SOLITONS IN NON-LINEAR CYCLIC SYSTEM

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Abstract: In this paper we consider the case of a non-linear structure with cyclic symmetry which can be seen as approximation for bladed disk type structures. Using the multiple scale method, we show that soliton solutions are possible within such structure. The theoretical developments are illustrated with numerical simulations on a simple system with cubic non-linearities.
Keywords: *non-linear dynamics, cyclic system, travelling waves, soliton, multiple scales*

1 Introduction

In aeronautics, cyclic structures are of particular interest especially for turbo machinery. Due to increased flexibility, reduced clearance and other design imperatives, non-linear effects have to be taken into account in order to model the system's behaviour adequately.

In the linear case, it is well known that most of the eigen-modes are doublets and can be seen as two stationary waves (or two travelling waves with with opposite velocities). In the non-linear case, mode shapes may be affected by nonlinearity (non-similar modes) so that motion can localize on a small number of sectors Vakakis (1996); Georgiades et al. (2009). Those stationary solutions are often referred to as intrinsic localized modes (ILM) or discrete breathers (DB), and can be computed with numerical method such as the Harmonic Balance Method (HBM). In contrast to those stationary solutions, travelling waves can also be computed using the HBM, and it has been shown that such solution can bifurcate leading to localized travelling waves solutions (see eg Grolet and Thouverez (2012)).

Following results of solid state physics, another type of travelling waves solution, composed of a carrier wave modulated by an envelope, may also exist and are often referred to as solitons Remoissonet (1994). In mechanics, there is a small number of studies about the numerical or experimental generation of solitons. Most of them focus on simple systems such as discrete lattices Davies and Moon (1997); Konotop (1996). Here we aim at searching for soliton-like solutions in simple non-linear cyclic structure. This constitutes an attempt to introduce the concept of soliton in the structural mechanics community, and to show their relation with classical solutions such as obtained with the harmonic balance method.

2 Description of the procedure

We consider a cyclic lattice with on site cubic nonlinearity (simple approximation for bladed disk structure Vakakis (1993)). The equation of motion can be written as follow (with $a = \frac{2\pi}{N_s}$):

$$\ddot{u}_n + \omega_0^2 u_n + \xi u_n^3 = \frac{c_0^2}{a^2}(u_{n-1} + u_{n+1} - 2u_n) \tag{1}$$

linear analysis: Using cyclic symmetry condition leads to the following expression for the dispersion relation, the group velocity $c_g = \frac{dw_k}{dk}$ and the group velocity dispersion (GVD) $2\alpha = \frac{d^2 w_k}{dk^2}$:

$$w_k^2 = \omega_0^2 + 2\frac{c_0^2}{a^2}(1 - \cos(ka)), \quad c_g = \frac{dw_k}{dk} = \frac{1}{w_k}\frac{c_0^2}{a}\sin(ka), \quad 2\alpha = \frac{d^2 w_k}{dk^2} = \frac{1}{w_k}(c_0^2 \cos(ka) - c_g^2) \tag{2}$$

soliton solution: Here we search for a solution of the cyclic lattice under the form of a travelling wave with modulated amplitude Remoissonet (1994):

$$u_n = \Psi_n(X,T)e^{i(k(n-1)a-\omega_k t)} + c.c. = 2Re\big(\Psi_n(X,T)e^{i(k(n-1)a-w_k t)}\big) \tag{3}$$

where $X = \epsilon x$, $T = \epsilon t$ and $\Psi_n = \Psi(na) = \Psi(x)$. Using an order 2 Taylor expansion for Ψ_{n+1} (continuum approximation), keeping only one harmonic and supposing the non-linearity is of the order ϵ^2, we can show (see e.g Remoissonet (1994)) that the envelope should verify the following Non-Linear Schroedinger Equation (NLSE):

$$i\frac{d\Psi}{d\tau} + \alpha\frac{d^2\Psi}{d\eta^2} + \beta|\Psi|^2\,\Psi = 0 \tag{4}$$

