

# Uniqueness of iterative positive solutions for the singular infinite-point $p$ -Laplacian fractional differential system via sequential technique\*

Limin Guo<sup>a,1</sup>, Lishan Liu<sup>b,c</sup>, Yanqing Feng<sup>a</sup>

<sup>a</sup>School of Science, Changzhou Institute of Technology,  
Changzhou 213002, Jiangsu, China  
[guolimin811113@163.com](mailto:guolimin811113@163.com); [fengyq@czu.cn](mailto:fengyq@czu.cn)

<sup>b</sup>School of Mathematical Sciences, Qufu Normal University,  
Qufu 273165, Shandong  
[mathlls@163.com](mailto:mathlls@163.com)

<sup>c</sup>Department of Mathematics and Statistics, Curtin University,  
Perth, WA6845, Australia

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**Abstract.** By sequential techniques and mixed monotone operator, the uniqueness of positive solution for singular  $p$ -Laplacian fractional differential system with infinite-point boundary conditions is obtained. Green's function is derived, and some useful properties of Green's function are obtained. Based on these new properties, the existence of unique positive solutions is established, moreover, an iterative sequence and a convergence rate are given, which are important for practical application, and an example is given to demonstrate the validity of our main results.

**Keywords:** fractional differential system, iterative positive solution, sequential techniques, mixed monotone operator, singular problem.

## 1 Introduction

Boundary value problems for nonlinear fractional differential equations arise from the studies of complex problems in many disciplinary areas such as aerodynamics, fluid flows, electrodynamics of complex medium, electrical networks, rheology, polymer rheology, economics, biology chemical physics, control theory, signal and image processing, blood flow phenomena, and so on. There has been a significant development in the study of fractional differential equations in recent years. For an extensive collection of such literature, readers can refer to [1, 3, 4, 7, 11–17] and the references therein. Multi-point boundary

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<sup>1</sup>Corresponding author.

value problems are a significant development for fractional differential equation, and the system in this paper is infinite-points boundary value problem, and about values at infinite-points are involved in the boundary conditions that we refer the reader to [3, 4, 11] and the references therein. For  $p$ -Laplacian fractional differential equation, we refer the reader to [9, 12]. In this paper, we consider the following singular infinite-point  $p$ -Laplacian nonlinear fractional differential equation system:

$$\begin{aligned}
 & D_{0+}^{\alpha} (\varphi_{p_1} (D_{0+}^{\gamma} u))(t) + \lambda^{1/(q_1-1)} f(t, u(t), D_{0+}^{\mu_1} u(t), D_{0+}^{\mu_2} u(t), \dots, \\
 & \quad D_{0+}^{\mu_{n-2}} u(t), v(t)) = 0, \quad 0 < t < 1, \\
 & D_{0+}^{\beta} (\varphi_{p_2} (D_{0+}^{\delta} v))(t) + \mu^{1/(q_2-1)} g(t, u(t), D_{0+}^{\eta_1} u(t), D_{0+}^{\eta_2} u(t), \dots, \\
 & \quad D_{0+}^{\eta_{m-2}} u(t)) = 0, \quad 0 < t < 1, \\
 & u(0) = D_{0+}^{\mu_i} u(0) = 0, \quad D_{0+}^{\gamma} u(0) = 0, \quad i = 1, 2, \dots, n - 2, \\
 & D_{0+}^{\mu_{n-2} + r_1} u(1) = \sum_{j=1}^{\infty} \eta_j D_{0+}^{\mu_{n-2} + r_2} u(\xi_j), \\
 & v(0) = D_{0+}^{\eta_i} v(0) = 0, \quad D_{0+}^{\delta} v(0) = 0, \quad i = 1, 2, \dots, m - 2, \\
 & D_{0+}^{\eta_{m-2} + \bar{r}_1} v(1) = \sum_{j=1}^{\infty} \bar{\eta}_j D_{0+}^{\eta_{m-2} + \bar{r}_2} v(\bar{\xi}_j),
 \end{aligned} \tag{1}$$

where  $\alpha, \beta, \gamma, \delta, \mu_{\kappa}, \eta_{\varrho} \in \mathbb{R}_+^1$  ( $\kappa = 1, 2, \dots, n - 2$ ;  $\varrho = 1, 2, \dots, m - 2$ ),  $n, m \in \mathbb{N}$  (natural number set),  $n, m \geq 2$ , and  $1/2 < \alpha, \beta \leq 1, n - 1 < \gamma \leq n, m - 1 < \delta \leq m, p_1, \bar{p}_1 \in [2, n - 2]$ .  $p$ -Laplacian operator  $\varphi_{p_i}$  is defined as  $\varphi_{p_i}(s) = |s|^{p_i-2}s, p_i, q_i > 1, 1/p_i + 1/q_i = 1$  ( $i = 1, 2$ ),  $n - 1 - \kappa < \gamma - \mu_{\kappa} \leq n - \kappa, m - 1 - \varrho < \delta - \eta_{\varrho} \leq m - \varrho$  ( $\kappa = 1, 2, \dots, n - 2$ ;  $\varrho = 1, 2, \dots, m - 2$ ),  $\eta_i \leq \mu_{n-2}$  ( $i = 1, 2, \dots, m - 2$ ), and  $0 < \eta, \vartheta \leq 1$ .  $\lambda, \mu, \chi, t > 0$  are parameters,  $f \in C((0, 1) \times (0, +\infty)^n, \mathbb{R}_+^1)$  and  $f(t, x_1, x_2, \dots, x_n)$  has singularity at  $x_i = 0$  ( $i = 1, 2, \dots, n$ ) and  $t = 0, 1, g \in C((0, 1) \times \mathbb{R}_+^{m-1}, \mathbb{R}_+^1), h, a \in C(0, 1)$  with  $\int_0^{\eta} \chi t^{\gamma - \mu_{n-2} - 1} h(t) dA(t) < 1, \int_0^{\vartheta} \mu t^{\delta - \eta_{m-2} - 1} a(t) dB(t) < 1$ .  $A, B$  are functions of bounded variation,  $\int_0^{\eta} h(t) D_{0+}^{\mu_{n-2}} u(t) dA(t), \int_0^{\vartheta} a(t) D_{0+}^{\eta_{m-2}} v(t) dB(t)$  denote the Riemann–Stieltjes integral with respect to  $A$  and  $B$ .  $D_{0+}^{\alpha} u, D_{0+}^{\beta} v, D_{0+}^{\gamma} u, D_{0+}^{\delta} v, D_{0+}^{\mu_{\kappa}} u, D_{0+}^{\eta_{\varrho}} v$  are the standard Riemann–Liouville derivative.

Motivated by the results above, we utilize fixed point theorem to investigate the existence results of positive solution of BVP (1). Compared with our paper [6],  $p_1 \neq p_2$  in  $p$ -Laplacian system of (1), but  $p_1 = p_2$  in  $p$ -Laplacian system in [6]. Compared with our papers [5, 6], values at infinite points are involved in the boundary conditions of the boundary value problem (1), and the method that we used in this paper is sequential techniques. Compared with [3, 4, 11], the method, which we used is sequential techniques, and the positive solution, which we obtained, is iterative solution. Compared with [16, 17], fractional derivatives are involved in the nonlinear terms, and the solution we obtained is iterative solution, the result is accurate.

Next we list the following assumptions for convenience.

- (S1)  $f(t, x_1, x_2, \dots, x_n) = \phi(t, x_1, x_2, \dots, x_n) + \psi(t, x_1, x_2, \dots, x_n)$ , where  $\phi : (0, 1) \times [0, +\infty)^n \rightarrow \mathbb{R}_+^1$  and  $\psi : (0, 1) \times (0, +\infty)^n \rightarrow \mathbb{R}_+^1$  are continuous, and

for any fixed  $t \in [0, 1]$ ,  $\phi(t, x_1, x_2, \dots, x_n)$  is nondecreasing and  $\psi(t, x_1, x_2, \dots, x_n)$  is nonincreasing on  $x_i > 0$  ( $i = 1, 2, \dots, n$ ), respectively.

(S2) There exists  $\sigma \in (0, 1)$  such that, for  $x_i > 0$  ( $i = 1, 2, \dots, n$ ) and for any  $l \in (0, 1), t \in (0, 1)$ ,

$$\begin{aligned} \phi(t, lx_1, lx_2, \dots, lx_n) &\geq l^{\sigma^{1/(q_1-1)}} \phi(t, x_1, x_2, \dots, x_n), \\ \psi(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_n) &\geq l^{\sigma^{1/(q_1-1)}} \psi(t, x_1, x_2, \dots, x_n). \end{aligned}$$

(S3)  $g \in C([0, 1] \times \mathbb{R}_+^{m-1}, \mathbb{R}_+)$  is nondecreasing on  $x_i > 0$  ( $i = 1, 2, \dots, m - 1$ ), and  $g(t, 1, \dots, 1) \neq 0$ . Moreover, for  $t \in (0, 1)$ , there exists  $\varsigma \in [0, 1]$  such that

$$g(t, lx_1, lx_2, \dots, lx_{m-1}) \geq l^{\varsigma^{1/(q_2-1)}} g(t, x_1, x_2, \dots, x_{m-1}), \quad l \in (0, 1).$$

(S4)  $0 < \int_0^1 \phi^2(\tau, 1, 1, \dots, 1) d\tau < +\infty, 0 < \int_0^1 \tau^{-2(\gamma-1)\sigma^{1/(q_1-1)}} \psi^2(\tau, 1, 1, \dots, 1) d\tau < +\infty, 0 < \int_0^1 g^2(\tau, 1, 1, \dots, 1) d\tau < +\infty$ .

