On counting problems in nonstandard models of Peano arithmetic with applications to groups

by

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Abstract

Coding devices in Peano arithmetic (PA) allow complicated finite objects such as groups to be encoded in a model $M \vDash \mathrm{PA}$. We call such coded objects M-finite. This thesis concerns M-finite abelian groups, and counting problems for M-finite groups. We define a notion of cardinality for non-M-finite sets via the suprema and infima of appropriate M-finite sets, if these agree we call the set M-countable.

We investigate properties of M-countable sets and give examples which demonstrate marked differences to measure theory. Many of the pathologies are related to the arithmetic of cuts and we show what can be recovered in special cases. We propose a notion of measure that mimics the Carathéodory definition.

We show that an M-countable subgroup of any M-finite group has an M-countable transversal of appropriate cardinality.

We look at M-finite abelian groups. After discussing consequences of the basis theorem we concentrate on the case of a single M-finite group $C(p^k)$ and investigate its external structure as an infinite abelian group. We prove that certain externally divisible subgroups of $C(p^k)$ have M-countable complements. We generalize this result to show that dG, the divisible part of G, has an M-countable complement for a general M-finite abelian G.

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Chapter 1

Introduction to Material and Review of Literature

1.1 Preliminaries

We shall outline in this section the areas that we explore in this thesis. The material is divided into three parts; M-countable sets, results on transversals and nonstandard Lagrange's theorem, and nonstandard abelian groups. We shall briefly describe the work carried out in each part, why we are interested in doing so, and cite analogies with previous work in the literature wherever appropriate.

Throughout, M will be a nonstandard $\mathcal{L}_{\mathrm{A}}$-structure satisfying Peano Arithmetic (PA) where $\mathcal{L}_{\mathrm{A}}$ is the usual first order language with $+, \cdot, <, 0, 1$. For background on models of PA see Kaye [8]. For a statement σ in this language we write $M \vDash \sigma$ for 'σ is true in M'.

Throughout this thesis, lower case Roman letters range over elements of M, internal subsets of M, or the extensions of such objects (taking care in situations where this might lead to confusion). Upper case roman letters refer to subsets of M that may or may not be M-finite. It will be stated which if not clear from context. When X is M-finite we will write $|X|$ to mean the number of elements of X in M.

The natural numbers of M contain all of the standard natural numbers and other nonstandard natural numbers. We use $\mathbb{N}$ to denote the set of standard natural numbers and ω^M for all the natural numbers of M, and we may assume that $\mathbb{N}$ is an initial segment of ω^M. The reader is directed to the definitions and results section of this introduction for more details on this and why these are valid assumptions.

1.2 M-countable Sets

The material presented in chapter 2 will appear in a forthcoming paper by Kaye and Reading. The results given here are not intended to be a complete account of the theory but rather they lay the foundations for the remainder of the thesis.

The idea of M-countability here is motivated by the observation that a bounded external subset X of M can be approximated by internal sets $x \subseteq X \subseteq x'$ and some measure of the 'size' of X can be gained. The sizes of the internal sets x, x' are simply their cardinality in the sense of M, and thus the inner and outer 'measures' of X are cuts corresponding to the supremum of $|x|$ over all internal $x \subseteq X$, and infimum of all $|x'|$ over all internal $x' \supseteq X$. In the case that X is M-finite the upper and lower cardinalities of X are simply equal to $|X|$. The idea of what we call here M-countability appeared in John Allsup's PhD thesis [1] and the author acknowledges that work. What we do here builds on and extends Allsup's work. It is written with the benefit of a better understanding of the arithmetic of cuts [9] for which the author's supervisor Richard Kaye must be acknowledged.

The idea is a modification of Loeb's construction of measure in nonstandard analysis. The reader is directed to *Nonstandard Analysis and its Applications* pages 27-34, [5], for more information on Loeb measure. Whilst there are some analogies with Loeb measure the notion of M-countability defined here is somewhat more awkward. The Loeb construction gives rise to an algebra of sets, the so called Loeb Algebra; whilst the M-countable sets do not form an algebra. Indeed we present results showing that even in the disjoint case M-countable sets X and Y can turn out to have non M-countable union, $X \cup Y$. The situation for intersections is also complex and we give an example to show that $X \cap Y$ where X is M-countable and Y is *M-finite* is not necessarily M-countable. Nevertheless we show that under certain conditions we can be sure that such $X \cap Y$ is M-countable for M-countable sets X and Y. On the other hand, by choosing to measure only to a degree of approximation corresponding to a given cut I one can, by a trick analogous to Carathéodory's, give a family of satisfactory measures with algebraic closure. This is the subject of section 2.3, and in some sense is analogous to Hausdorff measure, which is measure relative to a pre-chosen fractal dimension.

In one of the theorems in chapter 3 we shall need to make the assumption that two M-countable sets are separable in a certain sense. This motivates the study of separability in this chapter, and we give an example to show that not all M-countable sets are separable. However there are also plenty of examples of M-countable sets that are separable and this helps to justify the use of the assumption later on.

A detailed study of M-countable functions is carried out in the forthcoming paper on M-countability. The results given there are not needed for this thesis and so we omit M-countable functions.

Whilst there are some difficulties with this theory of M-countable sets there are also several things which are natural about it. The theorems on transversals in chapter 3 are very natural and the fact the cardinalities turn out as expected provides justification for this approach. The first of the transversal theorems is proved without the need for any additional assumptions (other than countability of M), a result which lends credence to the notion. The subgroups that arise naturally in chapter 4 also turn out to be M-countable in the sense studied in chapter 2. Monotonically definable sets are external sets that can be defined by varying a parametrised formula over a cut. Thus the resulting set is a monotone union or intersection of definable sets. We show that monotonically definable sets are M-countable (the converse is not true in general). Another good reason for using this notion of M-countable is that a rich class of sets turn out to be M-countable. The most natural examples of all, the initial segments, have cardinality equal to themselves. Although not all subsets of M are M-countable, any subset has a lower and an upper cardinality even if they are not the same. Much of the work we do on M-countable sets can be applied to the upper and lower cardinalities separately and so in a sense is applicable to any subset of M.

1.3 Results on Transversals and Nonstandard Lagrange's Theorem

In chapter 3 we shall apply some of the ideas of chapter 2 to groups. We take a nonstandard model $M \vDash \mathrm{ZF}^* - \mathrm{inf}$ (Zermelo-Fraenkel set theory with the axiom of infinity negated together with the axiom of transitive containment). This theory is equivalent to PA (see the definitions and results section) and so we may regard M as being a model of PA also. Our model M will often be countable, but sometimes not. Of course the theory $\mathrm{ZF} - \mathrm{inf}$ is strong enough to describe complicated finite objects, such as finite groups. Therefore it makes sense to consider an object $G \in M$ which, as far as M is concerned, is a finite group of size n. We say G is an M-finite group.

To motivate this study of M-finite groups in nonstandard models we define the notion of LEF group. The reader is directed to Pestov and Kwiatkowska [14] for more information. A group G is said to be locally embeddable into finite groups (a LEF group, for short) if for every finite subset $F \subseteq G$ there is a partially defined monomorphism $i : F \cup FF \to H$ for some finite group H. It is the case that any countable LEF group is a subgroup of some M-finite group G for $M \vDash Th(\mathbb{N})$. This means that our abstract study could potentially be used in the future to say something about LEF groups in general.

G may have an interesting external subgroup H. By external we mean that the underlying set of H is not M-finite but H is closed (in an external sense) under the group operation of G. We will be particularly interested in the case when the underlying set of H is M-countable and we begin the chapter by proving a closure condition on the cardinality of such H.

We define the index of a subgroup H in an M-finite group G and then prove some basic results in the case when that subgroup is M-countable. Some of this work has analogies with this paper: *On the orbit-sizes of permutation groups containing elements separating finite subsets* [3]. Indeed we prove a combinatorial lemma which is essentially a nonstandard version of a result found in that paper i.e. here we allow the sets concerned to be M-finite rather than actually finite.

In the case that M is countable we generalize the results to construct an M-countable transversal, T, for H in G under the assumption that G is an M-finite group, and $H \subseteq G$ is an M-countable subgroup. We construct the transversal by an external induction on $\mathbb{N}$ carefully ensuring that T has all the desired properties by including suitable conditions in the inductive hypothesis. T is constructed to be M-countable and the cardinality turns out to be what one would expect $|G|/I$, where $I = \mathrm{card}(H)$ is the cardinality of H. The assumption that M is countable is vital for the proof as we carry out various enumerations by omega sequences during the construction. All the lemmas required for the proof are true in the uncountable case but it is an open question whether the theorem holds in this case. This theorem is motivated by the links with the work on nonstandard abelian groups and also provides some justification for the abstract study of M-countable sets (in this sense) in first chapter.

We prove two propositions showing that it is also possible to violate the M-countability of a transversal T under the same set up as above. The first proposition shows that there is no restriction (other than obvious ones) on the lower cardinality of such a T. The second proposition does something similar for the upper cardinality. Between them, these results strengthen the transversal theorem because they imply that the careful inductive construction really was necessary in order to ensure M-countability.

The last theorem in this section is a generalization of the original transversal theorem. Here we assume G is an M-finite group and $H < K \leqslant G$ are M-countable subgroups with $\mathrm{card}(H) = I \subsetneq J = \mathrm{card}(K)$. We show that under certain assumptions there is a M-countable transversal T_K for H in K and that $\mathrm{card}(T) = J/I$ for a certain definition of J/I. The result here is sensitive to the definition of J/I, and also to technical concerns about the 'separability' of the underlying sets of H and K. The latter is a motivation for brief study in the next chapter. As before, we also need the assumption that M is countable.

These theorems are closely related to, and provide motivation for, the work on M-countable complements in chapter 4.

1.4 Nonstandard Abelian Groups

We look at nonstandard finite abelian groups in a nonstandard model of arithmetic. The set up here is analogous to that already discussed except that the M-finite group a is additionally assumed to be abelian. There will be such internal abelian groups of some nonstandard size n in a definable class $\mathscr{A}$ whenever the model M contains arbitrarily large standard abelian groups in $\mathscr{A}$. This is due to the principle known as *overspill* - see the definitions and results section. Therefore we may assume the number n describing the size of a is nonstandard, and we may additionally assume that a is a member of any particular definable class $\mathscr{A}$ provided it satisfies the overspill requirement. The group a can then be regarded externally as an abelian group A in its own right. Its underlying set is the set of objects x in M such that $M \vDash x \in a$ and its addition operation is defined by: $x + y$ is the object in M that M thinks is the sum of x and y in a. The external group A

is an infinite abelian group and its structure can be investigated.

We fix an M-finite abelian group a in M of nonstandard size $n \in \omega^M$, and let A be the group a viewed from outside the model. Some further assumptions on M, a or n, such as which class of groups the group a belongs to, will be made in the different sections of the chapter.

We investigate which aspects of A depend on the choice of a and M and which instead follow from the finite nature of a. As an example of the latter we prove the following easy theorem.

Theorem 1.4.1 *If A is torsion-free then it is divisible.*

Proof. Fix $m \in \mathbb{N}$ and consider the map $f_m \colon A \to A$ defined by $y \mapsto my$. This is 1–1 since if $my = mz$ then $m(y - z) = 0$ so $y = z$ as A is torsion-free. But f_m is a definable map of an M-finite set to itself in M. Therefore since f_m is 1–1 it is onto, so for each $x \in A$ there is $y \in A$ such that $x = f_m(y) = my$, as required.

This is of course is not true for an arbitrary abelian group and so demonstrates a 'finite-like' property which A inherits from a.

Not every property of an abelian group has the same meaning internally and externally. An easy example is finiteness: $M \vDash a$ is finite but A will be infinite externally whenever $|a|$ is some nonstandard natural number. The same applies to the property of being cyclic: If $M \vDash$ 'a is cyclic' then there is x in a such that each $y \in a$ is kx for some $k \in \omega^M$. However this k may be nonstandard so externally y is some nonstandard multiple of x, but for A to be cyclic k would need to be a standard natural number.

There are nonstandard primes p in ω^M (by overspill). So there is a nonstandard a where a is the cyclic group with p elements in the sense of M, and we will see examples of these groups later. Since p is nonstandard and every nonzero element of a has order p it follows that A is torsion-free. By the theorem mentioned above A is divisible and so cannot be cyclic in the external sense.

The basis theorem for finite abelian groups is provable in PA and hence true in our nonstandard model M. However direct sums require care: If $M \vDash a = b \oplus c$ then it is true that, externally, $A = B \oplus C$, where A, B, C are the corresponding external abelian groups a, b, c. However if a is an arbitrary direct sum of M-finitely many subgroups b_i in M then this number may be nonstandard. a may then be a sum of nonstandard-many components and this does not translate to saying that the external version A of a is a direct product or a direct sum of the external versions B_i of the b_i. This point is discussed in more detail in the section of the chapter on direct sums but it provides some motivation for breaking the study of A down into different cases of single direct summands. We shall often blur the distinction between A and a but it should be clear from context which we are referring to.

For a single direct summand the most interesting case turns out to be $a = C(p^k)$, where p is a standard prime and k is a nonstandard natural number. Firstly we note a similar approach that one might describe as 'folklore'. Consider the standard model $\mathbb{N} = (\mathbb{N}, +, ., 0, 1, <)$ of PA. Let D_{cof} denote the cofinite filter $D_{\text{cof}} = \{A \subseteq \omega : \omega \setminus A \text{ is finite}\}$ on $\mathcal{P}(\omega)$. It is a straightforward proof by Zorn's Lemma to check that every filter can be extended to an ultrafilter. Let $D \supseteq D_{\text{cof}}$ be an ultrafilter on $\mathcal{P}(\omega)$. Consider the ultrapower $\Pi_D \mathbb{N}$. Then $\mathbb{N} \prec \Pi_D \mathbb{N}$ by Corollary 4.1.10 in Chang and Keisler [4]. For a fixed standard prime p let $M_i = C(p^i)$ for $i \in \omega$ and consider the structure $\Pi_D M_i$. It is straightforward to show that $\Pi_D M_i$ is an M-finite $C(p^k)$ in $\Pi_D \mathbb{N}$. $\Pi_D \mathbb{N}$ is ω_1-saturated by theorem 6.1.1 in Chang and Keisler [4]. However our set up is more general than this as we can consider M-finite $C(p^k)$ in non-saturated models of arithmetic. In particular M can be countable.

By the apparatus of encoding in M it may be that a collection of objects is described in M as a single object. We say that the collection is *M-coded*. Thus the abelian group a can be thought of as a family of cyclic groups rolled up into one and this provides some motivation for its study. There are weak analogies here with pseudofinite groups. 'A group is said to be pseudofinite if it is an infinite model for the first order theory of finite groups' - *On simple pseudofinite groups* by John Wilson [16]. Equivalently a pseudofinite group is one that is elementarily equivalent (in the first order language of group theory) to $\Pi_D G_i$ for a family of finite groups G_i.

We define, for a cut $I < k$, external subgroups A_I and A^I of A. It is traditional to explore an abelian group by identifying its torsion and divisible parts. We show that $A_{k-\mathbb{N}}$ is the torsion subgroup of A and moreover it is isomorphic to the Prüfer group $C(p^\infty)$. We also show that $A_\mathbb{N}$ is divisible, and moreover that A_I is divisible for all cuts $I < k$.

It is a standard theorem of infinite abelian group theory that extensions of divisible groups are split. In our case we use this to conclude that for each group A_I there is a subgroup B_I of A such that $A_I \bigoplus B_I = A$. In analogy to Kaye and Allsup [2] we ask if B_I can be monotonically definable? We prove that no B_I can be so defined and as such B_I is non constructive in a certain sense. It is the aim of section 4.4 to show that we *can* construct an M-countable B_I. This is a major new result of this thesis. Countability of M is assumed for this and use is made of a condition on I. The proof is highly technical and requires several delicate lemmas proved in the section. It proceeds by an external induction on $\mathbb{N}$ (like the transversal theorems of chapter 3) using the various lemmas to satisfy the conditions in the external induction hypothesis.

We show how the group $A/A_\mathbb{N}$ can be made into a topological group and prove that it is topologically isomorphic to a subgroup of the group of p-adic integers $\mathbb{Z}_p = \lim\limits_{\leftarrow} C(p^m)$. This together with another standard result is used to deduce that A is a direct sum of a subgroup of $\mathbb{Z}_p$ with a $\mathbb{Q}$-vector space and $C(p^\infty)$. Under a suitable saturation assumption ($\aleph_0$-saturation) $A/A_\mathbb{N}$ turns out to be $\mathbb{Z}_p$ itself. In this case we deduce that $A = \mathbb{Z}_p \bigoplus V \bigoplus C(p^\infty)$ where V is $\mathbb{Q}$-vector space of full dimension.

Groups such as $\mathbb{Z}_p$ belong to a special class of infinite groups known as profinite groups.

A profinite group is a topological group that is isomorphic to the inverse limit of an inverse system of finite groups each endowed with the discrete topology. Profinite groups appear in the literature and the reader is directed to a survey of the subject by Dan Segal [15].

A result from this paper: *Some model theory of abelian groups* [6] uses so called Szmielew invariants (which we define in the chapter) to characterize the first order theory of an abelian group. We calculate the Szmielew invariants for our group $A = C(p^k)$ and show that they agree with our earlier result on the structure of A.

Another area in the literature with which there are analogies is the paper *Normal subgroups of nonstandard symmetric and alternating groups*[2] by Allsup and Kaye. This paper looks at the M-finite permutation groups S_n and A_n for n a nonstandard element in some nonstandard model of $M \vDash \mathrm{PA}$. In analogy with our set up these groups behave like ordinary finite permutation groups when viewed internally, but the interest lies in their external structure. The paper defines the groups $S_n^{[I]} := \{g \in S_n : |\mathrm{support}(g)| \in I\}$ and $A_n^{[I]} := \{g \in A_n : |\mathrm{support}(g)| \in I\}$. This is in some ways analogous to the external subgroups of $C(p^k)$ which we study. The paper goes onto prove that for a cut $I < n$, closed under addition, $S_n^{[I]} \lhd S_n$, $A_n^{[I]} \lhd S_n$ and together with A_n these comprise all the normal subgroups of S_n. They also prove that the analogous result holds for A_n i.e. for I as above $A_n^{[I]} \lhd A_n$ and these comprise all the normal subgroups of A_n. The authors then consider the question of whether S_n or A_n can be a split extension of some $S_n^{[I]}$ or $A_n^{[I]}$. They conjecture that answer is no which contrasts with our result on A_I. The authors show that no complement of $A_n^{[I]}$ or $S_n^{[I]}$ can be monotonically definable which is analogous to our result on B_I.

In section 4.5 we generalize the main result of section 4.4 to the case of an arbitrary M-finite abelian group G. To this end we introduce a 'pseudo complement' H to the divisible part of dG of G. H is shown to have the desired cardinality but it is not a full complement because $dG + H \neq G$. We generalize some of the lemmas of section 4.4 and use these to extend H to a full complement via an external induction. As before we use countability of M and a closure condition on the cardinality of H which is discussed in the final subsection. Several of the results in this section and the definition of H itself require us to use the basis theorem in its full generality. Whilst we show earlier in the chapter that the basis theorem has several limitations as a tool to describe M-finite abelian groups externally it turns out to be quite useful for making internal definitions of subgroups and proving results about them.

1.5 Background Results

In this section we describe some of the background technical machinery that is used in this thesis. We state results without proofs but with references to where the proofs may be found in the literature.

1.5.1 Background Model Theory

For a full introduction to the subject of models of Peano arithmetic (PA) the reader is directed to Kaye [8]. In this introduction we give brief details of the set up.

Definition 1.5.1 *Let* $\mathscr{L}_A$ *denote the* first-order language of arithmetic*. The nonlogical symbols of* $\mathscr{L}_A$ *are constant symbols, 0 and 1; the binary relation symbol,* $<$*; and the two binary function symbols,* $+$ *and* $\cdot$*.*

The subject of PA is about $\mathscr{L}_A$ structures. The natural numbers $\mathbb{N}$ together with obvious interpretations for the symbols $0, 1, <, +, \cdot$ provide us with the standard model. Natural numbers can be expressed in $\mathscr{L}_A$ as canonical terms:

Definition 1.5.2 *For each* $n \in \mathbb{N}$ *we let* $\underline{n}$ *be the term* $(\cdots(((1 + 1) + 1) + 1) + \cdots + 1)$*(n 1's), of* $\mathscr{L}_A$*;* $\underline{0}$ *is just the constant symbol 0.*

PA is the theory of arithmetic with which we shall mostly be concerned. For a full definition see Kaye [8]. We shall want to work with nonstandard $M \models \text{PA}$, however many of the vital complex coding devices we will get automatically if we regard M as a model of the theory ZF $-$ inf (Zermelo-Fraenkel set theory with the axiom of infinity negated). That we can do this is justified by the fact that the two theories are mutually interpretable. For the technical details of this result the reader is directed to 'On interpretations of arithmetic and set theory' by Kaye and Wong [10]. We leave further details to the interested reader save for one remark. It is necessary to replace ZF $-$ inf by ZF $-$ inf plus the axiom of transitive containment in order to achieve an interpretation of PA in finite set theory in which the domain of the interpretation is the whole set theoretic universe - this is the key result of [10]. The axiom of transitive containment states that every set is contained in a transitive set. In the absence of the axiom of infinity this is needed to allow one to obtain the scheme of induction from the axiom of foundation. We use the notation $\mathrm{ZF}^* - \mathrm{inf}$ to denote the theory ZF $-$ inf plus the axiom of transitive containment. From now one we shall regard the two theories $\mathrm{ZF}^* - \mathrm{inf}$ and PA synonymously and may freely switch to the arithmetic view whenever appropriate. We shall appeal to the set theoretic view whenever we need to in order to avoid complex coding devices. As an example of this we may define a set $a \subseteq M$ to be M-finite if, in addition, $a \in M$. It will be clear from context whether some $a \in M$ is regarded as a number, i.e. an element of M, or an M-finite subset of M. The identification above shows that each such a can be regarded in either way. We write $y \in x$ for the PA-formula expressing that y is in the canonical M-finite set defined by x.

1.5.2 Initial Segments and Cuts

An important notion is that of initial segment. We give the following definition based on definitions in Kaye [8].

Definition 1.5.3 *If $M \vDash PA$ with $A \subseteq M$, then A is an* initial segment *of M, or*

$A \subseteq_e M$, *iff*

$$\text{for all } x \in A, \text{ for all } y \in M(M \models y < x \implies y \in A).$$

A is a proper *initial segment, if in addition,* $A \neq M$. *When M and N are models of PA and* $N \subseteq_e M$ *we say that M is an* end-extension *of N.*

We have actually defined two relations here, one between subsets of a model and one between models. Usually we shall mean the one between subsets of a model. Related to the notion of an initial segment is that of a cut which we define as follows.

Definition 1.5.4 *A non empty subset I of a model* $M \models PA$ *is called a* cut *of M iff*

$$(x < y \in I) \implies (x \in I) \text{ and } (x \in I) \implies (x + 1 \in I)$$

I is called a proper *cut, if in addition,* $I \neq M$. *Equivalently a cut is an initial segment that is also closed under the successor function.*

The following theorem from Kaye [8] gives an important property of a model of PA.

Theorem 1.5.5 *Let* $M \models PA$. *Then the map* $\mathbb{N} \to M$ *given by* $n \mapsto \underline{n}^M$ *is an embedding of* $\mathscr{L}_A$*-structures sending* $\mathbb{N}$ *onto an initial segment of M.*

Note that in the above $\underline{n}^M$ means the interpretation in M of the canonical term $\underline{n}$ of $\mathscr{L}_A$. Theorem 1.5.5 tells us that we may always identify $\mathbb{N}$ with the smallest initial segment $\{\underline{n}^M : n \in \mathbb{N}\}$ of a model of PA. We shall do this from now on.

We define the notions of supremum and infimum of a subset of a model.

Definition 1.5.6 *Let* $M \models PA$ *and* $A \subseteq M$. *We define*

$$\sup(A) = \{x \in M : \exists y \in A \; M \models x \leqslant y\}$$

$$\inf(A) = \{x \in M : \forall y \in A \; M \models x \leqslant y\}$$

We conclude this subsection with a brief note on notation. When a is an element of a model we shall sometimes have cause to identify a with its set of predecessors: $\{x \in M : M \models x < a\}$. When we wish to do so explicitly we will use the notation $< a$.

We define the following interesting properties an initial segment may possess. These are roughly analogous to a large cardinal properties in set theory. The definitions and results given here are due to Paris and Kirby [11].

Definition 1.5.7 *Let $I \subseteq M$ be a cut.*

(1) We say I is semi regular *if and only if for all a in I and every M-finite function $f : a \to M$ we have $f(< a) \cap I$ is bounded in I.*

(2) We say I is regular *if and only if every M-finite function f whose domain includes I is constant on a cofinal subset of I.*

(3) We say I is strong *if and only if for every M-finite function f there exists $a \in (M \setminus I)$, such that for all $x \in I$ $f(x) > I \iff f(x) > a$.*

In every countable nonstandard model of PA, $\mathbb{N}$ is semi-regular and regular but not necessarily strong. Moreover if $M \models$ PA is nonstandard then $\forall x \in M \exists y \in M$ such that there is a semi-regular cut I with $x \in I$ and $I < y$. These results are due to Paris and Kirby [11]. The same result holds for regular and also for strong. In general the result for strong requires the full consistency strength of PA whereas the analogous results for regular and semi-regular can be proved in some weaker fragment.

1.5.3 Overspill

A very important tool in the study of nonstandard models is the principle of overspill. The following is taken directly from Kaye [8], page 71.

Lemma 1.5.8 *Let $M \models PA$ be nonstandard and let I be a proper cut of M. Suppose $\overline{a} \in M$ and $\theta(x, (\overline{a}))$ is an $\mathscr{L}_A$-formula such that*

$$M \models \theta(b, \overline{a})$$

for all $b \in I$. Then there is $c > I$ in M such that

$$M \models \forall x \leqslant c\ \theta(x, \overline{a})$$

In particular this implies that no proper cut is definable. There are also two useful variants of this lemma (again see Kaye [8], page 71). We shall have cause to use all three throughout the thesis so we give the others here as well.

Lemma 1.5.9 *Let $M \models PA$ be nonstandard and let I be a proper cut of M. Suppose $\overline{a} \in M$, $\theta(x, \overline{a})$ is an $\mathscr{L}_A$-formula, and that for all $x \in I$ there exists $y \in I$ such that*

$$M \models y \geqslant x \wedge \theta(y, \overline{a})$$

(i.e. the set of y satisfying $\theta(y, \overline{a})$ is unbounded in I) Then for each $c > I$ in M there exists $b \in M$ with $I < b < c$ and

$$M \models \theta(b, \overline{a})$$

(i.e. there are arbitrarily small $b > I$ satisfying $\theta(b, \overline{a})$)

Lemma 1.5.9 will be particularly useful to us when we have a cut I and some definable sequence with arbitrarily large elements in I, as it will allow us to conclude there are members of the sequence above I.

The third 'overspill' lemma we present is known as *underspill*. As its name suggests it is used for concluding there are elements below a cut I with some definable property whenever there are arbitrarily small elements above with that property.

Lemma 1.5.10 *Let $M \models PA$ be nonstandard and let I be a proper cut of M.*

(1) Suppose $\overline{a} \in M$, $\theta(x, \overline{a})$ is an $\mathscr{L}_A$-formula such that

$$M \models \theta(b, \overline{a})$$

for all $b > I$ in M. Then there is $c \in I$ such that

$$M \models \forall x \geqslant c\ \theta(x, \overline{a}).$$

(2) Suppose $\overline{a} \in M$, $\theta(x, \overline{a})$ is an $\mathscr{L}_A$-formula and that for all $b > I$ in M there exists $x > I$ such that

$$M \models x < b \wedge \theta(x, \overline{a}).$$

Then for each $c \in I$ there exists $y \in I$ with

$$M \models y \geqslant c \wedge \theta(y, \overline{a}).$$

We will use these lemmas freely throughout the rest of this thesis.

1.5.4 Order type of a model of PA

We now discuss the order type of a nonstandard model of PA. The results given here can all be found (with proofs) in Kaye [8], pages 73-77. We begin with a definition.

Definition 1.5.11 *Let M be an $\mathscr{L}_A$ structure. Then $M \restriction <$ denotes the reduct of M to the language $\{<\}$. That is the structure with the same domain as M but only one relation, namely $<$.*

We present the following theorem from Kaye [8], page 75. In particular this result tells us that, in the countable case, there is only one possibility for the order type of $M \restriction <$.

Theorem 1.5.12 *Let $M \models PA$ be nonstandard. Then $M \restriction < \cong \mathbb{N} + \mathbb{Z} \cdot A$ for some linearly ordered set $(A, <_A)$ satisfying the theory DLO axiomatized by*

(1) $\forall x \neg x < x$

(2) $\forall x, y, z(x < y \wedge y < z \implies x < z)$

(3) $\forall x, y(x < y \vee x = y \vee y < x)$

(4) $\forall x, y(x < y \implies \exists x(x < z \wedge z < y))$

(5) $\forall x \exists y, z(y < x \wedge x < z)$

In particular, if M is countable, then $M \cong \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$, where $\mathbb{Q}$ is the set of rationals with its natural order.

This theorem is very useful and has the following important corollary (Kaye [8], page 76) showing that countable models of PA always have the maximum number of cuts.

Corollary 1.5.13 *Let $M \models PA$ be countable and nonstandard. Then M has $2^{\aleph_0}$ proper cuts I.*

This result can be modified to give the following version.

Theorem 1.5.14 *Let $M \models PA$ be countable and nonstandard. Then M has $2^{\aleph_0}$ proper cuts $I \subseteq_e M$ that are closed under $+, \cdot$.*

We shall use these results to conclude a variety of statements. For example given $a \in M$ with $a > \mathbb{N}$, we can always find an initial segment I closed under $+, \cdot$ with $\mathbb{N} < I < a$.

1.5.5 Saturation and the Standard System

In this subsection we present some definitions and results on saturation and the standard system of a model $M \models \mathrm{PA}$. We first define a type over a theory, T (but can be thought of as PA). This definition is based on Kaye [8].

Definition 1.5.15 *Given a theory T, a* type over T *is a set $p(\overline{x})$ of formulas in finitely many free-variables $\overline{x}$ such that $T \cup p(\overline{x})$ is consistent.*

If $M \models T$ then the type $p(\overline{x})$ is either realized or omitted by M. There is also a notion of *type over a subset* of a model M. We base the following on the definition in Marker [12] page 115, but stated here in the context of models of arithmetic.

Definition 1.5.16 *Given an $\mathscr{L}_A$ structure M and a subset $B \subseteq M$. We let $\mathscr{L}_A(B)$ denote the language $\mathscr{L}_A$ expanded by adding constant symbols for each $a \in B$. M can be viewed as an $\mathscr{L}_A(B)$ structure by interpreting the new symbols in the obvious way. Let $T_B(M)$ be the set of all $\mathscr{L}_A(B)$-sentences true in M. Let $p(\overline{x})$ be a set of $\mathscr{L}_A(B)$-formulas in finitely many free-variables. We call $p(\overline{x})$ a* type over B *if $T_B(M) \cup p(\overline{x})$ is consistent.*

We now define the notion of $\aleph_0$-saturation. See page 138 in Marker [12].

Definition 1.5.17 *M is $\aleph_0$-saturated means for all $A \subseteq M$ such that $|A| < \aleph_0$ and for all types $p(\overline{x})$ over A; $p(\overline{x})$ is realized by M.*

The following standard theorem of model theory helps to justify the use of this concept. For a similar result see page 141 of Marker [12].

Theorem 1.5.18 *For all countable M_0 there is an $\aleph_0$-saturated $M \succ M_0$ with $|M| = 2^{\aleph_0}$.*

This will be important for an application in the section on nonstandard abelian groups and uses the material on types and saturation we have just presented. We now define the standard system $\mathrm{SSy}(M)$. For more details about $\mathrm{SSy}(M)$ see Kaye [8], page 141 onwards. For this definition it will be convenient to switch to the set theoretic view point.

Definition 1.5.19 *Let* $M \models \mathrm{ZF}^* - inf$*. Then* ω^M *is a class of* M *and* $\mathbb{N} \subseteq_e \omega^M$*. We define* $SSy(M)$ *as follows*

$$SSy(M) = \{S \subseteq \mathbb{N} : S = \mathbb{N} \cap a \textit{ for some } a \in M\}$$

Note that we must regard a externally for this intersection to make sense.

