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# On Characterizations of Exponential Distribution through Order Statistics and Record Values with Random Shifts

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**Abstract:** Distributional relations of the form $Y \stackrel{d}{=} X + T$ where $X$, $Y$, and $T$ are record values or order statistics and the random translator $T$ is independent from $X$ are considered. Characterizations of the exponential distribution when the ordered random variables are non-neighboring are proved. Corollaries for Pareto and power function distributions are also derived.

**Keywords:** record values, order statistics, random translation, characterizations, exponential distribution, Pareto distribution, power function distribution

## 1 Introduction and Main Results

A number of known characterization results are based on the distributional equation $Y \stackrel{d}{=} X + T$ involving pair random variables (r.v.'s) $(X,Y)$ and a random translator (shift) variable $T$, independent of $X$. The sum $X+T$ is called random translation of $X$. We study a particular case when the pair $(X,Y)$ is a pair of possibly non-neighboring order statistics or record values. Characterizations based on the above distribution equation in the context of order statistics and record values were obtained by Wesołowski and Ahsanullah in [11] and Beutner and Kamps in [6], among others. Moreover, Ahsanullah et al. in [2] and Ahsanullah et al. in [3] studied two-sided translations. Recently Castaño-Martínez et al. in [7] generalized some existing results by exploring a new technique based on uniqueness results for non-linear Volterra integral equations. Alternatively, the proofs in this article use some recurrent relations for order statistics and record values and as a result the assumptions we make differ from those in Castaño-Martínez et al. in [7]. In general, the characterizations via random translations are subject to three groups of conditions: the distributional equation(s), the form of the parent distribution, and the distribution of the random translator T. Comparing our results with those in Castaño-Martínez et al. in [7], we impose on the ordered variables some restrictive conditions: two distributional equations and monotonicity of the hazard rate of the parent distribution. However, we do not assume the translator variable to have certain known distribution.

We begin with a characterization involving record values. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed (iid) random variables (r.v.'s) with cumulative distribution function (cdf) $F$, probability density $f$, and hazard rate $h(x) := f(x)/(1-F(x))$. Define (upper) record times by $t_1 = 1$ and $t_n = \min\{j : X_j > X_{t_{n-1}}\}$ for $n \geq 2$. The r.v.'s $R_n := R_n(X) = X_{t_n}$, for $n \geq 1$ are called (upper) record values of the sequence $\{X_n\}_{n\geq 1}$ ([5]). We write $X \sim Exp(\lambda)$ when $X$ has an exponential distribution with $F(x) = 1 - e^{-\lambda x}$ for $x \geq 0$ and $\lambda > 0$.

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**Theorem 1.1.** Let $X$ be a positive random variable with absolutely continuous cdf $F$ and $\lim_{x\to 0^+} F(x) = 0$. Suppose the hazard rate $h(x) > 0$ for all $x$ and $h(x)$ is either non-increasing or non-decreasing. For fixed integers $1 \le r < s$, assume that the (translator) r.v.'s $R'_{s-r}$ and $R''_{s-r}$ satisfy

(i) $R'_{s-r}$ is independent from $R_r$ and $R''_{s-r}$ is independent from $R_{r+1}$;
(ii) $R'_{s-r} \stackrel{d}{=} R''_{s-r} \stackrel{d}{=} R_{s-r}$.

Then both

$$R_s \stackrel{d}{=} R_r + R'_{s-r} \qquad \text{and} \qquad R_{s+1} \stackrel{d}{=} R_{r+1} + R''_{s-r} \tag{1}$$

hold true if and only if $X \sim Exp(\lambda)$ for some positive $\lambda$.

To obtain some corollaries of Theorem 1.1, observe that if $g(y)$ is a measurable non-decreasing function and $X = g(Y)$, then the record values with parents $X$ and $Y$ satisfy $R_k(X) \stackrel{d}{=} g(R_k(Y))$ for $k = 1,2,\ldots$. Moreover, if $X = \log Y \sim Exp(\lambda)$ then $Y$ has Pareto distribution, $Y \sim Par(\lambda)$ say, with cdf $F_Y(y) = 1 - y^{-\lambda}$, for $y \ge 1$ and $\lambda > 0$ ([8]). Setting $g(y) = \log y$ and $R_k(X) \stackrel{d}{=} \log R_k(Y)$ we convert (1) into $\log R_s(Y) \stackrel{d}{=} \log R_r(Y) + \log R'_{s-r}(Y)$, which is equivalent to $R_s(Y) \stackrel{d}{=} R_r(Y) R'_{s-r}(Y)$. This, in view of Theorem 1.1, implies the following characterization of Pareto distribution.