**Remark 1.** According to (S2) and (S3), for all  $x_i > 0$  ( $i = 1, 2, \dots, n$ ),  $t \in (0, 1), l \geq 1$ , we have

$$\begin{aligned} g(t, lx_1, lx_2, \dots, lx_{m-1}) &\leq l^{\varsigma^{1/(q_2-1)}} g(t, x_1, x_2, \dots, x_{m-1}), \\ \phi(t, lx_1, lx_2, \dots, lx_n) &\leq l^{\sigma^{1/(q_1-1)}} \phi(t, x_1, x_2, \dots, x_n), \\ \psi(t, l^{-1}x_1, l^{-1}x_2, \dots, l^{-1}x_n) &\leq l^{\sigma^{1/(q_1-1)}} \psi(t, x_1, x_2, \dots, x_n). \end{aligned}$$

## 2 Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas, which are useful for the following research and can be found in the recent literature such as [8, 10].

In what follows, we will give the expression of the linear problems.

**Lemma 1.** Let  $y, \bar{y} \in L^1(0, 1) \cap C(0, 1)$ , then the equation of the BVPs

$$\begin{aligned} -D_{0+}^{\gamma-\mu_n-2} u(t) &= y(t), \quad 0 < t < 1, \\ u(0) = 0, \quad D_{0+}^{\tau_1} u(1) &= \sum_{j=1}^{\infty} \eta_j D_{0+}^{\tau_2} u(\xi_j), \\ -D_{0+}^{\delta-\eta_m-2} v(t) &= \bar{y}(t), \quad 0 < t < 1, \\ v(0) = 0, \quad D_{0+}^{\bar{\tau}_1} v(1) &= \sum_{j=1}^{\infty} \bar{\eta}_j D_{0+}^{\bar{\tau}_2} v(\bar{\xi}_j) \end{aligned} \tag{2}$$

has integral representation

$$u(t) = \int_0^1 G(t, s)y(s) ds, \quad v(t) = \int_0^1 H(t, s)\bar{y}(s) ds, \tag{3}$$

respectively, where

$$G(t, s) = \frac{1}{\Delta \Gamma(\gamma - \mu_{n-2})} \times \begin{cases} \Gamma(\gamma - \mu_{n-2})t^{\gamma - \mu_{n-2} - 1}P(s)(1 - s)^{\gamma - \mu_{n-2} - r_1 - 1} \\ - \Delta(t - s)^{\gamma - \mu_{n-2} - 1}, & 0 \leq s \leq t \leq 1, \\ \Gamma(\gamma - \mu_{n-2})t^{\gamma - \mu_{n-2} - 1}P(s)(1 - s)^{\gamma - \mu_{n-2} - r_1 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (4)$$

$$H(t, s) = \frac{1}{\bar{\Delta} \Gamma(\delta - \eta_{m-2})} \times \begin{cases} \Gamma(\delta - \eta_{m-2})t^{\delta - \eta_{m-2} - 1}\bar{P}(s)(1 - s)^{\delta - \eta_{m-2} - \bar{r}_1 - 1} \\ - \bar{\Delta}(t - s)^{\delta - \eta_{m-2} - 1}, & 0 \leq s \leq t \leq 1, \\ \Gamma(\delta - \eta_{m-2})t^{\delta - \eta_{m-2} - 1}\bar{P}(s)(1 - s)^{\delta - \eta_{m-2} - \bar{r}_1 - 1}, & 0 \leq t \leq s \leq 1, \end{cases} \quad (5)$$

in which

$$P(s) = \frac{1}{\Gamma(\gamma - \mu_{n-2} - r_1)} - \frac{1}{\Gamma(\gamma - \mu_{n-2} - r_2)} \sum_{s \leq \xi_j} \eta_j \left( \frac{\xi_j - s}{1 - s} \right)^{\gamma - \mu_{n-2} - r_2 - 1} (1 - s)^{r_1 - r_2},$$

$$\bar{P}(s) = \frac{1}{\Gamma(\delta - \eta_{m-2} - \bar{r}_1)} - \frac{1}{\Gamma(\delta - \eta_{m-2} - \bar{r}_2)} \sum_{s \leq \bar{\xi}_j} \bar{\eta}_j \left( \frac{\bar{\xi}_j - s}{1 - s} \right)^{\delta - \eta_{m-2} - \bar{r}_1 - 1} (1 - s)^{\bar{r}_1 - \bar{r}_2},$$

$$\Delta = \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_1)} - \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_2)} \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma - \mu_{n-2} - r_2 - 1},$$

$$\bar{\Delta} = \frac{\Gamma(\delta - \eta_{m-2})}{\Gamma(\delta - \eta_{m-2} - \bar{r}_1)} - \frac{\Gamma(\delta - \eta_{m-2})}{\Gamma(\delta - \eta_{m-2} - \bar{r}_2)} \sum_{j=1}^{\infty} \bar{\eta}_j \bar{\xi}_j^{\delta - \eta_{m-2} - \bar{r}_2 - 1}.$$

*Proof.* We only need to prove (4), the proof of (5) is similar with the proof of (4). By means of the definition of fractional differential integral, we can reduce (2) to an equivalent integral equation

$$u(t) = -I_{0+}^{\gamma - \mu_{n-2}} y(t) + C_1 t^{\gamma - \mu_{n-2} - 1} + C_2 t^{\gamma - \mu_{n-2} - 2}$$

for  $C_1, C_2 \in \mathbb{R}$ . From  $u(0) = 0$  we have  $C_2 = 0$ . Consequently, we get

$$u(t) = C_1 t^{\gamma - \mu_{n-2} - 1} - I_{0+}^{\gamma - \mu_{n-2}} y(t).$$

By some properties of the fractional integrals and fractional derivatives, we have

$$\begin{aligned}
 D_{0+}^{r_1} u(t) &= C_1 \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_1)} t^{\gamma - \mu_{n-2} - r_1 - 1} - I_{0+}^{\gamma - \mu_{n-2} - r_1} y(t), \\
 D_{0+}^{r_2} u(t) &= C_1 \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_2)} t^{\gamma - \mu_{n-2} - r_2 - 1} - I_{0+}^{\gamma - \mu_{n-2} - r_2} y(t).
 \end{aligned}
 \tag{6}$$

On the other hand, combining  $D_{0+}^{r_1} u(1) = \sum_{j=1}^{\infty} \eta_j D_{0+}^{r_2} u(\xi_j)$  with (6), we get

$$\begin{aligned}
 C_1 &= \int_0^1 \frac{(1-s)^{\gamma - \mu_{n-2} - r_1 - 1}}{\Gamma(\gamma - \mu_{n-2} - r_1) \Delta} y(s) \, ds - \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} \frac{(\xi_j - s)^{\gamma - \mu_{n-2} - r_2 - 1}}{\Gamma(\gamma - \mu_{n-2} - r_2) \Delta} y(s) \, ds \\
 &= \int_0^1 \frac{(1-s)^{\gamma - \mu_{n-2} - r_1 - 1} P(s)}{\Delta} y(s) \, ds,
 \end{aligned}$$

where  $P(s)$ ,  $\Delta$  are as (4). Hence,

$$\begin{aligned}
 u(t) &= C_1 t^{\gamma - \mu_{n-2} - 1} - I_{0+}^{\gamma - \mu_{n-2}} y(t) \\
 &= - \int_0^t \frac{\Delta(t-s)^{\gamma - \mu_{n-2} - 1}}{\Gamma(\gamma - \mu_{n-2}) \Delta} y(s) \, ds + \int_0^1 \frac{(1-s)^{\gamma - \mu_{n-2} - r_1 - 1} t^{\gamma - \mu_{n-2} - 1} P(s)}{\Delta} y(s) \, ds \\
 &= \int_0^1 G(t, s) y(s) \, ds.
 \end{aligned}$$

Therefore, (4) holds, similarly, (5) holds. □

**Lemma 2.** Let  $\Delta, \bar{\Delta} > 0$  for  $s \in [0, 1]$ , then the Green functions defined by (3) satisfies:

(i)  $G, H : [0, 1] \times [0, 1] \rightarrow \mathbb{R}_+$  are continuous, and  $G(t, s), H(t, s) > 0$  for all  $t, s \in (0, 1)$ ;

(ii) 
$$\frac{1}{\Gamma(\gamma - \mu_{n-2})} t^{\gamma - \mu_{n-2} - 1} j(s) \leq G(t, s) \leq a^* t^{\gamma - \mu_{n-2} - 1}, \tag{7}$$

$$\frac{1}{\Gamma(\delta - \eta_{m-2})} t^{\delta - \eta_{m-2} - 1} \bar{j}(s) \leq H(t, s) \leq \bar{a}^* t^{\delta - \eta_{m-2} - 1}, \tag{8}$$

where

$$\begin{aligned}
 j(s) &= (1-s)^{\gamma - \mu_{n-2} - r_1 - 1} [1 - (1-s)^{r_1}], \\
 \bar{j}(s) &= (1-s)^{\delta - \eta_{m-2} - \bar{r}_1 - 1} [1 - (1-s)^{\bar{r}_1}], \\
 a^* &= \frac{1}{\Delta \Gamma(\gamma - \mu_{n-2} - r_1)}, \quad \bar{a}^* = \frac{1}{\Delta \Gamma(\delta - \eta_{m-2} - \bar{r}_1)},
 \end{aligned}$$

$\Delta, \bar{\Delta}$  are defined as in Lemma 1.

