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## Multiple positive solutions for singular higher-order semipositone fractional differential equations with $p$ -Laplacian\*

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**Abstract.** In this article, together with Leggett–Williams and Guo–Krasnosel’skii fixed point theorems, height functions on special bounded sets are constructed to obtain the existence of at least three positive solutions for some higher-order fractional differential equations with  $p$ -Laplacian. The nonlinearity permits singularities both on the time and the space variables, and it also may change its sign.

**Keywords:** fractional differential equations, height functions, singularity on space variable, semipositone, triple positive solutions.

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### 1 Introduction

In this article, we are devoted to investigating the existence of multiple positive solutions for the following fractional differential equation with  $p$ -Laplacian (FPDE for short):

$$\begin{aligned}
 -D_{0+}^{\mu}(\varphi_p(-D_{0+}^{\alpha}u(t))) &= f(t, u(t)), \quad 0 < t < 1, \\
 u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^{\alpha}u(0) = D_{0+}^{\alpha}u(1) = 0, \\
 D_{0+}^{\beta}u(1) &= \lambda \int_0^{\eta} D_{0+}^{\gamma}u(t) \, dA(t),
 \end{aligned} \tag{1}$$

where  $D_{0+}^{\mu}$ ,  $D_{0+}^{\alpha}$ ,  $D_{0+}^{\beta}$  and  $D_{0+}^{\gamma}$  denote the standard Riemann–Liouville derivatives of orders  $\mu$ ,  $\alpha$ ,  $\beta$  and  $\gamma$ , respectively,  $f \in C(I \times \mathbb{R}, \mathbb{R})$ ,  $J = [0, 1]$ ,  $I = (0, 1)$ ,  $\mathbb{R} = (-\infty, +\infty)$ ,  $\mathbb{R}_+ = (0, +\infty)$ ,  $1 < \mu \leq 2$ ,  $n - 1 < \alpha \leq n$ ,  $n \geq 3$ ,  $\beta \geq 1$ ,  $\alpha - \beta - 1 > 0$ ,  $\alpha - \gamma - 1 > 0$ ,  $0 < \eta \leq 1$ ,  $\lambda$  is a positive parameter with  $0 \leq \lambda \Gamma(\alpha - \beta) \times \int_0^{\eta} t^{\alpha - \gamma - 1} \, dA(t) < \Gamma(\alpha - \gamma)$ ,  $\varphi_p(s) = |s|^{p-2}s$ ,  $p > 1$ ,  $A$  is a function of bounded variation,  $\int_0^{\eta} D_{0+}^{\gamma}u(t) \, dA(t)$  denotes the Riemann–Stieltjes integral with respect to  $A$ . It is clear,  $\varphi_p(s)$  is invertible, and its inverse operator is  $\varphi_q(s)$ , where  $q > 1$  with  $1/p + 1/q = 1$ . In this paper, the nonlinearity permits singularities both on the time, and the space variables and it also may change its sign.

Nowadays, there is a sharp increase in investigation of fractional nonlocal problems for their wide successful applications in tackling various physical phenomena in natural sciences and engineering; see [1–5, 7–13, 15–26, 29–55]. Very recently, by constructing height functions in different bounded sets, researchers obtained existence results on positive solutions and multiple positive solutions for some fractional nonlocal problems [31, 33, 34, 46, 51]. Recently, when  $f$  is semipositone, Luca [31] investigated the existence of positive solutions for the following fractional differential equations:

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{2}$$

subject to Riemann–Stieltjes boundary conditions

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^p u(1) = \int_0^1 D_{0+}^q u(t) \, dH(t),$$

where the BCs is the generalization of multi-point BCs

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^p u(1) = \sum_{i=1}^m a_i D_{0+}^q u(\xi_i),$$

which is studied by Zhang et al. [46], Pu et al. [34], Henderson and Luca [25]. In a recent paper [50], by means of the property of  $u_0$ -positive linear operator and Banach contraction

map principle, we discussed the uniqueness of solution for BVP (2) under more general integral BCs

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad D_{0+}^\beta u(1) = \lambda \int_0^\eta h(t) D_{0+}^\gamma u(t) dt.$$

Motivated by papers mentioned above and the multiple solutions results such as in [3–5, 28], the purpose of this paper is to investigate the existence of at least triple positive solutions for FPDE (1). The discussion based on height functions constructed on special bounded sets together with Leggett–Williams and Guo–Krasnosel’skii fixed point theorems. As far as we know, there are relatively few results on multiple solutions for fractional differential equation nonlocal problems when the nonlinearity permits singularities both on the time and the space variables. This paper admits the following features. Firstly, Riemann–Stieltjes integral is involved in BCs, which makes the problems discussed in this paper be a generalization of [25, 34, 46]. Secondly, two parameters  $\lambda$ ,  $\eta$  and  $p$ -Laplacian operator are contained in FPDE (1), so the problems considered in this paper perform more general form compared with those in [7, 31, 47, 50, 51]. Thirdly, different from the traditional results obtained by Leggett–Williams, the nonlinearity permits singularity on space variable. It should be pointed out that the height function  $\widehat{\Psi}(t, r_1, r_2)$  employed in this article is quite different from  $\widehat{\psi}(t, r_1, r_2)$  used in [46]. This makes the verification of the condition easier and more efficient.

## 2 Auxiliary results

The basic Banach space used in this paper is  $E = C[0, 1]$  equipped with the maximum norm  $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ . By a positive solution of BVP (1) we mean a function  $u \in E$  satisfying BVP (1) with  $u(t) > 0$  for all  $t \in (0, 1]$ .

**Definition 1.** (See [28].) A functional  $\zeta : P \rightarrow [0, +\infty)$  is called a concave positive functional on a cone  $P$  if

$$\zeta(tx + (1 - t)y) \geq t\zeta(x) + (1 - t)\zeta(y) \quad \forall x, y \in P, 0 \leq t \leq 1.$$

**Lemma 1.** (See [6].) Given  $h \in L^1[0, 1]$  and  $1 < \mu \leq 2$ , the unique solution of the boundary value problems

$$-D_{0+}^\mu u(t) = h(t), \quad 0 < t < 1, \quad u(0) = u(1) = 0,$$

is

$$u(t) = \int_0^1 K_\mu(t, s)y(s) ds, \quad t \in [0, 1],$$

where

$$K_\mu(t, s) = \begin{cases} \frac{t(1-s)^{\mu-1} - (t-s)^{\mu-1}}{\Gamma(\mu)}, & 0 \leq s \leq t \leq 1, \\ \frac{t(1-s)^{\mu-1}}{\Gamma(\mu)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{3}$$

**Lemma 2.** Assume that  $\lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t) \neq \Gamma(\alpha - \gamma)$ . Then for any  $y \in C[0, 1]$ , the unique solution of the boundary value problems

$$\begin{aligned}
 -D_{0+}^\alpha u(t) &= y(t), \quad 0 < t < 1, \\
 u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^\beta u(1) = \lambda \int_0^\eta D_{0+}^\gamma u(t) dA(t),
 \end{aligned} \tag{4}$$

solves

$$u(t) = \int_0^1 G(t, s)y(s) ds, \quad t \in [0, 1],$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \tag{5}$$

$$G_1(t, s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{6}$$

$$G_2(t, s) = \frac{\lambda\Gamma(\alpha - \beta)t^{\alpha-1}}{\Gamma(\alpha - \gamma) - \lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \int_0^\eta H(t, s) dA(t),$$

$$H(t, s) = \begin{cases} \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Here  $G(t, s)$  is called the Green function of BVP (4). Obviously,  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$ .

