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### Recommended Citation

Semenov, Aggey, and Jean-Baptiste Tondji. "On the Dynamic Analysis of Cournot–Bertrand Equilibria." *Economics Letters* 183 (October 1, 2019): 108549. <https://doi.org/10.1016/j.econlet.2019.108549>.

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# On the Dynamic Analysis of Cournot-Bertrand Equilibria\*

Aggey Semenov<sup>†</sup> and Jean-Baptiste Tondji

July 14, 2019

## Abstract

We consider a setting where firms in the first stage invest in cost-reducing R&D. In the market stage, one firm sets a quantity, and another sets a price. We prove that the quantity-setting firm invests more in R&D, has a lower price, and produces higher quantity than the price-setting firm. We also consider welfare implications.

**JEL:** D43, L13, O32

**Keywords:** Cournot-Bertrand model, Product differentiation, R&D, Welfare.

## 1 Introduction

Static models of product differentiation mostly focus on quantity (Cournot) and price (Bertrand) types of competition. However, there are sectors where firms engage in mixed (Cournot-Bertrand) competition; some firms offer quantity contracts to customers, and other firms offer price contracts. Tremblay et al. (2013) and Tremblay & Tremblay (2011) give an example of the market for small cars, where Honda and Subaru set the quantities and Saturn and Scion set prices. Flath (2012) shows that for 30 out of 70 Japanese industries companies use some form of mixed competition. In the Japanese home electronics industry, Panasonic (formerly known as Matsushita) uses quantity competition while Sanyo employs pricing strategies (Sato (1996)). Klemperer & Meyer (1986) argue that in international consulting firms the management consulting division operates in a quantity-setting fashion, whereas the auditing division sets a strictly defined fee per hour.

Vives (2001) and Singh & Vives (1984) study the choice of contracts in a differentiated duopoly with linear demands. They show that with symmetric costs, firms prefer Cournot game to the Bertrand and mixed settings when products are substitutes. Several studies have also shown that firms often compete against each other by investing in R&D to reduce production cost (see, for instance, Qiu (1997) and the references therein). Such a dynamic framework changes the post-innovation demand and cost structures of the firms and might affect the market competition.

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In this paper, we consider a duopoly model, where in the first stage firms invest in cost-reducing R&D. In the second stage, firms engage in mixed competition. We provide new findings that the quantity-setting firm may undertake more investments than the Cournot firm, and sets a price which is lower than the price-setting firm's price. We show that mixed competition yields greater consumer surplus and social welfare than Cournot competition, and for close substitutes, there is over-production for the quantity setting firm.

## 2 The model

Consider a sector of an economy with two firms  $i = 1, 2$  producing differentiated goods  $q_1$  and  $q_2$  respectively. A representative consumer's utility function is:  $U(q_1, q_2) = \alpha(q_1 + q_2) - (q_1^2 + 2\beta q_1 q_2 + q_2^2)/2$ ,  $\alpha > 0$ ,  $\beta \in [0, 1]$ . The parameter  $\beta$  measures the degree of product differentiation, with differentiation increasing as  $\beta$  is close to zero. The inverse market demands are linear:  $p_i = \alpha - q_i - \beta q_j$ ,  $i, j = 1, 2; i \neq j$ .

Firms may invest in cost-reducing R&D. The pre-innovation costs for the two firms are  $c_i = c < \alpha$ . If firm  $i$  engages in R&D, then by spending  $V(x_i)$  on R&D, it lowers its marginal cost by  $x_i$ :  $c_i = c - x_i$ . We assume that  $V(x_i) = v \frac{x_i^2}{2}$ , where  $v$  relates to the productivity of the R&D technology (higher  $v$  means lower productivity).

Timing is the following. In the first stage, firms 1 and 2 simultaneously invest the amounts  $x_i, i = 1, 2$  respectively in R&D. Firms observe the amount invested by their rivals. In the second stage, firms compete in the market. Firm 1 chooses an output  $q_1$ , while firm 2 chooses a price  $p_2$ .