**Corollary 1.1 (random dilation).** Let $Y$ be a positive random variable with absolutely continuous cdf $F_Y$, such that $\lim_{x\to 1^+} F_Y(x) = 0$. Suppose the hazard rate $h_{\log Y}(y) > 0$ for all $y$ and $h_{\log Y}(y)$ is either non-increasing or non-decreasing. For fixed integers $1 \le r < s$, assume that the (dilator) r.v.'s $R'_{s-r}(Y)$ and $R''_{s-r}(Y)$ satisfy

(i) $R'_{s-r}(Y)$ is independent from $R_r(Y)$ and $R''_{s-r}(Y)$ is independent from $R_{r+1}(Y)$;
(ii) $R'_{s-r}(Y) \stackrel{d}{=} R''_{s-r}(Y) \stackrel{d}{=} R_{s-r}(Y)$ .

Then both

$$R_s(Y) \stackrel{d}{=} R_r(Y) R'_{s-r}(Y) \qquad \text{and} \qquad R_{s+1}(Y) \stackrel{d}{=} R_{r+1}(Y) R''_{s-r}(Y) \tag{2}$$

hold true if and only if $Y \sim Par(\lambda)$ for some positive $\lambda$.

Recall that if $X = -\log Z \sim Exp(\lambda)$ then $Z$ has the power function distribution, $Z \sim Pow(\lambda)$ say, with cdf $F_Z(z) = 1 - z^{\lambda}$, for $0 < z < 1$ and $\lambda > 0$ (see [8]). Clearly, if $q(z)$ is a measurable non-increasing function and $X = q(Z)$, then the lower record values $L_k(X)$ and $L_k(Z)$ (see [5]) with parents $X$ and $Z$, respectively, satisfy $L_k(X) \stackrel{d}{=} q(L_k(Z))$. Setting in (1), $q(z) = -\log z$ and $L_k(X) \stackrel{d}{=} -\log L_k(Z)$ for $k = 1,2,\ldots$, we obtain $-\log L_s(Z) \stackrel{d}{=} -\log L_r(Z) - \log L'_{s-r}(Z)$, which is equivalent to $L_s(Z) \stackrel{d}{=} L_r(Z) L'_{s-r}(Z)$. Now, Theorem 1.1 yields the following characterization of the power function distribution.

**Corollary 1.2 (random contraction).** Let $Z$ be a positive random variable with absolutely continuous cdf $F_Z$, such that $\lim_{x\to 1^-} F_Z(x) = 1$. Suppose the hazard rate $h_{-\log Z}(z) > 0$ for all $z$ and $h_{-\log Z}(z)$ is either non-increasing or non-decreasing. For fixed integers $1 \le r < s$, assume that the (contractor) r.v.'s $L'_{s-r}(Z)$ and $L''_{s-r}(Z)$ satisfy

(i) $L'_{s-r}(Z)$ is independent from $L_r(Z)$ and $L''_{s-r}(Z)$ is independent from $L_{r+1}(Z)$;
(ii) $L'_{s-r}(Z) \stackrel{d}{=} L''_{s-r}(Z) \stackrel{d}{=} L_{s-r}(Z)$.

Then both

$$L_s(Z) \stackrel{d}{=} L_r(Z) L'_{s-r}(Z) \qquad \text{and} \qquad L_{s+1}(Z) \stackrel{d}{=} L_{r+1}(Z) L''_{s-r}(Z) \tag{3}$$

hold true if and only if $Z \sim Pow(\lambda)$ for some positive $\lambda$.

Our next theorem concerns the order statistics $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ from a simple random sample with parent $X$.

**Theorem 1.2.** Let $X$ be a positive random variable with absolutely continuous cdf $F$ and $\lim_{x\to 0^+} F(x) = 0$. Suppose $h(x) > 0$ for all $x$ and $h(x)$ is either non-increasing or non-decreasing. For fixed integers r and s such that $1 \le r < s \le n-1$, assume that the (translator) r.v.'s $X'_{r:n}$ and $X''_{r:n}$ satisfy

(i) $X'_{r:n}$ is independent from $X_{s-r:n-r}$ and $X''_{r:n}$ is independent from $X_{s-r+1:n-r}$;
(ii) $X'_{r:n} \stackrel{d}{=} X''_{r:n} \stackrel{d}{=} X_{r:n}$.

Then both

$$X_{s:n} \stackrel{d}{=} X_{s-r:n-r} + X'_{r:n} \qquad \text{and} \qquad X_{s+1:n} \stackrel{d}{=} X_{s-r+1:n-r} + X''_{r:n} \tag{4}$$

hold true if and only if $X \sim Exp(\lambda)$ for some positive $\lambda$.

