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Functional analysis

# Joint spectra of spherical Aluthge transforms of commuting $n$ -tuples of Hilbert space operators



## *Spectres joints des transformées d'Aluthge sphériques de $n$ -uplets commutatifs d'opérateurs d'un espace de Hilbert*

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## ABSTRACT

Let  $\mathbf{T} \equiv (T_1, \dots, T_n)$  be a commuting  $n$ -tuple of operators on a Hilbert space  $\mathcal{H}$ , and let  $T_i \equiv V_i P$  ( $1 \leq i \leq n$ ) be its canonical joint polar decomposition (i.e.  $P := \sqrt{T_1^* T_1 + \dots + T_n^* T_n}$ ,  $(V_1, \dots, V_n)$  a joint partial isometry, and  $\bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker V_i = \ker P$ ). The spherical Aluthge transform of  $\mathbf{T}$  is the (necessarily commuting)  $n$ -tuple  $\hat{\mathbf{T}} := (\sqrt{P} V_1 \sqrt{P}, \dots, \sqrt{P} V_n \sqrt{P})$ . We prove that  $\sigma_{\mathbf{T}}(\hat{\mathbf{T}}) = \sigma_{\mathbf{T}}(\mathbf{T})$ , where  $\sigma_{\mathbf{T}}$  denotes the Taylor spectrum. We do this in two stages: away from the origin, we use tools and techniques from criss-cross commutativity; at the origin, we show that the left invertibility of  $\mathbf{T}$  or  $\hat{\mathbf{T}}$  implies the invertibility of  $P$ . As a consequence, we can readily extend our main result to other spectral systems that rely on the Koszul complex for their definitions.

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## R É S U M É

Soit  $\mathbf{T} \equiv (T_1, \dots, T_n)$  un  $n$ -uplet commutatif d'opérateurs sur un espace de Hilbert  $\mathcal{H}$ , et soient  $T_i \equiv V_i P$  ( $1 \leq i \leq n$ ) sa décomposition polaire jointe canonique (i.e.  $P := \sqrt{T_1^* T_1 + \dots + T_n^* T_n}$ ,  $(V_1, \dots, V_n)$  une isométrie partielle jointe et  $\bigcap_{i=1}^n \ker T_i = \bigcap_{i=1}^n \ker V_i = \ker P$ ). La transformée d'Aluthge sphérique de  $\mathbf{T}$  est le  $n$ -uplet (nécessairement commutatif)  $\hat{\mathbf{T}} := (\sqrt{P} V_1 \sqrt{P}, \dots, \sqrt{P} V_n \sqrt{P})$ . Nous démontrons que  $\sigma_{\mathbf{T}}(\hat{\mathbf{T}}) = \sigma_{\mathbf{T}}(\mathbf{T})$ , où  $\sigma_{\mathbf{T}}$  désigne le spectre de Taylor. Nous procédons pour cela en deux étapes : en dehors de l'origine, nous utilisons les outils et les techniques de la commutativité criss-cross ; à l'origine, nous prouvons que l'inversibilité à gauche de  $\mathbf{T}$  ou de  $\hat{\mathbf{T}}$  implique l'inversibilité

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de  $P$ . Comme conséquence, nous pouvons étendre notre résultat à d'autres systèmes spectraux définis à partir des complexes de Koszul.

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### 1. Introduction

Let  $\mathcal{H}$  be a complex infinite-dimensional Hilbert space, let  $\mathcal{B}(\mathcal{H})$  denote the algebra of bounded linear operators on  $\mathcal{H}$ , and let  $T \in \mathcal{B}(\mathcal{H})$ . For  $T \equiv V|T|$  the canonical polar decomposition of  $T$ , we let  $\tilde{T} := |T|^{1/2}V|T|^{1/2}$  denote the Aluthge transform of  $T$  [1]. It is well known that  $T$  is invertible if and only if  $\tilde{T}$  is invertible; moreover, the spectra of  $T$  and  $\tilde{T}$  are equal. Over the last two decades, considerable attention has been given to the study of the Aluthge transform; cf. [2–4,35], [10], [13,14,17–28], [32], [39–42]). Moreover, the Aluthge transform has been generalized to the case of powers of  $|T|$  different from  $\frac{1}{2}$  ([5], [8], [9], [29]) and to the case of commuting pairs of operators ([13], [14]).