We consider the subgame perfect Nash equilibrium. In the second stage, firms market profits are  $\pi_1(q_1, p_2; x_1) = p_1(q_1, p_2)q_1 - (c - x_1)q_1$  and  $\pi_2(q_1, p_2; x_2) = p_2 q_2(q_1, p_2) - (c - x_2)q_2(q_1, p_2)$ . The equilibrium in the market subgame is

$$q_1(x_1, x_2) = \frac{(2 - \beta)(\alpha - c) + 2x_1 - \beta x_2}{4 - 3\beta^2} \text{ and} \quad (1)$$

$$p_2(x_1, x_2) = \frac{\alpha(1 - \beta)(2 + \beta^2) + c(2 + \beta - 2\beta^2) - \beta x_1 - 2(1 - \beta^2)x_2}{4 - 3\beta^2}. \quad (2)$$

The equilibrium of the game  $(x_1^*, x_2^*, q_1^*, p_2^*)$  is described by best responses of each firm  $i$  given the outcome of the induced market game

$$x_i^* \in \arg \max_{x_i} \Pi_i(x_i, x_j^*) = \pi_i(q_1(x_i, x_j^*), p_2(x_i, x_j^*); x_i) - \frac{1}{2}v x_i^2,$$

and  $q_1^* = q_1(x_1^*, x_2^*)$ ,  $p_2^* = p_2(x_1^*, x_2^*)$ . Denote by  $\bar{v} = \frac{4(1+\beta)}{(2+\beta)(4-3\beta^2)}$  and  $\underline{v} = \frac{2(2-\beta^2)}{(2-\beta)(4-3\beta^2)}$ . Note that  $\bar{v} > \underline{v}$ . The equilibrium R&D levels are

$$x_1 = \frac{\chi}{\Delta} \frac{v - \underline{v}}{\underline{v}}, \text{ and } x_2 = \frac{\chi}{\Delta} \frac{v - \bar{v}}{\bar{v}}, \quad (3)$$

where  $\Delta \equiv 8(1 - \beta^2)(2 - \beta^2) - 2(4 - 3\beta^2)(8 - 8\beta^2 + \beta^4)v + (4 - 3\beta^2)^3 v^2$  and  $\chi = 8(\alpha - c)(1 - \beta^2)(2 - \beta^2)$ .

**Assumption 1:**  $v > \frac{3\alpha}{c}$ .

This assumption guarantees positive investments in R&D for both firms and positive post-innovation costs. Also, Assumption 1 is needed for the second-order and stability conditions. If R&D investments are very productive, the firms will invest more to gain a competitive advantage in the market game which will lead to zero and even negative post-innovation costs. Throughout the remaining of the paper, we assume that Assumption 1 holds.<sup>1</sup>

Using (1) - (3), we obtain the equilibrium prices and quantities

$$q_1 = q_1^0 + \frac{\chi}{(4 - 3\beta^2)\Delta} \left( 2\frac{v - \underline{v}}{\underline{v}} - \beta\frac{v - \bar{v}}{\bar{v}} \right), \quad (4)$$

$$p_1 = p_1^0 - \frac{\chi}{(4 - 3\beta^2)\Delta} \left( (2 - \beta^2)\frac{v - \underline{v}}{\underline{v}} + (1 - \beta^2)\beta\frac{v - \bar{v}}{\bar{v}} \right), \quad (5)$$

$$q_2 = q_2^0 + \frac{\chi}{(4 - 3\beta^2)\Delta} \left( (2 - \beta^2)\frac{v - \bar{v}}{\bar{v}} - \beta\frac{v - \underline{v}}{\underline{v}} \right), \text{ and} \quad (6)$$

$$p_2 = p_2^0 - \frac{\chi}{(4 - 3\beta^2)\Delta} \left( \beta\frac{v - \underline{v}}{\underline{v}} + 2(1 - \beta^2)\frac{v - \bar{v}}{\bar{v}} \right), \quad (7)$$

where  $p_i^0$  and  $q_i^0$  are equilibrium prices and quantities in the game without R&D. The equilibrium profits are

$$\Pi_1 = \frac{\chi^2 v}{2\Delta^2} \left( \frac{(4 - 3\beta^2)^2 v}{8(1 - \beta^2)} - 1 \right) \left( \frac{v - \underline{v}}{\underline{v}} \right)^2 \text{ and } \Pi_2 = \frac{\chi^2 v}{4\Delta^2} \left( \frac{(4 - 3\beta^2)^2 v}{(2 - \beta^2)^2} - 2 \right) \left( \frac{v - \bar{v}}{\bar{v}} \right)^2. \quad (8)$$

Comparisons based on (3) - (8) yield the following proposition.