Informally we regard $\mathrm{SSy}(M)$ as those subsets of $\mathbb{N}$ that are initial parts of some M-finite set. Let $S \subseteq \mathbb{N}$, and let $p(x)$ be the following set of formulas

$$\{i \in x : i \in S\} \cup \{i \notin x : i \notin S\}$$

If $p(x)$ is realized by $c \in M$ then $M \models i \in c$ for all $i \in S$ and $M \models i \notin c$ for all $i \notin S$, i.e., $S = \{i \in \mathbb{N} : M \models i \in c\}$ so $S \in SSy(M)$. Conversely if $S \in \mathrm{SSy}(M)$ then $S = \{i \in \mathbb{N} : M \models i \in c\}$ for some $c \in M$ and it easy to see that c is a realization of $p(x)$. So we see that M realizes $p(x)$ if and only if S is in $\mathrm{SSy}(M)$. Since all finite sets $F \subseteq S \subseteq \mathbb{N}$ are also elements of $M \models \mathrm{ZF}^* - \mathrm{inf}$ it follows that $p(x)$ is finitely satisfiable and hence is a type of M over S. Suppose M is $\aleph_0$-saturated. We are assuming that each natural number is given by a canonical term in the language $\mathscr{L}_A$ so that even when $S \subseteq \mathbb{N}$ is infinite, M will realize $p(x)$ as $p(x)$ only contains finitely many parameters from $M \setminus \mathbb{N}$ (none in fact). So if M is $\aleph_0$-saturated then $\mathrm{SSy}(M) = \mathcal{P}(\mathbb{N})$.

Chapter 2

M-countable Sets

2.1 M-countability

Many of the results presented in this chapter will appear in a forthcoming paper by Kaye and Reading and should be regarded as joint work. Throughout we let M be a nonstandard model of Peano Arithmetic (PA). For an internal (or M-finite) subset a of M there is a well-defined notion of the size of a, or the number of elements of a, defined internally in PA using the usual coding devices. We denote this by $\operatorname{card} a$ or $|a|$. This notion of cardinality can be used to define a notion of cardinality for external bounded sets as follows.

Definition 2.1.1 *Given a bounded set $X \subseteq M$ we define initial segments*

$$\overline{\operatorname{card}}\, X = \inf\{\operatorname{card} a : a \in M \text{ and } a \supseteq X\}$$

and

$$\underline{\operatorname{card}}\, X = \sup\{\operatorname{card} a : a \in M \text{ and } a \subseteq X\}.$$

If these initial segments are equal we say that X is M-countable and write $\operatorname{card} X$ *for this initial segment. The upper and lower cardinality of a set X will be cuts unless the* *set is M-finite in which case they will both be the initial segment with top element that corresponds to the internal cardinality of X.*

It is easy to check that all cuts I are M-countable, and $\operatorname{card} I = I$ for such I. If X is cofinal in M the definitions above also make sense, and $\operatorname{card} X = M$, but in this case the card notion only tells us about the cofinality of X in M, and we do not study this here. Note also that an M-finite set a has $\overline{\operatorname{card}}\, a = \underline{\operatorname{card}}\, a$ and it is equal to the normal (internal) cardinality $|a|$ of a in M. The following lemma is straightforward.

Lemma 2.1.2 *Let X be a subset of M. If $a \in \underline{\operatorname{card}}(X)$ then there is $x_a \subseteq X$ in M with $|x_a| = a$. If $a \geqslant \overline{\operatorname{card}}(X)$ then there is $x_a \supseteq X$ with $|x_a| = a$.*

Example 2.1.3 *In the case when $M \models \mathrm{PA}$ is countable we can start with two proper cuts $I \subseteq J \subsetneq M$, and build a set X with $\underline{\operatorname{card}}\, X = J$ and $\overline{\operatorname{card}}\, X = I$.*

A version of the following proof is given in Allsup's thesis [1].

Proof. By countability let $(a_0, \cdots, a_n, \cdots)_{n\in\mathbb{N}}$ be an increasing sequence cofinal in I. Let $(b_0, \cdots, b_n, \cdots)_{n\in\mathbb{N}}$ be a decreasing sequence cofinal in J from above. Let $(C_0 \cdots C_n \cdots)_{n\in\mathbb{N}}$ be an enumeration of the M-finite sets with $I < |C_n| < J$ (If $I = J$ then we omit this enumeration and the relevant parts of the induction are satisfied vacuously). Let $(d_0 \cdots d_n \cdots)_{n\in\mathbb{N}}$ be an enumeration of all the elements of M. We proceed by induction on $\mathbb{N}$. Suppose inductively that

(1) We have M-finite sets $x_0 \subseteq \cdots \subseteq x_n \subseteq x'_n \subseteq \cdots \subseteq x'_0$

(2) $a_n \leqslant |x_n| \leqslant I \leqslant J \leqslant |x'_n| \leqslant b_n$

(3) $x_n \nsubseteq C_n$ and $C_n \nsubseteq x'_n$

(4) $d_n \in x_n$ or $d_n \notin x'_n$

Building x_{n+1} inside x'_n. Given $|x_n| < a_{n+1} < I$ choose M-finite $y_n \subseteq x'_n \setminus x_n$ with $|y_n| = a_{n+1} - |x_n|$. Given the M-finite set $C_{n+1} \subseteq M$ choose $c \in x'_n \setminus C_{n+1}$. Such c exists because $|C_{n+1}| < J < |x'_n|$. Put $x_{n+1} = x_n \cup y_n \cup \{c\}$ and we also add d_{n+1} to x_{n+1} if $d_{n+1} \in x'_n$.

Building x'_{n+1} to contain x_{n+1}. Given $|x'_n| > b_{n+1} > J$ choose M-finite $z_n \subseteq x'_n \setminus x_{n+1}$ with $|z_n| = |x'_n| - b_{n+1}$. Given the M-finite set $C_{n+1} \subseteq M$ choose $d \in C_{n+1} \setminus x_{n+1}$. Such d exists by assumption on C_{n+1} as $|x_{n+1}| < I < |C_{n+1}|$. Put $x'_{n+1} = x'_n \setminus \{d\} \setminus \{z_n\}$ and we also remove d_{n+1} from x'_{n+1} if $d_{n+1} \notin x_{n+1}$.

Thus the induction hypothesis is satisfied at stage $n+1$. Set $X = \bigcup_{n\in\mathbb{N}} x_n = \bigcap_{n\in\mathbb{N}} x'_n$ (these are equal because of condition 4) and it straightforward to check that X satisfies all the desired criteria.

For example, we can have X such as $I \cup \{x_n : n \in \mathbb{N}\}$ where $\{x_n : n \in \mathbb{N}\}$ is a suitably chosen ω-sequence in $J \setminus I$ with limit J.

The argument used in example 2.1.3 does not work in the uncountable case, or would at least require careful assumptions on the cardinalities and cofinalities of I, J, M and X. This thesis will largely be concerned with the countable case and we leave the general details in the uncountable case for another time. Uncountable examples can be obtained as elementary extensions of countable 4-tuples (M, I, J, X) as constructed above.

We have the following definition due to Kaye [9].

Definition 2.1.4 *Let I, J be cuts. Then define*

$$I + J = \sup\{i + j : i \leqslant I, j \leqslant J\},$$

$$I \oplus J = \inf\{i' + j' : i' \geqslant I, j' \geqslant J\}.$$

In addition, if $I \geqslant J$, then we define

$$I - J = \sup\{i - j' : i \leqslant I, j' \geqslant J\},$$

$$I \ominus J = \inf\{i' - j : i' \geqslant I, j \leqslant J\}.$$

Note that if one of I or J is an initial segment with top element then the two notions of addition coincide, as do the two notions of subtraction. This is straightforward by overspill. In general the two notions do not coincide. The following example is taken from Kaye [9, Example 2.20]. See also Kaye [9, Example 2.21].

Example 2.1.5 *Let K be a cut closed under addition and let $a, b \in M$ with $b > K$. Let $I = a + K$ and $J = b - K$, then it is straightforward to calculate that $I + J = a + b - K$ and $I \oplus J = a + b + K$.*

The following lemma bounds the cardinality of a complement in terms of the operations of subtraction defined above and will be useful later.

Lemma 2.1.6 *Let X be a bounded subset of M and $a \supseteq X$ be M-finite. Then*

$$|a| \ominus \overline{\text{card}}\, X \leqslant \overline{\text{card}}(a \setminus X) \leqslant |a| \ominus \underline{\text{card}}\, X$$

and

$$|a| - \overline{\text{card}}\, X \leqslant \underline{\text{card}}(a \setminus X) \leqslant |a| - \underline{\text{card}}\, X.$$

Proof. For $|a| \ominus \overline{\text{card}}\, X \leqslant \overline{\text{card}}(a \setminus X)$ we must show that given $b \supseteq a \setminus X$ we have $|b| \geqslant |a| \ominus \overline{\text{card}}\, X$, i.e. $|b| \geqslant |a| - j'$ for all $j' \geqslant \overline{\text{card}}\, X$. Given such j' there is $c \supseteq X$ with $|c| = j'$ and $b \supseteq a \setminus X \supseteq a \setminus c$, so $|b| \geqslant |a \setminus c| \geqslant |a| - j'$ as required.

If $j \in \underline{\text{card}}\, X$ we show $|a| - j \geqslant \overline{\text{card}}(a \setminus X)$. For given $b \subseteq X$ with $|b| = j$ we have $a \setminus b \supseteq a \setminus X$ and $|a \setminus b| = |a| - j$. Thus $\overline{\text{card}}(a \setminus X) \leqslant |a| \ominus \underline{\text{card}}\, X$.

The cut $|a| - \overline{\text{card}}\, X$ is $\sup\{|a| - j' : j' > \overline{\text{card}}\, X\}$ and so we must show that any such $|a| - j'$ is in $\underline{\text{card}}(a \setminus X)$. But given $b \supseteq X$ with $|b| = j'$ we have $a \setminus X \supseteq a \setminus b$ and $|a \setminus b| \geqslant |a| - j'$ as required.

Finally, given $b \subseteq a \setminus X$ we have $|a| \geqslant |b| + j$ for all $j \in \underline{\text{card}}\, X$. So there is $j' \geqslant \underline{\text{card}}\, X$ with $|a| = |b| + j' \leqslant |a| - \underline{\text{card}}\, X$.

Corollary 2.1.7 *Let X be a bounded M-countable subset of M and $a \supseteq X$ be internal. Then $a \setminus X$ is M-countable.*

Proof. a is M-finite and so $|a| \ominus \text{card}\, X = |a| - \text{card}\, X$ as mentioned above. The previous result then gives,

$$\overline{\text{card}}(a \setminus X) \leqslant |a| \ominus \underline{\text{card}}\, X = |a| - \overline{\text{card}}\, X \leqslant \underline{\text{card}}(a \setminus X)$$

as required.

We will explore some of the properties and limitations of the notion of being M-countable in the next few propositions.

One case of a set that might be hoped to be M-countable is that of intervals, such as $(I, J) = \{x \in M : I < x < J\}$.

Example 2.1.8 *There are cuts I, J such that (I, J) is not M-countable.*

Proof. Let K be a cut closed under addition. Let $a \in M$ and choose $b \in M$ such that $a+K < b$. Set $I = a+K < b+K = J$. Then $\overline{\text{card}}(I, J) = \lim\{(b+k') - (a+k) : k' > K, k \in K\} = b - a + K$ whereas $\underline{\text{card}}(I, J) = \lim\{(b+k) - (a+k') : k \in K, k' > K\} = b - a - K$, so (I, J) is not M-countable.

For some of the results following, we shall use the notion of derivative of a cut. We also state here, although it is not needed in what follows in this chapter, the notion of second derivative of a cut.

Definition 2.1.9 *Let I be a cut. Then define the derivative of I to be the cut $\partial I = \inf\{i' - i : i' \geqslant I \geqslant i\}$. Define the second derivative of I to be the cut $\partial^2 I = \inf\{\lfloor \frac{i'}{i} \rfloor : i' \geqslant I \geqslant i\}$.*

Thus for any set X cuts $\partial\, \underline{\mathrm{card}}\, X$ and $\partial\, \overline{\mathrm{card}}\, X$ are defined. Even when X is not M-countable, it will also be convenient to define

$$\partial\, \mathrm{card}\, X = \inf\{i' - i : i' \geqslant \overline{\mathrm{card}}\, X,\ i \leqslant \underline{\mathrm{card}}\, X\}.$$

The operator here is a single operator '$\partial\, \mathrm{card}$' as when X is not M-countable the expression '$\mathrm{card}\, X$' does not have any meaning. Note that the definition of $\partial\, \mathrm{card}\, X$ does agree with the previous definition when X is M-countable. An alternative expression for $\partial\, \mathrm{card}\, X$ is the difference (in the inf form) of the upper and lower cardinalities:

$$\partial\, \mathrm{card}\, X = \overline{\mathrm{card}}\, X \ominus \underline{\mathrm{card}}\, X.$$

The following proposition is also obvious.

Proposition 2.1.10 *Let $X \subseteq M$ be an arbitrary bounded set. Then*

$$\partial\, \underline{\mathrm{card}}\, X, \partial\, \overline{\mathrm{card}}\, X \leqslant \partial\, \mathrm{card}\, X.$$

Unlike $\partial\, \underline{\mathrm{card}}\, X$ and $\partial\, \overline{\mathrm{card}}\, X$, the cut $\partial\, \mathrm{card}\, X$ need not be closed under $+$, though obviously it will be when X is M-countable. Counterexamples are easy to find using the technique of Example 2.1.3 as we may simply pick I and J such that their difference (in the inf form) is not closed under $+$. Nor is it true that $\partial\, \mathrm{card}\, X$ closed under $+$ implies X is M-countable. For example, we build X with $\overline{\mathrm{card}}\, X = \partial\, \overline{\mathrm{card}}\, X = I$ and $\underline{\mathrm{card}}\, X = J < I$. Then $\partial\, \mathrm{card}\, X = I \ominus J = I$ is closed under $+$ since I is.

Lemma 2.1.11 *Let X and Y be bounded disjoint sets. Then*

$$\underline{\mathrm{card}}(X) + \underline{\mathrm{card}}(Y) \leqslant \underline{\mathrm{card}}(X \cup Y) \text{ and } \overline{\mathrm{card}}(X) \oplus \overline{\mathrm{card}}(Y) \geqslant \overline{\mathrm{card}}(X \cup Y).$$

Proof. If $x \subseteq X$ and $y \subseteq Y$ then $x \cup y \subseteq X \cup Y$ and $|x \cup y| = |x| + |y|$. The proof of the second statement is similar.

Corollary 2.1.12 *Let X and Y be disjoint bounded sets and suppose that*

$$\underline{\mathrm{card}}(X) + \underline{\mathrm{card}}(Y) = \overline{\mathrm{card}}(X) \oplus \overline{\mathrm{card}}(Y)$$

then $X \cup Y$ is M-countable.

Proof. From the properties given,

$$\overline{\mathrm{card}}(X \cup Y) \leqslant \overline{\mathrm{card}}(X) \oplus \overline{\mathrm{card}}(Y) = \underline{\mathrm{card}}(X) + \underline{\mathrm{card}}(Y) \leqslant \underline{\mathrm{card}}(X \cup Y),$$

by Lemma 2.1.11.

Corollary 2.1.13 *Let X and Y be disjoint M-countable sets and suppose that $\partial\,\mathrm{card}\,X > \partial\,\mathrm{card}\,Y$. Then $X \cup Y$ is M-countable.*

Proof. A general result from the arithmetic of cuts, Kaye [9], says that $I + J = I \oplus J$ whenever $\partial I \neq \partial J$. The corollary then follows from 2.1.12.

For more detail on the disjoint union $X \cup Y$ without assuming X or Y are M-countable we can use the $\partial\,\mathrm{card}$ operator.

Lemma 2.1.14 *Let X, Y be bounded disjoint subsets of M.*

(a) If $\partial\,\mathrm{card}\,Y < \partial\,\underline{\mathrm{card}}\,X$ then $\underline{\mathrm{card}}\,X + \underline{\mathrm{card}}\,Y = \underline{\mathrm{card}}(X \cup Y)$.

(b) If $\partial\,\mathrm{card}\,Y < \partial\,\overline{\mathrm{card}}\,X$ *then* $\overline{\mathrm{card}}\,X \oplus \overline{\mathrm{card}}\,Y = \overline{\mathrm{card}}(X \cup Y)$*.*

Proof. For part (a) Let $u \subseteq X \cup Y$ and $y \subseteq Y \subseteq y'$ with $|y' \setminus y| \leqslant \partial\,\underline{\mathrm{card}}\,X$. Then $u \setminus y' \subseteq X$ and there's $v \subseteq X$ with $|v| = |u \setminus y'| + |y' \setminus y|$. So $v \cup y \subseteq X \cup Y$ and $|v \cup y| \geqslant |u|$ since $v \cap y = \varnothing$. Part (b) is similar: if $u' \supseteq X \cup Y$ and $y \subseteq Y \subseteq y'$ with $|y' \setminus y| \leqslant \partial\,\overline{\mathrm{card}}\,X$, then $u' \setminus y \supseteq X$ and there is $v' \supseteq X$ with $|v'| = |u' \setminus y| - |y' \setminus y|$. It follows that $|v' \cup y'| \leqslant |u'|$.

Example 2.1.15 *There are disjoint* M*-countable sets* X, Y *such that* $X \cup Y$ *is not* M*-countable.*

Proof. Let I, J be cuts such that $I \subsetneq J$ and $I + J \subsetneq I \oplus J$, I and J could be as given in Example 2.1.5. Let $n \in M$ be such that $I < n \in J$. Define sets $X := I$ and $Y := \{n + j : j \in J\}$, clearly $X \cap Y = \varnothing$ and it is immediate that both are M-countable with $\mathrm{card}(X) = I$ and $\mathrm{card}(Y) = J$. By Lemma 2.1.11 we have that $I + J \subseteq \underline{\mathrm{card}}(X \cup Y)$ and $\overline{\mathrm{card}}(X \cup Y) \subseteq I \oplus J$.

Let $x \in \underline{\mathrm{card}}(X \cup Y)$. Then it is easy to see that $x = |a \cup b|$ where $a \subseteq X$ and $b \subseteq Y$. That is, $x \leqslant \underline{\mathrm{card}}\,X + \underline{\mathrm{card}}\,Y$, and it follows that $\underline{\mathrm{card}}(X \cup Y) = I + J$. Similarly, given $x' \geqslant \overline{\mathrm{card}}(X \cup Y)$ we can write x' as $|a' \cup b'|$ where $a' \supseteq X$ and $b' \supseteq Y$ are disjoint. Thus $\overline{\mathrm{card}}(X \cup Y) = I \oplus J > \underline{\mathrm{card}}(X \cup Y)$ and hence $X \cup Y$ is not M-countable.

The notion of a pair of M-countable sets being separable will be used in the next chapter. We make the following definition.

Definition 2.1.16 *Let* $A \subsetneq B$ *be* M*-countable sets with* $\mathrm{card}(A) = I \subsetneq J = \mathrm{card}(B)$*. We say that* A *and* B *are* separable *if there is an* M*-finite set* x *with* $A \subseteq x \subseteq B$*.*

Any two cuts $I \subsetneq J$ are separable. The set a for any $I < a < J$ is clearly an M-finite set separating I and J. That not all M-countable sets are separable is demonstrated by the following example.

Example 2.1.17 *Let $J \supsetneq \mathbb{N}$ be a cut with $\partial(J) > \mathbb{N}$. Let $\alpha \in M$ be such that $\mathbb{N} < \alpha \in \partial(J)$. Define sets A, B as follows.*

$$A := \mathbb{N}$$

$$B := \mathbb{N} \cup \{j \in J : j \geqslant \alpha\}$$

Clearly A is M-countable with $\mathrm{card}(A) = \mathbb{N}$. *Since $B \subseteq J$ it is immediate that* $\overline{\mathrm{card}}(B) \leqslant J$. *Let $j^* \in J$. Then by the definition of $\partial(J)$ and the choice of α we have that $j^* + \alpha \in J$. Define an M-finite set $X_{j^*} = \{j \in J : \alpha \leqslant j < j^* + \alpha\} \subseteq B$. Moreover we have that $|X_{j^*}| = j^* \in J$, and so $J \leqslant \underline{\mathrm{card}}(B)$. We see that B is M-countable with* $\mathrm{card}(B) = J$. *Now suppose we have an M-finite set X such that $A \subseteq X$. By overspill there is some $\beta > \mathbb{N}$ such that $X \subseteq \beta$. Pick some $\delta \in M$ such that $\mathbb{N} < \delta < min(\beta, \alpha)$. Then $\delta \in X$ but $\mathbb{N} < \delta < \alpha$ and so $\delta \notin B$. Hence $X \nsubseteq B$.*

We now consider intersections of the form $A \cap b$ where A is M-countable and b is M-finite. One might hope that such an $A \cap b$ will always turn out to M-countable but unfortunately this is not the case as the following example shows.

Example 2.1.18 *There are sets A, b such that A is M-countable and b is internal and $A \cap b$ is not M-countable.*

Proof. Let $I > \mathbb{N}$ be a cut closed under addition and let $b \in M$ be such that $\mathbb{N} < b \in I$. Take X to be any non-M-countable subset of b. Since I is closed under addition the set $I \setminus b$ is M-countable with $\mathrm{card}(I \setminus b) = I$. Finally set $A = X \cup (I \setminus b)$ and note that A is M-countable with $\mathrm{card}(A) = I$ for the same reason, but $A \cap x = X$ is not M-countable.

Some positive results can, however, be given concerning sets of the form $A \cap b$, and more generally sets of the form $A \cap B$ where $\partial\,\mathrm{card}\,B$ is small compared to $\partial\,\underline{\mathrm{card}}\,A$ or $\partial\,\overline{\mathrm{card}}\,A$. This situation is closely related to Lemma 2.1.14, and a version of that lemma for intersections follows.

Lemma 2.1.19 *Let $A, B \subseteq M$ be bounded.*

(a) If $\partial \operatorname{card} B < \partial \underline{\operatorname{card}} A$ then $\underline{\operatorname{card}}(A \cap B) + \underline{\operatorname{card}}(A \setminus B) = \underline{\operatorname{card}} A$.

(b) If $\partial \operatorname{card} B < \partial \overline{\operatorname{card}} A$ then $\overline{\operatorname{card}}(A \cap B) \oplus \overline{\operatorname{card}}(A \setminus B) = \overline{\operatorname{card}} A$.

Proof. For the first, we note that $\leqslant$ is generally true and we show $\underline{\operatorname{card}}(A \cap B) + \underline{\operatorname{card}}(A \setminus B) \geqslant \underline{\operatorname{card}} A$. Let $x \in \underline{\operatorname{card}} A$ and $b \subseteq B \subseteq b'$ with $|b' \setminus b| \in \partial \underline{\operatorname{card}} A$. Take $a \subseteq A$ with $|a| = x + |b' \setminus b|$. Then $|a \cap b| + |a \setminus b'| \geqslant |a| - |b' \setminus b| = x$. Part (b) is similar.

Corollary 2.1.20 *If A is M-countable and B is bounded with $\partial \operatorname{card} B < \partial \operatorname{card} A$ then at least one of $A \cap B$ and $A \setminus B$ is M-countable.*

Proof. Assume otherwise. If $j, k \in M$ with $\underline{\operatorname{card}}(A \cap B) < j < \overline{\operatorname{card}}(A \cap B)$ and $\underline{\operatorname{card}}(A \setminus B) < k < \overline{\operatorname{card}}(A \setminus B)$ so

$$\operatorname{card} A = \underline{\operatorname{card}}(A \cap B) + \underline{\operatorname{card}}(A \setminus B) < j + k < \overline{\operatorname{card}}(A \cap B) \oplus \overline{\operatorname{card}}(A \setminus B) = \operatorname{card} A$$

which is impossible.

The last result leaves open the obvious question of which of the two subsets $A \cap B$ and $A \setminus B$ is M-countable. This is (partially) answered next.

Theorem 2.1.21 *Let A be M-countable and let B be bounded with $\partial \operatorname{card} B < \partial \operatorname{card} A$. Suppose also that $\partial \underline{\operatorname{card}}(A \cap B) > \partial \underline{\operatorname{card}}(A \setminus B)$. Then $A \cap B$ is M-countable.*

Proof. We must show that there is no α in M with $\underline{\operatorname{card}}(A \cap B) < \alpha < \overline{\operatorname{card}}(A \cap B)$. So suppose there is such α and try to obtain a contradiction. First take $bandb'$ such that $b' \supseteq B \supseteq b$ with $|b' \setminus b| < \partial \operatorname{card} A$, and as $\operatorname{card} A = \underline{\operatorname{card}}(A \cap B) + \underline{\operatorname{card}}(A \setminus B)$ (by lemma 2.1.19) we have $\partial \operatorname{card} A = \partial \underline{\operatorname{card}}(A \cap B) + \partial \underline{\operatorname{card}}(A \setminus B) = \partial \underline{\operatorname{card}}(A \cap B)$, the last equality here following by properties of the arithmetic of ∂ using the fact that $\partial\,\underline{\mathrm{card}}(A\cap B) \geqslant \partial\,\underline{\mathrm{card}}(A\setminus B)$. As $|b'\setminus b| < \partial\,\mathrm{card}\,A = \partial\,\underline{\mathrm{card}}(A\cap B)$ we have $\alpha - |b'\setminus b| > \underline{\mathrm{card}}(A\cap B)$. So by replacing α with $\alpha - |b'\setminus b|$ we shall assume that $\underline{\mathrm{card}}(A\cap B) < \alpha < \alpha + |b'\setminus b| < \overline{\mathrm{card}}(A\cap B)$.

Now, using $\partial\,\underline{\mathrm{card}}(A\cap B) > \partial\,\underline{\mathrm{card}}(A\setminus B)$, take $\beta \in \partial\,\underline{\mathrm{card}}(A\cap B)$ with $\beta > \partial\,\underline{\mathrm{card}}(A\setminus B)$, and using the definition of ∂ take also $j \in \underline{\mathrm{card}}(A\setminus B)$ with $\beta + j > \underline{\mathrm{card}}(A\setminus B)$. Then $\alpha > \underline{\mathrm{card}}(A\cap B)$ and $\beta \in \partial\,\underline{\mathrm{card}}(A\cap B)$ so $\alpha - \beta > \underline{\mathrm{card}}(A\cap B)$. Adding, we have

$$\alpha + j = (\alpha-\beta) + (j+\beta) > \underline{\mathrm{card}}(A\cap B) + \underline{\mathrm{card}}(A\setminus B) = \mathrm{card}\,A > \mathrm{card}(A\setminus B) > j.$$

We can use this to obtain internal approximations

$$a_{\alpha+j} \supseteq A \supseteq A\setminus B \supseteq a_j$$

where $|a_{\alpha+j}| = \alpha + j$ and $|a_j| = j$. Thus we can approximate $A\cap B$ by

$$A\cap B \subseteq a_{j+\alpha}\cap b' \subseteq (a_{j+\alpha}\setminus a_j)\cup(b'\setminus b)$$

with size

$$|a_{j+\alpha}\cap b'| \leqslant |a_{j+\alpha}\setminus a_j| + |b'\setminus b| \leqslant \alpha + |b'\setminus b| < \overline{\mathrm{card}}(A\cap B).$$

This is our contradiction.

A dual result result to the previous one can now be deduced.

Theorem 2.1.22 *Let A be M-countable and let B be bounded with $\partial\,\mathrm{card}\,B < \partial\,\mathrm{card}\,A$. If $\partial\,\overline{\mathrm{card}}(A\cap B) > \partial\,\overline{\mathrm{card}}(A\setminus B)$ then $A\cap B$ is M-countable.*

Proof. Assume $A\cap B$ is not M-countable, so $\overline{\mathrm{card}}(A\cap B) > \underline{\mathrm{card}}(A\cap B)$. It follows from Corollary 2.1.20 that $A\setminus B$ is M-countable, i.e. $\overline{\mathrm{card}}(A\setminus B) = \underline{\mathrm{card}}(A\setminus B)$.

Now as $\partial\,\overline{\mathrm{card}}(A\cap B) > \partial\,\overline{\mathrm{card}}(A\setminus B)$ we have $\mathrm{card}\,A = \overline{\mathrm{card}}(A\cap B)\oplus\overline{\mathrm{card}}(A\setminus B)$ so $\partial\,\mathrm{card}\,A = \partial\,\overline{\mathrm{card}}(A\cap B)\oplus\partial\,\overline{\mathrm{card}}(A\setminus B) = \partial\,\overline{\mathrm{card}}(A\cap B)$. In particular $\partial\,\overline{\mathrm{card}}(A\setminus B) = \partial\,\underline{\mathrm{card}}(A\setminus B) < \partial\,\mathrm{card}\,A$. Also, $\mathrm{card}\,A = \underline{\mathrm{card}}(A\cap B) + \underline{\mathrm{card}}(A\setminus B)$ so $\partial\,\mathrm{card}\,A = \partial\,\underline{\mathrm{card}}(A\cap B) + \partial\,\underline{\mathrm{card}}(A\setminus B) = \partial\,\underline{\mathrm{card}}(A\cap B)$ since $\partial\,\mathrm{card}\,A = \partial\,\underline{\mathrm{card}}(A\setminus B)$ is impossible. Therefore $\partial\,\underline{\mathrm{card}}(A\setminus B) < \partial\,\underline{\mathrm{card}}(A\cap B)$ and theorem 2.1.21 tells us that $A\cap B$ is M-countable.

Question 2.1.23 *What happens if* $\partial\,\overline{\mathrm{card}}(A\cap B) = \partial\,\overline{\mathrm{card}}(A\setminus B)$ *and* $\partial\,\underline{\mathrm{card}}(A\cap B) = \partial\,\underline{\mathrm{card}}(A\setminus B)$*? Corollary 2.1.20 tells us that at least one of* $(A\cap B)$ *and* $(A\setminus B)$ *is* M*-countable but the question is are there any further conditions that may tell us which one or are they always both* M*-countable in this case?*

As an application of these last two results we see that if X is M-countable and a is internal then *either* $X\cap a$ *or* $X\setminus a$ is M-countable–indeed the 'larger' of these two will always be M-countable. One might hope that they are both M-countable, but this is not true in general as shown by example 2.1.18.

2.2 Monotonically Definable Sets

The collection of M-countable sets includes the M-finite sets. An important family of non-definable sets that can easily be shown to be M-countable are the monotonically definable sets obtained by varying a parameterized definition over a cut.

Definition 2.2.1 *Let* $B\subseteq M$ *and let* I *be a cut of* M*. A set* $B\subseteq M$ *is said to be* monotonically definable *by* I *if there exists a monotonic formula* $\theta(x,y)$ *of* $\mathcal{L}_{\mathrm{A}}$ *possibly with parameters from* M *such that one of the following holds.*

(a) $b\in B \Leftrightarrow \forall x\in I\ M\vDash\theta(b,x)$

(b) $b\in B \Leftrightarrow \exists x> I\ M\vDash\theta(b,x)$

(c) $b \in B \Leftrightarrow \forall x > I \ M \models \theta(b, x)$

(d) $b \in B \Leftrightarrow \exists x \in I \ M \models \theta(b, x)$

In cases (a) and (b) θ is monotonic means that for each $b \in M$ and for all $x, y \in M$

$$x \leqslant y \to (\theta(b, y) \to \theta(b, x))$$

In cases (c) and (d) θ is monotonic means that for each $b \in M$ and for all $x, y \in M$

$$x \leqslant y \to (\theta(b, x) \to \theta(b, y))$$

By overspill in one direction and monotonicity in the other (a) is equivalent to (b), and by underspill in one direction and monotonicity in the other (c) is equivalent to (d).

We shall call a set defined by a type (c)/(d) formula *up monotonically definable*, and a set defined by a type (a)/(b) formula *down monotonically definable*. We shall call the set *monotonically J-definable* when we want to give reference to the particular cut J involved.

For example, if $g \in M$ is a nonstandard finite group, then for each M-finite $H \subseteq g$ the group $\langle H \rangle$ externally generated by elements of H is monotonically definable, for some appropriate ϕ, where the cut in question is $\mathbb{N}$. The formula ϕ simply counts the number of multiplications and inverses required to obtain x from elements of H.

Definition 2.2.2 *If B is monotonically definable by some formula β and cut I then denote by β_i the set $\{x : M \models \beta(x, i)\}$.*

Note that $B = \bigcup_{i \in I} \beta_i = \bigcap_{i' > I} \beta_{i'}$ in the case that β is a type (c)/(d) formula and $B = \bigcup_{i' > I} \beta_{i'} = \bigcap_{i \in I} \beta_i$ in the case that β is a type (a)/(b) formula. We have the following easy propositions.

Proposition 2.2.3 *Suppose $X \subseteq M$ is down monotonically J-definable for some initial segment J, and formula α. Then there is a formula β and initial segment $\widehat{J}$ such that X is up monotonically $\widehat{J}$-definable by β.*

Proof. Let $j' > J$ and for $j \leqslant j'$ define $\beta(x, j)$ as follows; $M \vDash \beta(x, j) \iff M \vDash \alpha(x, j' - j)$. Note that for $j > j'$ we can define $\beta(x, j)$ in any way that respects monotonicity. Let $j_1 \leqslant j_2 \leqslant j'$ and suppose that for some x we have $M \vDash \beta(x, j_1)$. By definition of β we have $M \vDash \alpha(x, j' - j_1)$, and since $j' - j_1 \geqslant j' - j_2$, we have by the monotonicity of α that $M \vDash \alpha(x, j' - j_2)$, whence $M \vDash \beta(x, j_2)$. So β is a type (c)/(d) formula. Set $\widehat{J} = j' - J$. Let $B = \{b \in M : \exists \widehat{j} \in \widehat{J} \text{ such that } M \vDash \beta(b, \widehat{j})\}$. Suppose $x \in X$. Then there is $j' > j^* > J$ such that $M \vDash \alpha(x, j^*)$. So $M \vDash \beta(x, j' - j^*)$, and $j' - j^* \in \widehat{J}$ so $x \in B$. Conversely if $b \in B$ then there is $\widehat{j} \in \widehat{J}$ such that $M \vDash \beta(b, \widehat{j})$, whence $M \vDash \alpha(b, j' - \widehat{j})$ but $j' - \widehat{j} > J$ and so $b \in X$. Therefore $B = X$ and this completes the proof.