*Proof.* The proof is similar to that in Lemma 2.3 in [50]. To complete the proof, we need only to replace  $\int_0^\eta h(t)t^{\alpha-\gamma-1} dt$  and  $\int_0^\eta h(t)H(t, s) dt$  with  $\int_0^\eta t^{\alpha-\gamma-1} dA(t)$  and  $\int_0^\eta H(t, s) dA(t)$ , respectively. We omit the details.  $\square$

**Lemma 3.** Let  $r \in L^1[0, 1]$ . Then the unique solution of fractional boundary value problem

$$\begin{aligned}
 -D_{0+}^\mu (\varphi_p(-D_{0+}^\alpha u(t))) &= r(t), \quad 0 < t < 1, \\
 u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^\alpha u(0) = D_{0+}^\alpha u(1) = 0, \\
 D_{0+}^\beta u(1) &= \lambda \int_0^\eta D_{0+}^\gamma u(t) dA(t),
 \end{aligned} \tag{7}$$

is

$$u(t) = \int_0^1 G(t, s)\varphi_q\left(\int_0^1 K_\mu(s, \tau)r(\tau) d\tau\right) ds.$$

*Proof.* Let  $v = \varphi_p(-D_{0+}^\alpha u)$ . Then we have by (7)

$$-D_{0+}^\mu v(t) = r(t), \quad 0 < t < 1, \quad v(0) = v(1) = 0. \tag{8}$$

It follows from Lemma 1 that the unique solution of (8) is

$$v(t) = \int_0^1 K_\mu(t, s)r(s) \, ds, \quad t \in [0, 1]. \tag{9}$$

We know from (9) that the solution of (7) meets

$$\begin{aligned} -D_{0+}^\alpha u(t) &= \varphi_q \left( \int_0^1 K_\mu(t, s)r(s) \, ds \right), \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^\beta u(1) = \lambda \int_0^\eta D_{0+}^\gamma u(t) \, dA(t). \end{aligned} \tag{10}$$

By means of Lemma 2 the solution of (10) admits the following form:

$$u(t) = \int_0^1 G(t, s)\varphi_q \left( \int_0^1 K_\mu(s, \tau)r(\tau) \, d\tau \right) \, ds. \quad \square$$

**Lemma 4.** (See [27].) *The Green function  $K_\mu(t, s)$  defined by (3) has the following properties:  $K_\mu(t, s) = K_\mu(1 - s, 1 - t)$  and for  $t, s \in (0, 1)$ ,*

$$\frac{\mu - 1}{\Gamma(\mu)} t^{\mu-1}(1 - t)(1 - s)^{\mu-1}s \leq K_\mu(t, s) \leq \frac{1}{\Gamma(\mu)} t^{\mu-1}(1 - t)(1 - s)^{\mu-2}.$$

**Lemma 5.** *The functions  $G_1(t, s)$  and  $G(t, s)$  given by (5) and (6), respectively, admit the following properties:*

- (a1)  $G_1(t, s) \geq \frac{1}{\Gamma(\alpha)} t^{\alpha-1}s(1 - s)^{\alpha-\beta-1} \quad \forall t, s \in [0, 1];$
- (a2)  $G_1(t, s) \leq \frac{1}{\Gamma(\alpha)} (\alpha - 1)s(1 - s)^{\alpha-\beta-1} \quad \forall t, s \in [0, 1];$
- (a3)  $G(t, s) \leq J(s),$   

$$J(s) = \frac{1}{\Gamma(\alpha)} (\alpha - 1)s(1 - s)^{\alpha-\beta-1} + \frac{\lambda\Gamma(\alpha - \beta) \int_0^\eta H(t, s) \, dA(t)}{\Gamma(\alpha - \gamma) - \lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} \, dA(t)} \quad \forall t, s \in [0, 1];$$
- (a4)  $\frac{1}{(\alpha - 1)} t^{\alpha-1}J(s) \leq G(t, s) \leq \frac{1}{\Gamma(\alpha)} \lambda t^{\alpha-1}(1 - s)^{\alpha-\beta-1},$

here

$$\Lambda = (\alpha - 1) + \frac{\lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)}{\Gamma(\alpha - \gamma) - \lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \quad \forall t, s \in [0, 1].$$

*Proof.* The proof is similar to the proof of Lemma 2.4 in [50], we need only to replace  $\int_0^\eta h(t)t^{\alpha-\gamma-1} dt$ ,  $\int_0^\eta h(t)H(t, s) dt$  with  $\int_0^\eta t^{\alpha-\gamma-1} dA(t)$  and  $\int_0^\eta H(t, s) dA(t)$ , respectively. We omit it here. □

**Lemma 6.** Suppose that  $w \in C[0, 1]$  be the solution of

$$\begin{aligned} -D_{0+}^\mu(\varphi_p(-D_{0+}^\alpha u(t))) &= k(t), \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^\alpha u(0) = D_{0+}^\alpha u(1) = 0, \\ D_{0+}^\beta u(1) &= \lambda \int_0^\eta D_{0+}^\gamma u(t) dA(t), \end{aligned}$$

where  $k \in L^1(0, 1)$ ,  $k(t) > 0$ . Then  $w(t) \leq \vartheta t^{\alpha-1}$ ,  $0 \leq t \leq 1$ , where

$$\vartheta = \frac{\Lambda}{\Gamma(\alpha)} \cdot \left(\frac{1}{\Gamma(\mu)}\right)^{q-1} \int_0^1 (1-s)^{\alpha-\beta+q-2} s^{(\mu-1)(q-1)} ds \cdot \left(\int_0^1 (1-\tau)^{\mu-2} k(\tau) d\tau\right)^{q-1}.$$

*Proof.* For any  $0 \leq t \leq 1$ , we get by Lemmas 3, 4 and 5,

$$\begin{aligned} w(t) &= \int_0^1 G(t, s)\varphi_q\left(\int_0^1 K_\mu(s, \tau)k(\tau) d\tau\right) ds \\ &\leq \frac{\Lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\frac{1}{\Gamma(\mu)} \int_0^1 s^{\mu-1}(1-s)(1-\tau)^{\mu-2}k(\tau) d\tau\right) ds t^{\alpha-1} \\ &= \frac{\Lambda\left(\frac{1}{\Gamma(\mu)}\right)^{q-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1}\varphi_q\left(\int_0^1 s^{\mu-1}(1-s)(1-\tau)^{\mu-2}k(\tau) d\tau\right) ds t^{\alpha-1} \\ &= \vartheta t^{\alpha-1}. \end{aligned} \quad \square$$

Denote

$$K = \left\{ u \in E: u(t) \geq \frac{1}{\alpha-1}t^{\alpha-1}\|u\|, t \in [0, 1] \right\}.$$

Then  $K$  is a cone in  $E$ . For simplicity, write  $K_r = \{u \in K: \|u\| < r\}$  and

$$\begin{aligned} K(\zeta, a, b) &= \{u \in K: a \leq \zeta(u), \|u\| \leq b\}, \\ \mathring{K}(\zeta, a, b) &= \{u \in K: a < \zeta(u), \|u\| \leq b\}. \end{aligned} \tag{11}$$

We formulate assumptions used throughout this paper.

