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Supermajority Politics: Counting Equilibria and Gauging Diversity[⋆]

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Abstract

The standard Bowen (Bowen, 1943) model of political competition with single-peaked preferences predicts party convergence to the ideal point supported by the median voter, with the number of equilibrium policies not exceeding two. This fundamental result assumes majority rule and unidimensional policy space. Relaxing these two assumptions, we extend this model to *static* and *dynamic* political economies where the voting rule is a *supermajority rule*, and the policy space is *totally ordered*, although it may not be unidimensional. Voters' strategic behavior is captured by the core in static environments and by the largest consistent set in dynamic environments. In these settings, we determine the exact number of equilibria and show that it is an increasing *correspondence* of the supermajority's size. This result has implications for the depth of policy diversity across structurally identical political economies governed by supermajority rules. We develop an application to immigration policies.

Keywords: Supermajority rules; Single-peakedness; Policy diversity; Static games; Dynamic games; Core; Largest consistent set *JEL*: C71; D72; P16

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1. Introduction

The standard Bowen model (Bowen, 1943; Black, 1948; Downs, 1957) of political competition with single-peaked preferences under majority rule generally predicts party convergence to the ideal point supported by the median voter. This fundamental result assumes *majority rule* and a *unidimensional* policy space. Relaxing these two assumptions, we extend this model to a spatial model of political competition between an incumbent policy and an alternative under two different decision-making environments, one being *static* and the other being *dynamic*. For adoption, the alternative must obtain the support of a supermajority of voters who, by assumption, hold single-peaked preferences over a *totally ordered* policy space, which need not be a uni-dimensional set. In the *static* setup, we focus on the set of equilibrium policies in the **core** (Gillies, 1959). *Equilibrium policies* are those which, if already the status quo, are never defeated in a pairwise supermajoritarian election against alternatives in the policy space.[4] In the *dynamic* setup, agents make amendments sequentially and the game can go on indefinitely. In this setting, we focus on the set of equilibrium policies in the **largest consistent set** (Chwe, 1994). This solution concept assumes that agents are farsighted.[5] In both cases, our main finding is to determine the **number of equilibria** and show how it depends on the supermajority's size.[6] We discuss implications for the depth of policy diversity and divergence across structurally identical political economies, and develop an application to immigration policies.

We consider a voting body, $N = \{1, 2, ..., n\}$, composed of a finite number of agents and endowed with a *supermajority rule*, $\mathcal{L}_\alpha$, and a non-empty *totally ordered* policy space, Z. The policy space Z represents the set of possible policies—the number of points or ideological approaches—to a given policy problem. We assume that Z is *totally ordered* by a binary relation denoted $\geq^Z$ (i.e,

[4] The core, like the Nash equilibrium, is regarded as a pioneer solution concept. In fact, it is the equilibrium concept used in the pioneering works of Bowen (1943), Black (1948), and Downs (1957), although it is not called as such in these studies. Note, however, that all our results are highly robust to alternative solution concepts (for example, top cycle, the uncovered set, the Banks set, etcetera). Proofs are available upon request.

[5] To our knowledge, our paper is the first to examine this question in a dynamic setting where agents are farsighted.

[6] While we generalize the Median Voter Theorem as a special case of our result, this is not our main finding. Our main goal is to count the number of equilibria as a function of the supermajority rule and discuss its empirical implications.

$\geq^Z$ is reflexive, transitive, antisymmetric, and complete), and we denote by $>^Z$ the strict part of this relation. We assume that agents' preferences are *single-peaked* with respect to the strict order $>^Z$ on Z. The *supermajority rule* $\mathcal{L}_\alpha$ is a function which maps each coalition $C \subseteq N$ into 1 or 0. Given a threshold $\alpha \in [1/2, 1]$, $\mathcal{L}_\alpha(C) = 1$ when either the coalition C consists of more than αn members if $\alpha < 1$, or C consists consists of n members if $\alpha = 1$; we say such C is a *winning coalition*, and it holds power to amend a policy under consideration in the election process. Coalitions for which $\mathcal{L}_\alpha(C) = 0$ are *losing* coalitions, and they do not hold the right to amend policies.

We examine the minimum and the maximum number of equilibrium policies under a pairwise supermajoritarian election between a status quo policy, z_0, chosen by Nature, a lottery or an agent, and an alternative policy z_1 chosen by an agent from the set $Z \setminus \{z_0\}$. Agents have equal probabilities of being selected, by Nature or a lottery, to make proposals against the status quo z_0. If z_0 wins, meaning that no winning coalition under $\mathcal{L}_\alpha$ chooses z_1 over z_0, then it remains in place and the contest ends. If z_0 loses (z_1 wins), then z_1 replaces z_0 and the contest ends. An equilibrium policy is never defeated in the election process. we find that the *minimum number* of equilibrium policies is a constant function of the supermajority's size. However, the *maximum number* of equilibria is an increasing function of the supermajority needed to pass legislation and is a function of the way the incumbent policy is selected. This number represents the depth of policy diversity across structurally *identical* political economies under supermajority rules. More precisely, if Nature randomly selects a legislator to make a proposal, the maximum number of policies is finite, and it is a non-decreasing function of the supermajority's size α (Theorem 1).[7] It follows that the number of equilibrium policies is a non-decreasing *correspondence* of the supermajority's size needed to pass a policy (see Figure 1). One corollary of this result is the existence of a unique equilibrium under majority rule ($\alpha = \frac{1}{2}$) when there are an odd number of voters and the existence of, at most, two equilibria when there are an even number of voters (Corollary 1). The familiar Median Voter Theorem (MVT) (Black, 1948; Downs, 1957), then, is a particular case of our result, which extends it to a more general setting. If Nature randomly selects

[7]The idea of selecting legislators and policymakers by a lottery system, also called "sortition", is old; it dates to the fourth century BC and is still practiced today (Manin, 1997; Wantchekon and Neeman, 2002; Procaccia, 2019).

the incumbent policy from the policy space, the set of equilibrium policies becomes a continuum—a convex and compact subset of the policy space—and we determine its exact bounds (Theorem 2). These findings are robust, in that they continue to hold when legislators display farsighted behavior in the dynamic setting (Theorem 3).