Proposition 2.2.4 *A bounded monotonically definable set X is M-countable, but not all M-countable sets arise in this way.*

Proof. Suppose X is bounded and monotonically definable. By proposition 2.2.3 we may suppose that there is a cut I and a formula θ such that X is up monotonically I definable by θ. Thus $X = \bigcup_{i \in I} \theta_i = \bigcap_{i' > I} \theta_{i'}$ and so $\underline{\text{card}}(X) = \sup\{|\theta_i| : i \in I\} \leqslant \inf\{|\theta_{i'}| : i' > I\} = \overline{\text{card}}(X)$. If we don't have equality here then there is x such that $\sup\{|\theta_i| : i \in I\} < x < \inf\{|\theta_{i'}| : i' > I\}$ whence by overspill on the first inequality there is $i' > I$ such that $|\theta'_i| < x$, which contradicts the second inequality. It is valid to use overspill here because each θ_i is an M-finite set and so there is a first order formula $\phi(i, y)$ for $|\theta_i| = y$.

To see that not all M-countable sets arise in this way, take M countable X and modify the construction in Example 2.1.3 so that for any upwards monotonic $\phi(x, y)$ (crucially there are only countably many) there is some $x \in X$ with $\phi(x, y') \wedge \neg\phi(x, y)$

for some $y < y'$, and for any downwards monotonic $\psi(x,y)$ there is some $x \in X$ with $\psi(x,y') \wedge \neg\psi(x,y)$ for some $y > y'$.

2.3 I-measurable Sets

One problem with this theory of M-countability is that the M-countable sets are not an algebra of sets, i.e. are not closed under intersections and unions. A natural suggestion is to mimic the Carathéodory definition of measure. Using $\overline{\text{card}}$ as an 'outer measure' the direct translation of this idea is that X is measurable if and only if for all $A \subseteq M$ we have $\overline{\text{card}}(A \setminus X) + \overline{\text{card}}(A \cap X) = \overline{\text{card}}(A)$. Unfortunately this doesn't work even for very well behaved X such as $\mathbb{N}$. If $X = \mathbb{N}$ and $A = a$ is a nonstandard M-finite initial segment then

$$\overline{\text{card}}(A \setminus X) + \overline{\text{card}}(A \cap X) = (a - \mathbb{N}) + \mathbb{N} = a - \mathbb{N} < \overline{\text{card}}(A)$$

and

$$\overline{\text{card}}(A \setminus X) \oplus \overline{\text{card}}(A \cap X) = (a - \mathbb{N}) \oplus \mathbb{N} = a + \mathbb{N} > \overline{\text{card}}(A).$$

However, note that inequalities

$$\overline{\text{card}}(A) \leqslant \overline{\text{card}}(A \setminus X) + \overline{\text{card}}(A \cap X) \leqslant \overline{\text{card}}(A \setminus X) \oplus \overline{\text{card}}(A \cap X)$$

always hold.

A version of this idea does make sense when taken relative to a cut I.

Definition 2.3.1 *Given a cut I, we say that a bounded set $X \subseteq M$ is I-Carathéodory-measurable (or I-measurable) if for all bounded $A \subset M$ we have*

$$\overline{\text{card}}(A \setminus X) \oplus \overline{\text{card}}(A \cap X) \leqslant \overline{\text{card}}(A) \oplus I.$$

We have motivated this definition as an analogy with Carathéodory's but the following gives a potentially more straightforward equivalent characterisation in terms of the $\partial\,\mathrm{card}$ operator defined earlier.

Proposition 2.3.2 *Let X be a bounded set. Then X is I-measurable if and only if its cardinality derivative ($\partial\,\mathrm{card}\,X$ or $\overline{\mathrm{card}}\,X \ominus \underline{\mathrm{card}}\,X$) is at most I, i.e. if and only if $\partial\,\mathrm{card}\,X \leqslant I$.*

Proof. Given bounded $A \subseteq M$, $a' \supseteq A$ internal and $i' > I$ we use $\overline{\mathrm{card}}\,X \ominus \underline{\mathrm{card}}\,X \leqslant I$ to choose x and x' such that $x \subseteq X \subseteq x'$ with $|x' \setminus x| < i'$. Then $A \cap X \subseteq a' \cap x'$ and $A \setminus X \subseteq a' \setminus x$ so $|a' \cap x'| + |a' \setminus x| = |a'| + j$ for some $j \leqslant |x' \setminus x| < i'$. Thus $\overline{\mathrm{card}}(A \cap X) \oplus \overline{\mathrm{card}}(A \setminus X) \leqslant \overline{\mathrm{card}}\,A \oplus I$.

Conversely, given I-measurable X, let $a \supseteq X$. By Lemma 2.1.6 we have $|a| \ominus \underline{\mathrm{card}}\,X \leqslant \overline{\mathrm{card}}(a \setminus X)$ and of course $\overline{\mathrm{card}}(a \cap X) = \overline{\mathrm{card}}\,X$. So by I-measurability of X we have

$$(|a| \ominus \underline{\mathrm{card}}\,X) \oplus \overline{\mathrm{card}}\,X \leqslant |a| \oplus I.$$

The left hand side here is the inf of $|a| - |x| + |x'|$ taken over all $x \subseteq X \subseteq x'$, which is just $|a| + \inf_{x,x'}(|x'| - |x|) = |a| + (\overline{\mathrm{card}}\,X \ominus \underline{\mathrm{card}}\,X)$ and $|a| \oplus I = |a| + I$ since $|a|$ is a number. Hence

$$|a| + (\overline{\mathrm{card}}\,X \ominus \underline{\mathrm{card}}\,X) \leqslant |a| + I,$$

and hence (using again the fact that $|a|$ is a number) $\overline{\mathrm{card}}\,X \ominus \underline{\mathrm{card}}\,X \leqslant I$, as required.

Corollary 2.3.3 *Given I-measurable $X \subseteq M$, we have $\partial\,\overline{\mathrm{card}}\,X \leqslant I$ and $\partial\,\underline{\mathrm{card}}\,X \leqslant I$.*

Proof. We have

$$\partial\,\overline{\mathrm{card}}\,X = \inf\{i' - i : i' \geqslant \overline{\mathrm{card}}\,X \geqslant i\} \leqslant \inf\{|x'| - |x| : x' \supseteq X \supseteq x\},$$

and the last cut here is precisely $\overline{\mathrm{card}}\, X \ominus \underline{\mathrm{card}}\, X$. Similarly for $\partial\, \underline{\mathrm{card}}\, X$,

$$\partial\, \underline{\mathrm{card}}\, X = \inf\{i' - i : i' \geqslant \underline{\mathrm{card}}\, X \geqslant i\} \leqslant \inf\{|x'| - |x| : x' \supseteq X \supseteq x\}.$$

Not surprisingly, since the cut I in 'I-measurable' is linked to a derivative $\partial\, \mathrm{card}\, X$ which is often closed under addition, the theory of I-measurable sets is smoother when I is closed under addition.

Proposition 2.3.4 *Let I be a cut closed under addition. Then the set of I-measurable subsets of M forms an algebra, i.e. is closed under unions, intersections and bounded complements.*

Proof. Let $X \subseteq M$ be I-measurable. Let $a \in M$ be such that $X \subseteq< a$. Then $\overline{\mathrm{card}}(a \setminus X) \ominus \underline{\mathrm{card}}(a \setminus X) \leqslant (|a| \ominus \underline{\mathrm{card}}(X)) \ominus (|a| - \overline{\mathrm{card}}(X))$ by lemma 2.1.6. It follows that this last expression is equal to $\overline{\mathrm{card}}(X) \ominus \underline{\mathrm{card}}(X)$ by the arithmetic of cuts because $|a|$ is a number. Finally $\overline{\mathrm{card}}(X) \ominus (\underline{\mathrm{card}}(X)) < I$ by proposition 2.3.2 and thus $a \setminus X$ is I-measurable also by 2.3.2.

Now suppose A, B are bounded and I-measurable. Fix $i' > I$, then there are M-finite sets $a \subseteq A \subseteq a'$ and $b \subseteq B \subseteq b'$ with $\mathrm{card}(a' \setminus a) < i'$ and $\mathrm{card}(b' \setminus b) < i'$. Then $a' \cup b' \supseteq A \cup B \supseteq a \cup b$ and

$$(a' \cup b') \setminus (a \cup b) \subseteq (a' \setminus a) \cup (b' \setminus b)$$

so

$$\mathrm{card}((a' \cup b') \setminus (a \cup b)) \leqslant \mathrm{card}(a' \setminus a) + \mathrm{card}(b' \setminus b) \leqslant 2i'.$$

This suffices since I is closed under addition. The rest follows by proposition 2.3.2 and the usual set constructions of intersection, etc.

Chapter 3

Index, Transversals and Nonstandard Lagrange

3.1 M-countable Groups

In the last chapter we studied the notion of an M-countable set in a nonstandard model of PA. In this chapter we shall apply some of those ideas to groups. In particular we shall look at the case of an M-finite group G having an M-countable subgroup H. In the special case that H is M-finite Lagrange's theorem works as normal but we will be concerned with what can be said when H is not M-finite. The results here are given in this kind of generality with extra assumptions stated where necessary. If the reader would like specific examples in mind then G could be a permutation group S_n for n nonstandard and H could be a subgroup $S_n^I = \{g \in S_n : |\text{support}(g)| \in I\}$ for some cut $I < n$ as mentioned in chapter 1. The specific case where G is abelian is studied in more detail in the next chapter.

The first question to ask is what are the possibilities for card(H), when H is an M-countable group? It turns out the only restriction on what card(H) can be is given by the next lemma. In fact for the following result it is not necessary to suppose that an M-finite supergroup $G \geqslant H$ exists. All we need is that the group operation on H is definable in the following sense.

Definition 3.1.1 *Suppose $M \vDash PA$ and $H \subseteq M$ is a group. Then we say the group operation on H is* definable *if for any M-finite $H' \subseteq H$ the sets $\{h_1h_2 : h_1, h_2 \in H'\}$ and $\{h^{-1} : h \in H'\}$ are also M-finite.*

Of course this situation would be guaranteed by the presence of an M-finite supergroup G. We ask the following question.

Question 3.1.2 *Is there a bounded group $H \subseteq M$ which has* definable *group operation but is not contained in any M-finite group G?*

The main thrust of the rest of this thesis is understanding subgroups of M-finite groups so the answer to this question will not trouble us greatly. We make the following remark on notation. Let $I \subseteq M$ be a cut of M, then we will write $I \vDash +$ to mean I is closed under addition and $I \vDash \cdot$ to mean I is closed under multiplication. Now the promised lemma.

Lemma 3.1.3 *Let $M \vDash PA$ and suppose H is an M-countable group with* definable *group operation but H is not M-finite. Then* $\mathrm{card}(H) = I \vDash +$.

Proof. Let $i \in I$ and select M-finite $H_i \subseteq H$ with $|H_i| \geqslant i$. Without loss we may suppose that H_i contains the identity and is closed under $^{-1}$. Since H is not M-finite, but has definable group operation, we have that $H_iH_i \subsetneq H$. Let $h \in H \setminus H_iH_i$, then it follows that $h \notin H_i$. If $H_ih \cap H_i \neq \varnothing$ then $\exists h_1, h_2 \in H_i\ h_1h = h_2$ and hence $h = h_1^{-1}h_2 \in H_iH_i$ is a contradiction. Thus $H_ih \cap H_i = \varnothing$ and so $|H_ih \cup H_i| \geqslant 2i$. Since $H_ih \cup H_i \subseteq H$ and is M-finite it follows that $2i \in I$ and so $I \vDash +$.

It turns out lemma 3.1.3 is the only restriction on a cut I as the potential cardinality of some group H. To see this simply consider the following example.

Example 3.1.4 *Let $M \models PA$ and let I be any cut of M such that I is closed under addition. Take H to be the group $\{x : \pm x \in I\}$ under addition in M. This is clearly a group and has definable group operation. It is easy to check that H is M-countable with* $\mathrm{card}(H) = I$.

3.2 A Notion of Index

In this section we extend our idea from the previous chapter of counting subsets and supersets to 'upper transversals' and 'lower transversals' in order to define a notion of index for M-countable subgroups of M-finite groups. We define the upper and lower index of a subgroup H in an M-finite group G.

Definition 3.2.1 *Let G be an M-finite group. Let $H \leqslant G$ be a subgroup. We define the upper index (1) and the lower index (2) as follows.*

(1) $\overline{[G : H]} = \overline{\mathit{Index}}(G : H) = \inf\{\mathrm{card}(T) : T \textit{ is } M\textit{-finite and } \forall g \in G\ \exists t \in T\ \exists h \in H : ht = g\}$

(2) $\underline{[G : H]} = \underline{\mathit{Index}}(G : H) = \sup\{\mathrm{card}(T) : T \textit{ is } M\textit{-finite and } (Ht_1) \cap (Ht_2) = \varnothing \textit{ for all } t_1 \neq t_2 \textit{ in } T\}$

We will refer to the property of H and T expressed in (2) above as unique product. Note that although we have stated the above definition in the context of H being a subgroup of G, the definition still makes sense if H is an arbitrary subset.

We now turn our attention to the case when H is M-countable. Suppose $G \in M$ is an M-finite group and $H \leqslant G$ is a M-countable subgroup with $\mathrm{card}(H) = I$. We now state and prove some propositions concerning this set up but first we require the following definition.

Definition 3.2.2 *Let $I \subseteq M$ be a cut and $a \in M$ an element of the model, then we define* $\frac{a}{I} := \inf\{\lfloor \frac{a}{i} \rfloor : i \in I\}$.

It is easy to check that $\inf\{\lfloor \frac{a}{i} \rfloor : i \in I\} = \sup\{\lfloor \frac{a}{i'} \rfloor : i' > I\}$.

Example 3.2.3 *Let n be nonstandard and let $G = S_n$, thus G is the M-finite permutation group of cardinality $n!$. Let $I < n$ be a cut closed under addition. Recall that $A_n^I = \{g \in A_n : |support(g)| \in I\}$ and $S_n^I = \{g \in S_n : |support(g)| \in I\}$. We know A_n^I and S_n^I are monotonically definable and therefore are M-countable. Intuitively A^I is half the size of S^I, but since the cardinality of the latter must be closed under addition by 3.1.3, it follows that $\mathrm{card}(A^I) = \mathrm{card}(S^I)$. There is a classical formula for the number of permutations of $\{0, 1, \cdots, n-1\}$ that fix $k < n$ points, it works the same for k, n nonstandard. Using this formula the cardinality of S^I can be worked out in terms of the cut I, it turns out that for I closed under addition this is equal to $n^I := \sup_{i \in I}\{n^i\}$. With these facts in mind we have that $\overline{[G : A_n^I]} = \underline{[G : A_n^I]} = \frac{n!}{\mathrm{card}(S_n^I)} = \frac{n!}{n^I}$ and also $\overline{[G : S_n^I]} = \underline{[G : S_n^I]} = \frac{n!}{\mathrm{card}(A_n^I)} = \frac{n!}{n^I}$. The reasons for these equalities will be explained by the following propositions.*

We have the following straightforward proposition.

Proposition 3.2.4 $\underline{[G : H]} \leqslant \frac{|G|}{I}$.

Proof. Suppose to the contrary that $\underline{[G : H]} > \frac{|G|}{I}$. Then by the definition of $\underline{[G : H]}$ there exists an M-finite T, with HT unique product, and $|T| > \frac{|G|}{I}$. Let $i \in I$ be arbitrary. Since $\mathrm{card}(H) = I$ there exists an M-finite $H_i \subseteq H$ with $|H_i| \geqslant i$. Since $H_i \subseteq H$ it must be that $H_i T$ is also unique product. Therefore $|H_i||T| \leqslant |G|$ and so $|T| \leqslant \frac{|G|}{i}$. But i was arbitrary and so this is true for all $i \in I$, hence $|T| \in \frac{|G|}{I}$ contradicting the choice of T.

Analogously we have the following proposition concerning the upper index $\overline{[G : H]}$.

Proposition 3.2.5 $\overline{[G : H]} \geqslant \frac{|G|}{I}$.

Proof. Suppose that $\overline{[G : H]} < \frac{|G|}{I}$. Then there exists an M-finite T, with $HT \supseteq G$, and $|T| < \frac{|G|}{I}$. Let $i' > I$ be arbitrary. Since $\mathrm{card}(H) = I$ there exists an M-finite $H_{i'} \supseteq H$

with $|H_{i'}| \leqslant i'$. Also $H_{i'}T \supseteq G$ so $|H_{i'}||T| \geqslant |G|$. Therefore $|T| \geqslant \frac{|G|}{i'}$, and this holds for all $i' > I$. So by underspill there is $i \in I$ such that $|T| \geqslant \frac{|G|}{i}$, hence $|T| \geqslant \frac{|G|}{I}$ contradicting the choice of T.

Having established an upper bound for the lower index and a lower bound for the upper index we now seek a converse to propositions 3.2.4 and 3.2.5. To this end we will require the following combinatorial lemma. For further details see [3].

Lemma 3.2.6 *Let G be an M-finite group. Suppose $\Delta = \{a_0 \cdots a_{n-1}\}$ and $\Sigma = \{b_0 \cdots b_{m-1}\}$ are M-finite subsets of G. If $|G| < \frac{nm}{k}$ (for $k \in \omega^M$) then there is $x \in G$ such that $|\Delta x \cap \Sigma| \geqslant k + 1$.*

Proof. Firstly we note that we can always find x such that $|\Delta x \cap \Sigma| \geqslant 1$. Just take $x = a_i^{-1} b_j$ for any i, j. Find x such that $|\Delta x \cap \Sigma| \geqslant 2$. We need $x = a_{i_1}^{-1} b_{j_1} = a_{i_2}^{-1} b_{j_2}$ for $i_1 \neq i_2$ and $j_1 \neq j_2$. We can always find such an x unless the set $\Delta^{-1}\Sigma = \{a_i^{-1} b_j : i < n, j < m\}$ is unique product. However if the cardinality of the set of possible pairs (i, j) is greater than the cardinality of G then this can't happen. So if $nm > |G|$ there exists such an x. This establishes the lemma for $k = 1$. Now suppose $|G| < \frac{nm}{k}$ for some $k \in \omega^M$, and we must show that there exists x such that $|\Delta x \cap \Sigma| \geqslant k+1$. We need $x = a_{i_1}^{-1} b_{j_1} = \cdots = a_{i_{k+1}}^{-1} b_{j_{k+1}}$. There will be such an x unless the set $\Delta^{-1}\Sigma = \{a_i^{-1} b_j : i < n, j < m\}$ is at most k covered i.e. for all g in $\Delta^{-1}\Sigma$ there are at most k pairs $a_i \in \Delta, b_j \in \Sigma$ such that $a_i^{-1} b_j = g$. However in this case $|\Delta^{-1}\Sigma| \geqslant \frac{nm}{k} > |G|$ which is impossible since $\Delta^{-1}\Sigma \subseteq G$, and we conclude that such an x does exist.

We may now prove some further results concerning $\overline{[G : H]}$ and $\underline{[G : H]}$.

Theorem 3.2.7 *Let G be an M-finite group and let H be an M-countable subgroup, with* $\mathrm{card}(H) = I$. *Then* $\underline{[G : H]} = \frac{|G|}{I}$.

Our first proof of this result will assume that I is closed under multiplication but achieve a slightly stronger conclusion. We shall then prove the theorem as stated above without the closure condition.

Proof. By proposition 3.2.4 we have that $\underline{[G:H]} \leqslant \frac{|G|}{I}$. It remains to show that $\underline{[G:H]} \geqslant \frac{|G|}{I}$. Recall that $\frac{|G|}{I} = \sup\{\frac{|G|}{i'} : i' > I\}$. So we must show that for every $i' > I$ there is an M-finite $T \subseteq G$ such that HT is unique product, and $|T| \geqslant \frac{|G|}{i'}$. Let $i' > I$ and set $i = \lfloor \sqrt{i'} \rfloor$. Since I is closed under multiplication it follows that $i > I$. Since H is M-countable there exists an M-finite $H_i \supseteq H$ with $|H_i| = i$. We use lemma 3.2.6 to build an M-finite T with H_iT unique product and $|T|$ as large as possible. We proceed as follows.

(1) We seek $x_1 \in G$ such that $Hx_1 \cap H = \varnothing$ i.e. x_1 such that $|H_ix_1 \cap (G \setminus H_i)| \geqslant i$. By lemma 3.2.6 we can find such an x_1 provided $|G| < \frac{|H_i||G\setminus H_i|}{i-1} = \frac{i(|G|-i)}{i-1}$. Note $|G| < \frac{i(|G|-i)}{i-1} \iff i|G| - |G| < i|G| - i^2 \iff 1 < \frac{|G|}{i^2}$.

(2) We seek $x_2 \in G$ such that $H_ix_2 \cap (H_i \cup H_ix_1) = \varnothing$ i.e. $x_2 \in G$ such that $|H_ix_2 \cap (G \setminus (H_i \cup H_ix_1))| \geqslant i$. By lemma 3.2.6 we can find such an x_2 provided $|G| < \frac{|H_i||G\setminus(H_i\cup H_ix_1)|}{i-1} = \frac{i(|G|-2i)}{i-1}$. Note $|G| < \frac{i(|G|-2i)}{i-1} \iff i|G| - |G| < i|G| - 2i^2 \iff 2 < \frac{|G|}{i^2}$.

(n) We seek $x_n \in G$ such that $H_ix_n \cap (H_i \cup H_ix_1 \cup H_ix_2 \cup \cdots \cup H_ix_{n-1}) = \varnothing$ i.e. $x_n \in G$ such that $|H_ix_2 \cap (G \setminus (H_i \cup H_ix_1 \cup H_ix_2 \cup \cdots \cup H_ix_{n-1}))| \geqslant i$. By lemma 3.2.6 we can find such an x_n provided $|G| < \frac{|H_i||G\setminus(H_i\cup H_ix_1\cup H_ix_2\cup\cdots\cup H_ix_{n-1}|}{i-1} = \frac{i(|G|-ni)}{i-1}$. Note $|G| < \frac{i(|G|-ni)}{i-1} \iff i|G| - |G| < i|G| - ni^2 \iff n < \frac{|G|}{i^2}$.

So we set $T = \{id, x_1, \cdots, x_n\}$ where n is as large as possible i.e. $n = \lceil \frac{|G|}{i^2} \rceil - 1$. So $|T| = n + 1 = \lceil \frac{|G|}{i^2} \rceil \geqslant \frac{|G|}{i^2} \geqslant \frac{|G|}{i'}$ (since $i^2 \leqslant i'$). Suppose H_iT is not unique product, then $\exists x_l, x_m \in T$ with $l < m \leqslant n$ and $\exists h_l, h_m \in H_i$ such that $h_lx_l = h_mx_m$. But then at stage

m of the construction of T we would have $H_i x_m \cap H_i x_l \neq \varnothing$ contradicting the choice of x_m. Since $H_i T$ is unique product and $H \subseteq H_i$ it follows that HT is unique product and T is as required.

The assumption that I is closed under multiplication seems to be necessary for theorem 3.2.7 to work, however note that the proof above achieves the stronger conclusion that $H_i T$ is unique product for an M-finite superset $H_i \supseteq H$. If we do away with this stronger conclusion then we have an alternative proof of theorem 3.2.7 that does not need $I \vDash \cdot$.

Proof. Let $i' > I$ and since H is M-countable there exists an M-finite $H_{i'} \supseteq H$ with $|H_{i'}| = i'$. Set $l := \lceil \frac{|G|}{i'} \rceil$. We perform an internal induction to select l elements. Suppose inductively that we have already selected elements $\{t_0, \cdots, t_{r-1}\}$ for some $r < l$ and we show how to add t_r. $|H_{i'}||\{t_0, \cdots, t_{r-1}\}| = i'r < i'\frac{|G|}{i'} = |G|$. So there is $t_r \in G$ with $t_r \notin (H_{i'})(\{t_0, \cdots, t_{r-1}\})$. By induction we create a set $\{t_0, \cdots, t_{l-1}\}$. Set $T = \{t_0, \cdots, t_{l-1}\}$. It remains to check that T has the desired properties.

(1) $|T| \geqslant \frac{|G|}{i'}$: Follows from the choice of l and the fact that at each stage a new element is always selected (since $H_{i'}$ contains the identity).

(2) HT is unique product: Suppose to the contrary, then there are $h_i, h_j \in H$ and $t_i, t_j \in T$ for $0 \leqslant j < i \leqslant l-1$ such that $h_i t_i = h_j t_j$. Thus $t_i = h_i^{-1} h_j t_j$, since H is a group we get $h_i^{-1} h_j \in H \subseteq H_{i'}$ and so $t_i \in H_i'(\{t_0, \cdots, t_{i-1}\})$ which contradicts the choice of t_i.

Question 3.2.8 *Can an analogous result be proved in the case when I is not closed under multiplication? In other words given $i' > I \nvDash \cdot$ can it be shown that there are M-finite sets $H_{i'} \supseteq H$ and $T \subseteq G$ such that $|T| \geqslant \frac{|G|}{i'}$ and $H_{i'} T$ is unique product?*

In the case that H is monotonically definable we can construct T as above and then apply overspill to answer question 3.2.8 positively. Thus any counterexample to question 3.2.8 would probably require H to be constructed by an inductive process and $\mathrm{card}(H)$ would necessarily fail to satisfy multiplication.

Analogously to 3.2.7 we have the following theorem concerning the upper index $\overline{[G:H]}$.

Theorem 3.2.9 *Let G be an M-finite group and let H be a M-countable subgroup, with* $\mathrm{card}(H) = I$. *Then* $\overline{[G:H]} = \frac{|G|}{I}$.

Proof. By proposition 3.2.4 we have that $\overline{[G:H]} \geqslant \frac{|G|}{I}$. It remains to show that $\overline{[G:H]} \leqslant \frac{|G|}{I}$. We show that for arbitrary $i \in I$ there are M-finite sets $T \subseteq G$ and $H_i \subseteq H$ such that $H_i T = G$, and $|T| \leqslant \frac{|G|}{i}$.

By M-countability we may select M-finite $H^* \subseteq H$ such that $|H^*| \geqslant i$. Since G is an M-finite set it has an internal enumeration, that is $G = \{t_0, \cdots, t_{\alpha-1}\}$ for some $\alpha \in \omega^M$. For $0 \leqslant j,k < \alpha$ let us say $t_j \sim t_k$ if $\exists h_1, h_2 \in H^*\ h_1 t_j = h_2 t_k$. Note that $\sim$ is not transitive unless H^* happens to be a group but this will not matter for our purposes. We now inductively define an internal subset T of G.

(1) $t_0 \in T$.

(2) $t_k \in T$ if and only if $\forall j < k (t_j \in T \implies t_j \not\sim t_k)$.

This defines an M-finite set T. We define $H_i := \{h_2^{-1} h_1 : h_1, h_2 \in H^*\}$ which is M-finite also. It remains to check that T, H_i have the desired properties.

(i) $|T| \leqslant \frac{|G|}{i}$: If not then $|T||H^*| > |G|$ and so there is a unique product failure. For $0 \leqslant j < k < \alpha\ \exists t_j, t_k \in T \exists h_1, h_2 \in H^*$ such that $h_1 t_j = h_2 t_k$, but this contradicts $t_j \not\sim t_k$.

(ii) $H_i \subseteq H$: If $h \in H_i$ then $\exists h_1, h_2 \in H^*$ such that $h = h_2^{-1} h_1$. We have $H^* \subseteq H$ and H is a group, so $h \in H$ as required.

(iii) $H_iT = G$: Let $g \in G$. If $g \in T$ then we are done, so suppose that g was one of the things we removed. Then $g = t_k$ for some k such that $0 < k < \alpha$. Since $t_k \notin T$, for some $j < k$ and $t_j \in T \subseteq G$ we must have $\exists h_1, h_2 \in H^*\ h_1 t_j = h_2 t_k$ otherwise t_k would have gone into T. Whence $g = t_k = h_2^{-1} h_1 t_j \in H_i T$ as required.

So we know that $\underline{\text{Index}}(G : H) = \overline{\text{Index}}(G : H) = \frac{|G|}{I}$. This says that $\sup\{\text{card}(T) : HT \text{ is unique product}\} = \inf\{\text{card}(T) : HT = G\}$ but that in itself does not imply that there is a single M-countable T such that $HT = G$, HT is unique product and $\underline{\text{card}}(T) = \overline{\text{card}}(T) = \frac{|G|}{I}$.

Question 3.2.10 *Given M-countable $H \leqslant G$ with* $\text{card}(H) = I$ *is there an M-countable transversal T, with* $\text{card}(T) = \frac{|G|}{I}$*?*

It turns out we can answer this question positively under a countability assumption on M. This is the subject of the next section.

3.3 First Transversal Theorem

Throughout this section G will be an M-finite group and $H \leqslant G$ an M-countable subgroup with $\text{card}(H) = I$. We begin with a sequence of lemmas, which between them provide the main steps of the proof of our first transversal theorem. The first two lemmas below are about refining M-finite approximations for a transversal T for H in G. To these ends both require internal inductions.

The first lemma is about removing elements from an upper approximation to a transversal whilst preserving a subset which is unique product with respect to H. The proof is related to that of theorem 3.2.9.

Lemma 3.3.1 *Let $M \vDash PA$. Let G be an M-finite group and suppose $H \leqslant G$ is an M-countable subgroup with* $\text{card}(H) = I$. *Suppose there are M-finite sets $S \subseteq T \subseteq G$ and*

$C \subseteq H$ such that $CT = G$ and HS is unique product. If $i \in I$, then there are M-finite sets $C \subseteq C' \subseteq H$ and $S \subseteq T' \subseteq T$ such that $C'T' = G$ and $|T'| \leqslant \frac{|G|}{i}$.

Proof. By M-countability we may select M-finite C^* such that $C \subseteq C^* \subseteq H$ and $|C^*| \geqslant i$. Define $X = \{t \in T \setminus S : C^*t \cap C^*S \neq \varnothing\}$, and we set $T^* = T \setminus X$. Since T^* is an M-finite set it has an internal enumeration, that is $T^* = \{t_0, \cdots, t_{\alpha-1}\}$ for some $\alpha \in \omega^M$. For $0 \leqslant i, j < \alpha$ let us say $t_i \sim t_j$ if $\exists h_1, h_2 \in C^*\ h_1t_i = h_2t_j$. As before $\sim$ is not transitive unless C^* happens to be a group but again this will not matter. We now inductively define an internal subset T' of T^*.

(1) $t_0 \in T'$.

(2) $t_j \in T'$ if and only if $\forall i < j(t_i \in T' \implies t_i \not\sim t_j)$.

This defines an M-finite set T'. We define $C' := \{h_1h_2^{-1}h_3 : h_1, h_2, h_3 \in C^*\}$ which is M-finite also. It remains to check that T', C' have the desired properties.

(i) $|T'| \leqslant \frac{|G|}{i}$: If not then $|T'||C^*| > |G|$ and so there is a unique product failure. For some i, j such that $0 \leqslant i < j < \alpha\ \exists t_i, t_j \in T' \exists h_1, h_2 \in C^*$ such that $h_1t_i = h_2t_j$, but this contradicts $t_i \not\sim t_j$.

(ii) $S \subseteq T'$: Since $S \subseteq T$ by assumption it follows that $S \subseteq T^*$ as X contained only elements not in S. Then for any $s \in S$ we have $s = t_j \in T^*$ for some j such that $0 < j < \alpha$. If $t_j \notin T'$ then for some $i < j$, $t_i \in T' \subseteq T^*$ we must have $\exists h_1, h_2 \in C^*$ $h_1t_i = h_2t_j$ otherwise t_j would have gone into T'. But then $C^*t_i \cap C^*s \neq \varnothing$ and so $t_i \in X$ ($t_i \notin S$ since HS is unique product) which contradicts $t_i \in T^*$.

(iii) $C' \subseteq H$: If $h \in C'$ then $\exists h_1, h_2, h_3 \in C^*$ such that $h = h_1h_2^{-1}h_3$. We have $C^* \subseteq H$ and H is a group, so $h \in H$ as required.

(iv) $C'T' = G$: We know $C^*T = G$ since $CT = G$ and $C \subseteq C^*$. Let $g \in G$ then we have $\exists h \in C^* \ \exists t \in T \ g = ht$. If $t \in T'$ then we are done so suppose that t was one of the things we removed. If $t \in X$ then $\exists h_1, h_2 \in C^*$ and $\exists s \in S \subseteq T'$ such that $h_1 t = h_2 s$, whence $g = hh_1^{-1}h_2 s \in C'T'$ as required. If $t \notin X$, $t = t_j \in T^*$ for some j such that $0 < j < \alpha$. Since $t_j \notin T'$ then for some $i < j$ and $t_i \in T' \subseteq T^*$ we must have $\exists h_1, h_2 \in C^* \ h_1 t_i = h_2 t_j$ otherwise t_j would have gone into T'. Whence $g = hh_2^{-1}h_1 t_i \in C'T'$ as required.

