On The Henstock- Kurzweil Integral For Riesz-Spaces-Valued Functions Defined On Euclidean Space $\Re^n$

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Abstract

This paper is a partial result of our researchs in the main topic "On The Henstock-Kurzweil Integral for Riesz-Spaces-valued Functions Defined on Riesz Space L ". We construct Henstock-Kurzweil integral for Riesz-spaces-valued functions defined on Euclidean space $\Re^n$ and prove some basic properties among which the fact that our new integral is coincides with the Henstock-Kurzweil Integral for Banach-spaces valued functions defined on space $\Re^n$.
Keywords : *Riesz Space, Henstock-Kurzweil Integral*

1. INTRODUCTION

The Henstock-Kurzweil integral for Riesz-space-valued functions defined on bounded subintervals of the real line and with respect to operator-valued measures was investigated by Riecan(1989,1992) and Riecan and Brabelova(1996), with respect to (D)- convergence (that is a kind of convergence in which the ε-technique is replaced by a technique involving double sequences , see Riecan and Neubrunn(1997)), with respect to the order convergence, see Boccuto(1998) and in Boccuto and Riecan(2004) with respect to the order convergence but the Henstock-Kurzweil integral for Riesz-space-valued functions was defined on unbounded subintervals of the real line.

The Henstock-Kurzweil integral for real-valued functions defined on Euclidean space $\Re^n$ with respect to volume α was investigated in Pfeffer(1993) and Indrati(2002) and The Henstock-Kurzweil integral for bounded-sequence-space-valued functions defined on Euclidean space $\Re^n$ with respect to volume α was investigated in Muslim and Soeparna(2002) and Zachriwan(2004).

The main goal of this paper is to generalize the results above by constructing Henstock-Kurzweil integral for Riesz-valued functions defined on Euclidean space $\Re^n$ and we prove some fundamental properties.

2. PRELIMINARY

Let $\mathbb{N}$ be the set of all strictly positive integers, $\Re$ the set of the real numbers, $\Re^+$ be the set of all strictly positive real numbers. Moreover, we refer to (Pfeffer,1993)

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about the notions of cell, segmentation, partition, α-volume, and δ- fine Perron partition.

Definisi 2.1 (Zaanen,1996) **:** *A Riesz space L is said to be Dedekind complete if every nonempty subset of L, bounded from above, has supremum in L.*

Definisi 2.2 (Riecan, 1998) **:** *A bonded double sequence $\left(a_{i,j}\right)_{i,j} \in L$ is called <u>regulator</u> or $\left(D\right)$-sequence if, for each $i \in \mathbb{N}, a_{i,j} \downarrow 0$, that is $a_{i,j} \geq a_{i,j+1} \forall j \in \mathbb{N}$ and $\bigwedge_{j\in\mathbb{N}} a_{i,j} = 0$.*

Definisi 2.3 (Boccuto and Riecan, 2004) **:** *Given a sequence $\left(r_n\right)_n \in L$. Sequence $\left(r_n\right)_n$ is said to be $\left(D\right)$-convergence to an element $r \in L$ if there exist a regulator $\left(a_{i,j}\right)_{i,j}$, satisfying the following condition:*

for every mapping $\rho : L \to L$, there exists an integer n_0 sehingga $\left|r_n - r\right| \leq \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$

for all $n \geq n_0$. In this case, the notation is denoted by $\left(D\right)\lim_n r_n = r$.

Definition 2.4 (**Boccuto and Riecan,** 2004) **:** *A Riesz Space L is said to be weakly σ-distributive if for every $\left(D\right)$- sequence $\left(a_{i,j}\right)$, then*

$$\bigwedge_{\rho\in\mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\rho(i)} \right) = 0.$$

Throughout the paper, we shall always assume that L is Dedekind complete weakly σ – distributive Riesz space.

Main Results

In the principle, this integral is a generalization of Henstock-Kurzweil integral for Riesz-valued functions defined on subintervals of the real line by changing the length of $\left[a,b\right] \subset \Re$ with the general volume α of a cell $A \subset \Re^n$.See **Pfeffer**(1993) and **Muslim and Soeparna**(2002). Remember that the volume α on cell $A \subset \Re^n$ is an additive and non negative function from $\Im\left(A\right)$ into $\Re$, where $\Im\left(A\right)$ is a collection of all subcells in A.