**Proposition 1.** *For any  $\beta \in (0, 1)$*

- a)  $x_1 > x_2 > 0$ ,
- b)  $q_1 > q_2$  and  $p_1 < p_2$ ,
- c)  $\Pi_1 > \Pi_2 > 0$ .

Both firms use R&D to minimize costs when products are differentiated. Firm 1 invests more in R&D than firm 2. Consider the incentives of firms to invest in R&D. Using the first-order conditions and the Envelope Theorem we have

$$\frac{\partial \Pi_1}{\partial x_1} = \underbrace{\frac{\partial \pi_1}{\partial p_2} \frac{\partial p_2}{\partial x_1}}_{\text{strategic effect}} + \underbrace{q_1}_{\text{size effect}} - \underbrace{V'(x_1)}_{\text{cost effect}}, \text{ and } \frac{\partial \Pi_2}{\partial x_2} = \underbrace{\frac{\partial \pi_2}{\partial q_1} \frac{\partial q_1}{\partial x_2}}_{\text{strategic effect}} + \underbrace{q_2}_{\text{size effect}} - \underbrace{V'(x_2)}_{\text{cost effect}}.$$

The strategic effect reflects the fact that a firm's R&D reduces its production cost, and consequently affects the competitor's strategic choice. Investing  $x_1$  in R&D has a negative effect on  $p_2$ ,  $\frac{\partial p_2}{\partial x_1} < 0$ . Because goods are substitutes, the effect of  $p_2$  on firm 1's profit is positive:  $\frac{\partial \pi_1}{\partial p_2} = q_1 \frac{\partial p_1}{\partial p_2} > 0$ . It follows that the strategic effect is negative for firm 1. For firm 2, an increase in  $x_2$  induces firm 1 to decrease  $q_1$ ,  $\frac{\partial q_1}{\partial x_2} < 0$ . The effect of  $q_1$  on firm 2's profit is negative,  $\frac{\partial \pi_2}{\partial q_1} = (p_2 - (c - x_2)) \frac{\partial q_2}{\partial q_1} < 0$ . Thus, the strategic effect is positive for firm 2. Next,

<sup>1</sup>Note that Qiu (1997) uses a weaker assumption for Cournot competition,  $v > \frac{\alpha}{c}$ , and a stronger assumption (9) for Bertrand competition.

a firm's R&D reduces its unit cost of production. Therefore, *ceteris paribus*, the larger is the production, the higher is the size effect. Finally, investing in R&D is costly, i.e., the cost effect is negative for both firms.

Proposition 1 states that firm 1's size effect is stronger than the firm's 2 size effect. Moreover, the size effect dominates the negative strategic effect. To see the intuition, we note that the residual demands for firms 1 and 2 are  $q_1 = \frac{1}{1-\beta^2}(\alpha(1-\beta) - p_1 + \beta p_2)$  and  $q_2 = \alpha - p_2 - \beta q_1$  respectively. Firms choose price or quantity from the residual demands, which are defined by the competitor's strategy choice. Note that the residual demand for firm 1 is *more elastic* than for firm 2. In the first case, the absolute value of the slope is  $\frac{1}{1-\beta^2}$ , and in the second case, it is 1. Note also that investments in innovations have the opposite effect on residual demands. Investment  $x_1$  lowers costs for firm 1 and thus lowers  $p_2$ . This shifts the residual demand for firm 1 downwards. Investment  $x_2$  has a decreasing effect on  $q_1$  and, therefore, increases the residual demand for firm 2. Prices and quantities are defined by intersections of corresponding post-innovation costs  $c - x_i$  with marginal residual revenues. Consider an increase in  $x_1$ . Even though the strategic considerations shift the residual demand downwards, because it is sufficiently elastic, this increase in  $x_1$  has a large positive effect on  $q_1$ . Thus, the size effect is strong. Similarly, because the residual demand for firm 2 is relatively inelastic, an increase in  $x_2$  has a smaller size effect.

Without innovations, we have  $p_1^0 > p_2^0$ .<sup>2</sup> With innovations, the price  $p_1$  is lower than  $p_2$  because of two factors. First, firm 1 invests more in cost reduction than firm 2. Second, the residual demand curve is more elastic for firm 1 than for firm 2. Thus, firm 1 has greater incentives to lower the price. Finally, firm 1 is more efficient than firm 2 which leads to the ranking of profits.

