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The Weierstrass–Enneper system for constant mean curvature surfaces and the completely integrable sigma model

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The Weierstrass-Enneper System for
Constant Mean Curvature Surfaces and
the Completely Integrable Sigma Model

P. Bracken A. M. Grundland L. Martina

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Abstract

The integrability of a system which describes constant mean curvature surfaces by means of the adapted Weierstrass-Enneper inducing formula is studied. This is carried out by using a specific transformation which reduces the initial system to the completely integrable 2-dimensional Euclidean nonlinear sigma model. Through the use of the apparatus of differential forms and Cartan theory of systems in involution, it is demonstrated that the general analytic solutions of both systems possess the same degree of freedom. Furthermore, a new linear spectral problem equivalent to the initial Weierstrass-Enneper system is derived via the method of differential constraints. A new procedure for constructing solutions to this system is proposed and illustrated by several elementary examples, including a multi-soliton solution.

AMS subject classifications (1980). Primary 35P05; Secondary 35A25.

Résumé

On effectue une étude de l'intégrabilité d'un système décrivant les courbures moyennes constantes des surfaces à partir des formules induites de Weierstrass-Enneper. Une transformation particulière est proposée, qui réduit le système initial à un modèle sigma euclidien bidimensionnel complètement intégrable. En utilisant l'appareil des formes de Pfaff et la théorie de Cartan pour les systèmes en involution, il est démontré que ces deux systèmes ont une solution générale possédant le même degré de liberté. De plus, un nouveau problème spectral équivaut au système initial de Weierstrass-Enneper est dérivé à l'aide de la méthode des contraintes différentielles. Une nouvelle procédure pour la construction des solutions du système est proposée. Nous illustrons cette procédure par plusieurs exemples élémentaires incluant la solution multi-solitonique.

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1. INTRODUCTION. Since the last century, the problems of surfaces and their deformations under various types of dynamics have generated a great deal of interest and activity in several mathematical as well as physical fields of research ([1-9]). In particular, surfaces with constant mean curvature have been shown to play an essential role in several applications to nonlinear phenomena in such areas of physics as two-dimensional gravity [4,15], quantum field theory [4,16], statistical physics [3,17], and fluid dynamics [18,19]. The Weierstrass-Enneper formula for inducing minimal surfaces has been studied for several years [10,11,12], most recently by B. Konopelchenko and I. Taimanov [13,14]. They established a direct connection between certain classes of constant curvature surfaces and an integrable finite-dimensional Hamiltonian system (for a summary of their results, see [14]). In general, it was shown [13] that the following infinite-dimensional Hamiltonian system describes constant mean curvature surfaces,

$$(1.1) \quad \partial\psi_1 = 2H(|\psi_1|^2 + |\psi_2|^2)\psi_2, \quad \bar{\partial}\psi_2 = -2H(|\psi_1|^2 + |\psi_2|^2)\psi_1,$$

where ψ_1 and ψ_2 are complex functions of the complex variables (z, \bar{z}) . The bar denotes the complex conjugate, $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$, and H denotes the constant mean curvature of the surface. One can assume, without loss of generality, $H = 1/2$. Then, system (1.1) takes the form

$$(1.2a) \quad \partial\psi_1 = p\psi_2, \quad \bar{\partial}\psi_2 = -p\psi_1, \quad p = |\psi_1|^2 + |\psi_2|^2,$$

and its respective complex conjugate is

$$(1.2b) \quad \bar{\partial}\bar{\psi}_1 = p\bar{\psi}_2, \quad \partial\bar{\psi}_2 = -p\bar{\psi}_1.$$

The above system can be considered a variant of the original Weierstrass-Enneper (WE) system and we will refer to it as such.

The system (1.2) determines a set of constant mean curvature surfaces obtained by the following parametrization $(z, \bar{z}) \rightarrow (X_1, X_2, X_3)$

$$(1.3) \quad \begin{aligned} X_1 + iX_2 &= 2i \int_{z_0}^z (\bar{\psi}_1^2 - \bar{\psi}_2^2) dz', \\ X_1 - iX_2 &= 2i \int_{z_0}^z (\psi_2^2 - \psi_1^2) dz', \\ X_3 &= -2 \int_{z_0}^z (\psi_2\bar{\psi}_1 + \psi_1\bar{\psi}_2) dz'. \end{aligned}$$

Using the standard formulae, we find that the first fundamental form on the surface is given by

$$(1.4) \quad \Omega = 4p^2 dzd\bar{z}$$

and the Gaussian curvature is [13]

$$(1.5) \quad K = -\frac{\partial\bar{\partial}(\ln p)}{p^2}.$$

in isothermic coordinates.

The results obtained in [14] give a certain indication suggesting complete integrability of the WE system (1.2). However, a systematic approach to its integrability remains still an open problem. It will be shown here that the WE system (1.2) passes the Painlevé test, which means that it satisfies

the necessary condition for complete integrability. This fact will be deduced from the existence of a linear spectral problem for the WE system (1.2). Moreover, the Lie algebra of infinitesimal symmetries of the WE system (1.2) is spanned by the vector fields

$$(1.6) \quad \alpha_1 = \partial, \quad \alpha_2 = \bar{\partial}, \quad \beta = \psi_1 \partial_{\psi_1} - \bar{\psi}_1 \partial_{\bar{\psi}_1} + \psi_2 \partial_{\psi_2} - \bar{\psi}_2 \partial_{\bar{\psi}_2}$$

and

$$(1.7) \quad \alpha_\xi = \xi \partial + \bar{\xi} \bar{\partial} - \frac{1}{2} [\psi_1 (\bar{\partial} \bar{\xi}) \partial_{\psi_1} + \bar{\psi}_1 (\partial \xi) \partial_{\bar{\psi}_1} + \psi_2 (\partial \xi) \partial_{\psi_2} + \bar{\psi}_2 (\bar{\partial} \bar{\xi}) \partial_{\bar{\psi}_2}],$$

where $\xi(z)$ is an arbitrary analytic function and $\bar{\xi}(\bar{z})$ denotes its complex conjugate. Then, the set $\{\alpha_\xi\}$ generates an infinite dimensional Lie algebra, realizing the conformal symmetry property of the system (1.2). This algebra contains a Virasoro subalgebra [20], which has translations and dilations as special elements and $sl(2)$ as unique simple subalgebras. The vector field β commutes with the α_ξ and it describes a scaling transformation involving only the dependent variables ψ_i and $\bar{\psi}_i$.

To obtain a one-parameter subgroup, we integrate α_ξ . Thus, we obtain the transformations

$$z'(\lambda, z) = F^{-1}(\lambda + F(z)) = \int_{z_0}^z \frac{\xi[z'(\lambda, w)]}{\xi(w)} dw$$

$$\psi'_1(\lambda, \psi_1) = \psi_1 \left(\frac{\bar{\xi}(\bar{z})}{\xi(z'(\lambda, z))} \right)^{1/2}, \quad \psi'_2(\lambda, \psi_2) = \psi_2 \left(\frac{\xi(z'(\lambda, z))}{\xi(z)} \right)^{1/2},$$

where,

$$F(z) = \int_c^z \frac{ds}{\xi(s)},$$

with z_0 and c suitable complex numbers. Although the existence of a conformal symmetry does not imply in general complete integrability, it is, however, a strong indication in this direction.

Moreover, the WE system (1.2) possesses several conserved quantities [15]. The conservation of current which is defined by

$$(1.8) \quad J = \bar{\psi}_1 \partial \psi_2 - \psi_2 \partial \bar{\psi}_1.$$

leads to interesting consequences. Differentiation of J gives

$$(1.9) \quad \begin{aligned} \bar{\partial} J &= \bar{\partial}(\bar{\psi}_1 \partial \psi_2 - \psi_2 \partial \bar{\psi}_1) \\ &= (\bar{\partial} \bar{\psi}_1) \partial \psi_2 + \bar{\psi}_1 (\bar{\partial} \partial \psi_2) - (\bar{\partial} \psi_2) (\partial \bar{\psi}_1) - \psi_2 (\bar{\partial} \partial \bar{\psi}_1) \\ &= p \bar{\psi}_2 \partial \psi_2 + \bar{\psi}_1 (\bar{\partial} \partial \psi_2) + p \psi_1 \partial \bar{\psi}_1 - \psi_2 (\bar{\partial} \partial \bar{\psi}_1). \end{aligned}$$

The mixed derivatives obtained from (1.2) are

$$(1.10) \quad \begin{aligned} \partial \bar{\partial} \bar{\psi}_1 &= \partial(p \bar{\psi}_2) = (\partial p) \bar{\psi}_2 + p \partial \bar{\psi}_2 = (\partial p) \bar{\psi}_2 - p^2 \bar{\psi}_1, \\ \partial \bar{\partial} \psi_2 &= -\partial(p \psi_1) = -(\partial p) \psi_1 - p \partial \psi_1 = -(\partial p) \psi_1 - p^2 \psi_2. \end{aligned}$$

Consequently, the derivative of J vanishes,

$$\bar{\partial} J = p \bar{\psi}_2 \partial \psi_2 - |\psi_1|^2 (\partial p) - p^2 \bar{\psi}_1 \psi_2 + p \psi_1 \partial \bar{\psi}_1 - |\psi_2|^2 (\partial p) + p^2 \bar{\psi}_1 \psi_2$$

$$(1.11) \quad = -p(\partial p) + p(\bar{\psi}_2 \partial \psi_2 + \psi_1 \partial \bar{\psi}_1) = -p(\partial p) + p(\partial p) = 0.$$

Note that $\bar{\partial}J = 0$ holds even when no restriction has been placed on ∂p . Exactly the same situation occurs for the conjugate equation, $\partial\bar{J} = 0$.

In this paper, we examine certain aspects of complete integrability of the WE system (1.2) in the context of a two-dimensional Euclidean sigma-model. In particular, we focus on constructing a linear spectral problem for this system where the explicit form has not been known up to now.

This paper is organised as follows. In **Section 2**, we perform the reduction of the original system to a certain second order system of PDEs. **Section 3** presents an estimation of the degree of freedom of the general analytic solutions of both systems. This analysis is carried out by means of the Cartan theory of systems in involution. In **Section 4**, a linear spectral problem is derived for the WE system via a two-dimensional nonlinear sigma model based on the related second order system. This procedure amounts to a new technique for generating certain classes of solutions of the WE system which is illustrated with several examples in **Section 5**. **Section 6** contains final remarks and possible future developments.

2. THE SECOND-ORDER SYSTEM ASSOCIATED TO THE WEIERSTRASS-ENNEPER SYSTEM. In our investigation of the integrability of the WE system (1.2), we subject it to several transformations in order to simplify its structure.

