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DOI: https://doi.org/10.24297/jam.v19i.8833

Approximation properties For generalized S–Szasz Operators with Application

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Abstract

This work focuses on a class of positive linear operators of S–Szasz type; we establish some direct results, which include Voronovskaja type asymptotic formula for a sequence of summation–integral type, we find a recurrence relation of the m-the order moment and the convergence theorem for this sequence. Finally, we give some figures.

Keywords: Korovkin's theorem, Szász operator, m-th order moment, Voronovaskaja –asymptotic type formula.

1. **Introduction**

With a great potential for applications, approximation theory represents an old field of mathematical research. One of the most vital aspect in this field is the construction of these approximation processes by using various methods as algebraic and trigonometric identities, convolution products, and probability schemes. Actually, a probabilistic approach has the advantage of producing short, elegant proofs and going deeper into the subject under investigation [4, 5, 7, 9, ,12, 17, 18 and 19]. Use of linear positive operators has played a crucial role in approximation theory for the last seven decades. In 1950, Szasz [16] defined

$$S_n(f,x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),n > 0$$

Very recently Acar, et al. [1] introduced a sequence of linear positive operators based on the Sz´asz–Mirakyan operators having the form

$$\mathcal{R}_n(f;x) = \sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}f\left(\frac{k}{n}\right),n \in N, x \in R^+$$

Where α_n, β_n: $R^+ \to R$,

$\alpha_n(x) = \frac{(n-(a+b)}{n}x$ and $\beta_n(x) = \frac{(n-a)(n-b)}{n^2}x$

In the present paper, we study a generalization of some sequence of linear positive operators the S-Szász

$a,b \in R$, $n \in N^0$ and $-\frac{1}{2} > s > 0$ as follows:

For $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty): |f(t)| \le Me^{\gamma t}$, for some $\gamma > 0, t \in [0,\infty)\}$.

$$P^s_{n,a,b}(f;x) = \frac{(n-a)(n-b)}{n}\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}f\left(x+\frac{t-x}{n^s}\right)dt,$$

$$x \in [0,\infty), \quad (1.1)$$

The space $C_\gamma[0,\infty)$ is normed by the norm $\|f\|_{C_\gamma} = \sup_{t\in[0,\infty)}\frac{|f(t)|}{e^{\gamma t}}, f\epsilon C_\gamma[0,\infty)$.

Our new sequence is give us many other older sequence when.

In this paper, we assume that, $C[0,\infty)$ the space of all continuous real-valued functions on the interval $[0,\infty)$.

2. Basic Results

In this section, we need the following basic lemmas:

For $f: [0,\infty) \to R$, we define

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$$R_{n,a,b}^{s}(f,x)=\sum_{k=0}^{\infty}e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}f(\frac{k}{n}),x\in[0,\infty).$$

For $m\in N^0$, The m-th order moment of the sequence $R_{n,a,b}^{s}(f,x)$ is defined as

$$\mu_{n,m}(x)=\sum_{k=0}^{\infty}e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\left(\frac{k}{n}-x\right)^m.$$

Lemma 2.1. For the function $\mu_{n,m}(x)$, we have $\mu_{n,0}(x)=1$, $\mu_{n,1}(x)=\frac{-(a+b)x}{n}+\frac{abx}{n^2}$, and there holds the recurrence relation

$$n\mu_{n,m+1}(x)=x\left[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right],m\geq 1. \qquad (2.1)$$

Corollary 2.1. For the functions $\mu_{\alpha,m}(x)$ we have (i) $\mu_{\alpha,m}(x)$ are polynomials in x of degree exactly m; (ii) there holds the order $\mu_{n,m}(x)=O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, for all $x\in[0,\infty)$.

Lemma 2.2. For $P_{n,a,b}^{s}(f;x)$ and $m\in N^0$ the following conditions hold:

$P_{n,a,b}^{s}(1;x)=1$;

$$P_{n,a,b}^{s}(t;x)=x+\frac{1}{n^s(n-(a+b)+\frac{ab}{n})};$$

$$P_{n,a,b}^{s}(t^2;x)=x^2+\frac{2x}{n^s(n-(a+b)+\frac{ab}{n})}+\frac{2x}{n^{2s}(n-(a+b)+\frac{ab}{n})}+\frac{2}{n^{2s}(n^2-n(a+b)+2ab+(a+b)^2-\frac{2ab(a+b)}{n}+\frac{a^2b^2}{n^2})}.$$

