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Abstract

We consider a situation in which policymakers in a local community choose an *optimal* site for a controversial and essential project in a *dynamic* setting. Policymakers have either single-dipped or multi-dipped preferences over a Euclidean space of possible locations. We provide two existence results for this issue. Furthermore, we show that two optimal sites (at most) exist if policymakers have single-dipped preferences over a one-dimensional site space.

Keywords: Public bads, Single-dipped (Multi-dipped) preferences, Optimality.

JEL: C70, D72

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1. Introduction

Scholars and policymakers have demonstrated a growing interest in locating controversial facilities, as surveyed by Aldrich (2010) in his recent book. Governments often face fundamental siting issues as they attempt to build or expand public projects that serve residents' needs but potentially bring negative externalities into their targeted communities. Recent examples include, among others, the Enbridge Northern Gateway Pipelines, and the Trans Mountain Expansion Projects in Canada, and the fracking boom in the United States. States must choose among many possible policy instruments to handle local resistance, ranging from strong-arm tactics of coercion and rigid social control to more peaceful incentives and inducements. Addressing issues on *where to locate* and *how to overcome civil society* have significant implications for the design and implementation of public policies towards protecting both environment and human well-being.

In this paper, we look at how residents of a local community interact peacefully in a democratic institution to address the problem of locating a ***stable*** and ***efficient*** site for a public facility, such as a windmill park, a garbage dumping ground, or a nuclear plant. Such public projects are called *public bads*: residents agree about their value but typically do not want them in their backyards. We assume that the public bad is either within a one-dimensional space or a multi-dimensional space. In the former, each resident has a *single-dipped* preference over the set of possible (or feasible) sites: there is the worst point, called the *dip*, and the resident's satisfaction increases with the Euclidean distance from this dip. In the latter, we assume that each resident may have a *multi-dipped* preference: a resident may have multiple local dips over the site space. In the community, the *Council* (a finite group of political representatives, henceforth called legislators) adopts the policy through voting. A vote then consists of reporting one's dip. We assume that legislators are rational, truthful, and vote according to their constituents' preferences. A location for the public bad is adopted if (1) there is an endless cycle of voting: legislators agree on the public bad's site, and (2) the chosen site is ***efficient***: there is no other site which would be preferred by all legislators to the chosen site. When (1) and (2) are satisfied, the council has chosen an ***optimal*** site for the project. The main issue is that legislators may have antagonistic tastes upon locating the public facility, then render uncertain the outcome of this social decision by voting.

Legislators make decisions under a three-stage mechanism, L, described as follows. In the first stage, a *majority* coalition S can object against the status quo site x by proposing an alternative site y. Otherwise, the status quo remains, and the voting ends, or Nature ends the voting process after S proposes y. In the second stage, each legislator who is not a member of S can oppose y. Otherwise, y becomes the new site, which ends the process. However, if there is any opposition, S has the right to either withdraw or maintain y. If S withdraws y, the status quo x remains in place, and the process ends. However, if S maintains y, then the opposing legislators can invite other legislators to form a *majority* coalition T to replace y with another site z in the third and final stage. If T succeeds, z is elected, and the process ends. Otherwise, the *Council* elects y in the final stage.

We use *majority rule* to allocate decisive power among coalition of legislators. Thus, if n is the size of the council, then any coalition of legislators that consists of more than $n/2$ is a *majority*. We say that a given site x is ***stable*** if it cannot be overruled by a majority of legislators if it is submitted to a possible reform or amendment under the three-stage mechanism described above. If the decision-making process ends with a site x, and the latter is efficient, then x is optimal and will be adopted for the community. Otherwise, the council adjourns.

We prove that there exists an optimal (stable and efficient) site of the public bad when the location space is one-dimensional (Theorem 1) or multi-dimensional (Theorem 2). In addition, if preferences over a one-dimensional site space are single-dipped, then the council can select at most two optimal sites for the public bad (Theorem 1).

Scholars in social sciences have demonstrated interests in addressing where and how to locate a public bad. Early theory suggests a deep involvement of citizen participation through local referenda allowing residents to vote for or against the public facility (see, for example, Mitchell and Carson (1986)). In his book on siting issues and civil society, Aldrich (2010) debates the patterns by which most controversial projects are sited in France, Japan, and United States, and the policy instruments used by these countries to handle civil opposition when it arises. Other studies assume that individual preferences over the set of possible sites are single-dipped and focus on defining and characterizing collective rules, in which each agent has the incentives to reveal his or her sincere preferences (see, for example, Ehlers (2002), Barberà et al. (2012), Öztürk et al. (2013), Manjunath (2014), Öztürk et al. (2014), Lahiri et al. (2016), and Peters et al. (2017)). In a direct revelation mechanism setting, Yamamura (2016) analyzes the choice of locating a public bad from a coalitional standpoint using the equilibrium concepts of strong Nash equilibrium and coalition proof Nash equilibrium.

Our work contributes to this existing literature. Previous studies, however, have focused mostly on *static* decision-making mechanisms. While real-life institutions use static mechanisms, most institutions also employ *dynamic* procedures, wherein a proposed policy generally goes through a sequence of amendments before its final adoption (see, for example, Baron and Ferejohn (1989) and Chwe (1994)). In this paper, we use a *dynamic* legislative mechanism to determine the existence of an optimal (efficient and stable) site for a controversial facility when legislators are truthful and display specific preferences over the location space. While the result of Theorem 1 is consistent with early works on one-dimensional location space, we extend the analysis on multi-dimensional spaces, and we show that within such domains, the optimality is preserved.

We start by introducing notations and the main definitions in Section 2. In Section 3, we determine the existence of an optimal location for a controversial facility when legislators have either single-dipped or multi-dipped preferences over a Euclidean location space. Section 4 concludes.

