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SYSTEMS OF INTERVAL MIN-PLUS LINEAR EQUATIONS AND ITS APPLICATION ON SHORTEST PATH PROBLEM WITH INTERVAL TRAVEL TIMES

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Abstract

The travel times in a network are seldom precisely known, and then could be represented into the interval of real number, that is called interval travel times. This paper discusses the solution of the iterative systems of interval min-plus linear equations its application on shortest path problem with interval travel times. The finding shows that the iterative systems of interval min-plus linear equations, with coefficient matrix is semi-definite, has a maximum interval solution. Moreover, if coefficient matrix is definite, then the interval solution is unique. The networks with interval travel time can be represented as a matrix over interval min-plus algebra. The networks dynamics can be represented as an iterative system of interval min-plus linear equations. From the solution of the system, can be deter-mined interval earliest starting times for each point can be traversed. Furthermore, we can determine the interval fastest time to traverse the network. Finally, we can determine the shortest path interval with interval travel times by determining the shortest path with crisp travel times.

Key words: Min-Plus Algebra, Linear System, Shortest Path, Interval.

INTRODUCTION

Let $\mathbf{R}_\varepsilon := \mathbf{R} \cup \{\varepsilon\}$ with $\mathbf{R}$ the set of all real numbers and $\varepsilon := \infty$. In $\mathbf{R}_\varepsilon$ defined two operations : $\forall a,b \in \mathbf{R}_\varepsilon$, $a \oplus b := \min(a, b)$ and $a \otimes b := a + b$. We can show that $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a commutative idempotent semiring with neutral element $\varepsilon = \infty$ and unity element $e = 0$. Moreover, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a semifield, that is $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a commutative semiring, where for every $a \in \mathbf{R}$ there exist $-a$ such that $a \otimes (-a) = 0$. Thus, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a *min-plus algebra*, and is written as $\mathbf{R}_{\min}$. One can define $x^{\otimes 0} := 0$, $x^{\otimes k} := x \otimes x^{\otimes k-1}$, $\varepsilon^{\otimes 0} := 0$ and $\varepsilon^{\otimes k} := \varepsilon$, for $k = 1, 2, \ldots$ .. The operations $\oplus$ and $\otimes$ in $\mathbf{R}_{\min}$ can be extend to the matrices operations in $\mathbf{R}_{\min}^{m\times n}$, with $\mathbf{R}_{\min}^{m\times n} := \{A = (A_{ij}) \mid A_{ij} \in \mathbf{R}_{\min}$, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n\}$, the set of all matrices over max-plus algebra. Specifically, for $A, B \in \mathbf{R}_{\min}^{n\times n}$ we define $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$. We also define matrix $E \in \mathbf{R}_{\min}^{n\times n}$, $(E)_{ij} := \begin{cases} 0 & \text{, if } i = j \\ \varepsilon & \text{, if } i \neq j \end{cases}$ and $\varepsilon \in \mathbf{R}_{\min}^{m\times n}$, $(\varepsilon)_{ij} := \varepsilon$ for every i and j . For any matrices $A \in \mathbf{R}_{\min}^{n\times n}$, one can define $A^{\otimes 0} = E_n$ and $A^{\otimes k} = A \otimes A^{\otimes k-1}$ for $k = 1, 2, \ldots$. For any weighted, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ we can define a matrix $A \in$

$\mathbf{R}_{\min}^{n\times n}$, $A_{ij} = \begin{cases} w(j,i), & \text{if } (j,i)\in \mathcal{A} \\ \varepsilon, & \text{if } (j,i)\notin \mathcal{A}. \end{cases}$, called the *weight-matrix* of graph $\mathcal{G}$.

A matrix $A \in \mathbf{R}_{\min}^{n\times n}$ is said to be *semi-definite* if all of circuit in $\mathcal{G}(A)$ have nonnegative weight, and it is said *definite* if all of circuit in $\mathcal{G}(A)$ have positive weight. We can show that if any matrices A is semi-definite, then $\forall p \geq n,\ A^{\otimes p} \preceq_{\mathrm{m}} E \oplus A \oplus \ldots \oplus\ A^{\otimes n-1}$. So, we can define $A^*: = E \oplus A \oplus \ldots \oplus A^{\otimes n} \oplus A^{\otimes n+1} \oplus \ldots$. Define $\mathbf{R}_{\min}^{n} := \{\ \boldsymbol{x} = [\ x_1, x_2, \ldots, x_n]^{\mathrm{T}} \mid x_i \in \mathbf{R}_{\min},\ i = 1, 2, \ldots, n\}$. Notice that we can be seen $\mathbf{R}_{\min}^{n}$ as $\mathbf{R}_{\min}^{n\times 1}$. The elements of $\mathbf{R}_{\max}^{n}$ is called *vector* over $\mathbf{R}_{\min}$. In general, min-plus algebra is analogous to max-plus algebra. Further details about max-plus algebra, matrix and graph can be found in Baccelli *et.al* (2001) and Rudhito (2003).