- (H1)  $A : J \rightarrow \mathbb{R}$  is a nondecreasing function, and  $0 < \lambda \Gamma(\alpha - \beta) \int_0^\eta t^{\alpha - \gamma - 1} dA(t) < \Gamma(\alpha - \gamma)$ .
- (H2) The function  $f \in C(I \cdot \mathbb{R}_+, \mathbb{R})$ , and there exists a function  $k \in C(I, \mathbb{R}_+) \cap L^1(0, 1)$  such that  $f(t, u) \geq -k(t)$  for all  $t \in I, u \in \mathbb{R}_+$  with  $\int_0^1 (1 - \tau)^{\mu - 2} \times k(\tau) d\tau < +\infty$ .
- (H3) For any positive numbers  $r_1 < r_2$ , there exists a nonnegative continuous function  $\gamma_{r_1/(\alpha - 1), r_2} \in L^1(0, 1)$  such that  $f(t, u) \leq \gamma_{r_1/(\alpha - 1), r_2}(t), 0 < t < 1, r_1 t^{\alpha - 1}/(\alpha - 1) \leq u \leq r_2$  with  $\int_0^1 (1 - s)^{\mu - 2} \gamma_{r_1/(\alpha - 1), r_2}(s) ds < +\infty$ .

**Lemma 7.** (See [28].) *Suppose that  $T : \overline{K}_c \rightarrow K$  is completely continuous, and suppose there exist a concave positive functional  $\zeta$  with  $\zeta(u) \leq \|u\|, u \in K$ , and numbers  $b > a > 0, b \leq c$  satisfying the following conditions:*

- (i)  $\{u \in K(\zeta, a, b) : \zeta(u) > a\} \neq \emptyset$ , and  $\zeta(Tu) > a$  if  $u \in K(\zeta, a, b)$ ;
- (ii)  $Tu \in \overline{K}_c$  if  $u \in K(\zeta, a, c)$ ;
- (iii)  $\zeta(Tu) > a$  for all  $u \in K(\zeta, a, c)$  with  $\|Tu\| > b$ .

Then  $i(T, \overset{\circ}{K}(\zeta, a, c), \overline{K}_c) = 1$ .

**Lemma 8.** (See [14].) *Let  $K$  be a cone in Banach space  $X, T : K \rightarrow K$  be a completely continuous operator. Let  $a, b, c$  be three positive numbers with  $a < b < c$ .*

- (i) *If  $\|Tu\| > \|u\|$  for  $u \in \partial(K_a)$  and  $\|Tu\| \leq \|u\|$  for  $u \in \partial(K_b)$ , then*

$$i(T, \overline{K}_b \setminus \overline{K}_a, \overline{K}_b) = 1,$$

- (ii) *If  $\|Tu\| > \|u\|$  for  $u \in \partial(K_a)$  and  $\|Tu\| < \|u\|$  for  $u \in \partial(K_b)$ , then*

$$i(T, K_b \setminus \overline{K}_a, \overline{K}_c) = 1.$$

### 3 Main result

Suppose that  $0 < a^* < b^* \leq 1$ . In applications, we can choose  $a^*$  and  $b^*$  in terms of the properties of  $f(t, u)$ . Denote

$$\sigma^* = \min_{t \in [a^*, b^*]} \frac{1}{\alpha - 1} t^{\alpha - 1}.$$

For any  $r, r_1, r_2 > 0$  with  $r_1 < r_2$ , define the height functions as follows:

$$\widehat{\varphi}(t, r) = \max \left\{ f(t, u) : \left( \frac{1}{\alpha - 1} r - \vartheta \right) t^{\alpha - 1} \leq u \leq r \right\} + k(t),$$

$$\widehat{\psi}(t, r) = \min \left\{ f(t, u) : \left( \frac{1}{\alpha - 1} r - \vartheta \right) t^{\alpha - 1} \leq u \leq r \right\} + k(t),$$

$$\widehat{\varphi}(t, r_1, r_2) = \max \left\{ f(t, u) : \left( \frac{1}{\alpha - 1} r_1 - \vartheta \right) t^{\alpha - 1} \leq u \leq r_2 \right\} + k(t),$$

$$\widehat{\psi}(t, r_1, r_2) = \min \{ f(t, u) : (r_1 - \vartheta) \leq u \leq r_2 \} + k(t).$$

**Theorem 1.** *Suppose that (H1)–(H3) hold. In addition, there exist five positive numbers  $(\alpha - 1)\vartheta < e_1 < e_2 < e_3 < e_4 \leq e_5$  with  $e_4 \geq e_3\sigma^{*-1}$  satisfying*

$$(A1) \quad \int_0^1 (1 - \tau)^{\mu-2} \widehat{\varphi}(\tau, e_2) \, d\tau < \varphi_p \left[ \left( \frac{1}{\Gamma(\mu)} \right)^{1-q} \varrho^{-1} e_2 \right];$$

$$(A2) \quad \int_0^1 (1 - \tau)^{\mu-1} \tau \widehat{\psi}(\tau, e_1) \, d\tau \geq \varphi_p \left[ (\alpha - 1) \left( \frac{\mu - 1}{\Gamma(\mu)} \right)^{1-q} \varrho^{-1} e_1 \right];$$

$$(A3) \quad \int_0^1 (1 - \tau)^{\mu-2} \widehat{\varphi}(\tau, e_3, e_5) \, d\tau \leq \varphi_p \left[ \left( \frac{1}{\Gamma(\mu)} \right)^{1-q} \varrho^{-1} e_5 \right];$$

$$(A4) \quad \int_{a^*}^{b^*} (1 - \tau)^{\mu-1} \tau \widehat{\Psi}(\tau, e_3, e_4) \, d\tau > \varphi_p \left[ (\alpha - 1)\sigma^{*-1} \left( \frac{\mu - 1}{\Gamma(\mu)} \right)^{1-q} \varrho^{-1} e_3 \right].$$

Here  $\varrho = \int_0^1 J(s)\varphi_q(s^{\mu-1}(1-s)) \, ds$ . Then BVP (1) has at least three positive solutions  $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$  with  $e_1 - \vartheta \leq \|\widehat{u}_1\| \leq e_2, e_3 - \vartheta \leq \|\widehat{u}_2\| \leq e_5, e_2 - \vartheta \leq \|\widehat{u}_3\| \leq e_5$  and

$$\min_{t \in [a^*, b^*]} \widehat{u}_2(t) \geq e_3 - \vartheta, \quad \min_{t \in [a^*, b^*]} \widehat{u}_3(t) \leq e_3.$$

*Proof.* First, consider the following modified approximating BVP (MABVP for short):

$$\begin{aligned} D_{0+}^\mu (\varphi_p(-D_{0+}^\alpha u(t))) + f(t, \chi_n(u-w)(t)) + k(t) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \quad D_{0+}^\alpha u(0) = D_{0+}^\alpha u(1) = 0, \\ D_{0+}^\beta u(1) &= \lambda \int_0^\eta D_{0+}^\gamma u(t) \, dA(t), \end{aligned}$$

where  $w$  is the same as that in Lemma 6, and

$$\chi_n(u) = \begin{cases} u, & u \geq \frac{1}{n}, \\ \frac{1}{n}, & u < \frac{1}{n}, \end{cases} \quad n \in \mathbb{N}^+.$$

For  $t \in [0, 1], n \in \mathbb{N}^+$ , define operators  $T_n$  as follows:

$$(T_n u)(t) = \int_0^1 G(t, s)\varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] \, d\tau \right) \, ds.$$

In the sequel, we will give the proof by the following three steps.