In Propositions 2 and 3, we compare mixed competition with Bertrand and Cournot. In these comparisons, we assume that condition (9) below is satisfied to guarantee the regularity conditions for respective equilibria (see also Qiu (1997)).<sup>3</sup>

$$v > \frac{2(2-\beta^2)^2}{(1-\beta^2)(4-\beta^2)^2}. \quad (9)$$

Denoting by  $C$  and  $B$  the outcomes in Bertrand and Cournot games with R&D, the optimal R&D investments, prices, and quantities are

$$x_B = \frac{2(2-\beta^2)(\alpha-c)}{\Delta_B}, \quad x_C = \frac{4(\alpha-c)}{\Delta_C}, \quad p_B = \frac{(2-\beta)(1+\beta)(2+\beta)(\alpha(1-\beta)+c)v - 2\alpha(2-\beta^2)}{\Delta_B}, \quad (10)$$

$$p_C = \frac{(4-\beta^2)(\alpha+c+\beta c)v - 4\alpha}{\Delta_C}, \quad q_B = \frac{(4-\beta^2)(\alpha-c)v}{\Delta_B}, \quad \text{and} \quad q_C = \frac{(4-\beta^2)(\alpha-c)v}{\Delta_C}, \quad (11)$$

where  $\Delta_C = v(2+\beta)^2(2-\beta) - 4$ , and  $\Delta_B = (2-\beta)^2(1+\beta)(2+\beta)v - 2(2-\beta^2)$ . From (10) and (11), we immediately obtain  $x_C > x_B$ ,  $q_C < q_B$ , and  $p_C > p_B$ .

<sup>2</sup>See also Tremblay & Tremblay (2011).

<sup>3</sup>Note that condition (9) is involved only when we consider comparisons with Bertrand model.

**Proposition 2.** For any  $\beta \in (0, 1)$

- a)  $x_2 < x_B < x_C$ ,
- b) there exist  $0 < \beta_2 < \beta_1 < 1$  such that  $x_1 > x_C$  for  $\beta \in (0, \beta_2)$ ,  $x_B \leq x_1 \leq x_C$  for  $\beta \in [\beta_2, \beta_1]$  and  $x_1 < x_B$  for  $\beta \in (\beta_1, 1)$ ,
- c)  $p_B < p_1 < p_2 < p_C$ , and  $q_2 < q_C < q_B < q_1$ .

In Bertrand and Cournot models, the strategic effect is negative and positive respectively. The residual demands for Cournot competition are more inelastic than for Bertrand competition. Thus, the size effect is stronger for Bertrand competition than in Cournot competition. However, the size effect is not enough to overcome the differences in strategic effects (see Qiu (1997)). Thus, we have  $x_C > x_B$ . The relative importance of the positive strategic effect compared to the size effect for firm 2 leads to  $x_2 < x_B$ . The ranking of investment by firm 1 depends on the level of substitutability of goods. When  $\beta$  increases, the negative strategic effect becomes stronger (it may be even stronger than for Bertrand competition). This drives  $x_1$  down compared to  $x_C$ , and for large  $\beta$ ,  $x_1$  is below  $x_B$ . However, these differences in levels of innovations are not enough to change the ranking of prices and quantities.

It is clear that the consumer surplus under Bertrand competition is greater than under Cournot competition. Comparisons with the mixed competition are more intricate. Prices for the mixed competition are between prices for Cournot and Bertrand. However, the gap between quantities  $q_2$  and  $q_1$  is quite large. We consider welfare implications in the next section.

### 3 Consumer Surplus, Profits, and Welfare

The Bertrand and Cournot consumer surpluses and profits with R&D are

$$CS_B = \frac{(\alpha - c)^2 v^2}{\Delta_B^2} (1 + \beta)(4 - \beta^2)^2, \quad \Pi_B = \frac{(\alpha - c)^2 v}{\Delta_B^2} ((4 - \beta^2)^2 (1 - \beta^2)v - 2(2 - \beta^2)^2),$$

$$CS_C = \frac{(\alpha - c)^2 v^2}{\Delta_C^2} (1 + \beta)(4 - \beta^2)^2, \quad \Pi_C = \frac{(\alpha - c)^2 v}{\Delta_C^2} ((4 - \beta^2)^2 v - 8),$$

respectively. We have

**Proposition 3.** For any  $\beta \in (0, 1)$

- a)  $CS_C < CS < CS_B$ ,
- b)  $\Pi_2 < \Pi_B < \Pi_C$ , and there exists a unique  $v_1 = v_1(\beta)$  such that

$$\Pi_1 - \Pi_C = \begin{cases} \geq 0 & \text{if } v \leq v_1 \\ < 0 & \text{if } v > v_1 \end{cases},$$

- c)  $W_C < W < W_B$ .

