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The Dimension of Fractal Geometry and Its Applications

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Abstract

Geometry is one of the oldest mathematics sciences that never stop to be studied and be developed. The newest geometry that bring us some new perspective is Fractal Geometry. Fractal Geometry based on study about how look like a continous function that not differentiabel everywhere and how can be a bounded object has infinite perimeter. These search bring us to new concepts of dimension. Normally, dimension of an object is a non negative integer, but Fractal Geometry extends this concepts. It is not surprising that dimension of a fractal object can be a fraction number. Many methods are develop to calculate dimension of fractal objects. In this paper, we will study about some methods to calculate the fractal dimension and apply these methods for objects that we can find in our daily life, such as bread, tofu, sponge, etc.

Key words: Fractal Geometry, Fractal Dimension, Applications of Fractal Dimension.

1. Introduction

Geometry is one of the oldest mathematics sciences that never stop to be studied and be developed. The newest geometry that bring us some new perspective is Fractal Geometry. In the past, mathematics has been concerned with sets and functions that are sufficiently smooth and regular. In recent years, this idea has change. Some mathematicians realize that irregular sets provide much better representation in many natural phenomena than figures of classical geometry. It is difficult to find a classical geometry object to represent a sponge because it is not solid, every parts contain some holes.. In fact, the idea of fractal came out at 19th century when Carl Weierstrass (mathematician German) gave the continuous function that is not differentiable everywhere which is called by “monster function” (Weierstrass function) and now is known be a fractal object.

The term "fractal" was first used by French-America mathematician, Benoît Mandelbrot in 1975. Mandelbrot based it on the Latin, *frāctus* meaning "broken" or "fractured", and used it to extend the concept of theoretical fractional dimensions to geometric patterns in nature. He gave a method to construct a fractal set using iterasion on komplex function $f(z) = z^2+c$, known by Mandelbrot set. A fractal is a mathematical set that has a fractal dimension that usually exceeds its topological dimension and may fall between the integers. Fractals are typically self-similar patterns, where self-similar means they are "the same from near as from far" and exactly the same at every scale. Fractals are usually nowhere differentiable, which means that they cannot be measured in traditional ways. So, it is not impossible to find a fractal object that is bounded but has infinite perimeter.

ISBN : 978-602-1037-00-3

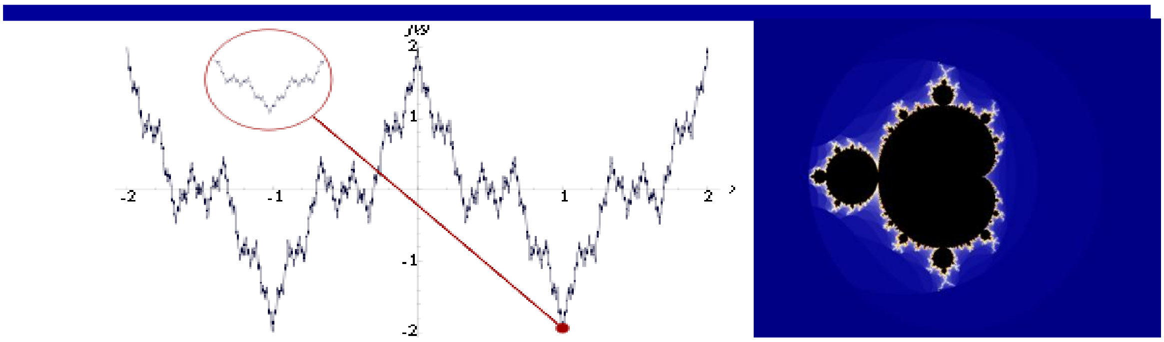


Figure 1.1 Graph of Weierstrass function and Mandelbrot set (200th iterasion using Ultrafractal).

Some methods is known to construct a fractal set, such as IFS (Iterated Function System), L-system, and Newton method.

A method of iterated function system is a method to create a fractal using iteration of functions. The example of this method is Mandelbrot set or Julia set. Mandelbrot set form by complex function $f(z) = z^2+c$. For every c complex number, the sequence $\left(0, f_c(0), f_c(f_c(0)), \ldots\right)$ is an iterated of $f_c(z)$ started from c = 0. Mandelbrot set is a set of point c such that the sequence started from c never vers to infininity, or $M = \{c \in \mathbb{C} : \exists s \in \mathbb{R}, \forall n \in \mathbb{N}, |f_c^n(c)| \leq s\}$.