(I) We show that for any  $(\alpha - 1)\vartheta < r_1 < r_2$  and sufficiently large  $n, T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \rightarrow K$  is completely continuous.



First, we show that  $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \rightarrow K$  is well defined. Let  $u \in \overline{K}_{r_2} \setminus K_{r_1}$ ,  $n \geq [1/r_1] + 1$ . Then for  $t \in [0, 1]$ , one has

$$u(t) - w(t) \geq \frac{1}{\alpha - 1} t^{\alpha-1} \|u\| - \vartheta t^{\alpha-1} \geq \left( \frac{r_1}{\alpha - 1} - \vartheta \right) t^{\alpha-1} \geq 0.$$

Therefore,

$$\left( \frac{r_1}{\alpha - 1} - \vartheta \right) t^{\alpha-1} \leq \max \left\{ u(t) - w(t), \frac{1}{n} \right\} \leq r_2, \quad n \geq \left[ \frac{1}{r_1} \right] + 1,$$

i.e.,

$$\left( \frac{r_1}{\alpha - 1} - \vartheta \right) t^{\alpha-1} \leq \chi_n(u - w)(t) \leq r_2, \quad n \geq \left[ \frac{1}{r_1} \right] + 1, \quad t \in (0, 1).$$

By Lemmas 5 and 2 we have

$$\begin{aligned} J(s) &\leq \frac{1}{\Gamma(\alpha)} + \frac{\lambda \Gamma(\alpha - \beta)}{\Gamma(\alpha - \gamma) - \lambda \Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \cdot \int_0^\eta \frac{1}{\Gamma(\alpha)} dA(t) \\ &\leq \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \cdot \frac{\lambda \Gamma(\alpha - \beta)}{\Gamma(\alpha - \gamma) - \lambda \Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \cdot (A(\eta) - A(0)) \\ &\triangleq M_0 \quad \forall s \in [0, 1]. \end{aligned}$$

This, together with (H1)–(H3) and Lemma 4 yields that

$$\begin{aligned} (T_n u)(t) &= \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u - w)(\tau)) + k(\tau)] d\tau \right) ds \\ &\leq \int_0^1 J(s) \varphi_q \left( \int_0^1 \frac{1}{\Gamma(\mu)} s^{\mu-1} (1 - s)(1 - \tau)^{\mu-2} \right. \\ &\quad \left. \times [f(\tau, \chi_n(u - w)(\tau)) + k(\tau)] d\tau \right) ds \\ &\leq M_0 \left( \frac{1}{\Gamma(\mu)} \right)^{q-1} \int_0^1 [s^{\mu-1} (1 - s)]^{q-1} ds \\ &\quad \times \varphi_q \left( \int_0^1 (1 - \tau)^{\mu-2} \left[ \gamma_{r_1/(\alpha-1)-\vartheta, r_2}(\tau) + k(\tau) \right] d\tau \right) \\ &< +\infty. \end{aligned} \tag{12}$$

Besides, for any  $u \in \overline{K}_{r_2} \setminus K_{r_1}$ ,  $t \in [0, 1]$ , by (12) and Lemma 5 we have

$$(T_n u)(t) \leq \int_0^1 J(s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds,$$

i.e.,

$$\begin{aligned} \|T_n u\| &\leq \int_0^1 J(s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds, \\ (T_n u)(t) &\geq \frac{1}{\alpha-1} t^{\alpha-1} \int_0^1 J(s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds \\ &\geq \frac{1}{\alpha-1} \|T_n u\|. \end{aligned} \tag{13}$$

It follows from (12) and (13) that  $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \rightarrow K$  is well defined.

Next, we prove that  $T_n$  is completely continuous. Given  $D \subset \overline{K}_{r_2} \setminus K_{r_1}$ , we deduce from (12) that  $T_n(D)$  is uniformly bounded. It follows from Arzelà–Ascoli theorem, we need only to show the equicontinuity of  $T_n(D)$ . We have from (5), (6), (H3) for  $t \in [0, 1]$ :

$$\begin{aligned} &|(T_n u)'(t)| \\ &= \left| \int_0^1 \frac{\partial}{\partial t} G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds \right| \\ &\leq \left| \left( \int_0^1 \frac{\partial}{\partial t} G_1(t, s) + \frac{(\alpha-1)\lambda\Gamma(\alpha-\beta)t^{\alpha-2} \int_0^\eta H(t, s) dA(t)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \right) \right. \\ &\quad \left. \times \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds \right| \\ &\leq \left( \int_0^1 \frac{(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} + \frac{(\alpha-1)\lambda\Gamma(\alpha-\beta)t^{\alpha-2} \int_0^\eta \frac{1}{\Gamma(\alpha)} dA(t)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \right) \\ &\quad \times \varphi_q \left( \int_0^1 \frac{1}{\Gamma(\mu)} (1-\tau)^{\mu-2} (\gamma_{r_1/(\alpha-1)-\vartheta, r_2}(\tau) + k(\tau)) d\tau \right) \\ &\leq (\alpha-1) \left[ \frac{1}{\Gamma(\alpha)} + \frac{\lambda\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma) - \lambda\Gamma(\alpha-\beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t)} \cdot \frac{A(\eta) - A(0)}{\Gamma(\alpha)} \right] \\ &\quad \times \left( \frac{1}{\Gamma(\mu)} \right)^{q-1} \varphi_q \left( \int_0^1 (1-\tau)^{\mu-2} (\gamma_{r_1/(\alpha-1)-\vartheta, r_2}(\tau) + k(\tau)) d\tau \right) \triangleq M_1. \end{aligned}$$

Then for any  $t_1, t_2 \in [0, 1]$ , we have

$$|T_n u(t_1) - T_n u(t_2)| = \left| \int_{t_1}^{t_2} (T_n u)'(s) \, ds \right| \leq M_1 |t_1 - t_2| \quad \forall t_1, t_2 \in [0, 1].$$

Thus,  $T_n(D)$  is equicontinuous. Therefore, we have proved that  $T_n : (\overline{K}_{r_2} \setminus K_{r_1}) \rightarrow K$  is completely continuous.

For sufficiently large  $n$ , the same conclusion is valid for  $T_n : \overline{K}_{e_5} \setminus K_{e_1} \rightarrow K$ . Define  $\zeta(u) = \min_{t \in [a^*, b^*]} u(t)$  for any  $u \in K$ . In the following,  $K(\zeta, e_3, e_4)$ ,  $\overset{\circ}{K}(\zeta, e_3, e_4)$ ,  $K(\zeta, e_3, e_5)$  have the same meaning as those in (11).

(II) We demonstrate that for sufficiently large  $n$ ,  $T_n$  has three fixed points.