In this note, we focus on the spherical Aluthge transform [14]. Although our results hold for arbitrary  $n > 2$ , for the reader's convenience we will focus on the case  $n = 2$ , that is, the case of commuting pairs of Hilbert space operators. Let  $\mathbf{T} \equiv (T_1, T_2)$  be a commuting pair of operators on  $\mathcal{H}$ . We now consider the canonical polar decomposition of the column operator  $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ ; that is,  $\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \equiv \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} P$ , where  $P := \sqrt{T_1^*T_1 + T_2^*T_2}$  and  $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$  is a (joint) partial isometry, and subject to the constraint  $\bigcap_{i=1}^2 \ker T_i = \bigcap_{i=1}^2 \ker V_i = \ker P$ .

The spherical Aluthge transform of  $\mathbf{T}$  is the (necessarily commuting)  $n$ -tuple

$$\widehat{\mathbf{T}} := (\sqrt{P}V_1\sqrt{P}, \dots, \sqrt{P}V_n\sqrt{P}) \quad ([13], [14]). \tag{1.1}$$

For a commuting pair  $\mathbf{T} \equiv (T_1, T_2)$  of operators on  $\mathcal{H}$ , the Koszul complex associated with  $\mathbf{T}$  is given as

$$K(\mathbf{T}, \mathcal{H}) : 0 \xrightarrow{0} \mathcal{H} \xrightarrow{\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}} \mathcal{H} \oplus \mathcal{H} \xrightarrow{(-T_2 \ T_1)} \mathcal{H} \xrightarrow{0} 0.$$

**Definition 1.1.** A commuting pair  $\mathbf{T}$  is said to be (Taylor) invertible if its associated Koszul complex  $K(\mathbf{T}, \mathcal{H})$  is exact. The Taylor spectrum of  $\mathbf{T}$  is

$$\sigma_T(\mathbf{T}) := \left\{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : K((T_1 - \lambda_1, T_2 - \lambda_2), \mathcal{H}) \text{ is not invertible} \right\}.$$

The pair  $\mathbf{T}$  is called Fredholm if each map in the Koszul complex  $K(\mathbf{T}, \mathcal{H})$  has closed range and all the homology quotients are finite-dimensional. The Taylor essential spectrum is

$$\sigma_{Te}(\mathbf{T}) := \left\{ (\lambda_1, \lambda_2) \in \mathbb{C}^2 : (T_1 - \lambda_1, T_2 - \lambda_2) \text{ is not Fredholm} \right\}.$$

J.L. Taylor showed in [36] and [37] that, if  $\mathcal{H} \neq \{0\}$ , then  $\sigma_T(\mathbf{T})$  is a nonempty, compact subset of the polydisc of multi-radius  $r(\mathbf{T}) := (r(T_1), r(T_2))$ , where  $r(T_i)$  is the spectral radius of  $T_i$  ( $i = 1, 2$ ). (For additional facts about these joint spectra, the reader is referred to [11,12,15,16] and [38].)

As shown in [11] and [16], the Fredholmness of  $\mathbf{T}$  can be detected in the Calkin algebra  $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ . (Here  $\mathcal{K}$  denotes the closed two-sided ideal of compact operators; we also let  $\pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{Q}(\mathcal{H})$  denote the quotient map.) Concretely,  $\mathbf{T}$  is Fredholm on  $\mathcal{H}$  if and only if the pair of left multiplication operators  $L_{\pi(\mathbf{T})} := (L_{\pi(T_1)}, L_{\pi(T_2)})$  is Taylor invertible when acting on  $\mathcal{Q}(\mathcal{H})$ . In particular,  $\mathbf{T}$  is left Fredholm on  $\mathcal{H}$  if and only if  $L_{\pi(\mathbf{T})}$  is bounded below on  $\mathcal{Q}(\mathcal{H})$ .