Vives (2001) and Singh & Vives (1984) show that without R&D, consumer surplus is the largest for Bertrand competition. Qiu (1997) finds the same result with cost-reducing R&D. We confirm the robustness of this result. Prices are still lower in the case of Bertrand competition. High quantity and low price for firm 1 are not enough to overcome the inefficiency generated by

firm 2. Singh & Vives (1984) establish that without R&D, Cournot competition leads to higher profits than Bertrand and mixed competition when products are substitutes. We show that this is not always the case when firms may invest in R&D. In the case of sufficiently productive innovations, firm 1's profit dominates the Cournot's profit. In this case, firm 1 invests more in cost-reducing innovation. Finally, Qiu (1997) shows that  $W_B > W_C$ ; Bertrand competition is always more efficient than Cournot. We find that mixed competition leads to intermediate welfare.

## Social Planner

The social planner's problem is

$$\max_{x_1, x_2, q_1, q_2} W_S = U(q_1, q_2) - \sum_{i=1}^2 (c - x_i)q_i - \frac{1}{2}v \sum_{i=1}^2 x_i^2.$$

**Assumption 2:**  $v > \frac{1}{1-\beta}$ .

Assumption 2 ensures that the second-order condition for the social planner's problem is satisfied. Assumption 1 guarantees that optimal post-innovation costs are positive. The social planner's optimal decision is symmetric and given by

$$x_i = x_S = \frac{\alpha - c}{(1 + \beta)v - 1}, \quad q_i = q_S = \frac{(\alpha - c)v}{(1 + \beta)v - 1}. \quad (12)$$

**Proposition 4.** *Assume Assumptions 1 and 2 hold. Then,*

- a) for any given  $\beta \in (0, 1)$   $x_S > x_1 > x_2$ ,
- b) for any given  $\beta \in (0, 1)$ ,  $q_S > q_2$ ,
- c) if  $\beta \in \left(0, \frac{\sqrt{17}-1}{4}\right)$ , then  $q_S > q_1$ ,
- d) there exist  $\beta_3 \in \left(\frac{\sqrt{17}-1}{4}, 1\right)$  and a unique  $v_2(\beta)$  such that  $q_S < q_1$  whenever  $\beta \in [\beta_3, 1)$  or  $[\beta < \beta_3$  and  $v > v_2(\beta)]$ .

In the social planner's model, the strategic effect vanishes. Even though firm 1 invests more than firm 2, it never overinvests. Interestingly, for close substitute products, firm 1 may produce more than the social optimum.<sup>4</sup> In this case, the negative strategic effect is strong and to overcome it, in equilibrium, firm 1 produces more than the socially optimal quantity.

## Acknowledgments

We are grateful to Roland Pongou, Elena Quercioli and especially Gamal Atallah for their insightful comments and useful discussions. We also acknowledge the editor and anonymous referee for their thoughtful suggestions.

<sup>4</sup>Qiu (1997) finds that  $q_S$  is always greater than  $q_B$ .

## Proofs

**Proof of Proposition 1.** a)  $\frac{x_1}{x_2} = \frac{v-\underline{v}}{v-\bar{v}} > 1$ , since  $\bar{v} > \underline{v}$ , and  $v > \bar{v}$ . b)  $q_1 - q_2 = q_1^0 - q_2^0 + \frac{\chi}{(4-3\beta^2)\Delta} \left\{ (2+\beta) \left( \frac{v-\underline{v}}{v} - \frac{v-\bar{v}}{v} \right) + \beta^2 \frac{v-\bar{v}}{v} \right\}$  and  $p_1 - p_2 = p_1^0 - p_2^0 - \frac{\chi}{(4-3\beta^2)\Delta} \left\{ (1-\beta) (2+\beta^2) \frac{v-\underline{v}}{v} + (2+\beta)(1-\beta^2) \frac{v-\bar{v}}{v} \right\}$ . Given that  $q_1^0 - q_2^0 > 0$ , and  $p_1^0 - p_2^0 < 0$ , it follows that  $q_1 > q_2$ , and  $p_1 < p_2$ . c) Since  $\frac{v-\underline{v}}{v} > \frac{v-\bar{v}}{v}$ ,  $\Pi_1 - \Pi_2 > \frac{\chi^2(4-3\beta^2)v^2}{32\Delta_C^2(1-\beta^2)(2-\beta^2)^2} \left( \frac{v-\bar{v}}{v} \right)^2 g(\beta)$ , with  $g(\beta) = 2(2-\beta^2)^2 - 8(1-\beta^2) \geq 0$ .