First, we are in position to show that for sufficiently large  $n$ ,

$$i(T_n, \overset{\circ}{K}(\zeta, e_3, e_5), \overline{K}_{e_5}) = 1. \tag{14}$$

Set  $u_0(t) \equiv (e_3 + e_4)/2$ . Then  $u_0 \in \overset{\circ}{K}(\zeta, e_3, e_4)$ , which means that  $\overset{\circ}{K}(\zeta, e_3, e_4) \neq \emptyset$ . If  $u \in K(\zeta, e_3, e_4)$ , we have  $e_3 \leq \min_{t \in [a^*, b^*]} u(t) \leq \max_{t \in [0, 1]} u(t) = \|u\| \leq e_4$ . Thus, for  $t \in [a^*, b^*]$ , by Lemma 6 we know that  $0 < (e_3 - \vartheta) \leq u(t) - w(t) \leq e_4$ , and  $(e_3 - \vartheta) \leq \max\{u(t) - w(t), 1/n\} \leq e_4 (n > N_1 = [1/e_3] + 1)$ , which means

$$(e_3 - \vartheta) \leq \chi_n(u - w)(t) \leq e_4, \quad a^* < t < b^*, \quad n > N_1.$$

It follows from Lemmas 4, 5 and (A4) that

$$\begin{aligned} \zeta(T_n u) &= \min_{t \in [a^*, b^*]} (T_n u)(t) \geq \min_{t \in [a^*, b^*]} \frac{1}{\alpha - 1} t^{\alpha-1} \|T_n u\| = \sigma^* \|T_n u\| \\ &= \sigma^* \max_{t \in [0, 1]} \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u - w)(\tau)) + k(\tau)] \, d\tau \right) \, ds \\ &\geq \sigma^* \max_{t \in [0, 1]} \frac{1}{\alpha - 1} t^{\alpha-1} \int_0^1 J(s) \varphi_q \left( \int_0^1 \frac{\mu - 1}{\Gamma(\mu)} s^{\mu-1} (1 - s)(1 - \tau)^{\mu-1} \tau \right. \\ &\quad \left. \times [f(\tau, \chi_n(u - w)(\tau)) + k(\tau)] \, d\tau \right) \, ds \\ &\geq \sigma^* \frac{1}{\alpha - 1} \left( \frac{\mu - 1}{\Gamma(\mu)} \right)^{q-1} \varrho \cdot \varphi_q \left( \int_{a^*}^{b^*} (1 - \tau)^{\mu-1} \tau \widehat{\Psi}(\tau, e_3, e_4) \, d\tau \right) > e_3. \end{aligned}$$

If  $u \in K(\zeta, e_3, e_5)$ , then by the construction of cone  $K$  and Lemma 6 we know that  $0 < (e_3/(\alpha - 1) - \vartheta)t^{\alpha-1} \leq u(t) - w(t) \leq e_5, t \in [0, 1]$ , and  $(e_3/(\alpha - 1) - \vartheta)t^{\alpha-1} \leq \max\{u(t) - w(t), 1/n\} \leq e_5, n > N_1$ , which means

$$\left( \frac{e_3}{\alpha - 1} - \vartheta \right) t^{\alpha-1} \leq \chi_n(u - w)(t) \leq e_5, \quad 0 < t < 1, \quad n > N_1. \tag{15}$$

This, together with Lemmas 4, 5 and (A3), means that

$$\begin{aligned} \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds \\ &\leq \int_0^1 J(s) \varphi_q \left( \int_0^1 \frac{1}{\Gamma(\mu)} s^{\mu-1} (1-s)(1-\tau)^{\mu-2} \widehat{\varphi}(\tau, e_3, e_5) d\tau \right) ds \\ &= \left( \frac{1}{\Gamma(\mu)} \right)^{q-1} \int_0^1 J(s) \varphi_q(s^{\mu-1}(1-s)) ds \cdot \varphi_q \left( \int_0^1 (1-\tau)^{\mu-2} \widehat{\varphi}(\tau, e_3, e_5) d\tau \right) \\ &\leq e_5, \quad n > N_1. \end{aligned} \tag{16}$$

Consequently,  $T_n u \in \overline{K}_{e_5}$ .

For  $u \in K(\zeta, e_3, e_5)$  with  $\|T_n u\| > e_4$ , noticing that  $e_4 \geq e_3 \sigma^{*-1}$ , we have  $\|T_n u\| > e_3 \sigma^{*-1}$ . Therefore,

$$\zeta(T_n u) = \min_{t \in [a^*, b^*]} (T_n u)(t) \geq \sigma^* \|T_n u\| > \sigma^* e_3 \sigma^{*-1} = e_3.$$

Thus, for  $n > N_1$ , we know from Lemma 7 that (14) holds.

If  $u \in \partial(K_{e_5})$ , then  $\|u\| = e_5$  and  $(e_3/(\alpha-1))t^{\alpha-1} \leq (e_5/(\alpha-1))t^{\alpha-1} \leq u(t) \leq e_5$ ,  $t \in [0, 1]$ . Thus, (15) holds. By (15), (A3), Lemmas 4 and 5, similar to the proof of (16), for any  $n > N_1$ , one gets

$$\begin{aligned} \|T_n u\| &\leq \left( \frac{1}{\Gamma(\mu)} \right)^{q-1} \int_0^1 J(s) \varphi_q(s^{\mu-1}(1-s)) ds \\ &\quad \times \varphi_q \left( \int_0^1 (1-\tau)^{\mu-2} \widehat{\varphi}(\tau, e_3, e_5) d\tau \right) \leq e_5 \quad \forall u \in \partial(K_{e_5}). \end{aligned} \tag{17}$$

If  $u \in \partial(K_{e_2})$ , then  $\|u\| = e_2$  and  $(e_2/(\alpha-1))t^{\alpha-1} \leq u(t) \leq e_2$ ,  $t \in [0, 1]$ . Thus, we have  $0 < (e_2/(\alpha-1) - \vartheta)t^{\alpha-1} \leq u(t) - w(t) \leq e_2$  for  $t \in [0, 1]$  and  $(e_2/(\alpha-1) - \vartheta)t^{\alpha-1} \leq \max\{u(t) - w(t), 1/n\} \leq e_2$ ,  $n > N_2 = [1/e_2] + 1$ , i.e.,

$$\left( \frac{e_2}{\alpha-1} - \vartheta \right) t^{\alpha-1} \leq \chi_n(u-w)(t) \leq e_2, \quad 0 < t < 1, \quad n > N_2.$$

By (A1), Lemmas 4 and 5, for any  $n > N_2$ , one has

$$\begin{aligned} \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u-w)(\tau)) + k(\tau)] d\tau \right) ds \\ &\leq \int_0^1 J(s) \varphi_q \left( \int_0^1 \frac{1}{\Gamma(\mu)} s^{\mu-1} (1-s)(1-\tau)^{\mu-2} \widehat{\varphi}(\tau, e_2) d\tau \right) ds \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{1}{\Gamma(\mu)}\right)^{q-1} \varrho \cdot \varphi_q \left( \int_0^1 (1-\tau)^{\mu-2} \widehat{\varphi}(\tau, e_2) \, d\tau \right) \\
 &< e_2 \quad \forall u \in \partial(K_{e_2}).
 \end{aligned} \tag{18}$$