A practical implication of our analysis is that economies that are *identical* in terms of their policy spaces, voters' preferences, and voting rule may end up diverging in terms of their policy choice. Theorems 1, 2, and 3 provide the possibility to *quantify* the extent of this divergence. In particular, under majority rule, no divergence is possible, unless the number of voters is even. Under a supermajority rule, policies can diverge, and the policy gap is an increasing function of the supermajority's size. Our results can also be used to determine the number of "competing" political parties in a given election under supermajority rules. Duverger (1963) was among the first scholars to examine the relationship between electoral systems and the number of parties in a political economy. He focused primarily on the *plurality rule*, whereas our focus is on *supermaority rules*. He proposed what are known today as the Duverger' law and the Duverger's hypothesis. The Duverger's law predicts that two major parties will form under the plurality rule (Duverger, 1963, 217), and the Duverger's hypothesis states that "the simple-majority system with second ballot and proportional representation favors multi-partyism" (Duverger, 1963, 239). Our analysis can be seen as an extension of the Duverger's ideas to supermajority rules in static and dynamic models of political competition. We find, in particular, that simple majority rule favours a two-party system, which offers an anology to Duverger's law. However, this law fails under larger majority requirement: the maximum equilibrium number of political parties is a non-decreasing function of the supermajority's size.

By focusing on the *number* of equilibria in a model of spatial political competition, we clearly depart from the extant literature that has primarily studied the question of equilibrium existence (see, for example, Feldman and Serrano (2006) for a thorough overview of these findings), but has completely overlooked the issue of *number* of equilibria. More generally, supermajority rules have been studied in terms of their equilibrium properties (see, for instance, Fey (2003), Tchantcho et al. (2010), Peleg (1978), and Freixas and Kurz (2019)) and as basis for generating more complex voting rules (see, for example, Taylor and Zwicker (1993), Freixas and Puente (2008), and Guemmegne and Pongou (2014)). Our analysis departs from these literatures by focusing on the number of equilibria instead and deriving implications for the depth of *policy diversity* across structurally similar *political economies*. In so doing, we also extend classical results to an environment that is more general.

The paper proceeds as follows. In Section 2, we introduce preliminary concepts. Section 3 examines the number and range of equilibrium policies under static and dynamic political settings. In Section 4, we use our results to explain policy diversity across identical political economies. We conclude in Section 5.

2. Preliminary Concepts

We model a *political economy* as a list $\mathcal{P} = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$, where: **(a)** $N = \{1, 2, ..., n\}$ is a voting body, composed of a finite number of agents (we assume that n is at least 2); **(b)** Z is a non-empty policy space, which is totally ordered by a binary relation $\geq^Z$ that is reflexive, transitive, antisymmetric, and complete (we denote by $>^Z$ the strict part of the binary relation $\geq^Z$); **(c)** $\succeq_i$ denotes agent i's preference relation over Z and $(\succeq_i)$ denotes a preference profile over Z; and **(d)** $\mathcal{L}_\alpha$ is a supermajority rule (or qualified majority) of size α ($\alpha \in [\frac{1}{2}, 1]$). A supermajority rule is a distribution of political decision-making power among the various coalitions of agents eligible to vote (for simplicity, we assume that each agent can vote). The aggregate function $\mathcal{L}_\alpha$ is a family of voting rules that includes from simple majority rule ($\alpha = \frac{1}{2}$) up to unanimity ($\alpha = 1$).

For any policies $x, y \in Z$, the intervals $[x, y]$ (and $]x, y[$) are subsets of Z defined as: $[x, y] = \{z \in Z : y \geq^Z z \geq^Z x\}$ (and $]x, y[= \{z \in Z : y >^Z z >^Z x\}$), respectively. For a given finite and non-empty set X, we denote by $|X|$, the cardinality of X (i.e., the number of elements contained in X), and n the cardinality of N. For $x, y \in Z$, $y \succeq_i x$ indicates that agent i weakly prefers y to x; $y \succ_i x$ indicates that agent i prefers y to x; and $y \sim_i x$ indicates that agent i is indifferent between y and x. Moreover, for $S \subseteq N$, $y \succ_S x$ indicates that $y \succ_i x$ for each $i \in S$ (we say S prefers y over x); and $y \succeq_S x$ indicates that $y \succ_i x$ for some $i \in S$ and $y \sim_j x$ for other $j \in S$ (we say S weakly prefers y over x).

Following the classical literature on spatial competition (see, for example, Bowen (1943) and Black (1948)), we assume that the profile $(\succeq_i)$ is **single-peaked** with respect to the strict order

$>^Z$ on Z. It means that each agent has an ideal policy in the policy space Z, and policies that are further from this ideal policy are preferred less. Formally, for each agent $i \in N$, there exists a policy $z_i^p \in Z$ such that: (1) for any other policy $z \neq z_i^p$, $z_i^p \succ_i z$; and (2) for any policy z, z', if $z >^Z z' >^Z z_i^p$, then $z' \succ_i z$, and, if $z_i^p >^Z z >^Z z'$, then $z \succ_i z'$.

3. Number and Range of Equilibrium Policies

In this section, we examine the existence and the maximum number of equilibrium policies under one-shot political games (Section 3.1) and dynamic political games (Section 3.2). To perform the analysis in one-shot games, we distinguish two cases: (i) an agent is randomly chosen to make a proposal (Section 3.1.1); or (ii) the status quo policy is chosen by Nature (Section 3.1.2). We assume that agents have equal probabilities of being selected, by Nature or a lottery as an agenda setter. Controlling for temporal and geographic factors that affect agents' preferences and status-quo policies, what is the relationship between a legislative body's voting rule and policy stability?