The proof above may seem a little fiddly. The main reason for this is that we are not allowed to refer to or use the external group H during the internal induction. The induction needs to be internal because we want C' and T' to be M-finite. Care has to be taken when expanding C to C' that we do not perform any operations that will take us outside H but without actually referring to H. To this end we first expand to C^* (this is done as C might not be large enough to get enough unique product failures) and systematically strip away enough elements from T to remove all unique product failures whilst simultaneously ensuring that $S \subseteq T'$. Since C^* is not necessarily a group we may have that $C^*T' \neq G$ but because of the careful way in which the removal of elements from T was done it suffices to define $C' := \{h_1 h_2^{-1} h_3 : h_1, h_2, h_3 \in C^*\}$ in order to restore $C'T' = G$.

The next lemma is about adding elements to a lower approximation to a transversal. It is shown that the elements added can be taken from a suitable upper approximation. The proof is noticeably more direct than that of lemma 3.3.1.

Lemma 3.3.2 *Let $M \models PA$. Let G be an M-finite group and suppose $H \leqslant G$ is an M-countable subgroup with* $\mathrm{card}(H) = I$. *Suppose there are M-finite sets $S \subseteq T \subseteq G$ and $\widehat{H} \subseteq H$ such that $\widehat{H}T = G$ and HS is unique product. Let $i' > I$ then there is an M-finite set S' such that $S \subseteq S' \subseteq T$ such that HS' is unique product and $|S'| \geqslant \frac{|G|}{i'}$.*

Proof. By M-countability we may select $H_{i'} \supseteq H$ with $|H_{i'}| \leqslant i'$. If $|S| \geqslant \frac{|G|}{i'}$ then we are done so suppose not and set $l := \lceil \frac{|G|}{i'} \rceil - |S|$. We perform an internal induction to add l elements to S. Suppose inductively that we have already added elements $\{t_0, \cdots, t_{r-1}\}$ for some $r < l$ and we show how to add t_r. Now $|H_{i'}||(S \cup \{t_0, \cdots, t_{r-1})| = i'(|S| + r) < i'[|S| + \frac{|G|}{i'} - |S|] = |G|$. So there is $g_r \in G$ with $g_r \notin (H_{i'})(S \cup \{t_0, \cdots, t_{r-1}\})$. Since $\widehat{H}T = G$, and both are M-finite, we may write $g_r = \widehat{h}_r t_r$ for $\widehat{h}_r \in \widehat{H}$ and $t_r \in T$. By induction we create a set $\{t_0, \cdots, t_{l-1}\} \subseteq T$. Set $S' = S \cup \{t_0, \cdots, t_{l-1}\}$. It remains to check that S' has the desired properties.

(1) $S' \subseteq T$ because $S \subseteq T$ and for each $j < l$, t_j is carefully chosen in the induction so that $t_j \in T$.

(2) $|S'| \geqslant \frac{|G|}{i'}$: Suppose $t_i = t_j$ for $0 \leqslant i < j \leqslant l - 1$. Then $g_j = \widehat{h}_j t_j = \widehat{h}_j t_i$, for $\widehat{h}_j \in \widehat{H}$, and so $g_j \in H(S \cup \{t_0, \cdots, t_i\}) \subseteq H_{i'}(S \cup \{t_0, \cdots, t_i\})$ which contradicts the choice of g_j. This means that $|\{t_0, \cdots, t_{l-1}\}| = l$ so $|S'| = |S| + l = \lceil \frac{|G|}{i'} \rceil \geqslant \frac{|G|}{i'}$.

(3) HS' is unique product: Suppose to the contrary, then there are $h_1, h_2 \in H$ and $s_1, s_2 \in S'$ such that $h_1 s_1 = h_2 s_2$. Since HS is unique product by assumption at least one of $s_1, s_2 \in \{t_0, \cdots, t_{l-1}\}$. We may suppose $s_2 = t_j$ for some $0 \leqslant j < l$ and $s_1 \in (S \cup \{t_0, \cdots, t_{j-1}\})$. Thus $h_1 s_1 = h_2 t_j$ and since $g_j = \widehat{h}_j t_j$ we get $g_j = \widehat{h}_j h_2^{-1} h_1 s_1$. Since H is a group we get $\widehat{h}_j h_2^{-1} h_1 \in H \subseteq H_{i'}$ and so $g_j \in H'_i(S \cup \{t_0, \cdots, t_{j-1}\})$ which contradicts the choice of g_j.

The following lemma is a straightforward statement in standard group theory. We include it here as it will be used in the results that follow.

Lemma 3.3.3 *Let G be a group and suppose $H \leqslant G$ is a subgroup. Suppose there are sets $S, T \subseteq G$ such that $HT = G$ and HS is unique product. Then for any $g \in G \setminus (HS)$ there is $t \in T$ such that $g \in H(S \cup \{t\})$ and $H(S \cup \{t\})$ is unique product.*

Proof. If $g \in HS$ then there is nothing to prove. So suppose $g \notin HS$. Since $HT = G$ there are $h \in H$ and $t \in T$ such that $ht = g$. Then $H(S \cup \{t\})$ is unique product for if not there are $h_1, h_2 \in H$ and $s_1, s_2 \in S \cup \{t\}$ such that $h_1 s_1 = h_2 s_2$. Since HS is unique product it must be that $s_1 = t$ or $s_2 = t$. Without loss $s_2 = t$ and then $t = h_2^{-1} h_1 s_1$ and so $g = hh_2^{-1} h_1 s_1$ which contradicts the assumption that $g \notin HS$.

We are now ready to prove the main theorem of this section.

Theorem 3.3.4 *Let $M \models PA$ and let M be countable. Let G be an M-finite group and let $H \leqslant G$ be an M-countable subgroup with* $\mathrm{card}(H) = I$*. Then there exists an M-countable transversal T with* $\mathrm{card}(T) = \frac{|G|}{I}$.

Proof. Since $G \subseteq M$ and M is countable it follows that G is also countable. Let $G = \{g_0, \cdots, g_n, \cdots\}_{n \in \mathbb{N}}$ be an external enumeration of G. Let $(a_0, \cdots, a_n, \cdots)_{n \in \mathbb{N}}$ be an increasing sequence cofinal in I. Let $(b_0, \cdots, b_n, \cdots)_{n \in \mathbb{N}}$ be a decreasing sequence cofinal in I from above. These are possible by countability. We shall build T by an external induction. We list our inductive assumptions as follows.

(1) For $0 \leqslant i \leqslant n$ there are M-finite sets H_i such that $H_0 \subseteq \cdots \subseteq H_n \subseteq H$.

(2) For $0 \leqslant i \leqslant n$ there are M-finite sets S_i, T_i such that $S_0 \subseteq \cdots \subseteq S_n \subseteq T_n \subseteq \cdots \subseteq T_0$.

(3) HS_i is unique product and $H_i T_i = G$ for $0 \leqslant i \leqslant n$.

(4) $|T_n| \leqslant \frac{|G|}{a_n}$, and $|S_n| \geqslant \frac{|G|}{b_n}$.

(5) $g_n \in HS_n$.

We carry out the induction in two stages. First we shall build $S_{n+1} \subseteq T_n$ and satisfying the appropriate parts of the induction hypothesis. Then we build $T_{n+1} \supseteq S_{n+1}$ satisfying the remainder of the induction hypothesis.

By lemma 3.3.2 we can build an M-finite set S_{n+1} such that $S_n \subseteq S_{n+1} \subseteq T_n$, HS_{n+1} is unique product and $|S_{n+1}| \geqslant \frac{|G|}{b_{n+1}}$. If g_{n+1} fails to lie in HS_{n+1} then we can make it so by simply adding an additional element $t_{n+1} \in T_n$ to S_{n+1} using lemma 3.3.3. This completes the construction of S_{n+1}.

By lemma 3.3.1 we can build M-finite sets T_{n+1} and H_{n+1} such that $S_{n+1} \subseteq T_{n+1} \subseteq T_n$, $H_n \subseteq H_{n+1} \subseteq H$, $H_{n+1}T_{n+1} = G$, and $|T_{n+1}| \leqslant \frac{|G|}{a_{n+1}}$. This completes the construction of T_{n+1} and H_{n+1}.

Note that $\bigcup_{n\in\mathbb{N}} S_n \subseteq \bigcap_{n\in\mathbb{N}} T_n$. Set $T = \bigcup_{n\in\mathbb{N}} S_n$ and we claim that T is an M-countable transversal for H with $\mathrm{card}(T) = \frac{|G|}{I}$.

(1) $\underline{\mathrm{card}}(T) \geqslant \sup\{|S_n| : n \in \mathbb{N}\} = \sup\{\frac{|G|}{b_n} : n \in \mathbb{N}\} = \frac{|G|}{I}$

(2) $\overline{\mathrm{card}}(T) \leqslant \inf\{|T_n| : n \in \mathbb{N}\} = \inf\{\frac{|G|}{a_n} : n \in \mathbb{N}\} = \frac{|G|}{I}$

(3) Suppose HT is not unique product. Then $\exists n, m \in \mathbb{N}$ and $\exists h', h'' \in H$ such that $h's_n = h''s_m$ for some $s_n \in S_n$ and $s_m \in S_m$. Take $k = \max\{n, m\}$ and then $s_n, s_m \in S_k$ so HS_k is not unique product which is a contradiction.

(4) Suppose $HT \neq G$. There is $g \in G$ with $g \notin HT$. But $g = g_n$ for some $n \in \mathbb{N}$ by the enumeration of G. So $g = g_n = hs_n$ for $s_n \in S_n \subseteq T$, $h \in H$ contradicting $g \notin HT$.

Thus T has all the desired properties and this completes the proof.

Example 3.3.5 *Suppose $M \vDash PA$ is countable. Let n, I, S_n, S_n^I and A_n^I be as in example 3.2.3. Then by theorem 3.3.4 there are M-countable transverals $T_S \subseteq S_n$ and $T_A \subseteq S_n$ such that $S_n^I T_S = S_n$, $A_n^I T_A = S_n$, $S_n^I T_S$ is unique product, $A_n^I T_A$ is unique product, and* $\mathrm{card}(T_S) = \mathrm{card}(T_A) = \frac{n!}{n^I}$.

The only assumption we make (beyond the basic set up) in theorem 3.3.4 is that the model M is countable. We have the following question.

Question 3.3.6 *The assumption that the model M is countable is vital in the proof of 3.3.4 for enumerating G and the cut I . Can any analogous statement be proved in the uncountable case or indeed demonstrated to be false?*

We give another straightforward lemma true in standard group theory. Its use will become clear in the proposition that follows.

Lemma 3.3.7 *Let G be a group and suppose $H \leqslant G$ is a nontrivial subgroup. Suppose there are sets $C, S \subseteq G$ such that $H(S \cup C)$ is unique product and $C \setminus S$ is nonempty and nontrivial. Then there is $g \in G$ such that $H(S \cup \{g\})$ is unique product but $H(S \cup \{g\} \cup C)$ is not unique product.*

Proof. Let $e \neq c \in C \setminus S$ and since $H(S \cup C)$ is unique product it follows that $c \notin HS$. Let $h \in H$ be anything other than the identity and set $g = hc$. Then $H(S \cup \{g\})$ must be unique product otherwise $c \in HS$ but $H(S \cup \{g\} \cup C)$ is not unique product because $g = hc$.

The next result strengthens this theorem by implying that the inductive argument used in the proof really is necessary. In other words not all transversals will be M-countable. Indeed we can force the lower cardinality to be anything we want and we show how to do likewise with the upper cardinality subject to a certain constraint.

Proposition 3.3.8 *Let $M \models PA$ and let M be countable. Let G be an M-finite group and let $H \leqslant G$ be an M-countable subgroup with* $\text{card}(H) = I$*. Let J be any cut such that $\mathbb{N} \leqslant J \leqslant \frac{|G|}{I}$. Then there exists a transversal $T \subseteq G$ with $\underline{\text{card}}(T) = J$.*

Proof. By countability of M we have the following sequences and enumerations. Let $G = \{g_0, \cdots, g_n, \cdots\}_{n \in \mathbb{N}}$ be an external enumeration of G. Let $(a_0, \cdots, a_n, \cdots)_{n \in \mathbb{N}}$ be an increasing sequence cofinal in J. Let $(C_0 \cdots C_n \cdots)_{n \in \mathbb{N}}$ be an enumeration of the M-finite subsets of G with $J < |C_n| < \frac{|G|}{I}$. We shall build T by an external induction. We list our inductive assumptions as follows.

(1) For $0 \leqslant i \leqslant n$ there are M-finite sets S_i such that $S_0 \subseteq \cdots \subseteq S_n$.

(2) HS_i is unique product for $0 \leqslant i \leqslant n$.

(3) $a_n \leqslant |S_n| < J$.

(4) $g_n \in HS_n$.

(5) $H(C_n \cup S_n)$ is not unique product.

Note that condition (5) is there to ensure that C_n cannot be a subset of the completed Transversal T.

By lemma 3.3.2 we can build an M-finite set S_{n+1} such that $S_n \subseteq S_{n+1} \subseteq G$, HS_{n+1} is unique product and $|S_{n+1}| \geqslant a_{n+1}$. We can ensure $|S_{n+1}| \in J$ as we simply terminate the internal induction in 3.3.2 as soon as S_{n+1} is sufficiently large. If g_{n+1} fails to lie in HS_{n+1} then we can make it so by simply adding an additional element $t_{n+1} \in T_n$ to S_{n+1} using lemma 3.3.3. Note $|S_{n+1}| \in J < |C_{n+1}|$ and so $C_{n+1} \setminus S_{n+1}$ is nonempty and nontrivial. If $H(C_{n+1} \cup S_{n+1})$ is unique product then by lemma 3.3.7 there is $g \in G$ such that $H(S_{n+1} \cup \{g\})$ is unique product but $H(S_{n+1} \cup \{g\} \cup C_{n+1})$ is not unique product. So in this case we simply add g to S_{n+1} in order to satisfy condition (5). These two potential additions are allowed since J is a cut closed under successor and so we don't make S_{n+1} too big. And since we add at most two extra elements we don't violate M-finiteness. This completes the construction of S_{n+1}.

Set $T = \bigcup_{n \in \mathbb{N}} S_n$ and we claim that T is a transversal for H in G with $\underline{\text{card}}(T) = J$.

(i) Suppose HT is not unique product. Then $\exists n, m \in \mathbb{N}$ and $\exists h', h'' \in H$ such that $h's_n = h''s_m$ for some $s_n \in S_n$ and $s_m \in S_M$. Take $k = \max\{n, m\}$ and then $s_n, s_m \in S_k$ so HS_k is not unique product which is a contradiction.

(ii) $\underline{\text{card}}(T) \geqslant \sup\{|S_n| : n \in \mathbb{N}\} \geqslant \sup\{a_n : n \in \mathbb{N}\} = J$. Suppose $\underline{\text{card}}(T) > J$. Then we can have an M-finite set $C \subseteq T$ with $J < |C| < \frac{|G|}{I}$ and so $C = C_m$ for some

$m \in \mathbb{N}$. Since S_m is also a subset of T we have that $C_m \cup S_m \subseteq T$, but $H(C_m \cup S_m)$ is not unique product by (5) of the induction hypothesis at stage m. This contradicts (i) above. So $\underline{\text{card}}(T) = J$.

(iii) Suppose $HT \neq G$. There is $g \in G$ with $g \notin HT$. But $g = g_n$ for some $n \in \mathbb{N}$ by the enumeration of G. So $g = g_n = hs_n$ for $s_n \in S_n \subseteq T$, $h \in H$ contradicting $g \notin HT$.

Thus T has all the desired properties and this completes the proof.

Note that we have not mentioned the upper cardinality of T at all in the above proof. However, it does alway follow that $\overline{\text{card}}(T) \geqslant \frac{|G|}{I}$ and so we have certainly succeeded in violating the M-countability of T. That we can build a transversal T with $\overline{\text{card}}(T) > \frac{|G|}{I}$ is the subject of the following proposition. This is a slightly trickier result and we need to impose a closure condition on the cut that we force the upper cardinality of T to equal. No restriction is imposed upon the lower cardinality of T as doing so would potentially limit further the choices for the upper cardinality.

Proposition 3.3.9 *Let $M \vDash PA$ and let M be countable. Let G be an M-finite group and let $H \leqslant G$ be a M-countable subgroup with* $\text{card}(H) = I$. *Let K be any cut subject to $\frac{|G|}{I} < K < |G|$ and $\partial(K) > I$ (recall definition 2.1.9). Then there exists a transversal $T \subseteq G$ with* $\overline{\text{card}}(T) = K$.

Proof. By countability of M we have the following external sequences and enumerations. Let $G = \{g_0, \cdots, g_n, \cdots\}_{n \in \mathbb{N}}$ be an external enumeration of G. Let $(a_0, \cdots, a_n, \cdots)_{n \in \mathbb{N}}$ be an decreasing sequence cofinal in K from above. Let $(C_0 \cdots C_n \cdots)_{n \in \mathbb{N}}$ be an enumeration of the M-finite subsets of G with $\frac{|G|}{I} < |C_n| < K$. We shall build T by an external induction. We list our inductive assumptions as follows.

(1) For $0 \leqslant i \leqslant n$ there are M-finite sets H_i such that $H_0 \subseteq \cdots \subseteq H_n \subseteq H$.

(2) For $0 \leqslant i \leqslant n$ there are M-finite sets S_i, T_i such that $S_0 \subseteq \cdots \subseteq S_n \subseteq T_n \subseteq \cdots \subseteq T_0$.

(3) HS_i is unique product and $H_iT_i = G$ for $0 \leqslant i \leqslant n$.

(4) $K < |T_n| \leqslant a_n$ and $|S_n| \in \mathbb{N}$.

(5) $g_n \in HS_n$.

(6) $S_n \nsubseteq C_n$.

We carry out the induction in two stages. First we shall build $S_{n+1} \subseteq T_n$ to ensure (5) and (6) are satisfied. Then we build $T_{n+1} \supseteq S_{n+1}$ satisfying the remainder of the induction hypothesis.

If $g_{n+1} \notin HS_n$ then we can make it so by simply adding an additional element $t_{n+1} \in T_n$ to S_n using lemma 3.3.3. Since $|T_n| > K$ and $|C_{n+1}| \in K$ it follows that $|T_n \setminus C_{n+1}| > \partial(K) \geqslant I$. Since S_n is finite and $I \vDash +$ (by lemma 3.1.3) it follows $\overline{\text{card}}(S_nH) \leqslant I$. Therefore there is $x_{n+1} \in T_n \setminus C_{n+1}$ with $x_{n+1} \notin S_nH$. Set $S_{n+1} = S_n \cup \{t_{n+1}, x_{n+1}\}$.

By lemma 3.3.1 we can build M-finite sets T_{n+1} and H_{n+1} such that $S_{n+1} \subseteq T_{n+1} \subseteq T_n$, $H_n \subseteq H_{n+1} \subseteq H$, $H_{n+1}T_{n+1} = G$, and $|T_{n+1}| \leqslant a_{n+1}$. We can ensure $|T_{n+1}| > K$ as we simply terminate the internal induction in lemma 3.3.1 as soon as T_{n+1} is sufficiently small. This completes the construction of T_{n+1} and H_{n+1}.

Set $T = \bigcup_{n\in\mathbb{N}} S_n$ and we claim that T is a transversal for H in G with $\overline{\text{card}}(T) = K$.

(i) Suppose HT is not unique product. Then $\exists n, m \in \mathbb{N}$ and $\exists h', h'' \in H$ such that $h's_n = h''s_m$ for some $s_n \in S_n$ and $s_m \in S_m$. Take $k = \max\{n, m\}$ and then $s_n, s_m \in S_k$ so HS_k is not unique product which is a contradiction.

(ii) $\overline{\text{card}}(T) \leqslant \inf\{|T_n| : n \in \mathbb{N}\} \leqslant \inf\{a_n : n \in \mathbb{N}\} = K$. Suppose $\overline{\text{card}}(T) < K$. Then we can have an M-finite set $C \supseteq T$ with $\frac{|G|}{I} < |C| < K$ and so $C = C_m$ for some $m \in \mathbb{N}$. Since S_m is also a subset of T we have that $S_m \subseteq C_m$, but $S_m \nsubseteq C_m$ by (6) of the induction hypothesis at stage m. So $\overline{\text{card}}(T) = K$.

(iii) Suppose $HT \neq G$. There is $g \in G$ with $g \notin HT$. But $g = g_n$ for some $n \in \mathbb{N}$ by the enumeration of G. So $g = g_n = hs_n$ for $s_n \in S_n \subseteq T$, $h \in H$ contradicting $g \notin HT$.

Thus T has all the desired properties and this completes the proof.

Question 3.3.10 *Can we do away with the condition $\partial(K) > I$ by developing a slicker proof of proposition 3.3.9.*

3.4 Second Transversal Theorem

In this section we prove a generalization of theorem 3.3.4. Suppose G is an M-finite group and $H < K \leqslant G$ are M-countable subgroups with $\mathrm{card}(H) = I \subsetneq J = \mathrm{card}(K)$ for cuts I, J. Theorem 3.3.4 gives us a transversal T for H in G with $\mathrm{card}(T) = \frac{|G|}{I}$ and it is straightforward to check that $T_K = T \cap K$ is a transversal for H in K. Our goal is to show that under suitable conditions T_K can also be made M-countable. To this end we begin by proving generalised version of lemmas 3.3.1 and 3.3.2.

The following lemma is a generalization of lemma 3.3.1. It turns out that the proof of lemma 3.3.1 is already good enough to give a 'small' M-finite upper approximation of $T_{n+1} \cap K$. The following lemma is really about making this precise and then verifying the fact.

Lemma 3.4.1 *Let $M \models PA$. Let G be an M-finite group and let $H < K \leqslant G$ be M-countable subgroups with* $\mathrm{card}(H) = I \subseteq J = \mathrm{card}(K)$*. Suppose there are M-finite sets $S, T \subseteq G$ and $C \subseteq H$ such that $CT = G$ and HS is unique product. Let $i \in I$ and $j' > J$ then there are M-finite sets C', T' and Y such that:*

(1) $C \subseteq C' \subseteq H$

(2) $S \subseteq T' \subseteq T$

(3) $Y \supseteq T' \cap K$

(4) $C'T' = G$

(5) $|T'| \leqslant \frac{|G|}{i}$

(6) $|Y| \leqslant \frac{j'}{i}$

Proof. By lemma 3.3.1 there are sets C' and T' satisfying conditions 1,2,4 and 5. By M-countability of K we may choose an M-finite set $K_{j'} \supseteq K$ with $|K_{j'}| = j'$. Let C^* be as in the proof of lemma 3.3.1 with $|C^*| \geqslant i$. Define $Y := \{t \in T' : C^*t \subseteq K_{j'}\}$. Suppose $k \in T' \cap K$ then $C^*k \subseteq K \subseteq K_{j'}$ since $C^* \subseteq H$ and K is a group. Thus we see that $k \in Y$ and so condition 3 is satisfied. It was shown in the proof of lemma 3.3.1 that C^*T' is unique product. Since $Y \subseteq T'$ it follows $C^*Y \subseteq K_{j'}$ is unique product also. Thus $|C^*||Y| \leqslant j'$ and so $|Y| \leqslant \frac{j'}{|C^*|} \leqslant \frac{j'}{i}$ as required.

The following lemma is generalization of lemma 3.3.2. We wish to show that there is a 'large' M-finite lower approximation of $S' \cap K$. Here we construct this set as part of the construction of S' and so the proof of lemma 3.3.2 is reworked with this in mind. In order to get the construction off the ground the assumption that H and K are separable, in the sense of definition 2.1.16, is used.

Lemma 3.4.2 *Let $M \vDash PA$. Let G be an M-finite group and let $H < K \leqslant G$ be M-countable subgroups with* $\text{card}(H) = I \subsetneq J = \text{card}(K)$ *with H and K separable. Suppose there are M-finite sets $S, T \subseteq G$ and $\widehat{H} \subseteq H$ such that $\widehat{H}T = G$ and HS is unique product. Let $j \in J$ and $i' > I$ then there are M-finite sets S' and X such that:*

(1) $S \subseteq S' \subseteq T$

(2) $X \subseteq S' \cap K$

(3) *HS' is unique product.*

(4) $|S'| \geqslant \frac{|G|}{i'}$.

(5) $|X| \geqslant \frac{j}{i'}$

Proof. By M-countability of K we may choose an M-finite set K_j such that $K_j \subseteq K$ and $|K_j| = j$. By separability of H, K we may choose an M-finite set $H \subseteq L_{i'} \subseteq K$ and by M-countability of H we may arrange that $|L_{i'}| \leqslant i'$. Define an M-finite set $A = \{s \in S : L_{i'}s \cap K_j \neq \varnothing\}$. Let $s \in A$ then by definition there are $l \in L_{i'}$ and $k \in K_j$ such that $ls = k$. Whence $s = l^{-1}k \in K$ and so $A \subseteq K$.

We build S' in two parts. Our first task is to add appropriate elements in order to satisfy condition (5), secondly we add further elements to satisfy condition (4). Both parts must be done carefully to avoid violating the other criteria. If $|A| \geqslant \frac{j}{i'}$ then A is already a perfectly good candidate for the set X. So suppose $|A| < \frac{j}{i'}$ and set $r := \lceil \frac{j}{i'} \rceil - |A|$. We now perform an internal induction to 'add' r elements. Suppose inductively that for $p < r$ we have selected elements $x_0, \cdots, x_{p-1}$ such that $x_i \in K_j$ for $i < p$, $x_i \notin L_{i'}x_j$ for $j < i < p$ and $x_i \notin L_{i'}A_j$ for $i < p$.

We want to find a suitable element x_p. We have the following counting argument: $|L_{i'}A \cup L_{i'}\{x_0, \cdots, x_{p-1}\} :\leqslant i'(|A| + p) < i'(\frac{j}{i'}) = j$. It follows that there is an $x_p \in K_j$ such that $x_p \notin L_{i'}\{x_0, \cdots, x_{p-1}\}$ and $x_p \notin L_{i'}A$. By this internal induction we create a set $\{x_0, \cdots, x_{r-1}\} \subseteq K_j$ satisfying the properties above. Now note that $\widehat{H}T = G$ and $\widehat{H}, T$ are internal sets, therefore we can perform another internal induction to write $x_i = h_i t_i$, where $h_i \in \widehat{H}, t_i \in T$ for all $0 \leqslant i < r$. Set $X = A \cup \{t_0, \cdots, t_{r-1}\}$ and $S^* = \{t_0, \cdots, t_{r-1}\}$. We now check that conditions (1–5) are met.

(1) That $S \subseteq S^* \subseteq T$ holds by construction.

(2) $X = A \cup \{t_0, \cdots, t_{r-1}\}$. We already know $A \subseteq S \cap K \subseteq S^* \cap K$. Let $t_i \in \{t_0, \cdots, t_{r-1}\}$ then $t_i = h_i^{-1}x_i$ for $h_i \in \widehat{H} \subseteq H \subseteq K$ and $x_i \in K_j \subseteq K$. Since K is a group it follows $t_i \in K$ and $t_i \in S^*$ by definition. Hence $X \subseteq S^* \cap K$ as required.

(3) Suppose HS^* is not unique product. By assumption HS is unique product so one of the following must occur.

 (a) $h's = h''t_i$ for some $h', h'' \in H, s \in S$ and $0 \leqslant i < r$.

 (b) $h't_i = h''t_j$ for some $h', h'' \in H$ and $0 \leqslant i < j < r$.

 If (a) occurs then $h''^{-1}h's = t_i \implies x_i = h_it_i = h_ih''^{-1}h's$. Now $h_i \in \widehat{H} \subseteq H$ and $h', h''^{-1} \in H$ so $h_ih''^{-1}h' \in H \subseteq L_{i'}$. $x_i \in K_j$ by construction, and so $s \in A$. Hence $x_i \in L_{i'}A$ which contradicts the hypothesis of the internal induction. If (b) occurs then $h't_i = h''t_j$ and $t_i = h_i^{-1}x_i$, $t_j = h_j^{-1}x_j$ so $h'h_i^{-1}x_i = h''h_j^{-1}x_j$ and hence $x_j = h_jh''^{-1}h'h_i^{-1}x_i$. Note $h_i, h_j \in \widehat{H} \subseteq H$ so $h_jh''^{-1}h'h_i^{-1} \in H \subseteq L_{i'}$, so $x_j \in L_{i'}x_i$ which contradicts the hypothesis of the internal induction.

(4) It is possible that $|S^*| < \frac{|G|}{i'}$ but this can be rectified as we explain below.

(5) $|X| = |A \cup \{t_0, \cdots, t_{r-1}\}| = |A| + r = \lceil \frac{j}{i'} \rceil \geqslant \frac{j}{i'}$ as required. The second equality follows from the fact that if $t_i \in A$ then $x_i = h_it_i \in L_{i'}A$ contradicting the hypothesis of the internal induction.

It is possible that (4) fails for S^*. If this is the case then we simply use lemma 3.3.2 to expand S^* to S' with $|S'| \geqslant \frac{|G|}{i'}$. Properties (1) and(3) are preserved by lemma 3.3.2 and (2) is clearly preserved since $S^* \subseteq S'$. This completes the construction of S', X.

We shall need a notion of division for initial segments of a model. The following definitions are thanks to Richard Kaye. They appear, stated in slightly greater generality, in Kaye's paper [9].

Definition 3.4.3 *Let $I \subsetneq J \subseteq M$ be cuts then we define*

(1) $J \cdot I := \sup\{ji : j \in J, i \in I\}$.

(2) $J \odot I := \inf\{j'i' : j' > J, i' > I\}$.

(3) $J/I := \sup\{\lfloor \frac{j}{i'} \rfloor : j \in J, i' > I\}$.

(4) $J \oslash I := \inf\{\lfloor \frac{j'}{i} \rfloor : j' > J, i \in I\}$.

We are now ready to prove the promised result.

Theorem 3.4.4 *Let $M \vDash PA$ and let M be countable. Let G be an M-finite group and let $H < K \leqslant G$ be M-countable subgroups with* $\mathrm{card}(H) = I \subsetneq J = \mathrm{card}(K)$. *Suppose also that H, K are separable. Then there exists an M-countable transversal $T \subseteq G$ for H in G with* $\mathrm{card}(T) = \frac{|G|}{I}$. *Moreover $T_K = T \cap K$ is a transversal for H in K with $J/I \leqslant \underline{\mathrm{card}}(T_K) \leqslant \overline{\mathrm{card}}(T_K) \leqslant J \oslash I$.*

Proof. We proceed as in the proof of theorem 3.3.4 but we will add extra assumptions into the induction hypothesis in order to achieve the stronger conclusion. Since $G \subseteq M$ and M is countable it follows that G is also countable. Let $G = \{g_0, \cdots, g_n, \cdots\}_{n \in \mathbb{N}}$ be an external enumeration of G. Let $(a_0, \cdots, a_n, \cdots)_{n \in \mathbb{N}}$ be an increasing sequence cofinal in I. Let $(b_0, \cdots, b_n, \cdots)_{n \in \mathbb{N}}$ be a decreasing sequence cofinal in I from above. Let $(c_0, \cdots, c_n, \cdots)_{n \in \mathbb{N}}$ be an increasing sequence cofinal in J. Let $(d_0, \cdots, d_n, \cdots)_{n \in \mathbb{N}}$ be a decreasing sequence cofinal in J from above. These are possible by countability. We shall build T by an external induction. We list our inductive assumptions as follows. (1–5) are as in theorem 3.3.4 and (6) is an extra condition.

(1) For $0 \leqslant i \leqslant n$ there are M-finite sets H_i such that $H_0 \subseteq \cdots \subseteq H_n \subseteq H$.

(2) For $0 \leqslant i \leqslant n$ there are M-finite sets S_i, T_i such that $S_0 \subseteq \cdots \subseteq S_n \subseteq T_n \subseteq \cdots \subseteq T_0$.

(3) HS_i is unique product and $H_iT_i = G$ for $0 \leqslant i \leqslant n$.

(4) $|T_n| \leqslant \frac{|G|}{a_n}$, and $|S_n| \geqslant \frac{|G|}{b_n}$.

(5) $g_n \in HS_n$.

(6) There is an M-finite set $X_n \subseteq S_n \cap K$ with $|X_n| \geqslant \frac{c_n}{b_n}$ and an M-finite set $Y_n \supseteq T_n \cap K$ with $|Y_n| \leqslant \frac{d_n}{a_n}$.

We carry out the induction in two stages. First we shall build $S_{n+1} \subseteq T_n$ and $X_{n+1} \subseteq S_{n+1} \cap K$ satisfying the appropriate parts of the induction hypothesis. Then we build $T_{n+1} \supseteq S_{n+1}$ and $Y_{n+1} \supseteq T_{n+1} \cap K$ satisfying the remainder of the induction hypothesis.