**Remark.** Khan and Shah in (2012) give the statement of Theorem 1.2. However, the proof they provide is not correct.

Similarly to Corollaries 1.1 and 1.2 above, using Theorem 1.2 one can obtain characterizations of Pareto and power function distributions by means of random dilation and contraction equations for order statistics. For brevity, we omit the formulation of these results here.

## 2 Proofs

**Lemma** Let $X$ be a positive random variable with absolutely continuous cdf $F$, such that $\lim_{x\to 0^+} F(x) = 0$. Suppose the hazard rate $h(x) > 0$ for all $x$ and $h(x)$ is either non-increasing or non-decreasing. If $g(x,y) > 0$ for $y > 0$, $0 < x < y$ and

$$\int_0^y \left( \frac{1}{h(y)} - \frac{1}{h(y-x)} \right) g(x,y)\, dx = 0, \tag{5}$$

then $X \sim Exp(\lambda)$ for some $\lambda > 0$..

**Proof.** Since $h(x)$ is non-increasing or non-decreasing, we have $h^{-1}(y) - h^{-1}(y-x) \le 0$ or $h^{-1}(y) - h^{-1}(y-x) \ge 0$ , respectively. Therefore, (5) implies $h^{-1}(y) - h^{-1}(y-x) = 0$ for almost all $x$ such that $0 < x < y$ and all $y > 0$. Thus, $h(x)$ is a constant, which under the assumption $\lim_{x\to 0^+} F(x) = 0$, implies (e.g., [8]) that $X \sim Exp(\lambda)$ for some $\lambda > 0$.

### *2.1 Proof of Theorem 1.1*

**Necessity.** Recall that if $X \sim Exp(\lambda)$, then $R_k$ for $k \ge 1$ has a gamma distribution (e.g. [10]). Whence $M_{R_k}(t) := Et^{R_k} = (\lambda/(\lambda - t))^k$ for any $k \ge 1$. If $X \sim Exp(\lambda)$, under the theorem's assumptions, we obtain

$$M_{R_r}(t) M_{R'_{s-r}}(t) = \left( \frac{\lambda}{\lambda - t} \right)^r \left( \frac{\lambda}{\lambda - t} \right)^{s-r} = \left( \frac{\lambda}{\lambda - t} \right)^s = M_{R_s}(t),$$

which yields the first equation in (1). The second part of (1) verifies similarly.

**Sufficiency.** Denote by $F_k(x)$ and $f_k(x)$ for $k \ge 1$ the cdf and pdf of $R_k$, respectively. Assuming (1) we obtain

$$F_s(y) = \int_0^y F_r(y-x) f_{s-r}(x)\, dx \qquad \text{and} \qquad F_{s+1}(y) = \int_0^y F_{r+1}(y-x) f_{s-r}(x)\, dx. \tag{6}$$

Define the cumulative hazard rate function $H(x) := -\log(1 - F(x))$. Recall (e.g. [5]) that for $k \ge 1$

$$f_{k+1}(x) = f(x) \frac{H^k(x)}{k!}, \qquad -\infty < x < \infty. \tag{7}$$

It is also known (e.g. [1]) that for $k \ge 1$

$$F_k(x) - F_{k+1}(x) = (1 - F(x)) \frac{H^k(x)}{k!}. \tag{8}$$

Using (6)-(8), we obtain

$$\begin{aligned}F_s(y)-F_{s+1}(y) &= \int_0^y (F_r(y-x)-F_{r+1}(y-x))\, f_{s-r}(x)\,dx \\ &= \int_0^y (1-F(y-x))\frac{H^r(y-x)}{r!}\, f_{s-r}(x)\,dx \\ &= \int_0^y \frac{f(y-x)}{h(y-x)}\frac{H^r(y-x)}{r!}\, f_{s-r}(x)\,dx \\ &= \int_0^y \frac{1}{h(y-x)}\, f_{r+1}(y-x)\, f_{s-r}(x)dx\end{aligned} \tag{9}$$

On the other hand, (7), (8), and the second equality in (1) yield

$$\begin{aligned}F_s(y)-F_{s+1}(y) &= (1-F(y))\frac{H^s(y)}{s!} \\ &= \frac{1}{h(y)}\, f_{s+1}(y) \\ &= \int_0^y \frac{1}{h(y)}\, f_{r+1}(y-x)\, f_{s-r}(x)\,dx.\end{aligned} \tag{10}$$

Subtracting (9) from (10), we have

$$\int_0^y \left(\frac{1}{h(y)}-\frac{1}{h(y-x)}\right) f_{r+1}(y-x)\, f_{s-r}(x)\,dx = 0,$$

which, referring to the lemma, implies $X \sim Exp(\lambda)$ for some $\lambda > 0$.