We start by introducing the new complex variable

$$(2.1) \quad \rho = \frac{\psi_1}{\psi_2}.$$

Using equations (1.2) and the relation $p = |\psi_2|^2(1 + |\rho|^2)$, one obtains

$$(2.2) \quad \partial\rho = \frac{\partial\psi_1}{\psi_2} - \frac{\psi_1}{\psi_2^2}\partial\bar{\psi}_2 = \frac{p\psi_2}{\psi_2} - \frac{\psi_1}{\psi_2^2}(-p\bar{\psi}_1) = \frac{p^2}{\psi_2^2} = (1 + |\rho|^2)^2\psi_2^2.$$

Note that $\partial\rho$ and ψ_2^2 are related by a real function $(1 + |\rho|^2)^2$. Consequently, they have the same polar angle in the complex plane. Dividing (2.2) by $(1 + |\rho|^2)^2$ and taking the principal square root, one obtains ψ_2 . The complex conjugate of ψ_2 is found in the usual way by reflecting through the real axis. Using (2.1), ψ_1 can be obtained from the product of ρ and $\bar{\psi}_2$. This generates the following transformation from the variable ρ into the set of variables ψ_i ,

$$(2.3) \quad \psi_1 = \epsilon\rho \frac{(\bar{\partial}\bar{\rho})^{1/2}}{1 + |\rho|^2}, \quad \psi_2 = \epsilon \frac{(\partial\rho)^{1/2}}{1 + |\rho|^2}, \quad (\epsilon^2 = 1).$$

Let us now state the following proposition,

PROPOSITION 1. If ψ_1 and ψ_2 are solutions of the system (1.2), then the function ρ defined by (2.1) is a solution of the following second order system,

$$(2.4a) \quad \partial\bar{\partial}\rho - \frac{2\bar{\rho}}{1 + |\rho|^2}\partial\rho\bar{\partial}\rho = 0,$$

$$(2.4b) \quad \partial\bar{\partial}\bar{\rho} - \frac{2\rho}{1 + |\rho|^2}\partial\bar{\rho}\bar{\partial}\bar{\rho} = 0.$$

PROOF. Differentiation of equation (2.1) with respect to \bar{z} yields

$$(2.5) \quad \bar{\partial}\rho = \frac{\bar{\partial}\psi_1}{\psi_2} - \frac{\psi_1}{\psi_2^2}\bar{\partial}\bar{\psi}_2 = (\bar{\psi}_2)^{-2}[\bar{\psi}_2\bar{\partial}\psi_1 - \psi_1\bar{\partial}\bar{\psi}_2].$$

By an easy computation, one obtains from (2.2) and (2.5)

$$\begin{aligned}
\partial\bar{\partial}\rho &= \frac{\partial\bar{\partial}\psi_1}{\bar{\psi}_2} - \frac{\bar{\partial}\psi_1}{(\bar{\psi}_2)^2}\partial\bar{\psi}_2 - \frac{\partial\psi_1}{(\bar{\psi}_2)^2}\bar{\partial}\bar{\psi}_2 + 2\frac{\psi_1}{(\bar{\psi}_2)^3}\partial\bar{\psi}_2\bar{\partial}\bar{\psi}_2 - \frac{\psi_1}{(\bar{\psi}_2)^2}\partial\bar{\partial}\bar{\psi}_2 \\
(2.6) \quad &= (\bar{\psi}_2)^{-3}[\bar{\psi}_2^2(\partial\bar{\partial}\psi_1) - \bar{\psi}_2\bar{\partial}\psi_1\partial\bar{\psi}_2 - \bar{\psi}_2\partial\psi_1\bar{\partial}\bar{\psi}_2 + 2\psi_1(\partial\bar{\psi}_2)(\bar{\partial}\bar{\psi}_2) - \psi_1\bar{\psi}_2(\partial\bar{\partial}\bar{\psi}_2)] \\
&= (\bar{\psi}_2)^{-3}[\bar{\psi}_2^2(\partial\bar{\partial}\psi_1) + p\bar{\psi}_2\bar{\psi}_1\bar{\partial}\psi_1 - p|\psi_2|^2\bar{\partial}\bar{\psi}_2 - 2p|\psi_1|^2\bar{\partial}\bar{\psi}_2 - \psi_1\bar{\psi}_2(\partial\bar{\partial}\bar{\psi}_2)],
\end{aligned}$$

and its respective complex conjugate equation is

$$(2.7) \quad \partial\bar{\partial}\bar{\rho} = (\psi_2)^{-3}[\psi_2^2(\bar{\partial}\bar{\partial}\psi_1) + p\psi_2\psi_1\partial\bar{\psi}_1 - p|\psi_2|^2\partial\psi_2 - 2p|\psi_1|^2\partial\psi_2 - \bar{\psi}_1\psi_2(\bar{\partial}\bar{\partial}\psi_2)].$$

Using (1.2) the second derivatives (1.9) become

$$\begin{aligned}
\partial\bar{\partial}\bar{\psi}_1 &= (\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2)\bar{\psi}_2 - p^2\bar{\psi}_1, \\
\bar{\partial}\partial\psi_1 &= (\bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2)\psi_2 - p^2\psi_1, \\
(2.8) \quad \partial\bar{\partial}\psi_2 &= -(\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2)\psi_1 - p^2\psi_2, \\
\bar{\partial}\partial\bar{\psi}_2 &= -(\bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2)\bar{\psi}_1 - p^2\bar{\psi}_2.
\end{aligned}$$

Substituting (2.8) into (2.6) and (2.7), the following compact formulas for the mixed ρ derivatives can be obtained,

$$\begin{aligned}
\partial\bar{\partial}\rho &= \frac{2\bar{\psi}_1p}{\bar{\psi}_2^3}(\bar{\psi}_2\bar{\partial}\psi_1 - \psi_1\bar{\partial}\bar{\psi}_2) \\
(2.9) \quad \bar{\partial}\partial\bar{\rho} &= \frac{2\psi_1p}{\psi_2^3}(\psi_2\partial\bar{\psi}_1 - \bar{\psi}_1\partial\psi_2).
\end{aligned}$$

Substituting (2.1), (2.2), (2.5) and (2.9) into the left hand side of (2.4a), one obtains

$$\begin{aligned}
&\partial\bar{\partial}\rho - \frac{2\bar{\rho}}{1+|\rho|^2}\partial\rho\bar{\partial}\rho \\
&= \frac{2\bar{\psi}_1p}{\bar{\psi}_2^3}(\bar{\psi}_2\bar{\partial}\psi_1 - \psi_1\bar{\partial}\bar{\psi}_2) - \frac{2\bar{\psi}_1\psi_2}{\bar{\psi}_2^2}(1+|\rho|^2)(\bar{\psi}_2\bar{\partial}\psi_1 - \psi_1\bar{\partial}\bar{\psi}_2) \\
&= \frac{2\bar{\psi}_1p}{\bar{\psi}_2^3}(\bar{\psi}_2\bar{\partial}\psi_1 - \psi_1\bar{\partial}\bar{\psi}_2) - \frac{2\bar{\psi}_1p}{\bar{\psi}_2^3}(\bar{\psi}_2\bar{\partial}\psi_1 - \psi_1\bar{\partial}\bar{\psi}_2) = 0.
\end{aligned}$$

An analogous result holds for the conjugate equation (2.4b). Q.E.D.

Similar formulae to those of (2.4) can be found in the literature [15,16] in the context involving conformal immersions of a Riemann surface in \mathbb{R}^n .

The converse of Proposition 1 can be formulated as follows.

PROPOSITION 2. If ρ is a solution to the system (2.4), then the functions ψ_1 , and ψ_2 defined by (2.3) in terms of ρ satisfy the WE system (1.2).

PROOF. Differentiating (2.3) with respect to z , we obtain

$$\partial\psi_1 = \epsilon\left\{\partial\rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{1+|\rho|^2} + \rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{2(1+|\rho|^2)}\partial\bar{\partial}\bar{\rho} - \rho\frac{(\bar{\partial}\bar{\rho})^{1/2}}{(1+|\rho|^2)}(\rho\partial\bar{\rho} + \bar{\rho}\partial\rho)\right\}.$$

Substituting equation (2.4) into this expression, we get

$$(2.10) \quad \partial\psi_1 = \epsilon \left\{ \partial\rho \frac{(\bar{\partial}\bar{\rho})^{1/2}}{1+|\rho|^2} - |\rho|^2 \frac{(\bar{\partial}\bar{\rho})^{1/2}}{(1+|\rho|^2)^2} \partial\rho \right\} = \frac{(\partial\rho\bar{\partial}\bar{\rho})^{1/2}}{(1+|\rho|^2)} \psi_2.$$

Multiplying both equations in (2.3) together, the following expression for p results,

$$(2.11) \quad p = \frac{(\partial\rho\bar{\partial}\bar{\rho})^{1/2}}{(1+|\rho|^2)}.$$

Therefore,

$$\partial\psi_1 = \frac{(\partial\rho\bar{\partial}\bar{\rho})^{1/2}}{(1+|\rho|^2)} \psi_2 = p\psi_2.$$

Similarly, differentiation of (2.3) with respect to \bar{z} gives

$$(2.12) \quad \bar{\partial}\psi_2 = \epsilon \left\{ \frac{(\partial\rho)^{-1/2}}{2(1+|\rho|^2)} \bar{\partial}\partial\rho - \frac{(\partial\rho)^{1/2}}{(1+|\rho|^2)^2} (\rho\bar{\partial}\bar{\rho} + \bar{\rho}\bar{\partial}\rho) \right\}.$$

Substituting (2.4) and (2.11) into (2.12), we obtain,

$$\bar{\partial}\psi_2 = -\frac{(\bar{\partial}\bar{\rho}\partial\rho)^{1/2}}{(1+|\rho|^2)} \psi_1 = -p\psi_1.$$

which completes the proof. Q.E.D.

In some cases, it is more convenient to deal with (2.4) than the original system (1.2), since it consists of only two equations for two dependent variables ρ and $\bar{\rho}$. For example, a very large class of solutions of (2.4) can be found simply by requiring the holomorphicity ($\bar{\partial} = 0$) or antiholomorphicity ($\partial\rho = 0$) of the function ρ . We will show later in **Section 5** some examples of this type of solution. In the context of differential geometry, the system (2.4) was introduced by Kenmotsu in his seminal paper [10], and then often used by subsequent authors [13-15].

It is worth noting that, as in the case of system (1.2), the classical symmetry groups of (2.4) are conformal and scaling transformations. The corresponding symmetry algebra is spanned by

$$(2.13) \quad \alpha_1 = \xi(z)\partial, \quad \alpha_2 = \eta(\bar{z})\bar{\partial}, \quad \alpha_3 = \rho\partial_\rho - \bar{\rho}\bar{\partial}_{\bar{\rho}},$$

where ξ and η are arbitrary functions of their arguments. This algebra can be decomposed as a direct sum of two infinite dimensional simple Lie subalgebras with direct sum a one-dimensional algebra generated by α_3 . Assuming that the functions ξ and η are analytic in a proper open subset Ω of \mathbb{C} , they can be developed in a Laurent series so, we can provide a base for two centerless Virasoro algebras. Finite-dimensional subalgebras are spanned by $\{\partial\}$, $\{\partial, z\partial\}$, $\{\partial, z\partial, z^2\partial\}, \dots$ and $\{\bar{\partial}\}$, $\{\bar{\partial}, \bar{z}\bar{\partial}\}$, $\{\bar{\partial}, \bar{z}\bar{\partial}, \bar{z}^2\bar{\partial}\}, \dots$, respectively. In particular, the invariants of the one-dimensional subalgebra $\{\partial\}$ is given by $\{\bar{z}, \rho\}$. Then, the invariant solutions are any holomorphic functions ρ of \bar{z} . A detailed study of solutions invariant under vector fields (2.13) is beyond the scope of the present work, but there is no difficulty in treating them.

