any family $\{A_i\}$ of disjoint subsets of G, $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$.

Alternatively, we can normalise the measure μ as $\mu(A) = \operatorname{card} A/\operatorname{card} G$ or $\mu(A) = \operatorname{card} A/k$ for any other positive $k \in \mathbb{N}$.

We will be interested in the $\mathbb{C}$-vector space V_G of all complex-valued functions defined on domain G.

Addition and scalar multiplication on V_G are defined as:

(a) $(f+g)(x) = f(x)+g(x)$ for all $f,g \in V_G$;

(b) $\lambda(f(x)) = (\lambda f)(x)$ for all $f \in V_G$ and all $\lambda \in \mathbb{C}$.

The group G acts on the vector space V_G. This action is defined by setting $y \cdot f = f^y$ where $f^y(x) = f(x-y)$, and $f, f^y \in V_G$.

The vector space V_G has extra structure given by an inner product.

Definition 3.1.1. [11] Given functions f and g in V_G, we may form their *inner product* with normalisation (or other possible normalisation) as

$$\langle f,g \rangle = \frac{1}{|G|} \sum_{t \in G} \overline{f(t)} g(t),$$

where $\overline{f(t)}$ is the complex conjugate of $f(t)$.

The vector space V_G has dimension N. Rather than taking the 'usual' basis vectors v_g with value $v_g(h) = \delta_{gh}$ (where δ_{gh} is the Kronecker delta) it is better to reflect the group structure by taking as a basis a complete set of 1-dimensional representations of G.

Definition 3.1.2. [15] A function $f\colon G \to \mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$ is a 1-*dimensional representation* of G if the following holds:

(a) $f(x+y) = f(x)f(y)$ for all $x,y \in G$;

(b) $f(0) = 1$.

Theorem 3.1.3. [15] For an abelian group G of order N there is a basis of V_G consisting of 1-dimensional representations of G. These basis vectors are orthogonal with respect to the inner product $\langle x,y \rangle$ of Definition 3.1.1.

It follows easily that a 1-dimensional representation of a finite group G maps into $\mathbb{T}_\mathbb{C}$. The *dual group* G^* of G is defined to be the additive group of 1-dimensional representations $f\colon G \to \mathbb{T}_\mathbb{C}$ with pointwise multiplication of representations. That is, G^* is a subgroup of V_G.

Fix the orthonormal basis $\{e_0, \ldots e_{N-1}\}$ of V_G, where each e_j is a 1-dimensional representation of G. Then given $f \in V_G$ we can write f as linear combination of basis elements as

$$f = \sum_{j=0}^{N-1} \langle e_j, f \rangle e_j. \tag{1}$$

This expression describes f in terms of its Fourier coefficients,

$$\widehat{f}_j = \langle e_j, f \rangle = \frac{1}{|G|} \sum_{t \in G} \overline{e_j(t)} f(t). \tag{2}$$

The process of obtaining the Fourier coefficients $\widehat{f}_j$ from f is the operation of discrete Fourier transform (DFT) and the recovery of f from the collection of its Fourier coefficients as given in Equation (1) is the inverse DFT operation [36].

Example 3.1.4. Let $G = C_N$ be a cyclic group of order N, and $e_j(g) = e^{2\pi i j g/N}$. Given $f\colon G \to \mathbb{C}$, its DFT is

$$\widehat{f}_j = \langle e_j, f \rangle = \frac{1}{N} \sum_{g \in G} e^{-2\pi i j g/N} f(g).$$

f is recovered from DFT by the inverse operation using Kronecker delta ($\sum_{n \in G} e^{2\pi i k n/N} = N\delta_{k0}$)

$$f = \sum_{n=0}^{N-1} \langle e_n, f \rangle e_n = \sum_{n=0}^{N-1} \widehat{f}_n e_n.$$

One of the most important concepts of Fourier analysis is convolution. Let $f, g \in V_G$. The convolution of f and g is denoted by $f * g$ and defined by

$$(f * g)(x) = \frac{1}{|G|} \sum_{y \in G} f(y) g^y(x),$$

for every $x \in G$, where $g^y = y \cdot g$ is obtained from the group action defined above.

The convolution on V_G has the following properties:

(a) commutativity $f * g = g * f$;

(b) associativity $f * (g * h) = (f * g) * h$;

(c) linearity $f * (g + h) = (f * g) + (f * h)$ and $\lambda(f * g) = (\lambda f) * g = f * (\lambda g)$ for all $\lambda \in \mathbb{C}$.

We prove the associative property of the convolution and the other properties are easy to check.

Theorem 3.1.5. $((f*g)*h)(x) = (f*(g*h))(x)$ for all $f,g,h \in V_G$ and all $x \in G$.

Proof. By applying a change of variables we obtain

$$\begin{aligned}
((f*g)*h)(x) &= \frac{1}{|G|}\sum_{y\in G}(f*g)(y)h^y(x) \\
&= \frac{1}{|G|}\sum_{y\in G}\Big(\frac{1}{|G|}\sum_{z\in G} f(z)g(y-z)\Big)h(x-y) \\
&= \frac{1}{|G|}\sum_{z\in G} f(z)\Big(\frac{1}{|G|}\sum_{w\in G} g(w)h^w(x-z)\Big) \\
&= (f*(g*h))(x).
\end{aligned}$$

□

3.2 Abstract Harmonic Analysis on Topological Groups

In this section we start looking at classical harmonic analysis. The starting point is a topological abelian group G, written additively, and we will restrict attention throughout to the case where the topology on G is Hausdorff and locally compact. In many cases, this topology will be given by an invariant metric.

The vector space V_G of the last section is the space of all functions $G \to \mathbb{C}$. This space will typically have infinite dimension, and for many reasons it is too 'large' for analysis, and we need to consider more amenable subspaces of it. For example,

$$V_G^{\text{cts}} = \{f\colon G \to \mathbb{C} : f \text{ is continuous}\}.$$

It is easy to check that V_G^{cts} is indeed a subspace of V_G, that is, it is closed under addition and scalar multiplication.

In fact, the subspace V_G^{cts} has extra useful structure which is important: it is also closed under the G-action defined in the previous section $g \cdot f = f^g$, see Katznelson [16]. This groups action remains important throughout, and we will need to ensure that the subspaces of V_G we look at are closed under this action.

For more general function spaces we need to introduce the idea of measure.

Definition 3.2.1. [16] A *Haar measure* μ on a locally compact abelian group G is a positive regular Borel measure having the following properties:

(a) $\mu(K) < \infty$ for every compact set $K \subseteq G$;

(b) $\mu(g+A) = \mu(A)$ for all $g \in G$ and all measurable sets $A \subseteq G$. Here $g+A = \{g+a : a \in A\}$.

The property (b) of Definition 3.2.1 is called the *invariant property* of a Haar measure μ.

Theorem 3.2.2. [16] Any locally compact group G has a Haar measure. This measure is unique up to multiplication by a positive constant.

Example 3.2.3. If G is the additive real group $\mathbb{R}$. We can take the Haar measure μ to be any scalar multiple of Lebesgue measure.

Example 3.2.4. Let G be the additive complex group $\mathbb{C}$. Its Haar measure μ is the usual Lebesgue measure in the Argand diagram.

Example 3.2.5. Let G be the circle group $\mathbb{T} = [0, 2\pi)$ with addition operation modulo 2π. The usual Lebesgue measure μ can be taken on $\mathbb{T}$. So according to Definition 3.2.1, this measure is the Haar measure, up to a multiplicative positive constant.

Example 3.2.6. Let G be the group of integers $\mathbb{Z}$. We take the counting measure μ assigning a mass of 1 to each point on $\mathbb{Z}$. It is G-invariant. By Definition 3.2.1, up to a multiplicative positive constant this is the Haar measure.

Example 3.2.7. Let G be the group of p-adic integers $\mathbb{Z}_p$. $\mathbb{Z}_p$ is unlike $\mathbb{Z}$; it is compact. Take the natural probability measure μ on the smallest σ-algebra $\mathscr{B}$ of compact subsets of $\mathbb{Z}_p$ such that $\mu(\mathbb{Z}_p) = 1$. Therefore it has a finite measure which is G-invariant. The conditions of Definition 3.2.1 hold. Hence, this measure is Haar.

So we can fix a Haar measure μ on our group G. This gives rise to an integral for functions $f\colon G \to \mathbb{C}$,

$$\int_G f(t)\,\mathrm{d}\mu,$$

which is the Lebesgue integral of f with respect to the Haar measure $d\mu$ on G [16].

Definition 3.2.8. [32] A function $f\colon G \to \mathbb{C}$ is *integrable* (L^1), if $\int_G f(t)d\mu$ is finite and for every measurable subset $A \subseteq \mathbb{C}$, $f^{-1}(A)$ is a measurable subset of G.

It is clear that the set of L^1 functions from G to $\mathbb{C}$ is closed under vector addition and scalar multiplication. It is also closed under the group action. As the convolution of any two L^1 functions from G to $\mathbb{C}$ is an L^1 function from G to $\mathbb{C}$ as well, this is also closed under convolution.

Definition 3.2.9. [16] $V_G^{L^1}$ is defined to be the space of all equivalence classes with respect to the equivalence relation $\sim$ defined by $f \sim g$ if and only if $f = g$ almost everywhere. The set of representatives of these equivalence classes regarded as the set of L^1 functions in V_G, which is a subspace of V_G.

Observe also that we can regard V_G^{cts} is a subspace of $V_G^{L^1}$. This follows from the fact that every continuous complex-valued function on G is also an L^1 function on G.

Next, we use Haar measure to define an inner product of $V_G^{L^1}$.

Definition 3.2.10. For $f, g \in V_G^{L^1}$ define

$$\langle f, g \rangle = \int_G \overline{f(t)} g(t) \, \mathrm{d}\mu.$$

As before, a normalisation factor may be applied here. In the case where G is compact, $\mu(G)$ exists and is finite from the definition of Haar measure and this inner product could be naturally normalised as

$$\langle f, g \rangle = \frac{1}{\mu(G)} \int_G \overline{f(t)} g(t) \, \mathrm{d}\mu.$$

Proposition 3.2.11. For $f, g \in V_G^{L^1}$, $\langle f, g \rangle$ is finite and $\langle , \rangle$ is an inner product on $V_G^{L^1}$.

Example 3.2.12. Let $G = \mathbb{T}$. Then $V_G^{L^1}$ is the set of L^1 functions on $\mathbb{T}$.

For $j \in \mathbb{Z}$, let $e_j(t) = \mathrm{e}^{ijt}$. Then $\langle e_j, e_k \rangle = \delta_{jk}$ where δ_{jk} is the Kronecker delta. Given $f \in V_G^{L^1}$, its *Fourier coefficients* are defined analogously to the discrete case (2), by

$$\widehat{f}_j = \langle e_j, f \rangle = \int_G \overline{e_j(t)} f(t) d\mu$$

and in the case, where G is compact,

$$\widehat{f}_j = \langle e_j, f \rangle = \frac{1}{\mu(G)} \int_G \overline{e_j(t)} f(t) d\mu.$$

One of the first questions in Fourier analysis is whether the corresponding inverse formula

$$f(t) = \sum_j \widehat{f}_j e_j(t) \tag{3}$$

is true. Note that the series in the last expression has infinitely many terms.

Example 3.2.13. Continuing on from the last example, one easy case where the inverse formula is correct is that of exponential polynomials. For $G = \mathbb{T}$ let $V_{\mathbb{T}}^{\text{expp}}$ be the subspace of $V_{\mathbb{T}}^{L^1}$ generated by the functions e_j for $j \in \mathbb{Z}$. An element of $V_{\mathbb{T}}^{\text{expp}}$ is called an exponential polynomial. The space $V_{\mathbb{T}}^{\text{expp}}$ is a vector subspace of $V_{\mathbb{T}}^{L^1}$ which is closed under the group action. For $f \in V_{\mathbb{T}}^{\text{expp}}$, f can be recovered from its Fourier coefficients by Equation (3). [16]

Definition 3.2.14. Since G is a topological group and $\mathbb{C}$ has its usual topology, there is a topology on V_G^{cts}. We take the *compact-open topology* on V_G^{cts}, i.e. the topology having the family of sets $O(K,U) = \{\gamma \in \mathbb{C}^G : \gamma(K) \subseteq U\}$ as the base of neighbourhoods of 0, where K is a compact subset of G and U is an open subset of $\mathbb{C}$.

In the case of topological groups, we will be interested in *continuous* 1-dimensional representations $f\colon G \to \mathbb{T}_{\mathbb{C}}$. The standard terminology in this context is to call these *characters*.

Note 3.2.15. The dual group G^* of G is the group of (continuous) characters $f\colon G \to \mathbb{T}_{\mathbb{C}}$. This is topologised using the compact-open topology as before.

Example 3.2.16. $\mathbb{T}^* \cong \mathbb{Z}$, see Morris [23].

The importance of the dual group is that, for general Fourier Analysis over G, we wish to find a 'basis' of our vector space ($V_G^{L^1}$, perhaps) from elements of G^*, in analogy with the finite case, where we could take a basis of 1-dimensional representations. Of course, in the infinite dimensional case, we might expect the Fourier series to have infinitely many terms so this 'basis' is not a basis in the pure sense of vector spaces but has an analytic meaning involving convergence of series.

Notice that $V_{\mathbb{T}}^{\text{expp}}$ is the vector space spanned by $\mathbb{T}^*$. That is, for general G is $\text{span}(G^*)$. Thus each vector is written as a finite linear combination $\sum_{j=0}^{k-1} \lambda_j \chi_j$ where $\lambda_j \in \mathbb{C}$ and χ_j is a character in G^*.

The *convolution* of two functions f and g in V_G is defined to be the integral which expresses the amount of the area overlap between f and g by shifting g over f.

Definition 3.2.17. The *convolution* of functions f and g in V_G is the function $f * g \in V_G$ given by

$$(f * g)(t) = \int_{\tau \in G} f(\tau) g^{\tau}(t) d\mu.$$

If G is compact , then $\mu(G)$ exists and finite. This convolution could be naturally normalised as follows

$$(f * g)(t) = \frac{1}{\mu(G)} \int_{\tau \in G} f(\tau) g^{\tau}(t) d\mu.$$

Also the convolution on V_G, where G is a locally compact abelian group, has the following properties: commutativity; associativity; and linearity, as we mentioned in the last section.

For the proof of commutativity and associativity properties see Katznelson [16], while the proof of linearity follow readily from the basic properties of integration.

In addition, it is easy to check that the space $V_G^{L^1}$ is closed under convolution.

3.3 Fourier Coefficients and Fourier Series

The space $L^1(\mathbb{T})$ is defined to be the space of all equivalence classes with respect to the relation $\sim$ ($f \sim g$ if and only if $f = g$ almost everywhere) of complex-valued Haar integrable functions on $\mathbb{T}$. Given a function $f \in L^1(\mathbb{T})$, the Fourier coefficients, the Fourier series, the ℓth partial sum of the Fourier series and the average of the first $n+1$ partial sums of the Fourier series are defined by Katznelson [16], as follows:

Definition 3.3.1. [16] Let $f \in L^1(\mathbb{T})$. Then the nth *Fourier coefficient* of f is denoted by $\widehat{f}(n)$, and defined as

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{-int} dt, \tag{4}$$

where $n \in \mathbb{Z}$.

Definition 3.3.2. The *Fourier series* $S[f]$ of a function $f \in L^1(\mathbb{T})$ is the trigonometric series

$$S[f] = \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{int} \tag{5}$$

Definition 3.3.3. [37] The ℓth *partial sum* of the Fourier series of a function $f \in L^1(\mathbb{T})$ is denoted by $S_\ell(f,t)$ and defined by

$$S_\ell(f,t) = \sum_{k=-\ell}^{\ell} \widehat{f}(k)e^{ikt}. \tag{6}$$

Definition 3.3.4. [37] The nth *Cesàro mean* (or the *average* of the first $n+1$) of the partial sums of the Fourier series of a function $f \in L^1(\mathbb{T})$, is denoted by $\sigma_n(f,t)$ and defined by

$$\sigma_n(f,t) = \frac{1}{n+1}\sum_{\ell=0}^{n} S_\ell(f,t). \tag{7}$$

The idea is that, under "smoothness" conditions on f, we expect the series (7) in some way approximates the original function f. If we regard the integral in (4) as a limit of summation, the equation (4) formally is very similar to the equation (2) of the definition of the discrete Fourier transformation. The connection is even closer due to the way we chose the coefficients. (This was the reason for taking as data sequences indexed by negative as well as positive indices). We shall see later, using NSA, that the connection between these two is exact when N is a nonstandard *finite integer.

Chapter 4

Nonstandard Representations of Metric Groups

In this chapter we will look at ways in which a topological group G (usually an abelian group with invariant metric) can be represented and studied using a nonstandard *finite group F with a metric on F and a normal subgroup $\mathscr{N}$ of F.

The general set-up starts with the definition of a nice subset $\mathscr{N}$ of F named monotonically definable set which contributes to the metrisation of nonstandard finite groups F. The counting measure of such groups and quotient groups $F/\mathscr{N}$ is defined and it yields some useful results.

We define the notion of locally embeddable into finite metric groups (LEFM) and give results showing that a large number of topological groups can be represented in this way. In the final part of this chapter, we generalise the notion of LEFM to locally embeddable into (uniform) discrete metric groups (LEDM).

4.1 A Metrisation Theorem

The initial object of study is F, a *finite group in the nonstandard universe. (This F is of course internal and well described by nonstandard means.) In the most general set-up, we also have a fixed normal subgroup $\mathscr{N} \lhd F$, where $\mathscr{N}$ need not be internal.

Definition 4.1.1. We define the *standard part map*

$$\mathrm{st}_{\mathscr{N}} : F \to F/\mathscr{N}$$

to be the canonical natural quotient map.

We shall look at conditions on F and $\mathscr{N}$ making $F/\mathscr{N}$ into a topological group or a metric group, and conditions on the definability of $\mathscr{N}$.

We start with conditions relating to the way $\mathscr{N}$ is defined. Of these, the most important by far seems to be the following.

Definition 4.1.2. $\mathscr{N}$ is *monotonically* $\mathbb{N}$*-downwards definable* if there is a monotonic family of internal sets

$$\mathscr{N}_0 \supseteq \mathscr{N}_1 \supseteq \mathscr{N}_2 \supseteq \cdots \supseteq \mathscr{N}_i \supseteq \cdots \qquad (i \in \mathbb{N})$$

such that $\mathscr{N} = \bigcap_{i\in\mathbb{N}} \mathscr{N}_i$.

We will use 'monotonically definable' as an abbreviation for 'monotonically $\mathbb{N}$-downwards definable' through this chapter. Of course, by saturation and overspill, if $\mathscr{N}$ is monotonically definable then there is a monotonic family $(\mathscr{N}_i)_{i<\nu}$ such that $\mathscr{N} = \bigcap_{i\in\mathbb{N}} \mathscr{N}_i = \bigcup_{j>\mathbb{N}} \mathscr{N}_j$.

Example 4.1.3. Given an F-invariant *metric d on F, one can define a subgroup $\mathscr{N}_d$ by

$$\mathscr{N}_d = \{x \in F : d(1,x) \approx 0\}.$$

In this case $\mathscr{N}_d$ is monotonically definable. Indeed, define

$$\mathscr{N}_i = \{x \in F : d(1,x) < 2^{-i}\}$$

then $\mathscr{N}_i$ are internal sets for all $i \in \mathbb{N}$. Furthermore,

$$\mathscr{N}_0 \supseteq \mathscr{N}_1 \supseteq \mathscr{N}_2 \supseteq \cdots \supseteq \mathscr{N}_i \supseteq \cdots \qquad (i \in \mathbb{N})$$

and $\mathscr{N}_d = \bigcap_{i\in\mathbb{N}} \mathscr{N}_i$.

Proposition 4.1.4. If d is a left or right F-invariant *metric then $\mathcal{N}_d$ is a subgroup of F. If d is two-sided F-invariant then $\mathcal{N}_d \lhd F$.

Proof. Assume d is left F-invariant *metric. Then $\mathcal{N}_d \neq \emptyset$, since $1 \in \mathcal{N}_d$. Assume $x, y \in \mathcal{N}_d$. Then $d(1, xy^{-1}) = d(y^{-1}, x^{-1}) \leqslant d(y^{-1}, 1) + d(1, x^{-1}) = d(1, y) + d(x, 1) \approx 0$. Therefore, $\mathcal{N}_d \leqslant F$. Similarly, if d is a right F-invariant *metric then $\mathcal{N}_d \leqslant F$. Assume $h \in \mathcal{N}$ and $g \in F$. Then $d(1, ghg^{-1}) = d(g^{-1}g, h) = d(1, h) \approx 0$. Therefore, $ghg^{-1} \in \mathcal{N}_d$. Hence $\mathcal{N}_d \lhd F$. $\square$

Proposition 4.1.5. If $\mathcal{N}_i$ is defined as above, where $\bigcap_{i \in \mathbb{N}} \mathcal{N}_i = \mathcal{N}_d$ and d is a left-invariant (or right-invariant) metric then $\mathcal{N}_{i+1}^2 \subseteq \mathcal{N}_i$ for each i. Also, $\mathcal{N}_i^{-1} = \mathcal{N}_i$ for each i. If d is two-sided invariant then we also have $\mathcal{N}_i^x = \mathcal{N}_i$ for all i and all $x \in F$.

Lemma 4.1.6. If $\mathcal{N} \lhd F$ is monotonically definable, there is $\nu > \mathbb{N}$ and an internal family of *finite subsets $\mathcal{N}_i$ of F with $\mathcal{N}_i \supseteq \mathcal{N}_{i+1}$ for $0 \leq i < \nu$, and $\bigcap_{i \in \mathbb{N}} \mathcal{N}_i = \mathcal{N}$, satisfying:

(a) $\mathcal{N}_0 = F$ and $\mathcal{N}_\nu = \{1\}$;

(b) $\mathcal{N}_i^{-1} = \mathcal{N}_i$ for all $0 \leq i \leq \nu$;

(c) $\mathcal{N}_i^x = \mathcal{N}_i$ for all $0 < i \leq \nu$ and all $x \in F$.

Proof. By saturation, there is an internal sequence $\mathcal{N}_0 \supseteq \mathcal{N}_1 \supseteq \mathcal{N}_2 \supseteq \cdots \supseteq \mathcal{N}_\nu$ extending this, where $\nu > \mathbb{N}$. It follows that $\mathcal{N}_\alpha \subseteq \mathcal{N}$ for all $\alpha > \mathbb{N}$. The family $\mathcal{N}_i$ with $\mathcal{N} = \bigcap \mathcal{N}_i$ is internal for $i \leqslant \nu$ for some nonstandard ν, and in particular, each $\mathcal{N}_i$ is *finite. By renumbering and replacing $\mathcal{N}_0$ by F and $\mathcal{N}_\nu$ by $\{1\}$ if necessary, we can arrange that (a) holds.

To arrange for (b), we can use an internal induction to define $\mathcal{N}'_\nu = \{1\}$ and

$$\mathcal{N}'_{\nu-(i+1)} = \mathcal{N}_{\nu-(i+1)} \cup \mathcal{N}^{-1}_{\nu-(i+1)} \cup \mathcal{N}'_{\nu-i}.$$

Then clearly $\mathcal{N}'_{i-1} \supseteq \mathcal{N}_{i-1} \cup \mathcal{N}'_i$ for all i. It follows by an internal induction that $\mathcal{N}'_i \supseteq \mathcal{N}'_j$ for all $i \leq j$ and from this (using overspill) that $\bigcap_{i \in \mathbb{N}} \mathcal{N}'_i = \bigcup_{j > \mathbb{N}} \mathcal{N}'_j$. By an easy internal induction each $\mathcal{N}'_i$ is closed under inverses. Moreover, given $1 \neq g \in \mathcal{N}'_n$ for some $n > \mathbb{N}$ there is some i with $g \in \mathcal{N}'_{\nu-(i+1)} \setminus \mathcal{N}'_{\nu-i}$ and $\nu - (i+1) > n$, so $g \in \mathcal{N}_{\nu-(i+1)} \cup \mathcal{N}^{-1}_{\nu-(i+1)} \subseteq \mathcal{N}$, hence $\bigcap_{n \in \mathbb{N}} \mathcal{N}'_n = \bigcup_{n > \mathbb{N}} \mathcal{N}'_n = \mathcal{N}$.

The other is similar. Given an internal family $\mathscr{N}_n'$ satisfying the required properties up to and including (b) define $\mathscr{N}_\nu'' = \{1\}$ and

$$\mathscr{N}''_{\nu-(i+1)} = \bigcup_{x \in F} (\mathscr{N}'_{\nu-(i+1)})^x \cup \mathscr{N}''_{\nu-i}$$

and observe this is internal *finite (since F is), monotonic, and has $\mathscr{N}_i'' \supseteq \mathscr{N}_i'$ for all i. By another internal induction $(\mathscr{N}_i'')^{-1} = \mathscr{N}_i''$ for all i. If $1 \neq g \in \mathscr{N}_n''$ for some $n > \mathbb{N}$ then there is i (with $\nu - i > n$) such that g is in $(\mathscr{N}'_{\nu-(i+1)})^x$ for some x, so $g \in \mathscr{N}$ as required. □

Proposition 4.1.7. If $\mathscr{N} \lhd F$ is monotonically definable, then there is a $\nu > \mathbb{N}$ and an internal family of *finite subsets $\mathscr{N}_i$ of F with $\mathscr{N}_i \supseteq \mathscr{N}_{i+1}$ for $0 \leq i < \nu$, and $\bigcap_{i \in \mathbb{N}} \mathscr{N}_i = \mathscr{N}$, satisfying the properties of Lemma 4.1.6 and also

$$\mathscr{N}_i^2 \subseteq \mathscr{N}_{i-1}$$

for all $0 < i \leq \nu$.

Proof. We take a monotonic decreasing family of subsets $\mathscr{N}_i$, as given by Lemma 4.1.6, so that $\bigcap_{i \in \mathbb{N}} \mathscr{N}_i = \mathscr{N}$ and each $\mathscr{N}_i$ is closed under inverses and conjugation, with $\mathscr{N}_0 = F$ and $\mathscr{N}_\nu = \{1\}$, and clearly we can assume that this sequence is proper, i.e. $\mathscr{N}_\alpha$ properly contains $\mathscr{N}_{\alpha+1}$ for each $\alpha < \nu$. We define inductively a subsequence $\mathscr{N}_{\alpha_i}$ of these. If $\mathscr{N}_{\alpha_i}$ is defined, let $\mathscr{N}_{\alpha_{i+1}}$ be the set $\mathscr{N}_\beta$ with largest index β such that $\mathscr{N}_\beta \supseteq \mathscr{N}^2_{\alpha_i - 1}$. To see that there is always such β, observe that $\mathscr{N}_0 = F$ will work. (Note that the '-1' was introduced here to guard against the case when $\mathscr{N}_{\alpha_i}$ is actually a subgroup of F and to ensure $\alpha_{i+1} < \alpha_i$.)