The existence and uniqueness of the solution of the iterative system of min-plus linear equation and its application to determine the shortest path in the with crisp (real) travel times had been discussed in Rudhito (2013). The followings are some result in brief. Let $A \in \mathbf{R}_{\min}^{n\times n}$ and $\boldsymbol{b} \in \mathbf{R}_{\min}^{n\times 1}$. If A is semi-definite, then $\boldsymbol{x}^* = A^* \otimes \boldsymbol{b}$ is a solution of system $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}$. Moreover, if A is definite, then the system has a unique solution. A *one-way path network S* with crisp activity times, is a directed, strongly connected, acyclic, crisp weighted graph $S = (\mathcal{V}, \mathcal{A})$, with $V = \{1, 2, , \ldots , n\}$ suct that if $(i, j) \in \mathcal{A}$, then $i < j$. In this network, point represent *crosspathway*, arc expresses a *pathway*, while the weight of the arc represent *travel time*, so that the weights in the network is always positive. Let x_i^e is *earliest starting times* for point i can be traversed and $\boldsymbol{x^e} = [\ x_1^e, x_2^e, \ldots,\ x_n^e\ ]^{\mathrm{T}}$. For the network with crisp travel times, with n nodes and A the weight matrix of graph of the networks, then

$$\boldsymbol{x^e} = (E \oplus A \oplus \ldots \oplus\ A^{\otimes n-1}\ ) \otimes \boldsymbol{b^e} = A^* \otimes \boldsymbol{b^e}$$

with $\boldsymbol{b^e} = [0, \varepsilon, \ldots, \varepsilon]^{\mathrm{T}}$. Furthermore, x_n^e is the *fastest times to traverse* the network. Let x_i^l is be *latest times left* point i and $\boldsymbol{x}^l = [\ x_1^l, x_2^l\ \ldots.,\ x_n^l\ ]$. For the network above, vector

$$\boldsymbol{x}^l = -\ (\ (A^{\mathrm{T}}\ )^* \otimes \boldsymbol{b}^l\ )$$

with $\boldsymbol{b^l} = [\varepsilon, \varepsilon, \ldots, -x_n^e\ ]^{\mathrm{T}}$. Define, a pathway $(i, j) \in \mathcal{A}$ in the one-way path network $\mathcal{S}$ is called *shortest pathway* if $x_i^e = x_i^l$ dan $x_j^e = x_j^l$. Define, A path $p \in P$ in the one-way path network $\mathcal{S}$ is called *shortest path* if all pathways belonging to p are shortest pathway. From this definition, we can show that a path $p \in P$ is a shortest path if and only if p has minimum weight, that is equal to x_n^e. Also, a pathway is a shortest pathway if and only if it belonging to a shortest path.

DISCUSSION

We discusses the solution of the iterative systems of interval min-plus linear equations its application on shortest path problem with interval travel times. The discussion begins by reviewing some basic concepts of interval min-plus algebra and matrices over interval min-plus algebra. Definition and concepts in the min-plus algebra analogous to the concepts in the max-plus algebra which can be seen in Rudhito (2011).

The (closed) interval x in $\mathbf{R}_{\min}$ is a subset of $\mathbf{R}_{\min}$ of the form

$$\mathrm{x} = [\ \underline{x}\ ,\ \overline{x}\ ] = \{x \in \mathbf{R}_{\min} \mid \underline{x} \preceq_{\mathrm{m}} x \preceq_{\mathrm{m}} \overline{x}\ \}.$$

The interval x in $\mathbf{R}_{\min}$ is called *min-plus interval*, which is in short is called *interval*. Define

$$\mathbf{I}(\mathbf{R})_\varepsilon := \{ x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbf{R}, \ \varepsilon \prec_m \underline{x} \preceq_m \overline{x} \} \cup \{ \varepsilon \}, \text{ where } \varepsilon := [\varepsilon, \varepsilon].$$

In the $\mathbf{I}(\mathbf{R})_\varepsilon$, define operation $\overline{\oplus}$ and $\overline{\otimes}$ as

$$x \,\overline{\oplus}\, y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}] \text{ and } x \,\overline{\otimes}\, y = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}], \ \forall\, x, y \in \mathbf{I}(\mathbf{R})_\varepsilon.$$

Since $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is an idempotent semiring and it has no zero divisors, with neutral element ε, we can show that $\mathbf{I}(\mathbf{R})_\varepsilon$ is closed with respect to the operation $\overline{\oplus}$ and $\overline{\otimes}$. Moreover, $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is a comutative idempotent semiring with neutral element $\varepsilon = [\varepsilon, \varepsilon]$ and unity element $0 = [0, 0]$. This comutative idempotent semiring $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is called *interval min-plus algebra* which is written as $\mathbf{I}(\mathbf{R})_{\min}$.