Finally, an interesting feature of the WE system (1.2) can be derived from the Gaussian curvature (1.5). It can be expanded in the following way

$$p^2K = -\bar{\partial}\partial \ln(p) = -\bar{\partial}\left(\frac{1}{p} \partial p\right) = \frac{1}{p^2} \partial p \bar{\partial} p - \frac{1}{p} \bar{\partial}\partial p.$$

Using the system of equations (1.2), the differentiation of the function p with respect to z and \bar{z} , respectively, yields

$$(2.14) \quad \partial p = \psi_1 \partial \psi_1 + \partial \psi_2 \bar{\psi}_2, \quad \bar{\partial} p = \bar{\psi}_1 (\bar{\partial} \psi_1) + \psi_2 (\bar{\partial} \bar{\psi}_2).$$

The mixed derivative of p becomes,

$$(2.15) \quad \bar{\partial} \partial p = \bar{\partial} \psi_1 \partial \bar{\psi}_1 + \psi_1 \bar{\partial} \partial \bar{\psi}_1 + \bar{\psi}_2 \bar{\partial} \partial \psi_2 + \partial \psi_2 \bar{\partial} \bar{\psi}_2 = \bar{\partial} \psi_1 \partial \bar{\psi}_1 + \partial \psi_2 \bar{\partial} \bar{\psi}_2 - p^3.$$

The product of the derivatives (2.14) is given by

$$\bar{\partial} p \partial p = |\psi_1|^2 (\bar{\partial} \psi_1) (\partial \bar{\psi}_1) + \psi_1 \psi_2 (\partial \bar{\psi}_1) (\bar{\partial} \bar{\psi}_2) + \bar{\psi}_1 \bar{\psi}_2 (\bar{\partial} \psi_1) (\partial \psi_2) + |\psi_2|^2 (\partial \psi_2) (\bar{\partial} \bar{\psi}_2).$$

Substituting these derivatives into the expression for $p^2 K$, we obtain the following result,

$$(2.16) \quad p^4 K = \psi_1 \psi_2 (\partial \bar{\psi}_1) (\bar{\partial} \bar{\psi}_2) + \bar{\psi}_1 \bar{\psi}_2 (\bar{\partial} \psi_1) (\partial \psi_2) - |\psi_1|^2 (\partial \psi_2) (\bar{\partial} \bar{\psi}_2) - |\psi_2|^2 (\bar{\partial} \psi_1) (\partial \bar{\psi}_1) + p^4.$$

This gives an explicit form for the Gaussian curvature K in terms of the functions ψ_1 and ψ_2 .

3. THE ESTIMATION OF DEGREE OF INDETERMINANCY OF GENERAL SOLUTIONS. Now, let us demonstrate that the general analytic solutions of the Weierstrass-Enneper system (1.2) and system (2.4) possess the same degree of freedom. To this end, we employ Cartan's theory of systems in involution [21]. For more information on this subject, see [22-24].

For computational purposes, it is useful to examine the systems of Pfaffian forms equivalent to the considered systems of equations (1.2) and (2.4). We determine the Cartan numbers of these systems and the numbers of arbitrary parameters admitted by the solutions of their polar equations [21].

3.1. The Weierstrass-Enneper System.

If one introduces the following notation,

$$x^1 = z, \quad x^2 = \bar{z}, \quad u^1 = \psi_1, \quad u^2 = \psi_2, \quad u^3 = \bar{\psi}_1, \quad u^4 = \bar{\psi}_2,$$

$$(3.1) \quad u^5 = u_{1,x^2}, \quad u^6 = u_{2,x^1}, \quad u^7 = u_{3,x^1}, \quad u^8 = u_{4,x^2},$$

then system (1.2) takes the form

$$u_{1,x^2} = (u_1 u_3 + u_2 u_4) u_2, \quad u_{3,x^1} = (u_1 u_3 + u_2 u_4) u_4,$$

$$(3.2) \quad u_{2,x^1} = -(u_1 u_3 + u_2 u_4) u_1, \quad u_{4,x^2} = -(u_1 u_3 + u_2 u_4) u_3.$$

If one chooses

$$(3.3) \quad \xi^1 = u_{1,x^1}, \quad \xi^2 = u_{2,x^2}, \quad \xi^3 = u_{3,x^2}, \quad \xi^4 = u_{4,x^1},$$

as parameters then, in terms of (3.1), equation (3.2) can be written as a system of differential one-forms

$$\begin{aligned} \omega_1 &= du_1 - (\xi^1 dx^1 + u_5 dx^2) = 0, \\ \omega_2 &= du_2 - (u_6 dx^1 + \xi^2 dx^2) = 0, \\ \omega_3 &= du_3 - (u_7 dx^1 + \xi^3 dx^2) = 0, \\ \omega_4 &= du_4 - (\xi^4 dx^1 + u_8 dx^2) = 0, \end{aligned}$$

$$\begin{aligned}
(3.4) \quad \omega_5 &= du_5 - \{[(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4)u_2 + (u_1 u_3 + u_2 u_4)u_6] dx^1 \\
&\quad + [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_2 + (u_1 u_3 + u_2 u_4)\xi^2] dx^2\} = 0, \\
\omega_6 &= du_6 + \{[(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4)u_1 + (u_1 u_3 + u_2 u_4)\xi^1] dx^1 \\
&\quad + [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_1 + (u_1 u_3 + u_2 u_4)u_5] dx^2\} = 0 \\
\omega_7 &= du_7 - \{[(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4)u_4 + (u_1 u_3 + u_2 u_4)\xi^4] dx^1 \\
&\quad + [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_4 + (u_1 u_3 + u_2 u_4)u_8] dx^2\} = 0 \\
\omega_8 &= du_8 + \{[(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4)u_3 + (u_1 u_3 + u_2 u_4)u_7] dx^1 \\
&\quad + [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_3 + (u_1 u_3 + u_2 u_4)\xi^3] dx^2\} = 0.
\end{aligned}$$

After exterior differentiation of (3.4) we obtain the following system of 2-forms, modulo (3.4),

$$\begin{aligned}
\Omega_1 &\equiv d\omega_1 = dx^1 \wedge d\xi^1 - [(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4)u_2 + (u_1 u_3 + u_2 u_4)u_6] dx^1 \wedge dx^2, \\
\Omega_2 &\equiv d\omega_2 = -[(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_1 + (u_1 u_3 + u_2 u_4)u_5] dx^1 \wedge dx^2 + dx^2 \wedge d\xi^2, \\
\Omega_3 &\equiv d\omega_3 = [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_4 + (u_1 u_3 + u_2 u_4)u_8] dx^1 \wedge dx^2 + dx^3 \wedge d\xi^3, \\
\Omega_4 &\equiv d\omega_4 = dx^1 \wedge d\xi^4 + [(\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_3 + (u_1 u_3 + u_2 u_4)u_7] dx^1 \wedge dx^2, \\
\Omega_5 &\equiv d\omega_5 = -u_2 u_3 d\xi^1 \wedge dx^1 + u_2^2 dx^1 \wedge d\xi^4 - u_1 u_2 d\xi^3 \wedge dx^2 - (u_1 u_3 + 2u_2 u_4) d\xi^2 \wedge dx^2 \\
(3.5) \quad &\quad + [u_1 u_2 ((u_3 u_5 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_4 + (u_1 u_3 + u_2 u_4)u_8) \\
&\quad - (u_1 u_3 + 2u_2 u_4) ((u_3 u_5 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_1 + (u_1 u_3 + u_2 u_4)u_5) \\
&\quad - u_2 u_3 ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_2 + (u_1 u_3 + u_2 u_4)u_6) \\
&\quad + u_2^2 ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_3 + (u_1 u_3 + u_2 u_4)u_7)] dx^1 \wedge dx^2, \\
\Omega_6 &\equiv d\omega_6 = -(2u_1 u_3 + u_2 u_4) dx^1 \wedge d\xi^1 - u_1 u_2 dx^1 \wedge d\xi^4 - u_1 u_4 dx^2 \wedge d\xi^2 - u_1^2 dx^2 \wedge d\xi^3 \\
&\quad - u_1^2 ((u_3 u_5 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_4 + (u_1 u_3 + u_2 u_4)u_8) \\
&\quad + u_1 u_4 ((u_3 u_5 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_1 + (u_1 u_3 + u_2 u_4)u_5) \\
&\quad - u_1 u_2 ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_3 + (u_1 u_3 + u_2 u_4)u_7) \\
&\quad + ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_2 + (u_1 u_3 + u_2 u_4)u_6) (2u_1 u_3 + u_2 u_4)] dx^1 \wedge dx^2, \\
\Omega_7 &\equiv d\omega_7 = u_3 u_4 dx^1 \wedge d\xi^1 + (u_1 u_3 + 2u_2 u_4) dx^1 \wedge d\xi^4 + u_4^2 dx^2 \wedge d\xi^2 + u_1 u_4 dx^2 \wedge d\xi^3 \\
&\quad + [u_1 u_4 ((u_3 u_5 + u_1 \xi^3 + u_4 \xi^2 + u_2 u_8)u_4 + (u_1 u_3 + u_2 u_4)u_8) \\
&\quad - u_4^2 ((u_3 u_5 + u_1 \xi^3 + u_4 \xi^2 + u_2 u_8)u_1 + (u_1 u_3 + u_2 u_4)u_5) \\
&\quad - u_3 u_4 ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_2 + (u_1 u_3 + u_2 u_4)u_6) \\
&\quad + (u_1 u_3 + 2u_2 u_4) ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_3 + (u_1 u_3 + u_2 u_4)u_7)] dx^1 \wedge dx^2, \\
\Omega_8 &\equiv d\omega_8 = -u_3^2 dx^1 \wedge d\xi^1 - u_2 u_3 dx^1 \wedge d\xi^4 - u_3 u_4 dx^2 \wedge d\xi^2 - (2u_1 u_3 + u_2 u_4) dx^2 \wedge d\xi^3 \\
&\quad + [u_3 u_4 ((u_3 u_5 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_1 + (u_1 u_3 + u_2 u_4)u_5) \\
&\quad - (2u_1 u_3 + u_2 u_4) ((u_3 u_5 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8)u_4 + (u_1 u_3 + u_2 u_4)u_8) \\
&\quad + u_3^2 ((\xi^1 u_3 + u_1 u_7 + u_4 u_6 + u_2 \xi^4)u_2 + (u_1 u_3 + u_2 u_4)u_6)
\end{aligned}$$

$$-u_2u_3((\xi^1u_3 + u_1u_7 + u_4u_6 + u_2\xi^4)u_3 + (u_1u_3 + u_2u_4)u_7)] dx^1 \wedge dx^2.$$

