COUPLING BY MEANS OF STRONG DISCONTINUITY APPROACH BETWEEN CRACK OPENING AND GAS PERMEABILITY FOR CONCRETE

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Abstract :

Due to industrial needs, one of the key issues nowadays is to develop numerical tools that are able to predict the leakage rate through a cracked concrete structure. This paper presents a validation of a numerical modelling of leakage rate through a mortar specimen in a splitting test versus experimental results. The mechanical state of the material is described by means of an enhanced non-local damage model which takes into account the stress state and provides a realistic damage field at failure. A semi discrete method based on a Strong Discontinuity (SD) approach isthen considered to study the coupling between the mechanical state of the material and its permeability. This methodconsists in first determining the crack opening field in the crack surface, then coupling the permeability with the crack opening by means of a modified Poiseuille's law. The assessed crack openings given by the SD method is compared to Digital Image Correlation measurements. The comparison between the two shows that the assessed crack openings given by SD are in good agreement(the maximum relative error is less than 20%). The results of the coupling compared to experimental data show a good estimation of the structural permeability for high level of cracking. Moreover, for lower levels of cracking,low differences between numerical and experimental permeabilities are observed.

Key words: Strong Discontinuity Approach, Damage, Gas Permeability, Splitting Test, Mortar / Concrete.

1 Introduction

During their service life, due to external loading (mechanical and/or environmental), concrete structures may undergo damage in a diffuse manner (microcracking) at the material scale and/or localized (macrocracking) at the structural level. The estimation of the evolution of transfer properties in such a cracked material is a key issue for structural durability analysis. Choinska et al. **[1]** observed in their experimental study three different regimes of permeability evolution. A first regime exhibiting relatively a slight permeability increase, which is due to the presence of microcracks spread out in the material. Coupling between permeability and microcracking (diffuse damage), has been proposed by Picandet et al. **[2]** when the material is subjected to compression and by Dal Pont et al. **[3]** when the material is subjected to high temperatures. The second regime is observed when the permeability of the material increases rapidly due to strain localization. This regime is an intermediate phase between diffuse damage and discontinuous macrocracking. A third regime is observed where macrocracks are formed and permeability is governed at the macrostructural level by Poiseuille's law (Permeability of parallel plates) and mainly depends on the crack opening. This regime is characterized by a slower rate of permeability increase with respect to the second regime. Many authors **[4,5, 6, 7]**has shown that Poiseuille's law overestimates a fluid flow in a real crack since secondary effects such as the crack roughness, opening variation and tortuosity are not taken into account in the perfect/smooth parallel plates. A correction factorξ is introduced in order to take into account crack roughness, aperture variation and tortuosity. Generally, this factor was supposed to be constant for a certain concrete, for instance load-induced cracks in HPC should be smoother than the tensile cracks in OC, and therefore the correction factor might be lower for OC **[4]**.Rastiello et al. **[5],**in their study on water permeability evolution of a localized crack in concrete, has proposed an empirical correction factor where two constant parameters were introduced. This correction factor is no longer constant but is function of the crack opening.

Consequently, crack opening assessment is the key factor for durability analysis of concrete structures and therefore numerical models for mechanical behavior and cracking assessment is needed. One can describe explicitly the cracking in the mechanical model as shown in X-FEM **[8]**, G-FEM **[9]**or E-FEM **[10]** for example. The location of the crack and the crack opening are directly quantified in these approaches. Nevertheless, the modelling of the crack initiation is still under discussion. Furthermore, the micro-cracking process and the macro-cracking failure can be modelled in a continuous framework using regularized damage models. The cracking description and crack opening assessment is then performed as a post-processing phase. Those models are proven to describe numerically the physical response of a material on the global scale as well as on the local one.

