Rayleigh limits for effective wavenumbers of randomly distributed porous cylinders in a fluid.

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Résumé :

Nous considérons la propagation d'ondes dans un fluide contenant des cylindres poreux parallèles aléatoirement répartis dans l'espace. Le nombre d'onde effectif de l'onde cohérente dans le milieu est dérivé à la limite de Rayleigh pour les formules explicites ISA (Independent Scattering Approximation), Waterman et Truell (WT) et Linton et Martin (LM) aussi bien que pour les formules implicites, i.e. CPA (Coherent Potential Approximation) et GSCM (General Self Consistent Method) appliquée à WT et à LM. Lorsque la porosité des cylindres tend vers zéro, on retrouve des situations connues correspondant à des distributions aléatoires de cylindres élastiques immergés.

Abstract :

We consider wave propagation in a fluid containing parallel porous cylinders randomly distributed in space. The effective wavenumber of the coherent wave in the medium is derived at the Rayleigh limit for explicit formulas ISA (Independent Scattering Approximation), Waterman and Truell (WT) and Linton and Martin (LM) as well as for implicit formulas, i.e. CPA (Coherent Potential Approximation) and GSCM (General Self Consistent Method) applied to WT and to LM. When the porosity of the cylinders tends to zero the well known cases of an assortment of random elastic cylinders in fluid is found.

Mots clefs : random medium, effective wavenumber, self consistent schemes, Rayleigh limit.

1 Introduction

The problem of sound propagation through fluid domains containing randomly distributed scatterers has been investigated in recent years. The scatterers can be as well bubbles, hard grains, elastic rods or contrast agents. The main interest is on the propagation of the coherent wave which represents a statistical average over all possible configurations of the scatterers. The wavenumber k_{eff} of the coherent wave which is also called the effective wavenumber is complex-valued.

Several explicit formulas have been proposed for the calculation of k_{eff} in the case of cylindrical scatterers, which include the Independant Scattering Approximation (ISA) [1]

$$k_{eff}^2 = k_{ISA}^2 = k_0^2 - 4in_0 f(0), \quad (1)$$

the Waterman and Truell [2] formula,

$$k_{eff}^2 = \left[k_0 - \frac{2in_0}{k_0} f(0)\right]^2 - \left[\frac{2in_0}{k_0} f(\pi)\right]^2 \quad (2)$$

and the Linton and Martin [3] formula

$$k_{eff}^2 = k_{LM}^2 = k_0^2 - 4in_0 f(0) + \frac{8n_0^2}{\pi k_0^2}\int_0^\pi \cot\left(\frac{\theta}{2}\right)\frac{d}{d\theta}\left[f(\theta)\right]^2 d\theta. \quad (3)$$

In the foregoing, $f(\theta)$ is the far-field scattered amplitude of each cylinder in the direction θ, n_0 the number of scatterers per unit area and k_0 the wavenumber in the fluid host. The accuracy of the three formulas depends on the value of the concentration $c = n_0 \pi a^2$ where a is the radius of circular cylinders, *i.e.* on the ratio n_0/k_0^2. The ISA is $O\left(n_0/k_0^2\right)$ whereas both WT and LM are $O\left(n_0^2/k_0^4\right)$.

Implicit methods have been also derived which include the Coherent Potential Approximation (CPA) [4] and the General Self Consistent Method (GSCM) [5] which originates from a self consistent scheme applied to the Waterman and Truell's formula. In these methods, the wavenumber k_{eff} is obtained by solving an equation containing the far-field scattered amplitude. In this paper, by considering fluid saturated porous cylinders, the Rayleigh limit for various effective wavenumbers are calculated and compared : those from explicit wavenumbers k_{ISA}, k_{WT}, k_{LM} and those from self-consistent schemes k_{CPA}, k_{G-WT} and k_{G-LM}.