3.1. One-shot Political Games

Political contests occur as follows:

1. At time $t = 0$, a policy z_0 is randomly chosen chosen by Nature, a lottery, or an agent from the policy space Z.

2. At time $t = 1$, a contest is organized between z_0 (the status quo) and an alternative z_1, chosen exogenously by an agent from the set $Z \setminus \{z_0\}$.

    a) If z_0 wins, meaning that no winning coalition under $\mathcal{L}_\alpha$ chooses z_1 over z_0, then it remains in place and the contest ends.

    b) If z_0 loses (z_1 wins), then z_1 replaces z_0 and the contest ends.

The rational behavior in the one-shot political game above is straightforward. Each agent chooses between the status quo policy and a political alternative. The incentive driving agents to vote for an opposition policy is the condition that it is preferable to the status quo. Formalizing this behavior, let $\gg_i$ denote the incentive by which agent i decides to support an opposition policy z_1 over the status quo z_0. If agent i prefers z_1 over z_0 (i.e., $z_1 \succ_i z_0$), agent i will vote for z_1

over z_0, denoted as $z_1 \gg_i z_0$. The policy z_1 wins the pairwise supermajoritarian election if there exists a winning coalition C that supports z_1 over z_0 ($z_1 \gg_C z_0$). We can now introduce the equilibrium set, defined as follows.

Definition 1 Let $\mathcal{P} = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy and C be a winning coalition.

1. z' defeats z (or $z' \gg z$) thanks to C (i.e., $z' \gg_C z$) if C prefers z' over z (i.e, $z' \succ_C z$).
2. z is defeated if there exists a policy z' and a winning coalition C' such that z' defeats z thanks to C'.
3. The *equilibrium set* $\mathcal{E}(\mathcal{P})$ consists of all undefeated policies. □

An equilibrium policy is one that, if chosen as the status quo, could not be defeated or replaced by another policy. In a pairwise contest between two policies, say z and z', the former receives votes from agents whose ideal points are closer than z to z', and vice versa. Each agent's payoff depends on the distance between her ideal policy and the winning policy.

3.1.1. Nature Randomly Selects a Proposer

At time $t = 0$, an agent is randomly selected, by Nature or a lottery, to make a proposal. Agents are identical with equal probabilities of being selected. The proposer chooses the status quo. Given single-peakedness and rational behavior, each agent's best choice is to propose the closest equilibrium policy ideal to his ideal point as the status quo.[8] The following theorem demonstrates the existence of a policy that cannot be defeated in a pairwise supermajoritarian election and provides the maximum number of policies that can be implemented. Before enunciating the result, we introduce the following notation: for any real number x, the value $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Theorem 1 *Let $\mathcal{P} = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy. Assume that voters are chosen randomly with positive probability to make a proposal. If voters have single-peaked preferences*

[8] One can trace a similar argument from the work of Downs (1957) and the seminal essay of Riker (1982) and the references therein. Even if we assume that the proposer is not rational and he or she proposes his or her ideal point as the status quo, our findings do not change. Throughout the paper, we assume that agents are rational in their decisions.

over Z, then there exists at least one equilibrium and the number of equilibria is finite. Formally:

$$1 \leq |\mathcal{E}(\mathcal{P})| \leq \min\{2\lfloor \alpha n \rfloor + 2 - n, n\}.$$

The maximum number of equilibria in $\mathcal{P}$ is n when $\alpha = 1$, and $2\lfloor \alpha n \rfloor + 2 - n$ when $\alpha \in [\frac{1}{2}, 1)$. □

Let $z \in Z$ be a policy and define $S(z)$ as the number of agents for whom z is the peak. A coalition of agents S has a veto right to amend a given status quo if S is a winning coalition, i.e., $|S| > \alpha n$. Consider the functions f and g defined on the policy space Z as follows: for any policy $z' \in Z$,

$$f(z') = \sum_{z \geq^Z z'} S(z) - \alpha n, \text{ and } g(z') = \sum_{z' \geq^Z z} S(z) - \alpha n.$$

Let define the following sets:

$$Z_f = \{z' \in Z : \ f(z') > 0\}, \text{ and } Z_g = \{z' \in Z : \ g(z') > 0\}.$$

To prove Theorem 1, the following lemmas proved useful.

Lemma 1 *Let $\mathcal{P} = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy, with $\alpha \in [0, 1)$. There exist two peaks z_1^* and z_2^* such that: z_1^* minimizes f over Z_f, and z_2^* minimizes g over Z_g.* □

Proof (Lemma 1) Notice that neither Z_f, nor Z_g is empty. In fact, given that z_{min} and z_{max} are respectively the smallest and the greatest peaks of Z, then $f(z_{min}) = n - \alpha n = (1-\alpha)n > 0$ and $g(z_{max}) = n - \alpha n = (1-\alpha)n > 0$, since $\alpha < 1$, which implies that $z_{min} \in Z_f$ and $z_{max} \in Z_g$, in turn implying that $Z_f \neq \emptyset$ and $Z_g \neq \emptyset$. Given that Z_f is finite and f is a strictly decreasing function, there exists a unique peak z_1^* which minimizes f over Z_f. In addition, for any peak $z' >^Z z_1^*$, $f(z') \leq 0$, which implies that $\sum_{z \geq^Z z'} S(z) \leq \alpha n$. Similarly, given that Z_g is finite and g is a strictly increasing function, there exists a unique peak z_2^* which minimizes g over Z_g. In addition, for any peak $z_2^* >^Z z'$, $g(z') \leq 0$, which implies that $\sum_{z' \geq^Z z} S(z) \leq \alpha n$. ■

Lemma 2 *Assume that $z_1^* = z_2^* = z^*$. Then, $\mathcal{E}(\mathcal{P}) = \{z^*\}$.* □