Therefore, by applying Korovkin theorem for a sequence of linear positive operators for algebraic function in the interval $[0,\infty)$, we have $P_{n,a,b}^{s}(f;x)\to f(x)$ as $n\to\infty$.[11]

We denote and define the m-th order moment for the sequence $P_{n,a,b}^{s}(f;x)$ by

$$T_{\alpha,m,s}(x)=P_{n,a,b}^{s}((t-x)^m;x)$$

$$=\frac{(n-a)(n-b)}{n}\sum_{k=0}^{\infty}e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty}e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left(\frac{t-x}{n^s}\right)dt.$$

Lemma 2.3. For the function $T_{n,m,s}(x)$, we have:

$T_{n,0,s}(x)=1$;

$$T_{n,1,s}(x)=\frac{1}{n^s(n-(a+b)+\frac{ab}{n})};$$

$$T_{n,2,s}(x)=\frac{2x}{n^{2s}(n-(a+b)+\frac{ab}{n})}+\frac{2}{n^{2s}(n^2-n(a+b)+2ab+(a+b)^2-\frac{2ab(a+b)}{n}+\frac{a^2b^2}{n^2})};$$

and we have the following recurrence relation

$$\left(n-(a+b)+\frac{ab}{n}\right)n^sT_{n,m+1,s}(x)=xT'_{n,m,s}(x)+\frac{2mx}{n^s}T_{\alpha,m-1,s}(x)+(m+1)T_{\alpha,m,s}(x). \qquad (2.3)$$

In addition, the function $T_{n,m,s}(x)$ is a polynomial in x of degree at most m, for every $x\in(0,\infty)$ and $m\in N^0=N\cup\{0\}$ then, $T_{n,m,s}(x)=O\left(n^{-\left[\frac{m+1}{2}\right]-ms}\right)$,where $\left[\frac{m+1}{2}+ms\right]$ most the integer part of $\frac{m+1}{2}+ms$

Proof: By direct computation, we have the values $T_{n,0,s}(x)$, $T_{n,1,s}(x)$ and $T_{n,2,s}(x)$ we prove (2.1). For $x=0$ it clearly holds. For $x\in(0,\infty)$, we have:

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Next,

$$T'_{\alpha,m,s}(x) = \frac{(n-a)(n-b)}{n}\sum_{k=0}^{\infty}\left(e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\right)'\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left(\frac{t-x}{n^s}\right)^m dt$$
$$-\frac{m(n-a)(n-b)}{n^{s+1}}\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left(\frac{t-x}{n^s}\right)^{m-1} dt.$$

Using the equation $x\left(e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\right)' = \left(k-(n-(a+b))x\right)\left(e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\right)$, we get

$$xT'_{n,m,s}(x) = \frac{(n-a)(n-b)}{n}\sum_{k=0}^{\infty}\left(e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\right)\left(k-(n-(a+b))x\right)$$
$$\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left(\frac{t-x}{n^s}\right)^m dt - \frac{mx}{n^s}\,T_{n,m-1,s}(x).$$

Let $\left(k-(n-(a+b))x\right) = \left(k-\left(n-(a+b)+\frac{ab}{n}\right)t\right)+\left(n-(a+b)+\frac{ab}{n}\right)(t-x)+\frac{abx}{n}$.

Since $t\left(e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\right)' = \left(k-\left(n-(a+b)+\frac{ab}{n}\right)t\right)\left(e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\right)$.

$$xT'_{n,m,s}(x) = \frac{(n-a)(n-b)}{n}\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty} t\left(e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\right)'\left(\frac{t-x}{n^s}\right)^m dt$$
$$+\left(n-(a+b)+\frac{ab}{n}\right)n^sT_{n,m+1,s}(x)-\frac{mx}{n^s}T_{\alpha,m-1,s}(x).$$

Since

$$\int_0^{\infty} t\left(e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\right)'\left(\frac{t-x}{n^s}\right)^m dt = -\frac{m}{\alpha^s}\int_0^{\infty} t\left(e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\right)\left(\frac{t-x}{n^s}\right)^{m-1} dt.$$

Let $t=(t-x)+x$, we get

$$xT'_{n,m,s}(x) = (m+1)T_{n,m,s}(x)+\left(n-(a+b)+\frac{ab}{n}\right)n^sT_{n,m+1,s}(x)-\frac{2mx}{n^s}T_{\alpha,m-1,s}(x).$$

Finally, we get the recurrence relation state above.