*Proof.* Let

$$G_0(t, s) = \frac{1}{\Gamma(\gamma - \mu_{n-2})} \times \begin{cases} t^{\gamma - \mu_{n-2} - 1} (1 - s)^{\gamma - \mu_{n-2} - r_1 - 1} - (t - s)^{\gamma - \mu_{n-2} - 1}, & 0 \leq s \leq t \leq 1, \\ t^{\gamma - \mu_{n-2} - 1} (1 - s)^{\gamma - \mu_{n-2} - r_1 - 1}, & 0 \leq t \leq s \leq 1. \end{cases}$$

From [7], for  $r_1 \in [2, n - 2]$ , we have

$$0 \leq t^{\gamma - \mu_{n-2} - 1} (1 - s)^{\gamma - \mu_{n-2} - r_1 - 1} [1 - (1 - s)^{r_1}] \leq \Gamma(\gamma - \mu_{n-2}) G_0(t, s) \leq t^{\gamma - \mu_{n-2} - 1} (1 - s)^{\gamma - \mu_{n-2} - r_1 - 1}. \tag{9}$$

By direct calculation, we get  $P'(s) \geq 0$ ,  $s \in [0, 1]$ , and so  $P(s)$  is nondecreasing with respect to  $s$ . For  $r_2 \leq r_1$ ,  $r_2 \in [2, n - 2]$ ,  $s \in [0, 1]$ , we get

$$\begin{aligned} \Gamma(\gamma - \mu_{n-2}) P(s) &= \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_1)} - \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_2)} \sum_{s \leq \xi_j} \eta_j \left( \frac{\xi_j - s}{1 - s} \right)^{\gamma - \mu_{n-2} - r_2 - 1} (1 - s)^{r_1 - r_2} \\ &\geq \Gamma(\gamma - \mu_{n-2}) P(0) \\ &= \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_1)} - \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{n-2} - r_2)} \sum \eta_j \xi_j^{\gamma - \mu_{n-2} - r_2 - 1} \\ &= \Delta. \end{aligned}$$

By (4) and (9), we have

$$\begin{aligned} \Delta \Gamma(\gamma - \mu_{n-2}) G(t, s) &\geq \Delta \Gamma(\gamma - \mu_{n-2}) G_0(t, s) \\ &\geq \Delta t^{\gamma - \mu_{n-2} - 1} (1 - s)^{\gamma - \mu_{n-2} - r_1 - 1} [1 - (1 - s)^{r_1}]. \end{aligned}$$

Clearly,  $\Delta \Gamma(\gamma - \mu_{n-2}) G(t, s) \leq \Gamma(\gamma - \mu_{n-2}) t^{\gamma - \mu_{n-2} - 1} P(s) (1 - s)^{\gamma - \mu_{n-2} - r_1 - 1}$ . So the proof of (7) is completed. Similarly, (8) also holds.  $\square$

To study the PFDE (1), in what follows, we consider the associated linear PFDE

$$\begin{aligned} D_{0+}^\alpha (\varphi_{p_1} (D_{0+}^{\gamma - \mu_{n-2}} x))(t) + \rho(t) &= 0, \quad 0 < t < 1, \\ x(0) = 0, \quad D_{0+}^\gamma x(0) = 0, \quad D_{0+}^{r_1} x(1) &= \sum_{j=1}^\infty \eta_j D_{0+}^{r_2} x(\xi_j), \end{aligned} \tag{10}$$

and

$$\begin{aligned} D_{0+}^\beta (\varphi_{p_2} (D_{0+}^{\delta - \eta_{m-2}} y))(t) + \bar{\rho}(t) &= 0, \quad 0 < t < 1, \\ y(0) = 0, \quad D_{0+}^\delta y(0) = 0, \quad D_{0+}^{\bar{r}_1} y(1) &= \sum_{j=1}^\infty \bar{\eta}_j D_{0+}^{\bar{r}_2} y(\bar{\xi}_j). \end{aligned} \tag{11}$$

**Lemma 3.** *The PFDE (10), (11) has the following unique positive solution:*

$$x(t) = \int_0^1 G(t, s) \left( \int_0^s \bar{a}(s - \tau)^{\alpha-1} \rho(\tau) \, d\tau \right)^{q_1-1} \, ds, \quad t \in [0, 1], \tag{12}$$

$$y(t) = \int_0^1 H(t, s) \left( \int_0^s \bar{b}(s - \tau)^{\beta-1} \bar{\rho}(\tau) \, d\tau \right)^{q_2-1} \, ds, \quad t \in [0, 1], \tag{13}$$

respectively, in which  $\bar{a} = 1/\Gamma(\alpha)$ ,  $\bar{b} = 1/\Gamma(\beta)$ .

*Proof.* Let  $h = D_{0+}^{\gamma-\mu_{n-2}}x, k = \varphi_{p_1}(h)$ , then the solution of the initial value problem

$$D_{0+}^{\alpha}k(t) + \rho(t) = 0, \quad 0 < t < 1, \quad k(0) = 0$$

is given by  $k(t) = C_1 t^{\alpha-1} - I_{0+}^{\alpha}\rho(t)$ ,  $t \in [0, 1]$ . By the relations  $k(0) = 0$ , we have  $C_1 = 0$ , and hence

$$k(t) = -I_{0+}^{\alpha}\rho(t), \quad t \in [0, 1]. \tag{14}$$

By  $D_{0+}^{\gamma}x = h, h = \varphi_{p_1}^{-1}(k)$ , we have from (14) that the solution of (10) satisfies

$$\begin{aligned} D_{0+}^{\gamma-\mu_{n-2}}x(t) &= \varphi_{p_1}^{-1}(-I_{0+}^{\alpha}\rho(t)), \quad 0 < t < 1, \\ x(0) = 0, \quad D_{0+}^{r_1}x(1) &= \sum_{j=1}^{\infty} \eta_j D_{0+}^{r_2}x(\xi_j). \end{aligned} \tag{15}$$

By (3), the solution of Eq. (15) can be written as

$$x(t) = - \int_0^1 G(t, s) \varphi_{p_1}^{-1}(-I_{0+}^{\alpha}\rho(s)) \, ds, \quad t \in [0, 1].$$

Since  $\rho(s) \geq 0, s \in [0, 1]$ , we have  $\varphi_{p_1}^{-1}(-I_{0+}^{\alpha}\rho(s)) = -(I_{0+}^{\alpha}\rho(s))^{q_1-1}, s \in [0, 1]$ , which implies that the solution of Eq.(10) is (12). Similarly, the solution of Eq.(11) is (13).  $\square$

Let  $u(t) = I_{0+}^{\mu_{n-2}}x(t), v(t) = I_{0+}^{\eta_{m-2}}y(t)$ , problem (1) can turn into the following modified problem of the PFDE (16):

$$\begin{aligned} D_{0+}^{\alpha}(\varphi_{p_1}(D_{0+}^{\gamma-\mu_{n-2}}x))(t) + \lambda^{1/(q_1-1)}f(t, I_{0+}^{\mu_{n-2}}x(t), I_{0+}^{\mu_{n-2}-\mu_1}x(t), \dots, \\ I_{0+}^{\mu_{n-2}-\mu_{n-3}}x(t), x(t), I_{0+}^{\eta_{m-2}}y(t)) = 0, \quad 0 < t < 1, \\ D_{0+}^{\beta}(\varphi_{p_2}(D_{0+}^{\delta-\eta_{m-2}}y))(t) + \mu^{1/(q_2-1)}g(t, I_{0+}^{\mu_{n-2}}x(t), I_{0+}^{\mu_{n-2}-\eta_1}x(t), \dots, \\ I_{0+}^{\mu_{n-2}-\eta_{m-2}}x(t)) = 0, \quad 0 < t < 1, \end{aligned} \tag{16}$$

$$x(0) = 0, \quad D_{0+}^{\gamma}x(0) = 0, \quad D_{0+}^{r_1}x(1) = \sum_{j=1}^{\infty} \eta_j D_{0+}^{r_2}x(\xi_j),$$

$$y(0) = 0, \quad D_{0+}^{\delta}y(0) = 0, \quad D_{0+}^{\bar{r}_1}y(1) = \sum_{j=1}^{\infty} \bar{\eta}_j D_{0+}^{\bar{r}_2}y(\bar{\xi}_j).$$

**Lemma 4.** Let  $u(t) = I_{0+}^{\mu_n-2}x(t), v(t) = I_{0+}^{\eta_{m-2}}y(t), x(t), y(t) \in C[0, 1]$ . Then (1) can be transformed into (16). Moreover, if  $(x, y) \in C[0, 1] \times C[0, 1]$  is a positive solution of problem (16), then  $(I_{0+}^{\mu_n-2}x, I_{0+}^{\eta_{m-2}}y)$  is a positive solution of problem (1).