**Problem 1.2.** Let  $\mathbf{T} \equiv (T_1, T_2)$  be a commuting pair of operators.

- (i) Assume that  $\mathbf{T}$  is (Taylor) invertible (resp. Fredholm). Is  $\widehat{\mathbf{T}}$  also (Taylor) invertible (resp. Fredholm)?
- (ii) Is the Taylor spectrum (resp. Taylor essential spectrum) of  $\widehat{\mathbf{T}}$  equal to that of  $\mathbf{T}$ ?

We first prove that  $\sigma_T(\widehat{\mathbf{T}}) = \sigma_T(\mathbf{T})$ . We do this in two stages: away from the origin, we use tools and techniques from criss-cross commutativity; at the origin we show that the left invertibility of  $\mathbf{T}$  or  $\widehat{\mathbf{T}}$  implies the invertibility of  $P$ ;  $P$  then helps to establish an isomorphism between the relevant Koszul complexes. As a consequence, we can readily extend the above result to other spectral systems that rely on the Koszul complex for their definitions, including spectral systems on  $\mathcal{Q}(\mathcal{H})$ .

## 2. Main results

Recall the joint polar decomposition of  $\mathbf{T}$  and the spherical Aluthge transform of  $\mathbf{T}$ ; cf. (1.1). We now state our first main result.

**Theorem 2.1.** *Assume that  $\mathbf{T}$  or  $\widehat{\mathbf{T}}$  is left invertible; that is, the associated Koszul complex is exact at the left stage, and the range of the corresponding boundary map is closed. Then the operator  $P$  is invertible.*

**Proof. Case 1.** If  $\mathbf{T}$  is left invertible, then  $T_1^*T_1 + T_2^*T_2$  is invertible, and therefore  $P$  is invertible.

**Case 2.** If  $\widehat{\mathbf{T}}$  is left invertible, then it is bounded below; that is, there exists a constant  $c > 0$  such that

$$\left\| \sqrt{P}V_1\sqrt{P}x \right\|^2 + \left\| \sqrt{P}V_2\sqrt{P}x \right\|^2 \geq c^2 \|x\|^2.$$

Since  $(V_1, V_2)$  is a joint partial isometry, it readily follows that

$$\left\| \sqrt{P}x \right\|^2 + \left\| \sqrt{P}x \right\|^2 \geq \frac{c^2}{\|P\|} \|x\|^2.$$

As a result,  $\sqrt{P}$  is bounded below, so  $P$  is invertible.  $\square$

We are now ready to state our second main result.

**Theorem 2.2.** *Let  $\mathbf{T} = (T_1, T_2)$  be a commuting pair of operators on  $\mathcal{H}$ . Then,*

$$\mathbf{T} \text{ is (Taylor) invertible} \iff \widehat{\mathbf{T}} \text{ is (Taylor) invertible}.$$

We now recall the notion of criss-cross commutativity.

**Definition 2.3.** Let  $\mathbf{A} \equiv (A_1, \dots, A_n)$  and  $\mathbf{B} \equiv (B_1, \dots, B_n)$  be two  $n$ -tuples of operators on  $\mathcal{H}$ . We say that  $\mathbf{A}$  and  $\mathbf{B}$  criss-cross commute (or that  $\mathbf{A}$  criss-cross commutes with  $\mathbf{B}$ ) if  $A_i B_j A_k = A_k B_j A_i$  and  $B_i A_j B_k = B_k A_j B_i$  for all  $i, j, k = 1, \dots, n$ . Observe that we do not assume that  $\mathbf{A}$  or  $\mathbf{B}$  is commuting.

**Definition 2.4.** Given two  $n$ -tuples  $\mathbf{A}$  and  $\mathbf{B}$  we define  $\mathbf{AB} := (A_1 B_1, \dots, A_n B_n)$  and  $\mathbf{BA} := (B_1 A_1, \dots, B_n A_n)$ .