From which (2.1) is immediate.

From the values of $T_{n,0,s}(x)$, $T_{n,1,s}(x)$, using the induction on m and the recurrence relation above, we can easily prove that $T_{n,m,s}(x) = O\left(\alpha^{-\left[\frac{m+1}{2}\right]-ms}\right)$ for every $x \in [0,\infty)$. ■

Lemma 2.4. Let $\gamma,\delta \in R^+$, be any two positive real number and $[a,b] \subset (0,\infty)$ be any bounded interval. Then, for any $m > 0$, $\exists\, M$ depending on m only and α independent of M.

We have

$$\left\|\int_{|t-x|\geq\delta} W_{n,s}(t,x)e^{\gamma t}dt\right\|_{C[a,b]} = O(\alpha^{-m}).$$

where $\|.\|_{C[a,b]}$ means the sup norm in the space $C[a,b]$.

Proof: By using of Schwartz inequality for summation-integration, the proof Lemma is easily follows [6, 8, 10 and 11]

Lemma 2.5. Let the coefficient in $P^s_{n,a,b}(t^m;x)$, $m \in N^0$ where

$$P^s_{n,a,b}(f;x) = \frac{(n-a)(n-b)}{n}\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!} f\left(x+\frac{t-x}{n^s}\right)dt.$$

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then, $P^s_{n,a,b}(t^m;x) = \frac{n^m}{(n-a)^m(n-b)^m}\sum_{i=0}^{m}\binom{n}{i}(n^s-1)^i x^i\left(\left(n\beta_n(x)\right)^m + \left(1+\frac{m(m+1)}{2}\right)\left(n\beta_n(x)\right)^{m-1} + o(n^{-2})\right)$

Proof:

$$P^s_{n,a,b}(t^m;x) = \frac{(n-a)(n-b)}{n}\sum_{i=0}^{m}\binom{n}{i}(n^s-1)^i x^i\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}t^m dt$$

Since $\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}t^m dt = \frac{(k+m)!\,n^{m+1}}{k!(n-a)^{m+1}(n-b)^{m+1}}$

By Lemma (2.2) and Lemma (2.1) we get

$$P^s_{n,a,b}(t^m;x) = \frac{n^m}{(n-a)^m(n-b)^m}\sum_{i=0}^{m}\binom{n}{i}(n^s-1)^i x^i\left(\left(n\beta_n(x)\right)^m + \left(\boldsymbol{n}\beta_n(x)\right)^{m-1} + L.p.o(x)\right)$$
$$+\frac{m(m+1)\,n^m}{2(n-a)^m(n-b)^m}\sum_{i=0}^{m}\binom{n}{i}(n^s-1)^i x^i\left(\left(n\beta_n(x)\right)^{m-1} + \left(\boldsymbol{n}\beta_n(x)\right)^{m-2} + L.p.o(x)\right)$$

Therefore, the conditions hold Lemma.

3. Main Results

In this section, we study Voronovskaja type theorem for the sequence $P_{\alpha,\rho,s}(f;x,c,r)$ in simultaneous approximation.

Theorem 3.1.

Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If f''exists and is continuous at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n^{ms+1}\left(P^s_{n,a,b}(f;x) - f(x)\right) = f'(x) + xf''(x). \qquad (2.3)$$

Proof:

By using Taylor's expansion, we write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \psi(t,x)(t-x)^2, t \in [0,\infty)$$

where the function $\psi(t,x) \longrightarrow 0$ as $t \longrightarrow x$.

$$P^s_{n,a,b}(f(t);x) = P^s_{n,a,b}(f(x);x) + P^s_{n,a,b}(f'(x)(t-x);x) + P^s_{n,a,b}\left(\frac{f''(x)}{2}(t-x)^2;x\right)$$
$$+P^s_{n,a,b}\left(\psi(t,x)(t-x)^2;x\right)$$

$$\lim_{n\to\infty} n^{ms+1}\left(P^s_{n,a,b}(f;x) - f(x)T_{n,0,s}(x)\right) = f'(x)T_{n,1,s}(x) + \frac{f''(x)}{2}T_{n,2,s}(x)$$
$$+\lim_{n\to\infty} nP^s_{n,a,b}\left(\psi(t,x)(t-x)^2;x\right).$$

Now, to prove $\lim_{n\to\infty} nP^s_{n,a,b}\left(\psi(t,x)(t-x)^2;x\right) = 0$.

