

Solutions of Young-Laplace equation for partially saturated porous media. Stability analysis of capillary bridges.

G rard Gagneux and Olivier Millet

*LaSIE, UMR-CNRS 7356, Universit  de La Rochelle, avenue Michel Cr peau,
17042 La Rochelle cedex 1, France.*

Email: olivier.millet@univ-lr.fr ; gerard.maryse.gagneux@gmail.com

Abstract

In this work, we propose an original resolution of Young-Laplace equation for capillary doublets from an inverse problem. We establish a simple explicit criterion based on the observation of the contact point, the wetting angle and the gorge radius, to classify in an exhaustive way the nature of the surface of revolution. The true shape of the admissible static bridges surface is described by parametric equations; this way of expressing the profile is practical and well efficient for calculating the binding forces, areas and volumes. A stability analysis of the resulting capillary bridges, revisited through Vogel's stability criteria, is also developed.

Keywords: capillary bridge, inter-particle force, Young-Laplace equation, missing data

1 Introduction

We revisit from an inverse problem method the exact resolution of the Young-Laplace equation for capillary doublets. The missing information on the pressure deficiency Δp (which is often an unknown of the problem) will be restored without experimental device of suction control. Only the use of a digital camera with macrozoom allows to measure the suction $s = -\Delta p$ according to the observed value of the gorge radius y^* , then compared to critical bounds. The sought value s results immediately *via* a set of available explicit formulas.

We establish a simple criterion based on the observation of the contact point, of the wetting angle and the gorge radius, to classify in an exhaustive way the nature of the surface of revolution: portion of nodoid, of unduloid, both with concave or convex meridian, of catenoid, of cylinder or of circular profiles (toroid). In every case, we propose an exact parametric representation of the meridian based on the observed geometry of the boundary conditions and on a first integral of

Young-Laplace equation that traduces a conservation energy principle. Moreover, we prove that the inter-particle force may be evaluated on any section of the capillary bridge and constitutes a kind of specific invariant for surfaces of revolution, as in [2]. The proposed method leads to very convenient analytical expressions easy to use. The parameterization chosen enables a direct link between the half-axis of the conics and the observed data on the boundary. This approach avoids to have recourse to the simple toroidal approximation or to spline functions that do not respect (except in an exceptional theoretical case¹) the Young-Laplace equation [5][6][7].

The pertinence of the addressed approach will be put in a prominent position on several experimental results obtained on various geometries of capillary bridges. Moreover, the stability of solutions of Young-Laplace equations will be analyzed, based on the second variation criterion of the associated potential (minimization problem under constraints), revisited through Vogel's stability criteria. A theoretical stability criterion and conjectures on breakage will be proposed and discussed.

2 General resolution of Young-Laplace equation as an inverse problem

For capillary bridges of revolution, using a Cartesian representation of the profile, Young-Laplace equation may be written as :

$$\frac{y''}{(1+y'^2)^{3/2}} - \frac{1}{y\sqrt{1+y'^2}} = -\frac{\Delta p}{\gamma} =: H \quad (1)$$

with the conditions

$$y'' \geq 0, 0 < y \leq r \sin \delta \quad (2)$$

for convex profiles and

$$y'' \leq 0, y \geq r \sin \delta \quad (3)$$

for concave profiles, where δ denotes the filling angle (Fig.1) of the capillary bridge.

It can be proved, from the integration of Young-Laplace equation, that for any capillary bridges whose profile of revolution is governed by equation (1), the quantity

$$\lambda = \frac{y}{\sqrt{1+y'^2}} + \frac{Hy^2}{2} \quad (4)$$

is constant at all points of the profile. Moreover, the expression $2\pi\gamma\lambda$ represents the inter-particle capillary force F^{cap} which can be evaluated at any point of the profile as follows:

$$F^{cap} = 2\pi\gamma \left(\frac{y}{\sqrt{1+y'^2}} + \frac{Hy^2}{2} \right) \quad (5)$$

Coming back to the integration of Young-Laplace equation, from (4), we get:

¹A negative suction device, as will be shown later.

