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# The dynamics of a delayed generalized fractional-order biological networks with predation behavior and material cycle*

**Yingkang Xie**[a], **Zhen Wang**[a,1], **Hao Shen**[b], **Yuxia Li**[c]

[a]College of Mathematics and Systems Science,
Shandong University of Science and Technology,
Qingdao, 266590, China
wangzhen_sd@126.com

[b]School of Electrical Engineering and Information,
Anhui University of Technology,
Anhui, 233100, China

[c]College of Electrical Engineering and Automation,
Shandong University of Science and Technology,
Qingdao 266590, China

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**Abstract.** In this paper, a delayed generalized fractional-order biological networks with predation behavior and material cycle is comprehensively discussed. Some criteria of stability and bifurcation for the present system is presented. Moreover some results of two delays are obtained. Finally, some numerical simulations are presented to support the analytical results.

**Keywords:** material cycle, fractional order, time delay, Hopf bifurcation, predation behavior.

## 1 Introduction

As is well known, predation behavior is widespread in nature, and it has been widely discussed due to the application value of it [38]. Mathematical method is a necessary instrument to study it [12]. The famous Lotka–Volterra model is one of the earliest models with predation behavior [20], which forms the basis of many models used today in the analysis of population dynamics. Since then, variety of realistic models with predation behavior have been established [2, 8, 14, 17, 29, 30, 32].

In the past few decades, fractional calculus theory has been improved significantly and has been successfully applied to various research fields [7,16,23,26,31,33,34,36]. In fact, most population systems have long-term memory. The integer derivative represents the

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[1]Corresponding author.

change at a particular moment, and the fractional derivative is related to the entire time domain of the biological process. Thus, the fractional-order systems are more suitable in describing population dynamics. Since then, more and more fractional-order population systems have been proposed, and some interesting results are obtained [1, 6, 9, 11, 15, 22, 24].

Time delay exists in population systems widely. The existence of time delay means that both the current state and the state of previous period of time will have an effect on the system's development [10]. Compared with the prey–predator model without time delay, the delayed prey–predator model is more suitable for describing nonlinear dynamical behaviors. In recent years, some significant achievements have been made in the study of the delayed population models [3, 4, 13, 21, 27, 35, 37].

As a matter of fact, material cycle plays an important role in the prey–predator system [19]. On the one hand, prey provides energy for the survival of predators. On the other hand, when the predator dies, the decomposition of the predator by the microorganisms promotes the growth of the prey. So material cycle should be considered in the realistic prey–predator models, but to the best of my knowledge, few prey–predator models consider it.

Biological networks with predation behavior have been receive a lot of attention [5, 18, 28]. Compared with the low-dimensional model, it is more universal and practical for the research of biological network. In this paper, a delayed generalized fractional-order biological networks with predation behavior and material cycle is considered.

The main contributions of this paper are summarized as follows: (i) A delayed generalized fractional-order biological networks with predation behavior and material cycle is proposed firstly. (ii) Some detailed criteria of stability and bifurcation of the proposed system are established. (iii) The impact of the order on dynamical behaviors for the proposed system is studied. (iv) Some numerical simulations are given for supporting the theoretical results.

The organization of this article is as follows. In Section 2, the detailed model description is presented. In Section 3, some theoretical results of stability and bifurcation of the positive equilibrium point of the present system are given. Section 4 focuses on numerical simulations to support the theoretical results. In Section 5, some conclusions are proposed.

## 2 Model description

In this paper, a generalized fractional-order $n$-species prey–predator model with different delays and cyclical effect will be considered. The mathematical model can be described by

$$\begin{aligned}
&D^{\alpha}x_1(t) = x_1(t)\left[f_{11}\big(x_1(t)\big) - \sum_{i=2}^{n} f_{1i}\big(x_i(t)\big) + \sum_{j=2}^{n} g_j\big(x_j(t-\tau_1)\big)\right],\\
&D^{\alpha}x_i(t) = x_i(t)\big[-f_{ii}\big(x_i(t)\big) + f_{i1}\big(x_1(t-\tau_2)\big)\big],\\
&x_1(\theta) = \phi_1(\theta), \quad -\tau_2 \leqslant \theta \leqslant t_0,\\
&x_i(\theta) = \phi_i(\theta), \quad -\tau_1 \leqslant \theta \leqslant t_0, \quad i = 2,\dots,n,
\end{aligned} \tag{1}$$

where $D^\alpha$ denotes the Caputo fractional derivative (see [25]), and $\alpha \in (0,1]$, $x_1(t)$ represents the population density of the producer(prey) at time $t$, $x_i(t)$ for $i = 2, 3, \dots, n$ represent the population density of the predator $x_i$ at time $t$, $\tau_i \geqslant 0$ for $i = 1, 2$ represent time delays.

The function $f_{11}(x_1(t))$ denotes the growth rate of the producer $x_1$ in the absence of other species, and the functions $-f_{ii}(x_i(t))$ for $i = 2, 3, \dots, n$ represent the growth rate of predators in the absence of other species. Because of the competition for resources, territory or mating partners, $\mathrm{d}f_{11}/\mathrm{d}x_1 < 0$ and $-\mathrm{d}f_{ii}/\mathrm{d}x_i < 0$.

The function $\sum_{j=2}^{n} g_j(x_j)$ represents the effect of biological matter cycle on the producer $x_1$ in a time unit. The greater $x_i$ is, the greater the impact on $x_1$ will be. This implies $\mathrm{d}g_j/\mathrm{d}x_j > 0$.

The functions $f_{i,1}(x_1)$ for $i = 2, \dots, n$ denote the effect of the predator species $x_i$ on the prey species $x_1$ in a time unit, and the functions $f_{1i}(x_i)$ denote the effect of the prey species $x_1$ on the predator species $x_i$ in a time unit. The greater $x_i$ is, the greater the impact on $x_1$ will be, and the greater $x_1$ is, the greater the impact on $x_i$ will be. This implies $\mathrm{d}f_{1i}/\mathrm{d}x_i > 0$ and $\mathrm{d}f_{i1}/\mathrm{d}x_1 > 0$. All of the functions are continuous, differentiable and positive.

Subsequently, to derive our main results, we make the hypothesis in model (1).

(H1) The following equations have a positive solution:

$$f_{11}\bigl(x_1(t)\bigr) - \sum_{i=2}^{n} f_{1i}\bigl(x_i(t)\bigr) + \sum_{j=2}^{n} g_j(x_j) = 0, \qquad -f_{ii}\bigl(x_i(t)\bigr) + f_{i1}\bigl(x_1(t)\bigr) = 0.$$

## 3 Main result

In this section, one will explore the local stability and cast about for the conditions on the occurrence of Hopf bifurcation for system (1).

In view of hypothesis (H1), system (1) has a positive equilibrium $E_1 = (x_1^*, x_2^*, \dots, x_n^*)$. Let $\bar{x}_i = x_i(t) - x_i^*$. Then system (1) can be written as

$$\begin{aligned} D^\alpha \bar{x}_1(t) &= \bigl(\bar{x}_1(t) + x_1^*\bigr)\Biggl[f_{11}\bigl(\bar{x}_1(t) + x_1^*\bigr) - \sum_{i=2}^{n} f_{1i}\bigl(\bar{x}_i(t) + x_i^*\bigr) \\ &\qquad + \sum_{j=2}^{n} g_j\bigl(\bar{x}_j(t-\tau_1) + x_j^*\bigr)\Biggr], \\ D^\alpha \bar{x}_i(t) &= \bigl(\bar{x}_i(t) + x_i^*\bigr)\bigl[-f_{ii}\bigl(\bar{x}_i(t) + x_i^*\bigr) + f_{i1}\bigl(\bar{x}_1(t-\tau_2) + \bar{x}_1^*\bigr)\bigr], \\ &\qquad i = 2, 3, \dots, n. \end{aligned} \tag{2}$$

Linearization of system (2) around the zero equilibrium reads

$$\begin{aligned} D^\alpha \bar{x}_1(t) &= x_1^*\Biggl[f_{11}'(x_1^*)\bar{x}_1(t) - \sum_{i=2}^{n} f_{1i}'(x_i^*)\bar{x}_i(t) + \sum_{j=2}^{n} g_j'(x_j^*)\bar{x}_j(t-\tau_1)\Biggr], \\ D^\alpha \bar{x}_i(t) &= x_i^*\bigl[-f_{ii}'(x_i^*)\bar{x}_i(t) + f_{i1}'(\bar{x}_1^*)\bar{x}_1(t-\tau_2)\bigr], \quad i = 2, 3, \dots, n. \end{aligned} \tag{3}$$