*Proof.* The proof is similar with Lemma 2.5 in [5], we omit it here. □

In order to establish the existence of positive solution for system (1), we shall consider the following problem:

$$\begin{aligned}
 & D_{0+}^{\alpha}(\varphi_{p_1}(D_{0+}^{\gamma-\mu_n-2}x))(t) + \lambda^{1/(q_1-1)}f\left(t, I_{0+}^{\mu_n-2}x(t) + \frac{1}{k}, I_{0+}^{\mu_n-2-\mu_1}x(t) + \frac{1}{k}, \dots, \right. \\
 & \quad \left. I_{0+}^{\mu_n-2-\mu_n-3}x(t) + \frac{1}{k}, x(t) + \frac{1}{k}, I_{0+}^{\eta_{m-2}}y(t) + \frac{1}{k}\right) = 0, \quad 0 < t < 1, \\
 & D_{0+}^{\beta}(\varphi_{p_2}(D_{0+}^{\delta-\eta_{m-2}}y))(t) + \mu^{1/(q_2-1)}g\left(t, I_{0+}^{\mu_n-2}x(t), I_{0+}^{\mu_n-2-\eta_1}x(t), \dots, \right. \\
 & \quad \left. I_{0+}^{\mu_n-2-\eta_{m-2}}x(t)\right) = 0, \quad 0 < t < 1, \\
 & x(0) = 0, \quad D_{0+}^{\gamma}x(0) = 0, \quad D_{0+}^{r_1}x(1) = \sum_{j=1}^{\infty} \eta_j D_{0+}^{r_2}x(\xi_j), \\
 & y(0) = 0, \quad D_{0+}^{\delta}y(0) = 0, \quad D_{0+}^{\bar{r}_1}y(1) = \sum_{j=1}^{\infty} \bar{\eta}_j D_{0+}^{\bar{r}_2}y(\bar{\xi}_j),
 \end{aligned} \tag{17}$$

where  $t \in (0, 1), k \in \{2, 3, \dots\}$ . Assume that  $f : [0, 1] \times (\mathbb{R}^1 \setminus \{0\})^n \rightarrow \mathbb{R}_+^1$  is continuous, then  $(x, y)$  is a solution of system (17) if and only if  $(x, y) \in C[0, 1] \times C[0, 1]$  is a solution of the following nonlinear integral equation system (18):

$$\begin{aligned}
 x(t) &= \lambda \int_0^1 G(t, s) \left( \int_0^s \bar{a}(s-\tau)^{\alpha-1} f\left(\tau, I_{0+}^{\mu_n-2}x(\tau) + \frac{1}{k}, I_{0+}^{\mu_n-2-\mu_1}x(\tau) + \frac{1}{k}, \dots, \right. \right. \\
 & \quad \left. \left. I_{0+}^{\mu_n-2-\mu_n-3}x(\tau) + \frac{1}{k}, x(\tau) + \frac{1}{k}, I_{0+}^{\eta_{m-2}}y(\tau) + \frac{1}{k}\right) d\tau \right)^{q_1-1} ds, \quad t \in [0, 1], \\
 y(\tau) &= \mu \int_0^1 H(\tau, s) \left( \int_0^s \bar{b}(s-w)^{\beta-1} g\left(w, I_{0+}^{\mu_n-2}x(w), I_{0+}^{\mu_n-2-\eta_1}x(w), \dots, \right. \right. \\
 & \quad \left. \left. I_{0+}^{\mu_n-2-\eta_{m-2}}x(w)\right) dw \right)^{q_2-1} ds, \quad t \in [0, 1].
 \end{aligned} \tag{18}$$

Easily, we get the following integral equation:

$$\begin{aligned}
 x(t) &= \lambda \int_0^1 G(t, s) \left( \int_0^s \bar{a}(s-\tau)^{\alpha-1} f\left(\tau, I_{0+}^{\mu_n-2}x(\tau) + \frac{1}{k}, I_{0+}^{\mu_n-2-\mu_1}x(\tau) + \frac{1}{k}, \dots, \right. \right. \\
 & \quad \left. \left. I_{0+}^{\mu_n-2-\mu_n-3}x(\tau) + \frac{1}{k}, x(\tau) + \frac{1}{k}, \right. \right.
 \end{aligned}$$



$$\begin{aligned}
 & I_{0^+}^{\eta_{m-2}} \left[ \mu \int_0^1 H(\tau, s) \left( \int_0^s \bar{b}(s-w)^{\beta-1} g(w, \dots, I_{0^+}^{\mu_{n-2}} x(w), I_{0^+}^{\mu_{n-2}-\eta_1} x(w), \right. \right. \\
 & \left. \left. I_{0^+}^{\mu_{n-2}-\eta_{m-2}} x(w)) dw \right)^{q_2-1} ds + \frac{1}{k} \right]^{q_1-1} d\tau \Big) ds, \quad t \in [0, 1]. \tag{19}
 \end{aligned}$$

Let  $P$  be a normal cone of a Banach space  $E$ , and  $e \in P$  with  $\|e\| \leq 1, e \neq \theta$  ( $\theta$  is a zero element of  $E$ ). Define  $Q_e = \{u \in P: \text{there exist constants } c, C > 0 \text{ such that } ce \leq u \leq Ce\}$ . Assume  $A : Q_e \times Q_e \rightarrow Q_e$ .  $A$  is said to be mixed monotone if  $A(u, y)$  is non-decreasing in  $u$  and nonincreasing in  $y$ , i.e.,  $u_1 \leq u_2$  ( $u_1, u_2 \in Q_e$ ) implies  $A(u_1, y) \leq A(u_2, y)$  for any  $y \in Q_e$ , and  $y_1 \leq y_2$  ( $y_1, y_2 \in Q_e$ ) implies  $A(u, y_1) \geq A(u, y_2)$  for any  $u \in Q_e$ . The element  $u^* \in Q_e$  is called a fixed point of  $A$  if  $A(u^*, u^*) = u^*$ . Now we give the following lemma.

**Lemma 5.** (See [2].) *Suppose that  $A : Q_e \times Q_e \rightarrow Q_e$  is a mixed monotone operator and there exists a constant  $\sigma$  satisfying  $0 < \sigma < 1$  such that*

$$A\left(lx, \frac{1}{l}y\right) \geq l^\sigma A(x, y), \quad x, y \in Q_e, \quad 0 < l < 1, \tag{20}$$

then  $A$  has a unique fixed point  $x^* \in Q_e$ , and for any  $x_0, y_0 \in Q_e$ , we have

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = x^*,$$

where

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \dots,$$

and convergence rate

$$\|x_n - x^*\| = o(1 - r^{\sigma^n}), \quad \|y_n - x^*\| = o(1 - r^{\sigma^n}),$$

$r$  is a constant,  $0 < r < 1$ , and dependent on  $x_0, y_0$ .