By lemma 3.4.2 we can build M-finite sets S_{n+1} and X_{n+1} such that $S_n \subseteq S_{n+1} \subseteq T_n$, $X_{n+1} \subseteq S_{n+1} \cap K$, HS_{n+1} is unique product, $|S_{n+1}| \geqslant \frac{|G|}{b_{n+1}}$ and $|X_{n+1}| \geqslant \frac{c_{n+1}}{b_{n+1}}$. If $g_{n+1} \notin HS_{n+1}$ then we can make it so by simply adding an additional element $t_{n+1} \in T_n$ to S_{n+1} using lemma 3.3.3. This completes the construction of S_{n+1}.

By lemma 3.4.1 we can build M-finite sets T_{n+1}, H_{n+1} and Y_{n+1} such that $S_{n+1} \subseteq T_{n+1} \subseteq T_n$, $H_n \subseteq H_{n+1} \subseteq H$, $Y_{n+1} \supseteq T_{n+1} \cap K$ $H_{n+1}T_{n+1} = G$, $|T_{n+1}| \leqslant \frac{|G|}{a_{n+1}}$, and $|Y_{n+1}| \leqslant \frac{d_{n+1}}{a_{n+1}}$. This completes the construction of T_{n+1} and H_{n+1}.

Note that $\bigcup_{n \in \mathbb{N}} S_n \subseteq \bigcap_{n \in \mathbb{N}} T_n$. Set $T = \bigcup_{n \in \mathbb{N}} S_n$ and we claim that T is a M-countable transversal for H in G with $\operatorname{card}(T) = \frac{|G|}{I}$. Moreover we claim that $T_K = T \cap K$ is a transversal for H in K with $J/I \leqslant \underline{\operatorname{card}}(T_K) \leqslant \overline{\operatorname{card}}(T_K) \leqslant J \oslash I$.

(1) $\underline{\operatorname{card}}(T) \geqslant \sup\{|S_n| : n \in \mathbb{N}\} = \sup\{\frac{|G|}{b_n} : n \in \mathbb{N}\} = \frac{|G|}{I}$ (Since $\inf_{n \in \mathbb{N}}(b_n) = I$).

(2) $\underline{\operatorname{card}}(T_K) \geqslant \sup\{|X_n| : n \in \mathbb{N}\} = \sup\{\frac{c_n}{b_n} : n \in \mathbb{N}\} = J/I$ (Since $\inf_{n \in \mathbb{N}}(b_n) = I$ and $\sup_{n \in \mathbb{N}}(c_n) = J$).

(3) $\overline{\operatorname{card}}(T) \leqslant \inf\{|T_n| : n \in \mathbb{N}\} = \inf\{\frac{|G|}{a_n} : n \in \mathbb{N}\} = \frac{|G|}{I}$ (Since $\sup_{n \in \mathbb{N}}(a_n) = I$).

(4) $\overline{\operatorname{card}}(T_K) \leqslant \inf\{|Y_n| : n \in \mathbb{N}\} = \inf\{\frac{d_n}{a_n} : n \in \mathbb{N}\} = J \oslash I$ (Since $\sup_{n \in \mathbb{N}}(a_n) = I$ and $\inf_{n \in \mathbb{N}}(d_n) = J$).

(5) Suppose HT is not unique product. Then $\exists n, m \in \mathbb{N}$ and $\exists h', h'' \in H$ such that $h's_n = h''s_m$ for some $s_n \in S_n$ and $s_m \in S_M$. Take $k = \max\{n, m\}$ and then $s_n, s_m \in S_k$ so HS_k is not unique product which is a contradiction.

(6) Suppose HT_K is not unique product. Then since $T_K \subseteq T$ it must be that HT is not unique product. This contradicts (5) above.

(7) Suppose $HT \neq G$. There is $g \in G$ with $g \notin HT$. But $g = g_n$ for some $n \in \mathbb{N}$ by the enumeration of G. So $g = g_n = hs_n$ for $s_n \in S_n \subseteq T$, $h \in H$ contradicting $g \notin HT$.

(8) Suppose $HT_K \neq K$. There is $k \in K$ with $k \notin HT_K$. But $k = ht$ for some $h \in H \subseteq K$ and $t \in T$ by (7). Whence $t = h^{-1}k \in K$, which is a contradiction.

Thus T and T_K have all the desired properties and this completes the proof.

This theorem is an interesting extension of theorem 3.3.4 although the result is perhaps not best possible in the sense that the assumption of separability is needed and the conclusion does not say that T_K is always M-countable. We do, however, have the following corollary.

Corollary 3.4.5 *Let M, H, K, G, I, and J be as in the statement of theorem 3.4.4. Suppose additionally that I, J are such that $J/I = J \oslash I$. Then there is T_K a transversal for H in K with $J/I = \underline{\mathrm{card}}(T_K) = \overline{\mathrm{card}}(T_K) = J \oslash I$. Hence T_K is M-countable.*

Proof. By 3.4.4 $J/I \leqslant \underline{\mathrm{card}}(T_K) \leqslant \overline{\mathrm{card}}(T_K) \leqslant J \oslash I$, but by assumption $J/I = J \oslash I$ and so we have equality everywhere.

It follows from Kaye [9] that $J/I = J \oslash I$ providing $\partial^2 I \neq \partial^2 J$. This fact will be used in the following example to deduce that the set which corresponds to T_K in Corollary 3.4.5 is M-countable.

Example 3.4.6 *Let $M \vDash PA$ be countable. Fix some nonstandard n in M and let I, J be cuts such that $I < J < n$ and $I, J \vDash +$. Consider the M-finite group S_n (recall example 3.2.3) and the M-countable subgroups S_n^I and S_n^J. Fix some j such that $I < j < J$, then the M-finite set $\{g \in S_n : |support(g)| < j\}$ separates S_n^I and S_n^J in the sense of definition 2.1.16. Furthermore $\text{card}(S_n^I) = n^I \vDash \cdot$ and $\text{card}(S_n^J) = n^J \vDash \cdot$ since I and J are closed under addition, thus $\partial^2 \text{card}(S_n^I) = n^I \neq n^J = \partial^2 \text{card}(S_n^J)$. We can apply theorem 3.4.4 to obtain an M-countable transversal T, for S_n^I in S_n with $\text{card}(T) = \frac{n!}{n^I}$. By Corollary 3.4.5 we have that $T \cap S_n^J$ is an M-countable transversal for S_n^I in S_n^J with $\text{card}(T \cap S_n^J) = n^J / n^I = n^J \oslash n^I = n^{J-I} = n^{J \ominus I}$.*

Corollary 3.4.5 tells us that $J/I = J \oslash I$ is a sufficient condition for T_K to be M-countable but this begs the following question.

Question 3.4.7 *Theorem 3.4.4 gives only a lower bound for $\underline{\text{card}}(T_K)$ and an upper bound for $\overline{\text{card}}(T_K)$. It seems plausible that $\underline{\text{card}}(T_K)$ and $\overline{\text{card}}(T_K)$ could be made to coincide even in cases where $J/I \subsetneq J \oslash I$. In other words is $J/I = J \oslash I$ not a necessary condition for T_K to be M-countable?*

The author suspects that $\underline{\text{card}}(T_K)$ cannot be larger than J/I and that $\overline{\text{card}}(T_K)$ cannot be smaller than $J \oslash I$, but has not proved this. If this is indeed the case then the answer to question 3.4.7 would be no.

A major assumption used in the proof of theorem 3.4.4 is that $H \subseteq K$ are separable. On inspection of the proof this assumption is only needed in lemma 3.4.2 to ensure $\underline{\text{card}}(T_K) \geqslant J/I$ and so T_k can be constructed such that $\overline{\text{card}}(T_K) \leqslant J \oslash I$ in the non-separable case.

Question 3.4.8 *Suppose $H \subseteq K$ are as in theorem 3.4.4 apart but with separability not assumed. What is the best lower bound that can be achieved for $\underline{\text{card}}(T_K)$? Is it sensitive to the combinatorics of the cuts I, J involved?*

Finally as with 3.3.4 we have the question over the cardinality assumption.

Question 3.4.9 *The assumption that the model M is countable is vital in the proof of theorem 3.4.4 for enumerating G and the cuts I and J. Can any analogous statement be proved in the uncountable case or indeed demonstrated to be false?*

Chapter 4

Abelian Groups

Here we extend some of the ideas of the previous chapters to M-finite abelian groups. In the first section we introduce the basic concept and analyze the implications of the basis theorem on the external (to M) structure of such a group. The basis theorem has some limitations in this regard as we explain below. In the second and third sections we prove results showing what these groups look like externally in the case of single M-finite direct summand. The final two sections then look at constructing M-countable complements for certain (externally) divisible subgroups of M-finite abelian groups. In the general case heavy use is made of the basis theorem and its implications for the internal structure of an M-finite abelian G are important.

4.1 Abelian Groups in General and Direct Sums

The set up in this chapter will be as follows. Let $M \models \mathrm{PA}$ and $M \models a$ is an abelian group. We shall use the notation A when we want to talk about the group a externally. The basis theorem for finite abelian groups is provable in PA and hence true in our nonstandard model M. This means that our group a is, from the point of view of M, a direct sum of cyclic groups: there is an M-coded sequence $(n_i)_{i<b}$ of (possibly nonstandard) length b of

(possibly nonstandard) integers n_i of M such that

$$M \models a = \bigoplus_{i<b} c(n_i)$$

where $c(n)$ is the group which from the point of view of M is cyclic of order n. Letting $C(n)$ be the external group corresponding to $c(n)$ this gives a reduction of the problem of describing A: we need to describe the individual groups $C(n)$, and say something about what the direct sum operation looks like from outside M. The first part, describing each $C(n)$ is somewhat easier and will be introduced here and discussed further in the following sections. The nature of this direct sum is somewhat harder.

Unfortunately $M \models a = \bigoplus_{i<b} c(n_i)$ does not say that A is the direct sum or the direct product of the C_{n_i}. It does follow that

$$\bigoplus_{i<b} C(n_i) \leqslant A \leqslant \prod_{i<b} C(n_i)$$

but this leaves many questions. We know from $M \models a = \bigoplus_{i<b} c(n_i)$ that each $x \in A$ is uniquely written as an M-coded sequence $\bigoplus_{i<b} x_i$ of elements $x_i \in c(n_i)$, so $A \leqslant \prod_{i<b} C(n_i)$, and each such sequence of actually finite support is M-coded, hence $\bigoplus_{i<b} C(n_i) \leqslant A$. In the case when b is finite the direct sum and direct product coincide so A is this product/sum. To see that this does not describe all situations and there are cases when b is nonstandard one only needs to observe that there are finite abelian groups made of arbitrarily many cyclic direct summands and apply overspill.

It will be helpful to have a precise description of the group $c(n)$ in M. This can be defined in the usual way by the additive structure of ω^M. We let $c(n)$ be the (internal, M-finite) set of elements $x \in \omega^M$ such that $0 \leqslant x < n$. We define addition on $c(n)$ by: $x + y$ is the unique $z \in c(n)$ such that $x + y \equiv z \bmod n$ in ω^M.

In M, $c(n)$ has n elements. This means that if $n \in \mathbb{N}$ is actually finite then $C(n)$ is the cyclic group of order n, and an element of $C(n)$ is a generator if and only if M thinks it is a generator of $c(n)$. In general, however, the sets of generators may look different: if n is nonstandard then M thinks $c(n)$ is cyclic, i.e. can be generated by a single element, whereas we will see in what follows that $C(n)$ is never cyclic in this case.

It may be helpful to split the direct sum above into parts. Given $i < b$ and working in M we have

$$M \vDash a = \bigoplus_{j<i} c(n_j) \oplus c(n_i) \oplus \bigoplus_{i<j<b} c(n_j)$$

writing a as a direct product of three groups. Thus

$$A = U \oplus C(n_i) \oplus V$$

for abelian groups $U, V \leqslant A$, which are the external versions of $\prod_{j<i} c(n_j)$ and $\prod_{i<j<b} c(n_j)$. In other words, each $C(n_j)$ is a direct summand of A, though A may not be the external direct sum of all of them taken together.

By another result in finite abelian groups, working in M each $c(n)$ is the direct sum of cyclic groups $c(p^k)$ of size a prime-power. Here the prime p and/or the exponent k may be finite or nonstandard. This decomposes a as a direct sum of such direct summands. In this thesis we shall put aside the difficult question of describing further such coded infinite direct sums, and how they embed in general products, but this is an important question for the future and a possibility for future research. This includes the question of the structure of $C(n)$ where the integer n is not a prime power. The evasion of this question means we can concentrate on the case of $C(p^k)$. We may assume that at least one of p and k is nonstandard, for otherwise $C(p^k)$ is a familiar finite cyclic group.

4.2 Cyclic Groups of Order a Power of a Nonstandard Prime

Here we shall look at the structure of particular cyclic summands of A. In particular we look at summands of the form $C(p^k)$ where p is a nonstandard prime and k is a (possibly nonstandard) integer.

Theorem 4.2.1 *If $A = C(p^k)$ where p is a nonstandard prime then A is a divisible abelian group.*

Proof. By theorem 1.4.1 it suffices to show that A is torsion-free. Let $x > 0$ in ω^M and suppose that $m \in \mathbb{N}$ has $mx \equiv 0 \bmod p^k$. So $mx = yp^k$ for some y. Then p^k divides the product mx so p^k divides x since p doesn't divide m, as p is nonstandard and larger than m. It follows that $x \equiv 0 \bmod p^k$ and hence in $C(p^k)$ the element x represents zero, as required.

More generally, we have the following.

Theorem 4.2.2 *Suppose $M \models a = \bigoplus_{i<b} c(p_i^{k_i})$ where each $p_i \in M$ is a nonstandard prime, and the sequences (p_i) and (k_i) are M-finite. Then the external group A corresponding to a is divisible.*

Proof. If not, there must be a torsion element $x \in A$ with $mx = 0$ in A, for some nonzero $m \in \mathbb{N}$. In M, this x is a coded sequence (x_i) where $x_i \in c(p_i^{k_i})$ and mx is $(mx_i) = 0$. So $mx_i \equiv 0 \bmod p_i^{k_i}$ and as p_i does not divide m we have $p_i^{k_i} | x_i$ i.e. $x_i = 0$.

Divisible abelian groups are $\mathbb{Q}$-vector spaces so have a dimension over $\mathbb{Q}$.

Definition 4.2.3 *Given $n \in \omega^M$, the cut $\log_{\mathbb{N}} n$ is the set of $x \in \omega^M$ such that for all $s \in \mathbb{N}$ we have $s^x < n$. This is the greatest initial segment below n that is closed under exponentiation by all standard bases.*

Theorem 4.2.4 *If $C(n)$ is divisible then for each $r \in \log_{\mathbb{N}} n$ the dimension of $C(n)$ over $\mathbb{Q}$ is greater than or equal to the cardinality (in the external sense) of the set $\{0, 1, \ldots, r\}$.*

Proof. As $r \in \log_{\mathbb{N}}(n)$ we have, for all standard primes q and standard integers k, $kq^r < n$. Therefore by overspill there is a nonstandard prime q such that $kq^r < n$ for all $k \in \mathbb{N}$. Choose such q. Then the set $X = \{q, q^2, q^3, \ldots, q^r\}$ is a set of numbers in $\{0, 1, \ldots, n-1\}$, so can be regarded as a subset of $C(n)$. The result will follow if we can show that X is independent over $\mathbb{Q}$. Suppose $\lambda_i \in \mathbb{Q}$, all but finitely many being zero, with $\sum_i \lambda_i q^i \equiv 0 \bmod n$. Then by multiplying through by the finitely many denominators occurring in nonzero values of the λ_is we obtain $\sum_i \mu_i q^i \equiv 0 \bmod n$ with all $\mu_i \in \mathbb{Z}$, where μ_i is the numerator of λ_i multiplied by all the denominators of the nonzero λ_js $(j \neq i)$. By our choice of q, each $\mu_i q^i$ is less than n and so $\sum_i \mu_i q^i = 0$. It follows that each $\mu_i = 0$ for if not, suppose j is least such that $\mu_j \neq 0$. Then by dividing $\sum_i \mu_i q^i = 0$ through by q^j we obtain the result that q divides μ_j, which is impossible as $\mu_j \in \mathbb{Z}$ and q is a nonstandard prime.

In the case when M is countable, this settles the question as to what the group $C(p^k)$ is, for p a nonstandard prime, as well as a number of other similar cases A (see theorem 4.2.2) corresponding to a, when no element x of a has actually finite order: it is the $\mathbb{Q}$-vector space of maximum possible dimension. This is by overspill, because if n is nonstandard there are always nonstandard elements in $\log_{\mathbb{N}}(n)$.

On the other hand, if M is uncountable, it is possible that the set of predecessors of n is of larger (external) cardinality than $\log_{\mathbb{N}}(n)$. In fact it may be that $\mathrm{card}\{0, 1, \ldots, n-1\} > 2^{\mathrm{card} \log_{\mathbb{N}}(n)}$. (This is a result due to Paris and Mills [13].)

Question 4.2.5 *Can the bound in the previous theorem be improved? What are the possibilities for the dimension of $C(n)$ when n has uncountably many predecessors?*

4.3 Cyclic Groups of Order a Power of a Standard Prime

In this section we shall look at the more interesting case of $A = C(p^k)$ where p is a standard prime and k is a nonstandard integer.

With the usual description of A via addition modulo p^k, a typical element of A can be written as

$$x_0 + x_1 p + \cdots + x_m p^m + \cdots + x_{k-1} p^{k-1}$$

where $0 \leqslant x_i < p$ for all i, in particular the x_i are standard integers for all i, but the indices range over a nonstandard initial segment $\{0, 1, 2, \ldots, k-1\}$ of ω^M. Since $x_i, i \in \omega^M$, for all $i < k$ it follows that the sequence $(x_i)_{i<k}$ is M-finite.

Using this representation of elements of A we can define a useful function.

Definition 4.3.1 *For $x = x_0 + \cdots + x_m p^m + \cdots + x_{k-1} p^{k-1}$ we define $\underline{v}(x)$ to be the least $i < k$ such that $x_i \neq 0$, if such i exists, ∞ if there is no such i. In other words $\underline{v}(x)$ picks out the smallest power of p in the p-adic representation of x which has nonzero coefficient.*

This function has the following properties.

Proposition 4.3.2 *The function $\underline{v}\colon A \to \{0, 1, \ldots, k-1, \infty\}$ has the following properties.*

(a) $\underline{v}(a+b) \geqslant \min\{\underline{v}(a), \underline{v}(b)\}$.

(b) $\underline{v}(-a) = \underline{v}(a)$.

(c) $\underline{v}(a) = \infty$ *if and only if* $a = 0$.

Proof. It is straightforward to carry out these checks. We do these below, mainly for illustrative purposes.

(a) Let $\underline{v}(b) \geqslant \underline{v}(a) = \gamma$. Then $a = x_{k-1}p^{k-1} + \cdots + x_\gamma p^\gamma$ and $b = y_{k-1}p^{k-1} + \cdots + y_\gamma p^\gamma$. Note $0 < x_\gamma < p$, and $0 \leqslant y_\gamma < p$. If $x_\gamma + y_\gamma = p$ then $\underline{v}(a+b) > \gamma = min\{\underline{v}(a), \underline{v}(b)\}$. If $x_\gamma + y_\gamma \neq p$ then $\underline{v}(a+b) = \gamma = min\{\underline{v}(a), \underline{v}(b)\}$. This establishes (a).

(b) Suppose $\underline{v}(a) = \gamma$, where $0 \leqslant \gamma < k$. Then $a = x_{k-1}p^{k-1} + \cdots + x_\gamma p^\gamma$, where $0 \leqslant x_i < p$ for $i > \gamma$, $0 < x_\gamma < p$, and $x_i = 0$ for $i < \gamma$. So $-a = (p-1-x_{k-1})p^{k-1} + \cdots + (p - x_\gamma)p^\gamma$ and it is easy to see that $0 < p - x_\gamma < p$ so $\underline{v}(-a) = \gamma = \underline{v}(a)$.

(c) is obvious.

Using 4.3.2 it is easy to see that $\underline{v}(x)$ gives rise to a chain of subgroups

$$0 = A_\infty < \cdots < A_\gamma < \cdots < A_0 = A$$

where $A_\gamma = \{x \in A : \underline{v}(x) \geqslant \gamma\}$. It is a straightforward fact that PA $\vdash$ if $0 < H < C(p^k)$ then $H = C(p^l)$ for some l such that $0 < l \leqslant k$. In our context $C(p^l)$ is A_{k-l} and so such $H = A_{k-n}$ for some n such that $0 < n \leqslant k$. This shows that $\underline{v}(x)$ gives rise to the unique maximal chain of M-finite subgroups of A.

There is another similar function which we define as follows.

Definition 4.3.3 *For $x = x_0 + \cdots + x_m p^m + \cdots + x_{k-1}p^{k-1}$ we define $\overline{v}(x)$ to be the greatest i such that $x < p^{k-i}$. In other words it is the greatest i such that $x_{k-j} = 0$ for all $j \leqslant i$; if there is no i with this property we write $\overline{v}(x) = 0$. We also write $\overline{v}(x) = \infty$ for $\overline{v}(x) = k$.*

Proposition 4.3.4 *The function $\overline{v} \colon A \to \{0, 1, \ldots, k-1, \infty\}$ satisfies the following.*

(a) For all $a, b \in A$, $\overline{v}(a+b) \geqslant \min\{\overline{v}(a), \overline{v}(b)\} - 1$.

(b) For all $a, b \in A$ if $\overline{v}(a), \overline{v}(b) > 0$, then $\overline{v}(a+b) \leqslant \min\{\overline{v}(a), \overline{v}(b)\}$.

(c) For all $a \in A$, $\overline{v}(a) = \infty$ if and only if $a = 0$.

Proof. (a) Suppose $m = \overline{v}(a) \geqslant n = \overline{v}(b)$. Then $a = \sum_{i=0}^{i=k-m-1} x_i p^i$ and $b = \sum_{i=0}^{i=k-n-1} y_i p^i$, with $0 \leqslant x_i, y_i < p$ and $0 < y_{k-n-1}, x_{k-m-1} < p$. There are now two cases to consider. If $a + b < p^{k-n}$ then $a + b = \sum_{i=0}^{i=k-m-1} x_i p^i + \sum_{i=0}^{i=k-n-1} y_i p^i = \sum_{i=0}^{i=k-n-1} z_i p^i$, for some z_i such that $0 \leqslant z_i < p$. So $\overline{v}(a + b) \geqslant n$. If $a + b \geqslant p^{k-n}$ then $a + b = \sum_{i=0}^{i=k-m-1} x_i p^i + \sum_{i=0}^{i=k-n-1} y_i p^i = \sum_{i=0}^{i=k-n-1} z_i p^i + p^{k-n}$, for some z_i such that $0 \leqslant z_i < p$. So $\overline{v}(a + b) = n - 1$. In either case we see that $\overline{v}(a + b) \geqslant n - 1$.

(b) Suppose $m = \overline{v}(a) \geqslant n = \overline{v}(b) > 0$. Then $a = \sum_{i=0}^{i=k-m-1} x_i p^i$ and $b = \sum_{i=0}^{i=k-n-1} y_i p^i$, with $0 \leqslant x_i, y_i < p$ and $0 < y_{k-n-1}, x_{k-m-1} < p$. Then $a + b = \sum_{i=0}^{i=k-n} z_i p^i$ for some z_i such that $0 \leqslant z_i < p$. If $z_{k-n-1} \neq 0$ then $\overline{v}(a + b) \leqslant n$. If $z_{k-n-1} = 0$ it must be the case that $z_{k-n} = 1$ and so $\overline{v}(a + b) = n - 1 \leqslant n$.

(c) is obvious.

We can use the functions $\underline{v}$ and $\overline{v}$ to define some interesting and useful subgroups and subsets of A.

Definition 4.3.5 *For $i \in \omega^M$, we define*

$$A_i = \{x \in A : \underline{v}(x) \geqslant i\}$$

and

$$A^i = \{x \in A : \overline{v}(x) \geqslant i \text{ or } \overline{v}(-x) \geqslant i\}.$$

Furthermore, for a cut I of ω^M, define

$$A_I = \bigcup_{i' > I} A_{i'} = \bigcap_{i \in I} A_i$$

and

$$A^I = \bigcup_{i' > I} A^{i'} = \bigcap_{i \in I} A^i.$$

For $i \in \omega^M$ the sets A^i are not groups because closure under addition fails (see part a) of proposition 4.3.4). However it is straightforward to check that A^I is a group for any cut I. We note that the groups A_I and A^I are down monotonically I-definable in the sense of definition 2.2.1. For the following proposition recall that $k - I := inf\{k - i : i \in I\} = sup\{k - i' : i' > I\}$.

Proposition 4.3.6 $A_I \cap A^{k-I} = \{0\}$.

Proof. If $x \in A_I$ then $p^i|x$ for all $i \in I$. We apply overspill to conclude that $p^\alpha|x$ for some $\alpha > I$. Hence $x = p^\alpha y$ for some $y \in \omega^M$. So if $x \neq 0$ we must have $p^{k-(k-\alpha)} = p^\alpha \leqslant x$ hence $\overline{v}(x) \leqslant k - \alpha - 1$ so $\overline{v}(x) < k - \alpha$. As A_I is a group $x \in A_I \implies -x \in A_I$ so we can repeat the argument to conclude $\overline{v}(-x) < k - \alpha$. As $\alpha > I$ and $A^{k-I} = \{x \in A : \overline{v}(x) > k - I \text{ or } \overline{v}(-x) > k - I\}$ it is clear that $x \notin A^{k-I}$.

It follows that $A_I \bigoplus A^{k-I} \leqslant A$. But it is not true that equality holds here. For example $\sum_{i<k} p^i \in A$ is not in $A_I \bigoplus A^{k-I}$.

Next we shall identify the torsion part and the divisible part of A. Firstly we identify the torsion part of A.

Proposition 4.3.7 *The subgroup $tA = A_{k-\mathbb{N}}$ consists of all the torsion elements of A. It is isomophic to the Prüfer group $C(p^\infty)$.*

Proof. An element of $A_{k-\mathbb{N}}$ is a finite sum of the form $x = x_{k-n}p^{k-n} + \cdots + x_{k-1}p^{k-1}$ where $0 \leqslant x_i < p$ for all i and $n \in \mathbb{N}$. It is easy to prove (by external induction on n) that all such elements are torsion, i.e. there is $m \in \mathbb{N}$ such that $mx \equiv 0 \bmod p^k$. Since we are only interested in $n \in \mathbb{N}$ external induction suffices. There is no contradiction to overspill here because $M \models$ all elements of A are torsion. It's just that for $x \notin A_{k-\mathbb{N}}$ the m such that $M \models \exists m : mx \equiv 0 \bmod p^k$ is nonstandard.

Conversely if $x \notin A_{k-\mathbb{N}}$ then $l = \underline{v}(x) \in k - \mathbb{N}$ so $x = x_l p^l$ + terms in higher powers with $x_l \neq 0$. If $m \in \mathbb{N}$ is written in p-adic form as $m_r p^r + m_{r+1}p^{r+1} \cdots + m_s p^s$ with $m_r \neq 0$

and $0 \leqslant m_i < p$ for all i then $mx = x_l m_r p^{l+r}$ + higher powers and as $x_l m_r \not\equiv 0 \bmod p$ and $l+r \in k-\mathbb{N}$ this shows that $\underline{v}(mx) = l+r < \infty$ i.e. $mx \neq 0$ hence x is not a torsion element.

It is straightforward to see (by dividing by an appropriate power of p) that $A_{k-l} \cong C(p^l)$ for each $l \in \mathbb{N}$ and that the embedding $A_{k-l} \to A_{k-l-1}$ is the natural one, $C(p^l) \to C(p^{l+1})$, hence $A_{k-\mathbb{N}}$ is the direct limit of these, i.e. the Prüfer group $C(p^\infty)$.

We now identify the divisible part of A. See Kaplansky [7, section 5] for more information on divisible groups.

Proposition 4.3.8 *The group $A_\mathbb{N}$ is divisible. More generally, A_I is divisible for all cuts I closed under successor not containing k. Also, for each element $x \in A \setminus A_\mathbb{N}$ there is $m \in \mathbb{N}$ such that for no $y \in A$ is $my = x$, so $A_\mathbb{N}$ is the divisible part dA of A.*

Proof. The first part is by long division in M, taking $x = x_r p^r$ + higher powers for some $r > \mathbb{N}$ and $m = m_0 + m_1 p + \cdots + m_s p^s$ where $0 \leqslant m_i < p$ for all i and $s \in \mathbb{N}$. Then long division gives $y \in A$ with $\underline{v}(y) \geqslant r - s > \mathbb{N}$ such that $my = x$. Hence $A_\mathbb{N}$ is divisible. The same works for any other cut I.

For the other part, if $x \notin A_\mathbb{N}$ then $x = x_r p^r$ + higher powers for some $r \in \mathbb{N}$ and $0 < x_r < p$. It follows that x is not divisible by p^{r+1}.

We now state the following standard theorem of infinite abelian groups.

Theorem 4.3.9 *Suppose A is an abelian group and $D \leqslant A$ is a subgroup which is divisible. Then there is $K \leqslant A$ such that $D \bigoplus K = A$, i.e. D is a direct summand of A.*

See theorem 2 in Kaplansky [7, section 5] for details of the general proof which requires Zorn's lemma.

It follows from 4.3.8 that A_I is divisible for each such I. Hence by 4.3.9 all the A_I are direct summands of A, i.e. for each such I there is B_I such that $A_I \bigoplus B_I = A$. The use of Zorn's lemma in the proof of theorem 4.3.9 suggests that such a B_I is somewhat non-constructive. It turns out that such a B_I can be M-countable which is the principal result of the next section.

Our next goal is to show that for certain I, no complement B_I of A_I is monotonically J-definable for any initial segment J. In the light of proposition 2.2.3 it suffices to show that B_I cannot be up monotonically J-definable for any initial segment J. We shall want to show that monotonically definable sets are enumerable by an initial segment. To make precise what we mean we give the following definition.

Definition 4.3.10 *We say B is enumerable if there are*

(1) a definable and 1-1 $f \in M$; and

(2) a cut or initial segment $I \subseteq_e M$

such that $B = \{f(i) : i \in I\}$. We shall say that B is I-enumerable when we want to refer to the particular cut involved.

Proposition 4.3.11 *Let $B \leqslant A$ be such that $B \cap A_I = 0$ and B is I-enumerable. Then $B + A_I \neq A$.*

Proof. $B = \{f(i) : i \in I\}$. Choose some small $\alpha > I$. Let $f(i)_i$ denote the i^{th} component in the p-adic expansion of $f(i)$. For each $i < \alpha$ define x_i so that $x_i \neq f(i)_i$. Then consider $x = \sum_{i=0}^{i=\alpha} x_i p^i$. Suppose $x \in B + A_I$. Then there are $j \in I$ and $y \in A_I$ such that $x = f(j) + y$. Since $y \in A_I$ the i^{th} components of $x - y$ are the same as the i^{th} components of x for all $i \in I$. But $x - y = f(j)$ and by definition $x_j \neq f(j)_j$ which is a contradiction.

Proposition 4.3.12 *Suppose $B \leqslant A$, $B \cap A_I = 0$, and B is up monotonically J-definable by some formula β. Then for all $j \in J$ we have $|\beta_j| \in p^I$.*

Proof. Let $j \in J$ so that $\beta_j \subseteq B$. Suppose $|\beta_j| > p^I$. Let $v = \lfloor \log_p |\beta_j| \rfloor - 1$. It follows that $v > I$. Since $|\{x : x = \sum_{0 \leqslant i < v} x_i p^i\}| = p^v < |\beta_j|$ two distinct elements a, b of β_j must agree on coordinates $x_0, \cdots, x_{v-1}$. Since $v > I$ we have $0 \neq a - b \in A_I \cap B_I$.

Proposition 4.3.13 *Suppose $B \leqslant A$, $B + A_I = A$, and B is up monotonically J-definable by some formula β. Then $\forall i \in I \exists y \in J$ such that $|\beta_y| \geqslant p^i$.*

Proof. Suppose to the contrary. Then there is $\alpha \in I$ such that for all $j \in J$ we have $|\beta_j| < p^\alpha$. By overspill there is some $j' > J$ such that $|\beta_{j'}| < p^\alpha$. But $\beta_{j'} \supseteq B$ and so $\beta_{j'} + A_I = A$. Since $A_i \supseteq A_I$ for all $i \in I$ we have that $\forall i \in I \beta_{j'} + A_i = A$ and so by overspill there is $i' > I$ such that $\beta_{j'} + A_{i'} = A$. Consider the M-finite set $X = \{x : x = \sum_{0 \leqslant i < \alpha} x_i p^i\}$. Now $|X| = p^\alpha$ and we may write each element $x \in X$ as a sum $x = a + b$ for $a \in A_{i'}$ and $b \in \beta_{j'}$. Suppose $y = b + a_1$ and $x = b + a_2$ for $x, y \in X$, $b \in \beta_{j'}$ and $a_1, a_2 \in A_{i'}$. But then $y - x = a_1 - a_2$. Note that $X \subseteq A^{k-I}$ and the latter is a group, also $A_{i'}$ is a subgroup of A_I. We now have a contradiction to proposition 4.3.6 because $0 \neq y - x = a_1 - a_2 \in A^{k-I} \cap A_I$. This says that $|\beta_{j'}| \geqslant p^\alpha$ which is the final contradiction we need.