L-system or now commonly known as *parametric* L systems, defined as a tuple *(V, S, R)*,
where

- *V* (*variables*) is a set of symbols containing elements that can be replaced
- *S* (*start*, axiom or *initiator*) defining the initial state of the system
- *R* is a set of *rules* defining the way variables can be replaced with combinations of constants and other variables.

The rules of the L-system grammar are applied iteratively starting from the initial state. To see the fractal images form by L-systems we can use *Fractint* program (uses turtle graphics), Turtle Graphics in Phyton, or L-system Turtrle graphics program to produce screen images.

Example 1.1:

Let L-system defined by Axiom : X
Angle : 30
Rule 1 : X=C0F-[C2[X]+C3X]+C1F[C3+FX]-X
Rule 2 : F = FF.

The symbol F means forward, - means turn left 30° , + means turn right 30°, Rule 1 for make a form to be repeated, rule 2 : F=FF means repeated the rule (the form) for every F (segmen line). The C0, C2 and C3 indicate the color for the segmen line , if the rule given without the C0, C2 andC3, the plant will be drawn in black.

Using L-system Turtle Graphics online in http://www.kevs3d.co.uk/dev/lsystems/, we have the image for iteration 1, 2 and 3 and 7, below.

 ISBN : 978-602-1037-00-3

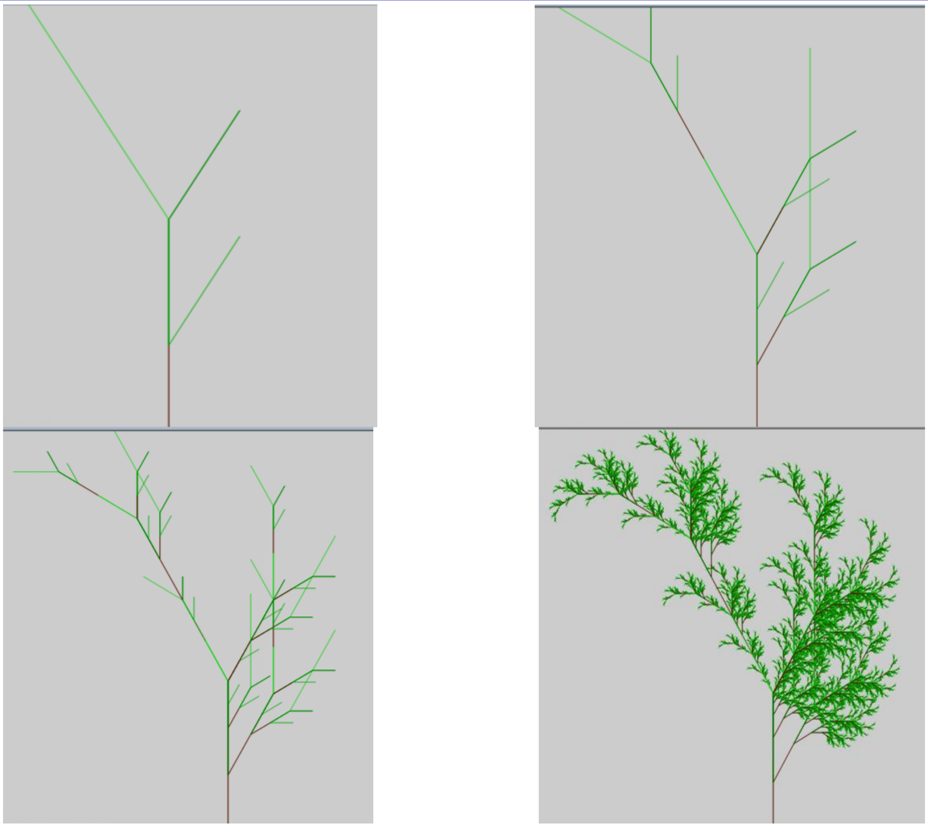


Figure 1.2 Iteration of L-system : 1st, 2 nd, 3rd and 7th respectively.

Newton method is a method to find a solution of a polynomial with approximation that we studied in Calculus. It's means to find an initial value x_0 and make iterasion process to find a sequence of x_0, x_1, x_2, x_3,... that converge to the solution using the equation The Newton for fractal is a boundary set in the complex plane which is characterized by Newton's method applied to a fixed polynomial *p(z)*. A generalization of Newton's iteration is $z_{n+1} = z_n - a\,\frac{p(z_n)}{p'(z_n)}$, with a w here a is any complex number.