**Remark 2.5.** It is an easy consequence of Definition 2.3 that, if  $\mathbf{A}$  and  $\mathbf{B}$  criss-cross commute and  $\mathbf{AB}$  is commuting, then  $\mathbf{BA}$  is also commuting.

**Lemma 2.6.** *Let  $\mathbf{T} \equiv (T_1, T_2)$  be a commuting pair of operators on  $\mathcal{H}$ , let  $P := \sqrt{T_1^*T_1 + T_2^*T_2}$ , and let  $\widehat{\mathbf{T}}$  be its spherical Aluthge transform. Then  $\mathbf{A} \equiv (A_1, A_2) := (\sqrt{P}, \sqrt{P})$  and  $\mathbf{B} \equiv (B_1, B_2) := (V_1\sqrt{P}, V_2\sqrt{P})$  criss-cross commute. As a consequence,  $\widehat{\mathbf{T}} (= \mathbf{BA})$  is commuting.*

**Lemma 2.7.** (cf. [6] and [7]) *Let  $\mathbf{A}$  criss-cross commute with  $\mathbf{B}$  on  $\mathcal{H}$ , and assume that  $\mathbf{AB}$  is commuting. Then  $\sigma_{\mathbf{T}}(\mathbf{BA}) \setminus \{0\} = \sigma_{\mathbf{T}}(\mathbf{AB}) \setminus \{0\}$ .*

We now prove our third main result.

**Theorem 2.8.** *Let  $\mathbf{T} = (T_1, T_2)$  be a commuting pair of operators on  $\mathcal{H}$ . Then*

$$\sigma_{\mathbf{T}}(\mathbf{T}) = \sigma_{\mathbf{T}}(\widehat{\mathbf{T}}).$$

**Proof.** Let  $\lambda \in \mathbb{C}^2$ . If  $\lambda = (0, 0)$ , use Theorem 2.2; if  $\lambda \neq (0, 0)$ , use Lemma 2.7.  $\square$

**Remark 2.9.** (i) Theorems 2.1, 2.2 and 2.8 can be easily extended to other spectral systems whose definition is given in terms of the Koszul complex; e.g., the left  $k$ -spectral systems  $\sigma_{\pi,k}$  defined by W. Słodkowski and W. Żelazko ([33], [34]). For, the Proof of Theorem 2.1 (which uses only left invertibility of the relevant Koszul complex) works well in case  $\mathbf{T}$  or  $\widehat{\mathbf{T}}$ . Once we know that  $\sqrt{P}$  is invertible, the Koszul complexes of  $\mathbf{T}$  and  $\widehat{\mathbf{T}}$  are isomorphic, so  $0 \notin \sigma_{\pi,k}(\mathbf{T})$  if and only if  $0 \notin \sigma_{\pi,k}(\widehat{\mathbf{T}})$ .

(ii) Similarly, Theorem 2.7 admits an easy extension to Słodkowski's left  $k$ -spectra (cf. [6], [7]), since the Proof of Theorem 2.7 relies on the isomorphism of the Koszul complexes for  $\mathbf{T}$  and  $\widehat{\mathbf{T}}$ , implemented by  $\sqrt{P}$ .

(iii) On the other hand, the above results cannot be extended to Słodkowski's right  $k$ -spectra; for, consider the adjoint  $U_+^*$  of the (unweighted) unilateral shift  $U_+$ . It is easy to see that  $U_+^*$  is onto while  $\widehat{U_+^*}$  is not.

Our final main result deals with Fredholmness.

**Theorem 2.10.** *Let  $\mathbf{T} = (T_1, T_2)$  be a commuting pair of operators on  $\mathcal{H}$ . Then*

$$\sigma_{Te}(\mathbf{T}) = \sigma_{Te}(\widehat{\mathbf{T}}).$$

Moreover, for each  $\lambda \notin \sigma_{Te}(\mathbf{T})$ , we have

$$\text{ind}(\mathbf{T} - \lambda) = \text{ind}(\widehat{\mathbf{T}} - \lambda),$$

where  $\text{ind}$  denotes the Fredholm index.

