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HYDROMECHANICS

AERODYNAMICS

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APPLIED MATHEMATICS

STRESSES IN A SPHERICAL SHELL (DOME)

(Spannungen in Kugelschalen (Kuppeln))

by

Hans Reissner (Aachen)

Translated by Werner B. Hinterthan
Reviewed and annotated by
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ABSTRACT

The stress distributions without bending resistance for asymmetrically and symmetrically loaded spherical shells are determined. With the aid of these limited displacements, the bending and/or torsional moments are calculated.

Also, the stress distribution is found for the basic problem of a spherical shell subjected at one boundary to a uniformly distributed boundary force or to a uniformly distributed bending moment, respectively, without any surface forces.

STRESSES IN SHPERICAL SHELLS (Domes)

Two extreme, especially elementary cases appear in the theory of elasticity for thin-walled shells. In the first case, the shell is supported so that it cannot undergo any inextensional bending; also it is so thin that the bending moments can be neglected, and thus the stresses can be assumed to be uniformly distributed across the wall thickness. In the second case, the shell can undergo inextensional bending, i.e., the length of the median surface remains constant during bending. The first case is the more important one in stress calculations of shell structures. Schwedler, for instance, has calculated the case of symmetrically loaded spherical arches in this manner. On the other hand, to my knowledge, the case of asymmetrically loaded spherical arches has not been considered up to now, and even for symmetrical loading there are no experimental data to indicate when and why the bending stresses may be neglected and what influence the boundary conditions have on the stress distribution.

Because we exclude the bending stresses, the problem is particularly simple in that the resultants of the tensile stresses and of the shear stresses may be obtained from the equilibrium conditions alone without considering the deformations. In this way, the problem becomes quite similar to that of statically determinate systems.

And so we shall first determine the stress distributions without bending resistance for both cases of symmetrically and asymmetrically loaded spherical shells. We shall then calculate the bending and/or torsional moments with the aid of these limited displacements, using them as a first approximation.

However, arbitrary boundary conditions would not be fully satisfied by this method. Therefore, we shall finally deal with the case of a shell whose surface is not loaded, and which is under the influence of edge forces. This case serves as a supplementary problem leading to a general solution.

THE ASYMMETRICALLY LOADED SPHERICAL SHELL WITHOUT BENDING RESISTANCE

If, according to Figure 1, we assume that the bending moments are initially neglected, the equations of equilibrium for an element which is bounded by two adjacent meridians and a pair of parallel latitude or circumferential circles are:[1]

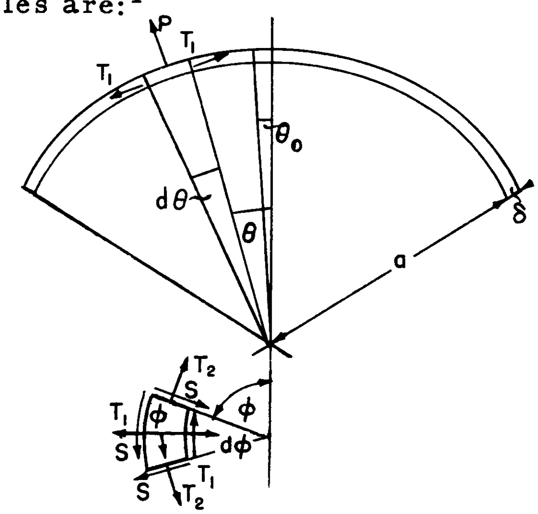

Figure 1

1 References are listed on page 22.

$$\left.\begin{aligned} &\frac{\partial T_1}{\partial \theta} + \cot\theta\,(T_1 - T_2) + \frac{1}{\sin\theta}\frac{\partial S}{\partial \phi} + a\Theta = 0 \\ &\frac{\partial S}{\partial \theta} + (2\cot\theta)\,S + \frac{1}{\sin\theta}\frac{\partial T_2}{\partial \phi} + a\Phi = 0 \\ &\qquad\qquad -(T_1 + T_2) + aP = 0 \end{aligned}\right\} \quad \ldots\ldots\ [1]$$

where T_1 and T_2 denote the normal stress resultants tangential to the middle surface of a spherical shell along the meridian and perpendicular thereto, respectively,

S denotes the corresponding shear stress resultants tangential to the middle surface,

θ is the complement of the latitude angle,

a is the radius to the middle surface,

ϕ is the meridian angle, and

Θ, Φ, and P denote the surface loading per unit area of the middle surface.