Let $k_1 = -x_1^* f_{11}'(x_1^*)$, $k_i = x_i^* f_{ii}'(x_i^*)$, $b_{1i} = x_i^* f_{1i}'(x_i^*)$, $c_{1i} = x_i^* g_i'(x_i^*)$ and $d_{i1} = x_i^* f_{i1}'(x_1^*)$ for $i = 2, 3, \ldots, n$. Then system (3) can be rewritten as

$$\begin{aligned} D^\alpha \bar{x}_1(t) &= -k_1(x_1^*)\bar{x}_1(t) - \sum_{i=2}^n b_{1i}\bar{x}_i(t) + \sum_{j=2}^n c_{1i}\bar{x}_j(t-\tau_1), \\ D^\alpha \bar{x}_i(t) &= -k_i\bar{x}_i(t) + d_{i1}\bar{x}_1(t-\tau_2), \quad i = 2, 3, \ldots, n. \end{aligned} \tag{4}$$

Hence, the associated characteristic equation of system (4) is obtained as

$$J = \begin{vmatrix} s^\alpha + k_1 & b_{12} - c_{12}\mathrm{e}^{-s\tau_1} & b_{13} - c_{13}\mathrm{e}^{-s\tau_1} & \cdots & b_{1n} - c_{1n}\mathrm{e}^{-s\tau_1} \\ -d_{21}\mathrm{e}^{-s\tau_2} & s^\alpha + k_2 & 0 & \cdots & 0 \\ -d_{31}\mathrm{e}^{-s\tau_2} & 0 & s^\alpha + k_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{n1}\mathrm{e}^{-s\tau_2} & 0 & 0 & \cdots & s^\alpha + k_n \end{vmatrix} = 0,$$

which is equal to

$$\begin{aligned} & a_0 s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_{n-1}s^\alpha + a_n \\ & \quad + \left[b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2}s^\alpha + b_{n-1}\right]\mathrm{e}^{-s\tau_2} \\ & \quad + \left[c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2}s^\alpha + c_{n-1}\right]\mathrm{e}^{-s(\tau_2+\tau_1)} = 0, \end{aligned} \tag{5}$$

where

$$a_0 = 1, \quad a_1 = \sum_{j=1}^n k_j, \quad a_2 = \sum_{1 \leqslant h < l \leqslant n} k_h k_l, \quad \ldots,$$

$$a_{n-1} = \sum_{1 \leqslant h < l < \cdots < p \leqslant n} k_h k_l \cdots k_p, \quad a_n = \prod_{j=1}^n k_j,$$

$$b_1 = \sum_{i=2}^n b_{1i}d_{i1}, \quad b_2 = \sum_{i=2}^n \sum_{\substack{2 \leqslant j \leqslant n \\ j \neq i}} b_{1i}d_{i1}k_j, \quad b_3 = \sum_{i=2}^n \sum_{\substack{h,l \neq i \\ 2 \leqslant h < l \leqslant n}} b_{1i}d_{i1}k_h k_l, \quad \ldots,$$

$$b_{n-2} = \sum_{i=2}^n \sum_{\substack{2 \leqslant h < l < m < \cdots < p \leqslant n \\ h,l,m,\ldots,p \neq i}} b_{1i}d_{i1}k_h k_l k_m \cdots k_p, \quad b_{n-1} = \sum_{i=2}^n b_{1i}d_{i1} \prod_{\substack{j=2 \\ j \neq i}}^n k_j$$

$$c_1 = -\sum_{i=2}^n c_{1i}d_{i1}, \quad c_2 = -\sum_{i=2}^n \sum_{\substack{2 \leqslant j \leqslant n \\ j \neq i}} c_{1i}d_{i1}k_j, \quad c_3 = -\sum_{i=2}^n \sum_{\substack{2 \leqslant h < l \leqslant n \\ h,l \neq i}} c_{1i}d_{i1}k_h k_l, \quad \ldots,$$

$$c_{n-2} = -\sum_{i=2}^n \sum_{\substack{2 \leqslant h < l < m < \cdots < p \leqslant n \\ h,l,m,\ldots,p \neq i}} c_{1i}d_{i1}k_h k_l k_m \cdots k_p, \quad c_{n-1} = -\sum_{i=2}^n c_{1i}d_{i1} \prod_{\substack{j=2 \\ j \neq i}}^n k_j.$$

For the sake of discussion, one defines $S_j$ as follows:

$$S_j = \begin{vmatrix} d_1 & d_3 & d_5 & \cdots & d_{2j-1} \\ 1 & d_2 & d_4 & \cdots & d_{2j-2} \\ 0 & d_1 & d_3 & \cdots & d_{2j-3} \\ 0 & 1 & d_2 & \cdots & d_{2j-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_j \end{vmatrix},$$

where $d_1 = a_1$, $d_i = a_i + b_{i-1} + c_{i-1}$ for $i = 2, 3, \dots, n$. For convenience, one gives the following hypothesis.

(H2) $S_k > 0$, $d_k > 0$ for $k = 1, 2, \dots, n$.

**Theorem 1.** *If* $\tau_1 = \tau_2 = 0$, (H1) *and* (H2) *are satisfied, then* $E_1$ *is locally asymptotically stable.*

*Proof.* If $\tau_1 = \tau_2 = 0$, then (5) can be rewritten as

$$s^{n\alpha} + d_1 s^{(n-1)\alpha} + d_2 s^{(n-2)\alpha} + \cdots + d_{n-1} s^{\alpha} + d_n = 0,$$

where $d_1 = a_1$, $d_i = a_i + b_{i-1} + c_{i-1}$ for $i = 2, 3, \dots, n$. Let $\lambda = s^{\alpha}$, one can see that

$$\lambda^n + d_1 \lambda^{(n-1)} + d_2 \lambda^{(n-2)} + \cdots + d_{n-1}\lambda + d_n = 0.$$

Due to $S_k > 0$ and $d_k > 0$ for $k = 1, 2, \dots, n$, according to the Routh–Hurwitz criterion, one can see that all the roots of (5) have negative real parts. Then $E_1$ is asymptotically stable. □

Assume that (5) has a purely imaginary root $s = \mathrm{i}\varphi = \varphi(\cos\pi/2 + \mathrm{i}\sin\pi/2)(\varphi > 0)$. Let

$$\begin{aligned} P_1(s) &= s^{n\alpha} + a_1 s^{(n-1)\alpha} + a_2 s^{(n-2)\alpha} + \cdots + a_{n-1}s^{\alpha} + a_n, \\ P_2(s) &= b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2}s^{\alpha} + b_{n-1}, \\ P_3(s) &= c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2}s^{\alpha} + c_{n-1}. \end{aligned} \tag{6}$$

Substituting $s = \mathrm{i}\varphi$ into $P_1(s)$, $P_2(s)$, $P_3(s)$, one can get

$$\begin{aligned} P_1(s) &= \sum_{j=0}^{n-1} a_j \varphi^{(n-j)\alpha} \cos\frac{\alpha(n-j)\pi}{2} + a_n \\ &\quad + \sum_{j=0}^{n-1} a_j \varphi^{(n-j)\alpha} \sin\frac{\alpha(n-j)\pi}{2}\mathrm{i}, \end{aligned} \tag{$7_1$}$$

$$\begin{aligned} P_2(s) &= \sum_{j=2}^{n-2} b_{j-1} \varphi^{(n-j)\alpha} \cos\frac{\alpha(n-j)\pi}{2} + b_{n-1} \\ &\quad + \sum_{j=2}^{n-2} b_{j-1} \varphi^{(n-j)\alpha} \sin\frac{\alpha(n-j)\pi}{2}\mathrm{i}, \end{aligned} \tag{$7_2$}$$

$$P_3(s) = \sum_{j=2}^{n-2} c_{j-1}\varphi^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2} + c_{n-1} + \sum_{j=2}^{n-2} c_{j-1}\varphi^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\mathrm{i}. \tag{$7_3$}$$

Let

$$A = \sum_{j=0}^{n-1} a_j\varphi^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2} + a_n, \tag{$8_1$}$$

$$B = \sum_{j=0}^{n-1} a_j\varphi^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}, \tag{$8_2$}$$

$$C = \sum_{j=2}^{n-2} b_{j-1}\varphi^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2} + b_{n-1}, \tag{$8_3$}$$

$$D = \sum_{j=2}^{n-2} b_{j-1}\varphi^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}, \tag{$8_4$}$$

$$E = \sum_{j=2}^{n-2} c_{j-1}\varphi^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2} + c_{n-1}, \tag{$8_5$}$$

$$F = \sum_{j=2}^{n-2} c_{j-1}\varphi^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}. \tag{$8_6$}$$

By (5), (6), (7) and (8), one can see that

$$A + B\mathrm{i} + (C + D\mathrm{i})\big(\cos\varphi\tau_2 - \sin\varphi\tau_2\mathrm{i}\big) + (E + F\mathrm{i})\big(\cos\varphi(\tau_1+\tau_2) - \sin\varphi(\tau_1+\tau_2)\mathrm{i}\big) = 0. \tag{9}$$

In the rest of this section, the stability and bifurcation of $E_1$ are discussed under the following cases.