Definition 3.1 : *Let* α *be a volume on* $\Re^n$ *and* $A \subset \Re^n$ *be a cell. A function* $f : \Re^n \to L$ *is said to be Henstock-Kurzweil integrable on* A *with respect to* α*, denoted by* $f \in HK(A, L, \alpha)$*, if there exists an element* $\Xi \in L$ *and* (D)*-sequence* $(a_{i,j})_{i,j} \in L$ *such that for every* $\rho \in \mathbb{N}^{\mathbb{N}}$ *we can find a function* $\delta : E \to \Re^+$ *such that*

$$\left|P\sum f(\overline{x})\alpha(I) - \Xi\right| = \left|\sum_{k=1}^{r} f(\overline{x}_k)\alpha(I_r) - \Xi\right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

for every δ*-fine Perron partition* $P = \{(I, \overline{x})\} = \{(I_1, \overline{x}_1), (I_2, \overline{x}_2), \ldots, (I_r, \overline{x}_r)\}$ *on* A.

We note that the Henstock-Kurzweil integral with respect to α is well- defined, that is there exists at most one element Ξ, satisfying Definition 3.1 and in this case we have $(HK)\int_A f d\alpha = \Xi$. The uniqueness is given in the following theorem.

Theorem 3.2 : *Let* α *be a volume on* $\Re^n$ *and* $A \subset \Re^n$ *be a cell. If function* $f \in HK(A, L, \alpha)$ *, then its* α*-integral is unique.*

Proof: Let $f \in HK(A, L, \alpha)$. If both Ξ_1 and Ξ_2 are Henstock-Kurzweil integral of function f , satisfying Definition 3.1, then there exists two (D)*-sequence* $(a_{i,j})_{i,j}$ and $(b_{i,j})_{i,j}$ in L such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$, we can find two positive function δ_1 and δ_2 on A, respectively, and for every δ_1-fine Perron partition $P_1 = \{(I, \overline{x})\}$ and δ_2-fine Perron partition $P_2 = \{(I, \overline{x})\}$ on A , we have

$$\left|P_1\sum f(\overline{x})\alpha(I) - \Xi_1\right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

and

$$\left|P_2\sum f(\overline{x})\alpha(I) - \Xi_2\right| \le \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

respectively. Let now $\delta(\overline{x}) = \min\{\delta_1(\overline{x}), \delta_2(\overline{x})\}$, for every $\overline{x} \in A$ and take any δ-fine Perron partition $P = \{(I, \overline{x})\}$ on A , then $P = \{(I, \overline{x})\}$ is both δ_1-fine Perron partition and δ_2-fine Perron partition on A, and thus we have

$$0 \le \left|\Xi_1 - \Xi_2\right| \le \left|P_1 \sum f(\bar{x})\alpha(I) - \Xi_1\right| + \left|P_2 \sum f(\bar{x})\alpha(I) - \Xi_2\right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)} + \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

$$\le \bigvee_{i=1}^{\infty}\left(a_{i,\rho(i)} + b_{i,\rho(i)}\right)$$

$$\le \bigvee_{i=1}^{\infty} c_{i,\rho(i)}$$

where $c_{i,j} = 2\left(a_{i,j} + b_{i,j}\right)$ $\forall i,j \in \mathbb{N}$. By arbitrariness of $\rho \in \mathbb{N}^{\mathbb{N}}$, we get

$$0 \le \left|\Xi_1 - \Xi_2\right| \le \bigwedge_{\rho \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} c_{i,\rho(i)} \right) = 0$$

since $c_{i,j}$ is (D)-sequence and thanks to weak σ-distributivity of L. Thus $\Xi_1 = \Xi_2$, and so our HK-integral is well-defined.□ `

Now, we give some fundamental properties of $HK(A,L,\alpha)$.