**Proof of Proposition 2.** a)  $x_2 - x_B = \frac{2\beta^4 v(2-\beta^2)(\alpha-c)}{\Delta\Delta_B} A(v; \beta)$ , with  $A(v; \beta) = a_1 + a_2 v$ , where  $a_1 = -2(2-\beta-2\beta^2)$ , and  $a_2 = -\beta^2(4-3\beta^2) < 0$ . Let  $\bar{a} = -\frac{a_1}{a_2}$ ,  $\bar{a}$  is an increasing function with  $\bar{a}(1) = 2$ , and  $\bar{a} \geq 0$  if  $\beta \geq (\sqrt{17}-1)/4$ . For any  $\beta$ ,  $A(v; \beta) < 0$  if  $v > \bar{a}$ . For any  $v$  we have  $v > \bar{a}$ . Therefore,  $A(v; \beta) < 0$ , and  $x_2 < x_B$ , because  $\text{sign}(x_2 - x_B) = \text{sign}(A(v; \beta))$ . Following that approach, we prove that  $x_2 < x_C$ . It is immediate that  $x_B < x_C$ . b)  $x_1 - x_B = \frac{2\beta^3 v(2-\beta^2)(\alpha-c)}{\Delta\Delta_B} B(v; \beta)$ , where  $B(v; \beta) = b_1 + b_2 v$ , with  $b_1 = -16 + 8\beta + 24\beta^2 - 8\beta^3 - 8\beta^4 + 2\beta^5$ , and  $b_2 = 32 - 16\beta - 64\beta^2 + 24\beta^3 + 38\beta^4 - 9\beta^5 - 6\beta^6$ . We have  $\text{sign}(x_1 - x_B) = \text{sign}(B(v; \beta))$ . Then, given any  $v$ , there exists a unique  $\beta_1$ —the root of order 4 of a polynomial of degree 6—, such that  $B(v; \beta) \geq 0$  if  $0 < \beta \leq \beta_1$  and  $B(v; \beta) < 0$  if  $\beta_1 < \beta < 1$ . Similarly, we compare  $x_1$  and  $x_C$ , and items in c).

**Proof of Proposition 3.** a)  $CS - CS_C = \frac{(\alpha-c)^2 v^2}{2\Delta^2 \Delta_C^2} C(v; \beta)$ , where  $G(v; \beta) = -2(1+\beta)(4-\beta^2)^2(8(2-3\beta^2+\beta^4)+2(-4+3\beta^2)(8-8\beta^2+\beta^4)v - (-4+3\beta^2)^3 v^2 + (4-3\beta^2)^2(4(8+\beta(8+\beta(2+\beta))(-6+\beta(-3+4\beta))))\Delta_C^2 + 4(-4+3\beta^2)\Delta^2$ . Using Mathematica (Wolfram Research Inc. (2019)), we can write  $C(v; \beta) = c_0 + c_1 v + c_2 v^2 + c_3 v^3 + c_4 v^4$ , where each  $c_i$  is a function of  $\beta$ . For any  $\beta$ , there are two real solutions to  $C = 0$ , which are  $\underline{c}(\beta)$  and  $\bar{c}(\beta)$ , with  $\bar{c} > \underline{c}$ . The latter is a decreasing function whereas the former is an increasing function with  $\bar{c}(1) = 4/3$ . Moreover for any  $v > \bar{c}$ ,  $C(v; \beta) > 0$ . Since  $v > \bar{c}$ , then, for any  $\beta \in (0, 1)$ ,  $CS > CS_C$ . Using the same approach, we prove that  $CS < CS_B$ , and it is immediate that  $CS_B > CS_C$ .