*Case 1:* $\tau_1 > 0, \tau_2 = 0$. In this case, (5) can be written as

$$s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_{n-1}s^{\alpha} + a_n + b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2}s^{\alpha} + b_{n-1} + \big[c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2}s^{\alpha} + c_{n-1}\big]\mathrm{e}^{-s\tau_1} = 0. \tag{10}$$

By (9), one can get

$$\big(E^2 + F^2\big)\big(\cos\varphi\tau_1 - \sin\varphi\tau_1\mathrm{i}\big) = -\big[AE + CE + BF + DF + (BE + DE - AF - CF)\mathrm{i}\big]. \tag{11}$$

Separating the real and imaginary parts of (11), then it follows that

$$\begin{aligned} \big(E^2 + F^2\big)\cos\varphi\tau_1 &= -(AE + CE + BF + DF), \\ \big(E^2 + F^2\big)\sin\varphi\tau_1 &= BE + DE - AF - CF. \end{aligned} \tag{12}$$

Add the squares of the corresponding sides of the above equation to get

$$\left(E^2+F^2\right)^2 = (BE+DE-AF-CF)^2 + (AE+CE+BF+DF)^2.$$

Let $B+D=M$, $A+C=N$, then

$$\left(E^2+F^2\right)^2 = M^2E^2+N^2F^2+N^2E^2+M^2F^2 = \left(M^2+N^2\right)\left(E^2+F^2\right).$$

If $E, F = 0$, then $\tau_1$ is not included in (10), so it can be omitted.

If $M^2+N^2-E^2-F^2=0$ has no real root, that is, (10) has no root with zero real parts for all $\tau_1 > 0$. One can see that the constant term of $M^2+N^2-E^2-F^2=0$ is $(a_n+b_{n-1})^2 - c_{n-1}^2$. If $(a_n+b_{n-1})^2 - c_{n-1}^2 < 0$, then (10) has at least one positive root. The delay $\tau_1$ can be used as a bifurcation parameter. From (12) one concludes

$$\tau_1^j = \frac{1}{\varphi(0)}\left[\arccos\frac{-(AE+CE+BF+DF)}{E^2+F^2} + 2j\pi\right], \quad j=0,1,2,\dots,n.$$

Let $\lambda(\tau_1) = \omega(\tau_1) + \mathrm{i}\varphi(\tau_1)$ be the eigenvalue of (10), so for some initial value of the bifurcation parameter $\tau_1$, one has $\omega(\tau_1^*) = 0$, $\varphi(\tau_1^*) = \varphi_0$, where $\tau_1^* = \min\{\tau_1^j\}$. Without loss of generality, one assumes $\varphi_0 > 0$.

To establish the Hopf bifurcation at $\tau_1^*$, one needs to prove that $\mathrm{Re}(\mathrm{d}s/\mathrm{d}\tau_1)|_{\tau_1=\tau_1^*} \neq 0$. Differentiating the characteristic equation (10) with respect to $\tau_1$ by means of the implicit function theorem, it is easy to get

$$\frac{\mathrm{d}s}{\mathrm{d}\tau_1} = \frac{sP_3(s)\mathrm{e}^{-s\tau_1}}{P_1'(s)+P_2'(s)+P_3'(s)\mathrm{e}^{-s\tau_1} - \tau_1P_3(s)\mathrm{e}^{-s\tau_1}}.$$

Then

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_1}\right]^{-1} = \frac{P_1'(s)+P_2'(s)}{sP_3(s)\mathrm{e}^{-s\tau_1}} + \frac{P_3'(s)}{sP_3(s)} - \frac{\tau_1}{s}.$$

By (6) and (10), one can see that $\mathrm{e}^{-s\tau_1} = -(P_1(s)+P_2(s))/P_3(s)$. Then

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_1}\right]^{-1} = -\frac{s(P_1'(s)+P_2'(s))}{s^2(P_1(s)+P_2(s))} + \frac{sP_3'(s)}{s^2P_3(s)} - \frac{\tau_1}{s}.$$

So

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_1}\right]^{-1}\bigg|_{s=\mathrm{i}\varphi_0} &= \mathrm{Re}\left[-\frac{s(P_1'(s)+P_2'(s))}{s^2(P_1(s)+P_2(s))} + \frac{sP_3'(s)}{s^2P_3(s)}\right]\bigg|_{s=\mathrm{i}\varphi_0} \\ &= \frac{N_1M - M_1N - E_1F + F_1E}{\varphi_0^2(E^2+F^2)},\end{aligned}$$

where

$$\begin{aligned}N_1 = \alpha\Bigg[&\sum_{j=0}^{n-1} a_j(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2} \\ &+ \sum_{j=2}^{n-2} b_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2}\Bigg],\end{aligned}$$

$$M_1 = \alpha\Bigg[\sum_{j=0}^{n-1} a_j(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2} + \sum_{j=2}^{n-2} b_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\Bigg],$$

$$E_1 = \alpha\Bigg[\sum_{j=2}^{n-2} c_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2}\Bigg],$$

$$F_1 = \alpha\Bigg[\sum_{j=2}^{n-2} c_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\Bigg].$$

Therefore, if $(N_1M - M_1N - E_1F + F_1E)/(E^2+F^2) \neq 0$, the transversality condition holds, and Hopf bifurcation occurs at $\tau_1 = \tau_1^*$, one has the following results.

**Theorem 2.** *Assume that* (H1) *and* (H2) *are satisfied.*

(i) *If $M^2 + N^2 - E^2 - F^2 = 0$ has no real root, then $E_1$ is locally asymptotically stable for $\tau_1 > 0$, $\tau_2 = 0$.*
(ii) *If $(a_n + b_{n-1})^2 - c_{n-1}^2 < 0$ and $(N_1M - M_1N - E_1F + F_1E)/(E^2+F^2) \neq 0$, then $E_1$ is locally asymptotically stable for $\tau_1 < \tau_1^*$, $\tau_2 = 0$; $E_1$ is unstable for $\tau_1 > \tau_1^*$, $\tau_2 = 0$; a Hopf bifurcation occurs at $\tau_1 = \tau_1^*$, $\tau_2 = 0$.*

*Case 2: $\tau_1 = 0, \tau_2 > 0$.* In this case, (5) can be written as

$$s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_{n-1}s^{\alpha} + a_n + \big[b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2}s^{\alpha} + b_{n-1} + c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2}s^{\alpha} + c_{n-1}\big]\mathrm{e}^{-s\tau_2} = 0. \tag{13}$$

Let $C + E = G$, $D + F = H$. By (9), one can get

$$\big(G^2 + H^2\big)(\cos\varphi\tau_2 - \sin\varphi\tau_2 \mathrm{i}) = -\big[AG + BH + (BG - AH)\mathrm{i}\big]. \tag{14}$$

Separating the real and imaginary parts of (14), then it follows that

$$\big(G^2 + H^2\big)\cos\varphi\tau_2 = -(AG + BH), \qquad \big(G^2 + H^2\big)\sin\varphi\tau_2 = BG - AH. \tag{15}$$

Add the squares of the corresponding sides of the above equation to get

$$\big(G^2 + H^2\big)^2 = (AG + BH)^2 + (BG - AH)^2 = \big(A^2 + B^2\big)\big(G^2 + H^2\big).$$

If $G, H = 0$, then $\tau_2$ is not included in (13), so it can be omitted.

If $A^2 + B^2 - (G^2 + H^2) = 0$ has no real root, that is, (13) has no roots with zero real parts for all $\tau_2 > 0$. One can see that the constant term of $A^2 + B^2 - (G^2 + H^2) = 0$ is $a_n^2 - (c_{n-1} + b_{n-1})^2$. If $a_n^2 - (c_{n-1} + b_{n-1})^2 < 0$, then (13) has at least one positive root. The delay $\tau_2$ can be used as a bifurcation parameter. From (15) one concludes

$$\tau_2^j = \frac{1}{\varphi(0)}\left[\arccos\frac{-(AG + BH)}{G^2 + H^2} + 2j\pi\right], \quad j = 0, 1, 2, \ldots, n.$$

Let $\lambda(\tau_2) = \omega(\tau_2) + \mathrm{i}\varphi(\tau_2)$ be the eigenvalue of (13), so for some initial value of the bifurcation parameter $\tau_2$, one has $\omega(\tau_2^*) = 0$, $\varphi(\tau_2^*) = \varphi_0$, where $\tau_2^* = \min\{\tau_2^j\}$. Without loss of generality, one assumes $\varphi_0 > 0$.