These partial differential equations allow for a simple integration if the external forces Θ, Φ, and P are considered as trigonometric series comprised of terms whose arguments are whole multiples of ϕ. Suppose we put $\Theta = \tau \sin n\phi$, $\Phi = \psi \cos n\phi$, $P = \rho \sin n\phi$, $T_1 = t_1 \sin n\phi$, $T_2 = t_2 \sin n\phi$, $S = s \cos n\phi$, where $\tau, \psi, \rho, t_1, t_2$, s are still functions of θ, and further for brevity we put $\frac{dt}{d\theta} \equiv t'$, etc., we then obtain total differential equations, with θ as the independent variable, in the form

$$\left.\begin{aligned} t_1' + \cot\theta\,(t_1 - t_2) - \frac{n}{\sin\theta}\,s + a\tau = 0 \\ s' + 2\cot\theta s + \frac{n}{\sin\theta}\,t_2 + a\psi = 0 \\ -(t_1 + t_2) + a\rho = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots [2]$$

The elimination of t_2 by means of the last of the above equations yields:

$$\left.\begin{aligned} t_1' + 2\cos\theta\, t_1 - \frac{n}{\sin\theta}\,s = a(\rho\cot\theta - \tau) \\ s' + 2\cot\theta\, s - \frac{n}{\sin\theta}\,t_1 = -a\left(\frac{\rho n}{\sin\theta} + \psi\right) \end{aligned}\right\} \quad \ldots\ldots\ldots [3]$$

The symmetrical form in which s and t_1 appear in [3] allows us to introduce new variables $t_1 + s \equiv u$ and $t_1 - s \equiv v$, and to then write [3] in the form

$$u' + u\left(2\cot\theta - \frac{n}{\sin\theta}\right) = a\left[\rho\left(\cot\theta - \frac{n}{\sin\theta}\right) - \tau - \psi\right]$$

$$v' + v\left(2\cot\theta + \frac{n}{\sin\theta}\right) = a\left[\rho\left(\cot\theta + \frac{n}{\sin\theta}\right) - \tau + \psi\right]$$

As is well known, the complete integral of each of these linear differential equations of first order with variable coefficients $k = 2\cot\theta \mp \frac{n}{\sin\theta}$ and disturbance functions

$$\ell = a\left[\rho\left(\cot\theta \mp \frac{n}{\sin\theta}\right) - \tau \mp \psi\right]$$

reads as follows:

$$u \text{ or } v \text{ (respectively)} = \left(c + \int \ell e^{\int k d\theta}\, d\theta\right) e^{-\int k d\theta}.$$

In this case, $e^{-\int k d\theta} = \tan^n \frac{\theta}{2} \sin^{-2}\theta$, and $\cot^n \frac{\theta}{2} \sin^{-2}\theta$, respectively, and finally we obtain: $t_1 = \frac{u+v}{2}$; $s = \frac{u-v}{2}$

Thus the general solution for asymmetrical loading is given; furthermore, the solution is of value only if one makes definite assumptions concerning the surface forces τ, ψ, and ρ. For instance, if one chooses $n = 1$, then $\tau = \psi = 0$, $\rho = \rho_o \sin\theta$ (corresponding to an approximate wind load upon the spherical shell with pressure on the windward side and negative (suction) pressure on the lee side) and the results will be:

$$u = \left[C_1 + a\rho_o\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\right]\frac{1}{\sin\theta\,(1+\cos\theta)}$$

$$v = \left[C_2 - a\rho_o\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\right]\frac{1}{\sin\theta\,(1-\cos\theta)}$$

The integration constants C_1 and C_2 in this case have to be determined from the boundary conditions at one boundary of the spherical zone, and the solution would seem to lead to infinitely large stresses T_1, T_2, and S if θ is assumed equal to zero, i.e., the sphere is closed at the apex. However, this is not the case; on the contrary, it has to be considered that with a closed sphere, the boundary conditions disappear altogether, and they have to be replaced by the requirement that the constants C_1 and C_2 are such that there will be no infinitely large stresses at the apex. It is easily seen that this can be satisfied by setting $C_1 = -\frac{2}{3}a\rho_o$ and $C_2 = +\frac{2}{3}a\rho_o$; the stresses are then given by

$$\left.\begin{aligned} T_1 &= t_1\sin\phi = a\rho_o\frac{\cos\theta}{\sin^3\theta}\left(\frac{2}{3} - \cos\theta + \frac{\cos^3\theta}{3}\right) \\ S &= s\cos\phi = -a\rho_o\frac{1}{\sin^3\theta}\left(\frac{2}{3} - \cos\theta + \frac{\cos^3\theta}{3}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots [4]$$