2. Setup

This section provides formal and brief notations of the basic concepts that we use throughout the paper.

2.1. The Council

The council is labeled as $\mathfrak{C} = (N, A, \mathcal{M}, L, (\succeq_i)_{i \in N})$, where N is the finite set of legislators (with $|N| = n \geq 2$), A is the Euclidean site space, $\mathcal{M}$ is the majority rule (or the set of majority coalitions), L is the legislative voting mechanism, and $(\succeq_i)_{i \in N}$ is the profile of legislators' preferences over the set A. The site space A represents the set of feasible sites at stake. We assume that A is a one-dimensional or multidimensional space.

2.2. Preferences

For $x, y \in A$, $y \succeq_i x$ indicates that legislator i weakly prefers y to x; $y \succ_i x$ indicates that legislator i prefers y to x; and $y \sim_i x$ indicates that legislator i is indifferent between y and x. Given the nature of the site space, we assume that preferences are *continuous*. Preference continuity is a natural assumption which means that a legislator who prefers a site x over another site y prefers any site close enough to x over any site close enough to y. This implies that a small perturbation of the site space does not radically change preferences. This structure of preferences is generally encountered in economic theory. Formally, the binary relation $\succeq_i$ is said to be continuous if for any site $x, y \in A$, such that y is preferred to x ($y \succ_i x$), there exists a neighborhood $S(x)$ of x and a neighborhood $S(y)$ of y such that site z is preferred to site t ($z \succ_i t$) for every sites $z \in S(y)$ and $t \in S(x)$. In this paper, we use two classes of continuous preferences: *single-dipped* and *multi-dipped* preferences.

2.3. Optimal Sites

The council decides on the choice of a site under the three-stage legislative mechanism, L, that we presented in the Introduction. Below, we describe the rational behavior of legislators within the mechanism.

Let us assume that a status quo site $x \in A$ is at stake in the Council. A group of legislators might gather together in a majority coalition, say S and *object* to the status quo if there exists another alternative, say y, which they prefer to x, i.e., $y \succ_S x$. This condition is a natural requirement that appears in all two-stage bargaining models of rationality in the literature (see, for example, Aumann and Maschler (1964) and all of the subsequent studies inspired by this paper). We can also view the condition as an expression of prudence or ambiguity aversion. It expresses that the majority S considers the presence of Nature in the voting process and cannot fully predict the future of the game after introducing the proposal y.

After the *objection* (y, S), a legislator i from the remaining group of council $(N \setminus S)$ may be worse off at y (i.e., not$(y \succeq_i x)$), and decides to join another majority coalition T which could propose another site z to the Council, i.e., $z \succ_T y$. In that case, a pair $(z, T) \in Z \times \mathcal{M}$ is a *counter-objection* against the objection (y, S) if $z \succ_T y$ and not$(y \succeq_T x)$. Also, legislators of T have to consider the fact that the first majority coalition S may withdraw their support for the change of the status quo x in case the new proposition z could hurt some of its legislators, i.e., not$(z \succeq_S x)$. If the latter occurs, then the counter-objection is *unfriendly* against S. In the Council, a majority S will initiate the first move against the status quo x by proposing a new site y against x if there is no unfriendly counter-objection against (y, S), i.e., $(z \succeq_S x)$; then the objection (y, S) is considered to be *justified*.

A *stable* site is an option in A against which no justified objection exists. In other words, a stable site is an alternative such that if it is the status quo, no majority coalition will seek to replace it. More formally, a site, say $y \in A$ *defeats* another site

x, labeled by $y \gg x$, if there exists a majority coalition $S \in \mathcal{M}$ $(y \gg_S x)$ such that:

1. $y \succ_S x$ and;

2. $\forall (z, T) \in A \times \mathcal{M},\ S \neq T,\ [z \succ_T y$ and $not(y \succeq_T x)]$ implies $[z \succeq_S x]$.

The set of stable sites, denoted by $O(\mathfrak{C})$, contains sites that are not defeated in the council $\mathfrak{C}$. A decision is adopted if legislators reach an agreement at the end of the legislative procedure, i.e., $O(\mathfrak{C}) \neq \emptyset$, and the outcome of this agreement is efficient. The set $O(\mathfrak{C})$ is also called the *reciprocity set* in Pongou and Tondji (2018b). By addressing a different question, they prove that the legislative procedure, L, in this study induces legislators to take reciprocal actions. The formalization of rational behavior in the Council follows the traditional *blocking approach* in coalitional games (see, for example, Harsanyi (1974), Chwe (1994), Dutta et al. (2005), Ray and Vohra (2014), and Dutta and Vohra (2017)). However, following the *bargaining approach* (see, for example, Nash (1953), Kimya (2020), and Serrano (2020)), the three-stage mechanism also implements the set of optimal outcomes in some subgame perfect Nash equilibrium concept. This additional extension and a comparison of the mechanism to other procedures that follows the *blocking approach* can be found in Pongou and Tondji (2018a,b). The following result reduces the selection of optimal sites.

Lemma 1 (Pongou and Tondji (2018b)) *Let $\mathfrak{C} = (N, A, \mathcal{M}, L, (\succeq_i)_{i \in N})$ be a council and $x \in A$ be a site. Assume that $O(\mathfrak{C}) \neq \emptyset$. If $x \in O(\mathfrak{C})$, then x is efficient.* □

The intuition behind this result follows from the fact that the dynamic mechanism, L, induces legislators to adopt farsighted and reciprocal behaviors. No potential second majority T will react against a first majority S who replaces a status quo site x with an alternative y that Pareto-dominates x. Opposing such a move would cause the majority S to withdraw y, thus allowing x to remain in place and inducing the persistence of a less preferred site by all the legislators.