To establish the Hopf bifurcation at $\tau_2^*$, one needs to prove that $\mathrm{Re}(\mathrm{d}s/\mathrm{d}\tau_2)|_{\tau_2=\tau_2^*} \neq 0$. Differentiating the characteristic equation (13) with respect to $\tau_2$ by means of the implicit function theorem, it is easy to arrive at

$$\frac{\mathrm{d}s}{\mathrm{d}\tau_2} = \frac{s(P_2(s) + P_3(s))\mathrm{e}^{-s\tau_2}}{P_1'(s) + (P_2'(s) + P_3')(s)\mathrm{e}^{-s\tau_2} - \tau_2(P_2(s) + P_3(s))\mathrm{e}^{-s\tau_2}}.$$

Then

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_2}\right]^{-1} = \frac{P_1'(s)}{s(P_2(s) + P_3(s))\mathrm{e}^{-s\tau_2}} + \frac{P_2'(s) + P_3'(s)}{s(P_2(s) + P_3(s))} - \frac{\tau_2}{s}.$$

By (6) and (13), one can see that $\mathrm{e}^{-s\tau_2} = -P_1(s)/(P_2(s) + P_3(s))$. Then

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_2}\right]^{-1} = -\frac{P_1'(s)}{sP_1(s)} + \frac{P_2'(s) + P_3'(s)}{s(P_2(s) + P_3(s))} - \frac{\tau_2}{s}.$$

So

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_2}\right]^{-1}\Bigg|_{s=\mathrm{i}\varphi_0} &= \mathrm{Re}\left[-\frac{sP_1'(s)}{s^2P_1(s)} + \frac{sP_2'(s) + sP_3'(s)}{s^2(P_2(s) + P_3(s))} - \frac{\tau_2}{s}\right]\Bigg|_{s=\mathrm{i}\varphi_0} \\ &= \frac{A_1B - B_1A - G_1H + H_1G}{\varphi_0^2(A^2 + B^2)},\end{aligned}$$

where

$$\begin{aligned}A_1 &= \alpha\left[\sum_{j=0}^{n-1} a_j(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2}\right], \\ B_1 &= \alpha\left[\sum_{j=0}^{n-1} a_j(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\right], \\ G_1 &= \alpha\left[\sum_{j=2}^{n-2} c_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2}\right. \\ &\quad \left.+ \sum_{j=2}^{n-2} b_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2}\right], \\ H_1 &= \alpha\left[\sum_{j=2}^{n-2} c_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\right. \\ &\quad \left.+ \sum_{j=2}^{n-2} b_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\right].\end{aligned}$$

Therefore, if $(A_1B - B_1A - G_1H + H_1G)/(A^2 + B^2) \neq 0$, the transversality condition holds, and Hopf bifurcation occurs at $\tau_2 = \tau_2^*$, one has the following results.

**Theorem 3.** *Assume that* (H1) *and* (H2) *are satisfied.*

(i) *If* $A^2 + B^2 - (G^2 + H^2) = 0$ *has no real root, then* $E_1$ *is locally asymptotically stable for* $\tau_1 = 0$, $\tau_2 > 0$.

(ii) *If* $a_n^2 - (c_{n-1} + b_{n-1})^2 < 0$ *and* $(A_1B - B_1A - G_1H + H_1G)/(A^2 + B^2) \neq 0$, *then* $E_1$ *is locally asymptotically stable for* $\tau_1 = 0$, $\tau_2 < \tau_2^*$; $E_1$ *is unstable for* $\tau_1 = 0$, $\tau_2 > \tau_2^*$; *a Hopf bifurcation occurs at* $\tau_1 = 0, \tau_2 = \tau_2^*$.

*Case 3:* $\tau_1 = \tau_2 = \tau > 0$. In this case, (5) can be written as

$$\begin{aligned} &s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_{n-1}s^\alpha + a_n \\ &\quad + \left[b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2}s^\alpha + b_{n-1}\right]\mathrm{e}^{-s\tau} \\ &\quad + \left[c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2}s^\alpha + c_{n-1}\right]\mathrm{e}^{-2s\tau} = 0. \end{aligned} \tag{16}$$

It can be seen that

$$\begin{aligned} &\left[s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_{n-1}s^\alpha + a_n\right]\mathrm{e}^{s\tau} \\ &\quad + b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2}s^\alpha + b_{n-1} \\ &\quad + \left[c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2}s^\alpha + c_{n-1}\right]\mathrm{e}^{-s\tau} = 0. \end{aligned} \tag{17}$$

Assume that (17) has a purely imaginary root $s = \mathrm{i}\varphi = \varphi(\cos\pi/2 + \mathrm{i}\sin\pi/2)(\varphi > 0)$. It is easy to see that

$$(A + B\mathrm{i})(\cos\varphi\tau + \sin\varphi\tau\mathrm{i}) + C + D\mathrm{i} + (E + F\mathrm{i})(\cos\varphi\tau - \sin\varphi\tau\mathrm{i}) = 0. \tag{18}$$

Separating the real and imaginary parts of (18), then it follows that

$$\begin{aligned} (A + E)\cos\varphi\tau + (F - B)\sin\varphi\tau + C = 0, \\ (A - E)\sin\varphi\tau + (F + B)\cos\varphi\tau + D = 0. \end{aligned}$$

Solve equation (19), one has

$$\begin{aligned} \sin\varphi\tau &= -\frac{AD - BC - CF + DE}{A^2 + B^2 - E^2 - F^2}, \\ \cos\varphi\tau &= -\frac{AC + BD - CE - DF}{A^2 + B^2 - E^2 - F^2}. \end{aligned} \tag{19}$$

Adding the squares of the corresponding sides of the above equation, one has

$$\begin{aligned} &\left(A^2 + B^2 - E^2 - F^2\right)^2 - (AD - BC - CF + DE)^2 \\ &\quad - (AC + BD - CE - DF)^2 = 0. \end{aligned} \tag{20}$$

If (20) has no real root, that is, (16) has no roots with zero real parts for all $\tau > 0$, one can see that the constant term of (20) is $(a_n - c_{n-1})^2 - (a_n b_{n-1} - b_{n-1}c_{n-1})^2$. If $(a_n - c_{n-1})^2 - (a_n b_{n-1} - b_{n-1}c_{n-1})^2 < 0$, then (16) has at least one positive root. The delay $\tau$ can be used as a bifurcation parameter. From (19) one concludes

$$\tau^j = \frac{1}{\varphi(0)}\left[\arccos\frac{-(AC+BD-CE-DF)}{A^2+B^2-E^2-F^2}+2j\pi\right], \qquad j=0,1,2,\dots,n.$$

Let $\lambda(\tau) = \omega(\tau) + \mathrm{i}\varphi(\tau)$ be the eigenvalue of (16), so for some initial value of the bifurcation parameter $\tau$, one has $\omega(\tau^*) = 0$, $\varphi(\tau^*) = \varphi_0$, where $\tau^* = \min\{\tau^j\}$. Without loss of generality, one can assume $\varphi_0 > 0$.

To establish the Hopf bifurcation at $\tau^*$, one needs to prove that $\mathrm{Re}(\mathrm{d}s/\mathrm{d}\tau)|_{\tau=\tau^*} \neq 0$. Differentiating the characteristic equation (17) with respect to $\tau$ by means of the implicit function theorem, it is easy to arrive at

$$\frac{\mathrm{d}s}{\mathrm{d}\tau_2} = \frac{2sP_3(s)\mathrm{e}^{-2s\tau}+sP_2(s)\mathrm{e}^{-s\tau}}{P_1'(s)+P_2'(s)\mathrm{e}^{-s\tau}-\tau_2P_2(s)\mathrm{e}^{-s\tau}+P_3'(s)\mathrm{e}^{-2s\tau}-2\tau P_3(s)\mathrm{e}^{-2s\tau}},$$

so

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\tau}\right]^{-1} = \frac{P_1'(s)+P_2'(s)\mathrm{e}^{-s\tau}+P_3'(s)\mathrm{e}^{-2s\tau}}{2sP_3(s)\mathrm{e}^{-2s\tau}+sP_2(s)\mathrm{e}^{-s\tau}} - \frac{\tau}{s}.$$