Theorem 3.3 : *If* $f_1, f_2 \in HK(A,L,\alpha)$ *and* $k_1, k_2 \in \Re$, *then* $k_1 f_1 + k_2 f_2 \in HK(A,L,\alpha)$ *and* $(HK)\int_A (k_1 f_1 + k_2 f_2) d\alpha = k_1 (HK)\int_A f_1 d\alpha + k_2 (HK)\int_A f_2 d\alpha$.

Proof : The proof is similar to the one of (**Muslim** , 2003), Theorem 3.1.3

Theorem 3.4 : *If* $f, g \in HK(A,L,\alpha)$ *and* $f(\bar{x}) \le g(\bar{x})$ *for every* $\bar{x} \in A$, *then*

$$(HK)\int_A f d\alpha \le (HK)\int_A g d\alpha .$$

Proof : By hypotesis, there exists two $(D)-$sequences, $\left(a_{i,j}\right)_{i,j}$ and $\left(b_{i,j}\right)_{i,j}$ such that, for every $\rho \in \mathbb{N}^{\mathbb{N}}$, we can find positive functions δ_1 dan δ_2 , respectively on A, and whenever $P_1 = \{(I,\bar{x})\}$ is δ_1-fine Perron partition and $P_2 = \{(I,\bar{x})\}$ is δ_2-fine Perron partition on A, we have

$$\left|P_1 \sum f(\bar{x})\alpha(I) - \int_A f d\alpha\right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)} \quad \Leftrightarrow$$

$$\int_A f d\alpha - \bigvee_{i=1}^{\infty} a_{i,\rho(i)} \le P_1 \sum f(\bar{x})\alpha(I) \le \int_A f d\alpha + \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

and

$$\left| P_2 \sum f(\overline{x})\alpha(I) - \int_A g d\alpha \right| \le \bigvee_{i=1}^{\infty} b_{i,\rho(i)} \Leftrightarrow$$

$$\int_A g d\alpha - \bigvee_{i=1}^{\infty} b_{i,\rho(i)} \le P_2 \sum g(\overline{x})\alpha(I) \le \int_A g d\alpha + \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

respectively.

For every $\overline{x} \in A$, let $\delta(\overline{x}) = \min\{\delta_1(\overline{x}), \delta_2(\overline{x})\}$, and take δ-fine Perron partition $P = \{(I, \overline{x})\}$ on A, then $P = \{(I, \overline{x})\}$ is both δ_i-fine Perron partition $(i = 1, 2)$ on A. Thus we get

$$\int_A f d\alpha - \bigvee_{i=1}^{\infty} a_{i,\rho(i)} \le P \sum f(\overline{x})\alpha(I) \le P \sum g(\overline{x})\alpha(I) \le \int_A g d\alpha + \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

and hence, for every $\rho \in \mathbb{N}^{\mathbb{N}}$,

$$\int_A f d\alpha - \int_A g d\alpha \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)} + \bigvee_{i=1}^{\infty} b_{i,\rho(i)} \le \bigvee_{i=1}^{\infty} c_{i,\rho(i)}$$

where $c_{i,j} = 2(a_{i,j} + b_{i,j}) \forall i, j \in \mathbb{N}$. By arbitrariness of $\rho \in \mathbb{N}^{\mathbb{N}}$, since $c_{i,j}$ is a (D) – sequence and taking into account of weak σ-distributivity of L, we get

$$\int_A f d\alpha - \int_A g d\alpha \le \bigwedge_{\rho \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} c_{i,\rho(i)} \right) = 0$$

that is $\int_A f d\alpha \le \int_A g d\alpha$. This concludes the proof.□

Definition 3.5 (Elementary Set**):** *A set* $A \subset \Re^n$ *which is union of finite cells is called an elementary set.*

Every elementary set can be segmented into non-overlapping cells. If A_1 and A_2 are elementary sets then $A_1 \cup A_2$ and $\overline{A_1 \setminus A_2}$ are also elementary sets. Integration on elementary set can be constructed through the following theorem.