In this paper the mechanical state of the material is described by means of an enhanced non-local damage model which takes into account the stress state and provides a realistic damage field representing micro cracking and macro cracking at failure **[11]**. A semi-discrete approach is considered inorder to numerically simulate the coupling between the mechanical state of the material and its permeability.A first possibility is to assess the crack path either using for instance a topological search **[12]** or the global tracking algorithm **[13]**. Once the crack path is found, the Crack Opening Displacement (COD) can be computed along the discretized crack surface by equivalence with strong discontinuity approach **[14]**. The final step is to prescribe the modified Poiseuille's law along the crack surface taking into account the roughness, opening variation and tortuosity of the crack to estimate the leakage rate while imposing a pressure gradient in the 2D crack surface.

An experimental campaign **[15]**has been performed on a mortar specimen subjected to splitting test; the gas permeability of the specimen is measured during the test at different load levels.As stated in **[7]**, it was noticed that the crack opened more on one of the faces than the other one, this heterogeneity comes from the slight conicity of samples.Crack openings are computed by means of digital image correlation which is performed on the face where the crack opened less. On the second face a displacement sensor is placed in the center in order to control the total displacement in the horizontal direction. The computed COD by the SD method will be compared to DIC data. The validation of the semi discrete approach against experimental results is performed on the leakage rate perpendicular to the disk for different load stages.

The numerical models (damage and semi discrete approach for leakage rate) will be described in the section 2 and the physical experiment of the splitting test will be detailed in section 3. In section 4 the application of the numerical models on the splitting test will be detailed.

2 Semi-Discrete Approach

In this approach continuous damage models are considered to simulate the cracking and a strong discontinuity discrete method is applied for crack opening calculations. The enhanced non-local damage model which is based on the stress state will be presented in the following section. In the second subsection the permeability-mechanical state coupling will be presented.

2.1 Regularized damage models

The loss of stiffness associated to mechanical degradation of the material is characterized by a scalar damage variable D. This internal variable links the Cauchy stress tensor $\boldsymbol{\sigma}$ to the strain tensor$\boldsymbol{\varepsilon}$, following equation (1).

$$\boldsymbol{\sigma} = (1 - D)\boldsymbol{C} : \boldsymbol{\varepsilon} \tag{1}$$

With C is the tensor of elastic moduli. The parameter D ranges from 0 for virgin material to 1 for completely damaged material. It is assumed that D depends on a state variable Y, which depends on the strains, (i.e. Y = Y (ε)).

The nonlocal regularization method on the internal variable Y is used in order to maintain the objectivity of the results by averaging the state variable Y in the neighborhood for each point. This method of regularization, proposed by Pijaudier-Cabot and Bazant in **[16]**, replaces a local variable Y by its nonlocal counterpart $\tilde{Y}$ following equation (2).

$$\tilde{Y} = \frac{\int_v \alpha(d) Y dV}{\int_v \alpha(d) dV} \tag{2}$$

The weight function α depends on the distance d to the point under consideration. Generally a Gaussian function is used:

$$\alpha(d) = \exp\left[-\left(\frac{2d}{l_{c0}}\right)^2\right] \tag{3}$$

Where l_{c0} is a material parameter of the nonlocal damage model called characteristic length, and Y is the state variable, that drives the damage (D=D($\tilde{Y}$)) according to Mazars law**[17]**:

$$Y = \sqrt{\sum_i [\max(0, \varepsilon_i)]^2} \tag{4}$$

Damage evolution follows a law which distinguishes tensile damage D_t and compressive damage D_c:

$$D = \alpha_t^{\beta} D_t + \alpha_c^{\beta} D_c \tag{5}$$

Where α_t and α_c are the weights computed from the strain tensor and β is a parameter which improve the shear behavior. The tensile damage D_t and compressive damage D_c are calculated as follows:

$$D_{t,c} = 1 - \frac{Y_{D0}(1 - A_{t,c})}{Y} - \frac{A_{t,c}}{e^{\left[B_{t,c}(Y - Y_{D0})\right]}} \tag{6}$$

Where Y_{D_0} is the strain at first crack in tension and is also called the threshold of damage.