More interestingly, if $\alpha_i > \mathbb{N}$ then $\mathscr{N}_{\alpha_i - 1} \subseteq \mathscr{N} \lhd F$ so

$$\forall g, h \in \mathscr{N}_{\alpha_i},\ gh \in \mathscr{N}_n$$

holds for all $n \in \mathbb{N}$ since $\mathscr{N} \subseteq \mathscr{N}_n$ and $\mathscr{N}$ is a group. Therefore, by overspill, there is $\beta > \mathbb{N}$ such that $\forall g, h \in \mathscr{N}_{\alpha_i},\ gh \in \mathscr{N}_\beta$ and hence $\alpha_i > \mathbb{N}$ implies $\alpha_{i+1} > \mathbb{N}$. Since the $\mathscr{N}_\alpha$s are closed under conjugation and inverses we may take $F = \mathscr{N}_0' \supseteq \mathscr{N}_1' \supseteq \cdots \supseteq \mathscr{N}_\mu' = \{1\}$ where $\mathscr{N}_i' =$

$\mathscr{N}_{\alpha_{\mu-i}}$ and μ is the length of the subsequence $\mathscr{N}_{\alpha_i}$. We have seen that if $\alpha_i > \mathbb{N}$ then $\alpha_{i+1} > \mathbb{N}$ so that there are nonstandard many sets $\mathscr{N}_j'$ in the subsequence containing $\mathscr{N}_{\alpha_i}$, and it follows that $\bigcap_{j>\mathbb{N}} \mathscr{N}_j = \bigcap_{k>\mathbb{N}} \mathscr{N}_k'$, as required. $\square$

The next result shows that the existence of a monotonically definable $\mathscr{N} \lhd F$ gives rise to an internal *metric d on F for which $\mathscr{N} = \mathscr{N}_d$. Before that, we will prove the following lemma, which we use implicitly in the proof of the next result (metrisation theorem).

Lemma 4.1.8 (Chaining)**.** Assume $F = \mathscr{N}_0 \supseteq \mathscr{N}_1 \supseteq \cdots \supseteq \mathscr{N}_\nu = \{1\}$, where each $\mathscr{N}_i$ is closed under conjugation and inverses, $\mathscr{N}_i^2 \subseteq \mathscr{N}_{i-1}$ for all $i > 0$ and $\nu > \mathbb{N}$. Define the nonstandard real-valued function f on F by $f(x) = 2^{-n}$ if $x \in \mathscr{N}_n \setminus \mathscr{N}_{n+1}$ and $f(1) = 0$. Suppose $x, y \in F$ and $x = x_0, x_1, \ldots, x_n = y$ is a sequence of elements such that $X = \sum_{i=0}^{n-1} f(x_i x_{i+1}^{-1})$. Then there is a sequence $x = y_0, y_1, \ldots, y_k = y$ of length $k \leq \nu$ with $\sum_{i=0}^{k-1} f(y_i y_{i+1}^{-1}) \leq X$.

Proof. Let $x = x_0, x_1, \ldots, x_n = y$ be given with $X = \sum_{i=0}^{n-1} f(x_i x_{i+1}^{-1})$ and suppose n is least possible, so that no shorter sequence $x = y_0, y_1, \ldots, y_k = y$ has $\sum_{i=0}^{k-1} f(y_i y_{i+1}^{-1}) \leq X$. Of course, $x_i \neq x_{i+1}$ for all i, else the sequence may be shortened by omitting one of x_i, x_{i+1}. So no $f(x_i x_{i+1}^{-1})$ is zero.

We use conjugation and $f(a) = f(a^b)$ to show that we can find a monotonic sequence of the same length, $x = y_0, y_1, \ldots, y_n = y$, with the same value $\sum_{i=0}^{n-1} f(y_i y_{i+1}^{-1})$ such that $f(y_i y_{i+1}^{-1}) \leq f(y_{i+1} y_{i+2}^{-1})$ for all i. Consider three consecutive points x_i, x_{i+1}, x_{i+2} in the original sequence and define $w = x_i x_{i+1}^{-1} x_{i+2}$. Then $f(w x_{i+2}^{-1}) = f(x_i x_{i+1}^{-1})$ and $f(x_i w^{-1}) = f(x_i x_{i+2}^{-1} x_{i+1} x_i^{-1}) = f(x_i x_{i+1}^{-1} x_{i+1} x_{i+2}^{-1} x_{i+1} x_i^{-1}) = f(x_{i+1} x_{i+2}^{-1})$. Thus, replacing x_{i+1} with w swaps $f(x_i x_{i+1}^{-1}) = f(w x_{i+2}^{-1})$ and $f(x_{i+1} x_{i+2}^{-1}) = f(x_i w^{-1})$. Applying swaps in this way we achieve the monotonic $x = y_0, y_1, \ldots, y_n = y$ as claimed.

The proof of the lemma is complete if we can show that the values $f(y_i y_{i+1}^{-1})$ are all distinct, since f has at most ν distinct non-zero values. Since $f(y_i y_{i+1}^{-1}) \leq f(y_{i+1} y_{i+2}^{-1})$ for all i, it suffices to show that this inequality is strict.

Because the length n is minimal, we have that, for all $0 \le i \le n-2$,

$$f(y_i y_{i+2}^{-1}) > f(y_i y_{i+1}^{-1}) + f(y_{i+1} y_{i+2}^{-1}),$$

for else replacing y_i, y_{i+1}, y_{i+2} with y_i, y_{i+2} and omitting y_{i+1} would make a shorter sequence with value no more than the original. So if $W = f(y_i y_{i+1}^{-1}) = f(y_{i+1} y_{i+2}^{-1}) = 2^{-w}$ then $f(y_i y_{i+2}^{-1}) > 2W$. On the other hand, since $y_i y_{i+2}^{-1} = (y_i y_{i+1}^{-1})(y_{i+1} y_{i+2}^{-1})$ and both $y_i y_{i+1}^{-1}$, $y_{i+1} y_{i+2}^{-1}$ are in $\mathscr{N}_w$, using $\mathscr{N}_w^2 \subseteq \mathscr{N}_{w-1}$, we obtain $f(y_i y_{i+2}^{-1}) \le 2W$. □

Theorem 4.1.9 (Metrisation)**.** An external normal subgroup $\mathscr{N} \lhd F$ is monotonically definable if and only if it is $\mathscr{N}_d$ for some *metric d on F.

Proof. If $\mathscr{N} = \mathscr{N}_d$ we have seen how to write $\mathscr{N}$ as a limit of a monotonic sequence. Conversely, if $\mathscr{N}$ is monotonically definable then by Proposition 4.1.7 (and taking alternate entries in the resulting sequence of subsets $\mathscr{N}_i$) we have $\mathscr{N} = \bigcap_{i \in \mathbb{N}} \mathscr{N}_i$ where $F = \mathscr{N}_0 \supseteq \cdots \supseteq \mathscr{N}_\nu = \{1\}$, each $\mathscr{N}_i$ is closed under conjugation and inverses, and $\mathscr{N}_i^2 \subseteq \mathscr{N}_{i-1}$ for all $i > 0$. Define the nonstandard real-valued function f on F by $f(x) = 2^{-i}$ if $x \in \mathscr{N}_i \setminus \mathscr{N}_{i+1}$ and $f(1) = 0$.

Now given $x, y \in F$ we define $d(x,y)$ from f by the chaining process. We define

$$d(x,y) = \min \sum_{i=0}^{n-1} f(x_i x_{i+1}^{-1})$$

where the nonstandard internal minimum is over all sequences $x_0, \ldots x_n$ of length $n \le \nu$ for which $x_0 = x$ and $x_n = y$. We have to show that d is a two-sided invariant metric on F and $\mathscr{N} = \mathscr{N}_d$. In the definition of f we have $f(x_i x_{i+1}^{-1}) \geqslant 0$ for all sequences $x = x_0, x_1, \ldots x_n = y$ in F. Then $d(x,y) = \min \sum_{i=0}^{n-1} f(x_i x_{i+1}^{-1}) \geqslant 0$ for all $x, y \in F$. Also $d(x,y) = 0$ if and only if $x = y$. In addition, by the property $\mathscr{N}_i^{-1} = \mathscr{N}_i$ we see $d(x,y) = d(y,x)$ for all $x, y \in F$. Finally, by applying Lemma 4.1.8 the triangle inequality holds as follows. Given $x, y, z \in F$, $d(x,y) = \sum_{i=0}^{n-1} f(x_i x_{i+1}^{-1})$ for some sequence $x = x_0, x_1, \ldots x_n = y$ and $d(y,z) = \sum_{i=0}^{m-1} f(x_{n+i} x_{n+i+1}^{-1})$ for some sequence $y = x_n, x_{n+1}, \ldots x_{n+m} = z$. So

$$\sum_{i=0}^{n-1} f(x_i x_{i+1}^{-1}) + \sum_{i=n-1}^{n+m-1} f(x_i x_{i+1}^{-1}) \geqslant \min\{\sum_{i=0}^{k-1} f(x_i x_{i+1}^{-1}) : x = x_0, x_1, \ldots x_k = z\},$$

that is, $d(x,y)+d(y,z) \leqslant d(x,z)$. Hence the function d is metric on F.

Obviously, d is right invariant, since for given $x_i, x_{i+1}, z \in F$, $f(x_i z(x_{i+1}z)^{-1}) = f(x_i z z^{-1} x_{i+1}^{-1}) = f(x_i x_{i+1}^{-1})$. Therefore $d(x_i z, x_{i+1}z) = d(x_i, x_{i+1})$ for all $x_i, x_{i+1}, z \in F$. Also d is left invariant, if $(zx_i)(zx_{i+1})^{-1} \in \mathscr{N}_{i-1} \setminus \mathscr{N}_i$, by using the property $\mathscr{N}_i^{-1} = \mathscr{N}_i$, $(zx_i(zx_{i+1})^{-1})^{-1} \in \mathscr{N}_{i-1} \setminus \mathscr{N}_i$. So $zx_{i+1}x_i^{-1}z^{-1} \in \mathscr{N}_{i-1} \setminus \mathscr{N}_i$. By using $\mathscr{N}_i^z = \mathscr{N}_i$, we obtain $x_{i+1}x_i^{-1} \in \mathscr{N}_{i-1} \setminus \mathscr{N}_i$. Then apply $\mathscr{N}_i^{-1} = \mathscr{N}_i$, we have $x_i x_{i+1}^{-1} \in \mathscr{N}_{i-1} \setminus \mathscr{N}_i$. Therefore, $f(zx_i(zx_{i+1})^{-1}) = f(x_i x_{i+1}^{-1})$. So $d(zx,zy) = d(x,y)$ for all $x,y,z \in F$.

Now to show that $\mathscr{N} = \mathscr{N}_d$, on one hand given $\mathscr{N} \lhd F$ and $F = \mathscr{N}_0 \supseteq \mathscr{N}_1 \supseteq \cdots \supseteq \mathscr{N}_\nu = \{1\}$ with $\mathscr{N} = \bigcap_{i\in\mathbb{N}} \mathscr{N}_i$ then we conclude that $\mathscr{N}$ is the monad of the identity. On the other hand by the Example 4.1.3 we have $\mathscr{N}_d$ is the monad of the identity 1 as well. Since the monad of each point is unique, so $\mathscr{N}_d = \mathscr{N}$. □

Let $\mathscr{N} = \mathscr{N}_d$ as in Theorem 4.1.9, where d is the *metric on F given above.

Definition 4.1.10. If d is a 2-sided invariant *metric on F and $\mathscr{N}_d$ is defined as above, then the natural induced metric $^\circ d$ on $F/\mathscr{N}$ is given by

$$^\circ d(x\mathscr{N}, y\mathscr{N}) = \operatorname{st} d(x,y).$$

Proposition 4.1.11. If d is a left or right F-invariant *metric, then $^\circ d$ is well-defined on the coset space $F/\mathscr{N}$, making it an $\mathbb{R}$-valued metric. If d is two-sided F-invariant then $^\circ d$ makes $F/\mathscr{N}$ into a topological (metric) group.

Proof. To show $^\circ d$ is well defined on $F/\mathscr{N}$, let $x\mathscr{N}, x'\mathscr{N}, y\mathscr{N}, y'\mathscr{N} \in F/\mathscr{N}$ such that $x\mathscr{N} = x'\mathscr{N}$ and $y\mathscr{N} = y'\mathscr{N}$. Then $xx'^{-1} \in \mathscr{N}$ and $yy'^{-1} \in \mathscr{N}$. So $d(xx'^{-1},1) \approx 0$ and $d(yy'^{-1},1) \approx 0$. Then $d(x,x') \approx 0$ and $d(y,y') \approx 0$. So $d(x,y) \approx d(x',y')$. Therefore $\operatorname{st} d(x,y) = \operatorname{st} d(x',y')$. Hence $^\circ d(x\mathscr{N}, y\mathscr{N}) = {}^\circ d(x'\mathscr{N}, y'\mathscr{N})$.

Moreover, the function $^\circ d$ is a metric on $F/\mathscr{N}$ since: (1) $\operatorname{st} d(x,y) \geqslant 0$ for all $x,y \in F$ then $^\circ d(x\mathscr{N}, y\mathscr{N}) \geqslant 0$ for all $x\mathscr{N}, y\mathscr{N} \in F/\mathscr{N}$; (2) $^\circ d(x\mathscr{N}, y\mathscr{N}) = 0 \Leftrightarrow \operatorname{st} d(x,y) = 0 \Leftrightarrow d(x,y) \approx 0 \Leftrightarrow d(xy^{-1},1) \approx 0 \Leftrightarrow xy^{-1} \in \mathscr{N} \Leftrightarrow x\mathscr{N} = y\mathscr{N}$, for all $x\mathscr{N}, y\mathscr{N} \in F/\mathscr{N}$; (3) $^\circ d(x\mathscr{N}, y\mathscr{N}) = \operatorname{st} d(x,y) = \operatorname{st} d(y,x) = {}^\circ d(y\mathscr{N}, x\mathscr{N})$ for all $x\mathscr{N}, y\mathscr{N} \in F/\mathscr{N}$, (4) $^\circ d(x\mathscr{N}, z\mathscr{N}) = \operatorname{st} d(x,z)$

$\leqslant \mathrm{st}\,(d(x,y)+d(y,z)) \leqslant \mathrm{st}\,d(x,y)+\mathrm{st}\,d(y,z) = {}^{\circ}d(x\mathscr{N},y\mathscr{N})+{}^{\circ}d(y\mathscr{N},z\mathscr{N})$ for all $x\mathscr{N}, y\mathscr{N}$, $z\mathscr{N} \in F/\mathscr{N}$. Therefore $(F/\mathscr{N}, {}^{\circ}d)$ forms a topological (metric) space.

To show that the multiplication function from $F/\mathscr{N} \times F/\mathscr{N}$ into $F/\mathscr{N}$ is continuous, assume $x,y \in F$. Given a standard real $\varepsilon > 0$, for every $m,n \in \mathscr{N}$, we have to find a standard real $\eta > 0$ such that if $d(xm,x) < \eta$ and $d(yn,y) < \eta$ then $d(xmyn,xy) < \varepsilon$. Consider

$$d(xmyn,xy) = d(xy^{m^{-1}}mn,xy) \leqslant d(xy^{m^{-1}}mn,xy^{m^{-1}}) + d(xy^{m^{-1}},xy) \approx 0,$$

where $y^{m^{-1}} = mym^{-1}$. So for all $\eta \approx 0$ if $d(xm,x) < \eta$ and $d(yn,y) < \eta$ then $d(xmyn,xy) < \varepsilon$. By overspill there is an appreciable $\eta > 0$ such that if $d(xm,x) < \eta$ and $d(yn,y) < \eta$ then $d(xmyn,xy) < \varepsilon$. Also since $\mathscr{N}$ is closed under inverse, then it is easy to show that the inverse function from $F/\mathscr{N}$ into $F/\mathscr{N}$ is continuous. Choosing $\eta = \varepsilon$, we obtain that if $d(x^{-1}m,x^{-1}) < \eta$ then $d(xm,x) < \varepsilon$. $\square$

Theorem 4.1.12. Any first-countable Hausdorff topological group G is metrisable with 2-sided invariant metric generating the same topology.

Proof. Let F be the nonstandard version ${}^*G$ of G. The standard part map $\mathrm{st}\colon F \to G$ gives inverse images $N_n = \mathrm{st}^{-1}\,U_n$ of a countable neighbourhood base of 1. By saturation, the subgroup $\mathscr{N} = \bigcap_{n\in\mathbb{N}} N_n$ is monotonically definable so is the d-monad of 1 for some *metric d, and this metric induces a metric ${}^{\circ}d$ on the original G, as required. The details are straightforward. $\square$

4.2 Measure on Nonstandard Sets F and $F/\mathscr{N}$

Definition 4.2.1. Since F is *finite internal, it has a natural *counting measure* which we can normalise to 1 if convenient. For $A \subseteq F$ internal,

$$\mu(A) = \frac{\mathrm{card}\,A}{\mathrm{card}\,F},$$

when we are normalising to 1, or use a different normalisation constant instead of $\mathrm{card}\,F$ if convenient. We will always choose the normalising constant so that $\mu(F) > 0$ possibly an infinite nonstandard number.

The idea is that this should extend naturally to a measure on F using the approach of Loeb (see Section 2.6).

Definition 4.2.2. For an arbitrary $A \subseteq F$ we may say A is *measurable* (with real-valued measure ${}^{\circ}\mu(A)$) if for each standard $\varepsilon > 0$ in $\mathbb{R}$ there are internal $B, C \subseteq F$ such that $B \subseteq A \subseteq C$ and $\mu(C \setminus B) < \varepsilon$ (and ${}^{\circ}\mu(A)$ is defined to be the supremum of $\operatorname{st}\mu(B)$ over such B, or equivalently the infimum of $\operatorname{st}\mu(C)$ over such C).

Proposition 4.2.3. The set of measurable subsets of F forms a σ-algebra on F containing $F, \emptyset$. ${}^{\circ}\mu$ is a (left and right) F-invariant measure on F.

Proof. Let $\mathscr{A}$ be the family of all measurable subsets of F. Obviously, F and $\emptyset$ are internal subsets of F of measure 1 and 0 respectively after normalising. Then $F, \emptyset \in \mathscr{A}$. Assume $B \in \mathscr{A}$. Then for a given standard real $\varepsilon > 0$ there are internal subsets $A, C \subseteq F$ such that $A \subset B \subset C$ with $\mu(C) - \mu(A) < \varepsilon$. Furthermore, A^c and C^c are internal subsets of F satisfying $C^c \subseteq B^c \subseteq A^c$ and $\mu(A^c) - \mu(C^c) < \varepsilon$, that is, B^c is measurable. Therefore $B^c \in \mathscr{A}$. Suppose $(B_i)_{i \in \mathbb{N}}$ is a sequence of measurable subsets in $\mathscr{A}$. There are two sequences $(A_i)_{i \in \mathbb{N}}$ and $(C_i)_{i \in \mathbb{N}}$ of internal sets in $\mathscr{A}$ such that $A_i \subseteq B_i \subseteq C_i$ with $\mu(C_i) - \mu(A_i) < \frac{\varepsilon}{2^{i+2}}$ for all i. We have to show that $B = \bigcup_{i \in \mathbb{N}} B_i$ is in $\mathscr{A}$. By saturation there are two internal sequences of measurable subsets $(A'_i)_{0 \leqslant i \leqslant \nu}$ and $(C'_i)_{0 \leqslant i \leqslant \nu}$ of internal subsets of F, where $\nu > \mathbb{N}$ such that $A'_i = A_i$ and $C'_i = C_i$ for all $i \in \mathbb{N}$. Let $\lim\limits_{n \to \infty} {}^{\circ}\mu(\bigcup_{i=0}^{n} A_i) = a$. Since F has a finite measure, a is finite. There is a standard $k \in \mathbb{N}$, such that $\mu(\bigcup_{i=0}^{k} A_i) > a - \frac{\varepsilon}{2}$. Let $A := \bigcup_{i=0}^{k} A_i$. Notice that $\mathscr{A}$, μ and $(C'_i)_{0 \leqslant i \leqslant \nu}$ are internal. So we can define the internal set $S := \{\ell \in {}^*\mathbb{N} : \bigcup_{i=0}^{\ell} C'_i \in \mathscr{A} \text{ and } \mu(\bigcup_{i=0}^{\ell} C'_i) < a + \frac{\varepsilon}{2}\}$.

Therefore, $\mu(\bigcup_{i=0}^{\ell} C'_i) < \mu(\bigcup_{i=0}^{\ell} A_i) + \sum_{i=0}^{\ell} \frac{\varepsilon}{2^{i+2}} < a + \frac{\varepsilon}{2}$ for all $\ell \in \mathbb{N}$. By overspill, there is $\omega > \mathbb{N}$ such that $C := \bigcup_{i=0}^{\omega} C_i$ and $A \subseteq B \subseteq C$ with $\mu(C) - \mu(A) < \varepsilon$. So $B = \bigcup_{i \in \mathbb{N}} B_i \in \mathscr{A}$. Hence $\mathscr{A}$ is a σ-algebra on F. To prove that ${}^{\circ}\mu$ is a left and right F-invariant measure on F, we note that

$$\operatorname{card} A = \operatorname{card} xA = \operatorname{card} Ax$$

for every $A \subseteq F$ and every $x \in F$. So

$$\mu(A) = \frac{\operatorname{card} A}{\operatorname{card} F} = \mu(xA) = \mu(Ax).$$

Therefore ${}^\circ\mu(A) = {}^\circ\mu(xA) = {}^\circ\mu(Ax)$. □

To define measure on $F/\mathscr{N}$ we need to take a little care, as there is no sensible notion of 'internal subset' of $F/\mathscr{N}$. However, by pulling back to F and noting that cosets of $\mathscr{N}$ are simply subsets of F, we can achieve something similar.

Definition 4.2.4. For $A \subseteq F/\mathscr{N}$, we say A is measurable if $\bigcup A \subseteq F$ is measurable and the measure of A is defined to be

$${}^\circ\mu(A) = {}^\circ\mu(\bigcup A).$$

(Some care needs to be taken with the overloading of the notation ${}^\circ\mu$ as both are notating a measure on F and a measure on $F/\mathscr{N}$.)

Proposition 4.2.5. Assume $\mu(F) \gtrapprox 0$ in ${}^*\mathbb{R}_{\text{fin}}$. If the normal subgroup $\mathscr{N}$ is measurable (as a subset of F) then ${}^\circ\mu(\mathscr{N}) = 0$ if and only if $\mathscr{N}$ has infinite index in F.

Proof. Assume $\mathscr{N}$ is a measurable subset of F and ${}^\circ\mu(\mathscr{N}) = 0$. Assume $[F : \mathscr{N}] = n$. Then for all $\varepsilon \gtrapprox 0$ there is an internal set A such that $A \supseteq \mathscr{N}$ and $\mu(A) < \varepsilon$. Note that $F = \bigcup_{i=1}^n a_i\mathscr{N}$, for some $a_i \in F$. Then $F \subseteq \bigcup_{i=1}^n a_iA$. So

$$\mu(F) \leqslant \sum_{i=1}^{n} \mu(a_iA) = n\mu(A) < n\varepsilon,$$

since $\mu(A_i) = \mu(A_j)$ for all i, j. Choose $\varepsilon = 1/(n+1)$. Then it contradicts $\mu(F) = 1$.

Conversely, suppose $[F : \mathscr{N}]$ is not finite. Also suppose $\mathscr{N}$ is measurable and $\mu(\mathscr{N}) \gtrapprox 0$. Then $[F : \mathscr{N}] > n$ for all $n \in \mathbb{N}$. Choose $\varepsilon \gtrapprox 0$ and an internal set A such that $A \subset \mathscr{N}$ with $\mu(A) > \varepsilon$. Then

$$\mu(A) < \mu(\mathscr{N}) = \frac{\mu(F)}{[F : \mathscr{N}]} < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus we get $\mu(A) \approx 0$ which is a contradiction. So ${}^\circ\mu(\mathscr{N}) = 0$. □

We are usually interested in the case where $\mathscr{N}$ is measurable of infinite index in F.

Proposition 4.2.6. If $\mathscr{N}$ is measurable of infinite index in F then the family of measurable subsets E of $F/\mathscr{N}$ is a σ-algebra and $^\circ\mu$ is an $F/\mathscr{N}$-invariant measure on $F/\mathscr{N}$.

Proof. Since $\bigcup(F/\mathscr{N}) = F$ and F is a measurable set, $\bigcup(F/\mathscr{N})$ is measurable with $^\circ\mu(F/\mathscr{N})$ $= {}^\circ\mu(\bigcup(F/\mathscr{N})) = {}^\circ\mu(F) = 1$ (after normalising to 1). Moreover, $\emptyset \subseteq F/\mathscr{N}$ is a measurable set of measure 0. Assume $E \subseteq F/\mathscr{N}$ is a measurable set. Then $\bigcup E \subseteq F$ is measurable. So $F \setminus (\bigcup E) = \bigcap E^c$ is a measurable set in F. Hence E^c is measurable in $F/\mathscr{N}$. Let $E_0, E_1, E_2, \ldots$ are measurable subsets of $F/\mathscr{N}$. Then $\bigcup E_n \subseteq F$ is measurable for each $n \in \mathbb{N}$.

Since F is closed under the countable union of measurable sets then $\bigcup_{n\in\mathbb{N}}(\bigcup E_n) \subseteq F$ is measurable. That is, $\bigcup(\bigcup_{n\in\mathbb{N}} E_n)$ is measurable in F. Therefore $\bigcup_{n\in\mathbb{N}} E_n \subseteq F/\mathscr{N}$ is a measurable set. Hence the set of measurable $E \subseteq F/\mathscr{N}$ is a σ-algebra.

To prove $^\circ\mu$ is an $F/\mathscr{N}$-invariant measure on $F/\mathscr{N}$ we have from the invariance property of $^\circ\mu$ on F, $^\circ\mu(\bigcup E) = {}^\circ\mu(\bigcup x\mathscr{N}E)$ for every measurable set $E \subseteq F/\mathscr{N}$ and every $x\mathscr{N} \in F/\mathscr{N}$. Therefore $^\circ\mu(E)) = {}^\circ\mu(x\mathscr{N}E)$. Hence, $^\circ\mu$ is $F/\mathscr{N}$-invariant measure on $F/\mathscr{N}$. $\square$

For the rest of this section, we suppose $\mathscr{N} \lhd F$ is measurable of infinite index and $F/\mathscr{N}$ as a group with an invariant measure $^\circ\mu$.

Proposition 4.2.8 below is a "taste" of the kind of results that can be obtained by measure-theoretic considerations.

The following lemma has no doubt been discovered many times, but was worked out by Kaye and Reading for a related problem in nonstandard finite groups and appears as Lemma 4.2.5 in Reading's MPhil thesis [28].

Lemma 4.2.7. Suppose F is a finite group and $\Sigma, \Delta \subseteq F$ have card $\Sigma = m$ and card $\Delta = n$, and suppose $k \in \mathbb{N}$ and $k < nm/\text{card } F$. Then there is $x \in F$ such that $\text{card}(x\Delta \cap \Sigma) \geq k+1$.

Proposition 4.2.8. Let F be a *finite group, $\mathscr{N} \lhd F$ and $0 < \varepsilon < \eta \leq 1$ be real numbers. Then there is $k \in \mathbb{N}$, a measurable set $A \subseteq F/\mathscr{N}$ with $\varepsilon < {}^\circ\mu A < \eta$ and elements $a_0, \ldots, a_{k-1} \in F$ such that $F/\mathscr{N} = \bigcup_{i<k} a_i A$. More specifically, this can be achieved whenever $(1-\varepsilon)^k < \eta - \varepsilon$.

Proof. Let $B \subseteq F$ be an internal *finite set with card $B = n$ for some $n > \mathbb{N}$ such that $\varepsilon \lesssim n/N \lesssim \eta$ where $N = \text{card}\, F$. Then there is some large k with $(1-\varepsilon)^k < \eta - \varepsilon$. By iterating Lemma 4.2.7 we need

$$\frac{N}{n}\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + 1\right) < \frac{N}{n}(\ln n + 1)$$

copies of B to cover the group F except the proportion $(1-\varepsilon)^k$, by k translates of B. Choose $b_0, \ldots, b_{k-1} \in F$ such that $\text{card}(\bigcup_{i<k}(b_i B)) \geqslant N(1-(1-\varepsilon)^k)$. Let $D := \bigcup_{i<k}(b_i B)$ and $C := F \backslash D$. Then the remaining part of the group F is C where C has measure smaller than $\eta - \varepsilon$ since card $C \leqslant N(1-\varepsilon)^k < N(\eta - \varepsilon)$. So $\text{card}(B \cup C) \lesssim N(\varepsilon + \eta - \varepsilon) = N\eta$. Now we can cover the whole F by k translates of $B \cup C$, that is, $F = (\bigcup_{i<k} b_i B) \cup C$ and $\varepsilon \leqslant {}^\circ\mu(B \cup C) \leqslant \eta$. Let A be the quotient for $B \cup C$. By pulling back every thing to the quotient group $F/\mathscr{N}$ using the fact that $\mathscr{N}$ has a measure 0, the proposition is proved. □

4.3 Locally Embeddable into Finite Metric Groups (LEFM)

An abstract group is embeddable in a *finite group if and only if it is *locally embeddable into finite groups* (*LEF*). These groups were studied in Pestov and Kwiatkowska's article [26], which is a good introduction to these matters. The work in Section 4.1 motivates a modification of this idea given in the following definition.