Lemma 1 shows that any stable site is efficient. It follows that, the optimal decision of the Council is guaranteed if stability succeeds. Throughout the remaining of the paper, the set $O(\mathfrak{C})$ consists of the optimal sites for the public bad. In the next section, we derive the main results of the paper.

3. Results

We first establish the existence of an optimal site when preferences are *single-dipped*. A preference relation over a space is said to be *single-dipped* if; the sites can be ordered as points on a line; the preference relation has a least preferred point–dip or worse point; and, points further away from the least point are more preferred. Formally, assume that all the sites are ordered by a binary relation denoted θ, and all individuals perceive them as being arranged in this order. An individual i of the council has a *single-dipped* preference $\succeq_i$, if there exists a site x_i such that: (1) for any other site $x \neq x_i$, $x \succ_i x_i$; and (2) for any site x, $y \in A$, it holds that: if $(x_i \ \theta \ y \ \theta \ x)$ or $(x \ \theta \ y \ \theta \ x_i)$, then $x \succ_i y$.

Theorem 1 *Let $\mathfrak{C} = (N, A, \mathcal{M}, L, (\succeq_i)_{i \in N})$ be a council where, A is a one-dimension site space, and where preferences are single-dipped. Then, an optimal site exists, and there are at most two optimal sites.* □

We relegate the proof of Theorem 1 in the appendix.

Theorem 1 is consistent with other studies on the site of a public bad under *strategy-proof* and efficient rules. Strategy-proofness avoids members' improvement of satisfaction or gains by misrepresenting their true preferences, and efficiency ensures that all members' satisfaction cannot improve by changing the chosen site of a public bad. As mentioned above, any stable site under the three-stage mechanism used in the Council is efficient. Manjunath (2014) studies the site of a public bad when agents have single-dipped preferences over a closed interval $[0, T]$. He proves that the range of an efficient and strategy-proof rule is the set $\{0, T\}$, the domain's two extremes. In another study, Barberà et al. (2012) proves that the range of strategy-proof rules under single-dipped preferences contains two alternatives at most. These studies focus exclusively on *static* decision-making mechanisms. Theorem 1 provides the same characterization in a dynamic voting mechanism. In our setting, each legislator has at most two *peak* (the most preferred) points and, those sites coincide with the minimum and maximum elements of A. Therefore, these two bounds are the optimal sites when preferences are single-dipped. Below, we illustrate our result in a Council consisting of three ($n = 3$) and four ($n = 4$) legislators.

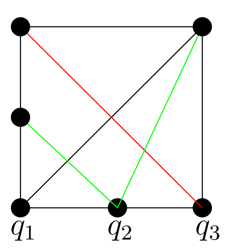
3.1. Illustrations

Let $x \in A$ be a site and define the number $S(x)$ as the number of legislators who have x as a peak point (the most preferred point). If no legislator has x as a peak point, then $S(x)$ takes the value 0. Let $V(q, q')$ denote the set of legislators who prefer q against q' in a choice between q and q': $V(q, q') = \{i \in N : q \succ_i q'\}$. We differentiate two cases: each legislator has only one peak over the set A, or at least one legislator has two peak sites. In what follows, q_m is legislator m's dip point, $m = 1, 2, 3, 4$. Legislators have single-dipped preferences. Therefore, they prefer to vote for sites far away from their dip points.

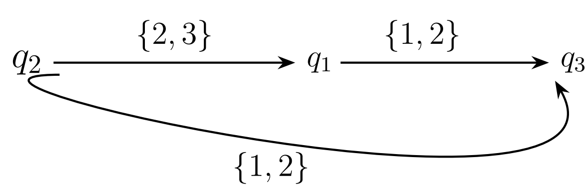
3.1.1. Illustration I: a 3-legislator Council

We consider two cases and determine the optimal sites.

Case a: $S(q_1) = 1$, $S(q_2) = 0$, and $S(q_3) = 2$ (see Figure 1).



(a) Preferences

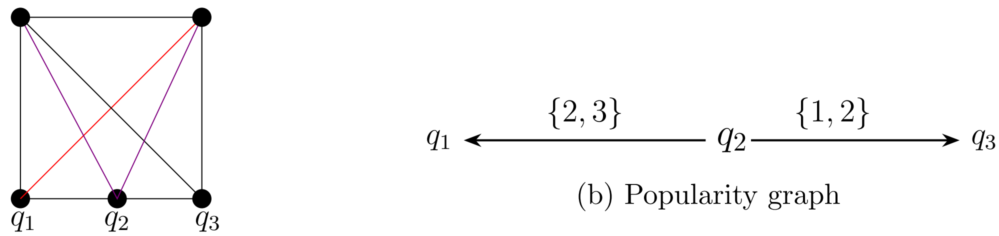


(b) Popularity graph

Figure 1: The Council's size is 3 and each legislator has a unique peak

The popularity graph among sites based on preferences is provided in Figure 1b (the arrows indicate the direction of the popularity relationship; for instance q_1 is a more popular site than q_2 because q_1 is preferred over q_2 by the majority coalition $\{2,3\}$). From Figure 1a and Figure 1b, we note that $V(q_1,q_2)=\{2,3\}$, $V(q_3,q_1)=\{1,2\}$, and $V(q_3,q_2)=\{1,2\}$. It follows that although, $q_1 \succ_{23} q_2$, the site q_1 does not defeat q_2 because there exists the alternative q_3 with $q_3 \succ_{12} q_1$, $not(q_1 \succeq_{12} q_2)$, and $not(q_3 \succeq_{23} q_2)$. Given that $q_3 \succ_{12} q_1$, and $q_3 \succ_{12} q_2$, and no other alternative is preferred to q_3, it follows that q_3 defeats q_1 and q_2 thanks to coalition $\{1,2\}$, and there is no site which defeats option q_3. Hence, $q_3 \gtrdot_{12} q_1$, $q_3 \gtrdot_{12} q_2$, and $O(\mathfrak{C})=\{q_3\}$.