This regularization method allows the objectivity of the results. However, it fails to properly describe both the strain field and the damage profile at complete failure as well as cracking initiation close to boundaries. In order to improve the description of the continuous fields, an evolution of non-local interactions should be introduced during computation. In the actual contribution, the method proposed by Giry, Dufour and Mazars in **[11]**is used. This regularized damage model, called the non local stress based (**NLSB**) is characterized by a regularization that takes into account the stress state of the material. A modification on the Gaussian function is applied and the function becomes:

$$\alpha(d) = \exp\left[-\left(\frac{2d}{l_{c0} * \rho}\right)^2\right] \tag{7}$$

Where $\rho$is a stress factor which is calculated for each integration point and depends on the principal stresses of the medium. More details can be found in **[11]**.
The application of the mechanical model on a 3D splitting test will be presented in section **4.1**.

2.2 Coupling permeability-SD approach

Assuming that the flow is laminar, Darcy's law is used to determine the global permeability of the sample. For a unidirectional flow, the mean permeability $k_m$is determined as:

$$k_m = \mu \frac{Q}{A}\left(\frac{\Delta P}{\Delta x}\right)^{-1} \tag{8}$$

Where A (m^2) is the cross sectional of the specimen normal to the flow, $\Delta P/\Delta x$is the gradient of pressure (Pa/m), μ is the dynamic viscosity of the fluid (N_2) at 20°C and Q is the volumetric flow rate through the sample (m^3/s).
In order to numerically find the flow rate using FE computations, a model for permeability has to be defined. The simplest model of an incompressible fluid flow through a crack is the model of parallel plates (Poiseuille's law). The Poiseuille's permeability for concrete is identified by solving Navier-Stokes equation for two fracture walls modelled as two smooth parallel plates, distant by an aperture (or crack opening)u. However, it is well known that the concrete fracture has a certain roughness and tortuosity; it means that a reduction factor should be introduced in Poiseuille's permeability. This reduction factor for a certain concrete is generally supposed constant for any crack opening**[4, 6]**. Nevertheless, in **[5]**it is shown that this reduction factor is not constant, but increases when the crack opening increases. An empirical equation with two variables γ and β were introduced. The variablesγ and βwere identified as equal to -1.19 and 5.625e-5 respectively. The equation (9) is used in the modelling.

$$\xi_u = \frac{[u]^{-\gamma}}{\beta} \tag{9}$$

The Poiseuille's permeability taken into account the roughness and the tortuosity of the crack, noted as modified Poiseuille's law, is given in equation (10).

$$k_{fr}^{mp} = \xi_u \frac{[u]^2}{12} \tag{10}$$

If we consider at the specimen scale, a crossing crack of length L and opening equal to $[u]$ in a specimen subjected to a pressure gradient, the total flow rate is obtained from Darcy's law equation by adding the flow rate through the crack and the one through the sound material of the specimen. The mean structural permeability is obtained as:

$$k_m = k_0 + \frac{A_{fr} k_{fr}^{mp}}{A} \tag{11}$$

Where $k_0 (m^2)$ is the permeability of a sound material, $A$is the cross section that is exposed to the pressure gradient and normal to the flow direction, A_{fr} andk_{fr}^{mp} are the crack surface and the modified Poiseuille's permeability respectively given in equation (10).One may suppose that besides cracking, diffuse damage in compression occurs and Picandet's permeability can be introduced instead of k_0. However, in case of splitting test, the damage is mainly generated in tension and is localized in the crack surface; consequently a negligible variation of the mean structural permeability due to diffuse damage is seen in the physical experiment.