Definition 4.3.1. A metric group $(G, \cdot, d)$ with a 2-sided invariant metric d is *locally embeddable into finite metric groups* (*LEFM*) if it is embeddable as a metric group (via an injective group homomorphism which is an isometry) into some $(\prod_D G_i)/\mathscr{N}$ where i is from an index set I, D is a nonprincipal ultrafilter on I, $(G_i, \cdot, d_i)_{i \in I}$ is a family of finite metric groups with 2-sided invariant metrics d_i, and

$$\mathscr{N} = \{(g_i)_{i \in I} \in \prod_D G_i : \forall \varepsilon > 0,\ \{i : d_i(g_i, 1) < \varepsilon\} \in D\}.$$

In other words, we take an ultraproduct of groups, and then factor this by the normal subgroup $\mathscr{N} = \mathscr{N}_d = \{x \in \prod_D G_i : d(x, 1) \approx 0\}$, the monad of 1, as given earlier in Example 4.1.3.

Theorem 4.3.2. A 2-sided metric group $(G,\cdot,d)$ embeds as a metric group into some $(\prod_D G_i)/\mathscr{N}$ as above if and only if for all $\varepsilon > 0$ and all finite subsets $A \subseteq G$ there is a finite 2-sided metric group H and a function $\phi\colon A \to H$ such that:

(a) $d(\phi(ab),\phi(a)\phi(b)) < \varepsilon$ whenever $a,b,ab \in A$;

(b) $|d(\phi(a),\phi(b)) - d(a,b)| < \varepsilon$ whenever $a,b \in A$.

Proof. Without loss of generality assume that the 2-sided metric group G is infinite. Let I be the set of all finite subsets A of G. For each $A \in I$, let $\varepsilon_A = \varepsilon = 2^{-|A|}$ and let G_A be a finite metric group with function $\phi_A = \phi\colon A \to G_A$ satisfying (a) and (b). Let D_0 be the filter on $\mathscr{P}(I)$ generated by all sets

$$U_A = \{B \in I : A \subseteq B\}.$$

Since

$$U_{A_1} \cap U_{A_2} \cap \ldots \cap U_{A_k} \subseteq U_{A_1 \cup A_2 \cup \ldots \cup A_k} \neq \emptyset,$$

we see that D_0 is nontrivial and so extends to an ultrafilter D. This ultrafilter D is nonprincipal as G is infinite. We must show that G embeds as a metric group into ${}^*G/\mathscr{N} = (\prod_D G_i)/\mathscr{N}$.

Let $f \in G$. For each $A \subseteq G$ finite with $f \in A$, let $f_A = \phi_A(f) \in G_A$, and for each $A \subseteq G$ with $f \notin A$, let $f_A = 1 \in G_A$. We map f to $\phi(f) = (\ldots, f_A, \ldots)/_D \in {}^*G$. Suppose $f \neq g \in G$, then $d(f,g) = \eta > 0$; let $A \supseteq \{f,g\}$ have size greater than $-\log_2(\eta) + 1$. Then, for all $B \in U_A$, from (b) we have $|d_B(\phi_B(f),\phi_B(g)) - d(f,g)| < \eta/2$ and so $\phi(f) \neq \phi(g)$, as $U_A \in D$. Or, with the same argument but replacing A with some $A' \supseteq \{f,g\}$ of size as large as we wish, we can see that $|d_B(\phi_B(f),\phi_B(g)) - d(f,g)| < \varepsilon$ for any $\varepsilon > 0$, so that $d(\phi(f),\phi(g)) \approx d(f,g)$ in the ultraproduct. Thus ϕ maps G into ${}^*G$ and factors by $\mathscr{N}$ as an isometry $\phi/\mathscr{N}\colon G \to {}^*G/\mathscr{N}$. The fact that this isometry $\phi/\mathscr{N}$ is also a group homomorphism follows by a similar argument using part (a), which shows that $\phi(fg) \approx \phi(f)\phi(g)$, $\phi(1) \approx 1$ and $\phi(f)^{-1} \approx \phi(f^{-1})$ for all $f,g \in G$.

Conversely, assume $\widehat{\psi}\colon G \to (\prod_D G_i/_D)/\mathscr{N}$ is an embedding of 2-sided metric groups, where each G_i is a finite 2-sided metric group and $\prod_D G_i$ is a *finite object in the universe ${}^*V = \prod_D V_i$, an elementary extension of the standard universe V. (This is to enable us to use Łoś's theorem for statements concerning ‘finiteness’.)

Given $\varepsilon > 0$ standard and $A \subseteq G$ finite, we have to show that there is a finite 2-sided metric group H satisfying (a) and (b). Suppose $A = \{a_0, a_1, \ldots, a_{n-1}\}$. Since n is finite we can use the finitely many elements of A as parameters. The embedding $\widehat{\psi}$ shows that

$$\widehat{d}(\widehat{\psi}(a_i a_j), \widehat{\psi}(a_i)\widehat{\psi}(a_j)) = 0 \text{ for all } a_i, a_j, a_i a_j \in A, \tag{1}$$

$$|\widehat{d}(\widehat{\psi}(a_i), \widehat{\psi}(a_j)) - d_{a_i a_j}| = 0 \text{ for all } a_i, a_j \in A. \tag{2}$$

From the definition of the metric $\widehat{d}$ on $(\prod_D G_i)/\mathscr{N}$ and the definition of $\mathscr{N}$, we can choose some function $\psi\colon G \to \prod_{i\in\mathbb{N}} G_i/_D$ such that $\widehat{\psi}(g) = \psi(g)\mathscr{N}$. (Once again, there is no way we can expect ψ to be a homomorphism.)

Thus, for all $a_i, a_j \in A$, we have $\widehat{\psi}(a_i) = \psi(a_i)\mathscr{N}$, $\widehat{\psi}(a_j) = \psi(a_j)\mathscr{N}$ and $\widehat{\psi}(a_i a_j) = \psi(a_i a_j)\mathscr{N}$. So

$$^*d(\psi(a_i a_j), \psi(a_i)\psi(a_j)) \approx 0 \tag{3}$$

$$|^*d(\psi(a_i), \psi(a_j)) - d_{a_i a_j}| \approx 0 \tag{4}$$

where $^*d\colon \prod_D G_i \times \prod_D G_i \to {}^*\mathbb{R}$ is defined by $^*d(a,b) = (d_i(a_i, b_i))/_D$ for all $a, b \in \prod_D G_i$ and is a *metric taking values in $^*\mathbb{R}$.

Thus from (3), (4) and $\varepsilon > 0$ standard we obtain

$$^*d(\psi(a_i a_j), \psi(a_i)\psi(a_j)) < \varepsilon \text{ for all } a_i, a_j, a_i a_j \in A$$

$$|^*d(\psi(a_i), \psi(a_j)) - d_{a_i a_j}| < \varepsilon \text{ for all } a_i, a_j \in A.$$

Thus our nonstandard universe $^*V$ satisfies the sentence “There is a finite 2-sided metric group

$H = \prod_D G_i$ and there are elements $\psi(a_0)$, $\psi(a_1), \ldots, \psi(a_{n-1})$ in H such that

$$^*d(\psi(a_i a_j), \psi(a_i)\psi(a_j)) < \varepsilon$$
$$|^*d(\psi(a_i), \psi(a_j)) - d_{a_i a_j}| < \varepsilon$$

for all $a_i, a_j, a_i a_j \in A = \{a_0, a_1, \ldots, a_{n-1}\}$".

This is a first order statement with parameters $d_{a_i a_j}$ for all $a_i, a_j \in A$. There are finitely many such parameters, all them in V.

Since the finite 2-sided metric group H and $\psi(a_0), \psi(a_1), \ldots, \psi(a_{n-1})$ are quantified out, by the Transfer Principle this statement is true in V. That is, there is a finite 2-sided metric group H and a function $\phi\colon A \to H$ satisfying (a) and (b), as required. $\square$

Corollary 4.3.3. A metric group $(G, \cdot, d)$ is LEFM if and only if every finitely generated subgroup of $(G, \cdot, d)$ is LEFM.

Definition 4.3.4. Let G and H be two groups and $A \subseteq G$. An injective function $f\colon A \to H$ is said to be a *partial homomorphism* on A if for all $x, y, z \in A$

$$x \cdot y = z \Rightarrow f(x) \cdot f(y) = f(z).$$

We now look to proving some abelian metric groups to be LEFM, and for this reason we switch notation throughout the remainder of this section to additive notation on abelian groups. Some familiar groups can be shown to be LEFM by direct construction. These include all countable subgroups of the additive group of the reals $\mathbb{R}$ and the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with their usual metrics. In Theorem 4.3.2 on LEFM groups, it clearly suffices to show that every finitely generated abelian metric group F is LEFM. Our proof will use the well-known structure theorem (see Theorem 1.2.11 in Section 1.2) for finitely generated abstract abelian groups.

Definition 4.3.5. Given an abelian group G and a nonempty subset $A \subseteq G$ closed under negation ($x \in A \to -x \in A$), a function $\|\ \|\colon A \to \mathbb{R}$ is a *partial norm* if $\|a\| \geq 0$ for all $a \in A$ with equality if and only if $a = 0$, and

$$\sum_i \|a_i\| \geq \|a\|$$

whenever $a, a_0, \ldots, a_{n-1} \in A$ and $a = \sum_i a_i$.

Proposition 4.3.6. Given an abelian group G, a nonempty finite subset $A \subseteq G$ closed under negation, and a partial norm $\| \ \|$ on A, there is an invariant metric d on $\langle A \rangle$ such that $\|a\| = d(a, 0)$ for all $a \in A$.

Proof. The canonical choice for d is

$$d(u, v) = \min \sum_i \|u_{i+1} - u_i\|$$

where the minimum is over all $u_0, \ldots, u_n$ such that $u_0 = u$, $u_n = v$ and each $u_{i+1} - u_i \in A$. The checking is straightforward. □

Theorem 4.3.7. Every abelian group with invariant metric is LEFM.

Proof. It suffices to show that every finitely generated abelian metric group is LEFM. By the structure theorem (Theorem 1.2.11), such a group F is of the form $\mathbb{Z}^k \oplus T$ where T is finite. Write elements of F as $\sum_{i=1}^k \lambda_i e_i + t$ where e_i is a generator for the ith $\mathbb{Z}$ and $t \in T$. A finite subset of F is contained in some

$$A = \left\{ \sum_{i=1}^k \lambda_i e_i + t : |\lambda_i| < M \text{ and } t \in T \right\},$$

where $M \in \mathbb{N}$.

We shall map $\phi : A \to \phi[A] \subseteq F_A = C_K^k \oplus T$ for suitably chosen large $K \in \mathbb{N}$. This map is given by $\phi(\sum_{i=1}^k \lambda_i e_i + t) = \sum_{i=1}^k \lambda_i f_i + t$ where $f_i = 1$ is the generator of the ith C_K. This object is given the partial norm $\|\phi(a)\|$ equal to the value $\|a\|$ in F. We show that, for a sufficiently large value of K, this is a partial norm and so it extends to a metric according to Proposition 4.3.6.

For $x \in \{0, 1, \ldots, K-1\} = C_K$, let $N(x) = \min\{x, K-1-x\}$ be its usual norm and $\operatorname{sign}(x) = 0$ if $x = 0$, $\operatorname{sign}(x) = +1$ if $0 < x < K/2$ and $\operatorname{sign}(x) = -1$ if $K/2 \leqslant x < K$. Let $B \subseteq F_A = C_K^k \oplus T$

defined by

$$B = \Big\{ \sum_i x_i f_i + t \in C_K^k \oplus T : N(x_i) < K/3 \text{ for all } i \Big\}.$$

Then there is a map $\psi\colon B \to \mathbb{Z}^k \oplus T$ given by

$$\psi(x_1 + \cdots + x_k + t) = \sum_i \operatorname{sign}(x_i) N(x_i) e_i + t.$$

By the use of $K/3$ in the definition of B, this is a partial homomorphism: if $u, v, w \in B$ and $u + v = w$ then $\psi(u) + \psi(v) = \psi(w)$. By arranging that $K > 3M$ we can ensure $\phi[A] \subseteq B$.

Let $\varepsilon_A = \min\{\|a\| : a \in A, a \neq 0\}$ and $E_A = \max\{\|a\| : a \in A\}$. Suppose that $K > 3ME_A/\varepsilon_A$. We consider an expression of the form $\sum_{j=0}^{r-1} \phi(a_j) = \phi(a)$ true in F_A of elements $\phi(a_j), \phi(a) \in \phi[A]$. If each partial sum $\sum_{j=0}^{s-1} \phi(a_j)$, for $s < r$, is in B then the partial homomorphism ψ applies to show that $\sum_{j=0}^{r-1} a_j = a$ and hence $\sum_{j=0}^{r-1} \|a_j\| \geq \|a\|$. So if $\sum_{j=0}^{r-1} \|a_j\| < \|a\|$ then some partial sum $\sum_{j=0}^{s-1} \phi(a_j)$ with $s < r$ is not in B. Since $N(\phi(a_j)) < M$ for all j, it follows that $sM \geq K/3$ and hence $r \geq K/(3M) > E_A/\varepsilon_A$. Thus $\sum_{j=0}^{r-1} \|a_j\| \geq r\varepsilon_A > E_A \geq \|a\|$. Therefore, for any $K > 3ME_A/\varepsilon_A$, the inequality $\sum_{j=0}^{r-1} \|a_j\| < \|a\|$ is impossible in F_A, and ϕ is a partial homomorphism and isometry, as required. $\square$

Rather than looking at examples that arise directly from metric ultraproducts, it is more interesting to show certain classical examples are LEFM groups directly. We present such examples in Chapter 5.

In the following theorem we notice that separable LEFM groups are compact under surjective embedding of metric groups.

Theorem 4.3.8. Assume $(G, \cdot, d_G)$ is a 2-sided invariant metric group and $(F, \cdot, d_F)$ is a *finite *metric group such that $\phi\colon G \to F/\mathcal{N}$ is an onto embedding of metric groups, where $\mathcal{N} \lhd F$ is the monad of the identity. If G is separable, then G is compact.

Proof. Assume G is separable. Then G has a countable base. Suppose $\{U_i : i \in \mathbb{N}\}$ is an open cover of G consisting of basic open sets U_i of G, for all $i \in \mathbb{N}$. Without loss of generality, assume $U_i = B_{\varepsilon_i}(g_i)$, where $B_{\varepsilon_i}(g_i)$ are open balls with centre $g_i \in G$ and radius standard real $\varepsilon_i > 0$, for all $i \in \mathbb{N}$. Take $f_i \in F$, such that $\phi(g_i) = f_i\mathcal{N}$. By saturation, the internal sequence $f_0, f_1, \ldots, f_\nu$ exists for $\nu > \mathbb{N}$.

Assume $F \neq \bigcup_{i<n} B_{\varepsilon_i}(f_i)$ for all standard $n \in \mathbb{N}$. Then by overspill there exists $\ell > \mathbb{N}$ such that $F \neq \bigcup_{i<\ell} B_{\varepsilon_i}(f_i)$. This means that there exists $c \in F \setminus \bigcup_{i<\ell} B_{\varepsilon_i}(f_i)$. That is, there exists $c\mathcal{N} \notin Im(\phi)$, contradicting the surjectivity of ϕ. Hence $G \subseteq \bigcup_{i<n} B_{\varepsilon_i}(g_i)$ for some standard $n \in \mathbb{N}$. □

4.4 Locally Embeddable into Discrete Metric Groups (LEDM)

When an invariant metric group is not, or is not known to be, LEFM, a modified weaker notion may be helpful.

Definition 4.4.1. A metric space (X, d) is said to be *discrete* if for all $x \in X$ there is a standard real $\varepsilon > 0$ such that the open ball $B_\varepsilon(x) = \{x\}$.

Definition 4.4.2. A metric space (X, d) is said to be *uniformly discrete* if there exists a standard real $\varepsilon > 0$ such that for all $x \in X$, $B_\varepsilon(x) = \{x\}$.

Definition 4.4.3. A 2-sided metric group $(G, \cdot, d)$ is *locally embeddable into uniformly discrete metric groups* (LEDM) if G embeds in $F/\mathcal{N}$, where F is a nonstandard invariant metric group that is uniformly discrete in the sense of the nonstandard universe; for example, an ultraproduct of standard uniformly discrete metric groups, and $\mathcal{N}$ is again the monad of the neutral element of F.

An easy variation of Theorem 4.3.2 gives the following.

Theorem 4.4.4. A 2-sided metric group $(G, \cdot, d)$ is LEDM if and only if, for all $\varepsilon > 0$ and all finite subsets $A \subseteq G$, there is a 2-sided metric group H which is uniformly discrete and a function $\phi : A \to H$ such that:

(a) $d(\phi(ab), \phi(a)\phi(b)) < \varepsilon$, whenever $a, b, ab \in A$;

(b) $|d(\phi(a), \phi(b)) - d(a, b)| < \varepsilon$, whenever $a, b \in A$.

Proof. The proof is similar to the proof of Theorem 4.3.2. □

Theorem 4.4.5. Every 2-sided metric group is LEDM.

Proof. Let $(G,\cdot,d_G)$ be a 2-sided metric group and $A \subseteq G$ is finite. Without loss of generality assume A contains the neutral element 1 and is closed under inverse. Define d_A on $\langle A\rangle$ by

$$d_A(x,y) = \min\Big\{ \sum_{i=1}^{n} d_G(1,a_i) : a_i \in A, \text{ for all } i = 1,2,\ldots,n, xa_1a_2\ldots a_n = y \text{ and } n \in \mathbb{N}\Big\}.$$

Since A is finite, there is some $a \in A$ of least positive distance $d_A(a,1)$. Then d_A is well-defined on $\langle A\rangle$. In addition, d_A is a metric on $\langle A\rangle$: (1) since $\sum_{i=1}^{n} d_G(1,a_i) \geqslant 0$, $d_A(x,y) \geqslant 0$ for all $x,y \in \langle A\rangle$, where $xa_1a_2\ldots a_n = y$; (2) if $d_A(x,y) = 0$, then $\sum_{i=1}^{n} d_G(1,a_i) = 0$ for some $n \in \mathbb{N}$. So $d_G(a_i,1) = 0$ for all i. That is, $a_i = 1$ for all i. Thus, $x = y$. Conversely, assume $x = y$. Then $a_i = 1$ for all i. So, $\min\big\{ \sum_{i=1}^{n} d_G(1,a_i) : a_i \in A, \text{ for all } i = 1,2,\ldots,n, xa_1a_2\ldots a_n = y \text{ and } n \in \mathbb{N}\big\} = 0$. That is, $d_A(x,y) = 0$; (3) if $xa_1a_2\ldots a_n = y$ then $ya_n^{-1}a_{n-1}^{-1}\ldots a_1^{-1} = x$. Also if $a_i \in A$ then $a_i^{-1} \in A$ and $d_G(a_i,1) = d_G(a_i^{-1},1)$. So $d_A(x,y) = d_A(y,x)$ for all $x,y \in \langle A\rangle$; (4) for all $x,y,z \in \langle A\rangle$, it is clear that $d_A(x,y) + d_A(y,z) \geqslant d_A(x,z)$.

To show that d_A is a 2-sided invariant metric on $\langle A\rangle$, let $zxa_1a_2\ldots a_n = zy$. Then $xa_1a_2\ldots a_n = y$. Therefore, $d_A(zx,zy) = d_A(x,y)$. If $xza_1a_2\ldots a_n = yz$, then by conjugation, we obtain that, $xz(z^{-1}a_1z)(z^{-1}a_2z)\ldots(z^{-1}a_nz) = yz$. So $xa_1a_2\ldots a_nz = yz$. Then $xa_1a_2\ldots a_n = y$. Therefore, $d_A(xz,yz) = d_A(x,y)$.

Notice that if $\min\{d_G(1,a) : a \in A, a \neq 1\} = \varepsilon$ then $\sum_{i=0}^{n} d_G(1,a_i) \geqslant n\varepsilon$. Therefore the metric d_A is uniformly discrete on $\langle A\rangle$. Then for each $x \in \langle A\rangle$, $B_\varepsilon(x) = \{x\}$. Hence $\langle A\rangle$ is a uniformly discrete metric group. Obviously, the identity function $\phi : A \to \langle A\rangle$ satisfies both conditions

(a) $d(\phi(ab),\phi(a)\phi(b)) < \varepsilon$, whenever $a,b,ab \in A$, and

(b) $|d(\phi(a),\phi(b)) - d(a,b)| < \varepsilon$, whenever $a,b \in A$.

By Theorem 4.4.4, G is LEDM. □

This theorem allows one to present the theory of 2-sided metric groups in general using discrete nonstandard groups.

Proposition 4.4.6. Let a 2-sided metric group G be LEDM via the function $\phi\colon G \to D/\mathcal{N}$, where D is a *discrete metric group and $\mathcal{N}$ is the monad of the identity of D. If $X \subseteq D_G$ is internal, where $D_G = \{x \in D : \exists g \in G \text{ such that } \phi(g) = x\mathcal{N}\}$, then $X/\mathcal{N}$ is bounded and closed.

Proof. Given $X \subseteq D_G$ is an internal set, assume $X/\mathscr{N}$ is not bounded. Then, for each $k \in \mathbb{N}$ there is $x_k \in X$, such that $d(x_k\mathscr{N}, 1\mathscr{N}) > k+1$ which implies $d(x_k, 1) \gtrapprox k$. By saturation, $x_0, x_1, \ldots$ is coded, that is, there is an internal sequence $x_1, x_2, \ldots, x_\alpha$, where $\alpha > \mathbb{N}$, agreeing on finite indices $i \in \mathbb{N}$. By overspill, there is $\nu > \mathbb{N}$ such that

$$\forall n \leqslant \nu (x_n \in X \wedge d(x_n, 1) > n).$$

Therefore, $d(x_\nu\mathscr{N}, 1\mathscr{N}) > \nu$, which contradicts the fact that d is a real-valued function on $D/\mathscr{N}$. Hence $X/\mathscr{N}$ is bounded.

Assume that $X/\mathscr{N}$ is not closed. There exists a cluster point $u\mathscr{N}$ in $D_G/\mathscr{N}$ with $u\mathscr{N} \notin X/\mathscr{N}$. Thus, for each $k \in \mathbb{N}$, there is $x_k \in X$ such that $d(u\mathscr{N}, x_k\mathscr{N}) < 1/k$. By saturation, $x_1, x_2, \ldots, x_\alpha$ coded in X, where $\alpha > \mathbb{N}$. By overspill, there is $\nu > \mathbb{N}$, such that $x_n \in X$ and $d(u, x_n) < 1/n$ for all $n \leqslant \nu$. This implies that $d(u\mathscr{N}, x_\nu\mathscr{N}) = 0$. So $u\mathscr{N} = x_\nu\mathscr{N}$. Since $x_\nu \in X$, we have $x_\nu\mathscr{N} \in X/\mathscr{N}$, which is a contradiction. Hence $X/\mathscr{N}$ is closed. □

Proposition 4.4.7. Let a 2-sided metric group G be LEDM via the function $\phi \colon G \to D/\mathscr{N}$, where D is *discrete and $\mathscr{N}$ is the monad of the identity. If G is separable and $X \subseteq D_G$ is internal, then $X/\mathscr{N}$ is compact.

Proof. Suppose that $X/\mathscr{N}$ is not compact. Without loss of generality, assume that the countable family $\{U_i : i \in \mathbb{N}\}$ is an open cover of $X/\mathscr{N}$ and $U_i = B_{\varepsilon_i}(g_i\mathscr{N})$, where $\varepsilon_i = 1/i$, for $i = 1, 2, \ldots$, and $g_i \in D_G$. Assume for each $k \in \mathbb{N}$ there is $x_k \in X$ such that $x_k\mathscr{N} \notin \bigcup_{i=1}^{k} B_{\varepsilon_i}(g_i\mathscr{N})$. That is, $d(x_k, g_i) \gtrsim \varepsilon_i$ for all $i \leqslant k$. By saturation, we can encode $x_1, x_2 \ldots, x_\alpha$ in X, encode $g_1, g_2 \ldots, g_\alpha$ in D, where $g_1, g_2 \ldots, g_k$ are all in D_G for all standard $k \in \mathbb{N}$ in X and encode the positive numbers $\varepsilon_1, \varepsilon_2 \ldots, \varepsilon_\alpha$ in ${}^*\mathbb{R}$ where $\alpha > \mathbb{N}$. By overspill, there exists $\nu > \mathbb{N}$, such that $x_\nu \in X$ and $x_\nu\mathscr{N} \notin B_{\varepsilon_i}(g_i\mathscr{N})$ for all $i \leqslant \nu$ which means that $\{U_i : i \in \mathbb{N}\}$ is not cover of $X/\mathscr{N}$, which contradicts the assumption. Hence $X/\mathscr{N}$ is compact. □

Chapter 5

Some Examples of LEFM Groups

Our motivation throughout this chapter is to present some interesting examples of classical abelian and nonabelian metric groups G represented by the quotient metric group of nonstandard *finite groups F factored by the monad $\mathscr{N}$ of the neutral element of F, to illustrate the use of nonstandard methods in Chapter 4.

The key theorems of Chapter 4 (Theorems 4.3.2 and 4.3.7) showed that such groups can be represented using a nonstandard finite group F but did not give further information on F.

The measure on these examples is also discussed. In particular, the usual counting measure on F, normalised in whatever way appropriate, can be used in the Loeb style to give a measure on $F/\mathscr{N}$ and hence on G, for example, as given by Lindstrøm in the volume edited by Cutland [4, pages 1–105]. We expect these ideas to be profitable.

5.1 Abelian LEFM Group Examples

The goal of this section is to show how most natural classical examples arise as LEFM groups, where the nonstandard *finite group F is cyclic. As all examples in this section will be abelian, we shall use additive notation throughout. We denote the cyclic group of order N as C_N and, where necessary, list its elements as $\lfloor -\frac{N}{2} \rfloor + 1, \lfloor -\frac{N}{2} \rfloor + 2, \ldots, 0, \ldots, \lfloor \frac{N}{2} \rfloor$. The usual metric on C_N is given by $d_N(x,y) = \min\{|x-y|, N-|x-y|\}$ and it corresponds to the number of steps from x to y in either a clockwise or anticlockwise direction. However, we will have to modify this metric (for example by dividing through by a constant) or replace it altogether in most of the examples that follow.

5.1.1 The Group of Integers $\mathbb{Z}$

The group of integers $\mathbb{Z}$ with the usual metric $d_{\mathbb{Z}}(x,y) = |x-y|$ is LEFM. Consider the *finite *cyclic group C_N with addition modulo N, where $N > \mathbb{N}$. We give C_N the usual metric $d(x,y) = \min\{|x-y|, N-|x-y|\}$; of course, d is an invariant metric on C_N. With this metric, the monad of 0 as given in Section 4.1 is

$$\mathcal{N} = \mathcal{N}_d = \{x \in C_N : d(x,0) \approx 0\} = \{0\}.$$

Therefore, $C_N/\mathcal{N} \cong C_N$ as abstract groups and $C_N/\mathcal{N}$ is *finite *cyclic as well. The function $\phi : \mathbb{Z} \to C_N/\mathcal{N}$ defined by

$$\phi(x) = x + \mathcal{N}$$

is easily seen to be well-defined, 1-1 and ϕ is a group homomorphism. In addition, ϕ is an isometry.