Case b: $S(q_1)=2$, $S(q_2)=0$, and $S(q_3)=2$ (see Figure 2).



(a) Preferences

(b) Popularity graph

Figure 2: The Council's size is 3; Legislators 1 and 3 have one peak site, while Legislator 2 has two peaks

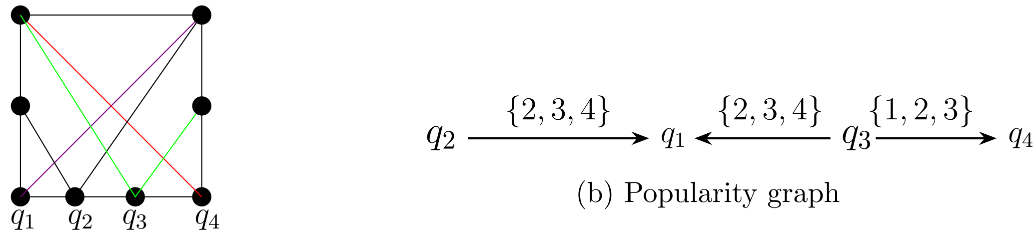
From Figure 2a and Figure 2b, we have what follows. The majority $\{2,3\}$ prefers q_1 to q_2 ($q_1 \succ_{23} q_2$), and the majority $\{1,2\}$ prefers q_3 to q_2 ($q_3 \succ_{12} q_2$). When it comes to the choice between q_1 and q_3, we have : $V(q_1,q_3)=\{3\}$ and $V(q_3,q_1)=\{1\}$. It is straightforward to note that no other alternative is preferred by a majority to either q_1 or q_3. Therefore, the site q_2 is defeated by either q_1 or q_3 and no site defeats either q_1 or q_3. Hence, $O(\mathfrak{C})=\{q_1,q_3\}$.

3.1.2. Illustration II: a 4-legislator Council

As in Section 3.1.1, we assume that legislators admit at most two peak sites.

Case c: Each legislator has only one peak point. We illustrate this case in Figure 3 and Figure 4. We start with the illustration displayed in Figure 3.

Using Figure 3, we get the following results: $V(q_1,q_2)=V(q_1,q_3)=\{2,3,4\}$ and



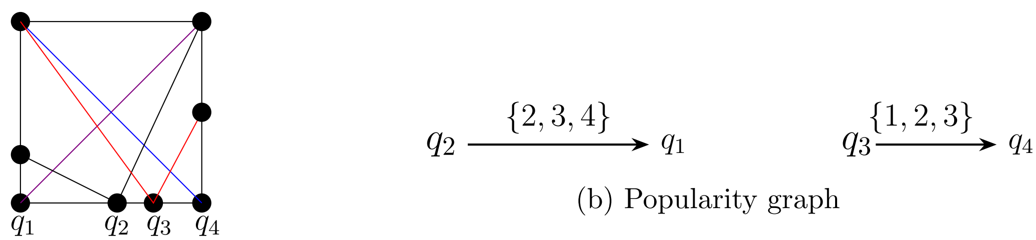
(a) Preferences

(b) Popularity graph

Figure 3: The Council's size is 4 and each legislator has a unique peak ($S(q_1) = 2$, $S(q_2) = S(q_3) = 0$, and $S(q_4) = 2$)

$V(q_2, q_1) = V(q_3, q_1) = \{1\}$. Given that there is no other alternative that is preferred to q_1, it follows that q_1 defeats q_3 and q_2 with the support of coalition $\{2, 3, 4\}$. Continuing with the same argument, given that $V(q_3, q_4) = \{4\}$ and $V(q_4, q_3) = \{1, 2, 3\}$, it follows that q_4 defeats q_3 with the support of majority $\{1, 2, 3\}$. Note that $V(q_1, q_4) = V(q_2, q_3) = V(q_2, q_4) = \{3, 4\}$, and $V(q_4, q_1) = V(q_3, q_2) = V(q_4, q_2) = \{1, 2\}$. As result, the only non-defeated sites are the two peak points q_1 and q_4. Hence, $q_1 \gtrdot_{234} q_2$, $q_1 \gtrdot_{234} q_3$, $q_4 \gtrdot_{123} q_3$, and $O(\mathfrak{C}) = \{q_1, q_4\}$.

We continue our analysis with the illustration in Figure 4.



(a) Preferences

(b) Popularity graph

Figure 4: The Council's size is 4, each legislator has a unique peak ($S(q_1) = 2$, $S(q_2) = S(q_3) = 0$, and $S(q_4) = 2$), and Legislator 2 is indifferent between sites q_1 and q_3

The main difference from Figure 3 and Figure 4 is that Legislator 2 is indifferent between sites q_1 and q_3 ($q_1 \sim_2 q_3$). Therefore, $V(q_1, q_3) = \{3, 4\}$, $V(q_3, q_1) = \{1\}$, and q_1 does not defeat q_3. However, as in the previous case, $V(q_1, q_2) = \{2, 3, 4\}$, $V(q_2, q_1) = \{1\}$, $V(q_3, q_4) = \{4\}$, and $V(q_4, q_3) = \{1, 2, 3\}$. Using Figure 4b, we conclude that q_1 defeats q_2 with the support of coalition $\{2, 3, 4\}$, and q_4 defeats q_3

with the support of majority $\{1,2,3\}$. Given that $V(q_1,q_4) = V(q_2,q_4) = \{3,4\}$, and $V(q_4,q_1) = V(q_4,q_2) = \{1,2\}$, it follows that the only non-defeated sites are the two peak points q_1 and q_4 so that $O(\mathfrak{C}) = \{q_1,q_4\}$.