$$\left|nP^s_{n,a,b}\left(\psi(t,x)(t-x)^2;x\right)\right|$$
$$\leq \frac{n(n-a)(n-b)}{n}\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_0^{\infty} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left|\psi(t,x)(t-x)^2\right|dt$$
$$= (n-a)(n-b)\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_{|t-x|<\delta} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left|\psi(t,x)(t-x)^2\right|dt$$
$$+(n-a)(n-b)\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{\left(n\beta_n(x)\right)^k}{k!}\int_{|t-x|\geq\delta} e^{-n\beta_n(t)}\frac{\left(n\beta_n(t)\right)^k}{k!}\left|\psi(t,x)(t-x)^2\right|dt$$

$:= I_1 + I_2$.

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Since $\psi(t,x) \to 0$ as $t \to x$, hence for given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|\psi(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. For $|t-x| \geq \delta$, there exists a constant $M > 0$, such that $|\psi(t,x)(t-x)^2| \leq Me^{\gamma t}$ therefore.

$$I_1 = (n-a)(n-b)\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\int_{|t-x|<\delta} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}|\psi(t,x)(t-x)^2|dt$$

$$\leq \varepsilon(n-a)(n-b)\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\int_{|t-x|<\delta} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}|(t-x)^2|dt$$

$$= \alpha\varepsilon T_{\alpha,2}(x) = \varepsilon O(1)$$

Since ε arbitrary, it follows that $I_1 \to 0$ as $\alpha \to \infty$.

Now,

$$I_2 = (n-a)(n-b)\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\int_{|t-x|\geq\delta} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}|\psi(t,x)(t-x)^2|dt$$

$$\leq \sup_{x\in[a,b]}\left|(n-a)(n-b)\sum_{k=0}^{\infty} e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\int_{|t-x|\geq\delta} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}Me^{\gamma t}dt\right|$$

by using Cauchy Schwarz inequality, and Lemma (2.1), (2.3)we obtain

$$\leq M\alpha\left(T_{n,2i}(x)\right)^{\frac{1}{2}} = M\alpha(O(\alpha^{-i}))^{\frac{1}{2}}$$

$$= O(\alpha^{-s}),\ s > 0.$$

Hence, $I_2 \to 0$ as $n \to \infty$. ■

Theorem 3.2

Let $f \in C_\gamma[0,\infty)$ for some $\alpha > 0$ and $r \leq q \leq r+2$. If $f^{(q)}$ exists and is continuous on $(a-\tau, b+\tau) \subset (0,\infty), \tau > 0$, then for sufficiently large α,

$$\|P_{n,a,b}^{s\ (r)}(f;x) - f^{(r)}(x)\|_{C[a,b]}$$

$$\leq Z_1\alpha^{-1}\sum_{i=s}^{q}\|f^{(i)}\|_{C[a,b]} + Z_2\alpha^{-\frac{1}{2}}\omega_{f^{(q)}}\left(\alpha^{-\frac{1}{2}}\right) + O(\alpha^{-2})$$

where Z_1, Z_2 are constant independent of f and $\alpha, \omega_f(\delta)$ is the modulus of continuity of f on $(a-\tau, b+\tau)$ and $\|.\|_{C[a,b]}$ denotes the sup-norm on [a,b].

Proof:

By a finite Taylor expansion of f, we have

$$f(t) = \sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q\chi(t)$$

$$+g(t,x)(1-\chi(t))$$

where ξ lies between t, x and $\chi(t)$ is the characteristic function of the interval $(a-\tau, b+\tau)$ and

$$g(t,x) = f(t) - \sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

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Now,

$P_{n,a,b}^{s\ (r)}(f;x) - f^{(r)}(x)$

$$= \left(\sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} P_{n,a,b}^{s\ (r)}((t-x)^i;x) - f^{(r)}(x)\right)$$

$$+P_{n,a,b}^{s\ (r)}\left(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t);x\right)$$

$$+P_{n,a,b}^{s\ (r)}(g(t,x)(1-\chi(t));x)$$

$$:= I_1 + I_2 + I_3.$$

By using Lemma 2.1 (iv) and Lemma 3.2, we get

$$I_1 = \sum_{i=r}^{q} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r}\left[\frac{n^j}{(n-a)^j(n-b)^j} \sum_{i=0}^{j} \binom{j}{i} (n^s-1)^i x^i \left((n\beta_n(x))^j\right.\right.$$

$$\left.\left. + \left(1+\frac{j(j+1)}{2}\right)(\boldsymbol{n}\beta_n(x))^{j-1} + O(n^{-2})\right)\right] - f^{(s)}(x)$$

Consequently,

$\|I_1\|_{C[a,b]} \leq Z_1\alpha^{-1}\left(\sum_{i=r}^{q}\|f^{(i)}\|_{C[a,b]}\right) + O(\alpha^{-2})$, uniformly on $[a,b]$.