If $\mathbb{Z}$ is given with the discrete metric $d(x,y) = 1$ for $x \neq y$ then we equip C_N with the same discrete metric and argue as before.

For the measure μ on C_N, we take the discrete measure $\mu(A) = \operatorname{card} A$ with a mass of 1 to each point. In other words, we use Definition 4.2.1 without normalisation by card C_N.

The measure on $\mathbb{Z}$, that we expect, is also the counting measure and the measurable sets should be precisely the finite sets. We will show that Definition 4.2.4 gives precisely this measure on $C_N/\mathcal{N}$.

According to Definition 4.2.2, a set $A \subseteq C_N$ is measurable if and only if for all $\varepsilon > 0$ there are *finite B, C such that $B \subseteq A \subseteq C$ and $\mu(C \setminus B) < \varepsilon$. By taking $\varepsilon = 1/2$ clearly this means we must have $B = A = C$, so a set $A \subseteq C_N$ is measurable if and only if it is *finite. Now suppose $A \subseteq \mathbb{Z}$ is measurable. Then it is *finite and hence finite (Section 2.3, Proposition 2.3.4). It follows that $A \subseteq \mathbb{Z}$ is measurable if and only if it is actually finite, and hence the measure of A as given in Definition 4.2.2 is card A.

5.1.2 The Direct Sum of the Group of Integers $\mathbb{Z} \oplus \mathbb{Z}$

The additive group of the direct sum $\mathbb{Z} \oplus \mathbb{Z}$ with discrete metric is LEFM. Take the discrete metric on $C_N = \{0, 1, 2, \ldots, N-1\}$, where $N > \mathbb{N}$. Obviously, d is invariant on C_N. The monad of 0, as defined in Section 4.1, is

$$\mathscr{N} = \mathscr{N}_d = \{x \in C_N : d(x,0) \approx 0\} = \{0\}.$$

Assume $k > \mathbb{N}$ with $kn < N$ for all $n \in \mathbb{N}$ (Proposition 2.3.5). We may now embed $\mathbb{Z} \oplus \mathbb{Z}$ into $C_N/\mathscr{N}$ via the function $\phi : \mathbb{Z} \oplus \mathbb{Z} \to C_N/\mathscr{N}$ defined by

$$\phi(x,y) = x + ky + \mathscr{N},$$

using the same enumeration of C_N.

The function ϕ is well-defined since $x + ky \in C_N$ for all $x, y \in \mathbb{Z}$. Obviously, ϕ is a group homomorphism. Furthermore, ϕ is 1-1. To see this, suppose $(x_1, y_1) \neq (x_2, y_2)$. Then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$, then for any values of $y_1, y_2 \in \mathbb{Z}$, $x_1 + ky_1 + \mathscr{N} \neq x_2 + ky_2 + \mathscr{N}$ since $|x_1 - x_2| < k$. Also, if $y_1 \neq y_2$ then for any values of $x_1, x_2 \in \mathbb{Z}$, $x_1 + ky_1 + \mathscr{N} \neq x_2 + ky_2 + \mathscr{N}$ since $|x_1 - x_2| < k$ and $k|y_1 - y_2| < N$. So $\phi(x_1, y_1) \neq \phi(x_2, y_2)$.

Also, one can easily check that ϕ is an isometry.

The measure μ defined on $\mathbb{Z} \oplus \mathbb{Z}$ is the usual counting measure.

The measure μ defined on $C_N/\mathscr{N}$ is the usual counting measure as well. In order to describe μ on the quotient set $C_N/\mathscr{N}$ we use both Definitions 4.2.4 and 4.2.2 and the same process as we applied on $\mathbb{Z}$ (in Section 5.1.1). Hence the embedding $\phi : \mathbb{Z} \oplus \mathbb{Z} \to C_N/\mathscr{N}$ preserves measurability.

5.1.3 The Additive Real Group $\mathbb{R}$

Consider the additive real group $\mathbb{R}$ with the usual metric $d_{\mathbb{R}}$ and the nonstandard *cyclic group C_N. Again take $k > \mathbb{N}$ with $nk < N$ for all finite n. Equip C_N with the usual metric normalised by k, that is,

$$d(x,y) = \frac{d_N(x,y)}{k}.$$

Clearly, d is C_N-invariant. We may say that an element $x \in C_N$ is *finite* if $d(x,0)$ is finite (as an element of ${}^*\mathbb{R}$). The set of all finite elements of C_N is denoted by C_N^{fin}, that is,

$$C_N^{\text{fin}} = \{x \in C_N : d(x,0) \text{ is finite}\} = \{x \in C_N : d_N(x,0) \leqslant kn \text{ for some } n \in \mathbb{N}\}.$$

In fact C_N^{fin} is a subgroup of C_N. For each $x \in C_N^{\text{fin}}$, the standard part $\operatorname{st}(x/k) \in \mathbb{R}$ in the usual sense, and the function $\psi\colon C_N^{\text{fin}} \to \mathbb{R}$ defined by $\psi(x) = \operatorname{st}(x/k)$ is a homomorphism. By saturation (Section 2.8), it is onto with kernel $\mathscr{N} \subseteq C_N$. Define the function $\phi\colon C_N^{\text{fin}}/\mathscr{N} \to \mathbb{R}$ by $\phi(x+\mathscr{N}) = \psi(x)$. Thus, by the first isomorphism theorem, $C_N^{\text{fin}}/\mathscr{N} \cong \mathbb{R}$ and it is easy to check that the canonical isomorphism is also an isometry, as

$$d_{\mathbb{R}}(\phi(x+\mathscr{N}), \phi(y+\mathscr{N})) = |\phi(x+\mathscr{N}) - \phi(y+\mathscr{N})| = |\operatorname{st}(x/k) - \operatorname{st}(y/k)| = \operatorname{st} d(x,y).$$

Hence, the metric group $\mathbb{R}$ embeds into the metric group $C_N^{\text{fin}}/\mathscr{N}$.

Assume $I = [a,b] \subseteq \mathbb{R}$. Then

$$\begin{aligned}
\psi^{-1}([a,b]) = \cup(\phi^{-1}([a,b])) &= \{x+\mathscr{N} \in C_N^{\text{fin}}/\mathscr{N} : \phi(x+\mathscr{N}) \in [a,b]\} \\
&= \{x \in C_N^{\text{fin}} : a \leqslant \operatorname{st}(x/k) \leqslant b\} \\
&= \{x \in C_N : ak \lesssim x \lesssim bk\}.
\end{aligned}$$

Consider the sets

$$A_{u,v} = \{x \in C_N : u \leqslant x \leqslant v\}$$

with $ak \leqslant u$ and $v \leqslant bk$, and

$$B_{u',v'} = \{x \in C_N : u' \leqslant x \leqslant v'\}$$

with $u' \leqslant ak$ and $bk \leqslant v'$. Then $\mu(A_{u,v}) = (v-u)/k$ and $\mu(B_{u',v'}) = (v'-u')/k$.

Therefore, the Loeb measure is $L(\mu)(\psi^{-1}[a,b]) = b-a$. So for all standard real $\varepsilon > 0$, there are internal subsets $A_{u,v}$ and $B_{u',v'}$ of C_N such that $A_{u,v} \subseteq \psi^{-1}(I) \subseteq B_{u',v'}$ with $\mu(B_{u',v'} \setminus A_{u,v}) < \varepsilon$.

By standard results on Loeb measure, the measure μ on $\mathbb{R}$, as defined in Section 4.2, is σ-additive. Therefore, every Lebesgue measurable set $A \subseteq \mathbb{R}$ has μ-measure and this measure is $\lambda(A)$.

On the other hand, the measure μ defined on C_N, is a normalised measure of the form $\mu(A) = \operatorname{card} A/k$ for every internal subset A of C_N.

Therefore, the embedding $\phi^{-1}\colon \mathbb{R} \to C_N^{\mathrm{fin}}/\mathscr{N}$ is measure-preserving.

5.1.4 The Additive Circle Group $\mathbb{T}$

The additive circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is identified with $[0,2\pi)$ as given in Definition 1.2.2 and is given the metric $d_{\mathbb{T}}$ defined by

$$d_{\mathbb{T}}(x,y) = \frac{1}{2\pi}\min\{|x-y|, 2\pi - |x-y|\}.$$

It is LEFM. Indeed, consider the nonstandard cyclic group $C_N = \{0,1,2,\ldots,N-1\}$ with the normalised *metric d defined as follows

$$d(x,y) = \frac{1}{N}\min\{|x-y|, N - |x-y|\}.$$

The monad of 0 in C_N is

$$\mathscr{N} = \mathscr{N}_d = \{n, N-1-n : kn < N \text{ for all } k \in \mathbb{N}\}.$$

The function $\phi\colon \mathbb{T} \to C_N/\mathscr{N}$ defined by

$$\phi(x) = \left\lfloor \frac{Nx}{2\pi} \right\rfloor + \mathscr{N}$$

is well-defined. Moreover, ϕ is 1-1. Suppose $x, y \in [0, 2\pi)$ such that $x \neq y$. Let $0 \leqslant x < y < 2\pi$. Then $y - x > 0$. By Archimedean property [8], there is $n \in \mathbb{N}$ such that $1/n < y - x$. Then $N/(2\pi n) < Ny/(2\pi) - Nx/(2\pi)$. Therefore, $\left\lfloor \frac{Nx}{2\pi} \right\rfloor$ and $\left\lfloor \frac{Ny}{2\pi} \right\rfloor$ are in distinct $\mathscr{N}$-cosets. So $\left\lfloor \frac{Nx}{2\pi} \right\rfloor + \mathscr{N} \neq \left\lfloor \frac{Ny}{2\pi} \right\rfloor + \mathscr{N}$. That is, $\phi(x) \neq \phi(y)$.

Obviously, ϕ is a group homomorphism since for all $x, y \in \mathbb{T}$,

$$\phi(x) + \phi(y) = \left\lfloor \frac{Nx}{2\pi} \right\rfloor + \mathscr{N} + \left\lfloor \frac{Ny}{2\pi} \right\rfloor + \mathscr{N} = \left\lfloor \frac{N(x+y)}{2\pi} \right\rfloor + \mathscr{N} = \phi(x+y)$$

due to $|\lfloor x \rfloor + \lfloor y \rfloor - \lfloor x - y \rfloor| \leqslant 1$, $1 \in \mathscr{N}$ and $k(1) < N$ for all $k \in \mathbb{N}$.

By Definition 4.1.10 of the *metric $d_{\mathscr{N}}$ on $C_N/\mathscr{N}$ and the definition of $d_{\mathbb{T}}$ on $\mathbb{T}$, we notice that ϕ is an isometry since

$$\begin{aligned} d_{\mathscr{N}}(\phi(x), \phi(y)) &= d_{\mathscr{N}}(\lfloor \frac{Nx}{2\pi} \rfloor + \mathscr{N}, \lfloor \frac{Ny}{2\pi} \rfloor + \mathscr{N}) \\ &= \mathrm{st}(d(\lfloor \frac{Nx}{2\pi} \rfloor, \lfloor \frac{Ny}{2\pi} \rfloor)) \\ &= \frac{1}{2\pi} \min\{|x-y|, 2\pi - |x-y|\} \\ &= d_{\mathbb{T}}(x, y). \end{aligned}$$

Hence $(\mathbb{T}, +, d_{\mathbb{T}})$ embeds into $(C_N/\mathscr{N}, +, d_{\mathscr{N}})$ as a metric group (Section 4.3). Actually, ϕ is onto, so $\mathbb{T} \cong C_N/\mathscr{N}$ as metric groups. Thus the image of $\mathbb{T}$ is the whole of $C_N/\mathscr{N}$.

The measure μ defined on $\mathbb{T} = [0, 2\pi)$ is the Lebesgue measure λ and the measure μ on the quotient set $C_N/\mathscr{N}$ is the Lebesgue measure λ as well. One can use the same process, as we applied in the previous section (Section 5.1.3), to prove our claim above about measurability of these sets. Hence the embedding $\phi : \mathbb{T} \to C_N/\mathscr{N}$ is measure-preserving.

5.1.5 The Additive 2-Torus of the Circles $\mathbb{T} \oplus \mathbb{T}$

The direct sum $\mathbb{T} \oplus \mathbb{T}$ is the torus which we endow with the 'taxicab' metric

$$d_{\mathbb{T} \oplus \mathbb{T}}((x, y), (x', y')) = d_{\mathbb{T}}(x, x') + d_{\mathbb{T}}(y, y').$$

This is clearly LEFM, by 5.1.4 and a direct sum. More interestingly, we can embed $\mathbb{T}\oplus\mathbb{T}$ into $C_N/\mathscr{N}$ for an appropriate choice of N and the metric on C_N.

Suppose $N = M(M+1)$, where M is nonstandard. We list the elements of C_N as $0,1,\dots,N-1$; similarly for C_M and C_{M+1}. The groups C_M and C_{M+1} are given normalised metrics $d_M(x,y)/M$ and $d_{M+1}(x,y)/(M+1)$ and $C_M\oplus C_{M+1}$ is given the corresponding 'taxicab' metric as follows

$$d((x,y),(x',y')) = d_M(x,x')/M + d_{M+1}(y,y')/(M+1).$$

Therefore, the monad of $(0,0)$ in $C_M\oplus C_{M+1}$ is

$$\mathscr{N}_d = \mathscr{N} = \{(x,y)\in C_M\oplus C_{M+1} : d((x,y),(0,0))\approx 0\}.$$

Now consider $a = (1,1)\in C_M\oplus C_{M+1}$. This is easily seen to generate $C_M\oplus C_{M+1}$ since $(M+1)a = (1,0)$. We identify $x\in C_N$ with $xa\in C_M\oplus C_{M+1}$ and define the metric d on C_N from the metric on $C_M\oplus C_{M+1}$ via this identification.

Define $\phi\colon \mathbb{T}\oplus\mathbb{T}\to C_N/\mathscr{N}$ by

$$\phi(x,y) = (\lfloor\frac{Mx}{2\pi}\rfloor, \lfloor\frac{(M+1)y}{2\pi}\rfloor) + \mathscr{N}.$$

It is now straightforward to see that $\mathbb{T}\oplus\mathbb{T}$ with the 'taxicab' metric is isomorphic to $C_N/\mathscr{N} = C_M\oplus C_{M+1}/\mathscr{N}$ with the induced metric, where $\mathscr{N}$ is the monad of $(0,0)\in C_M\oplus C_{M+1}$.

If required, other metrics are possible. For example, if $\mathbb{T}\oplus\mathbb{T}$ is given the metric corresponding to the distance on the surface of the torus, or the distance in 3-space corresponding to an embedding of $\mathbb{T}\oplus\mathbb{T}$ in $\mathbb{R}^3$, a modified metric on $C_M\oplus C_{M+1}$ could be given. Also, by similar arguments, $\mathbb{T}^k$ for any finite k is isomorphic to $C_N/\mathscr{N}$ for suitably chosen N and a metric on C_N.

By using the same process as given in Section 5.1.3, one can show that the measure μ on both sets $\mathbb{T}\oplus\mathbb{T} = [0,2\pi)\oplus[0,2\pi)$ and $C_N/\mathscr{N}$ is the Lebesgue measure λ. Therefore, the embedding $\phi'\colon C_N/\mathscr{N}\to\mathbb{T}\oplus\mathbb{T}$ preserves measurability.

5.1.6 The Additive Complex Group $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$

In a similar way we can embed the additive group $\mathbb{R} \oplus \mathbb{R}$ with the usual Euclidean metric into the quotient group $C_N/\mathscr{N}$ for some *metric on C_N. One way is to choose K nonstandard and $N = K(K+1)$ with C_N identified with $C_K \oplus C_{K+1}$ as before. Each of C_K and C_{K+1} is given a metric similar to that in Section 5.1.3 and the metric on $C_K \oplus C_{K+1}$ is given in the usual way by

$$d((u,v),(u',v')) = \sqrt{(u-u')^2/K^2 + (v-v')^2/(K+1)^2}.$$

Define $C_K^{\text{fin}} \oplus C_{K+1}^{\text{fin}} = \{(x,y) \in C_K \oplus C_{K+1} : d((x,y),(0,0)) \text{ is finite}\}$. The monad of $(0,0)$ is

$$\mathscr{N}_d = \mathscr{N} = \{(x,y) \in C_K \oplus C_{K+1} : d((x,y),(0,0)) \approx 0\}.$$

Now define $\phi \colon C_N \to \mathbb{R} \oplus \mathbb{R}$ by

$$\phi(x,y) = (\text{st}(x/K), \text{st}(y/(K+1))).$$

According to the first isomorphism theorem, $C_N/\ker(\phi) \cong \mathbb{R} \oplus \mathbb{R}$. But $\ker(\phi) = \mathscr{N}$. Therefore, the function $\phi' := \phi/\mathscr{N} : C_N/\mathscr{N} \to \mathbb{R} \oplus \mathbb{R}$ defined by

$$\phi'((x,y) + \mathscr{N}) = (\text{st}(x/K), \text{st}(y/(K+1)))$$

is an isomorphism. Also by taking the metric $d((u,v),(u',v')) = \sqrt{(u-u')^2 + (v-v')^2}$ on $\mathbb{R} \oplus \mathbb{R}$ we obtain that ϕ' is an isometry.

The same process, as described in Section 5.1.3, can be applied to show that the measure μ on the sets $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}$ and $C_N/\mathscr{N}$ is the Lebesgue measure λ. So the embedding $\phi \colon C_N/\mathscr{N} \to \mathbb{R} \oplus \mathbb{R}$ is measure-preserving.

5.1.7 The Additive Group of p-adic Integers $\mathbb{Z}_p$

The additive group of p-adic integers $\mathbb{Z}_p$ with the metric d_p defined by

$$d_p(x_0+x_1p+\cdots,y_0+y_1p+\cdots)=\begin{cases}2^{-k} & \text{if } k\in\mathbb{N} \text{ is the least such that } x_k\neq y_k,\\ 0 & \text{if } x_k=y_k \text{ for all } k\in\mathbb{N},\end{cases}$$

is LEFM.

Consider the *finite *cyclic additive group C_{p^N}, where p is standard prime and $N>\mathbb{N}$, with the *metric d defined in the same way as d_p. Define $\phi\colon C_{p^N}\to\mathbb{Z}_p$ by 'forgetting' the nonstandard elements:

$$\phi(x_0+x_1p+x_2p^2+\cdots+x_{N-1}p^{N-1})=x_0+x_1p+x_2p^2+\cdots.$$

Then ϕ is a homomorphism and the kernel of ϕ is the monad $\mathscr{N}$ of 0 in C_{p^N} since

$$\ker(\phi)=\{x_0+x_1p+\ldots+x_{N-1}p^{N-1}\in C_{p^N}:x_0+x_1p+x_2p^2+\ldots=0\}=\mathscr{N}.$$

Also, by saturation (Example 2.8.7), ϕ is onto.

Define $\phi'\colon C_{p^N}/\mathscr{N}\to\mathbb{Z}_p$ by $\phi'(x+\mathscr{N})=\phi(x)$. It follows by the first isomorphism theorem that $C_{p^N}/\mathscr{N}\cong\mathbb{Z}_p$ and an easy check shows that the induced isomorphism is an isometry.

In order to define the measure μ on the set of p-adic integers $\mathbb{Z}_p$, where p is a standard prime number, fix $k\in\mathbb{N}$. The basic open (which are closed and compact) sets of $\mathbb{Z}_p$ of length k are of the form

$$U_{z_0,z_1,\ldots,z_{k-1}}=\{w_0+w_1p+w_2p^2+\ldots\in\mathbb{Z}_p:w_i=z_i \text{ for all } 0\leqslant i<k\},$$

where $z_i\in\mathbb{Z}$ and $0\leqslant z_i<p$ for all $i\leqslant 0<k$. Then the measure μ of such a basic open set is

$$\mu(U_{z_0,z_1,\ldots,z_{k-1}})=p^{-k}.$$

If $\mathscr{B}$ is the family of all Borel sets generated by these basic open subsets of $\mathbb{Z}_p$, then $\mathscr{B}$ is the smallest σ-algebra containing such basic open sets. Thus the measure $\mu\colon \mathscr{B} \to \mathbb{R}$ has the following properties: $0 \leqslant \mu(B) \leqslant 1$ for all $B \in \mathscr{B}$ and particularly $\mu(\mathbb{Z}_p) = 1$; if $(B_i)_{i\in\mathbb{N}}$ is a countable family of pairwise disjoint Borel sets then $\mu(\bigcup_{i\in\mathbb{N}} B_i) = \sum_{i\in\mathbb{N}} \mu(B_i)$; for all $B \in \mathscr{B}$ and all $x \in \mathbb{Z}_p$, $\mu(xB) = \mu(Bx) = \mu(B)$; also, for each $B \in \mathscr{B}$ and each $\varepsilon > 0$, there is a closed set A and an open set C such that $A \subseteq B \subseteq C$ with $\mu(C \setminus A) < \varepsilon$.

So by viewing $\mathbb{Z}_p$, as a (locally) compact Hausdorff topological abelian group, we notice that $\mathbb{Z}_p$ has Haar measure μ as defined in Section 3.2. Also μ on $\mathbb{Z}_p$ is unique up to multiplication by a positive constant. The following steps show that $\mathbb{Z}_p$ and $C_{p^N}/\mathscr{N}$ have the measure μ which is the Haar measure.

Assume $I = U_{z_0,z_1,\ldots,z_{k-1}} \subseteq \mathbb{Z}_p$. Given a standard real $\varepsilon > 0$,

$$\begin{aligned}\phi^{-1}(U_{z_0,z_1,\ldots,z_{k-1}}) = \cup\phi'^{-1}(U_{z_0,z_1,\ldots,z_{k-1}}) &= \{w + \mathscr{N} \in C_{p^N}/\mathscr{N} : \phi'(w + \mathscr{N}) \in U_{z_0,z_1,\ldots,z_{k-1}}\} \\ &= \{w \in C_{p^N} : w_0 + w_1 p + w_2 p^2 + \ldots \in U_{z_0,z_1,\ldots,z_{k-1}}\}.\end{aligned}$$

Now consider the sets

$$A_{u,v} = \{w_0 + w_1 p + \ldots + w_{N-1} p^{N-1} \in C_{p^N} : u \leqslant w_0 + w_1 p + w_2 p^2 + \ldots \leqslant v\},$$

where $u = a + \varepsilon/4$ and $v = b - \varepsilon/4$, and

$$B_{u',v'} = \{w_0 + w_1 p + \ldots + w_{N-1} p^{N-1} \in C_{p^N} : u' \leqslant w_0 + w_1 p + w_2 p^2 + \ldots \leqslant v'\},$$

where $u' = a$ and $v' = b$. Notice that $A_{u,v} \subseteq \phi^{-1}(I) \subseteq B_{u',v'}$ with $\mu(B_{u',v'} \setminus A_{u,v}) = \varepsilon/2 < \varepsilon$.

By the standard results on Loeb measure, the measure μ on $\mathbb{Z}_p$, as defined in Section 4.2, is σ-additive. Therefore, every Haar measurable set $A \subseteq \mathbb{Z}_p$ has μ-measure and this measure is $\mu(A)$.

Hence, the embedding $\phi'\colon C_{p^N}/\mathscr{N} \to \mathbb{Z}_p$ is measure-preserving.

5.1.8 The Profinite Completion $\widehat{\mathbb{Z}}$ of $\mathbb{Z}$

The profinite completion of $\mathbb{Z}$, as an additive group, can be regarded as the set of sequences $(x_0, x_1, x_2, \ldots)$ with $x_n \in \mathbb{Z}/n\mathbb{Z}$ such that $x_n \equiv x_{nm} \bmod n$ for all $n, m \in \mathbb{Z}$. Its metric, $\widehat{d}$, is given by

$$\widehat{d}((x_0, x_1, x_2, \ldots), (y_0, y_1, y_2, \ldots)) = \begin{cases} 2^{-k} & \text{if } k \in \mathbb{N} \text{ is the least such that } x_k \neq y_k, \\ 0 & \text{if } x_k = y_k \text{ for all } k \in \mathbb{N}. \end{cases}$$

The metric group $(\widehat{\mathbb{Z}}, +, \widehat{d})$ is LEFM. Consider the *finite *cyclic additive group $C_{N!}$, where $N! = N(N-1)(N-2)\ldots(3)(2)(1)$. Also, consider the metric d given by $d(x,y) = 2^{-k}$, where k is the greatest, such that $x \equiv y \bmod n$, for all $n < k$. Define the function $\psi \colon C_{N!} \to \widehat{\mathbb{Z}}$ by

$$\psi(x) = (x_0, x_1, x_2, \ldots) \text{ such that } x \equiv x_n \bmod n \text{ for all } n \in \mathbb{N}.$$

Notice that

$$\ker(\psi) = \{x \in C_{N!} : (x_0, x_1, x_2, \ldots) = 0 \text{ and } x \equiv x_n \bmod n \text{ for all } n \in \mathbb{N}\} = \mathscr{N}.$$

Therefore, by the first isomorphism theorem $C_{N!}/\mathscr{N} \cong \widehat{\mathbb{Z}}$ and an easy check shows that the induced isomorphism is an isometry.

The measure μ defined on $\widehat{\mathbb{Z}}$ is the Haar measure and the measure μ on the quotient set $C_{N!}/\mathscr{N}$ is the Haar measure as well. One can use the same process, as we utilised in the previous section (Section 5.1.7), to prove the above claim about measurability of these sets. Hence, the embedding $\psi' \colon C_{N!}/\mathscr{N} \to \widehat{\mathbb{Z}}$ is measure-preserving as well.

Furthermore, according to Theorem 4.3.8, the metric space $(\widehat{\mathbb{Z}}, \widehat{d})$ is compact.

5.1.9 The Direct Sum of Finite Cyclic Groups $C_2 \oplus C_2$

The group $C_2 \oplus C_2$ with the discrete metric d is LEFM. This is obvious from the fact that it is finite. Our argument here will show that $C_2 \oplus C_2$ embeds into $C_N/\mathscr{N}$ for suitable N and $\mathscr{N}$.

Let N be a nonstandard prime, and take a nonstandard $\mathbb{N} < a$ such that $a^2 < N$. We define $\mathscr{N}$ by

$$\mathscr{N} = \{2\kappa + 4a\lambda : |\kappa|, |\lambda| < a/n \text{ for some } n > \mathbb{N}\}.$$

Notice that this is monotonically definable, since it is the intersection of all

$$\mathscr{N}_n = \{2\kappa + 4a\lambda : |\kappa|, |\lambda| < a/(n+1)\},$$

for $n \in \mathbb{N}$. Therefore, by the metrisation theorem (Theorem 4.1.9), $\mathscr{N}$ is the monad of 0 for some invariant *metric on C_N. By mapping

$$\kappa + 2a\lambda \mapsto (\kappa \bmod 2, \lambda \bmod 2) \in C_2 \oplus C_2,$$

we see that $C_2 \oplus C_2 \cong \mathscr{K}/\mathscr{N}$ where $\mathscr{K}$ is the subgroup of C_N consisting of all $\kappa + 2a\lambda$ with $|\kappa|, |\lambda| < a/n$ for some $n > \mathbb{N}$. Therefore, $C_2 \oplus C_2$ (with some discrete metric) embeds in $C_N/\mathscr{N}$.

Similar arguments apply to any finite abelian group, using the basis theorem, see for example Rotman [30, Chapter 4].