PROOF (LEMMA 2) We claim that z^* is the Condorcet winner. Indeed, let $z \in Z$ be a peak. If $z^* >^Z z$, by definition of z^*, $\sum_{z \geq^Z z'} S(z') \leq \alpha n$ and $\sum_{z' \geq^Z z^*} S(z') > \alpha n$, which implies that z^* defeats z in a pairwise supermajoritarian election. Similarly, if $z >^Z z^*$, we show in the same way that z^* defeats z. It follows that z^* defeats any other peak z. Since there is no other option which defeats z^*, then $\mathcal{E}(\mathcal{P}) = \{z^*\}$. ■

Lemma 3 *If $z_1^* \neq z_2^*$, then $z_2^* >^Z z_1^*$.* □

PROOF (LEMMA 3) Assume by contradiction that $z_1^* >^Z z_2^*$. By definition of z_1^* and z_2^*, we have $\sum_{z \geq^Z z_1^*} S(z) > \alpha n$ and $\sum_{z_2^* \geq^Z z} S(z) > \alpha n$, then $\sum_{z \in Z} S(z) > 2\alpha n$. Given that $\sum_{z \in Z} S(z) = n$, it follows that $n > 2\alpha n$, meaning that $\alpha < \frac{1}{2}$, a contradiction, since by assumption $\frac{1}{2} \leq \alpha < 1$. Hence, the only remaining possibility is $z_2^* >^Z z_1^*$. ■

Lemma 4 *There exists $z^* \in]z_1^*, z_2^*[$, with $S(z^*) \neq 0$.* □

PROOF (LEMMA 4) Assume the contrary. By the definition of policies z_1^* and z_2^*, we have $\sum_{z \geq^Z z_1^*} S(z) > \alpha n$ and $\sum_{z_2^* \geq^Z z} S(z) > \alpha n$. These imply that $\sum_{z \geq^Z z_1^*} S(z) + \sum_{z_2^* \geq^Z z} S(z) > 2\alpha n$ or $\sum_{z \in Z} S(z) > 2\alpha n$ leading to $\alpha < \frac{1}{2}$, which is a contradiction. It follows that there exists a policy $z^* \in]z_1^*, z_2^*[$, such that $S(z^*) \neq 0$. Note that, in this case, z_1^* and z_2^* are such that $\sum_{z_1^* \geq^Z z} S(z) < \alpha n$, and $\sum_{z \geq^Z z_2^*} S(z) < \alpha n$. ■

Lemma 5 *If $z \in Z \setminus [z_1^*, z_2^*]$, then $z \notin \mathcal{E}(\mathcal{P})$.* □

PROOF (LEMMA 5) Consider $z \in Z$ distinct to z_1^* and z_2^*. Assume that z is the closest peak to the left of z_1^*. In a pairwise supermajoritarian opposition between z and z_1^*, the former receives at most $\sum_{z \geq^Z z'} S(z')$ number of votes, while the latter receives at most $\sum_{z' \geq^Z z_1^*} S(z')$. Since, $\sum_{z \geq^Z z'} S(z') \leq \sum_{z_1^* \geq^Z z'} S(z') < \alpha n$, and $\sum_{z' \geq^Z z_1^*} S(z') > \alpha n$, then, z_1^* wins. We can also show that z_2^* defeats any peak z, with $z >^Z z_2^*$. ■

Lemma 6 *If $z \in [z_1^*, z_2^*]$, then $z \in \mathcal{E}(\mathcal{P})$.* □

PROOF (LEMMA 6) Consider a peak $z \in [z_1^*, z_2^*]$. Assume that there exists $z' \in]z_1^*, z_2^*[$ such that z' defeats z. Without loss of generality, assume that z' is the closest peak to z with $z >^Z z'$.

Policy z' defeats z implies that $\sum_{z' \geq^Z x} S(x) > \alpha n$, which is a contradiction, because by definition of z_2^*, $z_2^* >^Z z'$ implies that $\sum_{z' \geq^Z x} S(x) \leq \alpha n$. Thus, $\mathcal{E}(\mathcal{P}) = [z_1^*, z_2^*]$. ■

Now, we prove Theorem 1.

PROOF (THEOREM 1) First, if $\alpha = 1$, then the only winning coalition is the set N. Given that individuals make proposals against the status quo z_0, then each peak is a predicted outcome of the game. In fact, the maximum number of votes that an alternative policy z_1 (distinct than z_0) in a pairwise supermajoritarian opposition can receive is $n - 1$. If the supermajority rule requires n votes to win, then no alternative can be defeated, and the maximum number of predicted outcomes is the cardinality of N, i.e., n. Second, if n is odd, and $\alpha = 1/2$, then the median peak is the unique prediction of the pairwise supermajoritarian game, because it is the Condorcet winner, i.e., it defeats any other policy in a pairwise supermajoritarian opposition. Third, from Lemmas 2, 5, and 6, we show that any alternative which is not part of the interval bounded by the peaks z_1^* and z_2^* can be directly defeated by either z_1^* or z_2^*, and any peak in this interval cannot be defeated. Therefore, the maximal number of equilibria is equal to the number of individuals who have a peak between z_1^* and z_2^*. Given that the proportion of individuals required to form a winning coalition is at least $\frac{\lfloor \alpha n \rfloor + 1}{n}$, then the upper bound of $\mathcal{E}(\mathcal{P})$ is $n - 2(1 - \frac{\lfloor \alpha n \rfloor + 1}{n})n = 2\lfloor \alpha n \rfloor + 2 - n < n$. ■