Through series expansions of the cosine and sine functions, it can be shown that the stresses T_1 and S even become zero for $\theta = 0$.

A small table will show the distribution of the stresses along an arbritrarily chosen meridian calculated according to Equation [4]:

θ	$\frac{t_1}{a \rho_o}$	$\frac{s}{a \rho_o}$
0°	0	0
30°	0.13	-0.14
45°	0.156	-0.221
60°	0.161	-0.322
80°	0.0902	-0.518
90°	0	-0.667

This distribution is rather plausible. The horizontal components of the wind pressure cannot be taken up at the boundary $\theta = 90$ degrees (hemispherical shell) by the normal stresses which are perpendicular at that point. On the contrary, they are equilibrated by shear stresses tangential to the boundary circle since bending moments and transverse shear stresses are assumed not to occur. Therefore, if we assume $\theta = 90$ degrees, T_1 must become zero and S a maximum.

Thus we have solved the problem for a sufficiently thin-walled sphere, whose supporting boundary, perpendicular to the tangential plane, can adjust freely in the case of asymmetrical loading. A later investigation will establish the influence of wall thickness on stress distribution when the boundary is fixed. Most likely such an influence, as is shown below for symmetrical loading, will be noticeable only in the vicinity of the boundary itself.

THE SYMMETRICALLY LOADED SPHERICAL SHELL WITH NO BENDING RESISTANCE

By setting n equal to zero, the case of symmetrical loading can be expressed as the constant terms in the trigonometric series progressing as multiples of n. However, it will be more straight-forward to derive the solutions directly from the conditions of equilibrium which, for symmetrical loading, are given by the following:

$$\left.\begin{aligned} &T_1' + \cot\theta\,(T_1 - T_2) + a\Theta = 0 \\ &S' + (2\cot\theta)\,S + a\Phi = 0 \\ &-(T_1 + T_2) + aP = 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots [1a]$$

due to the disappearance of terms differentiated with respect to ϕ. Thus the equations break up into; (a) two for the normal stress resultants T_1 and T_2 and, (b) one independent one for the shear stress resultant S. If we substitute T_2 from the third into the first of Equations [1a] we obtain for T_1 a differential equation which is entirely similar to that for S, namely:

$$T_1' + (2\cot\theta)T_1 = a(P\cot\theta - \Theta) \qquad \ldots\ldots\ldots\ldots\ldots [1b]$$

The integrals of these linear differential equations of the first order with disturbance functions are:

$$\left.\begin{aligned} T_1 &= \frac{a}{\sin^2\theta}\left[C_1 + \int\left[P\,\frac{\sin 2\theta}{2} - \Theta\sin^2\theta\right] d\theta\right] \\ S &= \frac{a}{\sin^2\theta}\left[C_2 - \int \Phi\sin 2\theta\, d\theta\right] \end{aligned}\right\} \ldots\ldots [4a]$$

If the spherical shell is bounded by a pair of parallel latitude circles, the values of T_1 and S at one of the boundaries may be arbitrarily prescribed, and those at the other boundary are determined as a consequence of the equilibrium condition, as we can see by the existence of one integration constant in each of the expressions for T_1 and S. However, if the spherical shell is closed at the apex, the boundary values can no longer be prescribed arbitrarily; therefore it appears necessary to determine the integration constants from the condition that the stresses at the apex must not become infinite. It seems in fact, and also it is sometimes stated in the literature, that the stresses T_1 and S become infinite at the apex. But this is definitely not the case; on the contrary, C_1 and C_2 are to be determined in such a way that T_1 and S remain finite.