**Sketch of proof.** In Theorem 2.1, one can replace “left invertible” for the Koszul complex with “left Fredholm” and “invertible” for  $P$  with “Fredholm.” A similar adjustment works for Theorems 2.2 and 2.8. In the analog of Theorem 2.2, one first proves that  $\sqrt{P}$  is bounded below in the orthogonal complement of  $\ker T_1 \cap \ker T_2$ ; since this kernel is finite dimensional, it follows that  $\sqrt{P}$  is Fredholm. In Theorem 2.8, one needs to replace Lemma 2.7 with the results for Fredholmness proved in [6], [7], [30], and [31]. While Li’s results only guarantee that  $\text{ind}(\mathbf{T} - \lambda) = \text{ind}(\widehat{\mathbf{T}} - \lambda)$  whenever  $\lambda \neq (0, 0)$ , the continuity of the Fredholm index (cf. [16]) does the rest.  $\square$

## References

- [1] A. Aluthge, On  $p$ -hyponormal operators for  $0 < p < 1$ , *Integral Equ. Oper. Theory* 13 (1990) 307–315.
- [2] T. Ando, Aluthge transforms and the convex hull of the spectrum of a Hilbert space operator, in: *Recent Advances in Operator Theory and Its Applications*, in: *Operator Theory: Advances and Applications*, vol. 160, 2005, pp. 21–39.
- [3] T. Ando, T. Yamazaki, The iterated Aluthge transforms of a  $2 \times 2$  matrix converge, *Linear Algebra Appl.* 375 (2003) 299–309.
- [4] A.J. Antezana, P. Massey, D. Stojanoff,  $\lambda$ -Aluthge transforms and Schatten ideals, *Linear Algebra Appl.* 405 (2005) 177–199.
- [5] C. Benhida, Mind Duggal transforms, *Filomat*, to appear, arXiv:1804.00877.
- [6] C. Benhida, E.H. Zerouali, On Taylor and other joint spectra for commuting  $n$ -tuples of operators, *J. Math. Anal. Appl.* 326 (2007) 521–532.
- [7] C. Benhida, E.H. Zerouali, Spectral properties of commuting operations for  $n$ -tuples, *Proc. Amer. Math. Soc.* 139 (2011) 4331–4342.
- [8] C. Benhida, M. Chō, E. Ko, J. Lee, On the generalized mean transforms of (skew)-symmetric complex operators, preprint 2019.
- [9] F. Chabbabi, Product commuting maps with the  $\lambda$ -Aluthge transform, preprint 2016, submitted for publication.
- [10] M. Chō, I.B. Jung, W.Y. Lee, On Aluthge transforms of  $p$ -hyponormal operators, *Integral Equ. Oper. Theory* 53 (2005) 321–329.
- [11] R. Curto, Applications of several complex variables to multi-parameter spectral theory, in: J.B. Conway, B.B. Morrel (Eds.), *Surveys of Recent Results in Operator Theory, Vol. II*, Longman Publishing Co., London, 1988, pp. 25–90.
- [12] R. Curto, Spectral theory of elementary operators, in: M. Mathieu (Ed.), *Elementary Operators and Applications*, World Sci. Publishing, River Edge, NJ, USA, 1992, pp. 3–52.
- [13] R. Curto, J. Yoon, Toral and spherical Aluthge transforms of 2-variable weighted shifts, *C. R. Acad. Sci. Paris, Ser. I* 354 (2016) 1200–1204.
- [14] R. Curto, J. Yoon, Aluthge transforms of 2-variable weighted shifts, *Integral Equ. Oper. Theory* 90 (2018) 52.
- [15] R.E. Curto, On the connectedness of invertible  $n$ -tuples, *Indiana Univ. Math. J.* 29 (1980) 393–406.
- [16] R.E. Curto, Fredholm and invertible  $n$ -tuples of operators. The deformation problem, *Trans. Amer. Math. Soc.* 266 (1981) 129–159.
- [17] K. Dykema, H. Schultz, Brown measure and iterates of the Aluthge transform for some operators arising from measurable actions, *Trans. Amer. Math. Soc.* 361 (2009) 6583–6593.