Case d: There exists one legislator who has two peak points. We illustrate this case in Figure 5.

(a) Preferences for $n = 4$ (b) Preferences for $n = 4$ and $q_1 \sim_2 q_3$

Figure 5: The Council's size is 4, and Legislator 3 has two peaks

We use the similar argument in **Case c** to derive the optimal sites. From Figure 5, $V(q_1,q_2) = \{2,3,4\}$ and $V(q_2,q_1) = \{1\}$, then q_1 defeats q_2. Using Figure 5a, the site q_1 defeats q_3, because $V(q_1,q_3) = \{2,3,4\}$ and $V(q_3,q_1) = \{1\}$. However, given that, legislator 2 is indifferent between candidates q_1 and q_3 in Figure 5b, we have $V(q_1,q_3) = \{3,4\}$ and $V(q_3,q_1) = \{1\}$, and the latter relationship between q_1 and q_3 does not stand. Using Figure 5, $V(q_1,q_4) = \{4\}$, $V(q_4,q_1) = \{1,2\}$, $V(q_3,q_4) = \{4\}$ and $V(q_4,q_3) = \{1,2,3\}$. It follows that q_4 defeats q_3, and $O(\mathfrak{C}) = \{q_1,q_4\}$.

3.2. Optimal Sites in a Multi-dimensional Location Space

In real-life situations, legislators may have beliefs (or positions) over social issues that are not always ranked by a one-dimensional spectrum. In that case, the result of Theorem 1 is limited. Therefore, we assume that legislators in the Council may also display multi-dipped continuous preferences over a possibly multi-dimensional site space. In what follows, we assume that the site space A is a compact and convex subset of the k-dimensional Euclidean space ($k < \infty$). Following previous studies in economies with multi-dimensional spaces (see, for example, Debreu (1969), Hildenbrand (1974)), we endow the site space A with the topology of closed convergence. We start the analysis with the following definition.

Definition. Let $\succeq_i$ be a binary relation over a site space A. An alternative $x \in A$ is a *local minimizer* of $\succeq_i$ if there exists a neighborhood $S(x)$ of x such that $y \succeq_i x$, for every $y \in S(x)$.

Denote by $LM(\succeq_i, A)$ the set of all the local minimizers of the relation $\succeq_i$. Note that, if the preference $\succeq_i$ is a continuous binary relation over A, then $LM(\succeq_i, A)$ is a non-empty Borel subset of A. The concepts of Borel set and the Lebesgue measure (denoted here by $\mathfrak{L}$) have proven to be extremely useful in general equilibrium theory. Multi-dipped preferences have one or many local minimizers that form a null set. An interesting subclass of this class of preferences is the set of preferences that have countably many local minimizers. We have the following result.

Theorem 2 *Let $\mathfrak{C} = (N, A, \mathcal{M}, L, (\succeq_i)_{i \in N})$ be a Council where A is a k-dimensional Euclidean site space ($1 \leq k < \infty$), and where each preference $\succeq_i$ is such that:*

$$\mathfrak{L}[LM(\succeq_i, A)] = 0, \text{ for } i = 1, 2, ..., n.$$

Then, an optimal site exists. □

We relegate the proof of Theorem 2 in the appendix. Theorem 2 complements the result of Theorem 1, and proved useful by guaranteeing positive results in a framework where legislators may have several worst sites.

4. Conclusion

In this paper, we use a three-stage voting mechanism to derive two possible theorems for enacting an optimal location for a controversial public project when policymakers' preferences over the set of competing sites display specific structures. Specifically, we help a council to site a public bad in a local community where residents have either single-dipped or multi-dipped preferences over a set of feasible sites. We show that there exists at most two optimal sites if policymakers have single-dipped preferences over a one-dimensional site space.

Appendix A. Proof of Results

Proof (Theorem 1) **1. Assume that n is odd**. We distinguish two different cases: each legislator has a unique peak point or there exists at least one legislator who has two peak sites over A.

Case e: consider the situation where each legislator has a unique peak point over A. Assume q_1^* and q_3^* to be the two possible peak points (this is possible since A is bounded), then $S(q_1^*) + S(q_3^*) = n$, $S(q_1^*), S(q_3^*) \geq 0$. Without loss of generality, assume that $S(q_1^*) > S(q_3^*)$, then $S(q_1^*) > \frac{n}{2}$. Consider the following graph (Figure A.6) described below.

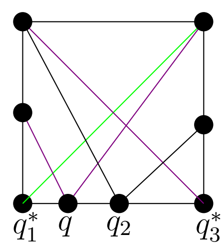


Figure A.6: The Council's size is odd, and each legislator has a unique peak

In a choice between between peaks q_1^* and q_3^* in the Council, the alternative q_1^* wins, because more than half of the council vote for q_1^* against q_3^*, since $S(q_1^*) > S(q_3^*)$ and $S(q_1^*) > \frac{n}{2}$. In a choice between q_1^* and any site $q \in (q_1^*, q_3^*)$ such that $S(q) = 0$, q loses. Indeed, the number of individuals who have q_1^* as dip point vote for q. It follows that, at least, $S(q_1^*)$ individuals vote for q_1^* against q. Given that $S(q_1^*) > \frac{n}{2}$, then q_1^* defeats all other sites in A and $O(\mathfrak{C}) = \{q_1^*\}$.

Case f: Assume that there exists at least one legislator who has two peaks over A. This situation can be illustrated by a graph similar to the one below (Figure A.7).