2 Wavenumbers from explicit theories

Let us consider circular porous cylinders of radius a immersed in a fluid of density ρ_0 and of sound velocity c_0. The wavenumber in the fluid is denoted by $k_0 = \omega/c_0$ with ω the angular frequency. In the saturated porous medium, according to Biot's theory three waves can propagate with respective wavenumbers $\ell_1 = \omega/c_1$ (fast longitudinal wave), $\ell_2 = \omega/c_2$ (slow longitudinal wave) and $\ell_t = \omega/c_t$ (shear or transverse wave). The magnitudes of k_0 and $\left|\ell_j\right|$ ($j = 1, 2, t$) being of the same order, the low frequency assumption is to consider normalized wavenumbers such that $k_0 a << 1$ and $\left|\ell_j a\right| << 1$. With cylinders of radius $a = 0.3\,\mathrm{cm}$, the assumption $k_0 a << 1$ is equivalent to considering that the frequency $f << 78.4\,\mathrm{kHz}$. In comparison, some authors [6] considered considered steel rods with diameter $0.8\,\mathrm{mm}$.

The far-field scattering amplitude in the direction θ for an individual cylinder can be written as

$$f(\theta)=\sum_{n=-\infty}^{+\infty}T_n e^{in\theta}\,, \tag{4}$$

where T_n is the scattering coefficient for the mode n that depends on k_0a and the $\left|\ell_j a\right|$ ($j=1,2,t$). It is calculated by using boundary conditions indicated in Ref. 9. For cylindrical scatterers, $T_{-n}=T_n$. Since the coefficients T_n become small for $n>>k_0a$, as it happens also for elastic cylinders [7], the infinite sum in Eq. (4) can be truncated to retain only those of the terms with $|n|\le 1$. Thus, for forward scattering the far-field scattering amplitude reduces to $f(\theta=0)=T_0+2T_1$. For backward scattering it reduces to $f(\pi)=T_0-2T_1$. By considering Bessel and Hankel functions for $k_0a<<1$ and $\left|\ell_j a\right|<<1$ [8], the following low frequency expansions are obtained

$$T_0=\frac{i\pi}{4}\left(k_0a\right)^2\left(B_0^{(0)}-1\right)+O\left(\left(k_0a\right)^4\right), \tag{5}$$

$$T_1=\frac{i\pi}{4}\left(k_0a\right)^2 B_1^{(0)}+O\left(\left(k_0a\right)^4\right), \tag{6}$$

where

$$B_0^{(0)}=-\rho_0c_0^2\,\frac{(1+\gamma_1)\left[\left(\rho_2-\rho_{f2}\right)c_2^2-\rho_tc_t^2\right]-(1+\gamma_2)\left[\left(\rho_1-\rho_{f1}\right)c_1^2-\rho_tc_t^2\right]}{\rho_{f2}c_2^2\left(\rho_1c_1^2-\rho_tc_t^2\right)-\rho_{f1}c_1^2\left(\rho_2c_2^2-\rho_tc_t^2\right)}, \tag{7}$$

$$B_1^{(0)}=-\frac{\rho_0\left[(1+\gamma_2)\left(\rho_1-\rho_{f1}\right)-(1+\gamma_1)\left(\rho_2-\rho_{f2}\right)\right]-\left(\rho_1\rho_{f2}-\rho_2\rho_{f1}\right)}{\rho_0\left[(1+\gamma_2)\left(\rho_1-\rho_{f1}\right)-(1+\gamma_1)\left(\rho_2-\rho_{f2}\right)\right]+\left(\rho_1\rho_{f2}-\rho_2\rho_{f1}\right)}, \tag{8}$$

In the foregoing, the $\gamma_j s$, $\rho_j s$, $\rho_{fj}s$ are quantities related to the porous medium and are detailed in [9]. When the porosity of the medium $\beta\to 0$, the porous medium becomes an elastic one (that of the solid grains) of density ρ_s, with longitudinal wave velocity c_L and transverse wave velocity c_T. It follows that

$$B_0^{(0)}\to\frac{\rho_0c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}, \qquad B_1^{(0)}\to\frac{\rho_s-\rho_0}{\rho_s+\rho_0}. \tag{9}$$

and Eqs. (5)-(6) then agree with Eqs. (10-11) of Ref. [10]. The scattering coefficients T_0 and T_1 for a fluid cylinder of density ρ_f in which sound propagates with the velocity c_f , follow from Eqs. (9) and Eqs. (5-6) by setting $\rho_s\to\rho_f$, $c_T\to 0$ and $c_L\to c_f$ (see Eqs. (73 a-b) of Ref. [11]).