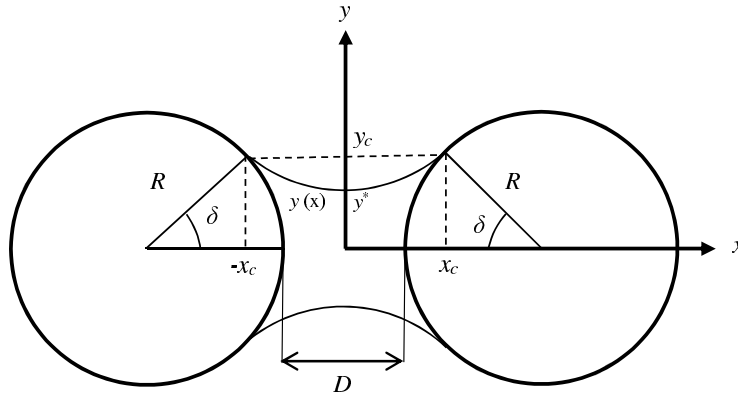


Figure 1: Cartesian representation of a convex profile of a capillary bridge.

$$1 + y'^2 = \frac{4y^2}{H^2 \left(y^2 - \frac{2\lambda}{H} \right)^2} \quad (6)$$

with $0 < y \leq r \sin \delta$ for convex profiles and $y \geq r \sin \delta$ for concave ones.

We then propose a simple criterion for identifying quickly with uniform treatment the nature of the meridian and providing parametric equations in order to calculate all the physical characteristics of the bridge (mean curvature, volume, free surface area, interparticle force). For given geometrical characteristics (grain radius r , interparticle distance D) of the grain-pair and given physical characteristics (surface tension γ , gorge radius y^* , filling angle δ , contact angle θ), an easy to use criterion allows instantaneously to identify the nature of the meridian and to give this parametric equations.

The results obtained are summarized in table 1. The other characteristics of the associated capillary bridges may be found in [3][4] for all the cases encountered. In the next section, we will only address the nodoid case with convex meridian, which is the most encountered case in practise.

3 Nodoid case with convex meridian

It is important to quote that we must perform a local study of Young-Laplace equation, involving Delaunay roulettes [1] (elliptic, parabolic and hyperbolic (with their particular cases)) which does not admit a global Cartesian representation. The key to understanding the link between analytical and geometrical approaches is the following result: concerning the Delaunay surfaces, the value of the constant mean curvature is given by the inverse of the semi-major axis a of the corresponding conic, with an appropriate specific sign. That yields consequently the strategic expression $\Delta p = \varepsilon \frac{\gamma}{a}$, $\varepsilon = -1$ for a nodoid, $\varepsilon = 1$ for an onduloid; the value a is taken infinite in the catenoid limit case, corresponding to the choice of conic as a parabola.

We limit our developments to the presentation of the nodoid case with convex meridian. We explicit the link between the geometric quantities characterizing the associated Delaunay roulette and the geometric data observed (y^* , δ , θ).

Criterion	Nature of the free surface	Nature of the meridian	Suction s
Convex meridian $r \sin \delta \sin(\delta + \theta) < y^* < r \sin \delta$	Nodoid	Delaunay hyperbolic roulette	$s > 0$
Convex meridian $y^* = r \sin \delta \sin(\delta + \theta)$	Catenoid	Catenary	$s = 0$
Convex meridian $0 < y^* < r \sin \delta \sin(\delta + \theta)$	Unduloid	Delaunay elliptic roulette	$s < 0$
Convex-concave meridian $y^* = r \sin \delta$	Right circular cylinder	Straight segment	$s < 0$
Concave meridian $r \sin \delta < y^* < \frac{r \sin \delta}{\sin(\delta + \theta)}$	Nodoid	Delaunay hyperbolic roulette	$s < 0$
Concave meridian $y^* = \frac{r \sin \delta}{\sin(\delta + \theta)}$	Portion of sphere (toroid)	Portion of a circle	$s < 0$
Concave meridian $y^* > \frac{r \sin \delta}{\sin(\delta + \theta)}$	Unduloid	Delaunay elliptic roulette	$s < 0$