b) It is also immediate that under (9),  $\Pi_B - \Pi_C < 0$ . Now, we write  $\Pi_2 - \Pi_C = \frac{(\alpha-c)^2 v}{\Delta^2 \Delta_C^2} D(v; \beta)$ , where  $D(v; \beta) = (1-\beta)^2[-2(2-\beta^2)^2 + (4-3\beta^2)^2 v][4(1+\beta) - (2+\beta)(4-3\beta^2)v]^2 \Delta_C^2 + (8 - (4-\beta^2)v)\Delta$ , and  $\text{sign}(\Pi_2 - \Pi_C) = \text{sign}(D(v; \beta))$ . Simplification and term collection yield  $D(v; \beta) = v(d_1 + d_2 v + d_3 v^2 + d_4 v^3 + d_5 v^4)$ , where each  $d_i$  is a function of  $\beta$ . Given any  $\beta$ , there exist four real solutions to  $D = 0$ , which we denote by  $\bar{d}_i(\beta), i = 1, 2, 3, 4$ . It can be shown that  $\bar{d}_1$  is a decreasing function with  $\bar{d}_1(0) = 0.5$ ,  $\bar{d}_2, \bar{d}_3$ , and  $\bar{d}_4$  are increasing functions with  $\bar{d}_2(1) = 8/9$ , and  $\bar{d}_3(1) = \bar{d}_4(1) = 2$ . Therefore, for any  $\beta \in (0, 1)$ ,  $D(v; \beta) < 0$  if  $0 < v < \bar{d}_1$  or  $\bar{d}_2 < v < \bar{d}_3$  or  $v > \bar{d}_4$ , and  $D(v; \beta) > 0$  if  $\bar{d}_1 < v < \bar{d}_2$  or  $\bar{d}_3 < v < \bar{d}_4$ . We have  $v > \bar{d}_4(\beta)$  for any  $\beta \in (0, 1)$ , and the result follows. We use the same reasoning to compare  $\Pi_1$  and  $\Pi_C$ , and prove that  $\Pi_2 - \Pi_B < 0$ .

c)  $W - W_B = \frac{(\alpha-c)^2 v}{2\Delta^2 \Delta_B^2} E(v; \beta)$ , where  $E(v; \beta) = e_0 + e_1 v + e_2 v^2 + e_3 v^3 + e_4 v^4 + e_5 v^5$ , with each  $e_i$  as a function of  $\beta$ . Under (9), there are two real solutions to  $E = 0$ , which we denote  $\underline{e}(\beta)$ , and  $\bar{e}(\beta)$ , with  $\bar{e} > \underline{e}$ . The inequality  $v < \bar{e}$  contradicts Assumption 1. Moreover, there exists a positive number  $s < \bar{\beta}$ , such that  $E(v; \beta) < 0$  for any  $v > \bar{e}$  and  $\beta < s$ . Also, if  $\beta \geq s$ , and  $v > \bar{e}$ , we can't have  $E(v; \beta) \geq 0$ . However, if  $\beta \geq s$ , then  $E(v; \beta) < 0$  if  $v > 3$  and  $\beta < 1$ . Since  $v > 3$ , we conclude that  $E(v; \beta) < 0$ , and  $W < W_B$ . Using the same reasoning, we prove that  $W > W_C$ .

**Proof of Proposition 4.** Part a) is straightforward.

b) We write,  $q_2 - q_S = \frac{v(\alpha-c)}{\Delta\Delta_S} F(v; \beta)$ , where  $\Delta_S = (1+\beta)v-1$ ,  $F(v; \beta) = f_0 + f_1 v + f_2 v^2$ , with  $f_0 = -4\beta^2(1-\beta^2) < 0$ ,  $f_1 = 16 - 20\beta^2 + 4\beta^3 + 2\beta^4 - 3\beta^5 + 3\beta^6 > 0$ ,  $f_2 = -32 + 16\beta + 64\beta^2 - 40\beta^3 - 42\beta^4 + 33\beta^5 + 9\beta^6 - 9\beta^7 < 0$ , and  $\text{sign}(q_2 - q_S) = \text{sign}(F(v; \beta))$ . Since the second derivative is  $F_{vv} = 2f_2 < 0$ ,  $F$  is strictly concave in  $v$ . Note that  $f_1^2 - 4f_0 f_2 > 0$  for all  $\beta \in (0, 1)$ . Thus, given any  $\beta$ , there are two solutions to  $F = 0$ , which are,