Proposition 4.3.14 *If B is monotonically J-definable by β and*

(a) $|\beta_y| \in I$ for all $y \in J$.

(b) $\forall i \in I \exists y \in J$ such that $|\beta_y| > i$

then B is I-enumerable.

Proof. List all elements of

(0) β_0

(1) $\beta_1 \backslash \beta_0$

(2) $\beta_2 \backslash (\beta_1 \cup \beta_0) \ \cdots$

(n) $\beta_n \backslash (\beta_{n-1} \cup \cdots \cup \beta_0)$

This is an internal construction since each β_n is definable for $n \in J$. Now set $f(i) = i^{th}$ element in the list. It follows from (b) that $\{f(i) : i \in I\} \subseteq B$. Conversely (a) says that all elements of β_y appear as $f(i)$ for some $i < |\beta_y|$, hence $B \subseteq \{f(i) : i \in I\}$.

Theorem 4.3.15 *Let I be any cut such that $p^I = I$, for example I could be closed under exponentiation by all standard basis. Let $B_I \leqslant A$ be a complement of A_I. Then B_I is not monotonically J-definable for any initial segment J.*

Proof. Suppose to the contrary that there is some monotonically J-definable B_I such that $B_I + A_I \neq A$ and $B_I \cap A_I = 0$. By proposition 4.3.12 we have that condition (a) in proposition 4.3.14 is satisfied and by proposition 4.3.13 we have that condition (b) is satisfied. Hence by proposition 4.3.14 we have that B is p^I-enumerable. By assumption $p^I = I$ and so by proposition 4.3.11 we have that $B_I + A_I \neq A$ which is a contradiction.

Question 4.3.16 *Is theorem 4.3.15 true in the case $I \subsetneq p^I$?*

The author strongly suspects the answer to this is yes. The proof would probably require a more sophisticated digitalization in proposition 4.3.11 to get it to work for p^I-enumerable B.

Question 4.3.17 *Can there be a monotonically definable transversal T that is not a group? I.e. a monotonically definable set T such that $T \cap A_I = 0$ and $T + A_I = A$ but T is not assumed to be closed under $+$ or $-$.*

$\mathbb{N}$ is the smallest cut closed under the successor and $k-\mathbb{N}$ the largest one not containing k. Notice that if $I \subseteq J$ then $A_J \subseteq A_I$, hence $A_{\mathbb{N}} \supseteq A_I \supseteq A_J \supseteq A_{k-\mathbb{N}}$ for each pair of cuts $I \subseteq J$ closed under the successor.

Question 4.3.18 *Can such a family of B_Is be chosen so that B_I is a complement of A_I for all I and $B_{\mathbb{N}} \subseteq B_I \subseteq B_J \subseteq B_{k-\mathbb{N}}$ for each pair of cuts $I \subseteq J$ closed under successor?*

To attempt to answer this question we first answer it in the case of a single pair of cuts $I \subseteq J$. It follows that $A_J \subseteq A_I$. By theorem 4.3.9 we have $B_I \leqslant A$ and $L_{JI} \leqslant A_I$ such that $A_I \bigoplus B_I = A$ and $A_J \bigoplus L_{JI} = A_I$. It follows that $A_J \bigoplus L_{JI} \bigoplus B_I = A$ and on setting $B_J = L_{JI} \bigoplus B_I$ we have $B_I \subseteq B_J$.

This shows that we can answer 4.3.18 positively in the discrete case. Suppose we have a decreasing family of cuts indexed by some set J, together with a chain of A_{I_η} and a corresponding chain of complements as below.

$$I_0 \supseteq I_1 \supseteq \cdots \supseteq I_\eta \supseteq \cdots$$

$$A_{I_0} \subseteq A_{I_1} \subseteq \cdots \subseteq A_{I_\eta} \subseteq \cdots$$

$$B_{I_0} \supseteq B_{I_1} \supseteq \cdots \supseteq B_{I_\eta} \supseteq \cdots$$

Set $I = \bigcap_{\eta \in J} I_\eta$ and it follows easily that I is a cut and $\bigcup_{\eta \in J} A_{I_\eta} = A_I$. We know there exists B_I such that $A_I \bigoplus B_I = A$ but if we are to answer 4.3.18 positively then we need $B_I \subseteq \bigcap_{\eta \in J} B_{I_\eta}$.

Question 4.3.19 *Does $\bigcap_{\eta \in J} B_{I_\eta} + A_I = A$?*

A common theme in nonstandard finite objects is that there is often a nonstandard metric or topology on the elements of such objects, which by factoring out by an equivalence relation yields a continuous structure that is of interest for other reasons. This can be illustrated nicely with the group $C(p^k)$. First, consider the function $\underline{v}$ of definition 4.3.1 on $C(p^k)$. This yields a nonstandard metric

$$\underline{d}(x, y) = \begin{cases} \frac{1}{\underline{v}(x-y)} & \text{if } \underline{v}(x-y) < \infty \\ 0 & \text{if } x = y. \end{cases}$$

To make this a standard metric we need to make elements $x, y \in A$ equivalent whenever $\underline{d}(x, y)$ is infinitesimal. This means factoring out by the subgroup $A_{\mathbb{N}} = \{x \in A : \underline{v}(x) \text{ is infinite}\}$.

Proposition 4.3.20 *The nonstandard metric $\underline{d}$ induces a standard metric structure on $A/A_{\mathbb{N}}$ making it into a topological group. The group $A/A_{\mathbb{N}}$ is topologically isomorphic to a subgroup of the group of p-adic integers $\mathbb{Z}_p = \varprojlim C(p^m)$.*

Proof. We define a map $\Phi : A \to \varprojlim C(p^m)$ as follows.

$$\Phi(x_0 + x_1 p^1 + \cdots + x_{k-1} p^{k-1}) = (x_0, x_0 + x_1 p^1, \ldots, x_0 + \cdots + x_i p^i, \ldots)_{i \in \mathbb{N}}.$$

The right hand side is clearly an element of $\varprojlim C(p^m)$ as

$$\varphi_{ij}(x_0 + \cdots + x_j p^j) = x_0 + \cdots + x_i p^i \text{ for all } i \leqslant j.$$

Φ is well defined as M can extract definably the i^{th} digits of the p-adic representation of a number less than p^k. It is straightforward to check Φ is a homomorphism of groups with kernel $A_{\mathbb{N}}$ and that it is continous with respect to the topology defined by the metric and the natural topology of $\mathbb{Z}_p$.

The group $A_{\mathbb{N}}$ is divisible by proposition 4.3.8 which means the extension A of $A_{\mathbb{N}}$ is split by theorem 4.3.9 i.e. $A = A_{\mathbb{N}} \oplus A/A_{\mathbb{N}}$. A divisible group is just a direct sum of its torsion subgroup with a $\mathbb{Q}$-vector space by theorem 4 in Kaplansky [7, section 5] and $tA_{\mathbb{N}} = A_{k-\mathbb{N}} \cong C(p^\infty)$ by 4.3.7. So $A_{\mathbb{N}} = V \bigoplus C(p^\infty)$, where V is a vector space over $\mathbb{Q}$. We have shown the following.

Proposition 4.3.21 *A is a direct sum of a subgroup of $\mathbb{Z}_p$ with a $\mathbb{Q}$-vector space and $C(p^\infty)$.*

The group $A/A_{\mathbb{N}}$ is a version of the standard system of the original model M. For more details about $\mathrm{SSy}(M)$ see Kaye [8], page 141 onwards. Real numbers $r \in \mathbb{R}$ (or $R \subseteq \mathbb{N}$) can be encoded by elements of $\mathbb{Z}_p$ and they are present in $A/A_{\mathbb{N}}$ if and only if they are coded in M in one of the more usual means from nonstandard arithmetic. The set of reals coded in M is an important invariant of M called the *standard system of* M and denoted by $\mathrm{SSy}(M)$, which is normally viewed as a subset of $\mathcal{P}(\mathbb{N})$. There are recursion theoretic closure-conditions on $\mathrm{SSy}(M)$ which characterize precisely which sets of reals $\mathscr{X} \subseteq \mathcal{P}(\mathbb{N})$ can arise as $\mathrm{SSy}(M)$ for some M, and from these closure-conditions the subgroups of $\mathbb{Z}_p$ that can arise in this way can be described too. The case $\mathrm{SSy}(M) = \mathcal{P}(\mathbb{N})$ does arise, in particular if M is $\aleph_0$-saturated then $\mathrm{SSy}(M) = \mathcal{P}(\mathbb{N})$. This gives the following.

Proposition 4.3.22 *If M is $\aleph_0$-saturated of cardinality $2^{\aleph_0}$ then $A = \mathbb{Z}_p \oplus V \oplus C(p^\infty)$ where V is a $\mathbb{Q}$-vector space of dimension $2^{\aleph_0}$.*

We have the following proposition concerning the structure of the group $A/A^{\mathbb{N}}$.

Proposition 4.3.23 *The group $A/A^{\mathbb{N}}$ is isomorphic to a subgroup of the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and contains the usual copy of the Prüfer group $C(p^\infty)$ in $\mathbb{T}$.*

Proof. Define $\Phi : A \to \mathbb{T}$ by $\Phi(x) = exp(2\pi i[\frac{x_{k-1}}{p} + \frac{x_{k-2}}{p^2} + \cdots + \frac{x_{k-n}}{p^n} + \cdots]_{n\in\mathbb{N}})$. Note $0 \leqslant \sum_{n=1}^{\infty} \frac{x_{k-n}}{p^n} \leqslant \sum_{n=1}^{\infty} \frac{p-1}{p^n} = 1$ so $\mathrm{im}(\Phi) \subseteq \mathbb{T}$. It is straightforward to check that Φ is a homomorphism of groups with kernel $A^{\mathbb{N}}$.

Again, elements of $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ code reals and the group $A/A^{\mathbb{N}}$ is a version of the standard system $\mathrm{SSy}(M)$ of M. If M is $\aleph_0$-saturated of cardinality $2^{\aleph_0}$ then $\mathrm{SSy}(M) = P(\mathbb{N})$, in which case $A/A^{\mathbb{N}} = \mathbb{T}$. It turns out that the extension A of $A^{\mathbb{N}}$ does not split. In the suitably saturated case $A/A^{\mathbb{N}} = \mathbb{T}$ which has elements of all finite orders, however A only has elements of infinite order and of order a power of p and so $A/A^{\mathbb{N}}$ cannot occur as a subgroup. In fact saturation is not really needed here because $\mathrm{im}(\Phi)$

will always contain elements of all finite orders. This is because all such elements arise as the image of an element coding a sum of the form $(x_{k-1}p^{k-1} + \cdots + x_{k-n}p^{k-n} + \cdots)_{n\in\mathbb{N}}$ for which there is $m \in \mathbb{N}$ such that the sum consists of some repeating block finite block of coefficients after x_{k-m}, and such sums are coded in arbitrary M by overspill.

4.3.1 Szmielew Invariants

In this short subsection we refer to Some Model Theory of Groups [6]. We state a theorem below which is taken directly from that paper.

Theorem 4.3.24 *If A and B are abelian groups, then A is elementarily equivalent to B if and only if*

(i) A is of finite exponent $\Leftrightarrow$ B is of finite exponent;

and for each prime p and integer $n \geqslant 0$:

(ii) dim $p^nA[p]/p^{n+1}A[p] =$ dim $p^nB[p]/p^{n+1}B[p]$

(iii) $\lim_{n\to\infty}$ dim $p^nA/p^{n+1}A = \lim_{n\to\infty}$ dim $p^nB/p^{n+1}B$

(iv) $\lim_{n\to\infty}$ dim $p^nA[p] = \lim_{n\to\infty}$ dim $p^nB[p]$

in the sense that the two sides of the equality are the same finite cardinal or are both infinite. (Here $p^nG = \{p^nx : x \in G\}$; $G[p] = \{x \in G : px = 0\}$: and dim means dimension over Z/pZ).

We have the following corollary, also from [6].

Corollary 4.3.25 *Any abelian group is A is elementarily equivalent to a group of the form*

$$\bigoplus_p [\bigoplus_n C(p^n)^{(\alpha_{p,n})} \oplus Z_p^{(\beta_p)} \oplus C(p^\infty)^{(\gamma_p)}] \oplus \mathbb{Q}^{(\delta)}.$$

It is easy to see that for a given $n \in \mathbb{N}$ and prime p

(ii) determines the value of $\alpha_{p,n}$;

and for a given prime p:

(iii) determines the value of β_p

(iv) determines the value of γ_p.

If the exponent is finite then $\delta = 0$ since otherwise $\mathbb{Q}^{(\delta)}$ would contain elements of infinite order. If the exponent is infinite then δ can be anything, provided at least one of the β_ps or γ_ps is nonzero, otherwise δ must be strictly greater than zero.

We calculate the Szmielew Invariants for the group $A = C(p^k)$ for p a standard prime and k a nonstandard integer.

(i) Exponent $= \infty$ because $1 \in A$ is of infinite order.

(ii) We calculate $\dim q^nA[q]/q^{n+1}A[q]$. If $q \neq p$ then $q^nA[q] = 0$, so $\dim q^nA[q]/q^{n+1}A[q] = 0$. If $q = p$ then $p^nA[p] = A_{k-1}$, and $p^{n+1}A[p] = A_{k-1}$. So $p^nA[p]/p^{n+1}A[p] = 0$, and $\dim p^nA[p]/p^{n+1}A[p] = 0$.

(iii) We calculate $\lim_{n\to\infty} \dim q^nA/q^{n+1}A$. If $q \neq p$ then $q^nA = A$ and $q^{n+1}A = A$, so $\lim_{n\to\infty} \dim q^nA/q^{n+1}A = 0$. If $q = p$ then $p^nA = A_n$ and $p^{n+1}A = A_{n+1}$, so $q^nA/q^{n+1}A = C(p)$ and $\lim_{n\to\infty} \dim p^nA/p^{n+1}A = 1$.

(iv) We calculate $\lim_{n\to\infty} \dim q^nA[q]$. If $q \neq p$ then $q^nA[q] = 0$, so $\lim_{n\to\infty} \dim q^nA[q] = 0$. If $q = p$ then $p^nA[p] = A_{k-1}$, so $\lim_{n\to\infty} \dim p^nA[p] = 1$.

Hence we see that $A = C(p^k)$ is elementarily equivalent to a group of the form $A = \mathbb{Z}_p \bigoplus V \bigoplus C(p^\infty)$. This agrees with proposition 4.3.22. We can add the V, a $\mathbb{Q}$-vector space, since it is consistent that $\delta > 0$.

4.4 Complement Theorem

We will study $G = C(p^k)$ for p a standard prime and k a nonstandard integer as in the previous section. Let A_I be as in definition 4.3.5. It is the aim of this section to show that under suitable conditions the corresponding B_I such that $A_I \bigoplus B_I = A$ can be M-countable. We begin by proving some lemmas that we shall need for the main result.

The next lemma is a straightforward property of general abelian groups. The proof is extracted from theorem 2 in Kaplansky [7, section 5].

Lemma 4.4.1 *Suppose $A, B \leqslant G$ are abelian groups such that $A \cap B = \{0\}$. Suppose there is $z \in G$ such that $qz = b \in B$ for some prime q.*

If $z \notin A + B$ then $\langle B \cup \{z\} \rangle \cap A = \{0\}$.

Proof. Suppose not. Then there are $m \in \mathbb{N}$, $b' \in B$ and $0 \neq a \in A$ such that $mz + b' = a$. If $q \mid m$ then $m = rq$ which implies $a = rb + b' \in B$ and this contradicts the assumption that $A \cap B = \{0\}$. If $q \nmid m$ then $(q, m) = 1$ and so there are $s, t \in \mathbb{Z}$ such that $sm + tq = 1$. Whence $z = smz + tqz = s(a - b') + tb \in A + B$ which contradicts that assumption that $z \notin A + B$.

The final lemma given here is more specific to models of arithmetic.

Lemma 4.4.2 *Suppose $G = C(p^k)$ for p a standard prime, k a nonstandard number. Let $g \in G$. If q is a prime $q \neq p$ then there is a unique $g_1 \in G$ such that $qg_1 = g$.*

Proof. Fix the prime q. Define $f_q : G \to G$ by $y \to qy$. If $qy = qz$ then $q(y - z) = 0$. Then $y = z$ since $p \nmid q$ and every element of G has order a power of p. But f_q is a definable $1-1$ map of an M-finite set to itself and so it is also onto. Thus for each $g \in G$ there is a unique $g_1 \in G$ such that $g = f_q(g_1) = qg_1$ as required.

Lemma 4.4.3 *Suppose G is an M-finite abelian group and $A \leqslant G$ is an M-countable subgroup with* $\text{card}(A) = I$. *Suppose there are M-finite sets $B, B' \subseteq G$ and $A_i \subseteq A$ such that $\langle B \rangle \cap A = \{0\}$ and $B' + A_i = G$. Denote by J the cardinality of the torsion subgroup $tG \subseteq G$. Note that the torsion subgroup of any M-finite abelian group is monotonically definable and therefore M-countable. For any $\alpha, \beta > \mathbb{N}$, $\gamma > J$ and $i' > I$ define $r \in \omega^M$ to be least such that $\frac{|G|}{i'} \leqslant (r+|B|)^{2\alpha}\gamma\beta$. Then there is an M-finite set $\{b_0, \cdots, b_{r-1}\} \subseteq B' \backslash B$ such that $\langle B \cup \{b_0, \cdots, b_{r-1}\} \rangle \cap A = \{0\}$.*

Proof. By M-countability of A select $A_{i'} \supseteq A$ with $|A_{i'}| = i'$. For $m \in \omega^M$ define $\langle B \rangle_m := \{\sum_{j=0}^{j=m-1} x_j : \pm x_j \in B\}$. It is straightforward to see that $\langle B \rangle = \bigcup_{m \in \mathbb{N}} \langle B \rangle_m$ and $|\langle B \rangle_m| \leqslant |B|^{2m}$. Fix some $\alpha > \mathbb{N}$ and consider $\langle B \rangle_\alpha$. Since tG is M-countable we may apply theorem 3.2.7 to define an M-finite set X_γ with $|X_\gamma| \geqslant \frac{|G|}{\gamma}$ and $X_\gamma + tG$ unique sum. Since $X_\gamma + tG$ is unique product it follows that $\forall m \in \mathbb{N} \forall g_1, g_2 \in X_\gamma (m(g_1 - g_2) = 0 \implies g_1 = g_2)$. Then by overspill there is $\beta > \mathbb{N}$ such that $\forall m < \beta \forall g_1, g_2 \in X_\gamma (m(g_1 - g_2) = 0 \implies g_1 = g_2)$. We can of course take β to be as small as we like subject to $\beta > \mathbb{N}$. Define r to be least such that $\frac{|G|}{i'} \leqslant (r + |B|)^{2\alpha}\beta\gamma$. We now perform an internal induction to add r elements to B. Suppose inductively that for $s < r$ we have already specified $b_0, \cdots, b_{s-1}$ satisfying: $mb_j \notin A_{i'} + \langle B \cup \{b_0\} \cup \cdots \cup \{b_{s-1}\} \rangle_\alpha$. Suppose that for all $g \in X_\gamma$ there is $m < \beta$ such that

(*) $mg \in A_{i'} + \langle B \cup \{b_0\} \cup \cdots \cup \{b_{s-1}\} \rangle_\alpha$

Define a function $f : (A_{i'} + \langle B \cup \{b_0\} \cup \cdots \cup \{b_{s-1}\} \rangle_\alpha) \times \beta \to X_\gamma$

$$f(x, n) = g \text{ if } g \in X_\gamma \text{ satisfies } ng = x$$

$$f(x, n) = 0 \text{ otherwise}$$

For a given pair (x, n) there can be at most one such g by the property of X_γ. (*) says that every $g \in X_\gamma$ is of this form and so f is surjective. But $|A_{i'}+\langle B\cup\{b_0\}\cup\cdots\cup\{b_{s-1}\}\rangle_\alpha|\beta \leqslant |A_{i'}|(|B|+s)^{2\alpha}\beta < \frac{i'|G|\beta}{i'\beta\gamma} = \frac{|G|}{\gamma} = |X_\gamma|$ so (*) can't happen. So there is $b_s \in X_\gamma$ such that for all $m < \beta$ we have $mb_s \notin A_{i'} + \langle B \cup \{b_0\} \cup \cdots \cup \{b_{s-1}\}\rangle_\alpha$. By this internal induction we create a set $b_0, \cdots, b_{r-1}$. Now from the statement of the lemma there are M-finite sets B', A_i such that $B' + A_i = G$. By another internal induction we may rewrite each $b_j = a_j + b'_j$ where $a_j \in A_i$ and $b'_j \in B'$.

Finally we claim that $\{b'_0, \cdots, b'_{r-1}\}$ is as desired. For brevity let X denote the set $B \cup \{b'_0, \cdots, b'_{r-1}\}$. Suppose $A \cap \langle X\rangle \neq \{0\}$. Then there are $\lambda_0, \cdots, \lambda_{r-1} \in \mathbb{Z}$ such that $\sum_{j=0}^{j=r-1} \lambda_j b'_j \in A+\langle B\rangle$ where $\lambda_j \neq 0$ for at most an actual finite number of j. At least one $\lambda_j \neq 0$ because $A\cap\langle B\rangle = \{0\}$ by assumption. Let k be largest for which λ_k is nonzero. By rearranging we have that $\lambda_k b'_k \in A + \langle B \cup \{b'_0, \cdots, b'_{k-1}\}\rangle$, and we can assume that λ_k is positive. Now since finitely many λ_js are nonzero and each $b'_j = b_j - a_j$ (where $a_j \in A_i \subseteq A$) we can conclude that $\lambda_k b_k \in A + \langle B \cup \{b'_0, \cdots, b'_{k-1}\}\rangle \subseteq A_{i'} + \langle B \cup \{b_0, \cdots, b_{k-1}\}\rangle_\alpha$ and since $\lambda_j \in \mathbb{N} < \beta$ this contradicts the choice of b_j.

Suppose $b'_k = b'_j$ for $j < k \leqslant r-1$. Then $b'_k + a_k = b_k$ and $b'_j + a_j = b_j$ gives $b_k = b_j - a_j + a_k \in A + \langle B \cup \{b_0, \cdots, b_{k-1}\}\rangle \subseteq A_{i'} + \langle B \cup \{b_0, \cdots, b_{k-1}\}\rangle_\alpha$ which is a contradiction. Thus $|\{b'_0, \cdots, b'_{r-1}\}| = r$ as required.

Before stating the next lemma we shall need a definition. Recall definition 4.3.5 from which the group $A^{k-\mathbb{N}} \leqslant G$ is defined for $G = C(p^k)$.

Definition 4.4.4 *Let $b \in G = C(p^k)$, for p a standard prime and k a nonstandard number. Then $X_b := \{g \in G : \exists r \in \mathbb{N}\ rg \in b + A^{k-\mathbb{N}}\}$. In other words an element of X_b is a translation of b by any element of $A^{k-\mathbb{N}}$ followed by a division by any $r \in \mathbb{N}$; if such a division is possible we may pick any of the possibilities for this division that we wish.*

We observe that X_b is up monotonically definable by some $(\theta^b, \mathbb{N})$. We know $A^{k-\mathbb{N}}$ is up-monotonically $\mathbb{N}$ definable. Thus we may define $\theta^b_n = \{g \in G : \exists r \leqslant n\ rg \in b + A^{k-n}\}$ and we see that $X_b = \bigcup_{n\in\mathbb{N}} \theta^b_n$.

Lemma 4.4.5 *Let $b \in G = C(p^k)$, for p a standard prime, k a nonstandard number. Then X_b is M-countable with* $\mathrm{card}(X_b) = \mathbb{N}$.

Proof. We claim that $\forall n \in \mathbb{N}(|\theta_n^b| \in \mathbb{N})$. Since, for fixed b, the set $\{b + \widehat{a} : (\exists m \leqslant n)\widehat{a} = x_0 + \cdots + x_m p^m\}$ is finite providing, for each $g \in G$ and $r \in \mathbb{N}$, the set $Y_{r,g} = \{y \in G : ry = g\}$ is finite. This is true because $t(G) = A_{k-\mathbb{N}}$ has $\mathrm{card}(A_{k-\mathbb{N}}) = \mathbb{N}$. Since X_b is monotonically definable, $X_b = \bigcup_{n\in\mathbb{N}} \theta_n^b$, it follows X_b is M-countable with $\mathrm{card}(X_b) = \mathbb{N}$.

Lemma 4.4.6 *Suppose $G = C(p^k)$ for p a standard prime, k a nonstandard number and $A_I \leqslant G$ is the subgroup defined in 4.3.5 for some cut I such that $\mathbb{N} \leqslant I < k$. Suppose $D \subseteq G$ is an M-finite set such that $D + A_I = G$. Suppose $C \subseteq \bigcup_{b\in D} X_b$ is a set such that $\langle C\rangle \cap A_I = \{0\}$. Let $g \in G$ be an arbitrary element. Then there is a set $\{z_1, \cdots, z_t\} \subseteq \bigcup_{b\in D} X_b$, for $t \in \mathbb{N}$, such that $\langle C \cup \{z_1, \cdots, z_t\}\rangle \cap A_I = \{0\}$ and $g \in \langle C \cup \{z_1, \cdots, z_t\}\rangle + A_I$.*

Proof. We will select a finite number of elements $\{z_1, \cdots, z_t\} \subseteq \bigcup_{b\in D} X_b$ such that $g \in \langle C \cup \{z_1, \cdots, z_t\}\rangle + A_I$. We must also take care to ensure that the elements we add do not violate $\langle C \cup \{z_1, \cdots, z_t\}\rangle \cap A_I = \{0\}$.

If $g \in A_I + \langle C\rangle$ then there is nothing to do, so suppose otherwise. If $\forall m \in \mathbb{N}$ $mg \notin A_I + \langle C\rangle$ then we simply write $g = a + b$ for $a \in A_I, b \in D$ and note that it suffices to add b. Otherwise suppose $m \in \mathbb{N}$ is least such that $mg \in A_I + \langle C\rangle$. Thus $mg = a + x$, for $a \in A_I, x \in \langle C\rangle$. Let q be a prime dividing m and by divisibility let $a_1 \in A_I$ be such that $qa_1 = a$; set $z := \frac{m}{q} g - a_1$ thus $qz = x$. By minimality $z \notin A_I + \langle C\rangle$.

Suppose $q = p$ (case 1): Since $x \in \langle C\rangle$ there are $r \in \mathbb{N}$ and $x_1, \cdots, x_r \in C$ such that $x = \pm x_1 \pm \cdots \pm x_r$. We will now translate each of the x_i to ensure they are divisible by p. For each $0 < i \leqslant r$ if $x_i = x_0^i + \cdots x_{k-1}^i p^{k-1}$ then set $\widehat{x}_i = x_i - x_0^i$. Thus $x = \pm\widehat{x}_1 \pm \cdots \pm \widehat{x}_r + \widehat{g}$, where $\widehat{g} = \pm x_0^1 \pm \cdots \pm x_0^r$. Now $\widehat{g} \in A^{k-\mathbb{N}}$ and $\widehat{g} \equiv 0 (\mathrm{mod}\ p)$, thus $\exists g \in A^{k-\mathbb{N}}$ such that $pg = \widehat{g}$. Since $\langle 1\rangle = A^{k-\mathbb{N}}$ we have $g \in \langle C\rangle$. Also we have $z_1, \cdots, z_r$

such that $pz_1 = \widehat{x}_1, \cdots, pz_r = \widehat{x}_r$. We see that $x = p(\pm z_1 \pm \cdots \pm z_r + g)$. Since $pz = x$ it follows that $p(z - [\pm z_1 \pm \cdots \pm z_r + g]) = 0$, so $p(z - [\pm z_1 \pm \cdots \pm z_r + g]) \in t(G) \subseteq A_I$. So if all of $z_1, \cdots, z_r$ lie in $A_I + \langle C \rangle$ then so does z, so this cannot happen. So at least one of $z_1, \cdots, z_r$ fails to lie in $A_I + \langle C \rangle$ and we may suppose that $z_1 \notin A_I + \langle C \rangle$. Now $pz_1 = \widehat{x_1} = x_1 - x_0^1 \in \langle C \rangle$ whence by lemma 4.4.1 we have $\langle C \cup \{z_1\} \rangle \cap A_I = \{0\}$. Thus adding z_1 will not violate $\langle C \cup \{z_1\} \rangle \cap A_I = \{0\}$. We check that $z_1 \in \bigcup_{b \in D} X_b$. Since $x_1 \in C \subseteq \bigcup_{b \in D} X_b$, there is $l \in \mathbb{N}$, $a' \in A^{k-\mathbb{N}}$ and $b' \in D$ such that $lx_1 = b' + a'$. Now $lpz_1 = l(x_1 - x_0^1) = b' + a' - x_0^1$ and $a' - x_0^1 \in A^{k-\mathbb{N}}$ so $z_1 \in \bigcup_{b \in D} X_b$ as required. If $z_1, \cdots, z_r \in A_I + \langle C \cup \{z_1\} \rangle$ then $\frac{m}{p} g \in A_I + \langle C \cup \{z_1\} \rangle$. Otherwise we may suppose that $z_2 \notin A_I + \langle C \cup \{z_1\} \rangle$ and we repeat the argument with z_2 in place of z_1 and $C \cup \{z_1\}$ in place of C. Continuing in this manner we may add sufficiently many $z_i' s$ so that $\frac{m}{p} g \in A_I + \langle C \cup \{z_1\} \cup \cdots \cup \{z_s\} \rangle$ and $A_I \cap \langle C \cup \{z_1\} \cup \cdots \cup \{z_s\} \rangle = 0$ for some $s \leqslant r \in \mathbb{N}$.

Suppose $q \neq p$ (case 2): We can use the same argument as above apart from noting that now $x_1, \cdots, x_r$ are all uniquely divisible by q by lemma 4.4.2. Therefore the translation part of the argument is not needed.

In either case we reduce the minimum value m to $\frac{m}{q}$ and so we may keep repeating the argument until $g \in A_I + \langle C \cup \{z_1\} \cup \cdots \cup \{z_t\} \rangle$ for some $t \in \mathbb{N}$ with $A_I \cap \langle C \cup \{z_1\} \cup \cdots \cup \{z_t\} \rangle = 0$ and $z_1, \cdots, z_t \in \bigcup_{b \in D} X_b$.

Lemma 4.4.7 *Suppose $G = C(p^k)$ for p a standard prime, k a nonstandard number and $A_I \leqslant G$ is the subgroup defined in 4.3.5 for some cut I such that $\mathbb{N} \leqslant I < k$. Suppose $D \subseteq G$ is an M-finite set such that $D + A_I = G$. Suppose $C \subseteq \bigcup_{b \in D} X_b$ is an M-finite set such that $\langle C \rangle \cap A_I = \{0\}$ and $1 = p^0 \in C$. It is straightforward to check that* $\mathrm{card}(A_I) = p^{k-I}$ *which we shall denote by J for convenience. Let $j \in J$ be arbitrary. Then there is $D' \subseteq D$ such that $D' + A_I = G$, $C \subseteq \bigcup_{b \in D'} X_b$ and $|D'| \leqslant \frac{|G|}{j}$.*

Proof. Recall that $A_I = \bigcup_{\alpha > I} A_\alpha$. Since $\mathrm{card}(A_I) = J$ it follows that there is some $\alpha > I$ such that $|A_\alpha| \geqslant j$. We have that $A_\alpha + D = G$. Set $r := |D| - \frac{|G|}{j}$. We perform an internal induction to remove r elements from D. Set $D_0 = D$ and suppose that for $l < r$ we have defined an M-finite set D_l such that:

(1) $|D_l| = |D| - l$

(2) $D_l + A_\alpha = G$

We will also ensure (by making a careful and definable choice using the monotonicity of X_b) that $C \subseteq \bigcup_{b \in D_l} X_b$. We are not allowed to assume this external condition in our induction hypotheses however. We shall simply show how to make the choices carefully and demonstrate at the end that the condition has been met. We have the following counting argument. $|D_l||A_\alpha| = (|D_l| - l)j > (|D| - r)j > j\frac{|G|}{j} = |G|$. So there must be elements $a_1, a_2 \in A_\alpha$ and $b_1, b_2 \in D_l$ such that $a_1 + b_1 = a_2 + b_2$. It is important that we choose the correct b to remove and we do this as follows. Let α_1 be least such that $\theta^{b_1}_{\alpha_1} \cap C \neq \varnothing$. Let α_2 be least such that $\theta^{b_2}_{\alpha_2} \cap C \neq \varnothing$. By relabelling if necessary we may suppose $\alpha_1 \geqslant \alpha_2$. We then set $D_{l+1} = D_l \setminus \{b_1\}$. It is clear that condition (1) holds for D_{l+1}. To establish (2) note that for each $g \in G$ we have $g = b + a$ for $b \in D_l$ and $a \in A_\alpha$. If $b \notin D_{l+1}$ then $b = b_1$, in which case $a_1 + b = a_2 + b_2$. Thus $g = b_2 + a_2 - a_1 + a$, A_α is a group and so $a_2 - a_1 + a \in A_\alpha$. Thus $g \in D_{l+1} + A_\alpha$. At the last stage of this internal induction we create D_r. Suppose that $C \nsubseteq \bigcup_{b \in D_r} X_b$. Since $C \subseteq \bigcup_{b \in D} X_b$ it follows that $\exists y_1 \in C$ such that $y_1 \notin \bigcup_{b \in D_r} X_b$ and $y_1 \in \bigcup_{b \in D} X_b$. Hence there is $b_1 \in D \setminus D_r$ such that $y_1 \in X_{b_1}$. Thus at some stage $l < r$ we had $b_1 \in D_l \setminus D_{l+1}$ and so there are $b_2 \in D_l$ $a_1, a_2 \in A_\alpha$ such that $a_1 + b_1 = a_2 + b_2$. Let α_1 be least such that $\theta^{b_1}_{\alpha_1} \cap C \neq \varnothing$. Let α_2 be least such that $\theta^{b_2}_{\alpha_2} \cap C \neq \varnothing$. Since $b_1 \notin D_{l+1}$ it follows that $\alpha_1 \geqslant \alpha_2$ but since $y_1 \in C \cap X_{b_1}$ it follows $C \cap X_{b_1} \neq \varnothing$. So for some m in $\mathbb{N}$ we have $\theta^{b_1}_m \cap C \neq \varnothing$ and since α_1 is the least element with this property it follows that $\alpha_1 \in \mathbb{N}$. So $\alpha_2 \in \mathbb{N}$ and thus $C \cap X_{b_2} \neq \varnothing$. Now by definition of X_{b_1}, X_{b_2} there are r_1, r_2 in $\mathbb{N}$ such that $r_1 y_1 \in b_1 + A^{k-\mathbb{N}}$ and $r_2 y_2 \in b_2 + A^{k-\mathbb{N}}$. It follows that $b_1, b_2 \in \langle C \rangle$ upon noting $A^{k-\mathbb{N}} = \langle 1 \rangle \leqslant \langle C \rangle$. Thus

$\langle C\rangle \ni b_2 - b_1 = a_2 - a_1 \in A_\alpha \leqslant A_I$ and this is a contradiction to the assumption in the lemma since $b_2 \neq b_1$. So we may set $D' = D_r$.