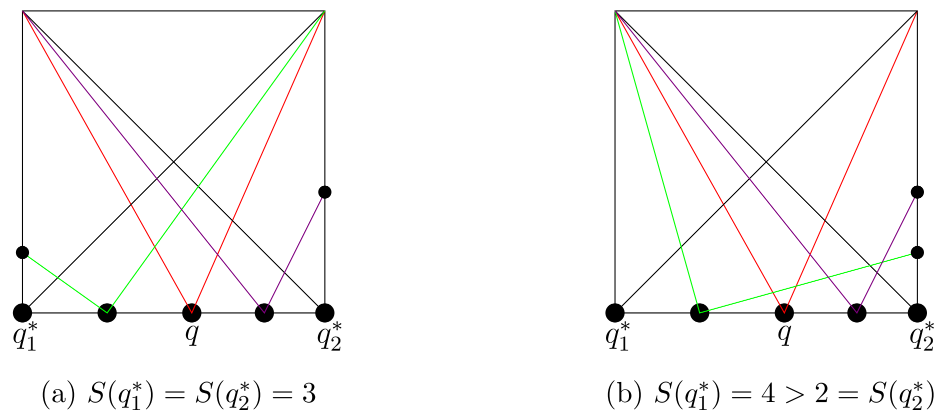


Figure A.7: The Council's size is odd, and legislator with dip q has two peaks

Note that any site q between alternative q_1^* and q_2^* is defeated by either q_1^* or q_2^*. So, the only interesting competition is between the extremes q_1^* and q_2^*. The legislator for whom q is a dip point has two peak points q_1^* and q_2^*. Thus he or she is indifferent between those peaks during a pairwise opposition when they are candidates. It follows that $|V(q_1^*, q_2^*)| + |V(q_2^*, q_1^*)| < n$, while $S(q_1^*) + S(q_2^*) > n$. In this situation, either $S(q_1^*) = S(q_2^*)$ (see, for example, Figure A.7a) or $S(q_1^*) \neq S(q_2^*)$. Without loss of generality, we can assume $S(q_1^*) > S(q_2^*)$ (see, for example, Figure A.7b). In cases similar to Figure A.7a, the only non-defeated sites are the extreme peaks, and $O(\mathfrak{C}) = \{q_1^*, q_2^*\}$. Otherwise, situations in Figure A.7b hold, q_1^* defeats q_2^*, and $O(\mathfrak{C}) = \{q_1^*\}$.

2. Assume that n is even. We study two different situations: each legislator has a unique peak point or there exists at least one legislator who has two peaks over A.

Case g: n is even, and each legislator has a unique peak point. Consider q_1^* and q_2^* as the two possible peak points, then $S(q_1^*) + S(q_2^*) = n$, with $S(q_1^*) \geq 0$ and $S(q_2^*) \geq 0$.

a) Assume that $S(q_1^*) > S(q_2^*)$, then $S(q_1^*) > \frac{n}{2}$, and $S(q_2^*) < \frac{n}{2}$. This item is similar to the situation already demonstrated, where n is odd (see **Case a**). As we proved previously, $O(\mathfrak{C}) = \{q_1^*\}$.

b) Suppose that $S(q_1^*) = S(q_2^*) = \frac{n}{2}$. In an opposition between in the Council q_1^* and q_2^*, there is no winner. Let $q \in A$ be a candidate such that $S(q) = 0$, and oppose this site to the peak point q_1^* in a pairwise opposition. Legislators for whom q_1^* is a dip point will vote for q and individuals for whom q_1^* is a peak point will vote for q_1^*. Also, individuals for whom q is the dip point will vote for q_1^*, although the latter is not their ideal point. Since $S(q_1^*) = \frac{n}{2}$, it follows that $|V(q_1^*, q)| > \frac{n}{2}$, and q_1^* defeats q. Therefore, all other sites between q_1^* and q_2^* are defeated either by q_1^* or q_2^*. Consequently, q_1^* and q_2^* are the only non-defeated sites, and $O(\mathfrak{C}) = \{q_1^*, q_2^*\}$.

Case h: n is even and there exists at least one legislator who has two peaks over A. Then, the one of the following situations hold: (a) $S(q_1^*) > S(q_2^*)$ (see, for example, Figure A.8a and Figure A.8b); (b) $S(q_1^*) < S(q_2^*)$ (see, for example, Figure A.8c); and (c) $S(q_1^*) = S(q_2^*)$ (see, for example, Figure A.8d). Individuals for whom q or q' is the dip point have two peak points q_1^* and q_2^*, thus there are indifferent between those peaks during a pairwise opposition where they are candidates.