2.1 Waterman and Truell's formula

WT's formula for the effective wavenumber k_{WT} is a second order correction to Foldy's formula [12] in terms of n_0 the number of scatterers *per* unit area. It provides a satisfactory estimate for effective wavenumbers if the separation distance between nearest scatterers is sufficiently large. Such an assumption means that the porous fraction Φ in a representative surface is small (Φ is defined as the ratio of the surface occupied by the porous scatterers to the total surface of a representative surface of the medium that contains the scatterers). By retaining only the two first scattering coefficients in $f\left(0\right)$ and $f\left(\pi\right)$ and by using next Eqs. (5)-(6) without the $O\left(\left(k_0 a\right)^4\right)$ terms, one finds that

$$k_{WT}^2 = k_0^2\left[1+c\left(B_0^{(0)}+2B_1^{(0)}-1\right)+2c^2\left(B_0^{(0)}-1\right)B_1^{(0)}\right]. \tag{10}$$

When the porosity $\beta \to 0$, the substitution of Eqs. (8-9) in Eq. (14) yields

$$k_{WT}^2 = k_0^2\left[1+c\left(\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}+2\frac{\rho_s-\rho_0}{\rho_s+\rho_0}-1\right)+2c^2\left(\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}-1\right)\frac{\rho_s-\rho_0}{\rho_s+\rho_0}\right]. \tag{11}$$

For fluid cylinders, it may be verified by setting $\rho_s \to \rho_f$, $c_T \to 0$ and $c_L \to c_f$ that Eq. (11) tends to Eq. (75) of Ref. [11].

2.2 Linton and Martin's formula

Norris and Conoir [11] showed that LM formula of the effective wavenumber for cylindrical scatterers Eq. (3) can be obtained alternatively (under the assumption $n_0/k_0^2 << 1$, up to the second order in n_0^2 and by considering the restricted case of symmetric scattering far-field amplitudes $f\left(\theta\right)=f\left(-\theta\right)$), as

$$k_{LM}^2 = k_0^2\left[1+\delta_1\frac{n_0}{k_0^2}+\delta_2\frac{n_0^2}{k_0^2}\right], \tag{12}$$

where

$$\delta_1 = -4i\sum_n T_n = -4if\left(0\right), \qquad \delta_2 = -\frac{8}{k_0^2}\sum_{m,n}\left|m-n\right|T_m T_n\,. \tag{13}$$

By retaining only the two first scattering coefficients of $f\left(0\right)$ and by substituting Eqs. (5-6) in Eqs. (13) it follows that

$$k_{LM}^2 = k_0^2\left[1+c\left(B_0^{(0)}+2B_1^{(0)}-1\right)+2c^2 B_1^{(0)}\left(B_0^{(0)}+B_1^{(0)}-1\right)\right]. \tag{14}$$

The case $\beta \to 0$ follows by substituting Eq. (9) in Eq. (14) :

$$k_{LM}^2 = k_0^2\left[1+c\left(\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}+2\frac{\rho_s-\rho_0}{\rho_s+\rho_0}-1\right)+2c^2\left(\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}+\frac{\rho_s-\rho_0}{\rho_s+\rho_0}-1\right)\frac{\rho_s-\rho_0}{\rho_s+\rho_0}\right]. \tag{15}$$

When $\rho_s \to \rho_f$, $c_T \to 0$ and $c_L \to c_f$ one finds the LM wavenumber for fluid cylinders given by Eq. (75) of [11].

2.3 Independent Scattering Approximation (ISA)

ISA's wavenumber is deduced from Eqs. (2) or (12) by neglecting n_0^2 terms. It follows at once that

$$k_{ISA}^2 = k_0^2\left[1+c\left(B_0^{(0)}+2B_1^{(0)}-1\right)\right], \tag{16}$$

a result readily obtainable from Eqs. (10) or (14) by neglecting c^2 terms. From the foregoing, WT's and LM's formulas can be written in term of k_{ISA} as

$$k_{WT}^2 = k_{ISA}^2 + 2k_0^2 c^2 \left(B_0^{(0)} - 1\right) B_1^{(0)}, \tag{17}$$

$$k_{LM}^2 = k_{ISA}^2 + 2k_0^2 c^2 B_1^{(0)} \left(B_0^{(0)} + B_1^{(0)} - 1\right). \tag{18}$$