We are now ready to prove the main result of this section.

Theorem 4.4.8 *Let $M \models PA$ be countable and nonstandard. Let p be a standard prime, k a nonstandard number. Let $G = C(p^k)$. Recall the definition (4.3.5) of the group A_I for a cut I such that $\mathbb{N} \leqslant I < k$. Recall also definition 2.1.9 of second derivative $\partial^2(I)$ of a cut I. Suppose $\partial^2(I) > \mathbb{N}$ or $I = \mathbb{N}$. Then there is an M-countable group $B_I \leqslant G$ such that $A_I \oplus B_I = G$ and* $\mathrm{card}(B_I) = \frac{|G|}{\mathrm{card}(A_I)} = \frac{p^k}{p^{k-I}} = p^I$.

Proof. The proof proceeds by an induction on $\mathbb{N}$. By countability let $G = \{g_0, \cdots, g_n, \cdots\}_{n\in\mathbb{N}}$, let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence cofinal in I from above, and let $(b_n)_{n\in\mathbb{N}}$ be an increasing sequence cofinal in I from below. Recall that $A^{k-\mathbb{N}} := \{\pm x : x = x_0 + \cdots + x_n p^n; n \in \mathbb{N}\}$, that $A_I := \{x : x = x_i p^i + \cdots + x_{k-1}p^{k-1}; i > I\}$ is divisible and that $A_{k-\mathbb{N}} = t(G)$. Note that $\mathrm{card}(A_{k-\mathbb{N}}) = \mathbb{N}$ and so in the notation of lemma 4.4.3 we may take $\mathbb{N} < \gamma < \partial^2(I)$ (see later remark for the case that $\mathbb{N} = I$). Again in the sense of lemma 4.4.3 we may take $\mathbb{N} < \alpha, \beta \leqslant \partial^2(I)$.

The construction of B_I will proceed by induction. We set out our inductive hypotheses.

(1) There are M-finite sets $\{1\} = B_0 \subseteq \cdots \subseteq B_n$; $B'_n \subseteq \cdots B'_0 = G$.

(2) $\langle B_n\rangle \cap A_I = 0$; $B'_n + A_{a_n} = G$

(3) $|B_n|^{2\alpha}\gamma\beta \geqslant p^{b_n}$; $|B'_n| \leqslant p^{a_n}$

(4) $B_n \subseteq \bigcup_{b\in B'_n} X_b$ where X_b is as defined in 4.4.4.

(5) $g_n \in \langle B_n\rangle + A_I$

$B_n \to B_{n+1}$ step:

Our first part of the construction of B_{n+1} is to ensure that the cardinality condition (condition 3 in the inductive hypotheses) is satisfied. Since A_I is M-countable with $\text{card}(A_I) = p^{k-I}$ it follows that $p^{k-b_{n+1}} > \text{card}(A_I)$. Setting $p^{k-b_{n+1}} = i'$ in the notation of lemma 4.4.3 it follows that $\frac{|G|}{i'} = p^{b_{n+1}}$. Define $r \in \omega^M$ to be least such that $p^{b_{n+1}} \leqslant (r + |B_n|)^{2\alpha}\beta\gamma$. Then by lemma 4.4.3 there is an M-finite set $\{b_0, \cdots, b_{r-1}\} \subseteq B'_n \setminus B_n$ such that $\langle B_n \cup \{b_0, \cdots, b_{r-1}\}\rangle \cap A_I = \{0\}$. It follows that $B_n \cup \{b_0, \cdots, b_{r-1}\}$ satisfies (1),(2) and (3) for $n+1$. Since $\{b_0, \cdots, b_{r-1}\} \subseteq B'_n$ (4) is satisfied too.

It remains to ensure that (5) is satisfied for $n+1$. Set $D = B'_n$ and $C = B_n \cup \{b_0, \cdots, b_{r-1}\}$ in the notation of lemma 4.4.6. By lemma 4.4.6 we have $\{z_1, \cdots, z_t\} \subseteq \bigcup_{b \in B'_n} X_b$, for $t \in \mathbb{N}$, such that $\langle C \cup \{z_1, \cdots, z_t\}\rangle \cap A_I = \{0\}$ and $g_{n+1} \in \langle C \cup \{z_1, \cdots, z_t\}\rangle + A_I$. So we may set $B_{n+1} := B_n \cup \{b_0, \cdots, b_{r-1}\} \cup \{z_1, \cdots, z_t\}$.

$B'_n \to B'_{n+1}$ step:

In order to fully satisfy the induction axioms at stage $n+1$ we must remove elements from B'_n to satisfy condition (3) for $n+1$ without violating conditions (2) or (4) for $n+1$. Setting $p^{k-a_{n+1}} = j$ in the notation of lemma 4.4.7 it follows that $\frac{|G|}{j} = p^{a_{n+1}}$. Set $C = B_{n+1}$ and $D = B'_n$. Then by lemma 4.4.7 there is $D' \subseteq B'_n$ such that $D' + A_I = G$, $B_{n+1} \subseteq \bigcup_{b \in D'} X_b$ and $|D'| \leqslant \frac{|G|}{a_{n+1}}$. Set $B'_{n+1} = D'$.

Definition of B_I:

Finally we set $B_I = \bigcup_{n \in \mathbb{N}} \langle B_n \rangle$. To complete the proof we must now prove that B_I satisfies each of the following conditions.

(i) $B_I \cap A_I = \{0\}$.

(ii) $B_I + A_I = G$.

(iii) $\underline{\text{card}}(B_I) \geqslant p^I$.

(iv) $\overline{\text{card}}(B_I) \leqslant p^I$.

For (i) suppose $0 \neq g \in B_I \cap A_I$ then it follows there is m in $\mathbb{N}$ such that $g \in \langle B_m \rangle$ but this contradicts condition (2) of the inductive hypotheses at stage m. For (ii) let $g \in G$ be arbitrary, then there is m in $\mathbb{N}$ such that $g = g_m$. By condition (5) of the induction hypotheses $g_m \in \langle B_m \rangle + A_I$. Since $\langle B_m \rangle \subseteq B_I$ it follows that $g_m \in B_I + A_I$. For (iii) we must show $p^I \leqslant \underline{\text{card}}(B_I) \leqslant \overline{\text{card}}(B_I) \leqslant p^I$. Note that for all n in $\mathbb{N}$ $B_n \subseteq B_I$ so the first inequality will follow if we can show that $\{|B_n|\}_{n\in\mathbb{N}} \subseteq_{cf} p^I$. Fix $i \in I$. Since $\alpha, \beta, \gamma \in \partial^2(I)$ it follows that $(i+\beta\gamma)2\alpha \in I$ and so there is n in $\mathbb{N}$ such that $b_n \geqslant (i+\beta\gamma)2\alpha$. Then we have $|B_n|^{2\alpha}\beta\gamma \geqslant p^{b_n} \geqslant p^{(i+\beta\gamma)2\alpha} = (p^i)^{2\alpha}(p^{\beta\gamma})^{2\alpha} \geqslant (p^i)^{2\alpha}\beta\gamma$. And so $|B_n| \geqslant p^i$ as required. Note that α, β, γ were chosen to be below $\partial^2(I)$ in order for this part of the argument to work. If $I = \mathbb{N}$ then this cannot be done but in this case the use of α, β, γ and lemma 4.4.3 is not required because $P^I = \mathbb{N}$ and the lower cardinality of any externally infinite set (including B_I) is guaranteed to be greater than or equal to the standard cut. For (iv) fix some $i' > I$. If $I = \mathbb{N}$ then $\forall n \in \mathbb{N}$ $\frac{i'-n}{2n} > n$ and so by overspill there is a greater than $\mathbb{N}$ such that $\frac{i'-a}{2a} > a > \mathbb{N} = I$. If $I \neq \mathbb{N}$ then by assumption $\partial^2(I) > \mathbb{N}$ so we may fix some $\partial^2(I) > a > \mathbb{N}$ which will also have the property that $\frac{i'-a}{2a} > I$ by the definition of $\partial^2(I)$. Clearly for all n in $\mathbb{N}$ $n^{2n} < a$ and so by overspill there is c greater than $\mathbb{N}$ such that $c^{2c} < a$, fix this c. By lemma 4.4.5 $\forall n \in \mathbb{N} \forall g \in G |\theta^g_n| \leqslant c$ so by overspill there is d greater than $\mathbb{N}$ such that $\forall n \leqslant d \forall g \in G |\theta^g_n| \leqslant c$. Note that $\langle \bigcup_{b\in B'_n} \theta^b_d \rangle_c \supseteq B_I$ for each $n \in \mathbb{N}$. $|\bigcup_{b\in B'_n} \theta^b_d| \leqslant |B'_n|c$ and so $|\langle \bigcup_{b\in B'_n} \theta^b_d \rangle_c| \leqslant |B'_n|^{2c}c^{2c} \leqslant |B'_n|^{2c}a$. Since $\frac{i'-a}{2c} > \frac{i'-a}{2a} > I$ there is $m \in \mathbb{N}$ such that $a_m \leqslant \frac{i'-a}{2c}$. So there is B'_m with $|B'_m| \leqslant p^{a_m}$. Finally $|\langle \bigcup_{b\in B'_m} \theta^b_d \rangle_c| \leqslant |B'_m|^{2c}a \leqslant (p^{\frac{i'-a}{2c}})^{2c}a \leqslant p^i$ as required. This establishes $\overline{\text{card}}(B_I) \leqslant p^I$ and so $\underline{\text{card}}(B_I) = \overline{\text{card}}(B_I) = p^I$.

Example 4.4.9 *Let p be a standard prime and fix some $k > \mathbb{N}$. For every n in $\mathbb{N}$*

we have that $n^n < k$ *and so by overpsill there is* $c > \mathbb{N}$ *with* $c^c < k$. *It follows that the cut* $I := sup_{n\in\mathbb{N}}\{c^n\}$ *is below* k. *Furthermore* I *is closed under multiplication and so* $\partial^2(I) = I > \mathbb{N}$. *Consider* $A_I < C(p^k)$, *by theorem 4.4.8 there is an* M*-countable complement* B_I*for* A_I *in* G *with* $\mathrm{card}(B_I) = p^I$.

4.5 Generalized Complement Theorem

In this section we prove a version of theorem 4.4.8 in the case of a general M-finite abelian group G. We shall restrict our attention to finding an M-countable complement for the divisible part dG of G. The reason for this is that in the previous section we had an obvious family of monotonically definable subgroups, the A_I for cuts I such that $\mathbb{N} \leqslant I \leqslant k - \mathbb{N}$, to study; however in this general case there is no obvious analogy to this family and so we shall stick to the study of the group dG and its complement. By the basis theorem $G = \bigoplus_{i<b} C(p_i^{k_i})$, for primes p_i, integers k_i and an integer b; some or all of which may be nonstandard. From time to time it will be useful for us to think of elements of G in the following way. If $g \in G$ then $g = g_0 + \cdots + g_{b-1}$ with $g_0 \in C(p_0^{k_0}), \cdots, g_{b-1} \in C(p_{b-1}^{k_{b-1}})$. In p_i-adic form each $g_i = x_0^i + \cdots + x_{k_i-1}^i p_i^{k_i-1}$ where $0 \leqslant x_j^i < p_i$.

4.5.1 Pseudo Complement

We begin by defining an external subgroup of G in this general setup.

Definition 4.5.1 *Let* $G = \bigoplus_{i<b} C(p_i^{k_i})$ *be as above. Let* $\overline{v}_i(x)$ *be the function from* $C(p_i^{k_i}) \to \{0, 1, \cdots, k_i - 1, \infty\}$ *as defined in 4.3.3. Using these definable functions* $\overline{v}_i(x)$ *let us define* $H_n = \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge g_i = 0 \vee ((p_i \leqslant n) \wedge (\overline{v}_i(\pm g_i) \geqslant k_i - n))\}$. *Then* $H := \bigcup_{n\in\mathbb{N}} H_n$.

It will be shown later how this subgroup plays the role of a 'pseudo complement' to the divisible part $dG \leqslant G$ and it will be our starting point for the construction of a full complement. The proof will follow the basic plan of the proof of theorem 4.4.8 with this 'pseudo complement' H taking over the role of the group $A^{k-\mathbb{N}}$ and also of lemma 4.4.3. We need to prove some basic facts about H and that is what we shall do in next few lemmas.

We have the following characterization of H.

Lemma 4.5.2 $H = \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A^{k_i-\mathbb{N}}))\}$, *where each group* $A^{k_i-\mathbb{N}}$ *is the subgroup of the* i^{th} *summand of* G *in the sense of definition 4.3.5.*

It is worth noting that if $k_i \in \mathbb{N}$ (as well as p_i) then the definition simply gives $A^{k_i-\mathbb{N}} = C(p_i^{k_i})$ so this case does not cause a problem.

Proof. If $x \in H$ then there is $n \in \mathbb{N}$ such that $x \in H_n$ and thus $x = g_0 + \cdots + g_{b-1}$ such that $g_i = 0 \vee (p_i \leqslant n \wedge \overline{v}_i(\pm g_i) \geqslant k_i - n)$. If $g_i \neq 0$ then this is to say $p_i \in \mathbb{N}$ and $g_i \in A^{k_i-\mathbb{N}}$ and so $x \in \{g \in G : g = g_0 + \cdots + g_{b-1} : g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A^{k_i-\mathbb{N}})\}$.

For the converse suppose $y = g_0 + \cdots + g_{b-1}$ where $g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A^{k_i-\mathbb{N}})$. Now we have that $\forall \alpha > \mathbb{N} \forall i < b(g_i \neq 0 \implies p_i \leqslant \alpha \wedge \overline{v}_i(\pm g_i) \geqslant k_i - \alpha)$ and so by underspill there is $n \in \mathbb{N}$ with this property. It follows that $y \in H_n \subseteq H$.

The description of H given by lemma 4.5.2 will generally be of more use to us than the original definition 4.5.1. We choose definition 4.5.1 to make clear that H is up monotonically $\mathbb{N}$ definable.

Proposition 4.5.3 $H \leqslant G$.

Proof. This follows easily from the fact that each $A^{k_i-\mathbb{N}}$ is a group.

The following lemma gives some more properties of this group H which will be useful.

Lemma 4.5.4 *Let H be defined as in definition 4.5.1. Then:*

(1) $H \cap dG = \{0\}$

(2) $(\forall g \in G)(\forall n \in \mathbb{N})(\exists h \in H)(\exists x \in G)(nx = g + h)$

(3) $\forall h \in H\ \forall n \in \mathbb{N}\ \exists x \in G\ nx = h \implies (\exists y \in H\ ny = h)$.

Proof. (1) Let $0 \neq h \in H$. Then $h = h_0 + \cdots + h_{b-1}$ where $h_i = 0 \vee (p_i \in \mathbb{N} \wedge h_i \in A^{k_i - \mathbb{N}})$. There must be some $0 \leqslant j < b$ such that $h_j \neq 0$ and so $p_j \in \mathbb{N} \wedge h_j \in A_j^{k_j - \mathbb{N}}$. Thus $\pm h_j = x_r p_j^r + \cdots + x_s p_j^s$ for some $r, s \in \mathbb{N}$. It follows that h_j and h are not divisible by $p_j^{r+1} \in \mathbb{N}$ and so $h \notin dG$.

(2) Let $g \in G$, $n \in \mathbb{N}$. We may we write g as an M-finite sum as follows: $g = g_0 + \cdots + g_{b-1}$, with $g_0 \in C(p_0^{k_0}), \cdots, g_{b-1} \in C(p_{b-1}^{k_{b-1}})$. If $n < p_i$ then there is x_i in $C(p_i^{k_i})$ such that $nx_i = g_i$, so we consider only those $g_i s$ for which $n \geqslant p_i$. Write $g_i = x_0^i + \cdots + x_{k_i-1}^i p_i^{k_{i-1}}$ and let m_i be largest such that $p_i^{m_i} | n$. We set $h_i = x_0^i + \cdots + x_{m_i}^i p_i^{m_i}$, i.e. the first m_i components of g_i. Since $n \in \mathbb{N}$ it follows that $p_i \in \mathbb{N}$ and $m_i \in \mathbb{N}$ thus each h_i is an actual finite sum and thus belongs to $A^{k_i - \mathbb{N}}$. Moreover we select each h_i in a uniformly definable way and so the sum $\Sigma_{i:n \geqslant p_i} h_i$ is M-finite and thus a member of G. Set $h = -(\Sigma_{i:n \geqslant p_i} h_i)$ and we see that $g + h$ is then divisible in G by n.

(3) Let $h \in H$ and suppose that $n \in \mathbb{N}$ divides h in G. We also have $h = h_0 + \cdots + h_{b-1}$ where $h_i = 0 \vee (p_i \in \mathbb{N} \wedge h_i \in A^{k_i - \mathbb{N}})$. It follows that if $h_i \neq 0$ then n divides h_i in $C(p_i^{k_i})$. We may suppose that $h_i \in A^{k_i - \mathbb{N}}$ through virtue of $\overline{v}_i(+h_i) > k_i - \mathbb{N}$, as the other case is simular. Consider $\{x \in C(p_i^{k_i}) : nx = h_i\}$ and selecting the x for which $\overline{v}_i(x)$ is greatest, it follows that this x is in $A^{k_i - \mathbb{N}}$, let us call it x_i. As before, we select each x_i in a uniformly definable way and it follows that $\Sigma_{i:h_i \neq 0} x_i$ is an element of G and also of H. Setting $y = \Sigma_{i:h_i \neq 0} x_i$ we see that $ny = h$ as required.

Consider now the torsion subgroup of $tG \leqslant G$ and note that $tG = \bigcup_{n \in \mathbb{N}} \{g \in G : ng = 0\}$ so we see that it is up monotonically $\mathbb{N}$ definable. As with H we have the following characterization of tG.

Lemma 4.5.5 *Let G be an M-finite abelian group. Then* $tG = \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A_{k_i - \mathbb{N}}))\}$.

Proof. If $g \in tG$, then $g = g_0 + \cdots + g_{b-1}$ and $\exists n \in \mathbb{N} \forall i < b \; ng_i = 0$. It follows that $g_i \in A_{k_i-\mathbb{N}}$ and $g_i \neq 0 \implies p_i \in \mathbb{N}$. Thus we see that $tG \subseteq \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A_{k_i-\mathbb{N}}))\}$. For the converse suppose that $g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A_{k_i-\mathbb{N}}))$. If $\forall n \in \mathbb{N}(ng \neq 0)$ then $\forall n \in \mathbb{N}(n!g \neq 0)$ and so by overspill there is β greater than $\mathbb{N}$ such that $\beta!g \neq 0$. Thus there must be some j such that $0 \leqslant j < b$ such that $\beta!g_j \neq 0$. But $g_j \in A_{k_i-\mathbb{N}}$ and so there is m in $\mathbb{N}$ such that $p_j^m g_j = 0$ and since $p_j \in \mathbb{N}$ we have $p_j^m | \beta!$ which is a contradiction. So $g \in tG$ as required.

Lemma 4.5.5 has the following corollary which will be useful in future counting arguments.

Corollary 4.5.6 $\mathrm{card}(tG) = \mathrm{card}(H)$.

Proof. By lemma 4.5.5 $tG = \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A_{k_i-\mathbb{N}}))\}$ and by lemma 4.5.2 $H = \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A^{k_i-\mathbb{N}}))\}$. If $k_i \in \mathbb{N}$ then $A^{k_i-\mathbb{N}} = A_{k_i-\mathbb{N}} = C(p_i^{k_i})$. If $k_i > \mathbb{N}$ then $\mathrm{card}(A^{k_i-\mathbb{N}}) = \mathrm{card}(A_{k_i-\mathbb{N}}) = \mathbb{N}$. The corollary follows.

That dG is also monotonically definable is the subject of the following proposition.

Proposition 4.5.7 *Let G be any M-finite abelian group. Then* $dG = \{x \in G : (\forall n \in \mathbb{N})(\exists y \in G)(ny = x)\}$.

Proof. It is clear that $dG \subseteq X := \{x \in G : (\forall n \in \mathbb{N})(\exists y \in G)(ny = x)\}$ as dG is divisible. Let $x \in X$. Then $(\forall n \in \mathbb{N})(\exists y \in G)(n!y = x)$. By overspill $(\exists \beta > \mathbb{N})(\exists y \in G)(\beta!y = x)$. Assuming we fix such a β and such a y this allows us to define a canonical choice for x/n

for each $n \in \mathbb{N}$, namely $x/n := \frac{\beta!}{n}y$. It is clear that $\{q\beta!y : q \in \mathbb{Q}\}$ is a divisible subgroup of G containing x and so $x \in dG$ as required.

Lemma 4.5.8 *Let G be an M-finite abelian group. Let $H \leqslant G$ be as defined in 4.5.1. Write* $\text{card}(dG) = I$. *Then* $\text{card}(H) = \frac{|G|}{I}$.

Proof. Since H is monotonically definable it follows that $\underline{\text{card}}(H) = \overline{\text{card}}(H)$, so it suffices to prove $\underline{\text{card}}(H) \leqslant \frac{|G|}{I}$ and $\overline{\text{card}}(H) \geqslant \frac{|G|}{I}$. For the first suppose $\underline{\text{card}}(H) > \frac{|G|}{I}$. Then there is $i \in I$ and $H_i \subseteq H$ such that $|H_i| > \frac{|G|}{i}$. There is also M-finite $G_i \subseteq dG$ such that $|G_i| = i$. Since $H \cap dG = \{0\}$ it follows that $|H_i + G_i| = |H_i||G_i| \geqslant \frac{|G|}{i} > |G|$ which is a contradiction. For the second suppose $\overline{\text{card}}(H) < \frac{|G|}{I}$. There are $i' > I$ and $H_{i'} \supseteq H$ such that $|H_{i'}| < \frac{|G|}{i'}$ and also $G_i \supseteq dG$ with $|G_{i'}| = i'$. Since $H_{i'} \supseteq H$ and is M-finite we can overspill the formula $\forall n \in \mathbb{N} \forall g \in G \exists h \in H_{i'} \exists x \in G(n!x = g - h)$ (which is true by lemma 4.5.4 (2)) to obtain $\beta > \mathbb{N}$ such that $\forall g \in G \exists h \in H_{i'} \exists x \in G(\beta!x = g - h)$. Since $\beta!$ is divisible by any $n \in \mathbb{N}$ it follows that $\forall n \in \mathbb{N} \forall g \in G \exists h \in H_{i'} \exists x \in G(nx = g - h)$ and by proposition 4.5.7 we have that $g - h \in dG$. So $\forall g \in G \exists h \in H_{i'} \exists d \in dG(g = d + h)$ i.e. $G = dG + H_{i'}$ and so $G_{i'} + H_{i'} = G$. Thus $|G| \leqslant |G_{i'}||H_{i'}| < \frac{|G|}{i'}i' = |G|$ which is a contradiction.

4.5.2 Proof of the Generalized Complement Theorem

What follows is a series of technical lemmas which between them provide the main steps of the proof of the generalized complement theorem. Many of the lemmas are analogues of the lemmas proved in the previous section. The proof itself will then just be an external induction on $\mathbb{N}$ using the lemmas below to satisfy the induction at stage $n + 1$. Throughout this section G will be an M-finite abelian group and H will be the subgroup of G in definition 4.5.1.

Definition 4.5.9 *Let $b \in G$. Then $X_b := \{g \in G : \exists r \in \mathbb{N}\ rg \in b + H\}$. In other words an element of X_b is a translation of b by any element of H followed by a division by any*

$r \in \mathbb{N}$*; if such a division is possible we may pick any of the possibilities for this division that we wish.*

We observe that X_b is up monotonically definable by some $(\theta^b, \mathbb{N})$. As observed above H is up monotonically $\mathbb{N}$ definable. Thus we may define $\theta^b_n = \{g \in G : \exists r \leqslant n \, rg \in b+H_n\}$ and we see that $X_b = \bigcup_{n\in\mathbb{N}} \theta^b_n$. The next lemma is the analogue of lemma 4.4.6, the main difference being that we specifically include the group H in the statement $\langle C \cup \{z_1, \cdots, z_t\} \cup H\rangle \cap dG = \{0\}$, and of course A_I is replaced by dG. It was not necessary to include $A^{k-\mathbb{N}}$ in lemma 4.4.6 because $\langle 1\rangle = A^{k-\mathbb{N}}$ and so it was included by implication.

Lemma 4.5.10 *Suppose* $D \subsetneq G$ *is an* M*-finite set such that* $D + dG = G$*. Suppose* $C \subseteq \bigcup_{b\in D} X_b$ *is an* M*-finite set such that* $\langle C \cup H\rangle \cap dG = \{0\}$*. Let* $g \in G$ *be an arbitrary element. Then there is a set* $\{z_1, \cdots, z_t\} \subseteq \bigcup_{b\in D} X_b$*, for* $t \in \mathbb{N}$*, such that* $\langle C \cup \{z_1, \cdots, z_t\} \cup H\rangle \cap dG = \{0\}$ *and* $g \in \langle C \cup \{z_1, \cdots, z_t\} \cup H\rangle + dG$*.*

Proof. Let us suppose that $g \notin \langle C \cup H\rangle + dG$, as otherwise we are done (with $t = 0$). If $\forall m \in \mathbb{N} \, mg \notin \langle C \cup H\rangle + dG$ then we may write $g = a+b$ for $a \in dG$ and $b \in D$. It is easy to check in this case that the set $\{b\}$ has the desired properties; $b \in X_b$ because we can take $r = 1$ and $0 \in H$. So let us suppose that $m \in \mathbb{N}$ is the least number for which $mg = a + b$ with $a \in dG$ and $b \in \langle C \cup H\rangle$. Let p be a prime dividing m and by divisibility let $a_1 \in dG$ be such that $pa_1 = a$. Set $z = \frac{m}{p} g - a_1$; by minimality of m, $z \notin dG + \langle C \cup H\rangle$ and $pz = b$. Now since $b \in \langle C \cup H\rangle$ $\exists r \in \mathbb{N} \exists b_1, \cdots, b_r \in C \exists h \in H$ such that $b = \pm b_1 \pm \cdots \pm b_r + h$. By lemma 4.5.4 for each $1 \leqslant i \leqslant r$ $\exists h_i \in H \exists z_i \in G$ $pz_i = b_i + h_i$. Set $\widehat{b_i} = b_i + h_i$ and then $b = \pm\widehat{b}_1 \pm \cdots \pm \widehat{b}_r + h_{new}$, where $h_{new} = \pm h_1 \cdots \pm h_r + h$ which belongs to H since the latter is a group and $r \in \mathbb{N}$. Since $p(z \mp z_1 \mp \cdots \mp z_r) = h_{new}$ it follows by lemma 4.5.4 that there is h' in H such that $ph' = h_{new}$. In general there will be more than one choice for each of the z_i as we can always add or subtract any x for which $px = 0$. Note that h' above does not depend on the choice of the z_i. Let us suppose that we picked $z_1, \cdots, z_{r-1}$

in an arbitrary fashion but we may choose $\pm z_r = z - h' \mp z_1 \mp \cdots \mp z_{r-1}$. It is easy to check that $pz_r = \widehat{b}_r$ and importantly $\pm z_1 \pm \cdots \pm z_r + h' = z$ so that if $z_1, \cdots, z_r \in dG + \langle C \cup H \rangle$ then so does z. So some z_i fails to lie in $dG + \langle C \cup H \rangle$ but $pz_i = \widehat{b}_i = b_i + h_i \in \langle C \cup H \rangle$. By lemma 4.4.1 $\langle C \cup H \cup \{z_i\} \rangle \cap dG = \{0\}$. Since $b_i \in \bigcup_{b \in D} X_b$ there are $l \in \mathbb{N}$, $\widehat{h} \in H$ and $b' \in D$ such that $lb_i = b' + \widehat{h}$. Now $pz_i = b_i + h_i$ and so $plz_i = b' + \widehat{h} + lh_i$ with $\widehat{h} + lh_i \in H$ and so $z_i \in \bigcup_{b \in D} X_b$. This establishes that we can add z_i. Now if $z \in dG + \langle C \cup \{z_i\} \cup H \rangle$ then fine and if not we simply repeat the procedure above to add a different z_j. After at most r steps we will have $z \in dG + \langle C \cup Z \cup H \rangle$ and $dG \cap \langle C \cup Z \cup H \rangle = \{0\}$ where $Z \subseteq \{z_1, \cdots, z_r\}$. By the definition of z we have $\frac{m}{p} g \in dG + \langle C \cup Z \cup H \rangle$ and so the original minimum value m has been reduced. Thus we may keep repeating this whole procedure to obtain the desired result.

The next lemma involves an internal induction and is the analogue of lemma 4.4.7. The main difference being again being that H has to be explicitly included in the statement and that A_I is replaced by dG.