- [18] G.R. Exner, Aluthge transforms and  $n$ -contractivity of weighted shifts, *J. Oper. Theory* 61 (2009) 419–438.
- [19] C. Foias, I.B. Jung, E. Ko, C. Pearcy, Complete contractivity of maps associated with the Aluthge and Duggal transforms, *Pac. J. Math.* 209 (2003) 249–259.
- [20] S.R. Garcia, Aluthge transforms of complex symmetric operators, *Integral Equ. Oper. Theory* 60 (2008) 357–367.
- [21] I.B. Jung, E. Ko, C. Pearcy, Aluthge transform of operators, *Integral Equ. Oper. Theory* 37 (2000) 437–448.
- [22] I.B. Jung, E. Ko, C. Pearcy, Spectral pictures of Aluthge transforms of operators, *Integral Equ. Oper. Theory* 40 (2001) 52–60.
- [23] I.B. Jung, E. Ko, C. Pearcy, The iterated Aluthge transform of an operator, *Integral Equ. Oper. Theory* 45 (2003) 375–387.
- [24] I.B. Jung, E. Ko, C. Pearcy, Complete contractivity of maps associated with Aluthge and Duggal transforms, *Pac. J. Math.* 209 (2003) 249–259.
- [25] J. Kim, J. Yoon, Aluthge transforms and common invariant subspaces for a commuting  $n$ -tuple of operators, *Integral Equ. Oper. Theory* 87 (2017) 245–262.
- [26] M.K. Kim, E. Ko, Some connections between an operator and its Aluthge transform, *Glasg. Math. J.* 47 (2005) 167–175.
- [27] F. Kimura, Analysis of non-normal operators via Aluthge transformation, *Integral Equ. Oper. Theory* 50 (2004) 375–384.
- [28] S.H. Lee, W.Y. Lee, J. Yoon, Subnormality of Aluthge transform of weighted shifts, *Integral Equ. Oper. Theory* 72 (2012) 241–251.
- [29] S.H. Lee, W.Y. Lee, J. Yoon, The mean transform of bounded linear operators, *J. Math. Anal. Appl.* 410 (2014) 70–81.
- [30] S. Li, On the commuting properties of Taylor’s spectrum, *Chin. Sci. Bull.* 37 (1992) 1849–1852.
- [31] S. Li, Taylor spectral invariance for crisscross commuting pairs on Banach spaces, *Proc. Amer. Math. Soc.* 124 (1996) 2069–2071.
- [32] K. Rion, Convergence properties of the Aluthge sequence of weighted shifts, *Houst. J. Math.* 42 (2016) 1217–1226.
- [33] Z. Słodkowski, An infinite family of joint spectra, *Stud. Math.* 61 (1977) 239–255.
- [34] Z. Słodkowski, W. Żelazko, On joint spectra of commuting families of operators, *Stud. Math.* 50 (1974) 127–148.
- [35] B. Taher, M. Rachidi, E.H. Zerouali, On the Aluthge transforms of weighted shifts with moments of Fibonacci type. Application to subnormality, *Integral Equ. Oper. Theory* 82 (2015) 287–299.
- [36] J.L. Taylor, A joint spectrum for several commuting operators, *J. Funct. Anal.* 6 (1970) 172–191.
- [37] J.L. Taylor, The analytic functional calculus for several commuting operators, *Acta Math.* 125 (1970) 1–48.
- [38] F.-H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions*, D. Reidel Publishing Co., Dordrecht–Boston–London, 1982.
- [39] P.Y. Wu, Numerical range of Aluthge transform of operator, *Linear Algebra Appl.* 357 (2002) 295–298.
- [40] T. Yamazaki, Parallels between Aluthge transformation and powers of operators, *Acta Sci. Math. (Szeged)* 67 (2001) 809–820.
- [41] T. Yamazaki, On numerical range of the Aluthge transformation, *Linear Algebra Appl.* 341 (2002) 111–117.
- [42] T. Yamazaki, An expression of spectral radius via Aluthge transformation, *Proc. Amer. Math. Soc.* 130 (2002) 1131–1137.