In all the examples here, we succeeded in giving a representation of the group using only the group $F = C_N$, cyclic of nonstandard finite order. We wonder whether all abelian groups have such a representation using C_N.

5.2 Nonabelian LEFM Group Examples

For a nonabelian example of an LEFM group, we observe that the dihedral groups with suitable metrics are interesting examples for this case.

5.2.1 The Dihedral Circle Group $D_{2\mathbb{T}}$

The dihedral circle group $D_{2\mathbb{T}}$ is LEFM. Consider the dihedral circle group $D_{2\mathbb{T}} := \mathbb{T} \rtimes C_2$ with metric d as given in Definition 1.2.5 and the *dihedral *metric group D_{2N} of order $2N > \mathbb{N}$ with the metric d as given in Definition 1.2.4. Define $\psi\colon D_{2N} \to D_{2\mathbb{T}}$ by

$$\psi(c,i) = (\mathrm{st}(\frac{2\pi c}{N}), i).$$

Then ψ is a well-defined 1-1 function. In addition, ψ is a group homomorphism. Let (c,i) and (c',i') be any two elements of D_{2N}. So:

(1) if $i = 0 = i'$ then $\psi((c,0)(c',0)) = \psi(c+c',0) = (\mathrm{st}(\frac{2\pi(c+c')}{N}),0) = (\mathrm{st}(\frac{2\pi c}{N}) + \mathrm{st}(\frac{2\pi c'}{N}),0)$ $= \psi(c,0)\psi(c',0)$;

(2) if $i = 0$ and $i' = 1$ then $\psi((c,0)(c',1)) = \psi(c+c',0+1) = (\mathrm{st}(\frac{2\pi(c+c')}{N}),0+1) = (\mathrm{st}(\frac{2\pi c}{N}) +$ $\mathrm{st}(\frac{2\pi c'}{N}),0+1) = \psi(c,0)\psi(c',1)$;

(3) if $i = 1$ and $i' = 0$ then $\psi((c,1)(c',0)) = \psi(c+N-c',1+0) = (\mathrm{st}(\frac{2\pi(c+N-c')}{N}),1+0) =$ $(\mathrm{st}(\frac{2\pi c}{N}) + 2\pi - \mathrm{st}(\frac{2\pi c'}{N}),1+0) = \psi(c,1)\psi(c',0)$; and

(4) if $i = 1 = i'$ then $\psi((c,1)(c',1)) = \psi(c+N-c',0) = (\mathrm{st}(\frac{2\pi(c+N-c')}{N}),0) = (\mathrm{st}(\frac{2\pi c}{N}) + 2\pi -$ $\mathrm{st}(\frac{2\pi c'}{N}),0) = \psi(c,1)\psi(c',1)$.

Thus, we obtain that $\psi((c,i)(c',i')) = \psi(c,i)\psi(c',i')$ for all $(c,i),(c',i') \in D_{2N}$.

Furthermore, the function ψ is an onto. Let $\alpha \in D_{2\mathbb{T}}$. Then $\alpha = (c',i')$ for some $c' \in \mathbb{T}$ and $i' \in C_2$. There is $\theta \in D_{2N}$, where $\theta = (\frac{Nc'}{2\pi},i')$, such that $\psi(\theta) = \psi(\frac{Nc'}{2\pi},i') = (\mathrm{st}(\frac{2\pi(Nc'/(2\pi))}{N}),i') =$ $(c',i') = \alpha$.

Therefore, by the first isomorphism theorem, $D_{2N}/\ker(\psi) \cong D_{2\mathbb{T}}$.

On the other hand,

$$\ker(\psi) = \{(c,i) \in D_{2N} : \psi((c,i)) = (0,0)\} = \{(c,i) : (\mathrm{st}(\frac{2\pi c}{N}),i) = (0,0)\} = \mathscr{N},$$

where $\mathscr{N}$ is the monad of $(0,0) \in D_{2N}$. Hence, $D_{2N}/\mathscr{N} \cong D_{2\mathbb{T}}$.

Now define $\psi' = \psi/\mathcal{N} : D_{2N}/\mathcal{N} \to D_{2\mathbb{T}}$ by

$$\psi'((c,i)\mathcal{N}) = (\mathrm{st}(\frac{2\pi c}{N}), i).$$

So ψ' is an isometry, since

$$\begin{aligned}
d(\psi'((c,i)\mathcal{N}), \psi'((c',i')\mathcal{N})) &= d((\mathrm{st}(\frac{2\pi c}{N}), i), (\mathrm{st}(\frac{2\pi c'}{N}), i')) \\
&= d_{\mathbb{T}}(\mathrm{st}(\frac{2\pi c}{N}), \mathrm{st}(\frac{2\pi c'}{N})) + d_2(i,i') \\
&= d_N(\frac{N}{2\pi}\mathrm{st}(\frac{2\pi c}{N}), \frac{N}{2\pi}\mathrm{st}(\frac{2\pi c'}{N})) + d_2(i,i') \\
&= {}^{\circ}d_N(c,c') + d_2(i,i') \\
&= d((c,i)\mathcal{N}, (c',i')\mathcal{N}).
\end{aligned}$$

Hence, $D_{2\mathbb{T}}$ embeds in $D_{2N}/\mathcal{N}$ as a metric group.

5.2.2 The Infinite Dihedral Group D_∞

The infinite dihedral group $D_\infty \cong \mathbb{Z} \rtimes C_2$, as given in Definition 1.2.6, taken with the discrete metric is LEFM. Consider the *dihedral *metric group D_{2N} of order $2N$, where $N > \mathbb{N}$, as given in Section 5.2.1, with the discrete metric. The monad of $(0,0)$ in D_{2N} is $\mathcal{N} = \{(0,0)\}$. The function $\psi\colon D_\infty \to D_{2N}/\mathcal{N}$ given by

$$\psi(c,i) = (c,i)\mathcal{N}$$

is well-defined and 1-1. In addition, ψ is a homomorphism. Let $(c,i),(c',i') \in D_\infty$. Then:

(1) if $i=0$ and $i'=0$ then $\psi((c,0)(c',0)) = \psi(c+c',0) = (c+c',0)\mathcal{N} = (c,0)\mathcal{N}(c',0)\mathcal{N} = \psi(c,0)\psi(c',0)$;

(2) if $i=0$ and $i'=1$ then we obtain $\psi((c,0)(c',1)) = \psi(c+c',0+1) = (c+c',0+1)\mathcal{N} = (c,0)\mathcal{N}(c',1)\mathcal{N} = \psi(c,0)\psi(c',1)$;

(3) if $i=1$ and $i'=0$ then $\psi((c,1)(c',0)) = \psi(c+(-c'),1+0) = (c+(-c'),1+0)\mathcal{N} =$

$(c,1)\mathscr{N}(c',0)\mathscr{N} = \psi(c,1)\psi(c',0)$; and

(4) if $i = 1 = i'$ then $\psi((c,1)(c',1)) = \psi(c+(-c'),0) = (c+(-c'),0)\mathscr{N} = (c,1)\mathscr{N}(c',1)\mathscr{N} = \psi(c,1)\psi(c',1)$.

Finally, an easy check shows that ψ is an isometry.

Chapter 6

Fourier Series Via a Nonstandard Approach

Throughout this chapter we study the nonstandard approach to some results in classical harmonic analysis (Sections 3.2 and 3.3) for a special case, where the topological group G is the additive circle group $\mathbb{T}$ (see Definition 1.2.2) with the usual topology. We represent such a topological group by a nonstandard *finite *cyclic group $F = C_N$ with addition modulo N, where $N > \mathbb{N}$ is the order of F with the *metric d defined by

$$d(x,y) = \frac{d_F(x,y)}{N},$$

where $d_F(x,y) = \min\{|x-y|, N-|x-y|\}$.

We attempt to use nonstandard techniques together with discrete Fourier analysis (Section 3.1) to translate as much as possible the classical results about the theory of Fourier series in $L^1(\mathbb{T})$ into nonstandard analysis.

6.1 The Inner Product Space in Nonstandard Universe

Our focus will be on the nonstandard universe as much as possible. We switch our view to $^*\mathbb{C}^F$, which is the set of all internal functions from F to $^*\mathbb{C}$, where F is defined as follows.

Definition 6.1.1. Let $F = \left\{\lfloor -\frac{N}{2} \rfloor + 1, \lfloor -\frac{N}{2} \rfloor + 2, \ldots, 0, \ldots, \lfloor \frac{N}{2} \rfloor\right\}$ where $N > \mathbb{N}$. If N is even, then $F = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\right\}$. And $F = \left\{-\frac{N-1}{2}, -\frac{N-1}{2}+1, \ldots, 0, \ldots, \frac{N-1}{2}\right\}$ if N is odd.

Throughout this chapter, we fix an even $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then $F = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\right\}$.

Note that work with finite summations is certainly easier than the work with integrals, due for example to the ease in swapping the order of summations. Whereas swapping order of integrations is sometimes problematic.

Let $X = {}^*\mathbb{C}^F$, where ${}^*\mathbb{C}^F = \{f\colon F \to {}^*\mathbb{C} \text{ is an internal function}\}$. Then $X \cong {}^*\mathbb{C}^N$. In fact X is a *vector space over the field ${}^*\mathbb{C}$.

Definition 6.1.2. Let f, g in $X = {}^*\mathbb{C}^F$. The *inner product* of f and g is defined by

$$\langle f, g\rangle = \frac{1}{N}\sum_{n\in F} f(n)\overline{g(n)}.$$

Example 6.1.3. We define an orthonormal basis $\{e_n : n \in F\}$ of $X = {}^*\mathbb{C}^F$ where $e_n\colon F \to {}^*\mathbb{C}$ is given by $e_n(k) := e^{2\pi ink/N}$ for all $n, k \in F$. Observe that

$$\langle e_n, e_m\rangle = \frac{1}{N}\sum_{k\in F} e^{2\pi ink/N} e^{-2\pi imk/N} = \frac{1}{N}\sum_{k\in F} e^{2\pi i(n-m)k/N} = \delta_{nm}.$$

Moreover, for every $f \in X$, we can write $f = \sum\limits_{n\in F} \lambda_n e_n$, for some $\lambda_n \in {}^*\mathbb{C}$. Therefore,

$$\langle f, e_m\rangle = \langle \sum_{n\in F} \lambda_n e_n, e_m\rangle = \sum_{n\in F} \langle \lambda_n e_n, e_m\rangle = \sum_{n\in F} \lambda_n \langle e_n, e_m\rangle = \lambda_m.$$

Note that,

$$\langle f, e_m\rangle = \frac{1}{N}\sum_{k\in F} f(k)\overline{e_m(k)} = \frac{1}{N}\sum_{k\in F} f(k)e^{-2\pi imk/N} = \widehat{f}(m)$$

is the finite discrete Fourier transform (DFT) in X, and

$$f(n) = \sum_{m\in F} \widehat{f}(m)e^{2\pi inm/N}$$

is the finite inverse discrete Fourier transform (IDFT) in X.

6.2 The DFT of Piecewise Continuous Functions in NSA

In this section we show that we can deal with piecewise continuous functions (Definition 1.2.20) defined on $\mathbb{T} = [-\pi, \pi]$ as discrete Fourier transform functions on the *finite set $F = C_N$ where $F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$ for $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, by using methods of NSA. In other words, the integration on $\mathbb{T} = [-\pi, \pi]$, which represents the Fourier coefficients can be written as a summation on F, which has a similar structure to the definition of DFT formula.

Consider a piecewise continuous function f defined on the closed interval $\mathbb{T} = [-\pi, \pi]$. Then f is Riemann integrable [3] and its integral on $\mathbb{T}$ is defined by

$$\int_{\mathbb{T}} f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=-\frac{N}{2}+1}^{N/2} f(t_k)\Delta x_k,$$

provided that the limit exists, where the limit is taken over all subdivisions $\{x_{-\frac{N}{2}}, x_{-\frac{N}{2}+1}, \ldots, x_{\frac{N}{2}}\}$ of $[-\pi, \pi]$. Here, $\Delta x_k = x_k - x_{k-1}$ and $x_{k-1} \leqslant t_k \leqslant x_k$, for every $k = -\frac{N}{2}+1, \ldots, 0, 1, \ldots, \frac{N}{2}$. If the limit is equal to A, then for a given $\varepsilon > 0$ there exists $\delta > 0$, such that for every subdivision $P = \{x_{-\frac{N}{2}}, x_{-\frac{N}{2}+1}, \ldots, x_0, \ldots, x_{\frac{N}{2}}\}$, with $\max \Delta x_k < \delta$, we have

$$\left| \sum_{k=-\frac{N}{2}+1}^{N/2} f(t_k)\Delta x_k - A \right| < \varepsilon,$$

for any points t_k, $x_{k-1} \leqslant t_k \leqslant x_k$, where $k = -\frac{N}{2}+1, \ldots, 0, 1, \ldots, \frac{N}{2}$.

We work in ${}^*V(\mathscr{C})_{\text{bdd}}$, where $\mathscr{C} = (\mathbb{C}, |\cdot|, +, \cdot, a, f, <)_{a \in \mathbb{C}}$.

The following result uses NSA to compute the Fourier coefficients in Definition 3.3.1.

Theorem 6.2.1. If $f\colon \mathbb{T} \to \mathbb{C}$ is a piecewise continuous function and $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, then

$$\widehat{f}(n) = \mathrm{st}\Big(\frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{N/2} {}^*f(\frac{2\pi k}{N}) e^{-2\pi i n k/N}\Big), \text{ for all } n \in \mathbb{Z}.$$

Proof. Suppose f is a piecewise continuous function on $\mathbb{T} = [-\pi, \pi]$. Then f is a Riemann integrable function on $[-\pi, \pi]$ [3]. So, $f(x)e^{-inx}$ is Riemann integrable function on $[-\pi, \pi]$, for all $n \in \mathbb{Z}$. In the Definition 3.3.1 of Fourier coefficients of f we have $\widehat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-inx}dx$. For a given real $\varepsilon > 0$ there is a real $\delta > 0$ such that for all N in the standard world

$$\max_{-\frac{N}{2}<k\leqslant\frac{N}{2}} \Delta x_k < \delta \implies \Big|\widehat{f}(n) - \frac{1}{2\pi}\sum_{k=-\frac{N}{2}+1}^{N/2} f(x_k)e^{-inx_k}\Delta x_k\Big| < \varepsilon,$$

where we are taking the subdivision $P = \{x_{-\frac{N}{2}}, x_{-\frac{N}{2}+1}, \ldots, x_0, \ldots, x_{\frac{N}{2}}\}$ of $[-\pi,\pi]$ such that $-\pi = x_{-\frac{N}{2}} < x_{-\frac{N}{2}+1} < \ldots < x_{\frac{N}{2}} = \pi,\ \Delta x_k = x_k - x_{k-1} = \frac{2\pi}{N}$ and $x_k = \frac{2\pi k}{N}$. That is,

$$\forall N \in \mathbb{N}, N > \frac{2\pi}{\delta}\Big(\Big|\widehat{f}(n) - \frac{1}{2\pi}\sum_{k=-\frac{N}{2}+1}^{N/2} f(\frac{2\pi k}{N})e^{-2\pi ink/N}(\frac{2\pi}{N})\Big| < \varepsilon\Big).$$

Now by the Transfer principle we get a similar statement true in the nonstandard world. Also, our $N \in {}^*\mathbb{N}\setminus\mathbb{N}$ is certainly greater than the standard quantity $\frac{2\pi}{\delta}$. So, we have

$$\Big|\widehat{f}(n) - \frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{N/2} {}^*f(\frac{2\pi k}{N})e^{-2\pi ink/N}\Big| < \varepsilon.$$

Since $N > \mathbb{N}$, it works for all possible standard real $\varepsilon > 0$, then

$$\widehat{f}(n) \approx \frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{N/2} {}^*f(\frac{2\pi k}{N})e^{-2\pi ink/N}.$$

Hence, $\widehat{f}(n) = \mathrm{st}\Big(\frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{N/2} {}^*f(\frac{2\pi k}{N})e^{-2\pi ink/N}\Big)$, for all $n \in \mathbb{Z}$. $\square$

Notice that, if $f\colon \mathbb{T} \to \mathbb{C}$ is a piecewise continuous function, then f yields an internal function $h_f\colon F \to {}^*\mathbb{C}$ given by

$$h_f(n) = {}^*f(\frac{2\pi n}{N}),\ \text{for all } n \in F,$$

where $F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$. Then h_f has a discrete Fourier transform

$$\widehat{h_f}(n) = \frac{1}{N}\sum_{k\in F} h_f(k)\overline{e_n(k)} = \frac{1}{N}\sum_{k\in F} h_f(k)e^{-2\pi ink/N} = \langle h_f, e_n\rangle.$$

Theorem 6.2.1 shows that $\widehat{h_f}(n) \approx \widehat{f}(n)$, for all standard $n \in \mathbb{Z}$. Furthermore, we can recover

h_f as defined in Section 6.1 by

$$h_f(n) = \sum_{k\in F} \langle h_f, e_k\rangle e_n(k) = \sum_{k\in F} \widehat{h_f}(k) e^{2\pi i nk/N}.$$

Since f is piecewise continuous, at all points of continuity t and $N > \mathbb{N}$ we have

$$f(t) = \mathrm{st}\Big(h_f(\lfloor \frac{Nt}{2\pi} \rfloor)\Big) = \mathrm{st}\Big(\sum_{k\in F} \widehat{h_f}(k) e^{2\pi i \lfloor \frac{Nt}{2\pi} \rfloor k/N}\Big).$$

6.3 The Dirichlet and Fejér Kernels in $X = {}^*\mathbb{C}^F$

Definition 6.3.1. The function $D_n \colon F \to {}^*\mathbb{C}$ is defined by

$$D_n = \sum_{k=-n}^{n} e_k,$$

in other words

$$D_n(m) = \sum_{k=-n}^{n} e_k(m),$$

where $m, n \in F$ and $0 \leqslant n \leqslant \frac{N}{2}$. The family of functions $\{D_n : 0 \leqslant n \leqslant \frac{N}{2}\}$ is analogous to the *Dirichlet kernel* of the classical harmonic analysis.

Definition 6.3.2. The function $\Phi_n \colon F \to {}^*\mathbb{C}$ is defined by

$$\Phi_n = \frac{1}{n+1} \sum_{k=0}^{n} D_k,$$

that is

$$\Phi_n(m) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(m),$$

where $m, n \in F$ and $0 \leqslant n \leqslant \frac{N}{2}$. The family of functions $\{\Phi_n : 0 \leqslant n \leqslant \frac{N}{2}\}$ is analogous to the *Fejér kernel* of the classical harmonic analysis.

It is interesting to study the behaviour of the Dirichlet and Fejér kernels, particularly when $n \in \mathbb{N}$ is large or $n \in {}^*\mathbb{N} \setminus \mathbb{N}$ is small. The Fejér kernel has the following useful and important properties.

Theorem 6.3.3. If $k \in F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$, $0 \leqslant k$ and $N > \mathbb{N}$, then

$$\frac{1}{N}\sum_{n\in F}\Phi_k(n) = 1.$$

Proof. Suppose $N \in \mathbb{N}$ and $X \cong \mathbb{C}^N$ is a standard complex vector space of dimension N. Then

$$\frac{1}{N}\sum_{n\in F}\Phi_k(n) = \frac{1}{N}\frac{1}{k+1}\sum_{n\in F}\sum_{\ell=0}^{k}(\sum_{m=-\ell}^{\ell} e_m(n)) = \frac{1}{N}\frac{1}{k+1}\sum_{\ell=0}^{k}\sum_{m=-\ell}^{\ell} N\delta_{m0} = 1$$

Then by the Transfer principle, the theorem is true for $N > \mathbb{N}$. That is, $\frac{1}{N}\sum\limits_{n\in F}\Phi_k(n) = 1$ in ${}^*\mathbb{C}^N$. □

Theorem 6.3.4. $D_k(r) \in {}^*\mathbb{R}$, for all $r,k \in F = \{-\frac{N}{2}+1, \ldots, 0, \ldots, \frac{N}{2}\}$, where $0 \leqslant k$ and $N > \mathbb{N}$.

Proof. Suppose $N \in \mathbb{N}$, and $X \cong \mathbb{C}^N$ is a standard complex vector space of dimension N. From the definition of Dirichlet kernel, we have

$$D_k(r) = \sum_{n=1}^{k}(e^{2\pi inr/N} + e^{-2\pi inr/N}) + 1 = 2\sum_{n=1}^{k}\mathrm{Re}(e^{2\pi inr/N}) + 1$$

since $e^{-2\pi inr/N}$ is a complex conjugate of $e^{2\pi inr/N}$. Therefore, $D_k(r) \in \mathbb{R}$. So, by Transfer, the theorem is true for $N > \mathbb{N}$. Hence, $D_k(r) \in {}^*\mathbb{R}$, for all $r,k \in F$, $0 \leqslant k$. □

Theorem 6.3.5. $\Phi_k(r) \in {}^*\mathbb{R}$, for all $r,k \in F = \{-\frac{N}{2}+1, \ldots, 0, \ldots, \frac{N}{2}\}$, where $0 \leqslant k$ and $N > \mathbb{N}$.

Proof. In Theorem 6.3.4, we have $D_n(r) \in {}^*\mathbb{R}$, for all $r,n \in F$, $0 \leqslant n$ and

$$\Phi_k(r) = \frac{1}{k+1}\sum_{n=0}^{k} D_n(r).$$

In addition, the operations of summation and division by $k+1$ are internal. So, $\Phi_k(r) \in {}^*\mathbb{R}$, for all $r,k \in F$, $0 \leqslant k$. □

The following theorem shows that the Fejér function is an even function on F,

Theorem 6.3.6. $\Phi_n(r) = \Phi_n(-r)$, for every $n,r \in F = \{-\frac{N}{2}+1, \ldots, 0, \ldots, \frac{N}{2}\}$ and $0 \leqslant n$.

Proof. Actually, the even property of Fejér function comes from the even function property of the Dirichlet function. That is, $D_n(r) = D_n(-r)$ for all $n, r \in F$ and $0 \leqslant n \leqslant \frac{N}{2}$. So,

$$\Phi_n(r) = \frac{1}{n+1}\sum_{k=0}^{n}\Big(\sum_{m=-k}^{k} e_m(r)\Big) = \frac{1}{n+1}\sum_{k=0}^{n}\Big(\sum_{m=-k}^{k} e_m(-r)\Big) = \Phi_n(-r). \qquad \square$$

Some useful equivalent formulas to the Fejér function are given in the following theorem.

Theorem 6.3.7. If $r,k \in F = \{-\frac{N}{2}+1,\ldots,0,\ldots,\frac{N}{2}\}$, $0 \leqslant k \leqslant \frac{N}{2}$ and $N > \mathbb{N}$ then:

(a) $\Phi_k(r) = \sum\limits_{m=-k}^{k}\big(1-\dfrac{|m|}{k+1}\big)\, e_m(r)$; and

(b) $\Phi_k(r) = \dfrac{1}{k+1}\left(\sin\dfrac{(k+1)\pi r}{N} \Big/ \sin\dfrac{\pi r}{N}\right)^2$.

Proof. Suppose $N \in \mathbb{N}$ and $X \cong \mathbb{C}^N$ is a standard complex vector space of dimension N.

(a) From the definition of Fejér kernel, for all $r \in F$, we have

$$\Phi_k(r) = \frac{1}{k+1}\sum_{n=0}^{k}\Big(\sum_{m=-n}^{n} e_m(r)\Big) = \frac{1}{k+1}\sum_{m=-k}^{k}(k+1-|m|)e_m(r) = \sum_{m=-k}^{k}\big(1-\frac{|m|}{k+1}\big)e_m(r).$$

Since the theorem is true for every $N \in \mathbb{N}$, then by Transfer it is true for every $N \in {}^*\mathbb{N}$.

(b) We consider the Dirichlet kernel and use the formula for partial sums of geometric series. Then

$$D_n(r) = \sum_{m=-n}^{n} e_m(r) = e^{-2\pi i r n/N}\sum_{m=0}^{2n} e^{2\pi i r m/N} = \frac{e^{-(2n+1)\pi i r/N} - e^{(2n+1)\pi i r/N}}{e^{-\pi i r/N} - e^{\pi i r/N}} = \frac{\sin\frac{(2n+1)\pi r}{N}}{\sin\frac{\pi r}{N}}.$$

So,

$$\begin{aligned}
\Phi_k(r) &= \frac{1}{k+1}\frac{1}{\sin\frac{\pi r}{N}}\sum_{n=0}^{k}\sin\frac{(2n+1)\pi r}{N} \\
&= \frac{1}{k+1}\frac{1}{\sin\frac{\pi r}{N}}\sum_{n=0}^{k}\frac{\cos\frac{2n\pi r}{N} - \cos\frac{2(n+1)\pi r}{N}}{2\sin\frac{\pi r}{N}} \\
&= \frac{1}{k+1}\frac{1}{(\sin\frac{\pi r}{N})^2}\frac{1-\cos\frac{2(k+1)\pi r}{N}}{2}
\end{aligned}$$

$$= \frac{1}{k+1}\left(\frac{\sin\frac{(k+1)\pi r}{N}}{\sin\frac{\pi r}{N}}\right)^2.$$

Thus the theorem is true for all $N \in \mathbb{N}$. So by the Transfer principle it is true for $N > \mathbb{N}$ as well. $\square$

Remark 6.3.8. $\Phi_k(0) = k+1$, for all $k \in F = \{-\frac{N}{2}+1,\ldots,0,\ldots,\frac{N}{2}\}$ and $0 \leqslant k$.

Proof. In Theorem 6.3.7(a), $\Phi_k(0) = \sum_{m=-k}^{k}(1-\frac{|m|}{k+1}) = k+1$, for all $k \in F$, $0 \leqslant k$. $\square$

Theorem 6.3.9. $\Phi_k(r) \geqslant 0$, for all $r,k \in F = \{-\frac{N}{2}+1,\ldots,0,\ldots,\frac{N}{2}\}$, $0 \leqslant k$ and $N > \mathbb{N}$.