Teorema 3.6 : *Let* α *be a volume on* $\Re^n$ *and* A_1 *and* A_2 be non-overlapping cells in $\Re^n$ *and* $A = A_1 \bigcup A_2$. *If* $f \in HK(A_1, L, \alpha)$ *and* $f \in HK(A_2, L, \alpha)$ *, then* $f \in HK(A, L, \alpha)$ *and*

$$(HK) \int_{A = A_1 \bigcup A_2} f d\alpha = (HK) \int_{A_1} f d\alpha + (HK) \int_{A_2} f d\alpha$$

Proof : Let $f \in HK(A_1, L, \alpha)$ and $f \in HK(A_2, L, \alpha)$. There exists two (D) – sequence $(a_{i,j})_{i,j}$ *and* $(b_{i,j})_{i,j}$, such that for every $\rho \in \mathbb{N}^{\mathbb{N}}$, we can find positive functions δ_1 and

δ_2 on A respectively. Whenever $P_1 = \{(I,\overline{x})\}$ is δ_1-fine Perron partition on A_1 and $P_2 = \{(I,\overline{x})\}$ is δ_2-fine Perron partition on A_2, we have

$$\left| P_1 \sum f(\overline{x})\alpha(I) - \int_{A_1} f d\alpha \right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

and

$$\left| P_2 \sum f(\overline{x})\alpha(I) - \int_{A_2} f d\alpha \right| \le \bigvee_{i=1}^{\infty} b_{i,\rho(i)}$$

Let now $\delta : A \to \Re^+$ be such that,

$$\delta(\overline{x}) = \begin{cases} \delta_1(\overline{x}) & \text{if } \overline{x} \in A_1 \text{ and } \overline{x} \notin A_2 \\ \delta_2(\overline{x}) & \text{if } \overline{x} \in A_2 \text{ and } \overline{x} \notin A_1 \\ \min\{\delta_1(\overline{x}), \delta_2(\overline{x})\} & \text{if } \overline{x} \in A_1 \cap A_2 \end{cases}$$

for every δ-fine Perron partition $P = \{(I,\overline{x})\}$ on A where $P = P_1 \cup P_2$. Therefore, we get

$$\left| P \sum f(\overline{x})\alpha(I) - \left(\int_{A_1} f d\alpha + \int_{A_2} f d\alpha \right) \right|$$

$$\le \left| P_1 \sum f(\overline{x})\alpha(I) - \int_{A_1} f d\alpha \right| + \left| P_2 \sum f(\overline{x})\alpha(I) - \int_{A_2} f d\alpha \right|$$

$$\le \bigvee_{i=1}^{\infty} a_{i,\rho(i)} + \bigvee_{i=1}^{\infty} b_{i,\rho(i)} \le \bigvee_{i=1}^{\infty} c_{i,\rho(i)}$$

where $c_{i,j} = 2(a_{i,j} + b_{i,j}) \forall i, j \in$ □ is a (D) – sequence, then assertion follows. □

Using Theorem 3.6 and Definition 3.5 above, we can see immediately that the following holds.

Corrolary 3.7 : *Given an elementary set* $A \subset \Re^n$ *and* α *volume on* A. *A function* $f : A \to L$ *is said to be Henstock-Kurzweil integrable on* A *with respect to* α, *denoted by* $f \in HK(A, L, \alpha)$, *if* $f \in HK(A_i, L, \alpha)$ *for every i, where* $A = \bigcup_{i=1}^{p} A_i$ *and* $\{A_1, A_2, ..., A_p\}$ *is any division on* A. *The Henstock-Kurzweil integral of function* f *on* A *is*

$$(HK)\int_A f d\alpha = \sum_{i=1}^{p} \int_{A_i} f d\alpha .$$

We now state version of the Cauchy criterion.

Theorem 3.8 : *A function $f : A \to L$ is Henstock-Kurzweil integrable if and only if there exists a $(D)-$sequence $\left(a_{i,j}\right)_{i,j}$ in L such that, for every $\rho \in \mathbb{N}^{\mathbb{N}}$ we can find a function $\delta : A \to \Re^{+}$ and for every $\delta-$fine Perron partition $P_1 = \left\{\left(A, \bar{x}\right)\right\}$ and $P_2 = \left\{\left(I, \bar{x}\right)\right\}$ on A, we have*

$$\left|P_1 \sum f\left(\bar{x}\right)\alpha\left(I\right) - P_2 \sum f\left(\bar{x}\right)\alpha\left(I\right)\right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

Proof : The proof is similar to the one of Theorem 3.1.8, p. 57 of **Muslim** (2003).