A number of important cases can easily be solved directly from the integrals of Equation [4a], for instance: constant excess pressure, hydrostatic loading, centrifugal loading of a shell which rotates around the axis of symmetry, snow loading, dead weight, evenly distributed forces as well as boundary torsion forces, etc. The case of loading of a shell by its own weight and with constant wall thickness may be chosen as an example. For this case we put:

$$\Phi = 0 \; ; \; P = -\gamma \delta \cos \theta \; ; \; \Theta = +\gamma \delta \sin \theta$$

where δ is the wall thickness and γ the specific weight of the material. It then follows from [4a]:

$$S = 0 \; ; \; T_1 = \frac{a}{\sin^2 \theta} \left[C_1 + \gamma \delta \cos \theta \right]$$

Assuming at $\theta = \theta_o$, i. e., that at an inner circumferential latitude circle (end ring), T_1 is equal to zero which means that no boundary force is applied, then C_1 becomes equal to $-\gamma\delta\cos\theta_o$ and therefore

$$T_1 = \frac{a\gamma\delta}{\sin^2\theta}\left[\cos\theta - \cos\theta_o\right] \quad\text{[4b]}$$

This solution is also valid for the shell closed at the apex, i.e., $\theta_o = 0$, at which point it leads to finite values for T_1 and T_2, namely

$$T_1 = T_2 = -\frac{a\gamma\delta}{2}.$$

At other points the tensile stress along circumferential latitude circles becomes:

$$T_2 = aP - T_1 = -a\gamma\delta\left[\cos\theta + \frac{\cos\theta - \cos\theta_o}{\sin^2\theta}\right] \quad\text{[4b]}$$

THE SYMMETRICALLY LOADED SPHERICAL SHELL WITH BENDING RESISTANCE AND MOVABLE SUPPORTING BOUNDARY

The question now is to what extent do the solutions already found require correction if we want to consider the effect of wall thickness δ, and the bending and the torsional moments as well as the shear stresses perpendicular to the middle surface. To answer this, it is obviously necessary to find additional equilibrium conditions by consideration of these forces. For the asymmetrical case, according to Figure 2, we then have two bending moments G_1, G_2; one torsional moment H; and two shear forces N_1, N_2; that is, five unknown forces, but only two new moment equilibrium equations. Thus the problem loses the character of <u>static determinateness</u>, and it becomes necessary to consider the <u>stress</u>

deformation conditions. The bending moments G_1 and G_2 are assumed positive if they converge towards one another; the shear force N_1 is positive if it is directed outward on the side of the larger θ values.

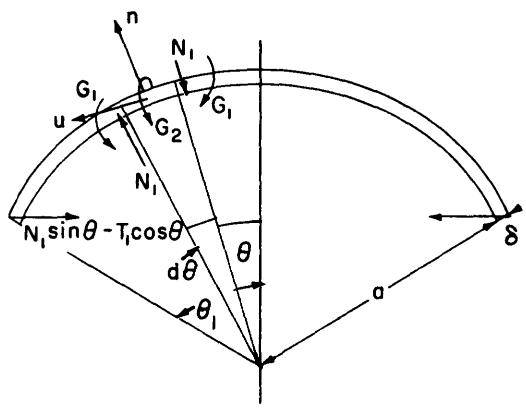

Figure 2

For the symmetrical case, which is the only one to be considered here, the torsional moment H and one of the shear forces N_2 are eliminated, but so is one of the equilibrium equations. It then follows that:

$$T_1' + \cot\theta\left(T_1 - T_2\right) + N_1 + a\Theta = 0$$

$$N_1' + (\cot\theta)\, N_1 - \left(T_1 + T_2\right) + aP = 0 \qquad \ldots\ldots\ldots [5]$$

$$G_1' + \cot\theta\left(G_1 - G_2\right) - aN_1 = 0$$

If ε_1 and ε_2 denote the strain of the longitudinal elements of the middle surface in the meridional and circumferential directions, respectively, κ_1 and κ_2 denote the corresponding curvature variations, and if D is set equal to

$$\frac{E\,\delta^3}{12} \cdot \frac{1}{1 - \sigma^2}$$

(where E represents the modulus of elasticity and σ the Poisson ratio), then the bending moments may be expressed by the changes in curvature, and the longitudinal forces by the strains in the following way:[2]

$$\left.\begin{aligned} T_1 &= \frac{12D}{\delta^2}\left(\varepsilon_1 + \sigma\varepsilon_2\right);\ G_1 = -D\left(\kappa_1 + \sigma\kappa_2\right) \\ T_2 &= \frac{12D}{\delta^2}\left(\varepsilon_2 + \sigma\varepsilon_1\right);\ G_2 = -D\left(\kappa_2 + \sigma\kappa_1\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots [6]$$

if the wall thickness is sufficiently thin and the deformations are small.