The tested specimen in the physical experiment is cylindrical therefore equation (11) can be rewritten as:

$$k_m = k_0 + \frac{[u_m]^{3-\gamma}}{3\pi D\beta} \quad (12)$$

Similarly, at the finite element scale equation (11) becomes:

$$k_m^e = k_0^e + \frac{A_{fr}^e \, k_{fr}^{e,mp}}{A^e} \quad (13)$$

Where k_m^e is the mean permeability of a cracked element at its integration point, A_{fr}^e and $k_{fr}^{e,mp}$ are the crack surface and the modified Poiseuille's permeability of an element respectively. Assuming that each element has a rectangular shape, equation (13) becomes:

$$k_m^e = k_0^e + \frac{[u^e]^{3-\gamma}}{12\, l^e\, \beta} \quad (14)$$

This formulation requires the calculation of the crack opening at each integration point $[u^e]$ of the crack surface and the element length as equal to the square root of its surface. To determine the total leakage rate, the permeability problem by imposing a pressure gradient has to be solved (More details can be found in section **4.3**). Once the leakage rate is determined, by means of Darcy's law, the mean structural permeability can be calculated and will be compared to the permeability measured experimentally.

3 Experimental Set-up

The experiment **[15]** consists of performing a Brazilian splitting test applied on mortar specimens. The Brazilian splitting test is an indirect tension test used to measure tensile strength of concrete, rocks and other geomaterials. It consists of loading a cylindrical specimen along a diametric plane by means of steel or wood bearing plates, as shown in **Fig**.**1**.Gas flow rate measurements are taken after a partial unloading to avoid brutal rupture (See **Fig**. **2**). The sample has a cylindrical shape of 40 mm of height and 110 mm of diameter. It is worth noting that the sample isn't a perfect cylinder, a difference of 0.1 mm in diameter was observed. The Young modulus is equal to 18 MPa and the Poisson's coefficient equal to 0.2.

Figure1: Permeability analysis while performing a Brazilian test on a mortar sample. **[15]**

3.1 Mechanical behavior

The splitting test is controlled and directed by the total horizontal displacement recorded by a displacement sensor located on the central horizontal line of the bigger face of the specimen. The sensor is 10 mm horizontally distant from the center of the surface. The position of the sensor is denoted by the point P. The other face (the smaller) is discretized by means of a speckle pattern in order to perform Digital Image Correlation and get the 2D displacement field on the surface.

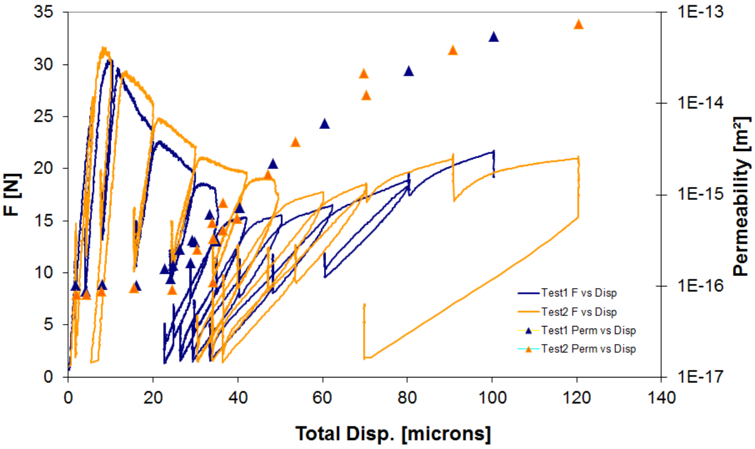


Figure2: Mechanical behavior and permeability evolution versus the total disp. for two mortar specimens. **[15]**

The mechanical response (Applied force versus the total horizontal displacement recorded by the displacement sensor) and the permeability evolution of two mortar specimens are given in **Fig**. **2**.The mechanical response is described byan elastic part for small displacements, that is followed by a peak and a softening behavior that ends by a complete split of the specimen. At total horizontal displacement equal to approximately 34 microns the specimen is split and the behavior afterwards is described by the half portion of the specimen subjected to compression loading. An analysis has shown that there is a 3D effect on the force-total displacement behavior of the specimen. The crack is initiated and propagated firstly on the bigger face (where the sensor is placed) and then is propagated in the longitudinal direction to reach the smaller face and propagate on it until the total split of the specimen. It is obvious that 2D simulations won't be sufficient to describe properly the mechanical behavior of the conic specimen, consequently 3D simulations are needed.
The permeability evolution can be described by the three regimes proposed by Choinska**[1]**: For crack openings varying from 0 to 30 microns a slight increase (almost none) in the permeability is observed which corresponds to the first regime. Afterwards the second regime is observed when a large increase in the permeability begins for crack openings higher than 30 microns. Finally, one can see a decrease of the rate permeability increase.