3 Wavenumbers from implicit methods

The Generalized Self Consistent Method (GSCM) applied by to WT's formula [5] and to LM's formula [11] follows the work of Ref. [13]. The far-field scattered amplitude $f\left(k_{eff}, \theta\right) = \sum_{n=-\infty}^{+\infty} T_n e^{in\theta}$ where T_n are the scattering coefficients corresponds to that of a three phase cylinder, *i.e.*, a porous cylinder of radius a coated by a fluid shell of outer radius $a_c > a$ immersed in an infinite outer medium of effective properties ρ_{eff} (density) and k_{eff} (wavenumber). The fluid shell has a density ρ_0 and wavenumber $k_0 = \omega / c_0$ (with c_0 the sound velocity). The concentration c of cylinders is defined as $c = n_0 \pi a^2 / \left(n_0 \pi a_c^2\right) = a^2 / a_c^2$ and $\rho_{eff} = (1-c)\rho_0 + c\rho$ with $\rho = (1-\beta)\rho_s + \beta\rho_0$ (ρ_s density of solid of the porous frame and β porosity of the porous medium). Three approaches are considered below, based upon self consistent schemes. The self consistent scheme applied to the ISA leads to the Coherent Potential Approximation (CPA). When applied to WT's or LM's formula, it generalizes the CPA.

The low frequency approximations of the first two scattering coefficients T_0 and T_1 can be written as

$$T_0 = \frac{i\pi}{4}\left(k_0 a_c\right)^2 \left(\bar{B}_0^{(0)} - \left(\frac{k_{eff}}{k_0}\right)^2\right) + O\left(\left(k_0 a_c\right)^4\right), \tag{19}$$

$$T_1 = \frac{i\pi}{4}\left(k_{eff} a_c\right)^2 \bar{B}_1^{(0)} + O\left(\left(k_0 a_c\right)^4\right), \tag{20}$$

where

$$\bar{B}_0^{(0)} = \frac{\rho_{eff}}{\rho_0}\left[1 + c\left(B_0^{(0)} - 1\right)\right], \tag{21}$$

$$\bar{B}_1^{(0)} = \frac{\left[(1+c)\rho_{eff} - (1-c)\rho_0\right] m^{(0)} + \left[(1-c)\rho_{eff} - (1+c)\rho_0\right] n^{(0)}}{\left[(1+c)\rho_{eff} + (1-c)\rho_0\right] m^{(0)} + \left[(1-c)\rho_{eff} + (1+c)\rho_0\right] n^{(0)}}. \tag{22}$$

In Eqs. (21)-(22), $m^{(0)}$ and $n^{(0)}$ do not depend on the concentration c since

$$m^{(0)} = \left(1+\gamma_1\right)\left(\rho_{f2} - \rho_2\right) - \left(1+\gamma_2\right)\left(\rho_{f2} - \rho_1\right), \qquad n^{(0)} = \rho_1 \rho_{f2} - \rho_2 \rho_{f1}. \tag{23}$$

Eqs. (4-5) (Eqs. (6-7), resp.) derive from Eqs. (21-22) if, at first, we set $\rho_{eff} = \rho_0$ in order to identify the outer effective fluid with the fluid itself, and then cancel the fluid shell by putting $c = 1$.

It may be noted that if $\beta \to 0$, Eqs. (21-22) become

$$\bar{B}_0^{(0)} = (1-c)\frac{\rho_{eff}}{\rho_0} + \frac{c\rho_{eff}c_0^2}{\rho_s\left(c_L^2 - c_T^2\right)}, \tag{24}$$

$$\bar{B}_1^{(0)} = -\frac{(1+c)\rho_0\left(\rho_{eff} - \rho_s\right) + (1-c)\left(\rho_s\rho_{eff} - \rho_0^2\right)}{(1+c)\rho_0\left(\rho_{eff} + \rho_s\right) + (1-c)\left(\rho_s\rho_{eff} + \rho_0^2\right)}, \tag{25}$$

with $\rho_{eff} = (1-c)\rho_0 + c\rho_s$. Furthermore, if $c=0$ then $\bar{B}_0^{(0)} = 1$ and $\bar{B}_1^{(0)} = 0$ implying $T_0 = T_1 = 0$, a result coherent with the absence of scattering. Returning to the general formulas, Eqs. (21-22), we see at first that Eq. (21) can be easily expanded in powers of c as

$$\bar{B}_0^{(0)} = 1 + c\left[\frac{\rho}{\rho_0} + \left(B_0^{(0)} - 2\right)\right] + c^2\left(\frac{\rho}{\rho_0} - 1\right)\left(B_0^{(0)} - 1\right). \tag{26}$$