To estimate I_2, we proceed as follows:

$$|I_2| \leq P_{n,a,b}^{s\ (r)}\left(\frac{|f^{(q)}(\xi)-f^{(q)}(x)|}{q!}|t-x|^q\chi(t);x\right) \leq \frac{\omega_{f^{(q)}}(\delta;(a-\tau,b+\tau)}{q!}\ P_{n,a,b}^{s\ (r)}\left(\left(1+\frac{|t-x|}{\delta}\right)|t-x|^q;x\right)$$

$$\leq \frac{\omega_{f^{(q)}}(\delta;(a-\tau,b+\tau)}{q!}\frac{(n-a)(n-b)}{n}$$

$$\sum_{k=0}^{\infty} \frac{dr}{dx^r}\left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)\int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^q+\delta^{-1}|t-x|^{q+1})dt\,,\delta>0$$

Now, for $u = 0,1,2,\ldots$, using Schwartz inequality for integration, summation, and Lemma 2.3 we have:

$$\sum_{k=0}^{\infty}\left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)|k-(n-(a+b))x|^j\int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^u)dt$$

$$\leq \sum_{k=0}^{\infty}\left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)|k-(n-(a+b))x|^j$$

$$\times\left(\int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^{2u})dt\right)^{\frac{1}{2}}\left(\int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}dt\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=0}^{\infty}\left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)|k-(n-(a+b))x|^{2j}\right)^{\frac{1}{2}}$$

$$\times\left(\frac{n}{(n-a)(n-b)}\int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^{2u})dt\right)^{\frac{1}{2}}$$

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$= O\left(n^{\frac{j}{2}}\right) O\left(n^{-\frac{u}{2}}\right) = O\left(n^{\frac{j-u}{2}}\right)$ uniformly on $[a,b]$. (3.7)

Hence, by Lemma 2.4 and (3.7), we get

$$\sum_{k=0}^{\infty} \frac{dr}{dx^r}\left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)|k-(n-(a+b))x|^j \int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^u)dt$$

$$\leq \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{n^i|Q_{i,j,r}(x)|}{x^r}\left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)|k-(n-(a+b))x|^j \int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^u)dt$$

$$\leq \left(\sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \sup_{x\in[a,b]} \frac{|Q_{i,j,r}(x)|}{x^r}\right) \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)|k-(n-(a+b))x|^j$$

$$\times \int_0^{\infty} e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}(|t-x|^u)dt$$

$\leq C\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\, O\left(n^{\frac{j-u}{2}}\right) = O\left(n^{\frac{r-u}{2}}\right)$, uniformly on $[a,b]$ (3.8)

(where $\sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \sup_{x\in[a,b]} \frac{|Q_{i,j,r}(x)|}{x^r} := C(x)$ but fixed)

By choosing $\delta = n^{-\frac{1}{2}}$ and applying (3.8), $m > 0$ we are led to:

$$\|I_2\|_{C[a,b]} \leq \frac{\omega_{f^{(q)}}(n^{-\frac{1}{2}};(a-\tau,b+\tau)}{q!}\left(O\left(n^{\frac{s-q}{2}}\right) + n^{\frac{1}{2}}O\left(\alpha^{\frac{r-q-1}{2}}\right) + O(n^{-m}\right),$$

(for any $m > 0$)

$$\leq Z_2 n^{\frac{-(q-r)}{2}} \omega_{f^{(q)}}\left(n^{-\frac{1}{2}}\right).$$

Since $t \in [0,\infty) \setminus (a-\tau, b+\tau)$, we can choose $\delta > 0$in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$. Thus, by Lemma 2.4, we get

$$|I_3| \leq \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r}|k-(n-(a+b))x|^j n^i \left(e^{-n\alpha_n(x)}\frac{(n\beta_n(x))^k}{k!}\right)$$

$$\times \int_{|t-x|\geq\delta}\left(e^{-n\beta_n(t)}\frac{(n\beta_n(t))^k}{k!}\right)|g(t,x)|dt$$

For $|t-x| \geq \delta$, we can find a constant $C > 0$ such that $|g(t,x)| \leq Ce^{nt}$. Hence, using Schwarz inequality for integration and for summation, (2.4) it easily follows that $I_3 = O(n^{-u})$ for any $u > 0$. Combining the estimate of I_1, I_2, I_3, the required result is immediate. ■

4. Graphical Representation

This section deals with graphical representation on some exponential functions of the cases treated in Theorem by parameters s, n.