3.2 Digital image Correlation

One of the two faces is discretized with speckles in order to assess on this face the crack opening field. The direction of the loading is vertical therefore the interests are given to the speckles' displacements in the horizontal direction. A window of interest is placed around the crack as shown in **Fig. 3**, and the crack opening displacement is calculated as the difference between the displacement of a point located on oneside of the crack and the displacement of its symmetric with respect to the crack.

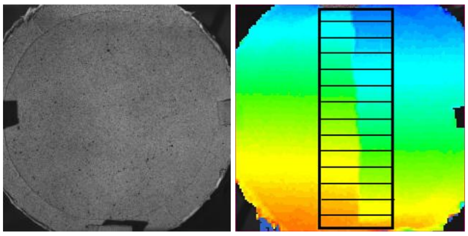
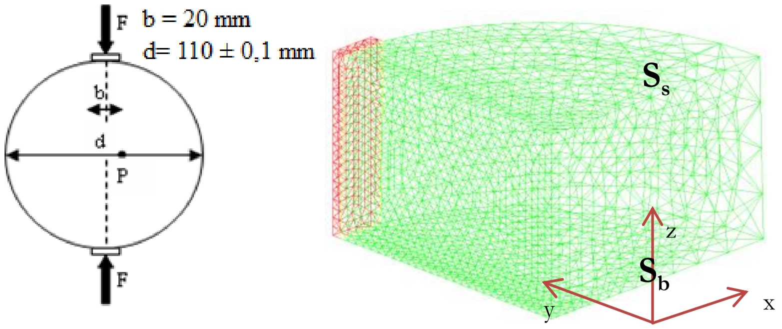


Figure3: Speckle pattern and the horizontal displacement field on the specimen face.**[15]**

The cross section of the specimen as well as the displacement field in the cross section at the last state (at total displacement recorded by the displacement sensor equal to 100 microns) are shown in **Fig. 3**. It is seen that the crack path is the surface located in the plane of symmetry that is parallel to the loading. Therefore this choice of numerical path is suitable. A comparison between experimental data and numerical one is presented in section **4.3**.

4 Application On A 3D Splitting Test

This numerical study is a simulation of the physical experiment presented in **section 3**. The steel bearing plates are modelled as rigid plates, with high Young's modulus (E = 300 GPa) and Poisson's ratio ν of mortar (ν=0.2) in order to avoid a confinement effect of mortar. Numerical simulations are performed in the FE code *Cast3m* with 4-nodes tetrahedral elements in 3D. Due to the symmetry of the problem, the computation domain consists of only quarter of the specimen. The mesh is shown in **Fig. 4**(b). It should be noted that the mesh is generated with the same conicity of the real specimen in order to reproduce the 3D effect in the simulations. $\mathbf{S_s}$ and $\mathbf{S_b}$ corresponds respectively to the smaller and bigger edge surfaces of the specimen. The post-peak behavior in splitting test includes a snap-back in the force displacement curve and therefore an arc-length control (by maximum strain or crack opening) is required to solve the numerical problem **[18, 19]**.



(a) (b)

Figure 4: Brazilian splitting test (a) problem statement (b) 3D conic FE mesh.

4.1 Mechanical description

In this subsection the mechanical description by the stress based non local **NLSB** is concerned. Young's Modulus and Poisson's ratio of the mortar are taken from physical experiment. A calibration of the parameters is done for the model after finding a good numerical description of the physical response and the set of Mazars' parameters is given in table 1.