Thus the changes of curvature κ_1 and κ_2 are positive in the diverging direction (aiming away from one another). On the other hand, the deformation components ε_1, ε_2, κ_1, and κ_2 can be expressed in the following form in terms of the displacements u along the meridian and w along the radius, and their derivatives with respect to θ, i.e., u', w', and w'':

$$\left.\begin{aligned} a\,\varepsilon_1 &= u' + w\ ;\ a^2\kappa_1 = w'' - u' \\ a\,\varepsilon_2 &= u\cot\theta + w\ ;\ a^2\kappa_2 = \cot\theta\,(w' - u) \end{aligned}\right\} \ldots\ldots\ldots\ldots [7]$$

In this way we have as many equations as we have unknowns.

These equations are not to be integrated immediately; instead, a successive approximation is attempted utilizing the solution of the preceeding section. In doing so, we proceed to calculate the displacements u and w from the values of T_1 and T_2 derived by neglecting the bending moments and to calculate from these displacements the corresponding bending moments G_1 and G_2. These values of G_1 and G_2 are then considered as external forces and are employed for the calculation of corrections to T_1 and T_2. The procedure is justified if these corrections are small and become increasingly smaller with each further iteration.

Hence, if we first calculate the displacements u and w from the simple differential Equations [7], we obtain:

$$u = a \sin\theta \left\{ \int \frac{\varepsilon_1 - \varepsilon_2}{\sin\theta} \, d\theta + c \right\} \; ; \; w = a \left\{ \varepsilon_2 - \cos\theta \left(\int \frac{\varepsilon_1 - \varepsilon_2}{\sin\theta} \, d\theta + c \right) \right\} \quad [8]$$

If we insert these values into the right-hand expressions of Equations [7], we obtain the following values for the curvature variations κ_1 and κ_2:

$$\left. \begin{aligned} a\kappa_1 &= \varepsilon_2'' - \left(\varepsilon_1' - \varepsilon_2'\right)\cot\theta + \left(\varepsilon_1 - \varepsilon_2\right)\frac{1}{\sin^2\theta} \\ a\kappa_2 &= \varepsilon_2' \cot\theta - \left(\varepsilon_1 - \varepsilon_2\right)\cot^2\theta \end{aligned} \right\} \ldots\ldots\ldots [9]$$

Now, ε_1, ε_2, and their derivatives can be obtained from Equations [6]. It follows, for the above case of a spherical shell under the action of its own weight and closed at the apex that with the values of Equations [4b] for T_1 and T_2, we obtain for the bending moments G_1 and G_2 the extremely simple result:

$$G_1 = G_2 = -\gamma \frac{\delta^3}{12} \frac{(1 + \sigma)(2 + \sigma)}{1 - \sigma^2} \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots [10]$$

The bending moments were neglected in the calculation of the longitudinal forces T_1 and T_2, and it can now be determined how justified this approximation was. The best criterion is probably given by the ratio of the bending moment to the longitudinal force, which indicates the amount of eccentricity between the line of action of the resultant force and the middle surface of the spherical shell.

It follows:

$$\frac{G_1}{T_1} = \frac{\delta^2}{6a} \frac{(1+\sigma)(2+\sigma)}{1-\sigma^2} \cos\theta \cos^2\frac{\theta}{2}$$

If we assume the Poisson ratio $\sigma = \frac{1}{4}$ and take for $\cos\theta \cos^2\frac{\theta}{2}$ its largest value corresponding to that at the apex, the largest eccentricity of the pressure resultant will be

$$\frac{G_1}{T_1} = \frac{\delta^2}{2a}$$

For instance, for a wall thickness to radius ratio of $\frac{\delta}{a} = \frac{1}{40}$, $\frac{G_1}{T_1}$ becomes equal to $\frac{\delta}{80}$, i.e., an eccentricity which in practice could not be represented graphically at all. The shearing stress N_1 perpendicular to the middle surface, which influences a circumferential element, becomes according to the third of the equilibrium Equations [5]:

$$N_1 = \frac{G_1'}{a} = \gamma \frac{\delta^3}{12a} \frac{(1+\sigma)(2+\sigma)}{1-\sigma^2} \sin\theta$$