Define $\mathbf{I}(\mathbf{R})_{\min}^{m\times n} := \{A = (A_{ij}) \mid A_{ij} \in \mathbf{I}(\mathbf{R})_{\min}$, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ }. The element of $\mathbf{I}(\mathbf{R})_{\min}^{m\times n}$ are called *matrices over interval min-plus algebra*. Furthermore, this matrices are called *interval matrices*. The operations $\overline{\oplus}$ and $\overline{\otimes}$ in $\mathbf{I}(\mathbf{R})_{\min}$ can be extended to the matrices operations of in $\mathbf{I}(\mathbf{R})_{\max}^{m\times n}$. Specifically, for A, B $\in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$ and $\alpha \in \mathbf{I}(\mathbf{R})_{\min}$ we define

$$(\alpha \,\overline{\otimes}\, A)_{ij} = \alpha \,\overline{\otimes}\, A_{ij}, \ (A \,\overline{\oplus}\, B)_{ij} = A_{ij} \,\overline{\oplus}\, B_{ij} \text{ and } (A \,\overline{\otimes}\, B)_{ij} = \overline{\bigoplus_{k=1}^{n}} A_{ik} \,\overline{\otimes}\, B_{kj}.$$

Matrices A, B $\in \mathbf{I}(\mathbf{R})_{\min}^{m\times n}$ are *equal* if $A_{ij} = B_{ij}$, that is if $\underline{A_{ij}} = \underline{B_{ij}}$ and $\overline{A_{ij}} = \overline{B_{ij}}$ for every i and j. We can show that $(\mathbf{I}(\mathbf{R})_{\min}^{n\times n}, \overline{\oplus}, \overline{\otimes})$ is a idempotent semiring with neutral element is matrix ε, with $(\varepsilon)_{ij} := \varepsilon$ for every i and j, and unity element is matrix E, with $(E)_{ij} := \begin{cases} 0, & \text{if } i = j \\ \varepsilon, & \text{if } i \neq j \end{cases}$. We can also show that $\mathbf{I}(\mathbf{R})_{\min}^{m\times n}$ is a semi-module over $\mathbf{I}(\mathbf{R})_{\min}$.

For any matrix A $\in \mathbf{I}(\mathbf{R})_{\min}^{m\times n}$, define the matrices $\underline{A} = (\underline{A_{ij}}) \in \mathbf{R}_{\min}^{m\times n}$ and $\overline{A} = (\overline{A_{ij}}) \in \mathbf{R}_{\min}^{m\times n}$, which is called *lower bound matrices* and *upper bound matrices of* A, respectively. Define *matrices interval of* A, that is

$$[\underline{A}, \overline{A}] = \{ A \in \mathbf{R}_{\min}^{m\times n} \mid \underline{A} \preceq_m A \preceq_m \overline{A} \} \text{ and } \mathbf{I}(\mathbf{R}_{\min}^{m\times n})^* = \{ [\underline{A}, \overline{A}] \mid A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n} \}.$$

Specifically, for $[\underline{A}, \overline{A}], [\underline{B}, \overline{B}] \in \mathbf{I}(\mathbf{R}_{\min}^{m\times n})^*$ and $\alpha \in \mathbf{I}(\mathbf{R})_{\min}$ we define

$$\alpha \,\overline{\otimes}\, [\underline{A}, \overline{A}] = [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}], \ [\underline{A}, \overline{A}] \,\overline{\oplus}\, [\underline{B}, \overline{B}] = [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$$

$$\text{and } [\underline{A}, \overline{A}] \,\overline{\otimes}\, [\underline{B}, \overline{B}] = [\underline{A} \otimes \underline{B}, \overline{A} \otimes \overline{B}].$$

The matrices interval $[\underline{A}, \overline{A}]$ and $[\underline{B}, \overline{B}] \in \mathbf{I}(\mathbf{R}_{\min}^{m\times n})^*$ are *equal* if $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$. We can show that $(\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*, \overline{\oplus}, \overline{\otimes})$ is an idempotent semiring with neutral element matrix interval $[\varepsilon, \varepsilon]$ and the unity element is matrix interval [E, E]. We can also show that $\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$ is a semimodule over $\mathbf{I}(\mathbf{R})_{\min}$.