It is easy to see

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\tau}\right]^{-1}\bigg|_{s=\mathrm{i}\varphi_0,\,\tau=\tau^*} &= \mathrm{Re}\left[\frac{s(P_1'(s)+P_2'(s)\mathrm{e}^{-s\tau}+P_3'(s)\mathrm{e}^{-2s\tau})}{2s^2P_3(s)\mathrm{e}^{-2s\tau}+s^2P_2(s)\mathrm{e}^{-s\tau}}\right]\bigg|_{s=\mathrm{i}\varphi_0,\,\tau=\tau^*}\\ &= \frac{J_1I_2-J_2I_1}{-\varphi_0^2(I_1^2+I_2^2)},\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \alpha\left[\sum_{j=2}^{n-2} b_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\cos\frac{\alpha(n-j)\pi}{2}\right],\\
D_1 &= \alpha\left[\sum_{j=2}^{n-2} b_{j-1}(n-j)\varphi_0^{(n-j)\alpha}\sin\frac{\alpha(n-j)\pi}{2}\right],\\
I_1 &= E\cos 2\varphi_0\tau^* + F\sin 2\varphi_0\tau^* + D\sin\varphi_0\tau^* + C\cos\varphi_0\tau^*,\\
I_2 &= -E\sin 2\varphi_0\tau^* + F\cos 2\varphi_0\tau^* - C\sin\varphi_0\tau^* + D\cos\varphi_0\tau^*,\\
J_1 &= A_1 + C_1\cos\varphi_0\tau^* + D_1\sin\varphi_0\tau^* + E_1\cos 2\varphi_0\tau^* + F_1\sin 2\varphi_0\tau^*,\\
J_2 &= B_1 + D_1\cos\varphi_0\tau^* - C_1\sin\varphi_0\tau^* + F_1\cos 2\varphi_0\tau^* - E_1\sin 2\varphi_0\tau^*.
\end{aligned}$$

Therefore, if $-(J_1I_2-J_2I_1)/(I_1^2+I_2^2) \neq 0$, the transversality condition holds, and Hopf bifurcation occurs at $\tau = \tau^*$, one has the following results.

**Theorem 4.** *Assume that* (H1) *and* (H2) *are satisfied.*

(i) *If* $(A^2+B^2-E^2-F^2)^2-(AD-BC-CF+DE)^2-(AC+BD-CE-DF)^2 = 0$ *has no real root, then* $E_1$ *is locally asymptotically stable for* $\tau_1=\tau_2=\tau>0$.

(ii) *If* $(a_n - c_{n-1})^2 - (a_n b_{n-1} - b_{n-1} c_{n-1})^2 < 0$ *and* $-(J_1 I_2 - J_2 I_1)/(I_1^2 + I_2^2) \neq 0$*, then* $E_1$ *is locally asymptotically stable for* $\tau_1 = \tau_2 < \tau^*$*;* $E_1$ *is unstable for* $\tau_1 = \tau_2 > \tau^*$*; a Hopf bifurcation occurs at* $\tau_1 = \tau_2 = \tau^*$*.*

*Case 4:* $\tau_1 \in [0, \tau_1^*)$, $\tau_2 > 0$. In this case, (5) can be written as

$$s^{n\alpha} + a_1 s^{(n-1)\alpha} + \cdots + a_{n-1} s^\alpha + a_n + \big[b_1 s^{(n-2)\alpha} + b_2 s^{(n-3)\alpha} + \cdots + b_{n-2} s^\alpha + b_{n-1} + \big(c_1 s^{(n-2)\alpha} + c_2 s^{(n-3)\alpha} + \cdots + c_{n-2} s^\alpha + c_{n-1}\big) e^{-s\tau_1}\big] e^{-s\tau_2} = 0, \quad (21)$$

Assume that (21) has a purely imaginary root $s = i\varphi = \varphi(\cos\pi/2 + i\sin\pi/2)$, $\varphi > 0$. One gets

$$\big(O^2 + Q^2\big)(\cos\varphi\tau_2 - \sin\varphi\tau_2 i) = -\big[AO + BQ + (BO - AQ)i\big], \quad (22)$$

where $O = (C + E\cos\varphi\tau_1 + F\sin\varphi\tau_1)$, $Q = (D + F\cos\varphi\tau_1 - E\sin\varphi\tau_1)$. Separating the real and imaginary parts of (22), then it follows that

$$\begin{aligned} \big(O^2 + Q^2\big)\cos\varphi\tau_2 &= -(AO + BQ), \\ \big(O^2 + Q^2\big)\sin\varphi\tau_2 &= -(BO - AQ). \end{aligned} \quad (23)$$

Add the squares of the corresponding sides of the above equation to get

$$\big(O^2 + Q^2\big)^2 = (AO + BQ)^2 + (BO - AQ)^2 = \big(O^2 + Q^2\big)\big(A^2 + B^2\big).$$

If $O, Q = 0$, then $\tau_2$ is not included in (21), thus it can be omitted.

If $A^2 + B^2 - (O^2 + Q^2) = 0$ has no real root, that is, (21) has no roots with zero real parts for all $\tau_2 > 0$, one can see that the constant term of $A^2 + B^2 - (O^2 + Q^2) = 0$ is $a_n^2 - 2b_n c_{n-1}\cos\varphi\tau_1 - b_{n-1}^2 - c_{n-1}^2$. If $a_n^2 - 2b_n c_{n-1} - b_{n-1}^2 - c_{n-1}^2 < 0$, then (21) has at least one positive root. The delay $\tau_2$ can be used as a bifurcation parameter. From (23) one concludes

$$\tau_2^j = \frac{1}{\varphi(0)}\left[\arccos\frac{-(AO + BQ)}{O^2 + Q^2} + 2j\pi\right], \quad j = 0, 1, 2, \ldots, n.$$

Let $\lambda(\tau_2) = \omega(\tau_2) + i\varphi(\tau_2)$ be the eigenvalue of (21), so that for some initial value of the bifurcation parameter $\tau_2$, one has $\omega(\tau_2^{**}) = 0$, $\varphi(\tau_2^{**}) = \varphi_0$, where $\tau_2^{**} = \min\{\tau_2^j\}$. Without loss of generality, one can assume $\varphi_0 > 0$.

To establish the Hopf bifurcation at $\tau_2^{**}$, one needs to prove that $\mathrm{Re}(ds/d\tau_2)|_{\tau_2 = \tau_2^{**}} \neq 0$. Differentiating the characteristic equation (21) with respect to $\tau_2$ by means of the implicit function theorem, it is easy to arrive at

$$\frac{ds}{d\tau_2} = \frac{s(P_2(s) + P_3(s)e^{-s\tau_1})e^{-s\tau_2}}{\Psi - \tau_2(P_2(s) + P_3(s)e^{-s\tau_1})e^{-s\tau_2}},$$

where $\Psi = P_1'(s) + (P_2'(s) + P_3(s)' e^{-s\tau_1})e^{-s\tau_2} - \tau_1 P_3(s) e^{-s\tau_1} e^{-s\tau_2}$. So

$$\left[\frac{ds}{d\tau_2}\right]^{-1} = \frac{\Psi}{s(P_2(s) + P_3(s)e^{-s\tau_1})e^{-s\tau_2}} - \frac{\tau_2}{s}.$$

It is easy to see

$$\begin{aligned}\operatorname{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_2}\right]^{-1}\bigg|_{s=\mathrm{i}\varphi_0,\,\tau_2\tau_2^{**}} &= \operatorname{Re}\left[\frac{s\Psi}{s^2(P_2(s)+P_3(s)\mathrm{e}^{-s\tau_1})\mathrm{e}^{-s\tau_2}}-\frac{\tau_2}{s}\right]\bigg|_{s=\mathrm{i}\varphi_0,\,\tau_2=\tau_2^{**}}\\ &= \frac{R_2T_1-T_2R_1}{-\varphi_0^2(R_1^2+R_2^2)},\end{aligned}$$

where

$$\begin{aligned}R_1 &= C\cos\varphi_0\tau_2^{**}+D\sin\varphi_0\tau_2^{**}+E\cos\varphi_0(\tau_1+\tau_2^{**})+F\sin\varphi_0(\tau_1+\tau_2^{**}),\\ R_2 &= D\cos\varphi_0\tau_2^{**}-C\sin\varphi_0\tau_2^{**}+F\cos\varphi_0(\tau_1+\tau_2^{**})-E\sin\varphi_0(\tau_1+\tau_2^{**}),\\ T_1 &= A_1+C_1\cos\varphi_0\tau_2^{**}+D_1\sin\varphi_0\tau_2^{**}+E_1\cos\varphi_0(\tau_1+\tau_2^{**})\\ &\quad+F_1\sin\varphi_0(\tau_1+\tau_2^{**})-\tau_1E\cos\varphi_0(\tau_1+\tau_2^{**})-\tau_1F\sin\varphi_0(\tau_1+\tau_2^{**}),\\ T_2 &= B_1+D_1\cos\varphi_0\tau_2^{**}-C_1\sin\varphi_0\tau_2^{**}+F_1\cos\varphi_0(\tau_1+\tau_2^{**})\\ &\quad-E_1\sin\varphi_0(\tau_1+\tau_2^{**})+\tau_1E\sin\varphi_0(\tau_1+\tau_2^{**})-\tau_1F\cos\varphi_0(\tau_1+\tau_2^{**}).\end{aligned}$$

Therefore, if $-(R_2T_1-T_2R_1)/(R_1^2+R_2^2)\neq 0$, the transversality condition holds, and Hopf bifurcation occurs at $\tau_2=\tau_2^{**}$, one has the following results.