Example 1. Let $f(x) = \sin(7x)$, $s = 0$, $n = 100, 500, 1000$ and $a, b = 0$. For $x \in [0,\infty)$, we get the figure (4.1-4.3) respectively.

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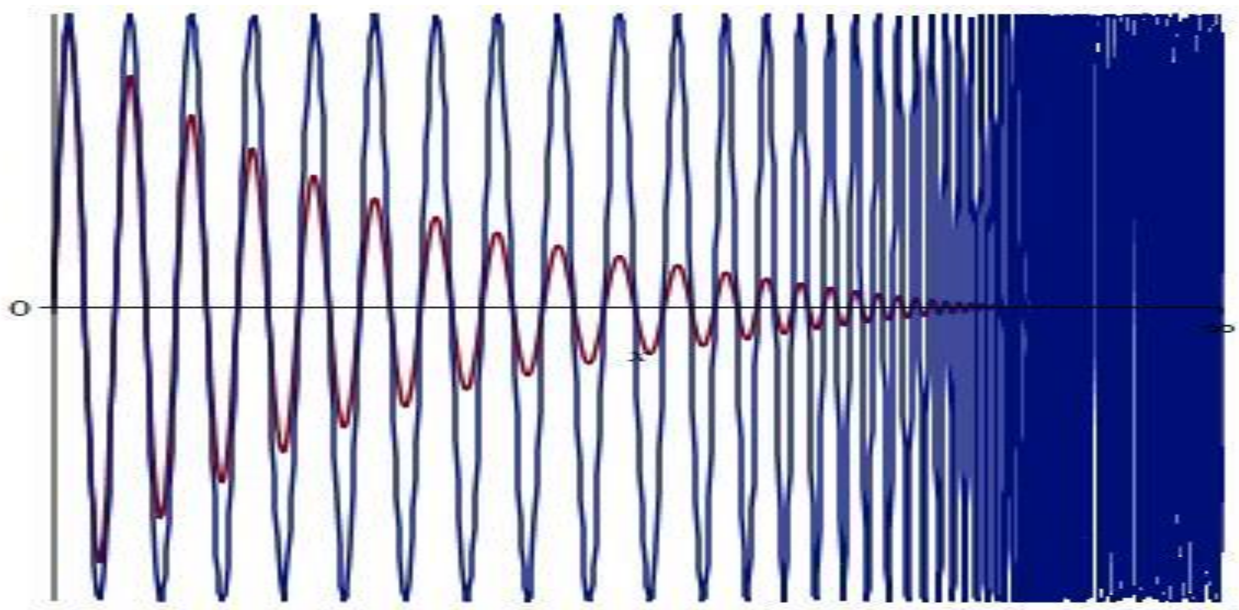

Figure (4.1)