Table 1: Synoptic table for identifying the capillary bridge and its properties

When the geometric data observed on the capillary bridge are such that the gorge radius y^* satisfies the following inequalities:

$$r \sin \delta \sin(\delta + \theta) < y^* < r \sin \delta \quad (7)$$

the free surface of revolution is a portion of nodoid whose meridian is an arc of Delaunay hyperbolic roulette. We have $H > 0$ and $\lambda > 0$ and associated parametric equations can be written as:

$$\left\{ \begin{array}{l} x(t) = \frac{b^2}{a} \int_0^t \frac{\cos u \, du}{(e + \cos u) \sqrt{e^2 - \cos^2 u}} \\ y(t) = b \sqrt{\frac{e - \cos t}{e + \cos t}}, \quad t \in [-\tau, \tau] \end{array} \right. \quad (8)$$

where $\tau = \tau(r, y^*, \delta, \theta)$ is the unique solution in $]0, \frac{\pi}{2}[$ of the equation $y(\tau) = r \sin \delta$, *i.e.*

$$\tau = \arccos \left(e \frac{b^2 - r^2 \sin^2 \delta}{b^2 + r^2 \sin^2 \delta} \right)$$

The values of the half-axis $a > 0$, $b > 0$, and of the eccentricity $e = \frac{c}{a}$ with $c = \sqrt{a^2 + b^2}$ of the associated hyperbola may be expressed with respect to (r, y^*, δ, θ) as:

$$a = a(r, y^*, \delta, \theta) = \frac{1}{2} \frac{r^2 \sin^2 \delta - y^{*2}}{y^* - r \sin \delta \sin(\delta + \theta)}, \quad \frac{\Delta p}{\gamma} = -\frac{1}{a} \quad (9)$$

$$\frac{\Delta p}{\gamma} < 0, \quad \Delta p = \Delta p(r, y^*, \delta, \theta) = -2\gamma \frac{y^* - r \sin \delta \sin(\delta + \theta)}{r^2 \sin^2 \delta - y^{*2}}$$

$$b^2 = b^2(r, y^*, \delta, \theta) = y^* r \sin \delta \frac{r \sin \delta - y^* \sin(\delta + \theta)}{y^* - r \sin \delta \sin(\delta + \theta)} \quad (10)$$

4 Stability analysis

A major focus should be done on determination of stability criteria for the capillary doublets whose exact geometry is determined through table 1. To do this, we must develop a strategy to identify exactly the local constrained minima of the free surface area and overcome a serious and former misunderstanding specifically on this topic. Very roughly speaking, a constant mean curvature free surface is called stable if the quadratic form arising from the second variation of the capillary surface area in volume preserving directions is positive. In fact, in Hilbert spaces of infinite dimension (the functional framework best suited through the Sobolev spaces rather than the incommodious Banach spaces \mathcal{C}^k), this is assuredly not correct and things are much finer and need to be clarified (*cf.* R. Finn's counterexample in an editorial cautionary comment on *Stability of a Catenoidal Liquid Bridge* by L. Zhou, pointing out the necessity to revisit the entire problem of stability and, by providing proofs *via* a relevance feedback approach, justify the results conceptually as analytical theorems on stability). This leads to Vogel's stability criteria established in 1996 [8].

Furthermore, alternative methods to spectral framework are available: the variational inequality approach to finer stability criteria, the bifurcation theory and the Poincaré-Maddocks theorem, *etc.*

5 Conclusions

We have revisited criteria for identifying quickly the nature of the meridian and providing exact parametric equations in order to calculate easily all the physical characteristics of capillary bridges between two grains (mean curvature, volume, surface area, inter-particle force).

For given geometrical and physical characteristics, an easy to use criterion based on the observation of the contact point, of the wetting angle and the gorge radius, allows instantaneously to identify the nature of the meridian and to give parametric equations. The approach proposed relies on an original inverse problem method to solve Young-Laplace equation. The missing information on the pressure deficiency Δp (which is often an unknown of the problem) is restored without experimental device of suction control. Only the use of a digital camera with macrozoom allows to measure the suction s according to the observed value of the gorge radius which is compared to critical bounds. The sought value of the suction results immediately *via* a set of available explicit formulas. Moreover, we have proved that the inter-particle force may be evaluated on any section of the capillary bridge and constitutes a kind of specific invariant for axisymmetric capillary bridges.

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