## 2.2 Proof of Theorem 1.2

**Necessity.** If $X \sim Exp(\lambda)$ then (e.g. [10]) the $k$th order statistic admits the representation $X_{k:n} \overset{d}{=} \sum_{i=1}^{k} W_i/(n-i+1)$ for $1 \le k \le n$, where $W_i$ are independent and $W_i \sim Exp(\lambda)$. Therefore, under the assumptions of the theorem,

$$X_{s-r:n-r} + X'_{r:n} \overset{d}{=} \frac{W_1}{n-r} + \frac{W_2}{n-r-1} + \ldots + \frac{W_{s-r}}{n-s+1} + \frac{W'_1}{n} + \frac{W'_2}{n-1} + \ldots + \frac{W'_r}{n-r+1} \overset{d}{=} X_{s:n},$$

which is the first equality in (4). Similarly one can verify the second part of (4).

**Sufficiency.** Let $F_{r,n}(x)$ and $f_{r,n}(x)$ for $1 \le k \le n$ denote the cdf and pdf of $X_{k:n}$, respectively. Assuming (7) we have

$$F_{s,n}(y) = \int_0^y F_{s-r,n-r}(y-x)\, f_{r,n}(x)\,dx \qquad \text{and} \qquad F_{s+1,n}(y) = \int_0^y F_{s-r+1,n-r}(y-x)\, f_{r,n}(x)\,dx. \tag{11}$$

It is known (e.g. [11]) that for $1 \le s \le n-1$

$$F_{s,n}(x) - F_{s+1,n}(x) = \frac{F(x)}{s f(x)}\, f_{s,n}(x). \tag{12}$$

Moreover, (e.g. [4]) for $1 \le k \le n-1$

$$\begin{aligned}f_{k,n}(x) &= \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x)(1-F(x))^{n-k} f(x) \\ &= \frac{k}{n-k}\frac{1-F(x)}{F(x)}\, f_{k+1,n}(x).\end{aligned} \tag{13}$$

Therefore, taking into account (11)-(13), we obtain

$$\begin{aligned}F_{s,n}(y) - F_{s+1,n}(y) &= \int_0^y (F_{s-r,n-r}(y-x) - F_{s-r+1,n-r}(y-x))\, f_{r,n}(x)\,dx \\ &= \int_0^y \frac{F(y-x)}{(s-r)f(y-x)}\, f_{s-r,n-r}(y-x)\, f_{r,n}(x)\,dx \\ &= \frac{1}{n-s}\int_0^y \frac{1-F(y-x)}{f(y-x)}\, f_{s-r+1,n-r}(y-x)\, f_{r,n}(x)\,dx.\end{aligned} \tag{14}$$

On the other hand, (13) yields

$$F_{s,n}(y) - F_{s+1,n}(y) = \frac{F(y)}{sf(y)} f_{s,n}(y) \tag{15}$$

$$= \frac{1}{n-s} \frac{1-F(y)}{f(y)} f_{s+1,n}(y)$$

$$= \frac{1}{n-s} \frac{1-F(y)}{f(y)} \int_0^y f_{s-r+1,n-r}(y-x) f_{r,n}(x)\, dx. \tag{16}$$

Therefore, subtracting (14) from (15),

$$\int_0^y \left( \frac{1}{h(y)} - \frac{1}{h(y-x)} \right) f_{s-r+1,n-r}(y-x) f_{r,n}(x)\, dx = 0. \tag{17}$$

It follows from (17) and the lemma that $X \sim Exp(\lambda)$ for some $\lambda > 0$, which completes the proof.

## 3 Concluding Remarks

The characterizations given in Section 1 can be used in developing goodness-of-fit tests for the corresponding probability distributions. Let us recall here a construction (see e.g. [4]) for implementing such tests. Suppose we have a large number of observations on a positive random variable $X$ and want to test whether $X$ is exponentially distributed with some unknown $\lambda$. Let us split the data into three independent samples: $X_1, X_2, \ldots, X_n$; $X_{n+1}, \ldots, X_{2n-r}$; $X_{2n-r+1}, \ldots, X_{3n-r}$, where $1 \le r < n-1$. Now, according to Theorem 2 for example, the data come from an exponential distribution if and only if for an integer $s$ such that $1 \le r < s \le n-1$ the equations (4) hold true, where the involved three order statistics come from the three sub-samples above.

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