Let  $e(t) = t^{\gamma-\mu_{n-1}-1}$ , we define a normal cone of  $C[0, 1]$  by

$$P = \{x \in C[0, 1]: x(t) \geq 0, 0 \leq t \leq 1\},$$

also define

$$Q_e = \left\{ x \in P: \text{there exists } D \geq 1, \frac{1}{D}e(t) \leq x(t) \leq De(t), t \in [0, 1] \right\}.$$

**Remark 2.** Let  $s = \tau t$ , by simple calculation, we have

$$I_{0^+}^{\mu_{n-2}} e(t) = \frac{1}{\Gamma(\mu_{n-2})} \int_0^t (t-s)^{\mu_{n-2}-1} s^{\gamma-\mu_{n-2}-1} ds = \frac{\Gamma(\gamma-\mu_{n-2})}{\Gamma(\gamma)} t^{\gamma-1}. \tag{21}$$

Similarly, we have

$$I_{0+}^{\eta_{m-2}} t^{\delta-\eta_{m-2}} = \frac{\Gamma(\delta - \eta_{m-2})}{\Gamma(\delta)} t^{\delta-1},$$

$$I_{0+}^{\mu_{n-2}-\mu_{\kappa}} e(t) = \frac{\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \mu_{\kappa})} t^{\gamma-\mu_{\kappa}-2}, \quad \kappa = 1, 2, \dots, n-3, \tag{22}$$

$$I_{0+}^{\mu_{m-2}-\eta_{\varrho}} e(t) = \frac{\Gamma(\gamma - \mu_{m-2})}{\Gamma(\gamma - \eta_{\varrho})} t^{\gamma-\eta_{\varrho}-1}, \quad \varrho = 1, 2, \dots, m-2. \tag{23}$$

### 3 Main results

**Theorem 1.** *Suppose that (S1)–(S4) hold. Then the PFDE (1) has a unique positive solution  $(u^*, v^*)$  for all  $t \in [0, 1]$ , which satisfies*

$$\frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma)} t^{\gamma-1} \leq u^*(t) \leq \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma)} t^{\gamma-1},$$

$$\frac{\mu t^{\delta-1} \bar{b}^{q_2-1}}{\Gamma(\delta)} K_1 \leq v^*(t) \leq \frac{\Gamma(\delta - \eta_{m-2}) \bar{a}^* \mu t^{\delta-1} \bar{b}^{q_2-1}}{\Gamma(\delta)(2\beta - 1)^{(q_2-1)/2}} K_2, \tag{*}$$

where

$$K_1 = \left( \frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma)} \right) \int_0^1 \bar{j}(s) \left[ \int_0^s w^{(\gamma-1)\zeta^{1/(q_2-1)}} (s-w)^{\beta-1} g(w, 1, \dots, 1) dw \right]^{q_2-1} ds,$$

$$K_2 = \left( \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_{m-2})} + 1 \right) \int_0^1 \left[ s^{(2\beta-1)/2} \left( \int_0^s g^2(w, 1, 1, \dots, 1) dw \right)^{1/2} \right]^{q_2-1} ds.$$

Moreover, for any  $u_0 \in Q_e$ , constructing successively sequences

$$u_{k+1}(t) = I_{0+}^{\mu_{n-2}} \left\{ \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha-1} (\phi(\tau, u_k(\tau), D_{0+}^{\mu_1} u_k(\tau), \dots, D_{0+}^{\mu_{n-2}} u_k(\tau), u_k(\tau), Au_k^{(\mu_{n-2})}(\tau)) + \psi(\tau, u_k(\tau), D_{0+}^{\mu_1} u_k(\tau), \dots, D_{0+}^{\mu_{n-2}} u_k(\tau), u_k(\tau), Au_k^{(\mu_{n-2})}(\tau))) d\tau \right]^{q_1-1} ds \right\}, \quad t \in [0, 1],$$

$$v_{k+1}(t) = I_{0+}^{\eta_{m-2}} \left\{ \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s - w)^{\beta-1} g(w, u_k(w), D_{0+}^{\eta_1} u_k(w), \dots, D_{0+}^{\eta_{m-2}} u_k(w)) dw \right]^{q_2-1} ds \right\}, \quad t \in [0, 1],$$

$k = 0, 1, 2, \dots$ , and we have  $\|u_k - u^*\| \rightarrow 0, \|v_k - v^*\| \rightarrow 0$  as  $k \rightarrow \infty$ , the convergence rate is  $\|u_k - u^*\| = o(1 - r^{\sigma_k})$ , where  $r$  is a constant,  $0 < r < 1$ , and dependent on  $u_0$ .

*Proof.* We first consider the existence of a positive solution to problem (17). From the discussion in Section 2 we only need to consider the existence of a positive solution to BVP (19). In order to realize this purpose, let

$$Ax(\tau) = I_{0+}^{\eta_{m-2}} \left\{ \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g(w, I_{0+}^{\mu_{n-2}} x(w), I_{0+}^{\mu_{n-2}-\eta_1} x(w), \dots, I_{0+}^{\mu_{n-2}-\eta_{m-2}} x(w)) dw \right]^{q_2-1} ds \right\}, \quad \tau \in [0, 1], \quad (24)$$

and define the operator  $T_k : Q_e \times Q_e \rightarrow P$  by

$$T_k(x, z)(t) = \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s-\tau)^{\alpha-1} \left( \phi \left( \tau, I_{0+}^{\mu_{n-2}} x(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}} x(\tau) + \frac{1}{k}, x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k} \right) + \psi \left( \tau, I_{0+}^{\mu_{n-2}} z(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}} z(\tau) + \frac{1}{k}, z(\tau) + \frac{1}{k}, Az(\tau) + \frac{1}{k} \right) \right) d\tau \right]^{q_1-1} ds, \quad t \in [0, 1].$$

Now we prove that  $T_k : Q_e \times Q_e \rightarrow P$  is well defined. For any  $x, z \in Q_e$ , by (24), (S3), (21), (23) and Remark 1, for all  $\tau \in [0, 1]$ , we have

$$\begin{aligned} & \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g(w, I_{0+}^{\mu_{n-2}} x(w), I_{0+}^{\mu_{n-2}-\eta_1} x(w), \dots, I_{0+}^{\mu_{n-2}-\eta_{m-2}} x(w)) dw \right]^{q_2-1} ds \\ & \leq \mu \bar{a}^* \tau^{\delta-\eta_{m-2}-1} \int_0^1 \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g \left( w, \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma)} w^{\gamma-1}, \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_1)} w^{\gamma-\eta_1-1}, \dots, \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_{m-2})} w^{\gamma-\eta_{m-2}-1} \right) dw \right]^{q_2-1} ds \\ & \leq \mu \bar{a}^* \tau^{\delta-\eta_{m-2}-1} \bar{b}^{q-1} \times \left( \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_{m-2})} + 1 \right) \int_0^1 \left[ \int_0^s (s-w)^{\beta-1} g(w, 1, 1, \dots, 1) dw \right]^{q_2-1} ds, \quad (25) \end{aligned}$$

$$\begin{aligned}
 & \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g(w, I_{0+}^{\mu_{n-2}} x(w), I_{0+}^{\mu_{n-2}-\eta_1} x(w), \dots, \right. \\
 & \quad \left. I_{0+}^{\mu_{n-2}-\eta_{m-2}} x(w)) dw \right]^{q_2-1} ds \\
 & \geq \frac{\mu}{\Gamma(\delta - \eta_{m-2})} \tau^{\delta - \eta_{m-2} - 1} \int_0^1 \bar{j}(s) \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g\left(w, \frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma)} w^{\gamma-1}, \right. \right. \\
 & \quad \left. \left. \frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma - \eta_1)} w^{\gamma - \eta_1 - 1}, \dots, \frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma - \eta_{m-2})} w^{\gamma - \eta_{m-2} - 1}\right) dw \right]^{q_2-1} ds \\
 & = \frac{\mu}{\Gamma(\delta - \eta_{m-2})} \tau^{\delta - \eta_{m-2} - 1} \bar{b}^{q_2-1} \left( \frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma)} \right)^\zeta \\
 & \quad \times \int_0^1 \bar{j}(s) \left[ \int_0^s w^{(\gamma-1)\varsigma^{1/(q_2-1)}} (s-w)^{\beta-1} g(w, 1, 1, \dots, 1) dw \right]^{q_2-1} ds. \tag{26}
 \end{aligned}$$

Hence, by (24), (25), (26) and Hölder inequality, we have

$$\begin{aligned}
 Ax(\tau) &= I_{0+}^{\eta_{m-2}} \left( \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g(w, I_{0+}^{\mu_{n-2}} x(w), \right. \right. \\
 & \quad \left. \left. I_{0+}^{\mu_{n-2}-\eta_1} x(w), \dots, I_{0+}^{\mu_{n-2}-\eta_{m-2}} x(w)) dw \right]^{q_2-1} ds \right) \\
 & \leq \frac{\Gamma(\delta - \eta_{m-2}) \bar{a}^* \mu \tau^{\delta-1} \bar{b}^{q_2-1}}{\Gamma(\delta) (2\beta-1)^{(q_2-1)/2}} \left( \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_{m-2})} + 1 \right)^\zeta \\
 & \quad \times \int_0^1 \left[ s^{(2\beta-1)/2} \left( \int_0^s g^2(w, 1, 1, \dots, 1) dw \right)^{1/2} \right]^{q_2-1} ds, \quad \tau \in [0, 1], \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 Ax(\tau) &= I_{0+}^{\eta_{m-2}} \left( \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s-w)^{\beta-1} g(w, I_{0+}^{\mu_{n-2}} x(w), \right. \right. \\
 & \quad \left. \left. I_{0+}^{\mu_{n-2}-\eta_1} x(w), \dots, I_{0+}^{\mu_{n-2}-\eta_{m-2}} x(w)) dw \right]^{q_2-1} ds \right) \\
 & \geq \frac{\mu \tau^{\delta-1} \bar{b}^{q_2-1}}{\Gamma(\delta)} \left( \frac{\Gamma(\gamma - \mu_{n-2})}{D\Gamma(\gamma)} \right)^\zeta \int_0^1 \bar{j}(s) \left[ \int_0^s w^{(\gamma-1)\varsigma^{1/(q_2-1)}} (s-w)^{\beta-1} \right. \\
 & \quad \left. \times g(w, 1, 1, \dots, 1) dw \right]^{q_2-1} ds, \quad \tau \in [0, 1].
 \end{aligned}$$