The vector fields $Y_j, j = 1, 2$ which annihilate the 1-forms ω_s and the 2-forms Ω_s , satisfy the polar equations,

$$(3.6) \quad \langle \omega_s \lrcorner Y_j \rangle = 0, \quad \langle \Omega_s \lrcorner Y_1, Y_2 \rangle = 0, \quad s = 1, \dots, 8, \quad j = 1, 2.$$

From conditions (3.6) one finds

$$\begin{aligned} Y_1 = & \partial_{x^1} + \sum_{r=1}^4 a^r \partial_{\xi^r} + \xi^1 \partial_{u_1} + u_6 \partial_{u_2} + u_7 \partial_{u_3} + \xi^4 \partial_{u_4} \\ & - [(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4) u_2 + (u_1 u_3 + u_2 u_4) u_6] \partial_{u_5} \\ & + [(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4) u_1 + (u_1 u_3 + u_2 u_4) \xi^1] \partial_{u_6} \\ & - [(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4) u_4 + (u_1 u_3 + u_2 u_4) \xi^4] \partial_{u_7} \\ & + [(\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4) u_3 + (u_1 u_3 + u_2 u_4) u_7] \partial_{u_8}, \end{aligned}$$

and,

$$\begin{aligned} Y_2 = & \partial_{x^2} + \sum_{r=1}^4 b^r \partial_{\xi^r} + u_5 \partial_{u_1} + \xi^2 \partial_{u_2} + \xi^3 \partial_{u_3} + u_8 \partial_{u_4} \\ & - [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8) u_2 + (u_1 u_3 + u_2 u_4) \xi^2] \partial_{u_5} \\ & + [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8) u_1 + (u_1 u_3 + u_2 u_4) u_5] \partial_{u_6} \\ & - [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8) u_4 + (u_1 u_3 + u_2 u_4) u_8] \partial_{u_7} \\ & + [(u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8) u_3 + (u_1 u_3 + u_2 u_4) \xi^3] \partial_{u_8}, \end{aligned}$$

where

$$\begin{aligned} a^2 = & -((u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8) u_1 + (u_1 u_3 + u_2 u_4) u_5), \\ a^3 = & ((u_5 u_3 + u_1 \xi^3 + \xi^2 u_4 + u_2 u_8) u_4 + (u_1 u_3 + u_2 u_4) u_8), \\ b^1 = & ((\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4) u_2 + (u_1 u_3 + u_2 u_4) u_6), \\ b^4 = & -((\xi^1 u_3 + u_1 u_7 + u_6 u_4 + u_2 \xi^4) u_3 + (u_1 u_3 + u_2 u_4) u_7), \end{aligned}$$

and, the quantities a^1, a^4, b^2, b^3 are arbitrary. Thus the number of free parameters in (3.6) is

$$(3.7) \quad N = 4.$$

Under the chosen notation (3.1) and (3.3) the Pfaffian system (3.4) takes the form

$$(3.8) \quad \omega_s = du_s - G_{s\mu}(x, \xi, u) dx^\mu, \quad s = 1, \dots, 8, \quad \mu = 1, 2,$$

where

$$x = (x^1, x^2), \quad \xi = (\xi^1, \dots, \xi^4), \quad u = (u^1, \dots, u^8).$$

and $G_{s\mu}$ depends linearly on ξ . The elements of the 8×4 matrix

$$(3.9) \quad a_{sr} = \left(\frac{\partial G_{s\mu}}{\partial \xi^r}(x, \xi, u) X^\mu \right), \quad X = (X^1, X^2) \in C^2,$$

determine the values of the Cartan quasicharacters $s_i, i = 1, 2$. The nonzero elements of the matrix (a_{sr}) are

$$\begin{aligned}
a_{11} &= X^1 & a_{51} &= u_3 u_2 X^1 & a_{61} &= -(2u_1 u_3 + u_2 u_4) X^1 \\
a_{22} &= X^2 & a_{52} &= (2u_2 u_4 + u_1 u_3) X^2 & a_{62} &= -u_1 u_4 X^2 \\
a_{33} &= X^2 & a_{53} &= u_1 u_2 X^2 & a_{63} &= -u_1^2 X^2 \\
a_{44} &= X^1 & a_{54} &= u_2^2 X^1 & a_{64} &= -u_1 u_2 X^1 \\
\\
a_{71} &= u_3 u_4 X^1 & a_{81} &= -u_3^2 X^1 \\
a_{72} &= u_4^2 X^2 & a_{82} &= -u_2 u_3 X^1 \\
a_{73} &= u_1 u_4 X^2 & a_{83} &= -(2u_1 u_3 + u_2 u_4) X^2 \\
a_{74} &= (u_1 u_3 + 2u_2 u_4) X^1 & a_{84} &= -u_3 u_4 X^2
\end{aligned}$$

Thus, the Cartan quasicharacters are given by,

$$s_1 = \max_{x \in \mathbb{C}} \text{rank}(a_{sr}) = 4, \quad s_2 = p - s_1 = 0,$$

where p is the number of coordinates ξ , that is, $p = 4$. From the definition of the Cartan number Q one has

$$(3.10) \quad Q = s_1 + 2s_2 = 4.$$

Since the number N of free parameters appearing in (3.6) equals 4, one has $Q = N$. Thus, according to Cartan's Theorem [21], system (3.4) (as well as (1.2)) is in involution. Its general analytic solution exists in some neighbourhood of a regular point (x_0, ξ_0, u_0) and depends on four arbitrary complex analytic functions of one complex variable.

3.2. The Second Order System of PDEs.

Now, for the system (2.4), a similar analysis is performed. We demonstrate that locally the solution space of (2.4) has the same dimension as the system (1.2). For computational purposes, it is useful to write equations (2.4) in the form

$$(3.11) \quad \begin{aligned} u_{1,x^1x^2} - \frac{2u_2}{1+u_1u_2}u_{1,x^1}u_{1,x^2} &= 0, \\ u_{2,x^1x^2} - \frac{2u_1}{1+u_1u_2}u_{2,x^1}u_{2,x^2} &= 0, \end{aligned}$$

where the following notation has been used

$$(3.12) \quad x^1 = z, \quad x^2 = \bar{z}, \quad u_1 = \rho, \quad u_2 = \bar{\rho}.$$

The system of differential one-forms corresponding to (3.11) can be written as follows

$$(3.13) \quad \begin{aligned} \omega_1 &= du_1 - u_3 dx^1 - u_5 dx^2 = 0, \\ \omega_2 &= du_2 - u_4 dx^1 - u_6 dx^2 = 0, \\ \omega_3 &= du_3 - \xi^1 dx^1 - \frac{2u_2}{1+u_1u_2}u_3u_5 dx^2 = 0, \\ \omega_4 &= du_4 - \xi^2 dx^1 - \frac{2u_1}{1+u_1u_2}u_4u_6 dx^2 = 0, \\ \omega_5 &= du_5 - \frac{2u_2}{1+u_1u_2}u_3u_5 dx^1 - \xi^3 dx^2 = 0, \\ \omega_6 &= du_6 - \frac{2u_1}{1+u_1u_2}u_4u_6 dx^1 - \xi^4 dx^2 = 0, \\ \omega_7 &= du_7 - u_{11} dx^1 - u_{12} dx^2, \\ \omega_8 &= du_8 - u_{21} dx^1 - u_{22} dx^2, \end{aligned}$$

where we use the standard notation

$$(3.14) \quad u^3 = u_{x^1}^1, \quad u^4 = u_{x^1}^2, \quad u^5 = u_{x^2}^1, \quad u^6 = u_{x^2}^2, \quad u^7 = u_{x^1x^2}^1, \quad u^8 = u_{x^1x^2}^2$$

and for the sake of simplicity, introduce the additional notation

$$\begin{aligned} u_{11} &= 2\left(\frac{u_2u_3u_5}{1+u_1u_2}\right)_{x^1} = 2\left[\frac{u_3u_4u_5 + u_2u_5\xi^1 + u_2u_3u_7}{1+u_1u_2} - \frac{(u_2u_3u_5)(u_2u_3 + u_1u_4)}{(1+u_1u_2)^2}\right], \\ u_{12} &= 2\left(\frac{u_2u_3u_5}{1+u_1u_2}\right)_{x^2} = 2\left[\frac{u_3u_5u_6 + u_2u_5u_7 + u_2u_3\xi^3}{1+u_1u_2} - \frac{(u_2u_3u_5)(u_2u_5 + u_1u_6)}{(1+u_1u_2)^2}\right], \\ u_{21} &= 2\left(\frac{u_1u_4u_6}{1+u_1u_2}\right)_{x^1} = 2\left[\frac{u_3u_4u_6 + u_1u_6\xi^2 + u_1u_4u_8}{1+u_1u_2} - \frac{(u_1u_4u_6)(u_2u_3 + u_1u_4)}{(1+u_1u_2)^2}\right], \\ u_{22} &= 2\left(\frac{u_1u_4u_6}{1+u_1u_2}\right)_{x^2} = 2\left[\frac{u_4u_5u_6 + u_1u_6u_8 + u_1u_4\xi^4}{1+u_1u_2} - \frac{(u_1u_4u_6)(u_2u_5 + u_1u_6)}{(1+u_1u_2)^2}\right]. \end{aligned}$$

We choose as parameters

$$(3.15) \quad \xi^1 = u_{x^1x^1}^1, \quad \xi^2 = u_{x^2x^1}^1, \quad \xi^3 = u_{x^2x^2}^1, \quad \xi^4 = u_{x^2x^2}^2.$$

As in the previous case, given the chosen notation (3.12), the Pfaffian system (3.13) takes the form (3.8). Note, that in this case the matrix (3.9) has the same dimension 8×4 as in the case of system (1.2). The nonzero elements of the matrix (a_{sr}) are

$$a_1^3 = X^1, \quad a_2^4 = -X^1, \quad a_3^5 = -X^2, \quad a_4^6 = -X^2.$$

Thus, the Cartan quasi-characters are given by

$$s_1 = \max_{X \in C^2} \text{rank}(a_{sr}) = 4, \quad s_2 = p - s_1 = 0,$$

where p is the number of coordinates ξ , that is, $p = 4$. Consequently, the Cartan number Q equals

$$(3.16) \quad Q = s_1 + 2s_2 = 4.$$

After exterior differentiation system (3.13) takes the form

$$\Omega_l \equiv d\omega_l \equiv 0, \quad l = 1, 2, 7, 8,$$

$$\Omega_3 \equiv d\omega_3 = -dx^1 \wedge d\xi^1 - u_{11} dx^1 \wedge dx^2,$$

$$\Omega_4 \equiv d\omega_4 = -dx^1 \wedge d\xi^2 - u_{21} dx^1 \wedge dx^2,$$

$$\Omega_5 \equiv d\omega_5 = u_{12} dx^1 \wedge dx^2 + dx^2 \wedge d\xi^3,$$

$$\Omega_6 \equiv d\omega_6 = u_{22} dx^1 \wedge dx^2 + dx^2 \wedge d\xi^4,$$

which is satisfied modulo (3.13).