This value of N_1 can now be used as a correction and yields for T_1, according to Equations [1b] and [5], the differential equation:

$$T_1' + (2\cot\theta)T_1 = -a\Theta + (\cot\theta)aP + (\cot\theta)N_1' + N_1\left(\cot^2\theta - 1\right) \ldots\ldots [11]$$

If we set T_1 equal to $T_1^\circ + t$ and consider T_1° to be the solution of the differential equations [1a] , in this case [4b], and if we consider t to be the correction and subtract Equation [1b] from Equation [11] then we obtain for the correction t the differential equation:

$$t_1' + (2 \cot \theta) t_1 = (\cot \theta) N_1' + \left(\cot^2 \theta - 1\right) N_1$$

$$= m \left(\frac{\cos^2 \theta}{\sin \theta} + \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta} \right)$$

$$= m \frac{2 \cos^2 \theta - \sin^2 \theta}{\sin \theta}$$

if, for brevity, we put:

$$\frac{\gamma \delta^3}{12a} \frac{(1 + \sigma)(2 + \sigma)}{1 - \sigma} \equiv m$$

The correction t_1 is obtained from this by integration:

$$t_1 = m \cos \theta + \frac{c}{\sin^2 \theta}$$

If this correction for $\theta = \theta_o$ is also to disappear, then c must be equal to $- m \cos \theta_o \sin^2 \theta_o$

Hence,

$$c = 0 \text{ for } \theta_o = 0$$

It then follows from the second of Equations [5] that

$$t_1 = t_2 = m \cos \theta$$

The relative values of the corrections are obtained as:

$$\frac{t_1}{T_1^o} = - \frac{\delta^2}{2a^2} \cos \theta \cos^2 \frac{\theta}{2} \quad ; \quad \frac{t_2}{T_2^o} = - \frac{\delta^2}{4a^2} \frac{1}{1 + \cfrac{1}{2 \cos \theta \cos^2 \frac{\theta}{2}}}$$

The largest relative values of the corrections are obtained for $\theta = 0$ namely,

$$\frac{t_1}{T_1^o} = -\frac{\delta^2}{2a^2} \qquad ; \qquad \frac{t^2}{T_2^o} = -\frac{\delta^2}{6a^2}$$

Thus for $\frac{\delta}{a} = \frac{1}{40}$, for instance, we obtain the extremely small values $-\frac{1}{3200}$ and $-\frac{1}{9600}$.

We could pursue this method of successive approximations to even greater accuracy, but we find that at least for the case of dead weight loading calculated in detail here, the accuracy obtained already satisfies all technical requirements. Thus we conclude the following:

> For a spherical shell under the action of its own weight with a supporting boundary which is free to move in a normal direction, the stress condition may be determined with great accuracy as a pure longitudinal stress without taking into consideration the bending moments. In other words, we obtain a pure longitudinal stress condition with a concentric pressure area, and we obtain it solely from the equilibrium conditions as a statically defined system.

Similarly, it would be necessary to develop a correction procedure also for the asymmetrical cases treated above in a purely statical manner and decide to what extent the concentrical pressure area is admissible.

THE BENDING-RESISTING SPHERICAL SHELL WITH ARBITRARY AXISYMMETRIC BOUNDARY FORCES

All cases considered so far had in common that at the boundary no shear forces N_1 normal to the boundary and no bending moments G_1 could exist owing to the support line which is free to deflect without resistance. If the boundary is firmly supported or even clamped, the previous solutions do not satisfy the boundary conditions. To satisfy these a further correction must be added, i.e., transverse forces N_1 and, in the case of the clamped boundary also bending moments G_1, of such magnitude that the displacements u and w, or the tangential rotations $\frac{u}{a} - \frac{dw}{d\theta}$, respectively, which result from previous solutions, are cancelled. It is obvious that to satisfy the later consideration of a supported boundary for symmetrical cases, the following basic problem has to be solved:

> A spherical shell is subjected at one boundary to a uniformly distributed boundary force or to a uniformly distributed bending moment, respectively, without any surface forces. The stress distribution is to be found.