Next, by assuming small enough c, the Taylor expansion of Eq. (22) up to the second order in c yields

$$\bar{B}_1^{(0)} = cM^{(0)} + c^2N^{(0)} + O\left(c^3\right), \tag{27}$$

where

$$M^{(0)} = \frac{\left(\rho+\rho_0\right)m^{(0)} + \left(\rho - 3\rho_0\right)n^{(0)}/\rho_0}{2\rho_0\left(m^{(0)} + n^{(0)}/\rho_0\right)}, \tag{28}$$

$$N^{(0)} = -\frac{\left(\rho-\rho_0\right)}{2\rho_0\left(m^{(0)} + n^{(0)}/\rho_0\right)}\left[m^{(0)} - n^{(0)}/\rho_0 - \frac{\left(\rho+\rho_0\right)m^{(0)} + \left(\rho-3\rho_0\right)n^{(0)}/\rho_0}{2\rho_0}\right]. \tag{29}$$

By evaluating these quantities when $\beta \to 0$, it is found that

$$M^{(0)} \to -\frac{\left(\rho_s - \rho_0\right)^2}{2\rho_0\left(\rho_s + \rho_0\right)}, \qquad N^{(0)} \to \frac{\left(\rho_s - \rho_0\right)^2}{4\rho_0^2}, \tag{30}$$

so that Eq. (27) becomes

$$\bar{B}_1^{(0)} = -\frac{c}{2\rho_0}\frac{\left(\rho_s - \rho_0\right)^2}{\rho_s + \rho_0} + c^2\frac{\left(\rho_s - \rho_0\right)^2}{4\rho_0^2} + \ldots \tag{31}$$

Eq. (31) agrees with the asymptotic expansion for the function $F\left(\rho_{eff}\right)$, given by [11]. These expansions will be used at subsequent points in order to compare effective wavenumbers obtained by an explicit approach with those from a self-consistent scheme (GSCM). The latter are calculated in the next sections.

3.1 Coherent Potential approximation (CPA)

The application of the GSCM to the ISA represented by Eq. (2) yields CPA's equation $f\left(k_{eff}, 0\right) = 0$. By considering only the first two scattering coefficients T_0 and T_1, one obtains

$$T_0 + 2T_1 = 0, \tag{32}$$

which, on account of Eqs. (19-20), leads to

$$\left(k_{eff}^2\right)_{CPA} = \frac{k_0^2\bar{B}_0^{(0)}}{1 - 2\bar{B}_1^{(0)}}, \tag{33}$$

with $\bar{B}_0^{(0)}$ and $\bar{B}_1^{(0)}$ given by Eqs. (24-25). Using next Eqs. (26-27), the Taylor expansion of $\left(k_{eff}\right)_{CPA}$ up to the second order of concentration can be expressed as

$$\begin{aligned}\left(k_{eff}^2\right)_{CPA} &= k_0^2\left[1+c\left(-2M^{(0)}+\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)\right.\\ &\left.+c^2\left(-2N^{(0)}-2M^{(0)}\left(2M^{(0)}+\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)+\left(B_0^{(0)}-1\right)\left(\frac{\rho}{\rho_0}-1\right)\right)\right]+O\left(c^3\right)\end{aligned}. \tag{34}$$