**Theorem 5.** *Assume that* (H1) *and* (H2) *are satisfied.*

(i) *If* $A^2+B^2-(O^2+Q^2)=0$ *has no real root, then* $E_1$ *is locally asymptotically stable for* $\tau_1\in[0,\tau_1^*)$, $\tau_2>0$.

(ii) *If* $a_n^2-2b_nc_{n-i}-b_{n-1}^2-c_{n-1}^2<0$ *and* $-(R_2T_1-T_2R_1)/(R_1^2+R_2^2)\neq 0$*, then* $E_1$ *is locally asymptotically stable for* $\tau_1\in[0,\tau_1^*)$, $\tau_2<\tau_2^{**}$; $E_1$ *is unstable for* $\tau_1\in[0,\tau_1^*)$, $\tau_2>\tau_2^{**}$; *a Hopf bifurcation occurs at* $\tau_1\in[0,\tau_1^*)$, $\tau_2=\tau_2^{**}$.

*Case 5:* $\tau_1>0,\tau_2\in[0,\tau_2^*)$. In this case, (5) can be written as

$$\begin{aligned}&s^{n\alpha}+a_1s^{(n-1)\alpha}+\cdots+a_{n-1}s^{\alpha}+a_n\\ &\quad+\left[b_1s^{(n-2)\alpha}+b_2s^{(n-3)\alpha}+\cdots+b_{n-2}s^{\alpha}+b_{n-1}\right]\mathrm{e}^{-s\tau_2}\\ &\quad+\left[\left(c_1s^{(n-2)\alpha}+c_2s^{(n-3)\alpha}+\cdots+c_{n-2}s^{\alpha}+c_{n-1}\right)\mathrm{e}^{-s\tau_2}\right]\mathrm{e}^{-s\tau_1}=0.\end{aligned}\tag{24}$$

Assume that (24) has a purely imaginary root $s=\mathrm{i}\varphi=\varphi(\cos\pi/2+\mathrm{i}\sin\pi/2)$, $\varphi>0$. One gets

$$\left(V^2+W^2\right)(\cos\varphi\tau_2-\sin\varphi\tau_2\mathrm{i})=-\left[VY+ZW+(ZV-WY)\mathrm{i}\right],\tag{25}$$

where $V=(E\cos\varphi\tau_2+F\sin\varphi\tau_2)$, $W=(F\cos\varphi\tau_2-E\sin\varphi\tau_2)$, $Y=A+C\cos\varphi\tau_2+D\sin\varphi\tau_2$, $Z=B+D\cos\varphi\tau_2-C\sin\varphi\tau_2$. Separating the real and imaginary parts of (25), then it follows that

$$\begin{aligned}\left(V^2+W^2\right)\cos\varphi\tau_2 &= -(VY+ZW),\\ \left(V^2+W^2\right)\sin\varphi\tau_2 &= -(ZV-WY).\end{aligned}\tag{26}$$

Adding the squares of the corresponding sides of the above equation, one has

$$\left(V^2+W^2\right)^2=(VY+ZW)^2+(ZV-WY)^2=\left(V^2+W^2\right)\left(Y^2+Z^2\right).$$

If $V, W = 0$, then $\tau_1$ is not included in (24), so it can be omitted.

If $Y^2 + Z^2 - (V^2 + W^2) = 0$ has no real root. That is (24) has no roots with zero real parts for all $\tau_1 > 0$. One can see that the constant term of $Y^2 + Z^2 - (V^2 + W^2)$ is $a_n^2 - 2b_n c_{n-1} \cos\varphi\tau_2 - b_{n-1}^2 - c_{n-1}^2$. If $a_n^2 - 2a_n b_{n-1} + b_{n-1}^2 - c_{n-1}^2 < 0$, then (24) has at least one positive root. The delay $\tau_1$ can be used as a bifurcation parameter. From (26), one concludes

$$\tau_1^j = \frac{1}{\varphi(0)}\left[\arccos\frac{-(VY + ZW)}{(V^2 + W^2)} + 2j\pi\right], \quad j = 0, 1, 2, \dots, n.$$

Let $\lambda(\tau_1) = \omega(\tau_1) + \mathrm{i}\varphi(\tau_1)$ be the eigenvalue of (24), so for some initial value of the bifurcation parameter $\tau_1$, one has $\omega(\tau_1^{**}) = 0$, $\varphi(\tau_1^{**}) = \varphi_0$, where $\tau_1^{**} = \min\{\tau_1^j\}$. Without loss of generality, one can assume $\varphi_0 > 0$.

To establish the Hopf bifurcation at $\tau_1^{**}$, one needs to prove that $\mathrm{Re}(\mathrm{d}s/\mathrm{d}\tau_1)|_{\tau_1=\tau_1^{**}} \neq 0$. Differentiating the characteristic equation (24) with respect to $\tau_1$ by means of the implicit function theorem, it is easy to arrive at

$$\frac{\mathrm{d}s}{\mathrm{d}\tau_1} = \frac{sP_3(s)\mathrm{e}^{-s(\tau_1+\tau_2)}}{\varPhi - (\tau_1 + \tau_2)P_3(s)\mathrm{e}^{-s(\tau_1+\tau_2)}},$$

where $\varPhi = P_1'(s) + P_2'(s)\mathrm{e}^{-s\tau_2} + P_3(s)'\mathrm{e}^{-s(\tau_1+\tau_2)} - \tau_2 P_2(s)\mathrm{e}^{-s\tau_2}$. So

$$\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_1}\right]^{-1} = \frac{\varPhi}{sP_3(s)\mathrm{e}^{-s(\tau_1+\tau_2)}} - \frac{\tau_1 + \tau_2}{s}.$$

It is easy to see

$$\begin{aligned}\mathrm{Re}\left[\frac{\mathrm{d}s}{\mathrm{d}\tau_1}\right]^{-1}\bigg|_{s=\mathrm{i}\varphi_0,\,\tau_1=\tau_1^{**}} &= \mathrm{Re}\left[\frac{s\varPhi}{s^2P_3(s)\mathrm{e}^{-s(\tau_1+\tau_2)}} - \frac{\tau_1+\tau_2}{s}\right]\bigg|_{s=\mathrm{i}\varphi_0,\,\tau_1=\tau_1^{**}} \\ &= \frac{S_2U_1 - U_2S_1}{-\varphi_0^2(U_1^2 + U_2^2)},\end{aligned}$$

where

$$\begin{aligned}U_1 &= E\cos\varphi_0(\tau_1^{**} + \tau_2) + F\sin\varphi_0(\tau_1^{**} + \tau_2),\\ U_2 &= F\cos\varphi_0(\tau_1^{**} + \tau_2) - E\sin\varphi_0(\tau_1^{**} + \tau_2),\\ S_1 &= A_1 + C_1\cos\varphi_0\tau_2 + D_1\sin\varphi_0\tau_2 + E_1\cos\varphi_0(\tau_1^{**} + \tau_2)\\ &\quad + F_1\sin\varphi_0(\tau_1^{**} + \tau_2) - \tau_2(C\cos\varphi_0\tau_2 + D\sin\varphi_0\tau_2),\\ S_2 &= B_1 + D_1\cos\varphi_0\tau_2 - C_1\sin\varphi_0\tau_2 + F_1\cos\varphi_0(\tau_1^{**} + \tau_2)\\ &\quad - E_1\sin\varphi_0(\tau_1^{**} + \tau_2) - \tau_2(D\cos\varphi_0\tau_2 - C\sin\varphi_0\tau_2).\end{aligned}$$

Therefore, if $-(S_2U_1 - U_2S_1)/(U_1^2 + U_2^2) \neq 0$, the transversality condition holds, and Hopf bifurcation occurs at $\tau_1 = \tau_1^{**}$, one has the following results.

**Theorem 6.** *Assume that* (H1) *and* (H2) *are satisfied.*

(i) *If* $Y^2 + Z^2 - (V^2 + W^2) = 0$ *has no real root, then* $E_1$ *is locally asymptotically stable for* $\tau_1 > 0, \tau_2 \in [0, \tau_2^*)$.