Proof. From Theorem 6.3.7 (b), we have $\Phi_k(r) = \frac{1}{k+1}\left(\frac{\sin\frac{(k+1)\pi r}{N}}{\sin\frac{\pi r}{N}}\right)^2$. Hence, $\Phi_k(r) \geqslant 0$, for all $k,r \in F = \{-\frac{N}{2}+1,-\frac{N}{2}+2,\ldots,0,\ldots,\frac{N}{2}\}$, $0 \leqslant k$ and $N > \mathbb{N}$. $\square$

Theorem 6.3.10. Let $F = \{-\frac{N}{2}+1,-\frac{N}{2}+2,\ldots,0,\ldots,\frac{N}{2}\}$ and $N > \mathbb{N}$. If $0 < k_0 < \frac{N}{2}$ with $\frac{k_0}{N}$ is not infinitesimal, then there is a standard $K \in \mathbb{N}$ such that for all $0 < n < \frac{N}{2}$,

$$\sum_{\substack{r\in F\\ |r|>k_0}} \Phi_n(r) < \frac{K}{n+1}N.$$

Proof. We will use Theorem 6.3.7 (b), which is an equivalent formula of the Fejér function, and Theorem 6.3.9. Then, for all $n > \mathbb{N}$, we have

$$0 \leqslant \Phi_n(r) = \frac{1}{n+1}\frac{(\sin\frac{(n+1)\pi r}{N})^2}{(\sin\frac{\pi r}{N})^2}.$$

Case (i): If

$$0 < k_0 < r < \frac{N}{2},$$

then

$$\sin\frac{\pi k_0}{N} < \sin\frac{\pi r}{N}.$$

So

$$\frac{1}{\sin^2\frac{\pi r}{N}} < \frac{1}{\sin^2\frac{\pi k_0}{N}}.$$

Since $\frac{k_0}{N}$ is not infinitesimal, then $\sin\frac{\pi k_0}{N}$ is not infinitesimal either. Therefore

$$\frac{\sin^2\frac{(n+1)\pi r}{N}}{\sin^2\frac{\pi r}{N}} < \frac{\sin^2\frac{(n+1)\pi r}{N}}{\sin^2\frac{\pi k_0}{N}}.$$

But $\sin^2\frac{(n+1)\pi r}{N} \leqslant 1$, for every $n \in F$, $n \geqslant 0$ and $N > \mathbb{N}$. Now, we choose

$$K = \left\lceil \frac{1}{\sin^2\frac{\pi k_0}{N}} \right\rceil.$$

Then, K is a required standard natural number, such that

$$\Phi_n(r) = \frac{1}{n+1}\frac{(\sin\frac{(n+1)\pi r}{N})^2}{(\sin\frac{\pi r}{N})^2} < \frac{1}{n+1}K.$$

Since, for all $r,n \in F = \{\frac{-N}{2}+1, \frac{-N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$, $0 \leqslant n \leqslant \frac{N}{2}$, $0 \leqslant \Phi_n(r)$. Therefore,

$$\sum_{\substack{r\in F\\ |r|>k_0}} \Phi_n(r) \leqslant \sum_{r\in F} \Phi_n(r) < \frac{K}{n+1}N.$$

Case (ii): If $-\frac{N}{2} < r < k_0 < 0$, then $0 < -k_0 < -r < \frac{N}{2}$. So, by using Case (i), we get the result. $\square$

Definition 6.3.11. The ℓth *partial sum* of the Fourier series of a function $f\colon F \to {}^*\mathbb{C}$ is denoted by $S_\ell(f,n)$ and defined by

$$S_\ell(f,n) = \sum_{k=-\ell}^{\ell} \widehat{f}(k)e_k(n),$$

where $\widehat{f_k} = \langle f, e_k\rangle$, and $\ell, n \in F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$, $\ell \geqslant 0$.

Definition 6.3.12. The *average* of the first $k+1$ partial sums of the Fourier series of a function $f\colon F \to {}^*\mathbb{C}$, for $k \in F = \{-\frac{N}{2}+1, \ldots, 0, \ldots, \frac{N}{2}\}$, $k \geqslant 0$ is denoted by $\sigma_k(f,n)$ and defined by

$$\sigma_k(f,n) = \frac{1}{k+1}\sum_{\ell=0}^{k} S_\ell(f,n),$$

where $S_\ell(f,n)$ is the ℓth partial sum, $n \in F$, and $F = \{\frac{-N}{2}+1, \frac{-N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$.

Note that, we can write the partial sums of the Fourier series and its average of a function

$f\colon F \to {}^*\mathbb{C}$ in terms of Dirichlet and Fejér functions, as shown in the following theorems.

Theorem 6.3.13. For every $\ell, n \in F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$, $\ell \geqslant 0$

$$S_\ell(f,n) = \frac{1}{N}\sum_{r\in F} f(n-r)D_\ell(r).$$

Proof. From the definition of the ℓth partial sum of the Fourier series of $f \in {}^*\mathbb{C}^F$, we have

$$S_\ell(f,n) = \sum_{k=-\ell}^{\ell} \widehat{f}(k)e_n(k) = \sum_{k=-\ell}^{\ell} (\frac{1}{N}\sum_{r\in F} f(r)e^{-2\pi irk/N})e^{2\pi ikn/N} = \frac{1}{N}\sum_{r\in F} f(r)D_\ell(n-r).$$

Notice that, for every $n, r \in F$, if we shift the set F to the left by r, we get $n-r \in F$. By applying a change of variables, we get $S_\ell(f,n) = \frac{1}{N}\sum\limits_{r\in F} f(n-r)D_\ell(r)$. □

Theorem 6.3.14. For every $k, n \in F$ where $F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$, $k \geqslant 0$

$$\sigma_k(f,n) = \frac{1}{N}\sum_{r\in F} f(n-r)\Phi_k(r).$$

Proof. By the definition of the average of the kth partial sum of the Fourier series of a function $f\colon F \to {}^*\mathbb{C}$ and Theorem 6.3.13, we obtain

$$\sigma_k(f,n) = \frac{1}{k+1}\sum_{\ell=0}^{k} S_\ell(f,n) = \frac{1}{k+1}\sum_{\ell=0}^{k}(\frac{1}{N}\sum_{r\in F} f(n-r)D_\ell(r)) = \frac{1}{N}\sum_{r\in F} f(n-r)\Phi_k(r).$$

Notice that, for every $n, r \in F$, if we shift the set F to the left by r, we get $n-r \in F$. □

6.4 Convolution

Definition 6.4.1. Let $f, g \in X = {}^*\mathbb{C}^F$ and $r \in F = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\right\}$. Then the *convolution* of f and g is denoted by $f * g$ and defined by

$$(f*g)(r) = \frac{1}{N}\sum_{s\in F} g(r-s)f(s).$$

Notice that $F = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\right\}$, has an abelian group structure under the addition operation modulo N. We write $+, -$ for these operations of F.

Definition 6.4.2. The abelian group $F = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\right\}$, *acts* on the space $X = {}^*\mathbb{C}^F$. The *action* $(\tau, s) \mapsto \tau^s \in X$ is defined by $\tau^s(f(r)) = f(r-s)$ for $\tau \in X$ and $s \in F$, where the operation $-$ is the binary operation of subtraction mod N in the group F.

Now we consider some properties of convolution of functions.

Theorem 6.4.3. $(f*g)(r) = (g*f)(r)$ for all $f, g \in {}^*\mathbb{C}^F$ and all $r \in F$.

Proof. By applying a change of variables, we get

$$(f*g)(r) = \frac{1}{N}\sum_{s\in F} g(r-s)f(s) = \frac{1}{N}\sum_{s\in F} g(s)f(r-s) = \frac{1}{N}\sum_{s\in F} f(r-s)g(s) = (g*f)(r). \quad \square$$

Theorem 6.4.4. $((f*g)*h)(r) = (f*(g*h))(r)$ for all $f, g, h \in {}^*\mathbb{C}^F$ and all $r \in F$.

Proof. By applying a change of variables, we see that

$$\begin{aligned}((f*g)*h)(r) &= \frac{1}{N}\sum_{s\in F}\Big(\frac{1}{N}\sum_{\ell\in F} f(\ell)g(s-\ell)\Big)h(r-s)\\ &= \frac{1}{N}\sum_{s\in F} f(r-s)\Big(\frac{1}{N}\sum_{\ell\in F} g(s-\ell)h(\ell)\Big)\\ &= (f*(g*h))(r).\end{aligned} \quad \square$$

Definition 6.4.5. Let $f \in X = {}^*\mathbb{C}^F$. Then $f*\colon X \to X$ is defined by $f*(g) = f*g$.

In fact, $f*\colon X \to X$ is a linear transformation. We can prove this as follows.

Theorem 6.4.6. If $f*\colon X \to X$, where $X = {}^*\mathbb{C}^F$, then:

(a) $f*(g+h) = (f*g)+(f*h)$, for all $g, h \in X$;

(b) $f*(ag) = a(f*g)$, for all $g \in X$ and $a \in {}^*\mathbb{C}$.

Proof. Let $f* \in X^X$, $g, h \in X$, $r \in F$ and $a \in {}^*\mathbb{C}$. Then:

(a)

$$f*(g+h)(r) = \frac{1}{N}\sum_{s\in F} f(r-s)g(s) + \frac{1}{N}\sum_{s\in F} f(r-s)h(s) = (f*g)(r)+(f*h)(r);$$

(b)

$$f*(ag)(r)=\frac{1}{N}\sum_{s\in F}f(r-s)(ag)(s)=a\Big(\frac{1}{N}\sum_{s\in F}f(r-s)g(s)\Big)=a(f*g)(r). \qquad \square$$

Theorem 6.4.7. The vector e_n is an eigenvector of $f*$, and the corresponding eigenvalue is $\langle f, e_n\rangle$ for all $n \in F$.

Proof. By using the definition of convolution of functions, we get

$$(f*e_n)(r)=\frac{1}{N}\sum_{s\in F}f(r-s)e_n(s)=\sum_{k\in F}\widehat{f}(k)e^{2\pi ikr/N}\delta_{nk}=\widehat{f}(n)e^{2\pi inr/N}=\langle f,e_n\rangle e_n(r). \qquad \square$$

Notice that the DFT of the convolution of two functions is equal to the product of their discrete Fourier transforms. This is proved in the following theorem.

Theorem 6.4.8. $\widehat{f*g}(n)=\widehat{f}(n)\widehat{g}(n)$, for all $f,g\in {}^*\mathbb{C}^F$, and $n\in F$.

Proof. By using the definitions of DFT of $f*g$ and convolution of functions, we obtain

$$\widehat{f*g}(n)=\frac{1}{N}\sum_{k\in F}(f*g)(k)e^{-2\pi ikn/N}=\frac{1}{N}\sum_{r\in F}f(r)e^{-2\pi inr/N}\frac{1}{N}\sum_{k\in F}g(k-r)e^{-2\pi in(k-r)/N}.$$

Let $k-r=s$. If $k=\frac{-N}{2}+1,\frac{-N}{2}+2,\dots,\frac{N}{2}$, then $s=\frac{-N}{2}+1-r,\frac{-N}{2}+2-r,\dots,\frac{N}{2}-r$. We shift the set F to the left by the integer r, we notice that

$$\sum_{s=-\frac{N}{2}+1-r}^{\frac{N}{2}-r}g(s)=\sum_{s=-\frac{N}{2}+1}^{\frac{N}{2}}g(s).$$

Therefore, $\widehat{f*g}(n)=\frac{1}{N}\sum_{r\in F}f(r)e^{-2\pi inr/N}\frac{1}{N}\sum_{s\in F}g(s)e^{-2\pi ins/N}=\widehat{f}(n)\widehat{g}(n)$ as required. $\square$

Theorem 6.4.9. $\sigma_n(f,s)=(\Phi_n*f)(s)$ for all $f\in {}^*\mathbb{C}^F$ and all $n,s\in F$, $n\geqslant 0$.

Proof. By using the definition of convolution of functions and Theorem 6.3.14, we obtain

$$\sigma_n(f,s)=\frac{1}{N}\sum_{t\in F}f(s-t)\Phi_n(t)=(\Phi_n*f)(s),$$

for all $n,s\in F$ and $0\leqslant n$. $\square$

Theorem 6.4.10. $\sigma_n(f,s) = \sum\limits_{k=-n}^{n}(1-\frac{|k|}{n+1})\widehat{f}(k)e^{2\pi iks/N}$, where $n,s \in F$ and $0 \leqslant n \leqslant \frac{N}{2}$.

Proof. By Theorems 6.3.7 (a) and 6.4.9, and the definition of convolution of functions, we get

$$\sigma_n(f,s) = \frac{1}{N}\sum_{t\in F}\Big(\sum_{k=-n}^{n}\big(1-\frac{|k|}{n+1}\big)e^{2\pi ik(s-t)/N}\Big)f(t) = \sum_{k=-n}^{n}\big(1-\frac{|k|}{n+1}\big)\widehat{f}(k)e^{2\pi iks/N}. \qquad \square$$

Definition 6.4.11. Let $F = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\right\}$ and $r,k \in F$ with $k \geq 0$. Then we define $B_k(r)$ by $B_k(r) = \{r-k, r-k+1, \ldots, r, \ldots, r+k\}$, addition being modulo N as usual. Also $B_k^c(r) = \left\{\frac{-N}{2}+1, \ldots, r-(k+1)\right\} \cup \left\{r+(k+1), \ldots, \frac{N}{2}\right\}$ is called the *complement* of $B_k(r)$ in F.

According to Definition 1.2.12, F is a precompact set with respect to the metric d given by $d(x,y) = \frac{2\pi}{N}\min\{|x-y|, N-|x-y|\}$, as shown in the following theorem.

Theorem 6.4.12. The *finite metric space (F,d) is a precompact space.

Proof. Since $\mathrm{dia}(F) \leqslant \pi$, then the diameter of F is limited. For any appreciable number $a > 0$, there exists a limited family $\{B_i(i\lceil\frac{aN}{8\pi}\rceil, \frac{a}{4}) : i = -n, \ldots, 0, \ldots, n\}$, $n = \lceil\frac{2\pi}{a/2}\rceil = \lceil\frac{4\pi}{a}\rceil$, of closed balls of F, such that

$$F = \bigcup_{i=-n}^{n} B_i\Big(i\lceil\frac{aN}{8\pi}\rceil, \frac{a}{4}\Big) \text{ and } \mathrm{dia}\Big(B_i\big(i\lceil\frac{aN}{8\pi}\rceil, \frac{a}{4}\big)\Big) = \frac{a}{2} < a$$

for all $-n \leqslant i \leqslant n$. Hence, (F,d) is a precompact space. $\square$

Theorem 6.4.13. Every internal subset E of the *finite metric precompact space F is a metric precompact space.

Proof. For every internal subset E of F, $\mathrm{dia}(E) \leqslant \mathrm{dia}(F) \leqslant \pi$. Then, diameter of E is limited. Moreover, given $a > 0$, take a limited cover $\{B_i(k_i, r) : i = 0, \ldots, n\}$ of F. Then, $\{B_i(k_i, r) \cap E : i = 0, \ldots, n\}$ is a limited cover of E. Hence, E is a precompact space. $\square$

6.5 The Relation Between $L^1(\mathbb{T})$ and $L^1(F)$

Our goal in this section is to show the relation between functions in the standard space $L^1(\mathbb{T})$ (Section 3.3) and functions in the nonstandard space $L^1(F)$.

Theorem 6.5.1. If $g \in L^1[-\pi,\pi]$, then there is $H \in L^1(F)$ such that $g(\mathrm{st}(\frac{2\pi\omega}{N})) = {}^\circ H(\omega)$ for almost all $\omega \in F$.

Proof. Suppose $g \in L^1[-\pi,\pi]$ is a (representative of) a classical L^1 (equivalence class of) functions. Then g is Lebesgue (λ) measurable and $\int_{[-\pi,\pi]} |g| d\lambda$ is finite. Define $h\colon F \to \mathbb{C}$ by

$$h(\omega) = g(\mathrm{st}(\frac{2\pi\omega}{N})).$$

(Note: h is not necessarily internal). Then, for a closed set $B \subseteq \mathbb{C}$, $g^{-1}(B)$ is a Lebesgue measurable set. In Note 2.7.4, $\mathrm{st}_N^{-1}(g^{-1}(B)) \in L(\mathscr{A})$ and we have

$$\mathrm{st}_N^{-1}(g^{-1}(B)) = \{\omega \in F : \mathrm{st}_N(\omega) \in g^{-1}(B)\} = \{\omega \in F : \mathrm{st}(\frac{2\pi\omega}{N}) \in g^{-1}(B)\} = h^{-1}(B)$$

Therefore, $h^{-1}(B) \in L(\mathscr{A})$. Hence, h is $L(\mathscr{A})$-measurable. In Definition 2.7.15, h has a lifting $H\colon F \to {}^*\mathbb{C}$ such that ${}^\circ H(\omega) = h(\omega)$ L(μ)-almost everywhere. In addition, we have $g \in L^1[-\pi,\pi]$, then $\int_{[-\pi,\pi]} |g| d\lambda$ is limited. Moreover, the standard part map is always limited. Then, $g(\mathrm{st}_N(\omega))$ is the composition of the functions $\mathrm{st}_N\colon F \to [-\pi,\pi]$ and $g\colon [-\pi,\pi] \to \mathbb{C}$. So, $\int_F |g(\mathrm{st}_N)| dL(\mu) = \int_F |h| dL(\mu)$ is limited. By Theorem 2.7.17, H is S-integrable on F.

To show that H is almost S-continuous on F, we have $h(\omega) = g(\mathrm{st}(\frac{2\pi\omega}{N}))$, for all $\omega \in F$. Hence,

$$h(\omega) = g(t), \text{ for all } \omega \in \mathrm{st}_N^{-1}(t).$$

Thus, h is constant on monads, $\mathrm{st}_N^{-1}(t)$. Moreover, we found S-integrable $H\colon F \to {}^*\mathbb{C}$ with $h(\omega) = \mathrm{st}(H(\omega))$, for all $\omega \in A$, where $A \subseteq F$ is $L(\mathscr{A})$-measurable with measure 2π. Thus, on a monad $\mathrm{st}_N^{-1}(t)$, the values of H vary by at most an infinitesimal. That is, for all $t \in [-\pi,\pi]$

and $\omega, \nu \in A$

$$\omega, \nu \in \mathrm{st}_N^{-1}(t) \Rightarrow H(\omega) \approx H(\nu).$$

Since t was arbitrary, then

$$\omega \approx \nu \Rightarrow H(\omega) \approx H(\nu), \text{ for all } \omega, \nu \in A.$$

Therefore, H is S-continuous on A. Hence, H is L-integrable on F. $\square$

The conclusion to Theorem 6.5.1 is that every function in $L^1([-\pi, \pi])$ can be represented as an $L^1(F)$ function.

6.6 Convergence in Norm

Definition 6.6.1. If $f\colon {}^*\mathbb{N} \to {}^*\mathbb{C}$ is a function, then the $^\circ$*limit* of $f(n)$ as $n \to \infty$, for $n \in \mathbb{N}$ is $\lambda \in \mathbb{C}$, if for all $\varepsilon \gtrapprox 0$, there exists a standard $K \in \mathbb{N}$ such that for all standard $n > K$, $|f(n) - \lambda| < \varepsilon$, written as $^\circ \lim\limits_{n \in \mathbb{N}} f(n) = \lambda$.

Lemma 6.6.2. Let $f\colon {}^*\mathbb{N} \to {}^*\mathbb{C}$ be a function and $n \in {}^*\mathbb{N}$. If $f(n)$ has a $^\circ$limit, then it is unique.

Proof. Same as in the usual real analysis. $\square$

Remark 6.6.3. Uniqueness follows because $\lambda \in \mathbb{C}$, not ${}^*\mathbb{C}$.

Theorem 6.6.4. If $f \in V = {}^*\mathbb{C}^F$ is limited on F, then $\|\sigma_n(f) - f\|_1$ is also limited, for every $n \in F = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, 0, \ldots, \frac{N}{2}\}$, $0 \leqslant n \leqslant \frac{N}{2}$.

Proof. Assume that f is limited on F. Then, there exists $M \in \mathbb{N}$, such that $|f(r)| < M$ for all $r \in F$. Since $\frac{1}{N}\sum_{r \in F} \Phi_n(r) = 1$ and $\Phi_n(r) \geqslant 0$ for all $r \in F$ and all $n \geqslant 0$, then

$$\|\sigma_n(f) - f\|_1 = \frac{2\pi}{N} \sum_{s \in F} \Big| \frac{1}{N} \sum_{r \in F} \Phi_n(r) f(s-r) - f(s) \Big| < \frac{2\pi}{N} \sum_{s \in F} \frac{1}{N} \sum_{r \in F} \Phi_n(r)(2M) = 4\pi M. \quad \square$$

Now, we shall show in the following theorem that the S-continuous functions are dense in $L^1(F)$. That is, the integrable functions can be approximated by S-continuous functions. The proof of this theorem will be done in two stages. First, L-integrable functions can be approximated by integrable averaging functions, relative to some nice partitions of F. Second, since each averaging function is a linear combination of constant functions, then it can be approximated by continuous functions on F [33].

Theorem 6.6.5. If $f\colon F \to {}^*\mathbb{C}$ is an L-integrable function, then for all appreciable numbers $a > 0$, there exists an S-continuous function $g\colon F \to {}^*\mathbb{C}$ such that, $\|f - g\|_1 < a$.

Proof. Assume $f\colon F \to {}^*\mathbb{C}$ is an L-integrable function. Then, for all appreciable numbers $a > 0$, we will choose a nice dissection $(\mathscr{P}_n)_{0 \leqslant n \leqslant \nu}, (\nu > \mathbb{N})$. That is, a dissection, in which $\mathscr{P}_n$ is infinitely fine, for all $n > \mathbb{N}$. Then, by Theorem 2.5.13 (c), we have

$$\|f - E^{\mathscr{P}_n}[f]\|_1 < \frac{a}{2}, \text{ for all } n > \mathbb{N}.$$

So by overspill, the above inequality is also true for some limited natural numbers m, where m depends on a. Therefore,

$$\|f - E^{\mathscr{P}_m}[f]\|_1 < \frac{a}{2}, \text{ for some } m \in \mathbb{N}.$$

Otherwise, we get

$$\|f - E^{\mathscr{P}_m}[f]\|_1 > \frac{a}{2}, \text{ for all } m \in \mathbb{N}.$$

So by overspill, there exist unlimited natural numbers m such that

$$\|f - E^{\mathscr{P}_m}[f]\|_1 > \frac{a}{2}, \text{ for some } m > \mathbb{N}$$

which is a contradiction to Theorem 2.5.13 (c). Therefore,

$$\|f - E^{\mathscr{P}_n}[f]\|_1 < \frac{a}{2}, \text{ for some } n \in \mathbb{N}.$$

Hence, the averaging functions $E^{\mathscr{P}_n}[f]$ relative to partitions $\mathscr{P}_n$, for some limited n, are dense in $L^1(F)$. Now, using Proposition 2.5.12, we can construct a partition $\mathscr{P}_n$, of F for limited n such that each atom of $\mathscr{P}_n$ is a near interval. So, we obtain finitely many convex atoms in $\mathscr{P}_n$, in which $E^{\mathscr{P}_n}[f]$ is limited.

Now we have to prove $E^{\mathscr{P}_n}[f]$ is approximated by a continuous function. Let $t_1, t_2, \ldots, t_w$ be points of F in which, $E^{\mathscr{P}_n}[f]$ is not continuous, where w is limited, and let $\eta > 0$ be an appreciable number.

Consider a point t_i, where $1 \leqslant i \leqslant w$. Then, to turn the function $E^{\mathscr{P}_n}[f]$ from discontinuous at t_i to continuous at t_i, we will take two points $(t_i - \eta, k)$ and $(t_i + \eta, l)$ from the left and the right of t_i, in the graph $E^{\mathscr{P}_n}[f]$. Join the two points by the line

$$g_i(t) = \Big(\frac{l-k}{2\eta}\Big)t + \Big(l - \frac{l-k}{2\eta}(t_i + \eta)\Big), \text{ for all } i = 1, 2, \ldots, w.$$

We define $g\colon F \to {}^*\mathbb{C}$ as follows

$$g(t) = \begin{cases} E^{\mathscr{P}_n}[f](t) & \text{if } t_{i-1} < t < t_i - \eta \text{ or } t_i + \eta < t < t_{i+1}, \\ \frac{(l-k)(t-t_i)}{2\eta} + \frac{l+k}{2} & \text{if } t_i - \eta \leqslant t \leqslant t_i + \eta, \end{cases}$$

where

$$k = E^{\mathscr{P}_n}[f](t), \text{ for all } t_{i-1} < t < t_i, \text{ and for all } i = 2, 3, \ldots, w-1,$$

and

$$l = E^{\mathscr{P}_n}[f](t), \text{ for all } t_i < t < t_{i+1}, \text{ and for all } i = 1, 2, \ldots, w-2.$$

Then g is the required continuous function such that for a small enough $\eta > 0$ we get

$$\|E^{\mathscr{P}_n}[f] - g\|_1 < \frac{a}{2}.$$

That is, $E^{\mathscr{P}_n}[f]$ approximated by an S-continuous functions g, for limited n. So, by overspill, there are some unlimited numbers w such that the approximation is true. Then

$$\|E^{\mathscr{P}_w}[f] - g\|_1 < \frac{a}{2}, \text{ for some } w > \mathbb{N}.$$

So, from the triangle inequality of norms we obtain

$$\|f-g\|_1 \leqslant \|f-E^{\mathscr{P}_w}[f]\|_1 + \|E^{\mathscr{P}_w}[f]-g\|_1 < \frac{a}{2}+\frac{a}{2} = a. \qquad \square$$

Theorem 6.6.6. If $f \in L^1(F)$, then ${}^{\circ}\lim_{n\in\mathbb{N}} \|\sigma_n(f)-f\|_1 = 0$.

Proof. Since $f\colon F \to {}^*\mathbb{C}$ is an L-integrable function, in Theorem 6.6.5, for all appreciable numbers $\varepsilon > 0$, there exists an S-continuous function $g\colon F \to {}^*\mathbb{C}$ such that $\|f-g\|_1 < \frac{\varepsilon}{2(2\pi+1)}$. Now consider

$$\|\sigma_n(f)-f\|_1 = \|\Phi_n * f - f\|_1 < (2\pi+1)\cdot\frac{\varepsilon}{2(2\pi+1)} + \|\Phi_n * g - g\|_1 = \frac{\varepsilon}{2} + \|\Phi_n * g - g\|_1,$$

since $\|\Phi_n\|_1 = \frac{2\pi}{N}\sum\limits_{t\in F}|\Phi_n(t)| = 2\pi(\frac{1}{N}\sum\limits_{t\in F}\Phi_n(t)) = 2\pi$.

Notice that, as g is S-continuous function on F, there exists $k \in F$ with $\frac{k}{N} \gtrapprox 0$, such that

$$t \in B_k(0) = \{-k,-k+1,\dots,0,\dots,k\} \Rightarrow \max_{s\in F}|g(s-t)-g(s)| < \frac{\varepsilon}{4\pi}.$$

Therefore,

$$\begin{aligned}
\|\Phi_n * g - g\|_1 &= \frac{2\pi}{N}\sum_{s\in F}|\Phi_n * g(s) - g(s)| \\
&\leqslant 2\pi\big(\max_{s\in F}|\Phi_n * g(s)-g(s)|\big) \\
&\leqslant 2\pi\big(\max_{s\in F}\frac{1}{N}\sum_{t\in F}\Phi_n(t)|g(s-t)-g(s)|\big) \\
&= 2\pi\big(\max_{s\in F}\frac{1}{N}\sum_{t\in B_k}\Phi_n(t)|g(s-t)-g(s)| + \max_{s\in F}\frac{1}{N}\sum_{t\in B_k^c}\Phi_n(t)|g(s-t)-g(s)|\big).
\end{aligned}$$

Since g is S-continuous on F and g approximates $f \in L^1(F)$, then g is limited on F. So, there is a standard $M \in \mathbb{N}$ such that $|g(s)| \leqslant M$, for all $s \in F$. Furthermore, in Theorem 6.3.10, there exists a standard K such that, $\frac{1}{N}\sum\limits_{t\in B_k^c}\Phi_n(t) < \frac{K}{n+1}$. So,

$$\|\Phi_n * g - g\|_1 < 2\pi\Big(\frac{1}{N}\sum_{t\in B_k}\Phi_n(t)(\frac{\varepsilon}{4\pi}) + \frac{1}{N}\sum_{t\in B_k^c}\Phi_n(t)(2M)\Big) < \frac{\varepsilon}{2} + \frac{4\pi KM}{n+1},$$

since $\frac{1}{N}\sum_{t\in F}\Phi_n(t)=1$. Therefore,

$$^\circ\lim_{n\in\mathbb{N}}\|\sigma_n(f)-f\|_1 <^\circ \lim_{n\in\mathbb{N}}\big(\frac{\varepsilon}{2}+\|\Phi_n * g-g\|_1\big) <^\circ \lim_{n\in\mathbb{N}}\big(\frac{\varepsilon}{2}+\frac{\varepsilon}{2}+\frac{4\pi KM}{n+1}\big)=\varepsilon.$$

Since ε is an arbitrary, it follows that $^\circ\lim\limits_{n\in\mathbb{N}}\|\sigma_n(f)-f\|_1=0$. □

Definition 6.6.7. Let $F=\big\{-\frac{N}{2}+1,-\frac{N}{2}+2,\ldots,0,\ldots,\frac{N}{2}\big\}$. Then a *trigonometric polynomial* on F is an expression of the form

$$p(t)=\sum_{n=-k}^{k}\widehat{f}(n)e^{2\pi itn/N}.$$

The numbers $n\in F$ in the summand are called the *frequencies* of p. The largest positive n such that $\widehat{f}(n)\neq 0$ or $\widehat{f}(-n)\neq 0$ is called the *degree* of p.