The vector fields Y_j , $j = 1, 2$ which annihilate the one-forms ω_s and the two-forms Ω_s satisfy the polar equations (3.6). From these equations we have

$$(3.17) \quad Y_1 = \partial_{x^1} + \sum_{r=1}^4 a^r \partial_{\xi^r} + u_3 \partial_{u^1} + u_4 \partial_{u^2} + \xi^1 \partial_{u^3} + \xi^2 \partial_{u^4} \\ + 2 \frac{u_2 u_3 u_5}{1 + u_1 u_2} \partial_{u^5} + 2 \frac{u_1 u_4 u_6}{1 + u_1 u_2} \partial_{u^6} + u_{11} \partial_{u^7} + u_{21} \partial_{u^8},$$

$$Y_2 = \partial_{x^2} + \sum_{r=1}^4 b^r \partial_{\xi^r} + u_5 \partial_{u^1} + u_6 \partial_{u^2} + 2 \frac{u_2 u_3 u_5}{1 + u_1 u_2} \partial_{u^3} + 2 \frac{u_1 u_4 u_6}{1 + u_1 u_2} \partial_{u^4} + \xi^3 \partial_{u^5} \\ + \xi^4 \partial_{u^6} + u_{12} \partial_{u^7} + u_{22} \partial_{u^8},$$

where

$$b^1 = -u_{11}, \quad b^2 = -u_{21}, \quad a^3 = u_{12}, \quad a^4 = u_{22}.$$

As in the previous case, we have four free parameters, a^1, a^2, b^3, b^4 . Thus, we get $Q = N$ and, according to Cartan's Theorem, system (3.11) (as well as (2.4)) is in involution. Its general analytic solution depends on four arbitrary complex analytic functions of one complex variable.

We have shown that systems (1.2) and (2.4) possess the same degree of freedom in terms of their general analytic solutions. Using Cartan's theorem, we can formulate the following conclusion.

PROPOSITION 3. Suppose the systems (1.2) and (2.4) are both in involution at regular points (z_0, ξ_0, ψ_0) and (z_0, ξ_0, ρ_0) , respectively. Then their general analytic solutions exist in some neighbourhood of these regular points and both depend on two arbitrary complex analytic functions of one complex variable, and their complex conjugate functions.

Note that the mapping given by (2.1), from the solution of (2.4) to the solution of (1.2), does not restrict the type of boundary value conditions imposed on (2.4) and (1.2).

4. COMPLETE INTEGRABILITY OF THE WEIERSTRASS-ENNEPER SYSTEM IN THE CONTEXT OF THE SIGMA-MODEL.

4.1 The Linear Spectral Problem Associated with the Weierstrass-Enneper System. The objective of this section is to demonstrate a connection between the Weierstrass-Enneper system (1.2) and the completely integrable Euclidean sigma-model in 2-dimensions, and next, to derive through this link the linear spectral problem for the Weierstrass-Enneper system.

Let us identify (2.3) with the stereographic coordinate representation [25] of the 2-dimensional Euclidean nonlinear sigma-model

$$(4.1) \quad [S, \partial \bar{\partial} S] = 0,$$

where the spin matrix

$$S = \begin{pmatrix} s_3 & \bar{s}_+ \\ s_+ & -s_3 \end{pmatrix}, \quad \det S = -1,$$

belongs to the hermitian space $SU(2)/U(1)$. In the stereographic coordinate representation, the matrix S is given by

$$(4.2i) \quad S = \frac{1}{1 + |\rho|^2} \begin{pmatrix} 1 - |\rho|^2 & 2\bar{\rho} \\ 2\rho & -1 + |\rho|^2 \end{pmatrix},$$

where

$$(4.2ii) \quad s_+ = \frac{2\rho}{1 + |\rho|^2}, \quad s_3 = \frac{1 - |\rho|^2}{1 + |\rho|^2}.$$

By substituting the matrix S given by (4.2) into (4.1), we obtain the following condition,

$$[\bar{\rho}(\bar{\partial}\partial\rho - \frac{2\bar{\rho}}{1 + |\rho|^2}\partial\rho\bar{\partial}\rho) - \rho(\partial\bar{\partial}\bar{\rho} - \frac{2\rho}{1 + |\rho|^2}\bar{\partial}\bar{\rho}\partial\bar{\rho})]I = 0,$$

where I is the unit matrix in this equation. This is identically satisfied whenever equations (2.4) hold.

In terms of the complex functions ψ_i and $\bar{\psi}_i$, $i = 1, 2$, which appear in (1.2), the spin matrix S takes the form

$$(4.3) \quad S = \frac{1}{p} \begin{pmatrix} -|\psi_1|^2 + |\psi_2|^2 & 2\bar{\psi}_1\bar{\psi}_2 \\ 2\psi_1\psi_2 & |\psi_1|^2 - |\psi_2|^2 \end{pmatrix}$$

From (2.3), we obtain that the inverse mapping of (4.3) is double-valued and is provided by

$$(4.4i) \quad \psi_1 = \frac{\epsilon}{2}s_+[\bar{\partial}(\frac{\bar{s}_+}{1 + s_3})]^{1/2}, \quad \psi_2 = \frac{\epsilon}{2}(1 + s_3)[\partial(\frac{s_+}{1 + s_3})]^{1/2}, \quad \epsilon^2 = 1.$$

where

$$(4.4ii) \quad \rho = \frac{s_+}{1 + s_3}, \quad \bar{\rho} = \frac{\bar{s}_+}{1 + s_3}.$$

PROPOSITION 4. If ψ_1 and ψ_2 are solutions of the WE system (1.2), then the spin matrix S given by (4.3) is a solution of the sigma model equation (4.1).

PROOF. The results are directly obtained by substituting the spin matrix S given by (4.3) into the commutator (4.1) and assuming that the functions ψ_i satisfy (1.2). This computation leads to a vanishing commutator (4.1). Q.E.D.

The procedure for constructing solutions to (1.2) can be reduced to the following. Take any solution of the sigma-model (4.1) and substitute it into equations (4.4ii). The function ρ thus obtained provides us by means of transformation (2.3), with solutions ψ_1 and ψ_2 of the WE system (1.2). The possibility of constructing such solutions is demonstrated in the next section.

We now consider the possibility of constructing a linear spectral problem for the WE system (1.2). Let us introduce a new set of complex functions φ_1 and $\varphi_2 : \mathbb{C} \rightarrow \mathbb{C}$ which are related to the complex functions ψ_1 and ψ_2 in the following way

$$(4.5) \quad \begin{aligned} \psi_1 &= f(z, \bar{z})\varphi_1, & \bar{\psi}_2 &= f(z, \bar{z})\bar{\varphi}_2, \\ \bar{\psi}_1 &= \bar{f}(\bar{z}, z)\bar{\varphi}_1, & \psi_2 &= \bar{f}(\bar{z}, z)\varphi_2, \end{aligned}$$

for any complex function $f : \mathbb{C} \rightarrow \mathbb{C}$. From the definition (2.1), it is evident that the transformation (4.5) leaves the functions ρ and $\bar{\rho}$ invariant

$$(4.6) \quad \rho = \frac{\varphi_1}{\varphi_2}, \quad \bar{\rho} = \frac{\bar{\varphi}_1}{\bar{\varphi}_2}.$$

and the structure of the spin matrix S given by (4.3) is also preserved. This means that there exists a freedom which resembles a type of gauge freedom in the definition of the ρ variable, since the numerator and denominator of (4.6) can be multiplied by any complex function. The crux of the matter is that it is not required that the set of functions φ_i satisfy the original system (1.2), but that the ratio of φ_1 over $\bar{\varphi}_2$ satisfy (2.4). Let us express equations (1.2) in terms of φ_i and f . The derivatives of ψ_1 and ψ_2 take the form

$$\partial\psi_1 = (\partial f)\varphi_1 + f(\partial\varphi_1), \quad \bar{\partial}\psi_2 = (\bar{\partial}\bar{f})\varphi_2 + f(\bar{\partial}\varphi_2).$$

We define the variable $q = |\varphi_1|^2 + |\varphi_2|^2$ and from (1.2a) it follows that $p = |f|^2 q$. Taking the above into account we can write (1.2) as

$$(\partial f)\varphi_1 + f(\partial\varphi_1) = p\bar{f}\varphi_2, \quad (\bar{\partial}\bar{f})\varphi_2 + \bar{f}(\bar{\partial}\varphi_2) = -p f\varphi_1,$$

Solving the above equations for $\partial\varphi_1$ and $\bar{\partial}\varphi_2$, respectively, we obtain the equations of motion,

$$(4.7) \quad \begin{aligned} \partial\varphi_1 &= q\bar{f}^2\varphi_2 - (\partial \ln f)\varphi_1, & \bar{\partial}\varphi_2 &= -qf^2\varphi_1 - (\bar{\partial} \ln \bar{f})\varphi_2, \\ \bar{\partial}\bar{\varphi}_1 &= qf^2\bar{\varphi}_2 - (\bar{\partial} \ln \bar{f})\bar{\varphi}_1, & \partial\bar{\varphi}_2 &= -q\bar{f}^2\bar{\varphi}_1 - (\partial \ln f)\bar{\varphi}_2. \end{aligned}$$

Using (4.7), the differentiation of equations (4.6) with respect to z and \bar{z} , respectively, yields a pair of relations similar to (2.3) which relate the functions φ_i to ρ and a nonzero f ,

$$(4.8) \quad \varphi_1 = \epsilon\rho \frac{(\bar{\partial}\bar{\rho})^{1/2}}{f(1 + |\rho|^2)}, \quad \varphi_2 = \epsilon \frac{(\partial\rho)^{1/2}}{\bar{f}(1 + |\rho|^2)}.$$

Relations (4.8) can also be obtained in a more straightforward way by substituting (4.5) into equations (2.3). Note that as well as (2.3), the transformations (4.8) are doubled valued. Now we can formulate the following,

PROPOSITION 5. If the function ρ defined by (4.6) is a solution of the system (2.4), then the functions φ_1 and φ_2 satisfy the following system of equations

$$(4.9) \quad \begin{aligned} \partial\varphi_1 &= q\bar{f}^2\varphi_2, & \bar{\partial}\bar{\varphi}_1 &= qf^2\bar{\varphi}_2, \\ \bar{\partial}\varphi_2 &= -qf^2\varphi_1, & \partial\bar{\varphi}_2 &= -q\bar{f}^2\bar{\varphi}_1. \end{aligned}$$

for any function $f : \mathbb{C} \rightarrow \mathbb{C}$ satisfying

$$(4.10) \quad \partial f = 0.$$

PROOF. Indeed, by an easy computation, one obtains from (4.6) and (4.7), the first derivatives of ρ and $\bar{\rho}$

$$(4.11a) \quad \begin{aligned} \partial\rho &= q^2|f|^2(\bar{\varphi}_2)^{-2}, & \partial\bar{\rho} &= (\varphi_2)^{-2}(\varphi_2\partial\bar{\varphi}_1 - \bar{\varphi}_1\partial\varphi_2), \\ \bar{\partial}\rho &= (\bar{\varphi}_2)^{-2}(\bar{\varphi}_2\bar{\partial}\varphi_1 - \varphi_1\bar{\partial}\bar{\varphi}_2), & \bar{\partial}\bar{\rho} &= q^2|f|^2(\varphi_2)^{-2} \end{aligned}$$

and the second derivatives of ρ and $\bar{\rho}$,

$$(4.11b) \quad \begin{aligned} \partial\bar{\partial}\rho &= (\bar{\varphi}_2)^{-3}[2q\bar{f}^2\bar{\varphi}_1\bar{\varphi}_2\bar{\partial}\varphi_1 - q\bar{f}^2(q + |\varphi_1|^2 - |\varphi_2|^2)\bar{\partial}\bar{\varphi}_2], \\ \partial\bar{\partial}\bar{\rho} &= (\varphi_2)^{-3}[2qf^2\varphi_1\varphi_2\partial\bar{\varphi}_1 - qf^2(q + |\varphi_1|^2 - |\varphi_2|^2)\partial\varphi_2]. \end{aligned}$$

Substituting expressions (4.11) into (2.4), we get a differential constraint for the function f and its respective complex conjugate

$$(4.12) \quad (\bar{f}^2 - 1)\bar{\partial}\bar{f} = 0, \quad (f^2 - 1)\partial f = 0.$$

Thus, the general solution of system (4.12) is given by any antiholomorphic function f such that relation (4.10) holds. Consequently, the equations of motion (4.7) become those in (4.9). Q.E.D.