(ii) *If* $a_n^2 - 2a_n b_{n-1} + b_{n-1}^2 - c_{n-1}^2 < 0$ *and* $-(S_2U_1 - U_2S_1)/(U_1^2 + U_2^2) \neq 0$*, then* $E_1$ *is locally asymptotically stable for* $\tau_1 < \tau_1^{**}$*,* $\tau_2 \in [0, \tau_2^*)$*;* $E_1$ *is unstable for* $\tau_1 > \tau_1^{**}$*,* $\tau_2 \in [0, \tau_2^*)$*; a Hopf bifurcation occurs at* $\tau_1 = \tau_1^{**}$*,* $\tau_2 \in [0, \tau_2^*)$*.*

**Remark 1.** Comparing Theorem 2 with Theorem 6 and Theorem 3 with Theorem 5, it is easy to know that $\tau_1$ and $\tau_2$ will influence each other.

**Remark 2.** From Theorems 2–6 one can see that all of the expression of $\tau_1^*$, $\tau_2^*$, $\tau^*$, $\tau_1^{**}$ and $\tau_2^{**}$ contain the order $\alpha$. So one can conclude that if $\tau_1$ and $\tau_2$ are determined, the order will become a bifurcation parameter.

## 4 Numerical simulation

In this section, an example will be proposed for numerical simulations to support the result mentioned above.

Considering the functions of system (1) as follows:

$$\begin{aligned} D^\alpha x_1(t) &= x_1(t)\big[1 - x_1(t) - x_2(t) + 0.5 * x_2(t - \tau_1)\big], \\ D^\alpha x_2(t) &= x_2(t)\big[3x_1(t - \tau_2) - x_2(t)\big] \end{aligned} \tag{27}$$

with initial condition $\alpha = 0.9$, $\phi_1(0) = 0.5$ and $\phi_2(0) = 1$, then the characteristic equation is

$$s^{2\alpha} + 1.6s^\alpha + 1.2 + 1.44\mathrm{e}^{-s\tau_2} - 0.72\mathrm{e}^{-s(\tau_1+\tau_2)} = 0.$$

It is easy to see that (H1) and (H2) are satisfied. From Fig. 1, one can see that $E_1$ is locally asymptotically stable for $\tau_1 = 0$, $\tau_2 = 0$. This conforms Theorem 1.

By calculation, it is easy to know that $M^2 + N^2 - E^2 - F^2 = 0$ has no real root. From Fig. 2 one can see that $E_1$ is locally asymptotically stable for $\tau_1 > 0$, $\tau_2 = 0$. This conforms Theorem 2.

By calculating, it is easy to know that $a_2^2 - (c_1 + b_1)^2 < 0$. One can get $\varphi(0) = 0.2915$ the critical value of system (27) $\tau_2^* \approx 5.2122$. By calculation, one obtains that $(A_1B - B_1A - G_1H + H_1G)/(\varphi^2(A^2 + B^2) \neq 0$. From Fig. 3 one can see that $E_1$ is locally asymptotically stable for $\tau_1 = 0$, $\tau_2 < \tau_2^*$, and Fig. 4 shows that $E_1$ is unstable for $\tau_1 = 0$, $\tau_2 > \tau_2^*$. This conforms Theorem 3.

By calculating, it is easy to know that $(a_2 - c_1)^2 - (a_2b_1 - b_1c_1)^2 < 0$. One can get $\varphi(0) = 0.9494$ the critical value of system (27) $\tau^* \approx 1.1492$. By calculation, one obtains that $-(J_1I_2 - J_2I_1)/\varphi^2(I_1^2 + I_2^2) \neq 0$. Figure 5 shows that $E_1$ is locally asymptotically stable for $\tau_1 = \tau_2 < \tau^*$, and from Fig. 6 one can see that $E_1$ is unstable for $\tau_1 = \tau_2 > \tau^*$. This conforms Theorem 4.

Let $\tau_2 = 2$, one can get the critical value of system (27) $\tau^{**} \approx 1.9733$. By calculation, one obtains that $-(S_2U_1 - U_2S_1)/(\varphi^2(U_1^2 + U_2^2) \neq 0$. Figure 7 shows that $E_1$ is locally asymptotically stable for $\tau_1 < \tau_1^{**}$, $\tau_2 \in [0, \tau_2^*)$, and Fig. 6 shows that $E_1$ is unstable for $\tau_1 > \tau_1^{**}$, $\tau_2 \in [0, \tau_2^*)$. This conforms Theorem 6.

Let $\tau_1 = 2$, $\tau_2 = 0$ and $\tau_1 = 0$, $\tau_2 = 2$, while keeping the other parameters constant, one can get Figs. 8 and 9. Comparing Fig. 6 with Figs. 8 and 9, one can get that two delays will effect each other.