Lateral forces and bending moments are obviously the important components here, and they must not be determined as correction terms. Instead we must start from the complete system of Equations [5], [6], and [7] and try to integrate them.

In the absence of surface forces, Equations [5] simplify to:

$$\left.\begin{aligned} T_1' + \left(T_1 - T_2\right)\cot\theta &= -N_1 \\ T_1 + T_2 &= N_1' + (\cot\theta)N_1 \\ G_1' + \left(G_1 - G_2\right)\cot\theta &= aN_1 \end{aligned}\right\} \qquad \text{[5a]}$$

These equations simplify even further if the shell is either closed at the apex or if one boundary line is loaded normal to the axis of symmetry. In that case also the other boundary line and, in fact, each circumferential latitude circle has only resultant stresses perpendicular to the axis of symmetry; this immediately suggests the following as an integral of the equilibrium equations:

$$N_1 \cot \theta - T_1 \sin \theta = 0$$

The three equilibrium equations then become:

$$\left.\begin{aligned} N_1 &= T_1 \tan \theta \\ T_2 &= \frac{d}{d\theta}\left(T_1 \tan \theta\right) \\ G_1' + \left(G_1 - G_2\right)\cot \theta &= aN_1 \end{aligned}\right\} \qquad \text{[5b]}$$

The governing differential equation in terms of a single dependent variable shall now be established on the basis of the observation that on one hand, the bending moments G_1 and G_2 may be expressed in terms of the change of curvature κ_2 alone, whereas on the other hand, T_1 can also be expressed in terms of κ_2, in the following manner:

From Equation [7]

$$\kappa_1 = \frac{d}{d\theta}\left(\kappa_2 \tan \theta\right)$$

and therefore

$$G_1 = -D\left(\frac{d\left(\kappa_2 \tan \theta\right)}{d\theta} + \sigma \kappa_2\right); \; G_2 = -D\left(\frac{d\left(\kappa_2 \tan \theta\right)}{d\theta}\sigma + \kappa_2\right)$$

$$G_1 - G_2 = -D(1 - \sigma)\left(\frac{d\left(\kappa_2 \tan \theta\right)}{d\theta} - \kappa_2\right)$$

The third of Equations [5b] then becomes

$$a N_1 = - D \left[\frac{d^2}{d\theta^2} \left(\kappa_2 \tan \theta \right) + \sigma \frac{d \kappa_2}{d\theta} + (1 - \sigma) \cot \theta \frac{d \left(\kappa_2 \tan \theta \right)}{d\theta} - \right.$$

$$\left. - (1 - \sigma) \kappa_2 \cot \theta \right] \quad \ldots\ldots\ldots\ldots [12]$$

On the other hand, we obtain according to Equations [9]

$$a \left(\kappa_2 \tan \theta \right) = \varepsilon_1' - \left(\varepsilon_1 - \varepsilon_2 \right) \cot \theta$$

and by Equations [6]

$$\varepsilon_2' = \frac{\delta^2}{12 D \left(1 - \sigma^2 \right)} \left(T_2' - T_1' \sigma \right)$$

$$\varepsilon_1 - \varepsilon_2 = \frac{\delta^2}{12 D (1 - \sigma)} \left(T_1 - T_2 \right)$$

If according to Equations [5b], we express T_2 in terms of T_1, and T_1 in terms of N_1 we obtain:

$$a \left(\kappa_2 \tan \theta \right) = \frac{1}{E \delta} \left[\frac{d^2}{d\theta^2} \left(N_1 \right) - \sigma \frac{d \left(N_1 \cot \theta \right)}{d\theta} + (1 + \sigma) (\cot \theta) \frac{dN_1}{d\theta} - \right.$$

$$\left. - (1 + \sigma) \left(\cot^2 \theta \right) N_1 \right] \quad \ldots\ldots\ldots [13]$$

Hence, N_1 may be expressed in a similar manner in terms of κ_2, and vice versa.