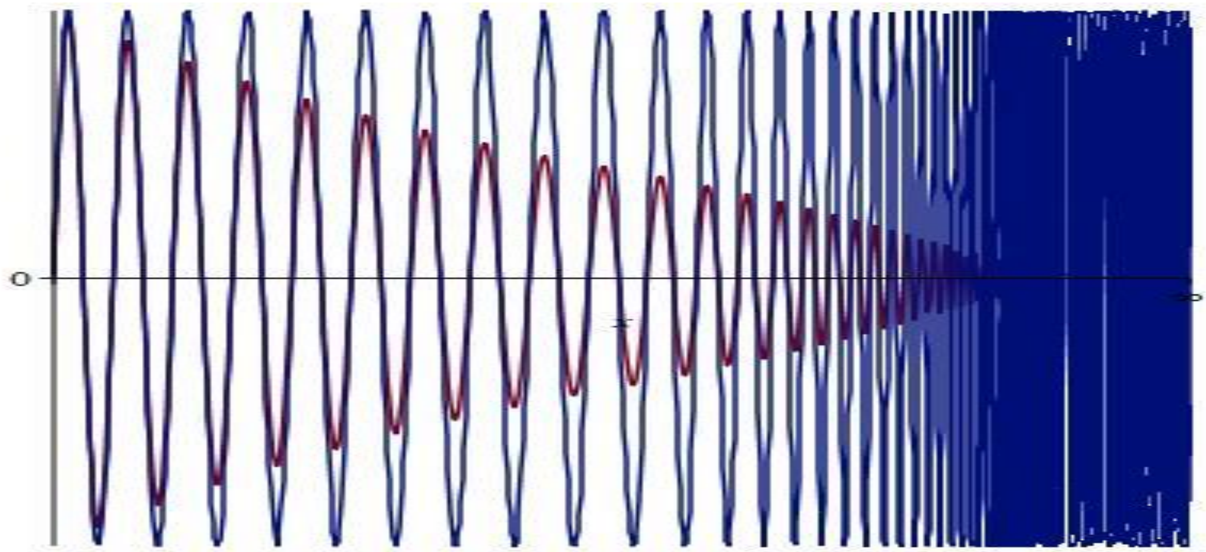

Figure (4.2)

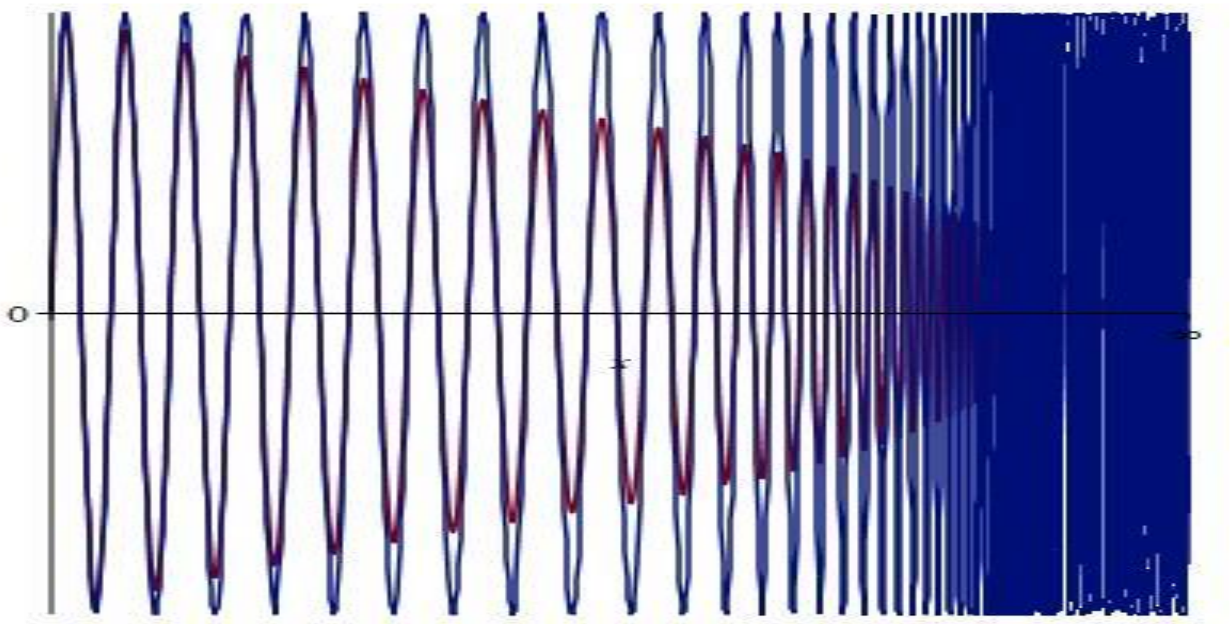

Figure (4.3)

Example 2. Let $f(x) = \sin(7x)$, $n = 30$, $s = 1,2,3$ and $a, b = 0$. For $x \in [0, \infty)$, we get the figure (4.4-4.5) respectively.