Lemma 4.5.11 *Suppose $D \subseteq G$ is an M-finite set such that $D + dG = G$. Suppose $C \subseteq \bigcup_{b \in D} X_b$ is an M-finite set such that $\langle C \cup H \rangle \cap dG = \{0\}$. Let* $\mathrm{card}(dG) = I$ *and let $i \in I$ be arbitrary. Then there is $D' \subseteq D$ such that $D' + dG = G$, $C \subseteq \bigcup_{b \in D'} X_b$ and $|D'| \leqslant \frac{|G|}{i}$.*

Proof. By proposition 4.5.7 $dG = \bigcup_{\alpha > \mathbb{N}} dG_\alpha$ where $dG_\alpha = \{g \in G : \forall n \leqslant \alpha \exists x \in G \; nx = g\}$. Since $\mathrm{card}(dG) = I$ it follows that there is some $\alpha > \mathbb{N}$ such that $|dG_\alpha| \geqslant i$. We have that $dG_\alpha + D = G$. Set $r := |D| - \frac{|G|}{i}$. We perform an internal induction to remove r elements from D. Set $D_0 = D$ and suppose that for $l < r$ we have defined an M-finite set D_l such that:

(1) $|D_l| = |D| - l$

(2) $D_l + dG_\alpha = G$

We will also ensure (by making a careful and definable choice using the monotonicity of X_b) that $C \subseteq \bigcup_{b \in D_l} X_b$. We are not allowed to assume this external condition in our induction hypotheses however. We shall simply show how to make the choices carefully and demonstrate at the end that the condition has been met. We have the following counting argument: $|D_l||dG_\alpha| = (|D_l| - l)i > (|D| - r)i > i\frac{|G|}{i} = |G|$. So there must be elements $a_1, a_2 \in dG_\alpha$ and $b_1, b_2 \in D_l$ such that $a_1 + b_1 = a_2 + b_2$. It is important that we choose the correct b to remove and we do this as follows. Let α_1 be least such that $\theta^{b_1}_{\alpha_1} \cap C \neq \varnothing$. Let α_2 be least such that $\theta^{b_2}_{\alpha_2} \cap C \neq \varnothing$. By relabelling if necessary we may suppose $\alpha_1 \geqslant \alpha_2$. We then set $D_{l+1} = D_l \setminus \{b_1\}$. It is clear that condition (1) holds for D_{l+1}. To establish (2) note that for each $g \in G$ we have $g = b + a$ for $b \in D_l$ and $a \in dG_\alpha$. If $b \notin D_{l+1}$ then $b = b_1$, in which case $a_1 + b = a_2 + b_2$. Thus $g = b_2 + a_2 - a_1 + a$, dG_α is a group and so $a_2 - a_1 + a \in dG_\alpha$. Thus $g \in D_{l+1} + dG_\alpha$. At the last stage of this internal induction we create D_r. Suppose that $C \nsubseteq \bigcup_{b \in D_r} X_b$. Since $C \subseteq \bigcup_{b \in D} X_b$ it follows that $\exists y_1 \in C$ such that $y_1 \notin \bigcup_{b \in D_r} X_b$ and $y_1 \in \bigcup_{b \in D} X_b$. Hence there is $b_1 \in D \setminus D_r$ such that $y_1 \in X_{b_1}$. Thus at some stage $l < r$ we had $b_1 \in D_l \setminus D_{l+1}$ and so there are $b_2 \in D_l$ $a_1, a_2 \in dG_\alpha$ such that $a_1 + b_1 = a_2 + b_2$. Let α_1 be least such that $\theta^{b_1}_{\alpha_1} \cap C \neq \varnothing$. Let α_2 be least such that $\theta^{b_2}_{\alpha_2} \cap C \neq \varnothing$. Since $b_1 \notin D_{l+1}$ it follows $\alpha_1 \geqslant \alpha_2$ but since $y_1 \in C \cap X_{b_1}$ it follows that $C \cap X_{b_1} \neq \varnothing$. So there is m in $\mathbb{N}$ such that $\theta^{b_1}_m \cap C \neq \varnothing$ and since α_1 is the least such thing $\alpha_1 \in \mathbb{N}$. So $\alpha_2 \in \mathbb{N}$ and thus $C \cap X_{b_2} \neq \varnothing$. Now by definition of X_{b_1}, X_{b_2} there are r_1, r_2 in $\mathbb{N}$ such that $r_1 y_1 \in b_1 + H$ and $r_2 y_2 \in b_2 + H$. It follows that $b_1, b_2 \in \langle C \cup H \rangle$. Thus $\langle C \cup H \rangle \ni b_2 - b_1 = a_2 - a_1 \in dG_\alpha \leqslant dG$ and this is a contradiction to the assumption in the lemma upon noting that $b_2 \neq b_1$. So we may set $D' = D_r$.

Lemma 4.5.12 *Let D, X_b and $\frac{|G|}{I}$ be defined as in lemma 4.5.11. Then the set $\langle \bigcup_{b \in D} X_b \rangle$ is up monotonically definable by $\mathbb{N}$. Moreover* $\operatorname{card}(\langle \bigcup_{b \in D} X_b \rangle) \leqslant |D|^{\mathbb{N}}$.

Proof. Recall from definition 4.5.1 that H is up monotonically $\mathbb{N}$ definable with $H = \bigcup_{m \in \mathbb{N}} H_m$. Define $\alpha_m := \{g \in G : \exists r \leqslant m \ rg \in D + H_m\}$. Define $\langle \alpha_m \rangle_m = \{g \in G :$

$g = \pm g_1 \pm \cdots \pm g_m$ where $g_1, \cdots, g_m \in \alpha_m\}$. It is easy to see that $\langle\alpha_m\rangle_m$ is definable and $\langle\bigcup_{b\in D} X_b\rangle = \bigcup_{m\in\mathbb{N}}\langle\alpha_m\rangle_m$. In order the verify the second claim of the lemma we must bound the size of α_m and $\langle\alpha_m\rangle_m$. We bound above the number of elements as follows. Firstly $|D + H_m| \leqslant |D||H_m|$. Then we are allowed to 'divide' by any $r \leqslant m$ so there are m choices. This is assuming the division is possible but we are calculating an upper bound so it doesn't matter if it is not. Finally we ask how many choices are there for a particular element divided by a particular natural number m. The answer is $|tG_m|$ where $tG_m := \{g \in G : mg = 0\} \subseteq tG$. Thus $|\alpha_m| \leqslant m|D||H_m||tG_m|$. Now it is easy to see that $|\langle\alpha_m\rangle_m| \leqslant (2m|D||H_m||tG_m|)^m$. Now $\mathrm{card}(tG) = \mathrm{card}(H) = \frac{|G|}{I}$ and $2m \in \mathbb{N} \subseteq \frac{|G|}{I}$, but $|D| > \frac{|G|}{I}$. This means that $|H_m|, |tG_m|, 2m < |D|$ for all $m \in \mathbb{N}$. Thus $|\langle\alpha_m\rangle_m| \leqslant |D|^{4m}$ for any $m \in \mathbb{N}$. It follows that $\mathrm{card}(\bigcup_{m\in\mathbb{N}}\langle\alpha_m\rangle_m) \leqslant |D|^{\mathbb{N}}$ as required.

We are now ready to prove the main result of this section.

Theorem 4.5.13 *Let $M \models PA$ be countable and nonstandard. Let G be an M-finite abelian group. Let dG be the divisible part of G. Let I be the initial segment such that* $\mathrm{card}(dG) = I$. *If $\frac{|G|}{I} \models \cdot$ then there is an M-countable group, B, such that $dG \oplus B = G$ and* $\mathrm{card}(B) = \frac{|G|}{I}$.

Proof. The proof proceeds by an induction on $\mathbb{N}$. By countability let $G = \{g_0, \cdots, g_n, \cdots\}_{n\in\mathbb{N}}$, and let $\{a_n\}_{n\in\mathbb{N}}$ be an increasing sequence cofinal in I. We now state the induction hypotheses.

(1) There are finite chains of M-finite sets $B_0 \subseteq \cdots \subseteq B_n$ and $B'_n \subseteq \cdots \subseteq B'_0$

(2) The sets B_n and B'_n satisfy the following conditions; $\langle B_n \cup H\rangle \cap dG = \{0\}$ and $B'_n + dG = G$.

(3) The set B'_n satisfies to the following cardinality condition; $|B'_n| \leqslant \frac{|G|}{a_n}$.

(4) The sets B_n and B'_n have the following relationship to each other; $B_n \subseteq \bigcup_{b\in B'_n} X_b$.

(5) At the n^{th} stage we add the element g_n to the sum; $g_n \in \langle B_n \cup H \rangle + dG$.

Firstly we add elements to B_n to create B_{n+1} such that $g_{n+1} \in \langle B_{n+1} \cup H \rangle + dG$. By lemma 4.5.10 we have $\{z_1, \cdots, z_t\} \subseteq \bigcup_{b \in B'_n} X_b$, for $t \in \mathbb{N}$, such that $\langle B_n \cup \{z_1, \cdots, z_t\} \cup H \rangle \cap dG = \{0\}$ and $g \in \langle B_n \cup \{z_1, \cdots, z_t\} \cup H \rangle + dG$. So we may set $B_{n+1} := B_n \cup \{z_1, \cdots, z_t\}$ and we may replace B_n by B_{n+1} in (1) – (5). In order to fully satisfy the induction axioms at stage $n+1$ we must remove elements from B'_n to satisfy condition (3) for $n+1$ without violating conditions (2) or (4) for $n+1$. In the notation of lemma 4.5.11 replace i by a_{n+1}, C by B_{n+1} and D by B'_n (this is fine as C, D were arbitrary sets satisfying the conditions that B_{n+1}, B'_n satisfy) then there is $D' \subseteq B'_n$ such that $D' + dG = G$, $B_{n+1} \subseteq \bigcup_{b \in D'} X_b$ and $|D'| \leqslant \frac{|G|}{a_{n+1}}$. Set $B'_{n+1} = D'$. This completes the induction. Finally we set $B = \bigcup_{n \in \mathbb{N}} \langle B_n \cup H \rangle$. To complete the proof we must now prove that B satisfies each of the following conditions.

(i) $B \cap dG = \{0\}$.

(ii) $B + dG = G$.

(iii) $\underline{\text{card}}(B) \geqslant \frac{|G|}{I}$.

(iv) $\overline{\text{card}}(B) \leqslant \frac{|G|}{I}$.

For (i) suppose $0 \neq g \in B \cap dG$ then it follows there is m in $\mathbb{N}$ such that $g \in \langle B_m \cup H \rangle$ but this contradicts condition (2) of the inductive hypothesis at stage m. For (ii) let $g \in G$ be arbitrary, then there is m in $\mathbb{N}$ such that $g = g_m$. By condition (5) of the induction hypothesis $g_m \in \langle B_m \cup H \rangle + dG$. Since $\langle B_m \cup H \rangle \subseteq B$ it follows that $g_m \in B + dG$. For (iii) we have, by lemma 4.5.8, that $\underline{\text{card}}(H) = \frac{|G|}{I}$ and since $H \leqslant B$ the result follows. (iv) requires significantly more work and this is where we shall need the assumption $\frac{|G|}{I} \models \cdot$ and lemma 4.5.12. By (4) of the inductive hypothesis we have that $B_n \subseteq \bigcup_{b \in B'_n} X_b$. Also $H \subseteq X_0 := \{g \in G : \exists r \in \mathbb{N}\ rg \in H\}$ and we can assume $0 \in B'_n$ so $B_n \cup H \subseteq \bigcup_{b \in B'_n} X_b$.

From (1) and (4) it follows that $\bigcup_{m\in\mathbb{N}}(B_m \cup H) \subseteq \bigcup_{b\in B'_n} X_b$ and so $B = \bigcup_{m\in\mathbb{N}}\langle B_m \cup H\rangle = \langle\bigcup_{m\in\mathbb{N}}(B_m \cup H)\rangle \subseteq \langle\bigcup_{b\in B'_n} X_b\rangle$. The latter has upper cardinality at most $|B'_n|^{\mathbb{N}}$ by lemma 4.5.12. Now by condition (3) of the inductive hypothesis given $i \in I$, we can take n large enough so that $|B'_n| \leqslant \frac{|G|}{i}$. For convenience let us denote $\frac{|G|}{I}$ by J. It follows that $\overline{\text{card}}(B) \leqslant \inf\{j'^{n'} : j' > J, n' > \mathbb{N}\}$. Now J is the cardinality of an up $\mathbb{N}$ monotonically definable set, namely tG and H. Observe that the definable function $f(n) = |H_n|$ has $\sup\{f(n) : n \in \mathbb{N}\} = J$. Let us suppose that $J < k < \inf\{j'^{n'} : j' > J, n' > \mathbb{N}\}$. Since $J \vDash \cdot$ it follows that $\forall n \in \mathbb{N}\ f(n)^n < k$ and so by overspill there is $n' > \mathbb{N}$ such that $f(n')^{n'} < k$. But $f(n') > J$ and so this contradicts $k < \inf\{j'^{n'} : j' > J, n' > \mathbb{N}\}$. Thus $J = \inf\{j'^{n'} : j' > J, n' > \mathbb{N}\}$ and we are done.

In the following subsection we will discuss the closure condition and present a few remaining questions. The following example demonstrates a non trivially application of theorem 4.5.13 and some of the preceding lemmas.

Example 4.5.14 *Let $b > \mathbb{N}$ and let $p_0, \cdots, p_{b-1}$ be an enumeration of the first b primes in M. Let $k > \mathbb{N}$ be any nonstandard element and set $G = \bigoplus_{i<b} C(p_i^{k_i})$ where $k_i = k$ for every $i < b$. It is not necessary for the powers of the p_i to all be the same but we do this for simplicity. By lemma 4.5.2 $H = \{g \in G : g = g_0 + \cdots + g_{b-1} \wedge (g_i = 0 \vee (p_i \in \mathbb{N} \wedge g_i \in A^{k_i - \mathbb{N}}))\}$ which in this case is described more simply as $H = \{g \in G : g = g_0 + \cdots + g_n \wedge n \in \mathbb{N} \wedge g_i \in A^{k_i - \mathbb{N}}\}$. Since each $A^{k_i-\mathbb{N}}$ has cardinality $\mathbb{N}$ it is relatively straightforward to see that $\text{card}(H) = \mathbb{N}$ in this case. By lemma 4.5.8 we have that $\text{card}(H) = \frac{|G|}{\text{card}(dG)}$ and so $\text{card}(dG) = \frac{|G|}{\text{card}(H)} = \frac{(p_0 p_1 \cdots p_b)^k}{\mathbb{N}}$. The element $g = 1_0 + \cdots + 1_b$ (where 1_i denotes the element 1 in the group $C(p_i^k)$) does not lie in $H + dG$. If $g = h + d$ for $h \in H$ and $d \in dG$ then $h = h_0 + \cdots + h_n$ for $n \in \mathbb{N}$ and $h_i \in A^{k_i - \mathbb{N}}$, but then the $(n+1)^{th}$ component of $d = g - h$ is 1_{n+1} which is not divisible by $p_{n+1} \in \mathbb{N}$. Thus $H + dG \neq G$ and so theorem 4.5.13 does real work for us in extending H to a full complement. Since $\mathbb{N} \vDash \cdot$ we can apply theorem 4.5.13 to conclude there is an M-countable group B such that*

$H < B < G$, $dG \oplus B = G$ *and* $\operatorname{card}(B) = \mathbb{N}$.

4.5.3 A Discussion of the Closure Condition

The aim of this section is to show that the assumption $J = \operatorname{card}(tG) = \operatorname{card}(H) = \frac{|G|}{I} \vDash \cdot$ in theorem 4.5.13 is relatively harmless and only reduces the generality slightly. Firstly it is worth noting that without this assumption most of the proof still goes through. It's just at the end all we can conclude is that $\overline{\operatorname{card}}(B) \leqslant J^{\mathbb{N}}$ rather than $\leqslant J$. The exponent $\mathbb{N}$ is essentially the result of an element of uncertainty when we have to pass from $\bigcup_{b\in B'_n} X_b$ to $\langle \bigcup_{b\in B'_n} X_b \rangle$ and it is quite possible there may be a more efficient counting argument or means of bounding B above that obviates this difficulty.

Question 4.5.15 *Is it possible to improve the bound* $\overline{\operatorname{card}}(B) \leqslant J^{\mathbb{N}}$ *to* $\overline{\operatorname{card}}(B) \leqslant J$ *by a counting argument or a slicker proof of theorem 4.5.13? One possibility might be to make more careful choices in lemma 4.5.11 in order to minimize the difference in size between* $\bigcup_{b\in B'_n} X_b$ *and* $\langle \bigcup_{b\in B'_n} X_b \rangle$.

Irrespective of the answer to 4.5.15, there are lots of cases where the choice of M-finite abelian group G forces $J \vDash \cdot$. The following proposition provides some of these cases.

Proposition 4.5.16 *Let* G *be an* M*-finite abelian group;* $G = \bigoplus_{i<b} C(p_i^{k_i})$. *If* $\operatorname{card}(\{i : p_i^{k_i} \in \mathbb{N}\}) \leqslant \mathbb{N}$ *then (assuming* tG *is not* M*-finite)* $J \vDash \cdot$.

In other words if the number of actually finite summands is sufficiently small then $J \vDash \cdot$. There are a number of possible conditions on G which will imply this is the case. For example G may have no actually finite direct summands or all primes might be distinct; $i \neq j \implies p_i \neq p_j$.

Proof. Let $j \in J$. Since $J = \operatorname{card}(tG)$ there is a least $m \in \mathbb{N}$ such that $|t_m G| \geqslant j$, where $t_m G = \{g \in G : mg = 0\}$. For each $i < b$ we define r_i to be greatest such that

$p_i^{r_i}|m$. Now we definably split G into two parts $G = G_L \bigoplus G_R$; $G_L = \bigoplus_{r_i > \frac{k_i}{2}} C(p_i^{k_i})$ and $G_R = \bigoplus_{r_i \leqslant \frac{k_i}{2}} C(p_i^{k_i})$ as i ranges over $\{0, \cdots, b-1\}$. We claim $t_mG \cap G_L = \{g \in G_L : mg = 0\}$ is actually finite. If g_i is a non-zero component of $g \in t_mG \cap G_L \subseteq tG$ it follows that $p_i \in \mathbb{N}$. Moreover it follows that $r_i \in \mathbb{N}$ and so $k_i < 2r_i \in \mathbb{N}$. Thus every summand of G_L which contains non-zero elements of t_mG has an actually finite order $p_i^{k_i} \in \mathbb{N}$, and since $\mathrm{card}(\{i : p_i^{k_i} \in \mathbb{N}\}) \leqslant \mathbb{N}$ it follows that there are only finitely many such summands. Let $|t_mG \cap G_L| = n \in \mathbb{N}$ say, then $|t_mG \cap G_R| = \frac{|t_mG|}{n}$. Our next claim is that every element of $t_mG \cap G_R$ is divisible by m. Let $g \in t_mG \cap G_R$ and it suffices to consider the i^{th} component $g_i \in C(p_i^{k_i})$. Since $mg_i = 0$ and r_i is largest such that $p_i^{r_i}|m$, g_i has the following p_i-adic form: $g_i = x_{k_i - r_i} p_i^{k_i - r_i} + \cdots + x_{k_i - 1} p_i^{k_i - 1}$. Since $k_i > 2r_i$ the element $g_i' = x_{k_i - r_i} p_i^{k_i - 2r_i} + \cdots + x_{k_i - 1} p_i^{k_i - r_i - 1} \in C(p_i^{k_i})$ and $p_i^{r_i} g_i' = g_i$. Since $(m/p_i^{r_i}, p_i) = 1$ it follows that g_i' is divisible by $m/p_i^{r_i}$ and so g_i is divisible by m. This completes the proof of the claim. We now form the set $Y := \{y \in G_R : my \in t_mG \cap G_R\}$. Suppose that $my_1, my_2, x_1, x_2 \in t_mG \cap G_R$ and $my_1 \neq my_2$, $x_1 \neq x_2$. If $y_1 + y_2 = x_1 + x_2$ then $m(y_1 - y_2) = m(x_2 - x_1) = 0$ so $my_1 = my_2$ which is a contradiction. Thus $|Y + t_mG \cap G_R| \geqslant |t_mG \cap G_R|^2 = \frac{|t_mG|^2}{n^2} \geqslant \frac{j^2}{n^2}$. Thus $\frac{j^2}{n^2} \in J$. Since $J \vDash +$ (by lemma 3.1.3) and $n^2 \in \mathbb{N}$ it follows $j^2 \in J$ and the proposition follows.

We shall now construct an example of an M-finite abelian group G for which $J = \mathrm{card}(tG)$ is not closed under multiplication. It is easy to do this by taking any M-finite (but not actually finite) abelian group G and adding a large M-finite number of actually finite direct summands as follows.

Example 4.5.17 *Let $G = \bigoplus_{i<b} C(p_i^{k_i})$ be any M-finite abelian group with* $\mathrm{card}(tG) = J$ *with J not M-finite. Let $\gamma > J$. Fix some prime $p \in \mathbb{N}$ and some (any) integer $k \in \mathbb{N}$. Let $G' = \bigoplus_{j<\gamma} C(p_j^{k_j}) \bigoplus G$ where $p_j = p$ and $k_j = k$ for $0 \leqslant j < \gamma$. Then* $\mathrm{card}(tG') = \gamma \cdot J$ *which is not closed under multiplication as $\gamma^2 \notin \gamma \cdot J$.*

This is not a particularly interesting example as an M-countable complement would just be a direct sum of an M-countable complement for the original group G together with all the finite direct summands. In order for the example to be more meaningful we shall ensure that for any pair of M-finite subgroups $G_1, G_2 \leqslant G$ such that $G_1 \bigoplus G_2 = G$, either $\mathrm{card}(tG_1) \nvDash \cdot$ and tG_1 is not M-finite or $\mathrm{card}(tG_2) \nvDash \cdot$ and tG_2 is not M-finite.

Example 4.5.18 *Fix some $\beta > \mathbb{N}$. We define a decreasing sequence $\alpha_1, \cdots, \alpha_\beta$. Let $\alpha_\beta = \beta$ and define $\alpha_{n-1} = \beta^{\alpha_n}$. Let $p_1, \cdots, p_\beta$ be an enumeration of the first β primes in M. Let $G = \alpha_1 C(p_1) \bigoplus \alpha_2 C(p_2) \bigoplus \cdots \bigoplus \alpha_\beta C(p_\beta)$, where $\alpha_n C(p_n)$ means a direct sum of α_n copies of $C(p_n)$. By lemma 4.5.5 we have that $tG = \{g \in G : g = g_1 + \cdots + g_b \wedge (g_i = 0 \vee p_i \in \mathbb{N})\}$ where $b = \alpha_1 + \cdots + \alpha_\beta$ and so discarding all cases where $g_i = 0$ we see that $tG = \{g \in G : g = g_1 + \cdots + g_r\}$ where $r = \alpha_1 + \cdots + \alpha_n$ for $n \in \mathbb{N}$, so $J = \mathrm{card}(tG) = \sup_{n \in \mathbb{N}} p_1^{\alpha_1} \cdots p_n^{\alpha_n}$. We claim that $J \nvDash \cdot$. Clearly $p_1^{\alpha_1} \in J$ and so also $\alpha_1 \in J$. If $\alpha_1 p_1^{\alpha_1} \in J$ then there is $m \in \mathbb{N}$ such that $\alpha_1 p_1^{\alpha_1} \leqslant p_1^{\alpha_1} \cdots p_m^{\alpha_m}$ whence $\alpha_1 \leqslant p_2^{\alpha_2} \cdots p_m^{\alpha_m}$. However $p_m^m \in \mathbb{N}$ and so $p_2^{\alpha_2} \cdots p_m^{\alpha_m} \leqslant (p_m^m)^{\alpha_2} < \beta^{\alpha_2} = \alpha_1$. Thus $\alpha_1 p_1^{\alpha_1} \notin J$ and so $J \nvDash \cdot$. Moreover suppose that $G_1, G_2 \leqslant G$ are M-finite subgroups such that $G_1 \bigoplus G_2 = G$. If tG_1 is M-finite then there exists $m \in \mathbb{N}$ and $m' > \mathbb{N}$ such that $G_1 \leqslant \alpha_1 C(p_1) \bigoplus \cdots \bigoplus \alpha_m C(p_m) \bigoplus \alpha'_m C(p'_m) \bigoplus \cdots \bigoplus \alpha_\beta C(p_\beta)$. Thus it follows that $\alpha_{m+1} C(p_{m+1}) \bigoplus \cdots \bigoplus \alpha_{m'-1} C(p_{m'-1}) = G'_2 \leqslant G_2$. Hence tG_2 is not M-finite and the same argument as above gives that $\mathrm{card}(tG'_2) \nvDash \cdot$. Since $tG_2 = tG'_2 \bigoplus H$ where H is an M-finite subgroup of $\alpha_1 C(p_1) \bigoplus \cdots \bigoplus \alpha_m C(p_m)$ it follows $\mathrm{card}(tG_2) = |H| \, \mathrm{card}(tG'_2)$ and so this cannot be closed under multiplication either. If both tG_1 and tG_2 are not M-finite then it follows $J = \mathrm{card}(tG_1) \cdot \mathrm{card}(tG_2)$ (where the multiplication is as defined in 3.4.3) and since J is not closed under multiplication at least one of $\mathrm{card}(tG_1)$, $\mathrm{card}(tG_2)$ must also not be closed under multiplication.*

The purpose of example 4.5.18 is to demonstrate that there are cases where an M-countable complement for dG is not guaranteed by theorem 4.5.13. Whether or not there is an M-countable complement for dG where G is as in example 4.5.18 is an open question closely related to question 4.5.15.

Chapter 5

Conclusion

In this chapter we review the work of the thesis. We describe the main results and mention where things could be taken further. We list some open questions together with brief explanations where necessary and give some details of the author's thoughts about the questions highlighting potential for future research.

5.1 A brief Synopsis of the Thesis

We began with an investigation of the notion of M-countability in chapter 2. The key idea being that a bounded set can always be approximated from above and below by M-finite ones and this gives us a notion of size. It was the goal of the first section of this chapter to investigate the behaviour of the class of M-countable sets. We showed that, at least in a countable nonstandard model, sets can be constructed with arbitrary upper and lower cardinality (example 2.1.3). We gave examples (2.1.15, 2.1.18) that demonstrated the class of M-countable sets is not closed under disjoint union or intersection. Not all M countable sets $X \subsetneq Y$, with $\mathrm{card}(X) < \mathrm{card}(Y)$, are separable (meaning there exists an M-finite set that is a superset of X and a subset of Y) as shown in example 2.1.17. The notion of the derivative ∂I of a cut I helps us to understand some of the reasons things go wrong; the union of disjoint M-countable sets X and Y is M-countable providing $\partial\,\mathrm{card}(X) \neq \partial\,\mathrm{card}(Y)$. In the case of intersections we don't even have that M-countable intersect M-finite is necessarily M-countable and we gave an example to show this. However by using ∂ we prove some results that show under certain conditions we can guarantee the M-countability of the intersection of two M-countable sets (theorem 2.1.21 and theorem 2.1.22). Monotonically definable sets were shown to be a particularly nice subclass of M-countable sets and continued to appear throughout the thesis. Finally we looked at I-measurable sets. The idea here was to mimic the Carathéodory definition of measure. We were able to prove that the collection of I-measurable sets forms an algebra. Unfortunately not all sets that we would like to be I-measurable; in fact only such X with $\partial\overline{\mathrm{card}}(X) \leqslant I$. In particular this rules out initial segments with a derivative greater than I.

In chapter 3 we extended the concept of M-countability to groups. The only restriction on the cardinality of an M-countable group is that it must be closed under addition. A notion of index was defined for an external subgroup H inside an M-finite group G, and we proved that if H is M-countable then the upper and lower versions of the index agree. We went on to prove some technical lemmas allowing us to refine to an arbitrary degree of accuracy upper and lower M-finite approximations to a transversal T whilst preserving an upper or lower transversal as a superset or subset respectively. Up to this point everything we proved is true in an arbitrary nonstandard model of PA but for the next theorem (3.3.4) it was necessary to assume countability of the model. Under this assumption we proved that H has an M-countable transversal in G with the expected cardinality. We followed this result with two diagonalisations allowing us to construct transversals for H with differing upper and lower cardinalities. In a sense these showed that the inductive argument in theorem 3.3.4 was doing real work for us. In the final section of this chapter we generalized 3.3.4 to the case of two M-countable groups H and K (theorem 3.4.4). For simplicity we assumed we were working inside an M-finite supergroup G but all that is actually needed is for the group operation to be definable on an M-finite superset of K. The theorem is sensitive to technical concerns about the division of initial segments (see Kaye [9]) and also the separability of H and K.

In chapter 4 we looked at a particular class of M-finite groups - the abelian groups. The basis theorem for finite abelian groups is provable in PA and we used it to (partially) justify an exploration of M-finite abelian groups consisting of a single internal direct summand $C(p^k)$. The case for nonstandard p turned out to be easy, being externally isomorphic to a vector space over $\mathbb{Q}$ of full dimension. In the case that p is standard and k is nonstandard we proved a more complex decomposition result 4.3.22. We looked briefly at the Szmielew invariants to characterize the theory of $C(p^k)$ and used to them check our structural result. Arising out of these investigations about the external structure of an M-finite abelian group G were questions about how certain complements could be described within the model and it was in this direction that the most progress was made. We showed that for a nice family of monotonically definable external subgroups $A_I < C(p^k)$ corresponding to cuts $I < k$ no complement B_I could be monotonically definable (at least in the case $I = p^I$, see theorem 4.3.15). This, of course, does not rule out the possibility of an M-countable complement and it was the goal of section 4.4 to construct such a complement. A sequence of technical lemmas paved the way for theorem 4.4.8 to answer this question positively in the case of countable M and I with $\partial^2(I) > \mathbb{N}$ or $I = \mathbb{N}$. Whether the theorem is true without these conditions is an open question. The group $A_{\mathbb{N}}$ is the divisible part of $C(p^k)$ and so theorem 4.4.8 provides an M-countable complement to the divisible part of $C(p^k)$. It was the goal of section 4.4 to generalize this result to an arbitrary M-finite abelian G. We used the basis theorem in order to define a monotonically definable 'pseudo' complement H with the same cardinality as an actual complement. We also proved that this is always equal to the cardinality of the torsion part of G. The idea was to mimic the lemmas of the previous section but with H in place of $A^{k-\mathbb{N}}$ in order to then extend H to a full complement by inductively adding suitable elements and closing under the group operation. In order for the resulting complement to be M-countable we had to build this construction inside a sequence of M-finite supersets. The result was theorem 4.5.13 in which a complement B is constructed. B turns out to be M-countable providing $\mathrm{card}(H) = \mathrm{card}(tG) \vDash \cdot$. As before this closure condition is needed because the M-finite supersets don't close down tightly enough upon B for M-countability to be guaranteed otherwise. Countability of M is also assumed. At the end of the chapter we proved a result (proposition 4.5.16) which shows $\mathrm{card}(H) = \mathrm{card}(tG) \vDash \cdot$ in all 'nice' cases and thus the generality of theorem 4.5.13 is affected only very slightly by assuming it. We gave an example 4.5.18 to demonstrate that it is possible to have $\mathrm{card}(H) = \mathrm{card}(tG) \nvDash \cdot$ although whether or not there is an M-countable complement in this case remains an open question.

5.2 Future Research

There are various directions in which the work on M-countable sets can be taken forward although much of this has been done and is due to appear in a forthcoming paper. Question 2.1.23, on determining which of $A \cap B$ and $A \setminus B$ is M-countable in the case that $\partial\,\underline{\mathrm{card}}(A \cap B) = \partial\,\underline{\mathrm{card}}(A \setminus B)$, remains unanswered.

The chief questions arising from chapter 3 concern generalizations to uncountable models of PA.

Question 5.2.1 *Are theorems 3.3.4 and 3.4.4 true in the uncountable case?*

The generalized transversal theorem also contains technical concerns over the cut division and separability.

Question 5.2.2 *Can theorem 3.4.4 be proved without assuming H and K are inseparable? What can be said if $J/I \neq J \oslash I$?*

Chapter 4 certainly contains areas that can be taken further. In proving theorem 4.5.13 considerable use was made of the basis theorem. This knowledge could be used to try and get more of a handle on what a general M-finite abelian G might look like externally. Question 4.3.18 remains unanswered. Even in the case of countable M there are $2^{\aleph_0}$ cuts less than k and so an inductive construction looks difficult. A positive answer to 4.3.18 would still leave open the question of whether all the B_I could be chosen to be M-countable. As before we have questions over generalizations to uncountable models.

Question 5.2.3 *Are theorems 4.5.13 and 4.4.8 true in the uncountable case?*

It should be possible to say more about the growth rate of $\langle X \rangle$ when X is an M-finite subset of (abelian) G. We would like to know more about what conditions can be placed on such an X to limit this growth rate. This would allow some light to be shed on questions such as 4.5.15 which the author suspects has a positive answer.

List of References

[1] John Allsup. *On groups and initial segments in nonstandard models of Peano Arithmetic*. Ph.D., University of Birmingham, 2007.

[2] John Allsup and Richard Kaye. Normal subgroups of nonstandard symmetric and alternating groups. *Arch. Math. Logic*, 46(2):107–121, 2007.

[3] B. J. Birch, R. G. Burns, Sheila Oates Macdonald, and Peter M. Neumann. On the orbit-sizes of permutation groups containing elements separating finite subsets. *Bull. Austral. Math. Soc.*, 14(1):7–10, 1976.

[4] C. C. Chang and H. J. Keisler. *Model theory*. North-Holland Publishing Co., Amsterdam, 1973. Studies in Logic and the Foundations of Mathematics, Vol. 73.

[5] Nigel Cutland. *Nonstandard Analysis and its Applications*, volume 10 of *Student Texts*. London Mathematical Society, London, 1988.

[6] Paul C. Eklof. Some model theory of abelian groups. *J. Symbolic Logic*, 37:335–342, 1972.

[7] Irving Kaplansky. *Infinite abelian groups*. University of Michigan Press, Ann Arbor, 1954.

[8] Richard Kaye. *Models of Peano arithmetic*, volume 15 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.

[9] Richard Kaye. The arithmetic of cuts in models of arithmetic. *MLQ Math. Log. Q.*, 59(4-5):332–351, 2013.

[10] Richard Kaye and Tin Lok Wong. On interpretations of arithmetic and set theory. *Notre Dame J. Formal Logic*, 48(4):497–510, 2007.

[11] L. A. S. Kirby and J. B. Paris. Initial segments of models of Peano's axioms. In *Set theory and hierarchy theory, V (Proc. Third Conf., Bierutowice, 1976)*, pages 211–226. Lecture Notes in Math., Vol. 619. Springer, Berlin, 1977.

[12] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.

[13] J. B. Paris and G. Mills. Closure properties of countable nonstandard integers. *Fund. Math.*, 103(3):205–215, 1979.

[14] Vladimir G. Pestov and Aleksandra Kwiatkowska. An introduction to hyperlinear and sofic groups, 2008.

[15] Dan Segal. Some aspects of profinite group theory. In *Essays in geometric group theory*, volume 9 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 27–60. Ramanujan Math. Soc., Mysore, 2009.

[16] John S. Wilson. On simple pseudofinite groups. *J. London Math. Soc. (2)*, 51(3):471–490, 1995.