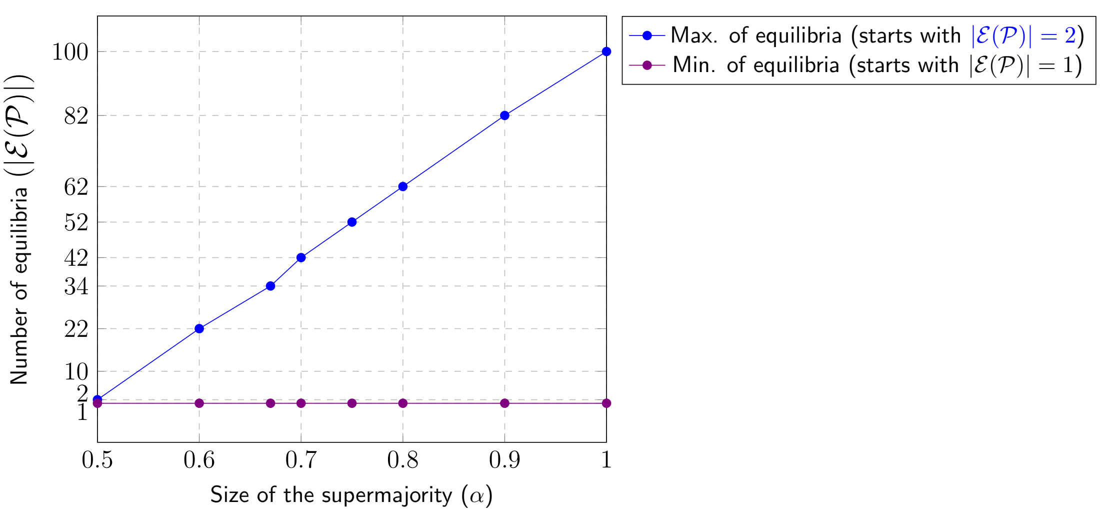

Figure 1: Number of equilibria and the size of supermajority rule in a voting body of 100 agents

As shown in Figure 1, the minimum number of equilibrium policies is 1 regardless of the supermajority rule. The maximum number of equilibrium policies is a non-decreasing function of the supermajority needed to replace them. It follows that, for a fixed size n of voters, the number of equilibrium policies is a non-decreasing *correspondence* of the supermajority's size α. A corollary of Theorem 1 is the following result, which derives the size of the equilibrium set under majority rule and thus clarifies the way Theorem 1 extends the MVT when the number of agents is even.

Corollary 1 *Let $\mathcal{P} = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy. Assume that voters are chosen randomly with positive probability to make a proposal. If preferences are single-peaked, and policies are chosen using majority rule ($\alpha = \frac{1}{2}$), then,*

1. *There is only one equilibrium if the size of voters is odd.*
2. *There exists at least one and at most two equilibria if the size of voters is even.*

PROOF (COROLLARY 1) From Theorem 1, if $\alpha = \frac{1}{2}$, then the size of equilibrium set $\mathcal{E}(\mathcal{P})$ depends on the size of n. If n is odd, the number $\lfloor \alpha n \rfloor = \frac{n-1}{2}$, therefore $2\lfloor \alpha n \rfloor + 2 - n = n - 1 + 2 - n = 1$, meaning that a unique equilibrium exists. It is, in fact, the ideal policy of the median voter. If n is even, there exist at most two equilibria since the number $\lfloor \alpha n \rfloor = \frac{n}{2}$ and $2\lfloor \alpha n \rfloor + 2 - n = n + 2 - n = 2$. ■

3.1.2. Nature Randomly Chooses a Status Quo

Next, suppose that Nature, rather than choosing the proposer in $t = 0$, instead chooses the status quo $z_0 \in Z$ from the set of all policies. Theorem 2 proves the existence of at least one and possibly an infinite number of equilibrium policies.

Theorem 2 *Let $\mathcal{P} = (N, Z, (\succeq_i), \mathcal{L}_\alpha)$ be a political economy and assume that Nature randomly chooses the status quo. Let z_1^* and z_2^* denote, respectively, the minimal and the maximal equilibria when Nature randomly selects a proposer. Then, $\mathcal{E}(\mathcal{P}) = [z_1^*, z_2^*]$.* □

PROOF (THEOREM 2) The proof is deduced from the proof of Theorem 1. The status quo, chosen randomly by Nature, can take any position in spatial space Z. From Theorem 1, any position between and including z_1^* and z_2^*, is invulnerable to pairwise supermajoritarian opposition. In this case, the interval bounded by the peaks z_1^* and z_2^* is the equilibrium set, $\mathcal{E}(\mathcal{P})$. ■

Under majority rule, the equilibrium set described in Theorem 2 exhibits an interesting property. When the number of voters is odd and $\alpha = \frac{1}{2}$, $z_1^* = z_2^*$, and the set $[z_1^*, z_2^*]$ is a singleton, which is the ideal point of the median voter. If the number of voters is even, however, the set of equilibria may be infinite. In this sense, Theorem 2 offers a more complete statement of the MVT compared to Black (1948).

3.2. **Dynamic Political Games**

In dynamic political games, contrary to one-shot games, agents (or coalitions) may vote indefinitely. Assume that, a status quo z_0 is randomly chosen from the set of policies. If no winning coalition replaces z_0, then it remains in place on an indefinite basis and the political opposition ends. If a winning coalition S replaces z_0, say with z_1, then z_1 becomes the new status quo, and the process restarts, continuing until a policy has been reached to which no winning coalition is willing to object. Once that policy has been reached, each agent earns and consumes his or her payoff and the political contest ends. We illustrate the predictions of such a game with the *largest consistent set* (Chwe, 1994), one of the prominent equilibrium concepts in infinite-horizon political games.