If  $u \in \partial(K_{e_1})$ , then  $(e_1/(\alpha - 1))t^{\alpha-1} \leq u(t) \leq e_1, t \in [0, 1]$ . Thus, we have  $0 < (e_1/(\alpha - 1) - \vartheta)t^{\alpha-1} \leq u(t) - w(t) \leq e_1$  for  $t \in [0, 1]$  and  $(e_1/(\alpha - 1) - \vartheta)t^{\alpha-1} \leq \max\{u(t) - w(t), 1/n\} \leq e_1, n > N_3 = [1/e_1] + 1$ , i.e.,

$$\left(\frac{e_1}{\alpha - 1} - \vartheta\right)t^{\alpha-1} \leq \chi_n(u - w)(t) \leq e_1, \quad 0 < t < 1, n > N_3.$$

By (A2), Lemmas 4 Lemma 5, for any  $n > N_3$ , one gets

$$\begin{aligned}
 \|T_n u\| &= \max_{t \in [0,1]} \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u - w)(\tau)) + k(\tau)] \, d\tau \right) \, ds \\
 &\geq \max_{t \in [0,1]} \frac{1}{\alpha - 1} t^{\alpha-1} \\
 &\quad \times \int_0^1 J(s) \varphi_q \left( \int_0^1 \frac{\mu - 1}{\Gamma(\mu)} s^{\mu-1} (1 - s)(1 - \tau)^{\mu-1} \tau \widehat{\psi}(\tau, e_1) \, d\tau \right) \, ds \\
 &= \frac{1}{\alpha - 1} \left(\frac{\mu - 1}{\Gamma(\mu)}\right)^{q-1} \varrho \cdot \varphi_q \left( \int_0^1 (1 - \tau)^{\mu-1} \tau \widehat{\psi}(\tau, e_1) \, d\tau \right) \\
 &\geq e_1 \quad \forall u \in \partial(K_{e_1}).
 \end{aligned} \tag{19}$$

By (17), (18), (19) and Lemma 8, for any  $n > N = \max\{N_1, N_2, N_3\}$ , we know (14) and the following two equalities hold simultaneously:

$$i(T_n, \overline{K}_{e_5} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) = 1, \quad i(T_n, K_{e_2} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) = 1. \tag{20}$$

It is clear, (A1) implies that  $T_n$  has no fixed point on  $\partial(K_{e_2})$ . In addition, for  $u \in K(\zeta, e_3, e_4)$ , we have that  $\zeta(T_n u) > e_3$  and for  $u \in K(\zeta, e_3, e_5)$  with  $\|T_n u\| > e_4$ , we also have that  $\zeta(T_n u) > e_3$ . This is to say,  $T_n$  has no fixed point on  $K(\zeta, e_3, e_5) \setminus \overset{\circ}{K}(\zeta, e_3, e_5)$ . Thus, for  $n > N$ , it follows from (14), (20) and the addition property of the topological degree that

$$\begin{aligned}
 &i(T_n, \overline{K}_{e_5} \setminus (K(\zeta, e_3, e_5) \cup \overline{K}_{e_2}), \overline{K}_{e_5}) \\
 &= i(T_n, \overline{K}_{e_5} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) - i(T_n, K_{e_2} \setminus \overline{K}_{e_1}, \overline{K}_{e_5}) \\
 &\quad - i(T_n, \overset{\circ}{K}(\zeta, e_3, e_5), \overline{K}_{e_5}) \\
 &= -1.
 \end{aligned}$$

As a consequence, for  $n > N$ ,  $T_n$  has at least three fixed points  $u_{1n}^* \in K_{e_2} \setminus \overline{K}_{e_1}$ ,  $u_{2n}^* \in \overset{\circ}{K}(\zeta, e_3, e_5)$ ,  $u_{3n}^* \in \overline{K}_{e_5} \setminus (K(\zeta, e_3, e_5) \cup \overline{K}_{e_2})$  satisfying  $e_1 \leq \|u_{1n}^*\| < e_2$ ,  $e_3 \leq \|u_{2n}^*\| \leq e_5$ ,  $e_2 \leq \|u_{3n}^*\| < e_5$  with  $\min_{t \in [a^*, b^*]} u_{2n}^*(t) > e_3$ ,  $\min_{t \in [a^*, b^*]} u_{3n}^*(t) < e_3$ .

(III) We prove that BVP (1) has at least triple positive solutions.

Taking into account the construction of the cone  $K$ , for  $t \in [0, 1]$ ,  $n > N$  ( $i = 1, 2, 3$ ), one has that

$$u_{in}^*(t) \geq \|u_{in}^*\| \frac{1}{\alpha - 1} t^{\alpha - 1} \geq \frac{e_1}{\alpha - 1} t^{\alpha - 1} \geq \vartheta t^{\alpha - 1} \geq w(t) \tag{21}$$

and

$$u_{in}^*(t) = \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, \chi_n(u_{in}^* - w)(\tau)) + k(\tau)] d\tau \right) ds. \tag{22}$$

It is easy to know from (H2) that  $\{u_{in}^* : n > N\}$  ( $i = 1, 2, 3$ ) are bounded and equicontinuous on  $[0, 1]$ . Thus, Arzelà–Ascoli theorem implies that there exists a subsequence  $N_0$  of  $N$  and corresponding continuous functions  $u_i^*$  ( $i = 1, 2, 3$ ) such that  $u_{in}^*$  converges to  $u_i^*$  ( $i = 1, 2, 3$ ) uniformly on  $[0, 1]$  as  $n \rightarrow \infty$  through  $N_0$ . Let  $n \rightarrow \infty$  on both sides of (22), for  $t \in [0, 1]$ ,  $i = 1, 2, 3$ , one has

$$u_i^*(t) = \int_0^1 G(t, s) \varphi_q \left( \int_0^1 K_\mu(s, \tau) [f(\tau, (u_i^* - w)(\tau)) + k(\tau)] d\tau \right) ds \tag{23}$$

and

$$e_1 \leq \|u_1^*\| \leq e_2, \quad e_3 \leq \|u_2^*\| \leq e_5, \quad e_2 \leq \|u_3^*\| \leq e_5 \tag{24}$$

with

$$\min_{t \in [a^*, b^*]} u_2^*(t) \geq e_3, \quad \min_{t \in [a^*, b^*]} u_3^*(t) \leq e_3. \tag{25}$$

It can be easily seen from (21) that  $u_i^*(t) \geq (1/(\alpha - 1))t^{\alpha - 1}\|u_i^*\| \geq (e_1/(\alpha - 1))t^{\alpha - 1} \geq \vartheta t^{\alpha - 1} \geq w(t)$  ( $i = 1, 2, 3$ ). Let  $\widehat{u}_i(t) = u_i^*(t) - w(t)$ , then we know from (23) that  $\widehat{u}_i(t)$  ( $i = 1, 2, 3$ ) are positive solutions for BVP (1). Combined with (24), (25) and Lemma 6, we know that  $e_1 - \vartheta \leq \|\widehat{u}_1\| \leq e_2$ ,  $e_3 - \vartheta \leq \|\widehat{u}_2\| \leq e_5$ ,  $e_2 - \vartheta \leq \|\widehat{u}_3\| \leq e_5$  and  $\min_{t \in [a^*, b^*]} \widehat{u}_2(t) \geq e_3 - \vartheta$ ,  $\min_{t \in [a^*, b^*]} \widehat{u}_3(t) \leq e_3$ .  $\square$