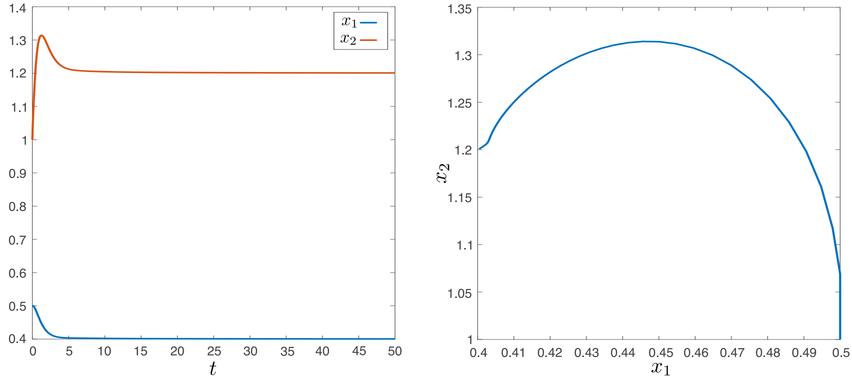


**Figure 1.** $E_1$ is asymptotically stable when $\tau_1 = 0$, $\tau_2 = 0$, $\alpha = 0.9$.

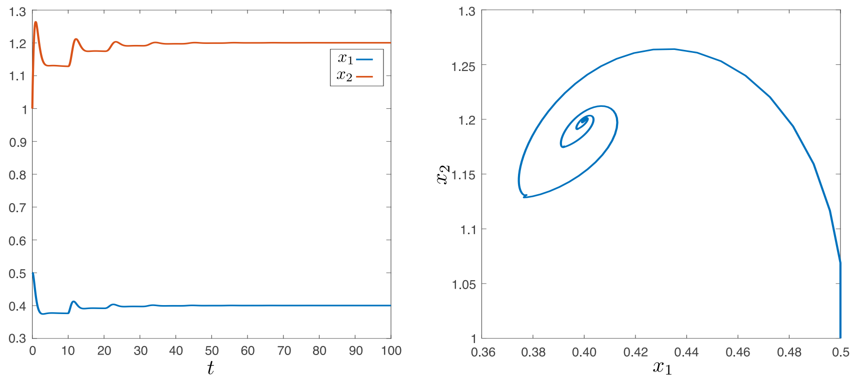


**Figure 2.** $E_1$ is asymptotically stable when $\tau_1 = 10$, $\tau_2 = 0$, $\alpha = 0.9$.

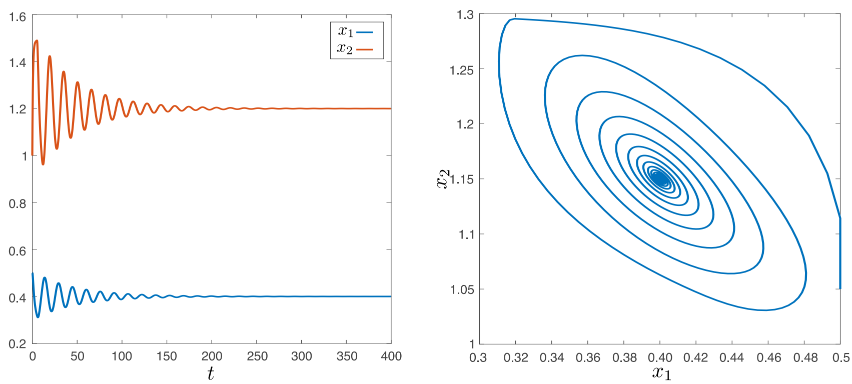


**Figure 3.** $E_1$ is asymptotically stable when $\tau_1 = 0$, $\tau_2 = 5$, $\alpha = 0.9$.

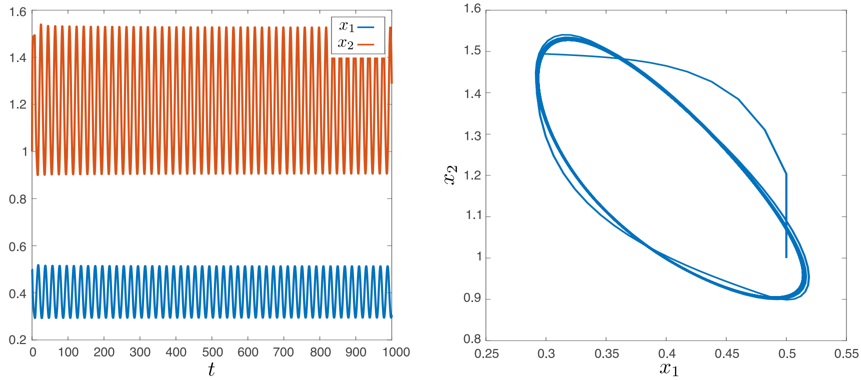


**Figure 4.** Stable periodic orbit of system (1) when $\tau_1 = 0$, $\tau_2 = 7$, $\alpha = 0.9$.

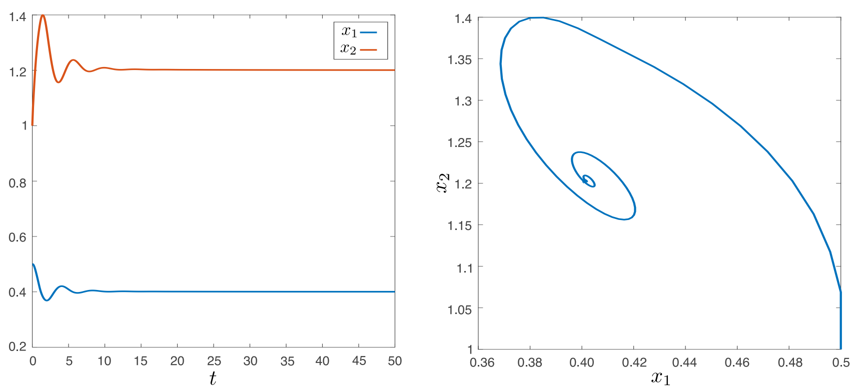


**Figure 5.** $E_1$ is asymptotically stable when $\tau_1 = 1$, $\tau_2 = 1$, $\alpha = 0.9$.

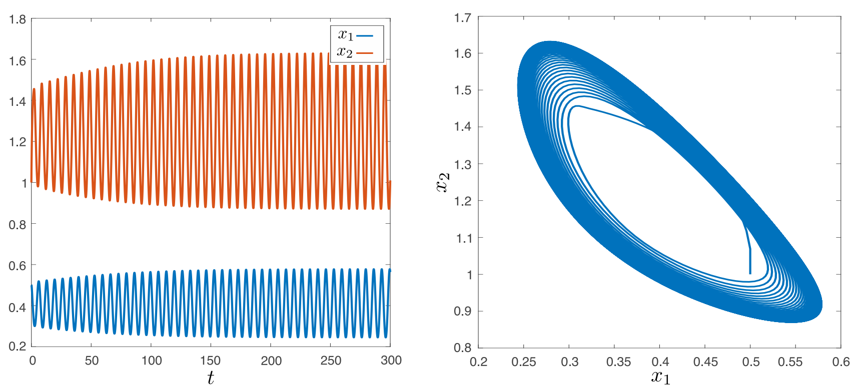


**Figure 6.** Stable periodic orbit of system (1) when $\tau_1 = 2$, $\tau_2 = 2$, $\alpha = 0.9$.

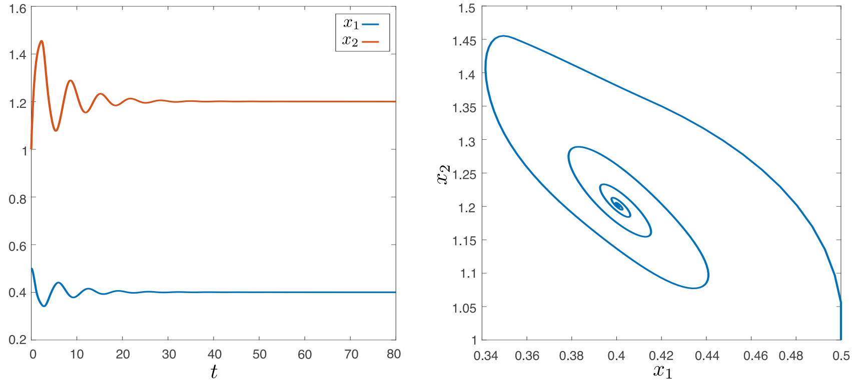


**Figure 7.** $E_1$ is asymptotically stable when $\tau_1 = 1$, $\tau_2 = 2$, $\alpha = 0.9$.

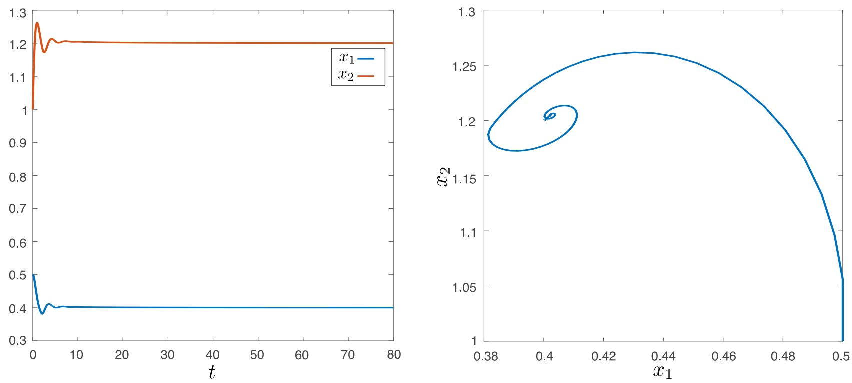


**Figure 8.** $E_1$ is asymptotically stable when $\tau_1 = 2$, $\tau_2 = 0$, $\alpha = 0.9$.

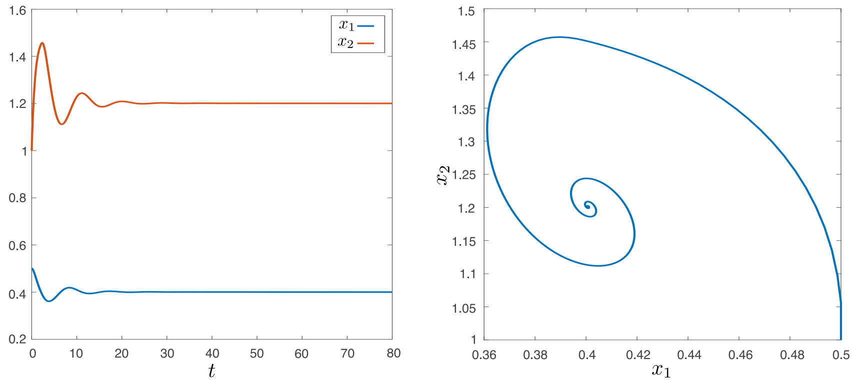


**Figure 9.** $E_1$ is asymptotically stable when $\tau_1 = 0$, $\tau_2 = 2$, $\alpha = 0.9$.

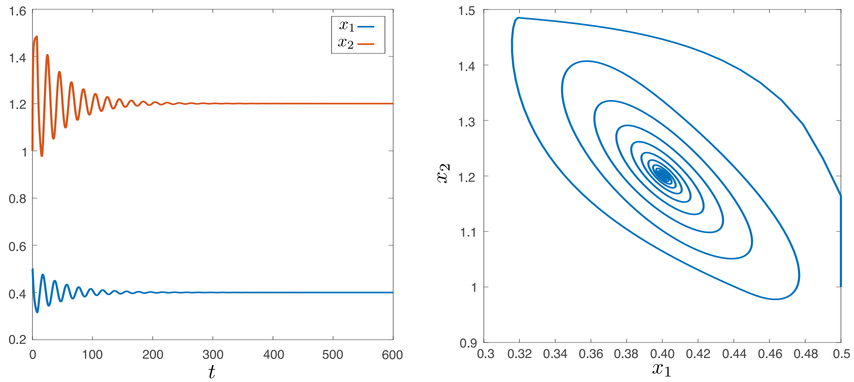


**Figure 10.** $E_1$ is asymptotically stable when $\tau_1 = 0$, $\tau_2 = 7$, $\alpha = 0.8$.

Let $\tau_1 = 0$, $\tau_2 = 7$, $\alpha = 0.8$, while keeping the other parameters constant, one can get Fig. 10. Comparing Fig. 4 with Fig. 10, one can get that whether or not the equilibrium of system (1) is stable, it is related to $\alpha$.

## 5 Conclusions

This paper considers a delayed generalized fractional-order biological networks with predation behavior and material cycle. The stability and bifurcation of the present model are studied and some theoretical results are given. It shows that the stability and bifurcation rely on time delays for the proposed system and the order also has a effect on it. In addition, it is displayed that the time delays will effect each other. Finally, some numerical simulations are presented for supporting them.

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