PROPOSITION 6. If the boundary value problem for the WE system (1.2) is given by two arbitrary complex analytic functions of one complex variable (and their complex conjugate functions), then the solution of (1.2) is unique up to a gauge transformation (4.10).

PROOF. By virtue of Propositions 1 and 2, the map from equations (1.2) to (2.4) is one-to-one. The map from equations (2.4) to the sigma-model (4.1) is also one-to-one because of the transformation (4.2ii). The solution of the boundary value problem for (4.1) possesses a unique solution [25]. Hence, from Proposition 3 and equations (4.5) and (4.8), it follows that the solution of the the boundary value problem for WE system (1.2) is unique up to multiplication by any function $f(\bar{z})$ satisfying (4.10). This means that the freedom of solutions to (1.2) and (2.4) is the same, up to a gauge function f . Q.E.D.

Now we examine certain aspects of complete integrability of the equations of motion (4.9) in the context of a two-dimensional Euclidean sigma model (4.1).

As it was shown by A. V. Mikhailov in [26], equation (4.1) is a compatibility condition for the two linear spectral problems

$$(4.13) \quad \partial\Phi = \frac{1}{\lambda + 1}\mathcal{U}\Phi, \quad \bar{\partial}\Phi = \frac{1}{\lambda - 1}\mathcal{U}^\dagger\Phi.$$

Here, $\mathcal{U} = \partial S S$, $\mathcal{U}^\dagger = S \bar{\partial} S$ with S given by (4.3), and $\Phi(z, \bar{z}, \lambda)$ is a matrix of fundamental solutions, while λ represents the spectral parameter. Note that there is a direct connection between

the matrix eigenfunction $\Phi(z, \bar{z}, \lambda)$ in expression (4.13) and the fields ψ_i , through the mapping (4.4) since there exists the relation [26]

$$(4.14) \quad S = \Phi(z, \bar{z}, 0).$$

Then we could say that the WE system (1.2) is completely integrable, because of the mappings (4.3) and (4.4). Indeed, by expressing the spin matrix S in terms of the functions φ_i and $\bar{\varphi}_i$, one obtains the explicit form of the linear spectral problem (4.13) for the equation of motion (4.9)

$$(4.15) \quad \partial\Phi = \frac{2}{\lambda+1}M\Phi, \quad \bar{\partial}\Phi = \frac{2}{\lambda-1}M^\dagger\Phi,$$

where

$$(4.16) \quad M = \begin{pmatrix} b/2 & a \\ -c & -b/2 \end{pmatrix}.$$

and we introduce the following notation,

$$(4.17) \quad \begin{aligned} a &= -\bar{f}^2\bar{\varphi}_1^2 + \frac{1}{\bar{f}q^2}[\bar{\varphi}_1\bar{\varphi}_2^2\partial(\bar{f}\varphi_2) - \bar{\varphi}_2|\varphi_2|^2\partial(\bar{f}\bar{\varphi}_1)], \\ b &= 2[-\bar{f}^2\bar{\varphi}_1\varphi_2 + \frac{1}{\bar{f}q^2}(\varphi_1|\varphi_2|^2\partial(\bar{f}\bar{\varphi}_1) - \bar{\varphi}_2|\varphi_1|^2\partial(\bar{f}\varphi_2))], \\ c &= -\bar{f}^2\varphi_2^2 + \frac{1}{\bar{f}q^2}[\varphi_1|\varphi_1|^2\partial(\bar{f}\varphi_2) - \varphi_1^2\varphi_2\partial(\bar{f}\bar{\varphi}_1)], \end{aligned}$$

Making use of (4.5), one can find an explicit form for the coefficients (4.17) in terms of the functions ψ_i . The matrix M can be written in the form

$$(4.18) \quad M = A + \frac{J}{p^2}A^\dagger, \quad \det M = -\frac{2J}{p^2},$$

where J is the current (1.8) and A is a degenerate nilpotent matrix which can be decomposed as follows,

$$(4.19) \quad A = -\bar{\psi}_1\psi_2\sigma_3 - \bar{\psi}_1^2\sigma_+ + \psi_2^2\sigma_-, \quad \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2),$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

However, in order to be able to use results from the inverse scattering method to construct soliton solutions of the WE system (1.2), it is convenient to simplify the form of the linear spectral problem (4.15).

PROPOSITION 7. For any bounded entire function J , the linear spectral problem for the WE system (1.2) has the form

$$(4.20) \quad \partial\Phi = \frac{2}{\lambda+1}A\Phi, \quad \bar{\partial}\Phi = \frac{2}{\lambda-1}A^\dagger\Phi$$

PROOF. Indeed, if we substitute (4.18) into the system (4.15), then the system takes the form

$$(4.21) \quad \partial\Phi = \frac{2}{\lambda+1} \left(A + \frac{J}{p^2} A^\dagger \right) \Phi$$

$$(4.22) \quad \bar{\partial}\Phi = \frac{2}{\lambda-1} \left(A^\dagger + \frac{\bar{J}}{p^2} A \right) \Phi.$$

From the conservation of the current (1.11), we obtain that the current J is a holomorphic function. According to Liouville's theorem if $J(z)$ is an entire function, and if $|J(z)| \leq M$ for all $z \in \mathbb{C}$, then $J(z) \equiv \text{constant}$. Consequently, one can take the current J to be equal to zero, hence equations (4.21) and (4.22) become (4.20). The compatibility condition for the two equations in (4.20), namely,

$$\bar{\partial}A - \partial A^\dagger + [A, A^\dagger] = 0,$$

is satisfied, whenever the WE system (1.2) holds. Under these circumstances, the linear spectral problem (4.20) holds for the WE system (1.2). So, matrices A and A^\dagger can be identified as the Lax pair for the WE system. Q.E.D.

Moreover, an interesting feature of the WE system (1.2) has been observed. Namely, the system (4.20) has the WE system of equations as compatibility conditions for any function $J(z)$, not necessarily bounded. This fact can be easily verified by direct calculation.

Note that the system of Riccati equations corresponding to (4.20) is

$$\partial y = -\frac{2}{\lambda+1} (\bar{\psi}_1 + \psi_2 y)^2, \quad \bar{\partial} y = \frac{2}{\lambda-1} (\bar{\psi}_2 - \psi_1 y)^2,$$

where y is a complex function (called the pseudopotential [29]) given by the ratio of the components of the vector Φ , that is, $y = \phi_1/\phi_2$. We conclude that the existence of the linear spectral problem (4.20) for the WE system implies that this system is completely integrable.

Finally, a property of the WE system (1.2) in the context of the sigma model, is the existence of a topological charge. Indeed, it is well known that the sigma-model (4.1) possesses a topological charge, [27-29] which we denote by I . Making use of the current J , the transformations (4.3) and the equations of motion (1.2), one finds that if the integral

$$(4.23) \quad I = \frac{i}{8\pi} \int_C \text{Tr}(S \cdot [\partial S, \bar{\partial} S]) dz d\bar{z} = -\frac{i}{2\pi} \int_C \frac{1}{p^2} [|J|^2 - p^4] dz d\bar{z},$$

exists, it is an integer, where J is given by equation (1.8).

4.2 Reduction of the Weierstrass-Enneper System to a Decoupled Linear System.

Now we discuss a set of conditions which allow the system (1.2) to become a linear decoupled system of equations.

PROPOSITION 8. If the functions ψ_1 and ψ_2 satisfy an overdetermined system composed of the equations of motion (1.2) and differential conditions

$$(4.24) \quad \bar{\psi}_1 \bar{\partial} \psi_1 + \psi_2 \bar{\partial} \bar{\psi}_2 = 0, \quad \bar{\psi}_2 \partial \psi_2 + \psi_1 \partial \bar{\psi}_1 = 0,$$

then the overdetermined system is equivalent to a linear decoupled system of the form

$$(4.25) \quad \bar{\partial} \partial \psi_i + p_0^2 \psi_i = 0, \quad \partial \bar{\partial} \bar{\psi}_i + p_0^2 \bar{\psi}_i = 0, \quad i = 1, 2,$$

$$|\psi_1|^2 + |\psi_2|^2 = p_0 \in \mathbb{R}.$$

PROOF. Making use of (1.2) and conditions (4.24) we obtain that the derivatives of p given by (2.14) vanish

$$(4.26) \quad \partial p = \psi_1(\partial\bar{\psi}_1) + \bar{\psi}_2(\partial\psi_2) = 0, \quad \bar{\partial} p = \bar{\psi}_1(\bar{\partial}\psi_1) + \psi_2(\bar{\partial}\bar{\psi}_2) = 0.$$

This means that if (4.24) holds, then p is a real constant, say p_0 . Thus,

$$(4.27) \quad |\psi_1|^2 + |\psi_2|^2 = p_0$$

is a conserved quantity. Hence, the Weierstrass-Enneper system (1.2) becomes a linear system which can be decoupled in terms of the functions ψ_i such that (4.25) holds. Q.E.D.

Let us now investigate the case in which all the derivatives of the functions ψ_i and $\bar{\psi}_i$ are specified. This means that we supplement the WE system (1.2) with some additional differential constraints, so we can formulate the following.