$\underline{f}(\beta) = \frac{-f_1 - \sqrt{f_1^2 - 4f_2f_0}}{2f_2}$ , and  $\bar{f}(\beta) = \frac{-f_1 + \sqrt{f_1^2 - 4f_2f_0}}{2f_2}$ , with  $\underline{f} < \bar{f}$ . Therefore,  $F(v; \beta) \geq 0$  if  $\underline{f} \leq v \leq \bar{f}$ , and  $F(v; \beta) < 0$  if  $0 < v < \underline{f}$  or  $v > \bar{f}$ . It can be shown that  $\bar{f}$  is an increasing function with  $\bar{f}(1) = 2$ . Under Assumption 1,  $v > \bar{f}$  for any given  $\beta$ . Therefore, for any  $\beta$ ,  $F(v; \beta) < 0$ , and  $q_2 < q_S$ .

c)  $q_1 - q_S = \frac{v(\alpha - c)}{\Delta_S \Delta} G(v; \beta)$ , where  $G(v; \beta) = g_0 + g_1 v + g_2 v^2$ , with  $g_0 = 2\beta^2(2 - \beta^2) > 0$ ,  $g_1 = (4 - 3\beta^2)[4 + \beta^2(-8 + \beta(-1 + 2\beta))]$ ,  $g_2 = (4 - 3\beta^2)^2(-2 + \beta + 2\beta^2)$ , and  $\text{sign}(q_1 - q_S) = \text{sign}(G(v; \beta))$ . Note that for any given  $\beta$ , there are two real solutions to  $G = 0$ , which we denote  $\underline{g}(\beta)$  and  $\bar{g}(\beta)$ , with  $\underline{g}(\beta) < 0$  for any  $\beta$ . Moreover,  $g_2(\beta) \geq 0$  if  $\beta \in (0, \frac{\sqrt{17}-1}{4}]$ , and  $g_2(\beta) < 0$  if  $\beta \in (\frac{\sqrt{17}-1}{4}, 1)$ . Assume that  $\beta \in (0, \frac{\sqrt{17}-1}{4}]$ . Then, the second derivative  $G_{vv} = 2g_2 \geq 0$ , and  $G$  is concave in  $v$ . Hence, for any given  $v$ ,  $G(v; \beta) \geq 0$  if  $0 < v \leq \bar{g}$ , and  $G(v; \beta) < 0$  if  $v > \bar{g}$ . Under Assumption 1,  $G$  is always negative, and this happens only if  $v > \bar{g}$ . Given  $\alpha > c$ , then  $v > 3$ . Hence, if  $v \leq 2.5(5 + \sqrt{17})$ , we have  $G < 0$ . Otherwise,  $v > 2.5(5 + \sqrt{17})$ , then we need  $\beta < \frac{\sqrt{17}-1}{4}$  to have  $G < 0$ , which is satisfied. It follows that for any  $\beta \leq \frac{\sqrt{17}-1}{4}$ ,  $G(v; \beta) < 0$ , and  $q_1 \leq q_S$ .

d) Assume that  $\beta \in (\frac{\sqrt{17}-1}{4}, 1)$ . Then, the second derivative  $G_{vv} = 2g_2 > 0$ , and  $G$  is strictly convex in  $v$ . Hence, for any given  $v$  which satisfies Assumption 2,  $G(v; \beta) \geq 0$  if  $v \geq \bar{g}$ , and  $G(v; \beta) < 0$  if  $\underline{g} < v < \bar{g}$ . Consider  $\beta_3$  the root of degree 3 of the polynomial  $-16 + 40x^2 + 8x^3 - 28x^4 - 16x^5 + 7x^6 + 6x^7$ . Given any  $\beta \geq \beta_3$ , and any  $v$  that satisfies Assumptions 1 and 2, then  $v > \bar{g}$ . Therefore,  $G(v; \beta) > 0$ , and  $q_1 - q_S > 0$ . If  $\beta < \beta_3$ , then  $G(v; \beta) > 0$  whenever  $v > \bar{g}$ . Take  $v_2 = \bar{g}$ , and  $q_1 > q_S$ .

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