Example : 1.2. Burton [3] give the image below for f(z) = z^4-1. The roots of the polynom are 1, -1, i, -i and colored by green, red, blue and teal respectively.

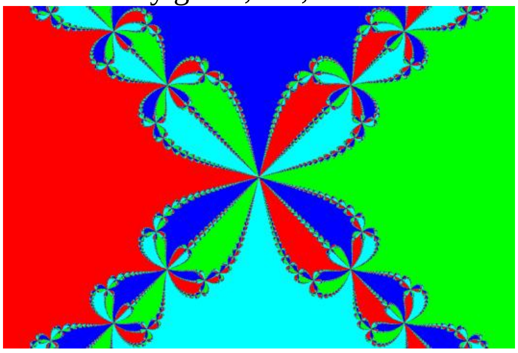


Figure 1.3. The bassins of attraction for f(z) = z^4-1.

2. Fractal Dimension of Fractal Geometry

When we talk about dimension there are several kind dimensions, such as topological dimension (or Lebesque covering dimension) and Haussdorf dimension (or

Hausdorff–Besicovitch dimension). The topological dimension of a set X is n if for every covering open balls of X, there at least one point where n+1 balls overlap.
The Haussdorf dimensionof X is defined by $dim_H(X) = \inf\{ d \geq 0 : C_H^d(X) = 0\}$,
where $C_H^d(S) = \inf \{ \sum_i {r_i}^d : there\ is\ a\ cover\ of\ S\ by\ balls\ with\ radii\ r_i > 0 \}$.
With these two definision, the dimension of a line is 1, 2 for a plane and 3 for a cube, but it will be different for a fractal set. For example, the curve that almost fill a plane has topological dimension 1, but it will be more than 1 and less than 2 on Haussdorf dimension.
The topological dimension for Cantor set is 0 but the Haussdorff dimension for this set is log2/log3. One method for calculating the fractal set is box counting that based on Haussdorf dimension.In fractal geometry, the fractal dimension, D, is the amount that describe how completely a fractal appear and fill a space.

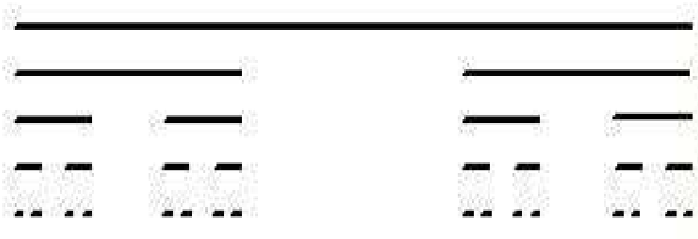
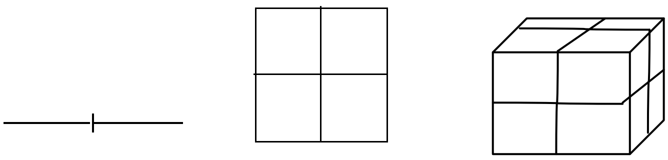


Figure 2.1. Cantor set

Box Counting Dimension

Box-counting or box dimension is one of the most widely used dimensions. It is popular due to its relative ease of mathematical calculation and empirical estimation. This methods based on traditional counting dimension and extends it for fractal object. In this method, we cover the fractal object with a grid, and then count how many boxes of the grid are covering part of the image. Then we do the same thing with smaller boxes. By changing the size of the grid repeatedly smaller and smaller, we will have structure of the pattern. The box means a segmen lines for line, a square or a circle for a plane and a cube or a ball for a space object.

It is known that the topological dimension of a point, a line and a square is 0, 1 and 2 respectively. Box counting fractal dimension calculation use this fact .
If a segmen line is divided with scale s = ½ , we have n= 2 new segmen lines,
If a square is divided with scale s = 1/2 , we have n = 4 new squares,
If a cube is divided with scale s = ½ , we have n=8 new cubes.
With various scales s, n will be change variously depends on line, square or cubes.