If the conditions (4.24) hold then we show that the system (1.2) can be extended to the system of the form

$$(4.28a) \quad \partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} p\psi_2 \\ \alpha \end{pmatrix}, \quad \bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \beta \\ -p\psi_1 \end{pmatrix}$$

and the respective conjugate system,

$$(4.28b) \quad \partial \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} \bar{\beta} \\ -p\bar{\psi}_1 \end{pmatrix}, \quad \bar{\partial} \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = \begin{pmatrix} p\bar{\psi}_2 \\ \bar{\alpha} \end{pmatrix}$$

where the quantities α and β are assumed to be some polynomial functions expressible in terms of ψ_i and $\bar{\psi}_i$, with constant coefficients. The system (4.28) will be called the augmented system. Our aim is to find an explicit form for α and β in such a way that they do not provide any additional differential constraints on ψ_i and $\bar{\psi}_i$ other than (1.2) and (4.24) when the compatibility conditions for (4.28) are added, namely (2.8) and

$$(4.29) \quad \begin{aligned} \partial\partial\psi_1 &= \partial(p\psi_2) = (\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2)\psi_2 + p\partial\psi_2, \\ \bar{\partial}\bar{\partial}\bar{\psi}_1 &= \bar{\partial}(p\bar{\psi}_2) = (\bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2)\bar{\psi}_2 + p\bar{\partial}\bar{\psi}_2, \\ \bar{\partial}\bar{\partial}\psi_2 &= -\bar{\partial}(p\psi_1) = -(\bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2)\psi_1 - p\bar{\partial}\psi_1, \\ \partial\partial\bar{\psi}_2 &= -\partial(p\bar{\psi}_1) = -(\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2)\bar{\psi}_1 - p\partial\bar{\psi}_1. \end{aligned}$$

Indeed, from the compatibility conditions (2.8) and (4.29), the analysis of the dominant terms in the ψ_i and $\bar{\psi}_i$ functions leads to the requirement that all unknown derivatives, $(\bar{\partial}\psi_1, \partial\bar{\psi}_1, \partial\psi_2, \bar{\partial}\bar{\psi}_2)$, other than those appearing in (1.2), have to be cubic in terms of the fields ψ_i and $\bar{\psi}_i$. Moreover, if one assumes that the discrete symmetry of the system (1.2), invariant under the reflection symmetry in the space of dependent and independent variables, namely,

$$(4.30) \quad \psi_i \rightarrow -\psi_j, \quad \bar{\psi}_i \rightarrow -\bar{\psi}_j, \quad i \neq j = 1, 2, \quad \partial \rightarrow -\bar{\partial}, \quad \bar{\partial} \rightarrow -\partial,$$

can be extended to the augmented system (4.28), then one obtains the following relations,

$$(4.31) \quad \begin{aligned} \partial\bar{\psi}_1 &= -p[\bar{c}_2\bar{\psi}_1 + \bar{c}_1\bar{\psi}_2 + \bar{c}_4\psi_1 + \bar{c}_3\psi_2], \\ \bar{\partial}\psi_1 &= -p[c_2\psi_1 + c_1\psi_2 + c_4\bar{\psi}_1 + c_3\bar{\psi}_2], \end{aligned}$$

$$\begin{aligned}\partial\psi_2 &= p[c_1\psi_1 + c_2\psi_2 + c_3\bar{\psi}_1 + c_4\bar{\psi}_2], \\ \bar{\partial}\bar{\psi}_2 &= p[\bar{c}_1\bar{\psi}_1 + \bar{c}_2\bar{\psi}_2 + \bar{c}_3\psi_1 + \bar{c}_4\psi_2].\end{aligned}$$

It is assumed that the c_i , $i = 1, \dots, 4$ are constants to be determined from the compatibility conditions for (2.8) and (4.29). Substituting (4.31) into (4.29) leads us to a system of equations which are polynomial in ψ_i and $\bar{\psi}_i$. The unique solution of this system has the form

$$(4.32a) \quad \partial \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = p \begin{pmatrix} \psi_2 \\ -\psi_1 \end{pmatrix} = \bar{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and its respective conjugate system

$$(4.32b) \quad \partial \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix} = p \begin{pmatrix} \bar{\psi}_2 \\ -\bar{\psi}_1 \end{pmatrix} = \bar{\partial} \begin{pmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \end{pmatrix}.$$

Note that the same formula (4.32) can be found in [15]. Under the conditions (4.24), we show that the system (1.2) admits a conserved quantity (4.27) with p a real constant. This means that in this case by virtue of (1.5), the Gaussian curvature $K = 0$, which implies the space is flat.

PROPOSITION 9. If the functions ψ_1 and ψ_2 satisfy an overdetermined system composed of the WE system (1.2) and the following differential constraint

$$(4.33) \quad -(\bar{\psi}_1\bar{\partial}\psi_1 + \psi_2\bar{\partial}\bar{\psi}_2) + (\psi_1\partial\bar{\psi}_1 + \bar{\psi}_2\partial\psi_2) = 0,$$

then the function p is a real valued function of a real argument $x = (z + \bar{z})/2$,

$$(4.34) \quad |\psi_1|^2 + |\psi_2|^2 = p(x).$$

PROOF. Indeed, using the derivatives of p , and taking into account (4.33), we obtain

$$(\partial - \bar{\partial})p = \psi_1(\partial\bar{\psi}_1) + \bar{\psi}_2(\partial\psi_2) - \bar{\psi}_1(\bar{\partial}\psi_1) - \psi_2(\bar{\partial}\bar{\psi}_2) = 0.$$

This completes the proof. Q.E.D.

Consequently, in the case when (4.33) holds, the first fundamental form (1.4) and the Gaussian curvature (1.5) take the following form

$$\Omega = 4p^2(x) dzd\bar{z}, \quad K = -\ddot{p}(x)/p^2(x),$$

respectively. Here, we introduce the notation $\ddot{p} = d^2p/dx^2$.

5. EXAMPLES AND APPLICATIONS. At this point, we would like to illustrate the proposed procedure for constructing solutions of the WE system (1.2) with several elementary examples.

Now, let us discuss some classes of solutions to the WE system (1.2), which can be obtained directly by applying the transformation (2.3). First, we consider the class of solutions which correspond to analytic choices of the function ρ . It is easy to check that for this class of solutions the conserved density J in (1.11) is identically equal to zero.

1. The simplest solutions of this type are given by

$$(5.1) \quad \rho = \left[\frac{(z - z_0)}{\lambda} \right]^n$$

where λ and z_0 are arbitrary real and complex numbers respectively. From the point of view of the sigma-model this form of ρ corresponds to the instanton of charge $I = -n$ located at z_0 and of size λ . By virtue of the invariance of the system (1.2) under conformal transformations, we can set

without loss of generality, $z_0 = 0$ and $\lambda = 1$. Then, using (2.3) we find that the solutions of (1.2) are given by

$$(5.2) \quad \psi_1 = \epsilon n^{1/2} \frac{z^n \bar{z}^{(n-1)/2}}{1 + |z|^{2n}}, \quad \psi_2 = \epsilon n^{1/2} \frac{z^{(n-1)/2}}{1 + |z|^{2n}}.$$

Each of these solutions belongs to a different topological sector of index n . Furthermore, notice that for all even n , the solutions are double valued. Nevertheless, this fact has no influence on the surfaces parametrized by the relations (1.3). Actually, the solutions (5.2) correspond to only one constant mean curvature surface, which is covered n times as z runs over the complex plane. This surface is obtained (modulo translations) by revolving the curve

$$(5.3) \quad X_2 = (X_3 - 2) \left(\frac{X_3}{4 - X_3} \right)^{1/2}$$

around the axis X_3 . It possesses a conic point in $(0, 0, 2)$.

2. Another class of solutions is provided by the analytic function

$$(5.4) \quad \rho = e^{\lambda z},$$

corresponding to a static domain wall in the isotropic $O(3)$ magnet. The associated solution of the system (1.2) is

$$(5.5) \quad \psi_1 = \epsilon \bar{\lambda}^{1/2} \frac{e^{\bar{\lambda} \bar{z}/2}}{e^{-\lambda z} + e^{-\bar{\lambda} \bar{z}}}, \quad \psi_2 = \epsilon \lambda^{1/2} \frac{e^{\lambda z/2}}{e^{-\lambda z} + e^{-\bar{\lambda} \bar{z}}}.$$

Also in this case, the whole class of solutions parametrized by λ represents a unique constant mean curvature surface (modulo translations), obtained from (1.3) by revolving the following curve around the X_3 axis

$$(5.6) \quad X_3 = 2 \frac{X_2^2}{1 \pm \sqrt{1 - X_2^2}}.$$

Many other solutions of (1.2) admitting ρ to be a meromorphic function can be found. For the present, we do not discuss them.

3. Let us assume now, as opposed to the previous cases, that the conserved density in (1.8) is a non-vanishing holomorphic function. In such a case, one can check that the matrix U in the spectral problem (4.13) has a non-vanishing $Tr \mathcal{U}^2$. The simplest choice is to put

$$(5.7) \quad \mathcal{U} = g(z) \sigma_3.$$

The solution of the corresponding spectral problem can be readily obtained [26]

$$(5.8) \quad \Phi = \exp[2i\chi \sigma_3] \sigma_1,$$

where σ_i are Pauli matrices, $\chi = Im \int_{\Gamma} g(z) dz$ and Γ is an arbitrary curve in the domain in which g is analytic. Then, resorting to the relations (4.14) and (4.4), one obtains the following solutions to equation (1.2)

$$(5.9) \quad \psi_1 = -\frac{\epsilon}{2} i e^{-i\chi} \bar{g}^{1/2}, \quad \psi_2 = \frac{\epsilon}{2} i e^{-i\chi} g^{1/2}.$$

The associated surface is given by the parametric equations

$$(5.10) \quad \begin{aligned} X_1 &= \sin 2\chi + X_{10}, & X_2 &= -\cos 2\chi + X_{20}, \\ X_3 &= \omega + X_{30} \quad (\omega = \operatorname{Re} \int_{\Gamma} f(z) dz). \end{aligned}$$

which describe a cylinder having X_3 as a symmetry axis. Non-trivial deformations of this type of solution can be found by using the recurrence N -soliton wave function formula [26] in expression (4.14).

Now, let us discuss a simple example to illustrate the construction introduced in **Section 4**.

4. Consider the possibility where f is real and $f = q^{-1/2}$ with φ_i chosen to make $\partial f = 0$. In this case, from (4.9) one has

$$(5.11) \quad \partial\varphi_1 = q\bar{f}^2\varphi_2 = \varphi_2, \quad \bar{\partial}\varphi_2 = -qf^2\varphi_1 = -\varphi_1.$$

This system reduces to two second order linear equations of the form (4.25)

$$\partial\bar{\partial}\varphi_i + \varphi_i = 0, \quad i = 1, 2.$$

For example, let us write a simple set of solutions to this equation

$$\begin{aligned} \varphi_1 &= -ie^{i(z+\bar{z})}, & \bar{\varphi}_1 &= ie^{-i(z+\bar{z})} \\ \varphi_2 &= e^{i(z+\bar{z})}, & \bar{\varphi}_2 &= e^{-i(z+\bar{z})}. \end{aligned}$$

Therefore, one has

$$q = |\varphi_1|^2 + |\varphi_2|^2 = 2, \quad f = \frac{1}{\sqrt{2}}.$$

Note that for these functions φ_i , conditions (4.24) are identically satisfied. From (4.6), we obtain

$$\rho = -ie^{2i(z+\bar{z})}.$$

It is also easy to show that for this class of solutions ρ , equations (2.4) are identically satisfied. Substituting the functions ρ and f into equations (4.8), one obtains an explicit solution of the WE system (1.2)

$$\psi_1 = -i\frac{\epsilon}{\sqrt{2}}e^{i(z+\bar{z})}, \quad \psi_2 = \frac{\epsilon}{\sqrt{2}}e^{i(z+\bar{z})}.$$

This solution represents a phase plane wave, since the argument is one dimensional $x = z + \bar{z}$, its absolute value is constant, and the solution has exponential form.