Parameter	NLSB
Y_{D0}	$3.5.10^{-4}$
A_c	1.4
A_t	0.88
A_c	800
A_t	4050
β	1.06
l_{c0}	7.5 mm

Table 1:Set of Mazars' parameters for the damage models

The point P on which the total horizontal displacement is calculated is located on the central horizontal line of the bigger face distant of 10 mm from the center (same as in the experiment) (See **Fig. 4** (a)). The numerical description of the mechanical response (Force versus displacement at P) of the physical experiment is shown in **Fig. 5.** This result shows a good agreement between the stress based non local model NLSB and the physical response even after the total split. However, the model is an elasto-damageable model that does not take into account the plastic deformation therefore the exaggerated snap-back where the displacement of point P decreases from about 35 to about 30.5

microns is probably due to elastic discharge of the face $\mathbf{S_b}$. Despite this drawback in the model, it will be shown that this drawback won't affect highly the coupling between the permeability and the mechanical state.

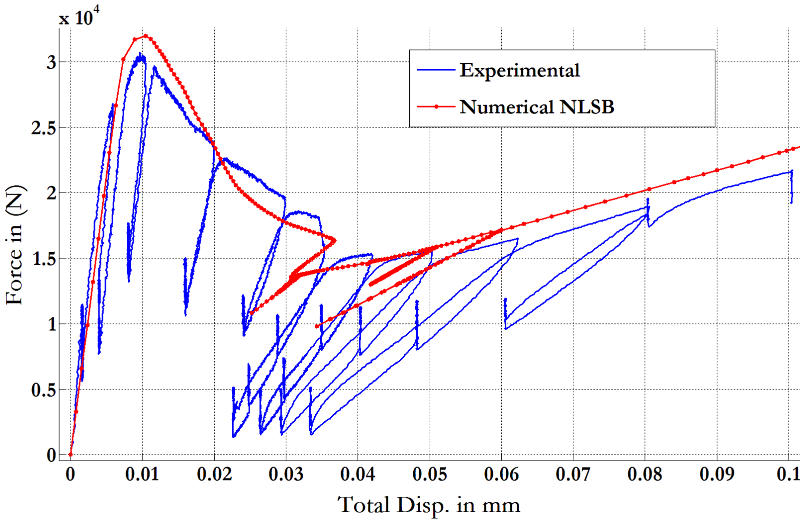
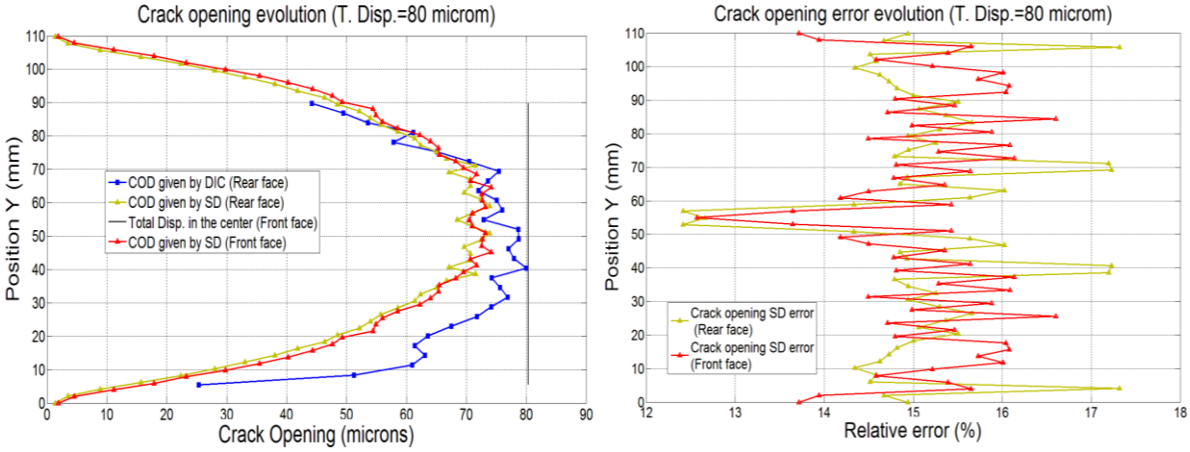


Figure 5: Mechanical response (Force versus total horizontal displacement of point P) described by NLSB compared to experimental response.