Equations [12] and [13] can be given in a still more symmetrical and clearer form if the following notations are introduced:

$$\lambda^4 \equiv -12\left(1-\sigma^2\right)\frac{a^2}{\delta^2}; \quad \frac{E\delta a}{\lambda^2}\left(\kappa_2 \tan\theta\right) \equiv K; \quad N_1 \equiv N$$

Equations [12] and [13] then take the following form:

$$\lambda^2 N = K'' + K' \cot\theta - K\left(\cot^2\theta + \sigma\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots [12a]$$

$$\lambda^2 K = N'' + N' \cot\theta - N\left(\cot^2\theta - \sigma\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots [13a]$$

The next step is to introduce the value of K from Equation [13a] into Equation [12a], or the other way around, N from [12a] into [13a] and thus obtain a fourth order linear differential equation with variable coefficients of rather unconventional form. The form of this equation would at first suggest integration with the aid of series-type solutions which progress according to powers of θ or $\sin\theta$. However, the convergence of these series will be so slow because of the magnitude of the coefficient λ, that it is impractical to obtain numerical results.

The reason for this phenomenon is probably based on the fact that we are dealing here with the determination of a stress condition which is considerably influenced by the lateral forces and the bending moments only in the vicinity of the boundary, and which effects must decrease rapidly towards the apex, similarly as in the case of the well-known stress condition for a cylindrical shell that is loaded by transverse boundary forces.[3]

The large value of the "parameter" λ, which prevents an elementary power-series solution, now leads to the so called asympototic integration with the aid of a series that progresses by descending powers of λ; the origin of this series is due to O. Blumenthal.[4]

The large factor λ^2 on the left-hand side of Equations [12a] and [13a] shows that N and K must be functions of θ whose derivatives must be large compared to N and K themselves. This observation makes it advisable to start with $e^{\lambda\theta}$ as a factor. Furthermore, Blumenthal proves that it is appropriate for the desired series development, useful for large values of λ, to eliminate the first derivatives on the right-hand side of Equations [12a] and [13a]. This is accomplished by the introduction of the new variables $K\sqrt{\sin\theta}$ and $N\sqrt{\sin\theta}$. It thus follows that for N and K themselves, the following expressions must be set up:

$$N = \left(\nu e^{\lambda\theta} \sin^{-1/2}\theta\right)\left(1 + \frac{f_1}{\lambda} + \frac{f_2}{\lambda^2} + \ldots\right)$$

$$K = \left(\kappa e^{\lambda\theta} \sin^{-1/2}\theta\right)\left(1 + \frac{\phi_1}{\lambda} + \frac{\phi_2}{\lambda^2} + \ldots\right)$$

where f_1, ϕ_1, f_2, ϕ_2 ... etc., represent functions of θ, and ν and κ represent constant values. If we introduce these values for N and K into Equations [12a] and [13a] and arrange the terms according to descending powers of λ, we obtain, by equating each coefficient to zero, the following relations which may be shown here only up to the values of f_1 and ϕ_1:

$$\nu = \kappa ; f_1 = \phi_1 ; \phi_2 - f_2 = 2\sigma$$

$$\frac{d}{d\theta}\left(f_1 \sin^{-1/2}\theta\right) = -\frac{1}{4}\left(1 + \frac{3}{2}\cos\theta \sin^{-2}\theta\right)\left(1 + \cot\theta\right) + \left(\cot^2\theta\right)\left(\sin^{-1/2}\theta\right)$$

.... etc.

In this manner f_1, f_2, ϕ_1, ϕ_2, etc., will at first be determined to within one integration constant each. The integration constants appearing in f_1, f_2, ϕ_1, ϕ_2 ... can be chosen arbitrarly; as a matter of fact, if these constants are chosen differently, the asymptotic integrals differ only by constant factors as indicated by Blumenthal. Thus, corresponding to the four (complex) values of λ we obtain four asymptotic particular integrals independent of one another which can be made real by appropriate combination in the usual manner. Of these integrals, two become small near the apex; the other two assume large values there. Since the latter are excluded from the present problem, the corresponding integration constants ν and κ must disappear, as can easily be proved, and thus there remain only the constants of the other two integrals for the satisfaction of the boundary conditions at the outer edge. These boundary conditions, in the problem considered, are: For $\theta = \theta_1$ the transverse lateral force N_1 and the bending moment G_1 are assumed to be given, where G_1 may again be expressed in terms of κ_2 or K, respectively, in accordance with Equations [6a].

The method described above readily provides a solution to problems where the usual power-series development fails completely. For the proof of the asymptotic convergence of the method the reader is referred to Blumenthal's paper.[4]

A numerical calculation and representation of the characteristics of this stress condition, which is important both in practice and in principle, will be published elsewhere.

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