If $\beta \to 0$, we find

$$-2M^{(0)}+\frac{\rho}{\rho_0}+B_0^{(0)}-2 \to \frac{-2\left(\rho_0-\rho_s\right)}{\left(\rho_s+\rho_0\right)}+\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}-1, \tag{35}$$

and

$$\begin{aligned}&-2N^{(0)}-2M^{(0)}\left(2M^{(0)}+\frac{\rho}{\rho_0}+\left(B_0^{(0)}-2\right)\right)+\left(B_0^{(0)}-1\right)\left(\frac{\rho}{\rho_0}-1\right)\to\\ &\frac{\left(\rho_s-\rho_0\right)^2}{2\rho_0^2}-\frac{\left(\rho_s-\rho_0\right)^2}{\rho_0\left(\rho_s+\rho_0\right)}\left[\frac{\left(\rho_s-\rho_0\right)^2}{\rho_0\left(\rho_s+\rho_0\right)}+\frac{\rho_s}{\rho_0}+\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}-2\right]+\left(\frac{\rho_0 c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}-1\right)\left(\frac{\rho_s}{\rho_0}-1\right)\end{aligned} \tag{36}$$

which, after substitution in Eq. (34) gives an explicit effective wavenumber for elastic cylindrical cores up to the second order in concentration.

3.2 GSCM applied to Waterman and Truell's formula

The GSCM applied to WT's formula, Eq. (2), leads after some calculations to the identity

$$T_0+2T_1=\frac{8in_0}{k_{eff}^2}T_0T_1\,, \tag{37}$$

from which WT's wavenumber is derived (note that $n_0\pi a_c^2=1$ has been accounted for)

$$\left(k_{eff}^2\right)_{WT}=k_0^2\bar{B}_0^{(0)}\left(1+2\bar{B}_1^{(0)}\right).$$

Using next Eqs. (26-27), one gets

$$\begin{aligned}\left(k_{eff}^2\right)_{WT} &= k_0^2\left[1+c\left(-2M^{(0)}+\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)\right.\\ &\left.+c^2\left(-2N^{(0)}-2M^{(0)}\left(\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)+\left(\frac{\rho}{\rho_0}-1\right)\left(B_0^{(0)}-1\right)\right)\right]+O\left(c^3\right)\end{aligned}. \tag{38}$$

It is only from the second order in c that WT's wavenumber diverges from CPA. When $\beta \to 0$, we have the following limit for the factor of c^2 in Eq. (38)

$$\left(-2N^{(0)}-2M^{(0)}\left(\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)+\left(\frac{\rho}{\rho_0}-1\right)\left(B_0^{(0)}-1\right)\right)$$
$$\rightarrow\frac{\left(\rho_s-\rho_0\right)^2}{2\rho_0^2}-\frac{\left(\rho_s-\rho_0\right)^2}{\rho_0\left(\rho_s+\rho_0\right)}\left[\frac{\rho_s}{\rho_0}+\frac{\rho_0c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}-2\right]+\left(\frac{\rho_s}{\rho_0}-1\right)\left(\frac{\rho_0c_0^2}{\rho_s\left(c_L^2-c_T^2\right)}-1\right) .\quad(39)$$

3.3 GSCM applied to Linton and Martin's formula

If applied to Eq. (12), the GSCM leads to

$$T_0+2T_1-\frac{8in_0}{k_{eff}^2}\left(T_0+T_1\right)T_1=0\,. \quad (40)$$

which, after the substitution of Eqs. (19-20) and using $n_0\pi a_c^2=1$ yields LM's formula

$$\left(k_{eff}^2\right)_{LM}=k_0^2\bar{B}_0^{(0)}\frac{1-2\bar{B}_1^{(0)}}{1-2\bar{B}_1^{(0)2}}\,. \quad (41)$$

Expansion in powers of c of this latter equation leads to

$$\left(k_{eff}^2\right)_{LM}=k_0^2\left[1+c\left(-2M^{(0)}+\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)\right.$$
$$\left.+c^2\left(-2N^{(0)}-2M^{(0)}\left(-2M^{(0)}+\frac{\rho}{\rho_0}+B_0^{(0)}-2\right)+\left(\frac{\rho}{\rho_0}-1\right)\left(B_0^{(0)}-1\right)\right)\right]+O\left(c^3\right) \quad (42)$$

4 Conclusion

Asymptotic forms of the effective wavenumbers in a fluid medium containing a random distribution of porous scatterers have been obtained both for explicit and implicit methods. At the first order in concentration, explicit WT and LM give the same wavenumber. The same conclusion holds for CPA, implicit WT and LM. All these formulas diverge from each other at the second order in concentration. As the porosity of the cylinders tends to zero the well known cases of random elastic cylinders in fluid is found.

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