 ISBN : 978-602-1037-00-3

	Line (D=1)	Square (D=2)	Cube (D=3)	…	D
scale	Number of parts (n)				
1	1	1	1	…	1
½	2	4	8	…	$\frac{1}{\left(\frac{1}{2}\right)^D}$
1/3	3	9	27	…	$\frac{1}{\left(\frac{1}{3}\right)^D}$
…					
s					$\boldsymbol{n = \frac{1}{s^D}}$

Figure 2.2 The number of parts depends on scale and dimension

In this last equation, the value of D depends on s and n. We have :

$$\log n = \log\left(\frac{1}{s^D}\right)$$

$$D = \frac{\log n}{\log\left(\frac{1}{s}\right)}.$$

Characteristic of fractal is similar for every scale, so extends this equation for fractal, we have the formula for fractal dimension :

$$D = \lim_{s \to 0} \frac{\log n}{\log\left(\frac{1}{s}\right)} \quad \text{or} \quad D = \frac{\log n}{\log\left(\frac{1}{s}\right)},$$

with n is the number of boxes that cover the fractal object and s is the scale.
In other way, using the box counting method, fractal dimension is the slope of the line when we plot the value of log(n) on the Y-axis against the value of log(1/s) on the X-axis.

Example 2.1.
Menger sponge is a fractal object. It is a three-dimensional generalization of the Cantor set and Sierpinski carpet. It is formed by dividing every face of the cube into 9 squares, the cube change into 27 smaller cubes. Remove the smaller cube in the middle of each face, and remove the smaller cube in the very center of the larger cube, leaving 20 smaller cubes. Repeated this process infinitely for the smaller cubes, we will have a Menger sponge.
The dimension of Menger sponge is

$$D = \lim_{s \to 0} \frac{\log n}{\log\left(\frac{1}{s}\right)}$$

$$D = \lim_{k \to \infty} \frac{\log(20)^k}{\log (3)^k}$$

ISBN : 978-602-1037-00-3

$$D = \frac{\log 20}{\log 3} = 2{,}7268.$$

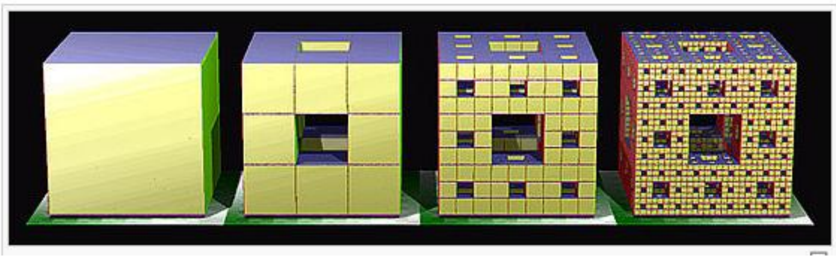
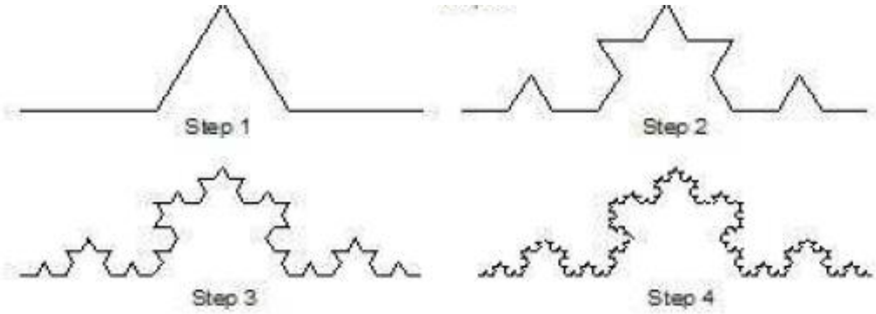


Figure 2.2 Menger Sponge
(source : Wikipedia)

Example 2.2. The dimension of Koch curve is log 4 /log 3, where the iteration of this set can be seen below.



There is a program called FDC (Fractal Dimension Calculator) see in [4] which calculates the fractal dimension of an object represented by an image. This program use the method of box counting, using various sizes of square (s) to calculate the number squares laid over the image (N(s)). The fractal dimension D is given by the slope of the linear portion of log (N(s)) vs log(1/s) graph.

3. The Application of fractal dimension

Fractal have been applied for many areas, such as fractal anthena, music fractal, batik fractal, and landscape fractal, etc. The result of the research in fractal dimension varied in manyt area domain. Such as, Fractal dimension used to detect of transients in the electrotencephalogrm (EEG) see in [7], dimension fractal for diagnosting cancer, dimension of surface human brain and lung surfaces has been calculated by Sapoval, etc.

In the previous section, we have obtained the formula to calculate the fractal dimension. We will use it for real object in our daily life. For example, sponge, bread, tofu etc.