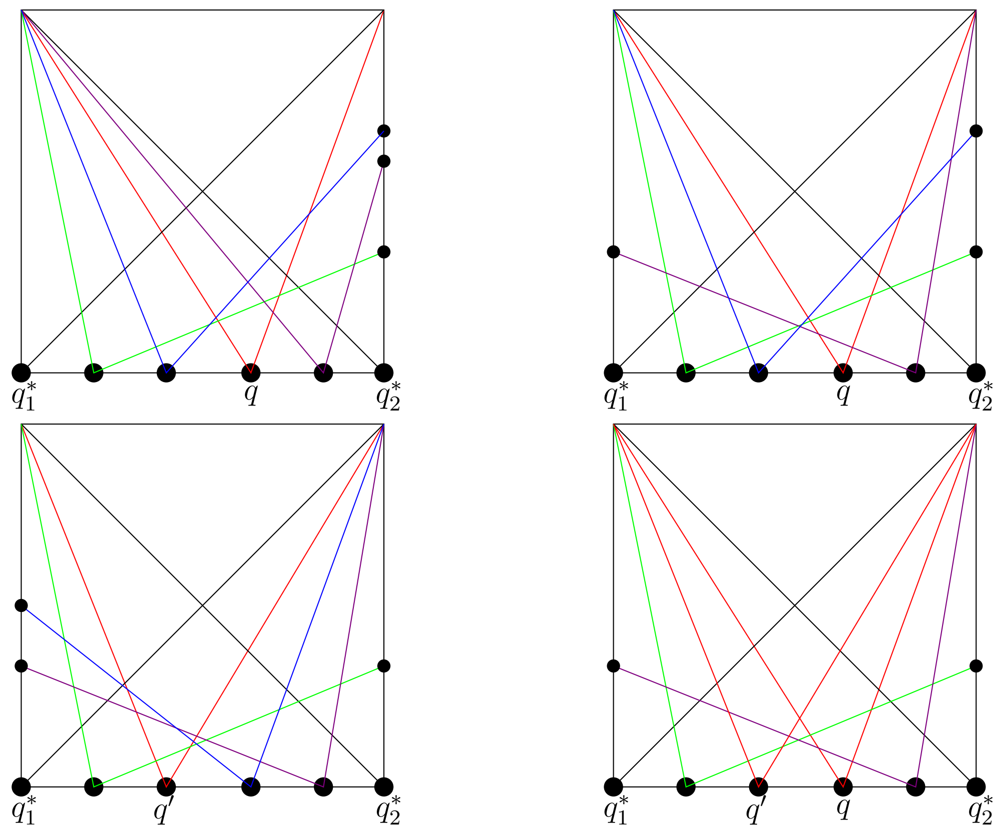
Given a situation in Figure A.8, we have $|V(q_1^*, q_2^*)| + |V(q_2^*, q_1^*)| < n$, while $S(q_1^*) + S(q_2^*) > n$.

i) However, in situations similar to the ones described in Figure A.8a, although, $|V(q_1^*, q_2^*)| < S(q_1^*)$, we have $|V(q_1^*, q_2^*)| > \frac{n}{2}$, and $|V(q_2^*, q_1^*) < S(q_2^*) < \frac{n}{2}$. Given that there is no other site that is majority-preferred to q_1^*, it follows that q_1^* defeats q_2^*, $O(\mathfrak{C}) = \{q_1^*\}$.

ii) In situations illustrated in Figure A.8b or Figure A.8c or Figure A.8d, it follows that $O(\mathfrak{C}) = \{q_1^*, q_2^*\}$. In fact, in Figure A.8b, $|V(q_1^*, q_2^*)| < S(q_1^*)$ and $|V(q_1^*, q_2^*)| = \frac{n}{2}$, while $|V(q_2^*, q_1^*)| < S(q_2^*) = \frac{n}{2}$. We have the reverse in Figure A.8c. In Figure A.8d, it holds that $|V(q_1^*, q_2^*)| < S(q_1^*)$, $|V(q_2^*, q_1^*)| < S(q_2^*)$, with $|V(q_1^*, q_2^*)| < \frac{n}{2} < S(q_1^*)$ and $|V(q_2^*, q_1^*)| < \frac{n}{2} < S(q_2^*)$. ∎

Proof (Proof of Theorem 2) For any $x \in A$, we denote $P_C(x) = \{y \in A : y \succ_C x\}$, $C \in \mathcal{M}$ and $P(x) = \{y \in A : y \succ x\}$, and $P(x) = \bigcup_{C \in \mathcal{M}} P_C(x)$. Consider h the mapping from A to the real numbers, and define $h(x) = \mathfrak{L}(P(x))$, for any $x \in A$, and $\mathfrak{L}$ the Lebesgue measure.

(a) $S(q_1^*) = 5 > 2 = S(q_2^*)$ (b) $S(q_1^*) = 4 > 3 = S(q_2^*)$



(c) $S(q_1^*) = 3 < 4 = S(q_2^*)$ (d) $S(q_1^*) = S(q_2^*) = 4$

Figure A.8: The Council's size is even, and legislator with dip q or q' has two peaks

1. h is lower semi-continuous.

   Let $x_0 \in A$. Consider a sequence x_l which converges to x_0 when l tends to infinity and show that $\liminf_{l\to\infty} h(x_l) \geq h(x_0)$. Let $y \in P(x_0)$, then there exists $C \in \mathcal{M}$ such that $y \in P_C(x_0)$. x_l converges to x_0 and P_C is continuous, thus for l sufficiently large, $y \in P_C(x_l)$. It follows that $y \in \liminf P_C(x_l)$ for some $C \in \mathcal{M}$, and then $y \in \liminf P(x_l)$, so $P(x_0) \subseteq \liminf P(x_l)$. Since there exists l_0 such that $\mathfrak{L}(\bigcup_{l\geq l_0} P(x_l)) < \infty$, then $\mathfrak{L}(\liminf_{l\to\infty} P(x_l)) \leq \liminf_{l\to\infty} \mathfrak{L}(P(x_l))$, i.e., $\mathfrak{L}(\liminf_{l\to\infty} P(x_l)) \leq \liminf_{l\to\infty} h(x_l)$, since $h(x_l) = \mathfrak{L}(P(x_l))$. Since $P(x_0) \subseteq \liminf P(x_l)$, then $\mathfrak{L}(P(x_0)) \leq \mathfrak{L}(\liminf_{l\to\infty} P(x_l))$ and therefore $h(x_0) \leq \lim_{l\to\infty} \inf h(x_l)$.

2. h is lower semi-continuous and A compact, then h attains a minimum on A, i.e.,

there exists $a \in A$ such that $h(a) = \inf_{x \in A} h(x)$.