5. A special class of exponential solutions to (1.2) can be found to hold when p is constant. According to Proposition 6, we have to solve (4.25). Thus, a vacuum solution takes the form

$$(5.12) \quad \begin{aligned} \psi_1 &= c_1 e^{i(hz+k\bar{z})}, \\ \psi_2 &= ic_1 \frac{h}{p} e^{i(hz+k\bar{z})}, \end{aligned}$$

where c_1 is a complex constant and h, k are real constants such that

$$|c_1|^2 = \frac{p^3}{(p^2 + h^2)}, \quad p = hk.$$

Due to the linearity of equations (4.25), we can look for a more general class of solutions which represent a superposition of exponential functions. The one soliton solution of (4.25) is given by

$$\begin{aligned} \psi_1 &= c_1 e^{i(h_1 z + k_1 \bar{z})} + c_2 e^{i(h_2 z + k_2 \bar{z})}, \\ (5.13) \quad \psi_2 &= \frac{i}{p} (h_1 c_1 e^{i(h_1 z + k_1 \bar{z})} + h_2 c_2 e^{i(h_2 z + k_2 \bar{z})}), \end{aligned}$$

where the c_i are complex constants and the h_i, k_i are real constants which satisfy,

$$h_1 k_1 = p^2, \quad h_2 k_2 = p^2, \quad h_1 h_2 = -p^2,$$

and

$$|c_2|^2 = \frac{p - |c_1|^2(1 + h_1^2/p^2)}{1 + p^2/h_1^2}.$$

From (2.1), we obtain the expression for ρ corresponding to the one soliton solutions,

$$\rho = -i \frac{c_1 e^{ih_1(z+\bar{z})} + c_2 e^{-ih_1(z+\bar{z})}}{\bar{c}_1 e^{ih_1(z+\bar{z})} - \bar{c}_2 e^{-ih_1(z+\bar{z})}}.$$

for which condition (2.4) is identically satisfied.

6. Now, let us discuss the construction of multi-soliton solutions to the WE system (1.2), which can be obtained by exploiting the linear spectral problem (4.15). According to the first step of the procedure, we choose an antiholomorphic function of the form

$$(5.14) \quad f = \epsilon(a-b) \frac{(2\bar{z} - a - b)z - a(\bar{z} - a) - b(\bar{z} - b)}{|z - a|^2 + |z - b|^2}, \quad \epsilon = \pm 1, \quad a, b \in \mathbb{R}$$

and look for a nontrivial solution ρ of the system (2.4),

$$\rho = \frac{z - a}{z - b}. \quad (5.15)$$

The substitution of (5.14) and (5.15) into the relations (4.8) gives

$$\begin{aligned} (5.16) \quad \varphi_1 &= \frac{z - a}{(2\bar{z} - a - b)z - a(\bar{z} - a) - b(\bar{z} - b)}, \\ \varphi_2 &= \frac{\bar{z} - b}{(2z - a - b)\bar{z} - a(z - a) - b(z - b)} \end{aligned}$$

Finally, from the solution of the linear spectral problem (4.15) and relations (4.14) and (4.4), we obtain an explicit one soliton solution of the WE system (1.2)

$$(5.17) \quad \psi_1 = \epsilon(a-b) \frac{z - a}{|z - a|^2 + |z - b|^2}, \quad \psi_2 = \epsilon(a-b) \frac{\bar{z} - b}{|z - a|^2 + |z - b|^2}.$$

A similar computation can be performed for the case in which ρ satisfies (2.4) and has a more general form than (5.15)

$$(5.18) \quad \rho = \prod_{j=1}^N \frac{z - a_j}{z - b_j}, \quad a_j, b_j \in \mathbb{R}.$$

with distinct parameters such that a and b are replaced by a_j and b_j , respectively. The same process is done with the function f in (5.14). Thus, we can determine explicitly the corresponding form of a multi-soliton solution by applying the recurrence N -soliton wave function formula [26] in the expression (4.14) to obtain,

$$(5.19) \quad \psi_1 = \epsilon \frac{\prod_{j=1}^N \frac{z - a_j}{z - b_j}}{1 + \prod_{j=1}^N \left| \frac{z - a_j}{z - b_j} \right|^2} \left(\sum_{s=1}^N \frac{1}{(\bar{z} - b_s)} \left(\prod_{\substack{j=1 \\ j \neq s}}^N \frac{(\bar{z} - a_j)}{(\bar{z} - b_j)} - \prod_{j=1}^N \frac{(\bar{z} - a_j)}{(\bar{z} - b_j)} \right) \right)^{1/2}$$

$$\psi_2 = \frac{\epsilon}{1 + \prod_{j=1}^N \left| \frac{z - a_j}{z - b_j} \right|^2} \left(\sum_{s=1}^N \frac{1}{(z - b_s)} \left(\prod_{\substack{j=1 \\ j \neq s}}^N \frac{(z - a_j)}{(z - b_j)} - \prod_{j=1}^N \frac{(z - a_j)}{(z - b_j)} \right) \right)^{1/2}$$

Note that this solution admits simple poles. The topological charge (4.23) for each of the instanton solutions (5.19) corresponds to $I = \epsilon N$.

6. Future Outlook. In this paper, we have shown that the adapted WE system (1.2) proposed by B. Konopelchenko and I. A. Taimanov as a tool to induce constant mean curvature surfaces, is closely related to the nonlinear Euclidean sigma-model $SU(2)$. This link enabled us to propose a new approach to the construction of solutions, based on the intermediate system of equations (4.9) with which the sigma model (4.1) is associated.

Let us now consider a system of the form

$$(6.1) \quad \begin{aligned} \partial \psi_1 &= h(z, \bar{z}) p \psi_2, & \bar{\partial} \bar{\psi}_1 &= h(z, \bar{z}) p \bar{\psi}_2, \\ \bar{\partial} \psi_2 &= -h(z, \bar{z}) p \psi_1, & \partial \bar{\psi}_2 &= -h(z, \bar{z}) p \bar{\psi}_1, \end{aligned}$$

where h is assumed to be a real function of z and \bar{z} . We are interested in conditions under which system (6.1) becomes a completely integrable one.

Making use of the transformation (2.1), by calculations similar to those done in **Section 2**, we find

$$(6.2) \quad \psi_1 = \epsilon \rho \frac{(\bar{\partial} \bar{\rho})^{1/2}}{h^{1/2} (1 + |\rho|^2)}, \quad \psi_2 = \epsilon \frac{(\partial \rho)^{1/2}}{h^{1/2} (1 + |\rho|^2)}, \quad \epsilon^2 = 1,$$

and system (6.1) becomes,

$$(6.3a) \quad \bar{\partial} \partial \rho = \frac{2\bar{\rho}}{1 + |\rho|^2} \partial \rho \bar{\partial} \rho + (\partial(\ln h))(\partial \rho),$$

$$(6.3b) \quad \partial \bar{\partial} \rho = \frac{2\rho}{1 + |\rho|^2} \bar{\partial} \bar{\rho} \partial \bar{\rho} + (\bar{\partial}(\ln h))(\bar{\partial} \bar{\rho})$$

The above form is more convenient to analyze than the equations (6.1). Employing the conditional symmetry method [30,31], we look for conditions necessary for solvability of a class of equations (6.3) which admit compatible first order differential constraints. We consider here the

simplest case where the differential constraints are based on an $sl(2, \mathbb{C})$ representation. So, we assume that they take the form of coupled Riccati equations (and their complex conjugate equations) with nonconstant coefficients,

$$\begin{aligned} \partial\rho &= A_1^0(z, \bar{z}) + A_1^1(z, \bar{z})\rho + A_1^2(z, \bar{z})\rho^2, \\ \bar{\partial}\rho &= A_2^0(z, \bar{z}) + A_2^1(z, \bar{z})\rho + A_2^2(z, \bar{z})\rho^2. \end{aligned} \tag{6.4}$$

The compatibility condition for the system (6.4) requires appending to it the zero curvature conditions,

$$A_{[\mu, \nu]}^l + \frac{1}{2}C_{ab}^l A_\mu^a A_\nu^b = 0, \quad a, b, l = 0, 1, 2 \quad \mu, \nu = 1, 2, \tag{6.5}$$

where $(z^\mu) = (z, \bar{z})$ and C_{ab}^l are structure constants of $sl(2, \mathbb{C})$. The brackets $[\mu, \nu]$ denote here the alternation with respect to the indices μ and ν . We look for conditions on the function h which ensures that the overdetermined system composed of the equations (6.3), differential constraints (6.4) and conditions (6.5) are in involution. These involutivity conditions give us the specific differential restrictions on the class of function h

$$\bar{\partial}\partial\left(\frac{1}{h}\right) = 0. \tag{6.6}$$

The general solution of (6.6) is given by

$$h(z, \bar{z}) = \frac{1}{r(z) + r(\bar{z})}. \tag{6.7}$$

Here, r is an arbitrary real function. Then, system (6.1) becomes

$$\begin{aligned} \partial\psi_1 &= \frac{p}{r(z) + r(\bar{z})}\psi_2, & \bar{\partial}\bar{\psi}_1 &= \frac{p}{r(z) + r(\bar{z})}\bar{\psi}_2, \\ \bar{\partial}\psi_2 &= -\frac{p}{r(z) + r(\bar{z})}\psi_1, & \partial\bar{\psi}_2 &= -\frac{p}{r(z) + r(\bar{z})}\bar{\psi}_1, \end{aligned} \tag{6.8}$$

and consequently, system (6.3) takes the form

$$\begin{aligned} \bar{\partial}\partial\rho &= \frac{2\bar{\rho}}{1 + |\rho|^2}\partial\rho\bar{\partial}\rho - \frac{\partial r}{r(z) + r(\bar{z})}(\partial\rho), \\ \partial\bar{\partial}\bar{\rho} &= \frac{2\rho}{1 + |\rho|^2}\bar{\partial}\bar{\rho}\partial\bar{\rho} - \frac{\bar{\partial}r}{r(z) + r(\bar{z})}(\bar{\partial}\bar{\rho}). \end{aligned} \tag{6.9}$$

It is easy to show that system (6.8) cannot be transformed into the original WE system (1.1), corresponding to constant mean curvature surfaces, by any change of independent variables.

An analysis of system (6.9) similar to the one carried out in **Section 4**, can provide us with an explicit form of the spectral problem for (6.9). Since system (1.1) constitutes a special case of system (6.1), it is evident that our approach can be applied to systems which describe much more diverse types of surfaces. This task will be undertaken in future work.

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