### 4 An example

Consider the following singular fractional differential equation:

$$\begin{aligned}
 -D_{0+}^{7/4}(\varphi_3(-D_{0+}^{11/3}u(t))) &= g(t, u(t)) - \frac{1}{10^5 \sqrt[4]{t}}, \quad 0 < t < 1, \\
 u(0) = u'(0) = u''(0) &= 0, \quad D_{0+}^{2/3}u(1) = \frac{1}{2} \int_0^{3/4} D_{0+}^{3/2}u(t) dA(t).
 \end{aligned} \tag{26}$$

Here  $\alpha = 11/3, n = 4, \beta = 2/3, \gamma = 1/3, \mu = 7/4, \eta = 3/4, p = 3, q = 3/2, \lambda = 1/2,$   
 $f(t, u(t)) = g(t, u(t)) - 1/(10^5 \sqrt[4]{t}), g(t, u(t)) = \theta(u(t))/(25 \sqrt[4]{t(1-t)}),$  where

$$\theta(u) = \begin{cases} u^{1/2} + u^{-1/6}, & 0 < u \leq 1, \\ u^{13} + 1, & 1 < u \leq 6, \\ u^{1/2} + 6^{13} + 1 - \sqrt{6}, & u > 6, \end{cases} \quad \text{and} \quad A(t) = \begin{cases} 1, & t \in [0, \frac{1}{4}), \\ 2, & t \in [\frac{1}{4}, \frac{1}{2}), \\ 4t, & t \in [\frac{1}{2}, \frac{3}{4}), \\ 4, & t \in [\frac{3}{4}, 1]. \end{cases}$$

Next, we are in position to check all the conditions of Theorem 1. Direct calculation shows that  $\lambda\Gamma(\alpha - \beta) \int_0^\eta t^{\alpha-\gamma-1} dA(t) = 0.3804 < 2.7782 = \Gamma(\alpha - \gamma) = \Gamma(10/3),$  which means (H1) holds.

Since  $\int_0^1 (1-\tau)^{\mu-2} k(\tau) d\tau = 10^{-5} \int_0^1 (1-\tau)^{-1/4} \tau^{-1/4} d\tau = 1.6944 \cdot 10^{-5} < +\infty,$  we know that (H2) meets for  $k(t) = 1/(10^5 \sqrt[4]{t}).$  It is easy to check that (H3) is valid for  $\gamma_{r_1/(\alpha-1), r_2}(t) = (1/(25 \sqrt[4]{t(1-t)})) [r_2^{1/2} + ((3/8)r_1 t^{8/3})^{-1/6} + r_2^{13} + r_2^{1/2} + 6^{13} + 1 - \sqrt{6}].$  Considering that  $\int_0^\eta H(t, s) dA(t) = H(1/4, s) + 4 \int_{1/2}^{3/4} H(t, s) dt,$  we have

$$\int_0^\eta H(t, s) dA(t) = \begin{cases} \begin{cases} 0.0946(1-s)^2 - 0.2492(\frac{1}{4}-s)^{7/3} \\ -0.2991(\frac{3}{4}-s)^{10/3} + 0.2991(\frac{1}{2}-s)^{10/3}, \end{cases} & 0 \leq s \leq \frac{1}{4}, \\ \begin{cases} 0.0946(1-s)^2 - 0.2991(\frac{3}{4}-s)^{10/3} \\ +0.2991(\frac{1}{2}-s)^{10/3}, \end{cases} & \frac{1}{4} \leq s \leq \frac{1}{2}, \\ 0.0946(1-s)^2 - 0.2991(\frac{3}{4}-s)^{10/3}, & \frac{1}{2} \leq s \leq \frac{3}{4}, \\ 0.0946(1-s)^2, & \frac{3}{4} \leq s \leq 1. \end{cases}$$

This implies that

$$J(s) = \begin{cases} \begin{cases} 0.6646s(1-s)^2 + 0.0394(1-s)^2 - 0.1039(\frac{1}{4}-s)^{7/3} \\ -0.1247(\frac{3}{4}-s)^{10/3} + 0.1247(\frac{1}{2}-s)^{10/3}, \end{cases} & 0 \leq s \leq \frac{1}{4}, \\ \begin{cases} 0.6646s(1-s)^2 + 0.0394(1-s)^2 - 0.1247(\frac{3}{4}-s)^{10/3} \\ +0.1247(\frac{1}{2}-s)^{10/3}, \end{cases} & \frac{1}{4} \leq s \leq \frac{1}{2}, \\ 0.6646s(1-s)^2 + 0.0394(1-s)^2 - 0.1247(\frac{3}{4}-s)^{10/3}, & \frac{1}{2} \leq s \leq \frac{3}{4}, \\ 0.6646s(1-s)^2 + 0.0394(1-s)^2, & \frac{3}{4} \leq s \leq 1. \end{cases}$$

Direct calculation means that

$$\int_0^1 J(s) \varphi_q(s^{\mu-1}(1-s)) ds = \int_0^1 J(s) s^{3/8}(1-s)^{1/2} ds$$

$$\begin{aligned}
 &= \int_0^1 0.6646s^{11/8}(1-s)^{5/2} + 0.0394s^{3/8}(1-s)^{5/2} \, ds \\
 &\quad - \int_0^{1/4} 0.1039\left(\frac{1}{4}-s\right)^{7/3} s^{3/8}(1-s)^{1/2} \, ds \\
 &\quad - \int_0^{3/4} 0.1247\left(\frac{3}{4}-s\right)^{10/3} s^{3/8}(1-s)^{1/2} \, ds \\
 &\quad + \int_0^{1/2} 0.1247\left(\frac{1}{2}-s\right)^{10/3} s^{3/8}(1-s)^{1/2} \, ds \\
 &\geq 0.0278 + 0.0058 - 0.1039 \int_0^{1/4} \left(\frac{1}{4}-s\right)^{7/3} \, ds \\
 &\quad - 0.1247 \int_0^{3/4} \left(\frac{3}{4}-s\right)^{10/3} \, ds \\
 &\approx 0.0250, \tag{27}
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^1 J(s)\varphi_q(s^{\mu-1}(1-s)) \, ds \\
 &= 0.0278 + 0.0058 - 0.1039 \int_0^{1/4} \left(\frac{1}{4}-s\right)^{7/3} s^{3/8}(1-s)^{1/2} \, ds \\
 &\quad - 0.1247 \int_0^{3/4} \left(\frac{3}{4}-s\right)^{10/3} s^{3/8}(1-s)^{1/2} \, ds \\
 &\quad + 0.1247 \int_0^{1/2} \left(\frac{1}{2}-s\right)^{10/3} s^{3/8}(1-s)^{1/2} \, ds \\
 &\leq 0.0278 + 0.0058 + 0.1247 \int_0^{1/2} \left(\frac{1}{2}-s\right)^{10/3} \, ds \approx 0.0350. \tag{28}
 \end{aligned}$$