Denote by $S = \left\{ a \in A : h(a) = \inf_{x \in A} h(x) \right\}$ and $\overline{LM} = \bigcup_{i=1}^{n} [LM(\succeq_i, A)]$. Consider the **binary relation** $\gg$ defined as follows: (1) for $x, y \in A$ and $C \subset N$, $y \gg_C x$ if $y \succ_C x$ and $z \succ_C x$ for all $z \notin \overline{LM}$ such that $z \succ y$; and (2) $y \gg x$ if there exists $C \in \mathcal{M}$ such that $y \gg_C x$. Let $\mathcal{V}(\mathfrak{C}) = \{x \in A : not(y \gg x), y \in A\}$.

3. $\mathcal{V}(\mathfrak{C})$ is non-empty.

   Assume the contrary and consider an arbitrary site $x_0 \in S$, and $x_0 \notin \mathcal{V}(\mathfrak{C})$. There exists an alternative site $x_1 \in A$ and $C \in \mathcal{M}$ such that $x_1 \gg_C x_0$ i.e., $x_1 \succ_C x_0$ and $z \succ_C x_0$ for any $z \in A$ such that $z \succ x_1$ and $z \notin \overline{LM}$. It follows that $P(x_1) \subseteq P(x_0) \cup (P(x_1) \cap \overline{LM})$, and $\mathfrak{L}(P(x_1)) \leq \mathfrak{L}(P(x_0)) + \mathfrak{L}(P(x_1) \cap \overline{LM}) \leq \mathfrak{L}(P(x_0)) + \mathfrak{L}(\overline{LM})$. Since $\overline{LM} = \bigcup_{i=1}^{n} [LM(\succeq_i, A)]$, then $\mathfrak{L}(\overline{LM}) \leq \sum_{i=1}^{n} \mathfrak{L}[LM(\succeq_i, A)]$ and $\mathfrak{L}[LM(\succeq_i, A)] = 0$ for all $i = 1, ..., n$ by hypothesis, therefore $\mathfrak{L}(\overline{LM}) = 0$ and $\mathfrak{L}(P(x_1)) \leq \mathfrak{L}(P(x_0))$ i.e., $h(x_1) \leq h(x_0)$. Since $h(x_0) = \inf_{x \in A} h(x)$, then the inequality $h(x_1) \leq h(x_0)$ implies that $h(x_1) = h(x_0)$ and $x_1 \in S$. We note two different situations described below.

   Situation 1: Assume there exists a neighborhood $S(x_1)$ for which there is no $y \in S(x_1)$ satisfying $y \succ x_1$. $x_1 \succ x_0$ and preferences are continuous, then there exists a neighborhood $S'(x_1) \subseteq S(x_1)$ such that $y \succ x_0$ for all $y \in S'(x_1)$. It follows that $S'(x_1) \subseteq P(x_0)$ and $S'(x_1) \subseteq (A - P(x_1))$. Thus, $P(x_1) \subsetneq P(x_0)$ and $h(x_1) < h(x_0)$, which is a contradiction.

   Situation 2: For all neighborhood $S(x_1)$ of x_1, there exists an alternative $y \in S(x_1)$ such that, $y \succ x_1$. Let x_2 be a site such that $x_2 \gg_{C'} x_1$. As previously, $P(x_2) \subseteq P(x_1) \cup (P(x_1 \cap \overline{LM}))$ and then $h(x_2) = h(x_1)$. We have $x_2 \succ x_1$ and preferences are continuous, thus there exists a neighborhood $S(x_1)$ such that $x_2 \succ a$ for any $a \in S(x_1)$. By assumption, there exists $y \in S(x_1)$ such that $y \succ x_1$. By continuity of preferences, there exists a neighborhood $S(y)$ of y such that $y' \succ x_1$ for any $y' \in S(y)$. Hence for all $y' \in S(y) \cap S(x_1)$, we have

$y' \succ x_1$ and $x_2 \succ y'$, so $h(x_1) > h(y')$ and $h(y') > h(x_2)$, then $h(x_1) > h(x_2)$, a contradiction.

Therefore, $\mathcal{V}(\mathfrak{C})$ is non-empty.

4. Let $x_0 \in \mathcal{V}(\mathfrak{C})$, prove that $x_0 \in O(\mathfrak{C})$.

Assume that $x_0 \notin O(\mathfrak{C})$, then there exists $x_1 \in A$ and $C \in \mathcal{M}$ such that $x_1 \gtrdot_C x_0$. Let $z \notin \overline{LM}$ such that $z \succ_T x_1$, for $T \in \mathcal{M}$. We would like to demonstrate that $z \succ_C x_0$. First, note that $z \succ_T x_1$, and $x_1 \gtrdot_C x_0$ implies that $z \succeq_C x_0$.

a) If $z \succ_i x_0$ for any $i \in C$, then $z \succ_C x_0$ and $x_1 \gg x_0$, which is a contradiction.

b) Assume that there exists $i \in C$ such that $z \sim_i x_0$. $z \succ_T x_1$ and preferences are continuous, then there exists a neighborhood $S(z)$ such that $z' \succ_T x_1$ for all $z' \in S(z)$. Because $z \notin \overline{LM}$, then $z \notin LM(\succeq_i, A)$ for any $i \in N$. Hence, there exists $z' \in S(z)$ such that $z \succ_i z'$. Since, $z \sim_i x_0$, then $x_0 \succ_i z'$ by transitivity. Also, $z' \in S(z)$, therefore $x_0 \succ_i z' \succ_T x_1 \succ_i x_0$ and by transitivity, $x_0 \succ_i x_0$ if $i \in C \cap T$. Otherwise, if $i \notin T$, then, given that $T \cap C \neq \emptyset$, there exists $i' \in T \cap C$ such that $x_0 \succ_{i'} z' \succ_{i'} x_1 \succ_{i'} x_0$ or $x_0 \succ_{i'} x_0$ by transitivity, which is a contradiction and $x_0 \notin \mathcal{V}(\mathfrak{C})$. In conclusion, $x_0 \in O(\mathfrak{C})$. ∎

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