By (S4), we get that  $Ax(\tau)$  is well defined. From (21), (22), (27), (S1), and Remark 1 we have

$$\begin{aligned} & \phi\left(\tau, I_{0+}^{\mu_{n-2}}x(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1}x(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}}x(\tau) + \frac{1}{k}, \right. \\ & \quad \left. x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k}\right) \\ & \leq \phi(\tau, Db + 1, Db + 1, \dots, D^\sigma b + 1) \\ & \leq 2^{\sigma^{1/(q_1-1)}} b^{\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \phi(\tau, 1, 1, \dots, 1), \quad \tau \in (0, 1), \end{aligned} \tag{28}$$

where  $D \geq 1, b$  are two positive constants. By (21), (22), (27), (S1), (S2), we also have

$$\begin{aligned} & \psi\left(\tau, I_{0+}^{\mu_{n-2}}x(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1}x(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}}x(\tau) + \frac{1}{k}, \right. \\ & \quad \left. x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k}\right) \\ & \leq \psi\left(\tau, \frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}, \frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}, \dots, \frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}\right) \\ & \leq c^{-\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \psi(\tau, 1, 1, \dots, 1), \quad \tau \in (0, 1), \end{aligned} \tag{29}$$

where  $c$  is a positive constant. Noting  $(c/D)\tau^{\gamma-1} < 1$  and by (21), (22), (27), (S1), (S2), we have

$$\begin{aligned} & \phi\left(\tau, I_{0+}^{\mu_{n-2}}x(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1}x(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}}x(\tau) + \frac{1}{k}, \right. \\ & \quad \left. x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k}\right) \\ & \geq \phi\left(\tau, \frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}, \frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}, \dots, \frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}\right) \\ & \geq \left(\frac{c}{D}\tau^{\gamma-1} + \frac{1}{k}\right)^{-\sigma} \phi(\tau, 1, 1, \dots, 1) \\ & = c^{\sigma^{1/(q_1-1)}} D^{-\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \phi(\tau, 1, 1, \dots, 1), \quad \tau \in (0, 1). \end{aligned} \tag{30}$$

By (21), (22), (27), (S1), and Remark 1, we also get

$$\begin{aligned} & \psi\left(\tau, I_{0+}^{\mu_{n-2}}x(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1}x(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_{n-2}-\mu_{n-3}}x(\tau) + \frac{1}{k}, \right. \\ & \quad \left. x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k}\right) \\ & \geq \psi(\tau, Db\tau^{\gamma-1} + 1, Db\tau^{\gamma-\mu_1-1} + 1, \dots, Db\tau^{\gamma-\mu_{n-2}-1} + 1, Db\tau^{\delta-1} + 1) \\ & \geq 2^{-\sigma^{1/(q_1-1)}} b^{-\sigma^{1/(q_1-1)}} D^{-\sigma^{1/(q_1-1)}} \psi(\tau, 1, 1, \dots, 1), \quad \tau \in (0, 1). \end{aligned} \tag{31}$$

For any  $x, z \in Q_e$ , it follows from (28), (29) that

$$\begin{aligned}
 T_k(x, z)(t) &\leq \lambda \alpha^* t^{\gamma - \mu_n - 2 - 1} D^\sigma \bar{a}^{q_1 - 1} \frac{1}{(2\alpha - 1)^{(q_1 - 1)/2}} \\
 &\quad \times \int_0^1 \left[ 2^{\sigma^{1/(q_1 - 1)}} b^{\sigma^{1/(q_1 - 1)}} \left( \int_0^s \phi^2(\tau, 1, 1, \dots, 1) d\tau \right)^{1/2} \right. \\
 &\quad \left. + c^{-\sigma^{1/(q_1 - 1)}} \left( \int_0^s \tau^{-2(\gamma - 1)\sigma^{1/(q_1 - 1)}} \psi^2(\tau, 1, 1, \dots, 1) d\tau \right)^{1/2} \right]^{q_1 - 1} ds \\
 &< +\infty, \quad t \in [0, 1].
 \end{aligned} \tag{32}$$

By (S4), (32), we have that  $T_k : Q_e \times Q_e \rightarrow P$  is well defined. Next, we will prove  $T_k : Q_e \times Q_e \rightarrow Q_e$ . Formula (32) implies that

$$T_k(x, z)(t) \leq Dt^{\gamma - \mu_n - 2 - 1} = De(t), \quad t \in [0, 1].$$

At the same time, by (30) and (31), we have

$$\begin{aligned}
 T_k(x, z)(t) &= \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha - 1} \left( \phi \left( \tau, I_{0+}^{\mu_n - 2} x(\tau) + \frac{1}{k}, I_{0+}^{\mu_n - 2 - \mu_1} x(\tau) + \frac{1}{k}, \dots, \right. \right. \right. \\
 &\quad \left. \left. I_{0+}^{\mu_n - 2 - \mu_n - 3} x(\tau) + \frac{1}{k}, x(\tau) + \frac{1}{k}, Ax(\tau) + \frac{1}{k} \right) + \psi \left( \tau, I_{0+}^{\mu_n - 2} z(\tau) + \frac{1}{k}, \right. \right. \\
 &\quad \left. \left. I_{0+}^{\mu_n - 2 - \mu_1} z(\tau) + \frac{1}{k}, \dots, I_{0+}^{\mu_n - 2 - \mu_n - 3} z(\tau) + \frac{1}{k}, z(\tau) + \frac{1}{k}, Az(\tau) + \frac{1}{k} \right) \right]^{q_1 - 1} ds \\
 &\geq \frac{\lambda}{\Gamma(\gamma - \mu_n - 2)} t^{\gamma - \mu_n - 2 - 1} D^{-\sigma} \bar{a}^{q_1 - 1} \\
 &\quad \times \int_0^1 j(s) \left[ \int_0^s (s - \tau)^{\alpha - 1} \left( c^{\sigma^{1/(q_1 - 1)}} \tau^{(\gamma - 1)\sigma^{1/(q_1 - 1)}} \phi(\tau, 1, 1, \dots, 1) \right. \right. \\
 &\quad \left. \left. + 2^{-\sigma^{1/(q_1 - 1)}} b^{-\sigma^{1/(q_1 - 1)}} \psi(\tau, 1, 1, \dots, 1) \right) d\tau \right]^{q_1 - 1} ds, \quad t \in [0, 1].
 \end{aligned} \tag{33}$$

Formula (33) implies that

$$T_k(x, z)(t) \geq \frac{1}{D} t^{\gamma - \mu_n - 2 - 1} = \frac{1}{D} e(t), \quad t \in [0, 1].$$

Hence,  $T_k : Q_e \times Q_e \rightarrow Q_e$ . It is easy to prove that  $T_k : Q_e \times Q_e \rightarrow Q_e$  is a mixed monotone operator.

Finally, we show that the operator  $T_k$  satisfies (20). For any  $x, z \in Q_e$  and  $l \in (0, 1)$ , by (S2) and Remark 1, for all  $t \in [0, 1]$ , we have

$$\begin{aligned} & \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha-1} \left( \phi \left( \tau, I_{0+}^{\mu_{n-2}} l x(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1} l x(\tau) + \frac{1}{k}, \dots, \right. \right. \right. \\ & \quad \left. \left. \left. I_{0+}^{\mu_{n-2}-\mu_{n-3}} l x(\tau) + \frac{1}{k}, l x(\tau) + \frac{1}{k}, A l x(\tau) + \frac{1}{k} \right) \right. \right. \\ & \quad \left. \left. + \psi \left( \tau, I_{0+}^{\mu_{n-2}} \frac{1}{l} z(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1} \frac{1}{l} z(\tau) + \frac{1}{k}, \dots, \right. \right. \right. \\ & \quad \left. \left. \left. I_{0+}^{\mu_{n-2}-\mu_{n-3}} \frac{1}{l} z(\tau) + \frac{1}{k}, \frac{1}{l} z(\tau) + \frac{1}{k}, A \frac{1}{l} z(\tau) + \frac{1}{k} \right) \right) d\tau \right]^{q_1-1} ds \\ & \geq l^\sigma \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha-1} \left( \phi \left( \tau, I_{0+}^{\mu_{n-2}} x(\tau), I_{0+}^{\mu_{n-2}-\mu_1} x(\tau), \dots, \right. \right. \right. \\ & \quad \left. \left. \left. I_{0+}^{\mu_{n-2}-\mu_{n-3}} x(\tau), x(\tau), A x(\tau) \right) + \psi \left( \tau, I_{0+}^{\mu_{n-2}} z(\tau), I_{0+}^{\mu_{n-2}-\mu_1} z(\tau), \dots, \right. \right. \right. \\ & \quad \left. \left. \left. I_{0+}^{\mu_{n-2}-\mu_{n-3}} z(\tau), z(\tau), A z(\tau) \right) \right) d\tau \right]^{q_1-1} ds. \tag{34} \end{aligned}$$