We now prove a result about Hentock-Kurzweil integrability on subcells.

Theorem 3.9 : *Let α be a volume on a cell $A \subset \Re^{n}$. If $f \in HK\left(A, L, \alpha\right)$, then $f \in HK\left(B, L, \alpha\right)$, for every cell $B \subset A$.*

Proof : By virtue of Theorem 3.8**,** there exists a $(D)-$sequence $\left(a_{i,j}\right)_{i,j}$ in L such that, for every $\rho \in \mathbb{N}^{\mathbb{N}}$ we can find a function $\delta : A \to \Re^{+}$ and for every $\delta-$fine Perron partition $P_1 = \left\{\left(I, x\right)\right\}$ and $P_2 = \left\{\left(I, x\right)\right\}$ on A, we have

$$\left|P_1 \sum f\left(\bar{x}\right)\alpha\left(I\right) - P_2 \sum f\left(\bar{x}\right)\alpha\left(I\right)\right| \le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}$$

Since cell $B \subset A$, then there exists a collection of finite non-overlapping cells Γ such that $\overline{A \setminus B} = \bigcup_{C \in \Gamma} C$. By virtue of Cousin Lemma, there exists a $\delta-$fine Perron partion P_C on C, for every $C \in \Gamma$. Let $\delta-$fine Perron partion $P_B^{'}$ and $P_B^{''}$ on B. Put $P_0 = P_B^{'} \cup \left(\bigcup_{C \in \Gamma} P_C\right)$ and $P_0^{'} = P_B^{''} \cup \left(\bigcup_{C \in \Gamma} C\right)$. Then P_0 and $P_0^{'}$ are $\delta-$fine Perron partion on A. Moreover, we get

$$\begin{aligned}
&\left|P_B^{'} \sum f\left(\bar{x}\right)\alpha\left(I\right) - P_B^{''} \sum f\left(\bar{x}\right)\alpha\left(I\right)\right| \\
&= \left|P_B^{'} \sum f\left(\bar{x}\right)\alpha\left(I\right) + \sum_{C \in \Gamma} P_C \sum f\left(\bar{x}\right)\alpha\left(I\right) - \sum_{C \in \Gamma} P_C \sum f\left(\bar{x}\right)\alpha\left(I\right) - P_B^{''} \sum f\left(\bar{x}\right)\alpha\left(I\right)\right| \\
&= \left|P_B^{'} \sum f\left(\bar{x}\right)\alpha\left(I\right) + \sum_{C \in \Gamma} P_C \sum f\left(\bar{x}\right)\alpha\left(I\right) - \left\{P_B^{''} \sum f\left(\bar{x}\right)\alpha\left(I\right) + \sum_{C \in \Gamma} P_C \sum f\left(\bar{x}\right)\alpha\left(I\right)\right\}\right| \\
&= \left|P_0 \sum f\left(\bar{x}\right)\alpha\left(I\right) - P_0^{'} \sum f\left(\bar{x}\right)\alpha\left(I\right)\right| \\
&\le \bigvee_{i=1}^{\infty} a_{i,\rho(i)}
\end{aligned}$$

Again, from Theorem 3.8 and the last term, it follows that $f \in HK(B,L,\alpha)$ □

By virtue of Theorem 3.9, we define primitif function of Henstock-Kurzweil integrable function f on a cell $A \subset \Re^n$ with respect to a volume α as follows.

Definition 3.10 : *If* $f \in HK(A,L,\alpha)$ *and* $\Im(A)$ *is a collection of all subcells in* A*, then a function* $F:\Im(A)\to L$ *satisfying*

$$F(I)=(HK)\int_I f d\alpha \quad \text{and } F(\phi)=\overline{0}$$

for every cell $I \in \Im(A)$ *is called* α *- Primitif of Henstock-Kurzweil integrable function* f *on* $\Im(A)$.

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