Chwe (1994) defines the *largest consistent set*, an equilibrium concept for *social environments* where agents, acting in public, can freely form coalitions without binding agreements and are farsighted. Chwe (1994) assumes that agent i holds a *strict preference* relation $\succ_i$ over Z, and coalitions of agents may be endowed with the power to replace one policy by some other policies. If a coalition $S \subseteq N$ has the right to replace $z \in Z$ by some $z' \in Z$, we write $z \longrightarrow_S z'$. Following Chwe's notations, a *social environment* is represented by a list $(N, Z, \{\succ_i\}_{i\in N}, \{\longrightarrow_S\}_{S\subset N, S\neq\emptyset})$. To capture the idea of farsightedness, Chwe (1994) formalizes the notion of *indirect dominance* that was formally discussed by Harsanyi (1974) in his criticism of the von Neumann and Morgenstern (1944)'s solution concept which is based on *direct dominance*. For $z, z' \in Z$, z' is said to *indirectly dominate* z, or $z' \gg z$, if there exists a sequence of policies $z_0,\ z_1,\ ...,\ z_m \in Z$ $(where\ z_0 = z\ and\ z_m = z')$ and a sequence of winning coalitions $S_0,\ S_1,\ ..., S_{m-1}$ such that $z_i \longrightarrow_{S_i} z_{i+1}$ and $z' \succ_{S_i} z_i$ for $i = 0, 1, ..., m-1$. The case $m = 1$ yields the definition of the *direct dominance*. Chwe (1994, Proposition 2, P. 305) shows that the *largest consistent set* is non-empty if Z is finite or countably infinite, and there are no $\gg$-

chains, i.e, an infinite sequences of policies z_1, z_2, z_3,... such that $i < j \Longrightarrow z_j \gg z_i$. Xue (1997, Theorem, P. 455) extends Chwe (1994, Proposition 2, P. 305)'s non-emptiness result of the *largest consistent set* by removing the countability and by weakening the condition that there is no $\gg$-chains. As discussed by Xue (1997, P. 453), such an extension allows one to apply the largest consistent set to models with continuum of alternatives. Note however, that both Chwe (1994, Proposition 2, P. 305) and Xue (1997, P. 453) assume that agents have *strict preferences* over the policy space Z, a different assumption that we make in this paper.

In this section, we examine Chwe (1994, Proposition 2, P. 305)'s non-emptiness result of the *largest consistent set* when the distribution of veto rights among coalitions is given by a super-majority rule, the policy space Z is totally ordered, and agents have single-peaked preferences over Z. For $z, z' \in Z$, $z \longrightarrow_S z'$ if and only if S is a winning coalition (i.e., $|S| > \alpha n$). Therefore, a social environment $(N, Z, \{\succ_i\}_{i\in N}, \{\longrightarrow_S\}_{S\subset N, S\neq\emptyset})$ is equivalent to a political economy $\mathcal{P} = (N, Z, \{\succeq_i\}_{i\in N}, \mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ replaces $\{\longrightarrow_S\}_{S\subset N, S\neq\emptyset}$. We recall the definition of the *largest consistent set* below.

Definition 2 Let $\mathcal{P} = (N, Z, \{\succeq_i\}_{i\in N}, \mathcal{L}_\alpha)$ be a political economy, and X be a subset of Z.

1. X is said to be *consistent* if $x \in X$ if and only if $\forall y \in Z$ and $S \subset N$ such that $x \longrightarrow_S y$, there exists $z \in X$, where $y = z$ or $z \gg y$, and $not(x \succ_S z)$.
2. The *largest consistent set* of the political economy $\mathcal{P}$, denoted $LCS(\mathcal{P})$, is the union of all the consistent sets. □

The largest consistent set formalizes the notion that a coalition that moves from a status quo to an alternative policy anticipates the possibility that another coalition might react. A third coalition might in turn react, and so on, without limit. It is therefore important to act in a way that does not lead a coalition to ultimately regret its action, i.e., coalitions are "*fully farsighted*" (Chwe, 1994, 300). In the following result, we show that the *largest consistent set* is non-empty, and we derive the maximum number of equilibria in the largest consistent set when Nature randomly chooses agents with equal probability to propose a status quo.

Theorem 3 *Let* $\mathcal{P} = (N, Z, \{\succeq_i\}_{i\in N}, \mathcal{L}_\alpha)$ *be a political economy.*

1. *Assume that Nature randomly proposes the status quo. Then,* $LCS(\mathcal{P}) = [z_1^*, z_2^*]$.

2. *Assume that Nature randomly chooses agents with equal probability to propose a status quo. Then,* $1 \leq |LCS(\mathcal{P})| \leq \min\{2\lfloor \alpha n \rfloor + 2 - n, n\}$*. Thus, the maximum number of equilibria in* $\mathcal{P}$ *is* n *when* $\alpha = 1$*, and* $2\lfloor \alpha n \rfloor + 2 - n$ *when* $\alpha \in [\frac{1}{2}, 1)$. □

PROOF (THEOREM 3) Let $\mathcal{P} = (N, Z, \{\succeq_i\}_{i \in N}, \mathcal{L}_\alpha)$ be a political economy.

1. Let $z \in Z$. If $z_1^* >^Z z$, then z_1^* indirectly dominates z; If $z >^Z z_2^*$, then z_2^* indirectly dominates z. The only alternatives that are not indirectly dominated belong to the interval $[z_1^*, z_2^*]$. A subset $X \subseteq Z$ is consistent if

$$f(X) = \left\{ \begin{matrix} x \in Z : \forall y \in Z, \forall S,\ x \longrightarrow_S y,\ \exists z \in X,\ \text{where} \\ y = z \text{ or } z \gg y \text{ and } not(x \succ_S z) \end{matrix} \right\} = X.$$

For each agent $i \in N$, we denote by z_i his or her ideal point. By definition of z_1^* and z_2^*, the sets $S = \{i \in N : z_i \geq^Z z_1^*\}$ and $T = \{i \in N : z_2^* \geq^Z z_i\}$ are winning coalitions. Let $z \in Z$ be a proposal: (a) if $z_1^* >^Z z$, then any deviation from z by any winning coalition to z_1^* is not deterred. Similarly; (b) if $z >^Z z_2^*$, then any deviation from z by any winning coalition to z_2^* is not deterred. Hence, in these two cases, $z \notin f(Z)$. However, if $z \in [z_1^*, z_2^*]$, any deviation from z is deterred. Indeed, without loss of generality, assume $x = z_1^*$, and consider $y \in Z$ and a winning coalition S', such that $x \longrightarrow_{S'} y$. (c) If $z_1^* >^Z y$, then there exists $z = z_1^*$, with $z_1^* \gg y$ via T, and $not(z_1^* \succ_S z_1^*)$; (d) If $y \in]z_1^*, z_2^*[$, then, there exists $z = y$, such that $not(z \succ_{S'} z_1^*)$, with $|S'| > \alpha n$, because $z_2^* >^Z y$; (e) If $y >^Z z_2^*$, then, there exists $z = z_2^*$, with $z_2^* \gg y$ via S, and $not(z_2^* \succ_{S'} z_1^*)$, with $|S'| > \alpha n$. It follows that $f(Z) = [z_1^*, z_2^*]$. It is straightforward to check that $f(f(Z)) = f(Z)$; therefore $f(Z)$ is the largest consistent set, and point 1. of Theorem 3 is proved.