Sponge or Bread are objects that have many holes for every parts, so it is normal that the dimension of sponge or bread between two and three. Amaku et all in [1] give the formula for calculating the dimension of bread. The method for calculate the dimension of bread is cut in cubes the bread of decreasing size and squashed by hand into a spherical form. For each a squashed sphere, the diameter φ (in cm) was measured. Measure the sphere with analytic balance, we have M (in grams). Use the formula $M = k\ \varphi^{D}$, with k is the density (the mass per unit of fractal volume), and make it in the logarithm form. So, we have :

$$\text{Log } M = D \log \varphi + \log k,$$

change in the form of ; $y = ax + b$, with $y = \log M$, $a = D$, $x = \log \varphi$ and $b = \log k$.
The fractal dimension can be obtained from the slope of the line $y = ax + b$.
Here, we use this method to calculate the fractal dimension of sponge, bread, tofu and tempe (soyabean curd).

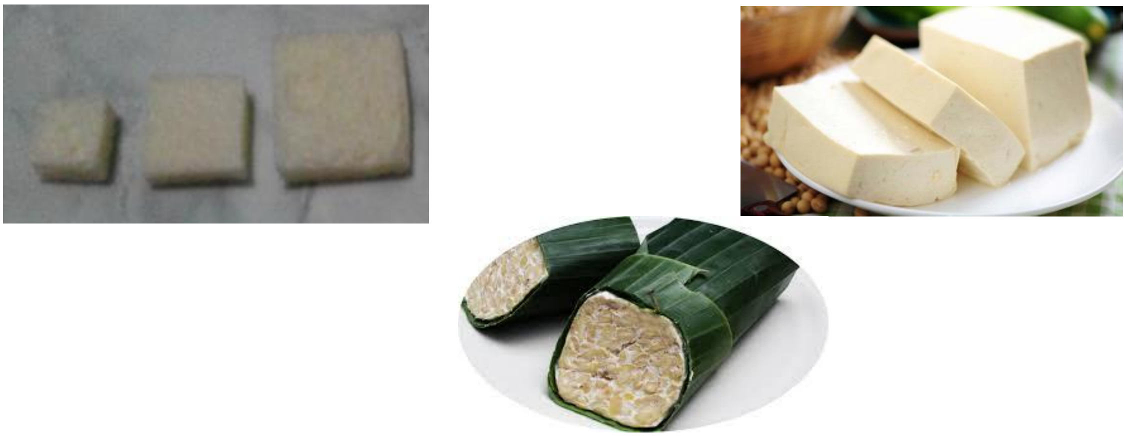


Figure 3.1 Bread, Tofu and tempe

We make several squahed sponge, bread, tofu and tempe for every size and measured the mass with Ohauss balance. For tofu, we separated the water from the tofu using a tissue. For bread we squashed to make a form of a ball, and for sponge, tofu and tempe, use the cube form. Make cub.es of different sides and measured the weight of these cubes. Here, we colected the data of 15 measurement for every object, and using SPSS software, to find the linear regression model with *ln m* for dependent variable and *ln diameter* for independent variable and take the slope for the dimension :
The model regression for sponge is : y = 2.934 – 1, 243
The model regression for bread is : y = 2. 275 x + 0.112,
The model regression for tofu is : y = 2. 519 x + 0.393,
The model regression for tempe is : y = 2. 496 x + 0.183,
For four model regression above, the model is significant with α = 5%, it means the models can be used.
So, the dimension of sponge = 2.934, the dimension of bread = 2.275, the dimension of tofu = 2.519 and the dimension of tempe = 2.496. this is reasonable, because the value is between two and three. We can use this method to calculate other fractal object that can be find in our real life.

Knowing the fractal dimension of object can be used to know more about the structure of an object, to quality control of the product t.hat be produced, to make a description more detail and specifique about an object.

References

[1] Amaku, M. Horodynski-Matsushigue, L. B.; Pascholati, P. R., (1999), *Fractal Dimension of bread*, Physics Teacher, v37 no.8, Eric.
[2] Barnsky, M. F., (1993), *Fractal Everywhere*, Academic Press, Elsevier.

[3] Burton, A., (2009), Newton Method and Fractal, Senior Project Archive from https://www.whitman.edu/mathematics.
[4] Bourke, P. (1993), Fractal Dimension Calculator User Manual,1993
[5] Falconer, K., (2003), *Fractal Geometry* : Mathematical Foundation and Applications, John Wiley and Sons, England.
[6] Sapoval, B., (2001), "Universalités et fractales", Flammarion-Champs.
[7] Simon, R and Arie,J., (1990), An Application of Fractal dimension to the detection of Transients in the Electroencephalogram, Electroencephalography and Clinical Neurophysiolоy, Vol. 75, Elsevier.