4.2 Crack opening displacements

A comparison is made between the digital image correlation technique and the strong discontinuity method at total horizontal displacement of point P equal to 80 microns.

The assessed crack opening profiles by the SD method on the edge vertical lines of the crack surface are compared to the DIC performed on the rear face only (See **Fig. 6** (a)).. The comparison shows that the numerical assessed crack opening is in a good agreement with the measured ones. One can notice that at displacement of point P equal to 80 microns the numerical 3D effect in terms of crack opening is almost disappeared. The error indicator which indicates how much the method is distant or close to the correct solution shows a maximum relative error that is below 18 %. Moreover it shows that the more the crack opening at an integration point is high, more the committed error decreases, for instance the relative error at the sample mid-heightwhere the crack opening is theoretically the maximum, is around 13%. This shows that this method is acceptable in order to assess crack opening.



(a) (b)

Figure 6: Crack opening profile(a) and error of the method (b)corresponding to total horizontal displacement of P equal to 80 microns.

4.3 Coupling Permeability-Mechanical State

Once the numerical description of the mechanical response is achieved and the crack opening assessment is done, one can calculate the local (element) permeabilitiesin order to compute numerically the total flow rate though the cracked material. It is supposed that the permeability of the elements that belong to the crack path is k_m^e, and k_0 elsewhere. Consequently the total flow rate is the sum of two flow rates; the first one is due to the isotropic permeability of the material (the initial permeability k_0 or Picandet's permeability if diffuse damage occurs), it can be computed by solving a permeability problem in the volume by applying a pressure gradient between the two faces $\mathbf{S_b}$and $\mathbf{S_s}$. The second one is due to the anisotropic permeability of cracking origin, it is found by solving a permeability problem in the crack surface by applying the same pressure gradient between the two crack lines that belong to $\mathbf{S_b}$and $\mathbf{S_s}$(**Fig. 7)**. Eventually the mean structural permeability is calculated by applying Darcy's law (See Equation 8).

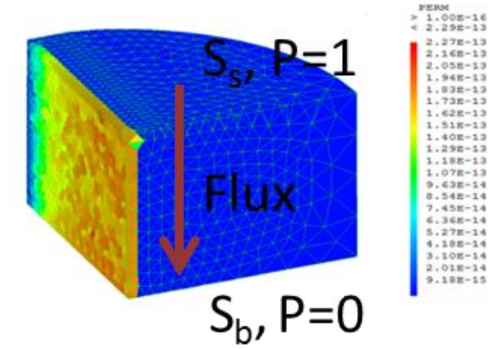


Figure 7: Solving the permeability problem by applying a pressure gradient.

The evolution of the mean structural permeability versus the total horizontal displacement at point P is presented in **Fig. 8**for two cases as well as the permeabilities calculated from flow rate measurements. The first case is supposing that the concrete crack surfaces are perfectly smooth and no tortuosity effect ($\xi = 1$), the second case is when considering the empirical relation for the correction factor proposed in **[5]** (See equation 9). One can clearly see, in the phase where there is an opened crack, that the non modified Poiseuille’s model of perfectly smooth parallel plates overestimates the mean structural permeability (flow rate through the cracked structure) with respect to the experimental results. Moreover, one can see that the more the crack opening increases the more the overestimation decreases with respect to the experimental results. In other words, the correction factor is non constant for one concrete but it decreases when the crack opening increases. This was already discovered by Rastiello et al. in his study **[5]**. Rastiello’s parameters γ and β are equal to -1.19 and 5.625E-5 respectively, and those parameters are adopted and introduced in the proposed model (See section **2.2**). The result after considering Rastiello’s parameters is shown in **Fig. 8**. A perfect match is found between the numerical modelling and the experiments for the phase where the cracking is important (total displacement of point P higher than 40 microns), and a low dispersion is found for less crack opening. However this dispersion is also seen in the experimental measurements in this phase of cracking. Nevertheless, dispersion in the crack opening assessment is expected when the cracking is not very important. Indeed, the hypothesis of the Strong Discontinuity method is based on a strong discontinuity along the crack, and consequently if the cracking is not significant the hypothesis is not correct. Moreover this result proves that the correction factor that intervenes in Poiseuille’s law is unique for a certain concrete and that is independent of the percolated fluid.