2. First, if $\alpha = 1$, then the only winning coalition is the set N. Given that agents propose the status quo z_0, the cardinality of the largest coalition than can propose an alternative policy z_1 against z_0 is $n - 1$. If the supermajority rule requires n agents to replace the status quo, then no alternative can be indirectly dominated, and the maximum number of predicted outcomes is the cardinality of N, i.e., n. Second, if n is odd, and $\alpha = 1/2$,

then the median peak is the unique prediction of the largest consistent set, because it is the Condorcet winner, i.e., it indirectly dominated any other policy in a the game. Third, from point. 1 above, we show that $LCS(\mathcal{P}) = [z_1^*, z_2^*]$. Therefore, the maximal number of equilibria is equal to the number of individuals who have a peak between z_1^* and z_2^*. Given that the proportion of agents required to form a winning coalition is at least $\frac{\lfloor \alpha n \rfloor + 1}{n}$, then the upper bound of $LCS(\mathcal{P})$ is $n - 2(1 - \frac{\lfloor \alpha n \rfloor + 1}{n})n = 2\lfloor \alpha n \rfloor + 2 - n < n$. ■

Having presented Theorems 1, 2, and 3, we now use the results to explain policy diversity across identical political economies. In Section 4, we propose an illustration which demonstrates how two countries with identical political, economic, and cultural preferences over immigration resettlement could implement different policies.

4. Application: Policy Diversity

The government of a country is developing a refugee resettlement program to help asylum seekers. Suppose that this decision fell to legislators representing the country's citizens and that the country derives utility from the number of refugees it admits. The utility can be in terms of the national and international "warm glow" that it receives, or in terms of the skills or cultural diversity brought by the refugees. We assume that the net utility received by each legislator i from z refugees being admitted (z is a positive real number) is $V_i(z) = v_i \ln(z) - z/n$, where $1/n$ is the fraction of the total cost of refugee admission incurred by each constituency (assuming n constituencies), and v_i is legislator i's valuation of the number of refugees. Suppose nine legislators ($n = 9$) collectively choose the number of refugees to be admitted using majority rule. Observe that V_i is single-peaked, and so voter i's peak is obtained by solving $V'(z_i) = 0$, leading to the solution $z_i^p = 9v_i$. We assume that $v_i = i$, where $i = 1, 2, ..., 9$. Then, the legislators' peaks are: $z_1^p = 9$, $z_2^p = 18$, $z_3^p = 27$, $z_4^p = 36$, $z_5^p = 45$, $z_6^p = 54$, $z_7^p = 63$, $z_8^p = 72$, and $z_9^p = 81$ (see also Figure 2). The legislator with valuation $v_i = 5$ is the median voter. The peak $z_5^p = 45$ defeats all other peaks in a pairwise majoritarian election, and becomes the only peak which is not defeated. Therefore, under the majority rule, the country grants permanent residency to 45 refugees.

Suppose that the legislators choose the number of refugees using a two-thirds supermajority

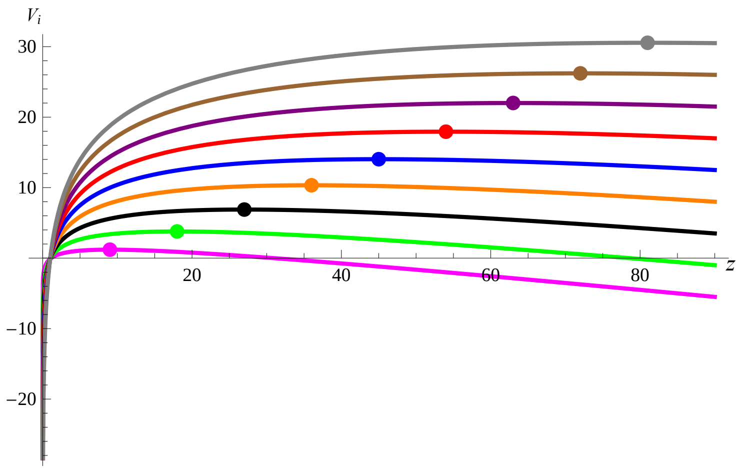

Figure 2: Legislators' utility functions. The bundle on each curve represents the coordinates of legislator i's peak.