with $\tau = \epsilon T = \epsilon^2 t$, $\eta = X - c_g T = \epsilon(x - c_g t)$, $\alpha = \frac{1}{2}\frac{d^2 w_k}{dk^2}$, $\beta = -\frac{1}{2\omega_k}3\xi$. There is a wide variety of solution for the NLS equation (envelope and hole solitons, discrete breathers, ..., see eg Karjanto and Van Groesen (2007)). Here we only consider envelope solitons which are known to exist for $\frac{\alpha}{\beta} \geq 0$ and which have the following expression: $\Psi(\eta,\tau) = V sech(V\sqrt{\frac{\beta}{2\alpha}}\eta)e^{i\frac{\beta}{2}V^2\tau}$. Going back to physical coordinates and dropping ϵ gives :

$$u_n(t) = 2V sech\big[V\sqrt{\frac{\beta}{2\alpha}}((n-1)a - c_g t)\big]\cos\big[k(n-1)a - \omega_k t + \frac{\beta}{2}V^2 t\big] \tag{5}$$

3 Numerical results, Discussion

Here we apply the previous developments to a numerical example of Eq.(1) with the following parameters: $n = 100$, $\omega_0^2 = 1$, $c_0^2 = 1$, $\xi = 0.1$. Using the expression in Eq.(5) one is able to compute initial conditions for generating envelope soliton in the cyclic lattice (provided that the wave number k is such that $\frac{\alpha}{\beta} \geq 0$).

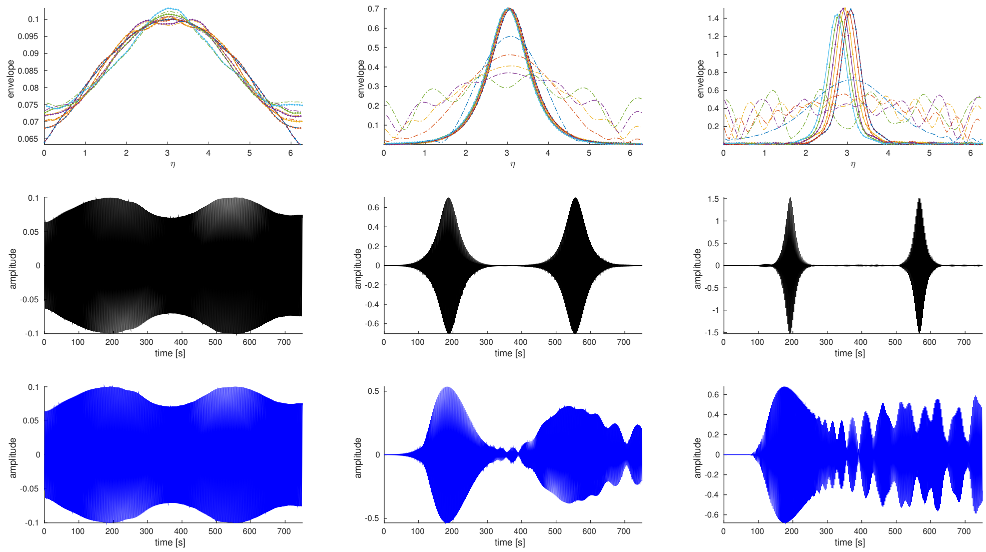

Figure 1: Exemple of soliton solution for system (1). line 1: envelope evolution in the moving frame (solid line: non linear, dash-dotted: linear). line 2: temporal evolution of nonlinear system. line 3: temporal evolution of linear system. (Results obtained for $k = 40$ for different amplitude col.1: $V = 0.1$. col.2: $V = 0.7$. col.3: $V = 1.5$)

Typical results are computed for $k = 40$ for three different amplitudes and are depicted on Fig.1. For low amplitude, the shape is not localized and non-linear results are close to the linear ones (the envelope behave almost like a constant). For middle amplitude, one can see clearly the effect of dispersion in the linear case: the envelope is flattening with time. In the non-linear case, the envelope does not change shape and is actually travelling with velocity c_g (envelope superimpose in the moving frame): it is a soliton type solution. For large amplitude, we can see that the shape remains rather well approximated by the envelope soliton (the envelope does not change), even if some error is made on the velocity (the envelope does not exactly superimpose in the moving frame).

The development presented here are valid as long as the continuum approximation holds (say for $N \geq 50$). For smaller number of sector, one would need to consider a discrete version of the non-linear Schroedinger equation (DNLSE). Further works should address this problem and should also investigate the link between stationary solution and soliton through variation of the group velocity.

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