Example 6.6.8. Given $\widehat{f}$ on $F=\big\{-\frac{N}{2}+1,-\frac{N}{2}+2,\ldots,0,\ldots,\frac{N}{2}\big\}$ by $\widehat{f}(n)=1$ if $|n|\leqslant 50$ and $\widehat{f}(n)=0$ otherwise. Then f is of degree 50 and $f(k)=\sum\limits_{n=-50}^{50}e^{2\pi ink/N}$.

For any $f\in L^1(F)$, the lower bound of $\|f\|_1$ can be determined, as shown in the following theorem.

Theorem 6.6.9. $|\widehat{f}(n)|\leqslant\|f\|_1$ for all $f\in L^1(F)$, and all $n\in F$.

Proof. Since $|e^{-2\pi irn/N}|=1$, then we have

$$|\widehat{f}(n)|=\Big|\frac{1}{N}\sum_{r\in F}f(r)e^{-2\pi irn/N}\Big|\leqslant\frac{1}{N}\sum_{r\in F}\big|f(r)\big|\big|e^{-2\pi irn/N}\big|\leqslant\frac{2\pi}{N}\sum_{r\in F}\big|f(r)\big|=\|f\|_1.$$ □

Theorem 6.6.10. The set of trigonometric polynomials of standard degree is dense in $L^1(F)$.

Proof. We use the sequence (Φ_n) of the Fejér kernel, which is a summability kernel in $L^1(F)$. By Theorem 6.6.6, for every $f\in L^1(F)$, the sequence $(\sigma_n(f))=(\Phi_n * f)$ converges to f in 1-norm. That is, $^\circ\lim\limits_{n\in\mathbb{N}}\|\sigma_n(f)-f\|_1=0$. Hence, trigonometric polynomials are dense in $L^1(F)$. □

Theorem 6.6.11. (***Riemann-Lebesgue Lemma***) If $f\in L^1(F)$, then $^\circ\lim\limits_{|n|\in\mathbb{N}}\widehat{f}(n)=0$.

Proof. Let $\varepsilon > 0$ be an appreciable number and $f \in L^1(F)$. By Theorem 6.6.6 there exists a standard $M \in \mathbb{N}$, such that $\|\sigma_k(f) - f\|_1 < \varepsilon$, for all $k > M$. Notice that for $|n| >$ the degree of $\sigma_k(f)$, we have

$$\widehat{f}(n) - \widehat{\sigma_k(f,n)} = \begin{cases} \widehat{f}(n) & \text{if } |n| > k, \\ 0 & \text{if } |n| \leqslant k. \end{cases}$$

So,

$$|\widehat{f}(n)| = |\widehat{\sigma_k(f,n)} - \widehat{f}(n)| = |(\widehat{\sigma_k(f) - f})(n)| \leqslant \|\sigma_k(f) - f\|_1 < \varepsilon.$$

Hence, ${}^\circ\lim\limits_{|n|\in\mathbb{N}} \widehat{f}(n) = 0$. $\square$

Theorem 6.6.12. (***Uniqueness theorem***) Let $f \in {}^*\mathbb{C}^F$. If $\widehat{f}(m) = 0$ for all $m \in F$, then $f = 0$.

Proof. In the vector space ${}^*\mathbb{C}^F$, we have $f(k) = \sum\limits_{m\in F} \widehat{f}(m) e_k(m)$. If $\widehat{f}(m) = 0$, for all $m \in F$, then $f(k) = 0$ for all $k \in F$. $\square$

Notice that in Theorem 6.6.12, if $f \in L^1(F)$, $\widehat{f}(m) = 0$, for all $m \in \mathbb{Z}$, and by Theorem 6.4.10 we have $\sigma_n(f,s) = \sum\limits_{m=-n}^{n} \left(1 - \frac{|m|}{n+1}\right) \widehat{f}(m) e^{2\pi i m s/N}$, then $\sigma_n(f,s) = 0$, for all $n \in \mathbb{N}$. In Theorem 6.6.6, we have ${}^\circ\lim\limits_{n\in\mathbb{N}} \|\sigma_n(f) - f\|_1 = 0$. So $\|f\|_1 \approx 0$.

Definition 6.6.13. Let $f \in V = {}^*\mathbb{C}^F$, and $s \in F$ then the ${}^\circ$*limit* of $f(r)$ as $r \to s$ is $\lambda \in \mathbb{C}$, if $\forall\, \varepsilon \gtrsim 0$, $\exists\, k$ such that $\frac{k}{N} \gtrsim 0$ and $\forall\, t \in B_k(s)$, $|f(t) - \lambda| < \varepsilon$. We also write this as ${}^\circ\lim\limits_{r\to s} f(r) = \lambda$.

Theorem 6.6.14. (***Fejér Theorem***) If $f \in L^1(F)$, $t_0 \in F$ and $y_0 \in {}^*\mathbb{C}$, such that

$$\mathrm{st}(y_0) = {}^\circ\lim_{h\to 0} \frac{f(t_0+h) + f(t_0-h)}{2}, \quad \text{then} \quad {}^\circ\lim_{n\in\mathbb{N}} \sigma_n(f,t_0) = \mathrm{st}(y_0).$$

Proof. Since ${}^\circ\lim\limits_{h\to 0} \frac{f(t_0+h)+f(t_0-h)}{2} = \mathrm{st}(y_0)$, then for a given $\varepsilon > 0$, there exists $k \in F$ with $\frac{k}{N} \gtrsim 0$, such that

$$t \in B_k(0) = \{-k, -k+1, \ldots, 0, \ldots, k\} \Rightarrow \left| \frac{f(t_0+t) + f(t_0-t)}{2} - \mathrm{st}(y_0) \right| < \varepsilon.$$

On the other hand, we have ${}^\circ\lim\limits_{n\in\mathbb{N}} (\max\limits_{t\in B_k^c} \Phi_n(t)) = 0$, by properties of Φ_n in Theorem 6.3.10. So, by overspill, there is some infinite $n \in {}^*\mathbb{N}$ with $\max\limits_{t\in B_k^c} \Phi_n(t) < \varepsilon$.

Notice that, $|\sigma_n(f,t_0) - \text{st}(y_0)|$

$$
\begin{aligned}
&= \Big|\frac{1}{N}\sum_{t\in F}\Phi_n(t)f(t_0-t) - \text{st}(y_0)\Big| \\
&= \Big|\frac{1}{N}\sum_{t\in F}\Phi_n(t)(f(t_0-t) - \text{st}(y_0))\Big| \;\; (\text{since } \frac{1}{N}\sum_{t\in F}\Phi_n(t) = 1) \\
&\leqslant \frac{1}{N}\sum_{t=0}^{N/2}\Phi_n(t)\Big|f(t_0-t)+f(t_0+t)-2\text{st}(y_0)\Big| \quad (\text{Theorem 6.3.6 } \Phi_n(t) = \Phi_n(-t), \forall t\in F) \\
&= \frac{2}{N}\sum_{t=0}^{k}\Phi_n(t)\Big|\frac{f(t_0-t)+f(t_0+t)}{2} - \text{st}(y_0)\Big| + \frac{2}{N}\sum_{t=k+1}^{N/2}\Phi_n(t)\Big|\frac{f(t_0-t)+f(t_0+t)}{2} - \text{st}(y_0)\Big| \\
&< \frac{2}{N}\sum_{t=0}^{k}\Phi_n(t)(\varepsilon) + \frac{2}{N}\sum_{t=k+1}^{N/2}(\varepsilon)\Big|\frac{f(t_0-t)+f(t_0+t)}{2} - \text{st}(y_0)\Big| \\
&\leqslant \frac{2\varepsilon}{N}\sum_{t\in F}\Phi_n(t) + \frac{2\varepsilon}{N}\sum_{t\in F}|f(t)| + \big(\frac{2\varepsilon}{N}\big)(N)|\text{st}(y_0)| \\
&= 2\varepsilon + \frac{\varepsilon}{\pi}\Big(\frac{2\pi}{N}\sum_{t\in F}|f(t)|\Big) + 2\varepsilon|\text{st}(y_0)| \\
&= 2\varepsilon(1+|\text{st}(y_0)|) + \frac{\varepsilon}{\pi}\|f\|_1
\end{aligned}
$$

Since ε is arbitrary and $\|f\|_1$, $\text{st}(y_0)$ are limited, therefore $\sigma_n(f,t_0) \to \text{st}(y_0)$.

Hence, ${}^{\circ}\lim_{n\in\mathbb{N}} \sigma_n(f,t_0) = \text{st}(y_0)$. $\square$

Chapter 7

Functionals in Nonstandard Analysis

An *exponential polynomial function* $f(z) = \sum_{k=-n}^{n} a_k e^{ikz}$ is a complex-valued function, where a_k are complex constants for all $-n \leqslant k \leqslant n$ and z is a complex variable. Let T be the test space of exponential polynomial functions $f\colon \mathbb{T} \to \mathbb{C}$. Obviously, the exponential polynomial functions are infinitely differentiable on $\mathbb{T}$. Since $\mathbb{T} = [-\pi, \pi]$ is a compact set, every closed subset of $\mathbb{T}$ is compact. That is, functions f defined on $\mathbb{T}$ are of compact support. We call a function $f \in T$ a *test function*.

Notice that, T is a vector space over the field of complex numbers $\mathbb{C}$. The componentwise addition and scalar multiplication are defined on T as follows:

$$(f+g)(t) = f(t) + g(t), \text{ and } (\lambda f)(t) = \lambda(f(t)) \text{ for all } f, g \in T \text{ and } \lambda \in \mathbb{C}.$$

In this chapter we would like to focus on the continuous linear functionals (generalised functions) $F\colon T \to \mathbb{C}$. The topology defined on T is the uniform convergence topology and the topology defined on $\mathbb{C}$ is the usual topology.

Note 7.0.15. To avoid confusion, we will denote the nonstandard *finite set F by X due to the use of the symbol of F for functionals, throughout this chapter.

For an internal function $h\colon X \to {}^*\mathbb{C}$, we define the continuous linear functional $F_h\colon {}^*T \to {}^*\mathbb{C}$ by

$$F_h({}^*f) = \frac{1}{N} \sum_{k \in X} {}^*f(\frac{2\pi k}{N}) h(k),$$

where ${}^*f\colon {}^*\mathbb{T} \to {}^*\mathbb{C}$ is the nonstandard natural extension of $f\colon \mathbb{T} \to \mathbb{C}$. Therefore, ${}^\circ F_h\colon T \to \mathbb{C}$ will be defined as

$$^\circ F_h(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X} {}^*f(\frac{2\pi k}{N})h(k)\Big),$$

provided that the standard part exists. In order to show that ${}^\circ F_h\colon T \to \mathbb{C}$ is well defined, we have to determine "nice" internal functions $h\colon X \to {}^*\mathbb{C}$.

The following example shows that the functional ${}^\circ F_h\colon T \to \mathbb{C}$ is not continuous, when we give the test space T of exponential polynomials the $\|\cdot\|_1$ topology.

Example 7.0.16. Let $\Phi_n(k) = \frac{1}{n+1}\sum\limits_{t=0}^{n} D_t(k)$, where $D_t(k)$ is the Dirichlet function. Consider the Fejér kernel sequence $(\frac{1}{n+1}\Phi_n)$. Then, $\frac{1}{n+1}\Phi_n$ is a function in T, for all $n \in \mathbb{N}$. Notice that $\|\frac{1}{n+1}\Phi_n\|_1 \to 0$ as $n \to \infty$. At the same time, $\frac{1}{N}\sum\limits_{k\in\Omega} \frac{1}{n+1}{}^*\Phi_n(\frac{2\pi k}{N})h(k) \to 1$ as $n \to \infty$, in the case where $h(0) = N$ and $h(k) = 0$ if $k \neq 0$. Therefore, ${}^\circ F_h$ is not a continuous function on T, when we use the 1-norm as our metric on the test space T.

Notice that, since every test function $f \in T$ is bounded on $\mathbb{T}$, so ${}^*f$ is limited on ${}^*\mathbb{T}$. Thus, the standard part ${}^\circ F_h(f)$ exists for every $f \in T$, if $\frac{1}{N}\sum\limits_{k\in X}|h(k)|$ is limited. Therefore, we obtain necessary and sufficient condition for ${}^\circ F_h(f)$ to exist, as shown in the following proposition.

Proposition 7.0.17. Let $h\colon X \to {}^*\mathbb{C}$ be an internal function. If $\frac{1}{N}\sum\limits_{k\in X}|h(k)|$ is limited, then

$$^\circ F_h(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X} {}^*f(\frac{2\pi k}{N})h(k)\Big)$$

exists for all $f \in T$.

Proof. Let $f\colon \mathbb{T} \to \mathbb{C}$ be a test function. Then f is a continuous function on a compact set $\mathbb{T}$. Therefore, f is a bounded function on $\mathbb{T}$. There exists $M \in \mathbb{N}$ such that $|f(t)| < M$, for all $t \in \mathbb{T}$. So, $|{}^*f(t)| < M$, for all $t \in {}^*\mathbb{T}$. To prove that ${}^\circ F_h(f)$ exists, Consider

$$\Big|\frac{1}{N}\sum_{k\in X} {}^*f(\frac{2\pi k}{N})h(k)\Big| \leqslant \frac{1}{N}\sum_{k\in X}\Big|{}^*f(\frac{2\pi k}{N})\Big|\big|h(k)\big| < \frac{M}{N}\sum_{k\in X}\big|h(k)\big|,$$

which is limited. Therefore, $\mathrm{st}\big(\frac{1}{N}\sum\limits_{k\in X} {}^*f(\frac{2\pi k}{N})h(k)\big)$ exists for all $f \in T$. Hence, ${}^\circ F_h(f)$ exists for all $f \in T$. □

Notice that in Proposition 7.0.17, if $h\colon X \to {}^*\mathbb{C}$ is an S-integrable function then ${}^\circ F_h(f)$ exists for all $f \in T$, since $\frac{1}{N}\sum\limits_{k\in X}|h(k)|$ is limited.

Proposition 7.0.18. Let $h\colon X \to {}^*\mathbb{C}$ be an internal function such that $\frac{1}{N}\sum\limits_{k\in X}|h(k)|$ is limited and T be the test space of exponential polynomial functions $f\colon \mathbb{T} \to \mathbb{C}$. Then the functional ${}^\circ F_h\colon T \to \mathbb{C}$ defined by

$$
{}^\circ F_h(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}{}^*f(\frac{2\pi k}{N})h(k)\Big)
$$

is continuous and linear.

Proof. There is a standard $M \in \mathbb{N}$ such that $\frac{1}{N}\sum\limits_{\omega\in X}|h(\omega)| \leqslant M$. To prove that ${}^\circ F_h\colon T \to \mathbb{C}$ is continuous, we have to show that for all $f \in T$ and all standard $\varepsilon > 0$, there is a standard $\delta > 0$ standard such that

$$
\forall g\ (\forall t\ |f(t)-g(t)| < \delta \Rightarrow |{}^\circ F_h(f) - {}^\circ F_h(g)| < \varepsilon).
$$

If $|f(t)-g(t)| < \delta$ for all $t \in \mathbb{T}$, then consider

$$
|{}^\circ F_h(f) - {}^\circ F_h(g)| = \mathrm{st}\Big(\frac{1}{N}\Big|\sum_{k\in X}({}^*f(\frac{2\pi k}{N}) - {}^*g(\frac{2\pi k}{N}))h(k)\Big|\Big) \leqslant \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}\delta|h(k)|\Big) \leqslant \mathrm{st}(\delta M).
$$

So, we achieve our goal by setting $\delta = \frac{\varepsilon}{2M}$. Now, to prove that ${}^\circ F_h$ is a linear on T, let $f, g \in T$ and $\lambda \in \mathbb{C}$. Then

$$
{}^\circ F_h(f+g) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}{}^*f(\frac{2\pi k}{N})h(k) + \frac{1}{N}\sum_{k\in X}{}^*g(\frac{2\pi k}{N})h(k)\Big) = {}^\circ F_h(f) + {}^\circ F_h(g).
$$

In addition,

$$
{}^\circ F_h(\lambda f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}{}^*(\lambda f)(\frac{2\pi k}{N})h(k)\Big) = \lambda\,\mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}{}^*f(\frac{2\pi k}{N})h(k)\Big) = \lambda\,{}^\circ F_h(f)
$$

Hence, ${}^\circ F_h$ is a linear functional on T. □

Example 7.0.19. Let T be the test space of exponential polynomial functions $f\colon \mathbb{T} \to \mathbb{C}$. The internal function $\delta\colon X \to {}^*\mathbb{C}$ defined by $\delta(0) = N$ and $\delta(k) = 0$ for all $k \neq 0$, is called the *Dirac* *delta (generalised) function*. Then

$$F_\delta(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X} {}^*f(\frac{2\pi k}{N})\delta(k)\Big)$$

exists and since $\delta(k) = 0$ for all $k \in X \setminus \{0\}$ we have

$$\frac{1}{N}\sum_{k\in X} {}^*f(\frac{2\pi k}{N})\delta(k) = \frac{1}{N}{}^*f(0)(N) = {}^*f(0) = f(0).$$

The main theorem of this chapter shows that any continuous linear functional F is equal to ${}^\circ F_h$ for some internal functions h.

Theorem 7.0.20. Let $F\colon T \to \mathbb{C}$ be a continuous linear functional. Then there is an internal $h\colon X \to {}^*\mathbb{C}$ such that for all $f \in T$,

$${}^\circ F_h(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})h(\omega)\Big)$$

is defined and ${}^\circ F_h(f) = F(f)$, for all $f \in T$.

Proof. Consider ${}^*F\colon {}^*T \to {}^*\mathbb{C}$ and look at the classical Fejér functions

$$\Phi_k(t) = \sum_{m=-k}^{k}(1 - \frac{|m|}{k+1})e^{imt}.$$

Note $\Phi_k \in {}^*T$ for all $k \in {}^*\mathbb{N}$. We choose $k > \mathbb{N}$ in ${}^*\mathbb{N}$, so that for all $g \in {}^*T$ of degree $\leqslant 2k$ and with coefficients $\widehat{g}(j)$ with $|\widehat{g}(j)| \leqslant 2k$, we have

$$\forall t,s \in {}^*\mathbb{T}\Big(|t-s| < \frac{1}{N} \Rightarrow |g(t) - g(s)| < \frac{1}{2k}\Big).$$

$\big($The existence of such k is by overspill, since for each $k \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that for all $g \in T$ of degree $\leqslant 2k$ with $|\widehat{g}(j)| \leqslant 2k$ for all j, we have $\forall t,s \in \mathbb{T}(|t-s| < \frac{1}{K} \Rightarrow |g(t)-g(s)| < \frac{1}{2k}).\big)$

As usual $\mathbb{T}$ (or ${}^*\mathbb{T}$) acts on T (or ${}^*T$) by $f \mapsto f^s$, where $f^s(t) = f(t-s)$.

We define $h\colon X \to {}^*\mathbb{C}$ by

$$h(\omega) = {}^*F(\Phi_k^{2\pi\omega/N}).$$

Now, let $f \in T$ be arbitrary, we show ${}^\circ F_h({}^*f)$ is defined and ${}^\circ F_h({}^*f) = F(f)$. Note

$$F_h({}^*f) = \frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})h(\omega) = \frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N}){}^*F(\Phi_k^{2\pi\omega/N}) = {}^*F\Big(\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}\Big)$$

by linearity of F and Transfer. To show $F_h({}^*f) \approx {}^*F({}^*f)$, it suffices, by continuity of F and Transfer, to prove that

$$\sup_{t\in{}^*\mathbb{T}}\Big|\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t) - {}^*f(t)\Big| \approx 0. \tag{1}$$

We use classical harmonic analysis notation as in Katznelson [16]. Recall that $f \in T$, that is,

$$f(t) = \sum_{j=-\ell}^{\ell} \widehat{f}(j)e^{ijt},$$

where $\ell \in \mathbb{N}$ is the degree of f, and $\widehat{f}(j) \in \mathbb{C}$. We also have

$$\sigma_n(f,t) = (\Phi_n * f)(t) = \sum_{j=-n}^{n}(1-\frac{|j|}{n+1})\widehat{f}(j)e^{ijt} \to f(t)$$

as $n \to \infty$. Given $n > \ell$, the error term here is

$$|f(t)-\sigma_n(f,t)| = \Big|\sum_{j=-\ell}^{\ell}\frac{|j|}{n+1}\widehat{f}(j)e^{ijt}\Big| \leqslant \frac{\ell(\ell+1)}{n+1}\max_j|\widehat{f}(j)|. \tag{2}$$

Note also that

$$\sigma_n(f,t) = (\Phi_n * f)(t) = \frac{1}{2\pi}\int_{\mathbb{T}}\Phi_n^\tau(t)f(\tau)d\tau. \tag{3}$$

By the triangle inequality, for any $t \in {}^*\mathbb{T}$, we have

$$\begin{aligned}\Big|\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t) - {}^*f(t)\Big| \leqslant & \Big|\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t) - (\Phi_k * {}^*f)(t)\Big| + \\ & \Big|(\Phi_k * {}^*f)(t) - {}^*f(t)\Big|.\end{aligned}$$

Inequality (2) already shows that $|(\Phi_k * {}^*f)(t) - {}^*f(t)| \leqslant \frac{\ell(\ell+1)}{k+1}\max_j|\widehat{f}(j)| \approx 0$, since ℓ and all the $\widehat{f}(j)$ are standard and $k > \mathbb{N}$.

For $\left|\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t) - (\Phi_k * {}^*f)(t)\right|$, note that, by (3) and Transfer, the convolution $\Phi_k * {}^*f$ is a function ${}^*\mathbb{T} \to {}^*\mathbb{C}$ defined by an integral

$$(\Phi_k * {}^*f)(t) = \frac{1}{2\pi}\int_{{}^*\mathbb{T}} \Phi_k^\tau(t)^*f(\tau)d\tau. \tag{4}$$

Indeed, $\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t)$ is the nonstandard approximation to this integral. We just need to check that this approximation is accurate.

As $\Phi_k^\tau(t)^*f(\tau)$ is an exponential polynomial of degree $k+\ell < 2k$ with $t \in \mathbb{T}$, all coefficients having finite modulus, our choice of k applies and for all $\frac{2\pi\omega}{N} \leqslant \tau < \frac{2\pi(\omega+1)}{N}$ we have

$$\left|\Phi_k^\tau(t)^*f(\tau) - \Phi_k^{2\pi\omega/N}(t)^*f(\frac{2\pi\omega}{N})\right| < \frac{1}{2k}.$$

Thus, summing and integrating (4),

$$\left|\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t) - \frac{1}{2\pi}\int_{{}^*\mathbb{T}} \Phi_k^\tau(t)^*f(\tau)d\tau\right| < \frac{1}{k},$$

since the range of integration and summation is normalised to 1 in both. This is as we required.

Hence,

$$\left|\frac{1}{N}\sum_{\omega\in X} {}^*f(\frac{2\pi\omega}{N})\Phi_k^{2\pi\omega/N}(t) - {}^*f(t)\right| \approx 0,$$

as required. $\square$

Chapter 8

Towards Abstract Harmonic Analysis by Nonstandard Methods

Our aim in this chapter is to investigate how the general theory in Chapter 4 might be applied to abstract harmonic analysis and whether the material in Chapters 6 and 7 can be extended to the more general case of an abelian group G equipped with an invariant metric. The work here is only at a preliminary stage.

Our starting point is an infinite abelian group G equipped with an invariant metric d_G (or any first countable Hausdorff topological group using Theorem 4.1.12). We write G additively. Where necessary, we suppose that the nonstandard universe is sufficiently saturated.

According to Theorem 4.3.2, there is a *finite nonstandard group F with an invariant metric d_F such that the embedding $\phi\colon G \to F/\mathscr{N}$ is an isometry, where $\mathscr{N}$ is the monad of the neutral element of F. We define

$$F_{\text{fin}} = \{f : f + \mathscr{N} \in \text{Im}(\phi)\}$$

by analogy to $^*\mathbb{R}_{\text{fin}}$ in the nonstandard real world $^*\mathbb{R}$ (Section 2.3).

Notice that F_{fin} is not necessarily internal. In Section 5.1.4, $F_{\text{fin}} = F$, so it is internal, whereas in Section 5.1.3, $F_{\text{fin}} \subsetneq F$ and is not internal. Also there is nothing necessarily 'finite' about the elements of F_{fin}.

Since $\phi\colon G \to F/\mathscr{N}$ is an embedding, $\phi\colon G \to F_{\text{fin}}/\mathscr{N}$ is an isomorphism.

We define the standard part map st as the function $\mathrm{st}\colon F_{\mathrm{fin}} \to G$ such that $\mathrm{st}(f) = \phi^{-1}(f+\mathscr{N})$. Without any confusion we shall also use the standard part maps $\mathrm{st}\colon {}^*\mathbb{R}_{\mathrm{fin}} \to \mathbb{R}$ and $\mathrm{st}\colon {}^*\mathbb{C}_{\mathrm{fin}} \to \mathbb{C}$.

The subject of harmonic analysis, as explained in Chapter 3, is functions from G to $\mathbb{C}$ especially functions that are continuous or integrable. Such functions often arise from internal functions $\psi\colon F \to {}^*\mathbb{C}$. We emphasise at this point that this study may be made easier by the fact that F behaves just as if it were finite group. We define ${}^\circ\psi\colon G \to \mathbb{C}$ by ${}^\circ\psi(g) = \mathrm{st}(\psi(f))$ whenever $g = \mathrm{st}(f)$.

An immediate question that arises here is: given an internal function $\psi\colon F \to {}^*\mathbb{C}$, when is ${}^\circ\psi(g) = \mathrm{st}(\psi(f))$ well-defined? Answering this question, we notice that ${}^\circ\psi$ is well-defined if ${}^\circ\psi(g)$ does not depend on the choice of f and $\psi(f) \in {}^*\mathbb{C}_{\mathrm{fin}}$, for all such f. That is, ${}^\circ\psi$ will be well-defined if:

(a) for all $f \in F_{\mathrm{fin}}$, $\psi(f) \in {}^*\mathbb{C}_{\mathrm{fin}}$;

(b) for all $f, f' \in F_{\mathrm{fin}}$, if $f \approx f'$ then $\psi(f) \approx \psi(f')$ in ${}^*\mathbb{C}$.

Note 8.0.21. For convenience, we call these two conditions the *well-definedness conditions*.

Another question here is: which functions $G \to \mathbb{C}$ arise as ${}^\circ\psi$ for some internal functions $\psi\colon F \to {}^*\mathbb{C}$? Theorem 8.0.23 below is the answer to this question. Following is an easy lemma.

Lemma 8.0.22. If $f_0 \in F_{\mathrm{fin}}$ and $f \approx f_0$ then $f \in F_{\mathrm{fin}}$.

Theorem 8.0.23. Assume $F_{\mathrm{fin}}/\mathscr{N} \subseteq F/\mathscr{N}$ is an open set. If $\psi\colon F \to {}^*\mathbb{C}$ satisfies the well-definedness conditions then ${}^\circ\psi\colon G \to \mathbb{C}$ is continuous. Conversely, any continuous function $\theta\colon G \to \mathbb{C}$ is ${}^\circ\psi$ for some internal function $\psi\colon F \to {}^*\mathbb{C}$ satisfying well-definedness conditions.