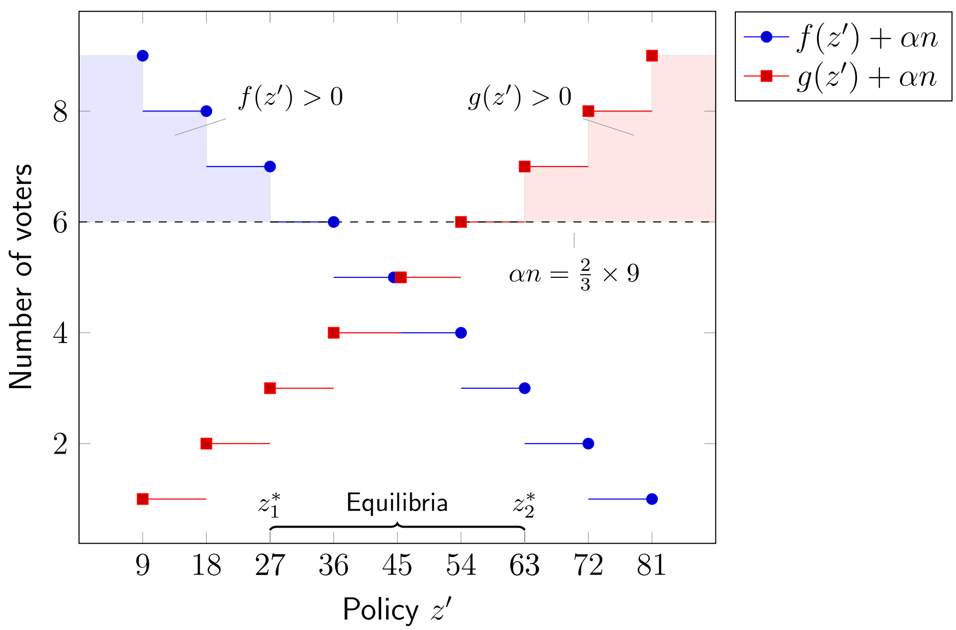

Figure 3: Illustration of equilibria for refugee resettlement program, assuming $\alpha = 2/3$. Equilibrium points are those between $z_1^* = 27$ and $z_2^* = 63$ inclusive. We note that for each policy z', $f(z') + \alpha n = \sum_{z \geq^Z z'} S(z)$ and $g(z') + \alpha n = \sum_{z' \geq^Z z} S(z)$.

rule ($\alpha = 2/3$). Any proposal in the set $\{27, 36, 45, 54, 63\}$ cannot be defeated in a pairwise supermajoritarian election, because all alternatives will fail to win support from the necessary supermajoritarian coalition. These proposals are shielded from the possibility of amendment on the legislative floor. Moreover, any outcome in the set $\{9, 18, 72, 81\}$ can be defeated by either $z_1^* = 27$ or $z_2^* = 63$ (see Figure 3).

It follows that two different countries that are identical in terms of the number of voters, voters' preferences, and voting rule are likely to diverge in policy choice if the voting is the two-thirds supermajority rule. For example, depending on the random voter that is chosen to make a proposal, one country may grant permanent residency to only 27 refugees while the other may grant this privilege to 54 refugees. Under majority rule, both countries will converge in their policy, and will grant permanent residency to 45 refugees.[9]

5. Conclusion and Remarks

In this study, we derive the minimum and the maximum number of equilibrium policies in static and dynamic political games under supermajority rules when agents have single-peaked preferences over a totally ordered policy space. Voters' strategic behavior is captured by the *core* (Black, 1948; Downs, 1957) in static environments, and by the *largest consistent set* (Chwe, 1994) in dynamic environments. We fully characterize the relationship between these numbers and a voting body's supermajority rule, showing that the minimum number is one regardless of the rule, and the maximum number increases in a nontrivial manner in the size of the supermajority coalition needed to change policy. The well-known Median Voter Theorem, which predicts party convergence to the median voter's ideal policy, is a particular case of our results. Our findings can explain why highly divergent policies may persist, even across democracies with identical political preferences and voting rules. Policy divergence increases as we move further from majority rule. Moreover, in deriving the minimum and the maximum number of equilibrium policies in a supermajoritarian setting, our results extend Duverger's propositions on institutions and political parties. In only imposing the assumption that voters hold single-peaked preferences over a totally ordered policy

[9] Beyond explaining policy developments in international negotiations, Theorem 2 is also validated by McGrath et al. (2018), who conduct a comparative study across US states. Leveraging cross-country variation in state legislative override requirements, they find that legislatures with higher override requirements demonstrate less ability to override an executive veto. Mapping the legislative process to our model, state governors first propose budgets and then legislatures pass their own. The budget is then sent to the governor for approval and, if vetoed, can only be enacted if a legislative supermajority overrides the veto. The supermajority thresholds used in the study—which, in this case, are the proportions of the legislature needed to override an executive veto—vary between $1/2$ and $2/3$. (Note, however, that three states with a $3/5$ or majority veto override were excluded from some models because they also had supermajority budgetary requirements (McGrath et al., 2018, 165).) In accordance with our results, budgets passed in states with higher override requirements were substantially closer to those proposed by the governor, with the strongest effects in states where executives' preferences diverged sharply from those of legislative veto players.

space, our model is quite general and applies to a variety of policies beyond those chosen from a unidimensional set.

Our theory generalizes dynamics in other theoretical work (Dixit et al., 2000), and its implications align with voting behavior in institutions ranging from state legislatures (McGrath et al., 2018) to international institutions (Stone, 2009). Additionally, we contribute to existing social choice literature. Focusing on supermajority voting rules—a topic that has, to date, received limited attention—the article raises and answers novel questions. What, precisely, is the relationship between supermajority thresholds and the number of equilibrium policies? And how does this relationship manifest in the diversity of policies across institutions with one threshold as opposed to another?

Our model also offers avenues for future empirical and theoretical research. Further extensions can consider proposal or amendment costs that vary based on legislators' ideal points or on the location of the proposed policy or amendment. The model is also amenable to accommodating "decision-costs" from policy gridlock (Buchanan and Tullock, 1962) and to introducing uncertainty in legislators' policy preferences. The latter extension would draw connections between policy diversity and the extensive literature examining the Condorcet Jury Theorem.[10]

Empirically, the model offers several testable predictions. Do reductions in amendment thresholds—such as revisions to the United States Senate requirements to invoke cloture, decrease policy diversity and increase the extent to which proposers (or political parties) compromise? And, comparing legislative bodies whose members have similar preferences, do those requiring high supermajoritarian thresholds to amend proposals generate more diverse policies than those with low thresholds? And how does the distribution of agenda power mediate the relationship between policy diversity and voting rules? Despite the challenges in finding variation in voting rules across otherwise comparable legislative bodies (Cameron, 2009), recent research has employed innovative data to discern such relationships, both globally and domestically (Blake and Payton, 2015; McGrath et al., 2018; Brutger and Li, 2019).

[10]The question of choosing the optimal system considering uncertainty was first formulated by Condorcet (1793). See Nitzan and Paroush (2017) for further details and extensions.

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