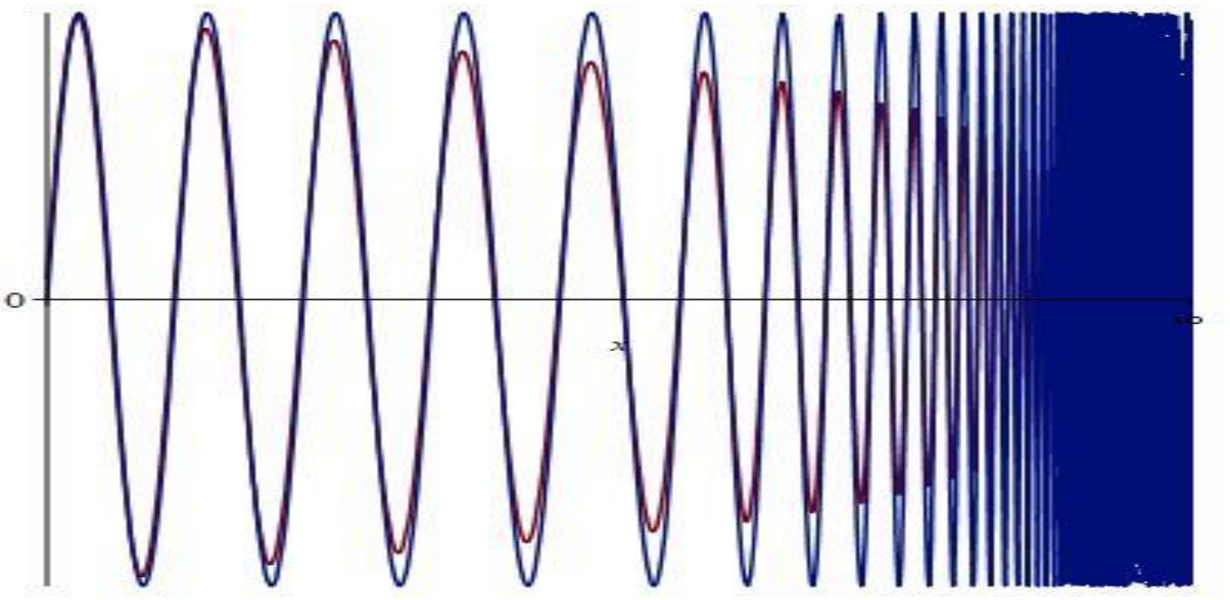

Figure (4.4)

Example 3. Let $f(x) = \sin(7x)$, $n = 100$, $s = 1$ and $a = 3, b = 1$. For $x \in [0, \infty)$, we get the figure (4-6) respectively.

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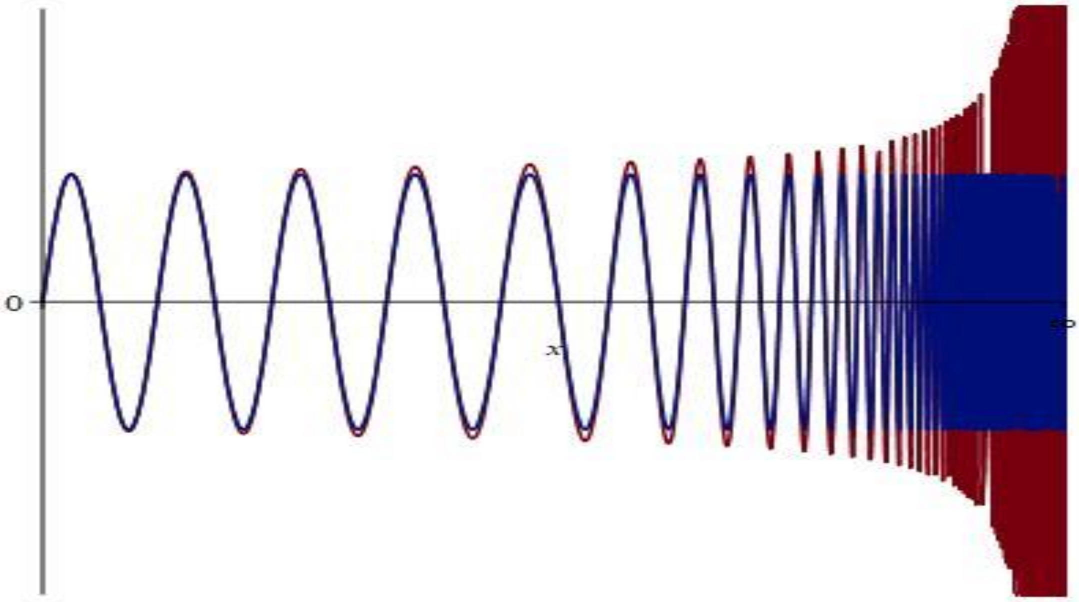

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Figure (4.5)

Discussion

The numerical results show parameter s by our sequence. It turns out that,

- The approximation increases whenever $s > 0$ increase.
- The approximation decreases whenever as the value $\frac{1}{2} < s < 0$.
- Put $s = 0$ we get the previous results.
- For $n = \infty$, We get a perfect approximation for graphics.

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