Formula (34) implies that

$$T_k \left( l x, \frac{1}{l} z \right) \geq l^\sigma T_k(x, z), \quad x, z \in Q_e. \tag{35}$$

Hence, Lemma 5 assumes that there exists a unique positive solution  $x_k^* \in Q_e$  such that  $T_k(x_k^*, x_k^*) = x_k^*$ . Consequently,  $x_k^*$  is a unique positive solution of (17) for every  $k \in \{2, 3, \dots\}$ . Since  $x_k^* \in Q_e$ , so  $x_k^*$  has uniform lower and upper bounds. Thus, in order to pass the solution  $x_k^*$  of (17) to that of (16), we need that the fact that  $\{x_k^*\}_{k \geq 2}$  is an equiconuous family on  $[0, 1]$ . In fact, by (28), (29),  $x_k^* \in Q_e$ , we have

$$\begin{aligned} & f \left( \tau, I_{0+}^{\mu_{n-2}} x_k^*(\tau) + \frac{1}{k}, I_{0+}^{\mu_{n-2}-\mu_1} x_k^*(\tau) + \frac{1}{k}, \dots, \right. \\ & \quad \left. I_{0+}^{\mu_{n-2}-\mu_{n-3}} x_k^*(\tau) + \frac{1}{k}, A x_k^*(\tau) + \frac{1}{k} \right) \\ & \leq 2^{\sigma^{1/(q_1-1)}} b^{\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \phi(\tau, 1, 1, \dots, 1) \\ & \quad + c^{-\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \psi(\tau, 1, 1, \dots, 1), \quad \tau \in (0, 1), \end{aligned}$$

and let

$$\begin{aligned} \varphi(\tau) &= 2^{\sigma^{1/(q_1-1)}} b^{\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \phi(\tau, 1, 1, \dots, 1) \\ & \quad + c^{-\sigma^{1/(q_1-1)}} D^{\sigma^{1/(q_1-1)}} \tau^{(\gamma-1)\sigma^{1/(q_1-1)}} \psi(\tau, 1, 1, \dots, 1), \quad \tau \in (0, 1), \end{aligned}$$

by (S4), we easily get that  $\varphi(s) \in L^1[0, 1]$ . Hence, for  $0 \leq t_1 \leq t_2 \leq 1$ , we have

$$\begin{aligned}
 & |(x_k^*)(t_2) - (x_k^*)(t_1)| \\
 &= \left| \int_0^1 (G(t_2, s) - G(t_1, s)) \left( \int_0^s \bar{a}(s - \tau)^{\alpha-1} f \left( \tau, I_{0+}^{\mu_n-2} x^*(\tau) + \frac{1}{k}, I_{0+}^{\mu_n-2-\mu_1} x_k^*(\tau) \right. \right. \right. \\
 &\quad \left. \left. \left. + \frac{1}{k}, \dots, I_{0+}^{\mu_n-2-\mu_n-3} x_k^*(\tau) + \frac{1}{k}, x_k^*(\tau) + \frac{1}{k}, Ax_k^*(\tau) + \frac{1}{k} \right) d\tau \right)^{q_1-1} ds \right| \\
 &\leq \left| \int_0^1 (G_1(t_2, s) + G_2(t_2, s) - G_1(t_1, s) - G_2(t_1, s)) \left( \int_0^s \bar{a}(s - \tau)^{\alpha-1} \varphi(\tau) d\tau \right)^{q_1-1} ds \right| \\
 &\leq \frac{\bar{a}^{q_1-1} \|\varphi\|_L}{\Delta(2\alpha - 1)^{(q_1-1)/2} \Gamma(\gamma - \mu_n - 2 - r_1)} (t_2^{\gamma-\mu_n-2-1} - t_1^{\gamma-\mu_n-2-1}) \\
 &\quad + \frac{\bar{a}^{q_1-1} \|\varphi\|_L}{\Delta(2\alpha - 1)^{(q_1-1)/2}} \left[ (t_2 - t_1)^{\gamma-\mu_n-2-1} \right. \\
 &\quad \left. + \int_0^{t_1} ((t_2 - s)^{\gamma-\mu_n-2-1} - (t_1 - s)^{\gamma-\mu_n-2-1}) ds \right]. \tag{36}
 \end{aligned}$$

Since  $(t - s)^{\gamma-\mu_n-2-1}$  is uniformly continuous on  $[0, 1] \times [0, 1]$  and  $t^{\gamma-\mu_n-2-1}$  is uniformly continuous on  $[0, 1]$ , so any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for  $0 \leq t_1 \leq t_2 \leq 1, t_2 - t_1 < \delta, 0 < s \leq t_1$ ,

$$\begin{aligned}
 & t_2^{\gamma-\mu_n-2-1} - t_1^{\gamma-\mu_n-2-1} < \varepsilon, \\
 & (t_2 - s)^{\gamma-\mu_n-2-1} - (t_1 - s)^{\gamma-\mu_n-2-1} < \varepsilon.
 \end{aligned}$$

Consequently, for all  $x \in D, 0 \leq t_1 \leq t_2 \leq 1$  and  $t_2 - t_1 < \min\{\delta, \gamma - \mu_n - 2 - \sqrt[q_1]{\varepsilon}\}$ , the inequality

$$|(x_k^*)(t_2) - (x_k^*)(t_1)| \leq \frac{\bar{a}^{q_1-1} \|\varphi\|_L}{\Delta(2\alpha - 1)^{(q_1-1)/2}} (2 + \Gamma(\gamma - \mu_n - 2 - r_1)) \varepsilon$$

holds. Hence, by Arzela–Ascoli theorem, we get that  $\{x_k^*\}_{k \geq 2}$  is an equiconuous family on  $[0, 1]$ . Hence,  $\{x_k^*\}_{k \geq 2}$  is relatively compact in  $P$ , then the sequence  $\{x_k^*\}$  has a subsequence converge to  $x^* \subset P$ . Without loss of generality, we still assume that  $\{x_k^*\}$  itself uniformly converges to  $x^*$ , that is  $\lim_{k \rightarrow \infty} x_k^* \rightarrow x^*$ , then  $(x^*, y^*)$  is the solution of (19), which can be easily get by the Lebesgue dominated convergence theorem.

Moreover, for any  $u_0(t) = I_{0+}^{\mu_n-2} x_0 \in Q_e$ , by Lemma 5, constructing a successively sequence

$$\begin{aligned}
 x_{m+1}(t) = \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha-1} (\phi(\tau, I_{0+}^{\mu_n-2} x_m(\tau), \dots, \right. \\
 I_{0+}^{\mu_n-2-\mu_n-3} x_m(\tau), x_m(\tau), Ax_m(\tau)) + \psi(\tau, I_{0+}^{\mu_n-2} x_m(\tau), \dots, \\
 \left. I_{0+}^{\mu_n-2-\mu_n-3} x_m(\tau), x_m(\tau), Ax_m(\tau)) \right) d\tau \Big]^{q_1-1} ds, \quad t \in [0, 1], m = 1, 2, \dots,
 \end{aligned}$$



by  $u_{m+1}(t) = I_{0+}^{\mu_{n-2}} x_{m+1}(t)$ , then

$$u_{m+1}(t) = I_{0+}^{\mu_{n-2}} \left\{ \lambda \int_0^1 G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha-1} (\phi(\tau, u_m(\tau), D_{0+}^{\mu_1} u_m(\tau), D_{0+}^{\mu_2} u_m(\tau), \dots, D_{0+}^{\mu_{n-2}} u_m(\tau), AD_{0+}^{\mu_{n-2}} u_m(\tau)) + \psi(\tau, u_m(\tau), D_{0+}^{\mu_1} u_m(\tau), D_{0+}^{\mu_2} u_m(\tau), \dots, D_{0+}^{\mu_{n-2}} u_m(\tau), AD_{0+}^{\mu_{n-2}} u_m(\tau))) \right]^{q_1-1} ds \right\}, \quad t \in [0, 1], \quad m = 1, 2, \dots,$$

and we have  $\|u_m - u^*\| = \|I_{0+}^{\mu_{n-2}} x_m - I_{0+}^{\mu_{n-2}} x^*\| \rightarrow 0$  as  $m \rightarrow \infty$ , convergence rate

$$\|u_m - u^*\| = \|I_{0+}^{\mu_{n-2}} x_m - I_{0+}^{\mu_{n-2}} x^*\| = o(1 - r^{\sigma^m}),$$

$r$  is a constant,  $0 < r < 1$ , and dependent on  $u_0$ . Hence, for  $t \in [0, 1]$ , by Lemma 4,  $u^*(t) = I_{0+}^{\mu_{n-2}} x^*(t)$ ,  $v^*(t) = I_{0+}^{\eta_{m-2}} y^*(t)$  is the unique positive solution of system (1), where

$$y^*(t) = \int_0^1 H(t, s) \left( \int_0^s \bar{b}(s - w)^{\beta-1} g(w, I_{0+}^{\mu_{n-2}} x^*(w), I_{0+}^{\mu_{n-2}-\eta_1} x^*(w), \dots, I_{0+}^{\mu_{n-2}-\eta_{m-2}} x^*(w)) dw \right)^{q_2-1} ds. \tag{37}$$

By (25), (26), (35), (36), and  $x^* \in Q_e$ , we get  $(u^*, v^*)$ , which satisfies (\*).