By simple computation, we have  $\int_0^\eta t^{\alpha-\gamma-1} \, dA(t) = \int_0^{3/4} t^{7/3} \, dA(t) = (1/4)^{7/3} + 4 \int_{1/2}^{3/4} t^{7/3} \, dt = 0.3804$ . Thus,  $\Lambda = 8/3 + (1/(2.7782 - 0.3804)) \cdot 0.3804 \approx 2.6667 + 0.1586 = 2.8253$ ,  $\vartheta = (2.8253/4.0122) \cdot (1/0.9191)^{1/2} \cdot 0.1483 \cdot 1.6944 \cdot 10^{-5} \approx 1.8460 \cdot 10^{-6}$ . Take  $e_1 = 10^{-3}$ ,  $e_2 = 1/2$ ,  $e_3 = 6$ ,  $e_4 = 55$ ,  $e_5 = 10^4$ . Next, we check



all the conditions of Theorem 1. We have

$$\begin{aligned} & \int_0^1 (1 - \tau)^{\mu-2} \widehat{\varphi}(\tau, e_2) \, d\tau \\ &= \int_0^1 (1 - \tau)^{-1/4} \left[ \max \left\{ f(\tau, u): 0.1875\tau^{8/3} \leq u \leq \frac{1}{2} \right\} + k(\tau) \right] \, d\tau \\ &= \frac{1}{25} \int_0^1 (1 - \tau)^{-1/4} \frac{1}{\sqrt[4]{\tau(1 - \tau)}} \left[ \left( \frac{1}{2} \right)^{1/2} + (0.1875\tau^{8/3})^{-1/6} \right] \, d\tau \\ &\approx 0.3052, \end{aligned}$$

and  $\varphi_q(\int_0^1 (1 - \tau)^{\mu-2} \widehat{\varphi}(\tau, e_2) \, d\tau) = 0.5524$ . This, together with (28) and  $(1/\Gamma(\mu))^{1-q} \times \varrho^{-1}e_2 \geq (1/\Gamma(7/4))^{-1/2}0.0350^{-1/2} = 13.6957$ , implies that (A1) holds. It can be easily known from (27) that

$$\begin{aligned} & \int_0^1 (1 - \tau)^{\mu-1} \tau \widehat{\psi}(\tau, e_1) \, d\tau \\ &= \int_0^1 (1 - \tau)^{\mu-1} \tau \left[ \frac{\min\{u^{1/2} + u^{-1/6}: 3.7315 \cdot 10^{-4}\tau^{8/3} \leq u \leq 10^{-3}\}}{25\sqrt[4]{\tau(1 - \tau)}} \right] \, d\tau \\ &= \frac{1}{25} \int_0^1 (1 - \tau)^{3/4} \tau \cdot \tau^{-1/4} \cdot (1 - \tau)^{-1/4} [(3.7315 \cdot 10^{-4}\tau^{8/3})^{1/2} + (10^{-3})^{-1/6}] \, d\tau \\ &\approx 0.0407, \end{aligned}$$

and  $(\alpha - 1)((\mu - 1)/\Gamma(\mu))^{1-q} \varrho^{-1}e_1 \leq (8/3) \cdot ((3/4)/\Gamma(7/4))^{-1/2} \cdot 0.0250^{-1} \cdot 10^{-3} \approx 0.1181$ . Thus,  $\varphi_q(\int_0^1 (1 - \tau)^{\mu-1} \tau \widehat{\psi}(\tau, e_1) \, d\tau) = 0.2017 > 0.1181$ , which means that (A2) holds. We also have

$$\begin{aligned} & \int_0^1 (1 - \tau)^{\mu-2} \widehat{\varphi}(\tau, e_3, e_5) \, d\tau \\ &= \frac{1}{25} \int_0^1 (1 - \tau)^{\mu-2} \tau \tau^{-1/4} (1 - \tau)^{-1/4} \\ &\quad \times \max\{u^{1/2} + 6^{13} + 1 - \sqrt{6}: 2.250\tau^{8/3} \leq u \leq 10^4\} \, d\tau \\ &= \frac{1}{25} \int_0^1 (1 - \tau)^{-1/4} \tau^{-1/4} (1 - \tau)^{-1/4} \, d\tau \cdot (10^2 + 6^{13} + 1 - \sqrt{6}) \\ &< \frac{1}{25} \cdot 2.3963 \cdot (6^{13} + 101) \approx 1.2519 \cdot 10^9, \end{aligned}$$

and  $\varphi_q(\int_0^1 (1-\tau)^{\mu-2} \widehat{\varphi}(\tau, e_3, e_5) d\tau) = (1.2519 \cdot 10^9)^{1/2} \approx 3.5382 \cdot 10^4$ . Considering (28) and  $(1/\Gamma(\mu))^{1-q} \varrho^{-1} e_5 \geq (1/\Gamma(7/4))^{-1/2} 0.0350^{-1} \cdot 10^4 \approx 2.2391 \cdot 10^5$ , we know that (A3) is valid. Take  $a^* = 2/3$ ,  $b^* = 1$ , then  $\sigma^* = \min_{t \in [a^*, b^*]} t^{\alpha-1}/(\alpha-1) = (3/8) \cdot (2/3)^{8/3} \approx 0.1272$ ,  $\sigma^{*-1} \approx 7.8616$ . Thus,  $\sigma^{*-1} e_3 = 47.1696 < 55 = e_4$ . Finally, we get from (27) that

$$\begin{aligned} & \int_{a^*}^{b^*} (1-\tau)^{\mu-1} \tau \widehat{\Psi}(\tau, e_3, e_4) d\tau \\ &= \int_{2/3}^1 (1-\tau)^{3/4} \tau \left[ \frac{\min\{u^{13} + 1: 6 - 1.8460 \cdot 10^{-6} \leq u \leq 55\}}{25 \sqrt[4]{\tau(1-\tau)}} \right] d\tau \\ &\geq \frac{1}{25} \int_{2/3}^1 (1-\tau)^{1/2} \tau^{3/4} d\tau \cdot 6^{13} \\ &\geq \frac{1}{25} \cdot \left(\frac{2}{3}\right)^{3/4} \int_{2/3}^1 (1-\tau)^{1/2} d\tau \cdot 6^{13} \approx 4.9453 \cdot 10^7, \end{aligned}$$

$\varphi_q(\int_{a^*}^{b^*} (1-\tau)^{\mu-1} \tau \widehat{\Psi}(\tau, e_3, e_4) d\tau) \approx 7.0323 \cdot 10^3$ , and  $(\alpha-1)\sigma^{*-1}((\mu-1)/\Gamma(\mu))^{1-q} \times \varrho^{-1} e_3 \leq (8/3) \cdot ((3/4)/\Gamma(7/4))^{-1/2} \cdot 7.8616 \cdot 0.0250^{-1} \cdot 6 \approx 5.5697 \cdot 10^3$ . Hence, (A4) is checked. It follows from Theorem 1 that FPDE (26) has at least three positive solutions  $\widehat{u}_1, \widehat{u}_2, \widehat{u}_3$  with  $9.9815 \cdot 10^{-4} \leq \|\widehat{u}_1\| \leq 1/2$ ,  $6 \leq \|\widehat{u}_2\| \leq 10^4$ ,  $0.5000 \leq \|\widehat{u}_3\| \leq 10^4$  with  $\min_{t \in [2/3, 1]} \widehat{u}_2(t) \geq 6$ ,  $\min_{t \in [2/3, 1]} \widehat{u}_3(t) \leq 6$ .

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