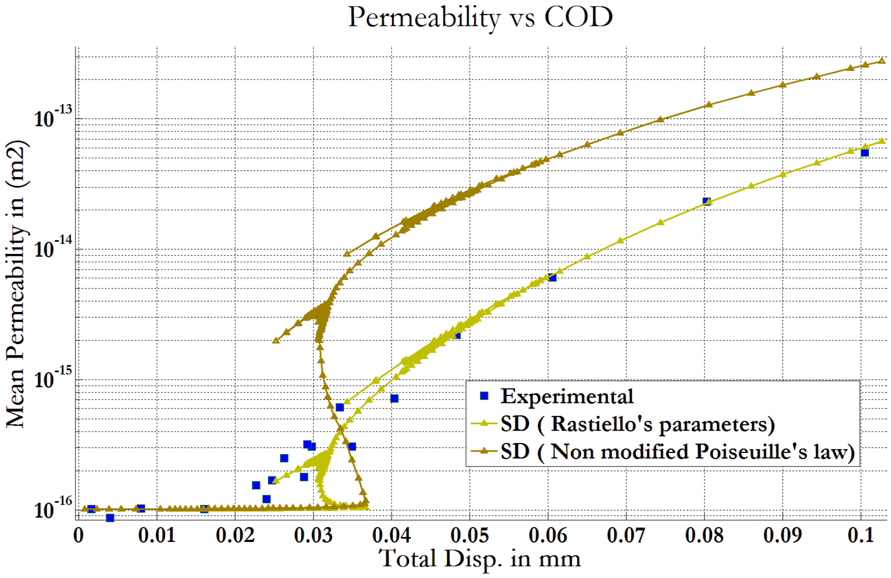


Figure 8: Mean permeability calculated by applying Matching law based on the first approach versus COD of point P.

5 Conclusions

This paper presents a validation of a numerical modelling of leakage rate through a mortar specimen in a splitting test versus experimental results. The mechanical state of the material was described by means of an enhanced non-local damage model which takes into account the stress state and provides a realistic damage field at failure. A semi discrete method based on a Strong Discontinuity (SD) approach was then considered to study the coupling between the mechanical state of the material and its permeability.The mechanical description using the non local stress based damage model of a splitting test in 3D has been achieved. The crack opening assessment is obtained accurately enough using the Strong Discontinuity approach applied in the post processing phase; the error indicator indicates a relative error of less than 20 % which is considered acceptable in numerical computations. Once the crack opening is assessed with the SD method, the coupling between the mean structural permeability and the mechanical state of a specimen subjected to a splitting test is achieved by means of a modified Poiseuille's law. A correction factor ξ is introduced in the model, as indicated by Rastiello in his study. This paper shows that the proposed parameters that intervene in the relation between the correction factor and the crack opening, γ and β, are valid for a certain concrete and that they are independent of the percolated fluid. Those parameters were validated for an ordinary concrete/mortar. Finally, the coupling is validated on the splitting test. The estimation of the mean permeability is in great agreement with the experimental measurements for high level of cracking. However a better assessment of the crack opening for lower level of cracking is needed, possibly by applying the weak form of the Strong discontinuity approach.

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