Moreover, by (37), we have

$$v^*(t) = I_{0+}^{\eta_{m-2}} \left\{ \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s - w)^{\beta-1} g(w, u^*(w), D_{0+}^{\eta_1} u^*(w), \dots, D_{0+}^{\eta_{m-2}} u^*(w)) dw \right]^{q_2-1} ds \right\}$$

and

$$\begin{aligned} v_{k+1} &= I_{0+}^{\eta_{m-2}} \left( \mu \int_0^1 H(\tau, s) \left[ \int_0^s \bar{b}(s - w)^{\beta-1} g(w, u_k(w), D_{0+}^{\eta_1}, \dots, D_{0+}^{\eta_{m-2}}(w)) dw \right]^{q_2-1} ds \right) \\ &\leq \frac{\Gamma(\delta - \eta_{m-2}) \bar{a}^* \mu \tau^{\delta-1} \bar{b}^{q_2-1}}{\Gamma(\delta) (2\beta - 1)^{(q_2-1)/2}} \left( \frac{D\Gamma(\gamma - \mu_{n-2})}{\Gamma(\gamma - \eta_{m-2})} + 1 \right)^\zeta \\ &\quad \times \int_0^1 \left[ s^{(2\beta-1)/2} \left( \int_0^s g^2(w, 1, 1, \dots, 1) dw \right)^{1/2} \right]^{q_2-1} ds < +\infty. \end{aligned}$$

By Lebesgue control convergence theorem, we have  $\|v_k - v^*\| \rightarrow 0 (k \rightarrow +\infty)$ . Therefore, the proof of Theorem 1 is completed. □

### 4 An example

Consider the following boundary value problem:

$$\begin{aligned}
 D_{0+}^{3/4}(\varphi_3(D_{0+}^{5/2-}u))(t) + \lambda^2 f(t, u(t), D_{0+}^{1/2}u(t), v(t)) &= 0, \quad 0 < t < 1, \\
 D_{0+}^{3/4}(\varphi_2(D_{0+}^{3/2}v))(t) + \mu^2 g(t, u(t)) &= 0, \quad 0 < t < 1, \\
 u(0) = 0, \quad D_{0+}^\gamma u(0) = 0, \quad D_{0+}^{r_1} u(1) &= \sum_{j=1}^\infty \eta_j D_{0+}^{r_2} u(\xi_j), \\
 v(0) = 0, \quad D_{0+}^\delta v(0) = 0, \quad D_{0+}^{\bar{r}_1} v(1) &= \sum_{j=1}^\infty \bar{\eta}_j D_{0+}^{\bar{r}_2} v(\bar{\xi}_j),
 \end{aligned} \tag{38}$$

where  $\gamma = 5/2, \delta = 3/2, \alpha = \beta = 3/4, r_1 = r_2 = 1/2, \bar{r}_1 = \bar{r}_2 = 1/2, \eta_j = \bar{\eta}_j = 1/(2j^5), \xi_j = \bar{\xi}_j = 1/j^2, p_1 = 3, q_1 = 3/2, p_2 = 2, q_2 = 1/2$ , and

$$\begin{aligned}
 \phi(t, x_1, x_2, x_3) &= (t^{-1/4} + \cos t)x_1^{1/9} + 2tx_2^{1/8} + 2x_3^{1/16}, \\
 \psi(t, x_1, x_2, x_3) &= t^{-1/16}x_1^{-1/8} + x_2^{-1/16} + (2-t)x_3^{-1/15}, \\
 g(t, u) &= (3t + t^2)u^{3/5} + (t \sin t + t)u^{2/3}.
 \end{aligned}$$

Hence, by simple calculation, we have  $(\Gamma(\gamma)/\Gamma(\gamma - p_2)) \sum_{j=1}^\infty \eta_j \xi_j^{\gamma-p_2-1} = 0.5412 \leq \Gamma(\gamma)/\Gamma(\gamma - p_1) = \Gamma(5/2)/\Gamma(3/2), (\Gamma(\delta)/\Gamma(\delta - \bar{p}_2)) \sum_{j=1}^\infty \bar{\eta}_j \bar{\xi}_j^{\delta-\bar{p}_2-1} = 0.5412 \leq \Gamma(\delta)/\Gamma(\delta - \bar{p}_1) = \Gamma(3/2)/\Gamma(1/2)$ . Moreover, for any  $(t, x_1, x_2, x_3) \in (0, 1) \times (0, \infty)^3$  and  $0 < l < 1$ , we have

$$\begin{aligned}
 \phi(t, lx_1, lx_2, lx_3) &= (t^{-1/4} + \cos t)(lx_1)^{1/9} + 2t(lx_2)^{1/8} + 2(lx_3)^{1/16} \\
 &\geq l^{1/8}((t^{-1/4} + \cos t)x_1^{1/9} + 2tx_2^{1/8} + 2x_3^{1/16}) \\
 &= l^{1/8}\phi(t, x_1, x_2, x_3) = l^{\sigma^{1/(q_1-1)}}\phi(t, x_1, x_2, x_3), \\
 \psi(t, l^{-1}x_1, l^{-1}x_2, l^{-1}x_3) &= t^{-1/16}(l^{-1}x_1)^{-1/8} + (l^{-1}x_2)^{-1/16} + (2-t)(l^{-1}x_3)^{-1/15} \\
 &\geq l^{1/8}(t^{-1/16}x_1^{-1/8} + x_2^{-1/16} + (2-t)x_3^{-1/15}) \\
 &= l^{1/8}\psi(t, x_1, x_2, x_3) = l^{\sigma^{1/(q_1-1)}}\psi(t, x_1, x_2, x_3), \\
 g(t, lu) &= (3t + t^2)(lu)^{3/5} + (t \sin t + t)(lu)^{2/3} \\
 &\geq l^{2/3}((3t + t^2)u^{3/5} + (t \sin t + t)u^{2/3}) \\
 &= l^{2/3}g(t, u) = l^{\sigma^{1/(q_2-1)}}g(t, u).
 \end{aligned}$$

Noting  $\sigma = 1/(2\sqrt{2}) < 1$ ,  $\varsigma = 2/3$ ,  $\psi(\tau, 1, 1, 1) = \tau^{-1/16} + 3 - \tau$ ,  $\phi(\tau, 1, 1, 1) = \tau^{-1/4} + \cos \tau + 2\tau + 2$ ,  $g(\tau, 1) = 3\tau + \tau^2 + \tau \sin \tau + \tau$ , we have

$$0 < \int_0^1 \phi^2(\tau, 1, 1, 1) \, d\tau = \int_0^1 (\tau^{-1/4} + \cos \tau + 2\tau + \tau)^2 \, d\tau$$

$$\leq 19 + 8 + \frac{16}{7} < +\infty,$$

$$0 < \int_0^1 \tau^{-2(\gamma-1)\sigma^{1/(q-1)}} \psi^2(\tau, 1, 1, 1) \, d\tau = \int_0^1 \tau^{-3/8} (\tau^{-1/16} + 3 - \tau)^2 \, d\tau$$

$$= 2 + \frac{96}{9} + 9 < +\infty,$$

$$0 < \int_0^1 g^2(\tau, 1, 1, 1) \, d\tau \leq \int_0^1 (3\tau + \tau^2 + \tau \sin \tau + \tau)^2 \, d\tau$$

$$\leq 17 \times \frac{1}{3} + 6\frac{1}{4} + \frac{1}{5} + 12 + 16 < +\infty.$$

Thus, assumptions (S1)–(S4) of Theorem 1 hold. Then Theorem 1 implies that problem (38) has a unique solution.

In addition, for any initial  $u_0 \in Q_e$ , we construct a successively sequence

$$u_{k+1}(t) = \int_0^1 \lambda G(t, s) \left[ \int_0^s \bar{a}(s - \tau)^{\alpha-1} (\phi(t, u_k(t), D_{0+}^{1/2} u_k(t), Au'_k(t)) \right. \\ \left. + \psi(t, u_k(t), D_{0+}^{1/2} u_k(t), Au'_k(t))) \, d\tau \right]^{q_1-1} \, ds, \quad t \in [0, 1], \quad k = 1, 2, \dots,$$

and we have  $\|u_k - u^*\| \rightarrow 0$  as  $k \rightarrow \infty$ , the convergence rate is  $\|u_k - u^*\| = o(1 - r^{\sigma^k})$ , where  $r$  is a constant,  $0 < r < 1$ , and dependent on  $u_0$ .

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