Proof. Assume $\psi\colon F \to {}^*\mathbb{C}$ satisfies the well-definedness conditions. To show that ${}^\circ\psi\colon G \to \mathbb{C}$ is continuous, let $\varepsilon > 0$ be in $\mathbb{R}$ and $g_0 \in G$. Then there is $f_0 \in F_{\mathrm{fin}}$ such that $g_0 = \mathrm{st}(f_0)$ and

$$\forall f \in F (f \approx f_0 \Rightarrow \psi(f) \approx \psi(f_0)).$$

Then

$$\forall f \in F (f \approx f_0 \Rightarrow d_{^*\mathbb{C}}(\psi(f), \psi(f_0)) < \varepsilon).$$

So

$$\forall f \in F(d_F(f, f_0) < 1/k \Rightarrow d_{^*\mathbb{C}}(\psi(f), \psi(f_0)) < \varepsilon)$$

is true for all $k \in {}^*\mathbb{N} \setminus \mathbb{N}$. By overspill, there is $k \in \mathbb{N}$ such that

$$\forall f \in F(d_F(f, f_0) < 1/k \Rightarrow d_{^*\mathbb{C}}(\psi(f), \psi(f_0)) < \varepsilon).$$

Therefore, $\forall g \in G(d_G(g, g_0) < 1/(k+1) \Rightarrow d_{\mathbb{C}}({}^\circ\psi(g), {}^\circ\psi(g_0)) < \varepsilon)$. Hence, ${}^\circ\psi$ is continuous on G.

Conversely, assume $\theta\colon G \to \mathbb{C}$ is continuous and the universe is $|G|^+-$saturated. So we can write down $|G|$ sentences. Enumerate $G = \{g_i : i \in \mathscr{G}\}$. So $|G| = |\mathscr{G}|$. For each g_i, choose $f_i \in F_{\text{fin}}$ such that $\text{st}(f_i) = g_i$.

Since θ is continuous, for all $g_i \in G$ and all $\varepsilon = 2^{-j} > 0$, there is $\delta = 2^{-k}$ such that

$$d_G(g, g_i) < \delta \Rightarrow d_{\mathbb{C}}(\theta(g), \theta(g_i)) < \varepsilon.$$

Define $k\colon \mathscr{G} \times \mathbb{N} \to \mathbb{N}$ so that $\delta = 2^{-k(i,j)}$, that is, k is chosen so that

$$\forall g \in G(d_G(g, g_i) < 2^{-k(i,j)} \Rightarrow d_{\mathbb{C}}(\theta(g), \theta(g_i)) < 2^{-j}) \text{ and } B_{2^{-k(i,j)}}(f_i) \subseteq F_{\text{fin}} \text{ for all } i, j.$$

The last condition here can be satisfied since $F_{\text{fin}}/\mathscr{N}$ is open. Now write down sentences describing some internal $\psi\colon F \to {}^*\mathbb{C}$,

$$\psi(f_i) = \theta(g_i) \text{ for all } i \in \mathscr{G}. \tag{1}$$

$$\forall f \in F(d_F(f, f_i) < 2^{-k(i,j)} \Rightarrow d_{^*\mathbb{C}}(\psi(f), \psi(f_i)) < 2^{-j+1}). \tag{2}$$

There are $|\mathscr{G}| + |\mathscr{G}||\mathbb{N}| = |\mathscr{G}|$ such sentences. So if we can show they are ((1) and (2)) finitely satisfied there will be some ψ satisfying them simultaneously.

Consider finitely many $i_1, i_2, \ldots, i_n$ in $\mathscr{G}$ and $j_1, j_2, \ldots, j_n$ in $\mathbb{N}$. Then we have finitely many balls of different radii about $f_{i_1}, f_{i_2}, \ldots, f_{i_n}$, $B_{2^{-k(i,j)}}(f_i)$ for $i \in \{i_1, i_2, \ldots, i_n\}$ and $j \in \{j_1, j_2, \ldots, j_n\}$.

Say $f,g \in F$ are in the same part if $f \in B \Leftrightarrow g \in B$ for each ball B as above. This gives a partition of F into finitely many parts p. Without loss of generality (if necessary choosing additional values $j \in \mathbb{N}$). We may assume that no two f_{i_α}, f_{i_β} are in the same part. For each part p, containing some $f \in F_{\text{fin}}$, choose one such $f_p \in F_{\text{fin}} \bigcap p$. If p contains one of the f_i (for $i \in \{i_1, i_2, \ldots, i_n\}$), we will choose that f_i for f_p.

Define ψ to be constant on parts of F, that is, $\psi(f) = \psi(g)$ if f and g are in the same part p, and define $\psi(f_p) = \theta(\text{st}(f_p))$. Because of finiteness, such ψ is internal.

If some part p has no $f \in F_{\text{fin}}$ inside it, we define ψ on p to be the constant function with value 1. In this case, p is not a subset of any open ball $B_{2^{-k(i,j)}}(f_i)$, since $B_{2^{-k(i,j)}}(f_i) \subseteq F_{\text{fin}}$ for all i, j.

To show that (1) and (2) hold for all $i \in \{i_1, i_2, \ldots, i_n\}$ and all $j \in \{j_1, j_2, \ldots, j_n\}$, we notice that $\psi(f_i) = \theta(g_i)$ for all $i \in \mathscr{G}$ was as defined. That is, ψ satisfied (1) finitely.

Now, let f_{p_1} and f_{p_2} belong to two different parts, say p_1 and p_2, respectively.

Suppose $p_1, p_2 \subseteq B_{2^{-k(i,j)}}(f_i)$, where $i \in \{i_1, i_2, \ldots, i_n\}$, and $j \in \{j_1, j_2, \ldots, j_n\}$. We have to show that

$$d_{\mathbb{C}}(\theta(\text{st}(f_{p_1})), \theta(\text{st}(f_{p_2}))) < 2^{-j}.$$

However,

$$d(\text{st}(f_{p_\alpha}), \text{st}(f_i)) < 2^{-k(i,j)}$$

for $\alpha = 1, 2$ in G. Then,

$$d_{\mathbb{C}}(\theta(\text{st}(f_{p_\alpha})), \theta(\text{st}(f_i))) < 2^{-j},$$

for $\alpha = 1, 2$. Therefore,

$$d_{\mathbb{C}}(\theta(\text{st}(f_{p_1})), \theta(\text{st}(f_{p_2}))) < 2(2^{-j})$$

by the triangle inequality. That is, (2) is satisfied finitely by ψ. Therefore, by $|G|^+$−saturation there is ψ satisfying (1) and (2).

Now, we have to show that ψ is S-continuous. Suppose $f, g \in F$ and $f \approx g$. So $f - g \in \mathscr{N}$. Then, either $f \notin F_{\text{fin}}$, in which case $g \notin F_{\text{fin}}$ and $\psi(f) = \psi(g) = 1$, or $f \in F_{\text{fin}}$ and then $g \in F_{\text{fin}}$. So $d(f,g) \approx 0$. Let $i \in \mathscr{G}$ such that if $f \approx f_i$ then $g \approx f_i$. By (2), $\psi(f) \approx \psi(f_i)$ and $\psi(g) \approx \psi(f_i)$.

Therefore, $\psi(f) \approx \psi(g)$, as required.

Finally, it is obvious that $\theta = {}^{\circ}\psi$ from (1). $\square$

We now consider the 1-dimensional representations (characters) that form a basis of the vector space of functions $F \to {}^{*}\mathbb{C}$.

Classical Fourier analysis on the circle group $\mathbb{T}$ uses exponential functions as a "basis" for vector spaces of functions $\mathbb{T} \to \mathbb{C}$. This is analogous to using the functions

$$e_k \colon C_N \to \mathbb{C}$$

defined by

$$e_k(j) = e^{2\pi i jk/N}$$

on finite cyclic groups C_N to study $\mathbb{C}^{C_N}$, as was done in Chapter 6. The present aim is to generalise this approach to other groups.

The functions $e_k \colon C_N \to \mathbb{C}$ form a complete set of 1-dimensional representations $C_N \to \mathbb{C}$. Thus a suitable generalisation to other groups F would be to replace these functions by a set of 1-dimensional representations. This is exactly what is happening in Fourier analysis on finite (discrete) groups (Section 3.1).

Let V_F be the vector space of internal functions $F \to {}^{*}\mathbb{C}$, and equip V_F with the inner product

$$\langle f, g \rangle = \frac{1}{|F|} \sum_{t \in F} f(t)\overline{g(t)}.$$

Note that V_F has dimension $|F| \in {}^{*}\mathbb{N}$.

Let E be a complete set of 1-dimensional (irreducible) representations of F. That is, each $e \in E$ is an internal function $e \colon F \to {}^{*}\mathbb{C}$, such that:

$$e(x+y) = e(x)e(y) \text{ and } e(0) = 1.$$

Then by Transfer and standard results in representation theory, it is known [15, Chapter 9] that there are exactly $|F|$ independent representations and these form a basis of V_F, which is orthonormal for the inner product $\langle\cdot,\cdot\rangle$.

It is easy to check that any 1-dimensional representation $e\colon F\to {}^*\mathbb{C}$ maps into the nonstandard version of the circle group ${}^*\mathbb{T}_{\mathbb{C}}=\{z\in {}^*\mathbb{C}:|z|=1\}$ and ${}^*\mathbb{T}_{\mathbb{C}}\subseteq {}^*\mathbb{C}_{\text{fin}}$.

Note also (again by transfer) that any internal function $F\to {}^*\mathbb{C}$ is a linear combination of functions in E:

$$f=\sum_{e\in E}\langle f,e\rangle e;$$

the coefficients $\langle f,e\rangle$ here may be regarded as the "Fourier coefficients" of the function f with respect to the set E.

However, there is an immediate problem at this point, which we are currently unable to resolve.

Question 8.0.24. If E is as above, when is it the case that each $e\in E$ S-continuous?

In general, functions $e\in E$ need not be S-continuous, as given in the following example.

Example 8.0.25. Let F be a *finite *cyclic group C_N with addition modulo N, where $N>\mathbb{N}$ and the *metric d defined on F by $d(x,y)=\frac{1}{N}\min\{|x-y|,N-|x-y|\}$. Note that the following 1-dimensional representation in E is not S-continuous. Consider $e_y\colon F\to {}^*\mathbb{T}_{\mathbb{C}}=\{z\in {}^*\mathbb{C}:|z|=1\}$ defined by

$$e_y(x)=e^{2\pi ixy/N}$$

where $0\leqslant x,y<N$. Notice that 0 and $\sqrt{N}$ are in the monad(0) of F and $0\approx\sqrt{N}$ since $d(\sqrt{N},0)=\frac{1}{N}\min\{|\sqrt{N}|,N-|\sqrt{N}|\}=\frac{1}{\sqrt{N}}\approx 0$. While, if $y=\lfloor\sqrt{N}/2\rfloor$, then

$e_y(0)=e^{2\pi i(0)\lfloor\sqrt{N}/2\rfloor/N}=e^0=1$ and $e_y(\lfloor\sqrt{N}\rfloor)=e^{2\pi i\lfloor\sqrt{N}\rfloor\lfloor\sqrt{N}/2\rfloor/N}=e^{\pi i}=-1.$

That is e_y is not S-continuous, where $y=\lfloor\sqrt{N}/2\rfloor$.

Even if we cannot use every 1-dimensional representation of F, we might hope to represent every continuous homomorphism $G\to\mathbb{T}$ as ${}^\circ\psi$ for some S-continuous homomorphism $\psi\colon F\to {}^*\mathbb{C}$. However, even this is not clear.

Question 8.0.26. Assume $F_{\text{fin}}/\mathscr{N}\subseteq F/\mathscr{N}$ is an open set and $\theta\colon G\to\mathbb{T}$ is a character (continuous homomorphism). Is there $\psi\colon F\to {}^*\mathbb{C}$ such that $\theta={}^\circ\psi$ and ψ a homomorphism?

Note that, adding

$$\forall f,g \in F(\psi(f+g) = \psi(f)\psi(g)) \tag{3}$$

to the proof of Theorem 8.0.23 is not possible: the resulting formulas are no longer finitely satisfied, and we do not know the answer to the above question.

However, according to Theorem 8.0.23, there is an internal S-continuous function $\psi\colon F \to {}^*\mathbb{C}$ such that $\theta = {}^\circ\psi$, and $\psi(f) = 1$, when $f \notin F_{\mathrm{fin}}$. If $\theta\colon G \to \mathbb{C}$ is a homomorphism then we have $\psi(f+g) \approx \psi(f)\psi(g))$ for all $f,g \in F$. So ψ is a "nearly" homomorphism.

Therefore, the approach outlined does indeed give nonstandard descriptions of all characters $\theta\colon G \to \mathbb{C}$. It would be worthwhile to try to reprove results in the classical harmonic analysis by nonstandard means. In particular, the following is a special case of a result by Peter, Weyl and van Kampen (see Higgins [12, page 100]).

Theorem 8.0.27. Let G be abelian with invariant metric and $x \in G \setminus \{0\}$. Then there is a continuous homomorphism $\theta\colon G \to \mathbb{T}_{\mathbb{C}}$ such that $\theta(x) \neq 1$.

It would be an interesting project to reprove this by nonstandard means. Given $x \in F$ with $x \notin \mathscr{N}$, one would have to construct an S-continuous nearly-homomorphism $\psi\colon F \to {}^*\mathbb{T}$ such that $\psi(x) \not\approx 1$. Theorems 8.0.23 and 8.0.27 show this is possible in principle. The goal would be to find a straightforward proof, using the structure of the *finite group F perhaps. We have as yet been unable to find a satisfactory simple proof along these lines. An alternative approach might be to attempt to enhance Theorem 4.3.7, so that, when F is written as a direct sum of cyclic groups $\bigoplus_i A_i$ (using the Basis theorem and Transfer), each component A_i has $\mathscr{N} \cap A_i$ equal to some 'convex' region about 0, so that the examples such as Example 8.0.25 fail, and all the usual representations of A_i are S-continuous.

Chapter 9

Conclusion

In this chapter we review the work done in this thesis. We refer to the main results obtained. In addition, we discuss where things could be taken further. We conclude this chapter with a number of open questions for further research.

9.1 A Brief Summary of the Thesis

Chapter 2 covered the basics of nonstandard analysis. We attempted to make this chapter self-contained, and we included basic and necessary materials on NSA, which were very helpful in applying NSA to both representations of classical topological (metric) groups by nonstandard metric groups and to convergence of Fourier transforms. Several interesting examples were given throughout the chapter in order that the subjects be more clear. Also, this chapter gave the basic tools for the construction of nonstandard structure of L^1 space of a nonstandard *finite set F with some relevant concepts.

Chapter 3 explained the main parts of the abstract harmonic analysis, such as abstract harmonic analysis on finite groups and abstract harmonic analysis on topological groups. We recalled the basic definitions of Fourier series of functions defined on the topological circle group $\mathbb{T}$ in view their importance. The main aim of this chapter was to present classical DFT and its inverse IDFT on finite groups, as well as Fourier coefficients and Fourier series on topological groups.

Certainly, nonstandard analysis works by using discrete methods with infinitesimal 'step size' (often, $\frac{1}{N}$) to approximate classical analysis via the standard part map. For Fourier analysis we can use the idea of the discrete Fourier transform in this way. In other words, a number of basic results in classical Fourier analysis follow directly from the analogous results in discrete Fourier analysis together with nonstandard techniques. Therefore, we can say that nonstandard analysis is a good way of doing Fourier analysis.

Chapter 4 started with the definition of monotonically definable subset $\mathscr{N}$ of the nonstandard *finite group F. We proved some nice properties of $\mathscr{N}$, such as: $\mathscr{N}_{i+1}^2 \subseteq \mathscr{N}_i$; $\mathscr{N}_i^{-1} = \mathscr{N}_i$; and $\mathscr{N}_i^x = \mathscr{N}_i$, for all i and all $x \in F$, where $\mathscr{N} = \cap_{i\in\mathbb{N}}\mathscr{N}_i$, for some sequence $\{\mathscr{N}_i\}_{i\in\mathbb{N}}$ of internal subsets of F. Also the following results were obtained:

(1) an external normal subgroup $\mathscr{N} \lhd F$ is monotonically definable if and only if it is $\mathscr{N}_d$ for some *metric d on F (the metrisation theorem);

(2) any first-countable Hausdorff topological group G is metrisable with 2-sided invariant metric generating the same topology.

The measure μ on subsets of nonstandard finite group F and the quotient $F/\mathscr{N}$ was defined. The main facts obtained through this kind of measure are:

(1) Assume $\mu(F) \gtrsim 0$ in $^*\mathbb{R}_{\text{fin}}$. If the normal subgroup $\mathscr{N}$ is measurable (as a subset of F) then $^\circ\mu(\mathscr{N}) = 0$ if and only if $\mathscr{N}$ has infinite index in F.

(2) Let F be a *finite group, $\mathscr{N} \lhd F$ and $0 < \varepsilon < \eta \leq 1$ be real numbers. Then there is $k \in \mathbb{N}$, a measurable set $A \subseteq F/\mathscr{N}$, with $\varepsilon < {}^\circ\mu A < \eta$, and elements $a_0, \ldots, a_{k-1} \in F$, such that $F/\mathscr{N} = \bigcup_{i<k} a_i A$. More specifically, this can be achieved whenever $(1-\varepsilon)^k < \eta - \varepsilon$.

We defined locally embeddability of classical metric groups with 2-sided metric into nonstandard finite metric groups (LEFM) and the following interesting results were proved.

(1) a 2-sided metric group $(G, \cdot, d)$ embeds as a metric group into some $(\prod_D G_i)/\mathscr{N}$ if and only if, for all $\varepsilon > 0$ and all finite subsets $A \subseteq G$, there is a finite 2-sided metric group H and a function $\phi \colon A \to H$ such that

  (a) $d(\phi(ab), \phi(a)\phi(b)) < \varepsilon$, whenever $a, b, ab \in A$; and

  (b) $|d(\phi(a), \phi(b)) - d(a, b)| < \varepsilon$, whenever $a, b \in A$;

(2) every abelian group with invariant metric is LEFM;

(3) assume $(G,\cdot,d_G)$ is a 2-sided invariant metric group and $(F,\cdot,d_F)$ is *finite *metric group, such that $\phi\colon G\to F/\mathcal{N}$ is a surjective embedding of metric groups, where $\mathcal{N}\lhd F$ is the monad of the identity. If G is separable, then G is compact.

Also we defined locally embeddability of classical metric groups with 2-sided metric into non-standard discrete metric groups (LEDM) and we proved the following results:

(1) The theorem of LEDM groups (Theorem 4.4.4).

(2) Every 2-sided metric group is LEDM.

(3) Let a 2-sided metric group G be LEDM via the function $\phi\colon G\to D/\mathcal{N}$, where D is a *discrete metric group and $\mathcal{N}$ is the monad of the identity of D. If $X\subseteq D_G$ is internal, where $D_G=\{x\in D:\exists g\in G \text{ such that } \phi(g)=x\mathcal{N}\}$, then $X/\mathcal{N}$ is bounded and closed.

(4) Let a 2-sided metric group G be LEDM via the function $\phi\colon G\to D/\mathcal{N}$, where D is *discrete and $\mathcal{N}$ is the monad of the identity. If G is separable and $X\subseteq D_G$ is internal, then $X/\mathcal{N}$ is compact.

Chapter 5 showed and explained that the following classical abelian metric groups are LEFM:

(1) the group of integers $\mathbb{Z}$ with the usual metric;

(2) the direct sum of $\mathbb{Z}\oplus\mathbb{Z}$ with the discrete metric;

(3) the additive real group $\mathbb{R}$ with the normalised usual metric;

(4) the additive circle group $\mathbb{T}$ with the normalised usual metric on circles;

(5) the additive 2-torus group $\mathbb{T}\oplus\mathbb{T}$ with the taxicab metric;

(6) the additive complex group $\mathbb{C}$ with the usual Euclidean metric;

(7) the additive group of p-adic integers $\mathbb{Z}_p$ with the metric d_p;

(8) the group $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, with the metric $\widehat{d}$;

(9) the additive group $C_2\oplus C_2$ with the discrete metric.

Also we showed the following classical nonabelian metric groups are LEFM:

(1) the dihedral circle group $D_{2\mathbb{T}}$ with a suitable metric d;

(2) the infinite dihedral group D_∞ with the discrete metric.

Chapter 6 used methods of NSA to study the representation of functions in $L^1(\mathbb{T})$ as series in Fourier analysis. As already indicated, NSA is an excellent tool to understand the Fourier transform. This chapter set up Fourier series by nonstandard methods, proved nonstandard versions of the main theorems on convergence. In this chapter we proved the following main results:

(1) If $f\colon \mathbb{T} \to \mathbb{C}$ is a piecewise continuous function and $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, then

$$\widehat{f}(n) = \mathrm{st}\Big(\frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{N/2} {}^*f(\frac{2\pi k}{N}) e^{-2\pi i n k/N}\Big), \text{ for all } n \in \mathbb{Z}.$$

(2) some useful properties of Dirichlet and Fejér functions;

(3) if $g \in L^1(\mathbb{T})$, then there is $H \in L^1(F)$, such that ${}^*g(\mathrm{st}(\frac{2\pi\omega}{N})) = {}^\circ H(\omega)$, for almost all $\omega \in F$, showing the relationship between the classical $L^1(\mathbb{T})$ and the nonstandard $L^1(F)$.

(4) if $f \in L^1(F)$, then ${}^\circ \lim\limits_{n\in\mathbb{N}} \|\sigma_n(f) - f\|_1 = 0$;

(5) the density of continuous functions in $L^1(F)$, that is, the approximation of Lebesgue integrable functions by S-continuous functions in $L^1(F)$;

(6) the density of trigonometric polynomials with standard degree in $L^1(F)$;

(7) the Riemann-Lebesgue theorem "if $f \in L^1(F)$, then ${}^\circ \lim\limits_{|n|\in\mathbb{N}} \widehat{f}(n) = 0$";

(8) the uniqueness theorem "if $f \in {}^*\mathbb{C}^F$ with $\widehat{f}(m) = 0$ for all $m \in F$, then $f = 0$";

(9) (Fejér Theorem) if $f \in L^1(F)$, $t_0 \in F$ and $y_0 \in {}^*\mathbb{C}$, such that $\mathrm{st}(y_0) = {}^\circ \lim\limits_{h\to 0} \frac{f(t_0+h)+f(t_0-h)}{2}$, then ${}^\circ \lim\limits_{n\in\mathbb{N}} \sigma_n(f, t_0) = \mathrm{st}(y_0)$.

Chapter 7 started to generalise functions to other cases such as functionals and generalised functions $F\colon T \to \mathbb{C}$, where T is the test space of exponential polynomial functions $f\colon \mathbb{T} \to \mathbb{C}$.

In this chapter we proved the following main results:

(1) let $h\colon X \to {}^*\mathbb{C}$ be an internal function. If $\frac{1}{N}\sum\limits_{k\in X}|h(k)|$ is limited, then for all $f \in T$, ${}^\circ F_h(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}{}^*f(\frac{2\pi k}{N})h(k)\Big)$ exist;

(2) let $h\colon X \to {}^*\mathbb{C}$ be an internal function. If $\frac{1}{N}\sum\limits_{k\in X}|h(k)|$ is limited, then the functional ${}^\circ F_h\colon T \to \mathbb{C}$ defined by ${}^\circ F_h(f) = \mathrm{st}\Big(\frac{1}{N}\sum_{k\in X}{}^*f(\frac{2\pi k}{N})h(k)\Big)$ is continuous and linear;

(3) if $F\colon T \to \mathbb{C}$ is a continuous linear functional then there is an internal $h\colon X \to {}^*\mathbb{C}$, such that for all $f \in T$, ${}^\circ F_h(f) = \mathrm{st}\big(\frac{1}{N}\sum\limits_{\omega\in X}{}^*f(\frac{2\pi\omega}{N})h(\omega)\big)$ is defined and ${}^\circ F_h(f) = F(f)$, for all $f \in T$.

Chapter 8 attempted to extend and generalise materials in Chapters 6 and 7 by using the general theory in Chapter 4 on abelian groups G with invariant metrics. We obtained the result: assume that $F_{\mathrm{fin}}/\mathscr{N}$ is an open subset of $F/\mathscr{N}$. If $\psi\colon F \to {}^*\mathbb{C}$ satisfies the well-definedness conditions, then ${}^\circ\psi\colon G \to \mathbb{C}$ is continuous. Conversely, any continuous function $\theta\colon G \to \mathbb{C}$ is ${}^\circ\psi$ for some internal function $\psi\colon F \to {}^*\mathbb{C}$ satisfying well-definedness conditions.

One advantage of using NSA is that NSA is good at producing "examples". For instance, we can easily define $\widehat{f}\colon F \to F$ by

$$\widehat{f}(t) = \begin{cases} 1 & \text{if } |t| < a \\ 0 & \text{if } |t| \geqslant a \end{cases}$$

and this $\widehat{f}$ is an internal function. Now, one might look at what one can say about the function $f = \sum\limits_{t\in F}\widehat{f}(t)e_t$. This is a nonstandard version of the Dirac delta function (see Example 7.0.19). In the same way NSA can be used to define other functionals by starting with other internal functions analogous to $\widehat{f}$ above.

So far, this shows that NSA is a good vehicle to study Fourier analysis. We would like to propose a plan of questions and areas for further study.

Central to the discrete Fourier transform is the cyclic group C_N and its action on spaces of functions. This corresponds to the role the circle group $\mathbb{T}$ plays in classical Fourier analysis.

9.2 Future Research

One obvious question is to extend the known examples of LEFM groups to other families of (non-abelian) 2-sided invariant metric groups. Clearly, a 2-sided metric group, that is locally finite as an abstract group, is LEFM.

Question 9.2.1. Is every 2-sided metric group, that is, residually finite as a topological group, LEFM? (G is residually finite as a topological group means that for all $g \in G$ with $g \neq 1$ there is a closed $N \lhd G$ such that $g \notin N$.)

We do not have an example of a 2-sided metric group which is not LEFM. Note that any sofic group $G \leq S_N/\mathscr{N}$ (where N is nonstandard) has a metric induced from the Hamming metric on S_N making it LEFM (for sofic groups, which are not defined here, see Pestov and Kwiatkowska [26]). Therefore, to find an abstract group, which is not LEFM for any metric on it, seems to be a particularly difficult question.

Finally, in reviewing the abelian examples of LEFM groups, an interesting question arises which we are not currently able to resolve. That is, the observation that, in all these examples, it sufficed to take C_N for F in the embedding $G \hookrightarrow F/\mathscr{N}$. We admit to being particularly surprised that this applies even to finite groups such as $C_2 \oplus C_2$. We therefore ask if this is generally true.

Question 9.2.2. Is it the case that every abelian group G with invariant metric embeds in $C_N/\mathscr{N}$ for some nonstandard cyclic group C_N, some nonstandard invariant metric d_N on C_N, where $\mathscr{N}$ is the monad of 0?

In view of the attempts in Chapter 8 to construct characters, it would seem interesting to ask if Theorem 4.3.7 could be extended so that the 1-dimensional representations of the *finite group F are automatically S-continuous.

Say, a subset $A \subseteq C_N = \{0, 1, \ldots, N-1\}$ is *convex* if, whenever $a, b \in A$, one of $\{a, a+1, \ldots, a+k-1, a+k = b\}$ or $\{b, b+1, \ldots, b+\ell-1, b+\ell = a\}$ is a subset of A, for k, ℓ chosen suitably and addition taken mod N. We ask the following question:

Question 9.2.3. Given abelian group G with an invariant metric, is there a *finite group F with metric d and embedding $\phi \colon G \hookrightarrow F/\mathscr{N}$ which is a homomorphism and isometry, such that for some description of F as $F = \bigoplus_{i=1}^{\nu} C_{N_i}$, given by Theorem 1.2.11, each of $\mathscr{N}_d \cap C_{N_i}$ is convex?

Question 9.2.4. Does the generalisation of functionals work on the other test spaces of functions defined on locally compact abelian groups G?

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