The semiring $(\mathbf{I}(\mathbf{R})_{\min}^{n\times n}, \overline{\oplus}, \overline{\otimes})$ is isomorfic with semiring $(\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*, \overline{\oplus}, \overline{\otimes})$. We can define a mapping f, where $f(A) = [\underline{A}, \overline{A}]$, $\forall A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$. Also, the semimodule $\mathbf{I}(\mathbf{R})_{\min}^{n\times n}$ is isomorfic with semimodule $\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$. So, for every matrices interval A $\in \mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$ we can determine matrices interval $[\underline{A}, \overline{A}] \in \mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$. Conversely, for every $[\underline{A}, \overline{A}] \in \mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$, then

$\underline{A}, \overline{A} \in \mathbf{R}_{\min}^{n\times n}$, such that $[\underline{A}_{ij}, \overline{A}_{ij}] \in \mathbf{I}(\mathbf{R})_{\min}$, $\forall i$ and j. The matrix interval $[\underline{A}, \overline{A}]$ is called *matrix interval associated with the interval matrix* A and which is written $A \approx [\underline{A}, \overline{A}]$. So we have $\alpha \overline{\otimes} A \approx [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}]$, $A \overline{\oplus} B \approx [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$ and $A \overline{\otimes} B \approx [\underline{A} \otimes \underline{B}, \overline{A} \otimes \overline{B}]$.

We define for any interval matrices $A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$, where $A \approx [\underline{A}, \overline{A}]$, is said to be *semi-definite (definite)* if every matrices $A \in [\underline{A}, \overline{A}]$ is semi-definite (definite). We can show that interval matrices $A \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$, where $A \approx [\underline{A}, \overline{A}]$ is semi-definite (definite) if and only if $\overline{A} \in \mathbf{R}_{\max}^{n\times n}$ semi-definite (definite).

Define $\mathbf{I}(\mathbf{R})_{\min}^{n} := \{ \mathbf{x} = [x_1, x_2, \ldots, x_n]^T \mid x_i \in \mathbf{I}(\mathbf{R})_{\min}, i = 1, 2, \ldots, n \}$. The set $\mathbf{I}(\mathbf{R})_{\min}^{n}$ can be seen as set $\mathbf{I}(\mathbf{R})_{\min}^{n\times 1}$. The Elements of $\mathbf{I}(\mathbf{R})_{\min}^{n}$ is called *interval vector over* $\mathbf{I}(\mathbf{R})_{\min}$. The interval vector $\mathbf{x}$ associated with *vector interval* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, that is $\mathbf{x} \approx [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$.

Definition 1. *Let* $A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$ *and* $\mathbf{b} \in \mathbf{I}(\mathbf{R})_{\min}^{n}$. *A interval vector* $\mathbf{x}^* \in \mathbf{I}(\mathbf{R})_{\min}^{n}$ *is called* ***interval solution*** *of iterative system of interval min-plus linear equations* $\mathbf{x} = A \overline{\otimes} \mathbf{x} \overline{\oplus} \mathbf{b}$ *if* $\mathbf{x}^*$ *satisfy the system*.

Theorem 1. *Let* $A \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$ *and* $\mathbf{b} \in \mathbf{I}(\mathbf{R})_{\min}^{n\times 1}$. *If* A *is semi-definite, then interval vector* $\mathbf{x}^* \approx [\underline{A}^* \otimes \underline{\mathbf{b}}, \overline{A}^* \otimes \overline{\mathbf{b}}]$, *is an interval solution of system* $\mathbf{x} = A \overline{\otimes} \mathbf{x} \overline{\oplus} \mathbf{b}$. *Moreover, if* A *is definite, then interval solution is unique*.

Proof. Proof is analogous to the case of max-plus algebra as seen in the Rudhito (2011)

Next will be discussed the *earliest starting times interval* for point *i* can be traversed. The discussion is analogous to the case of (crisp) travel time (Rudhito, 2013), using the interval min-plus algebra approach.

Let $ES_i = x_i^e$ is *earliest starting times interval* for point *i* can be traversed, with $x_i^e = [\underline{x}_i^e, \overline{x}_i^e]$.

$$A_{ij} = \begin{cases} \text{interval travel time from point } j \text{ to point } i & \text{if } (j,i) \in \mathcal{A} \\ \varepsilon (= [+\infty, +\infty]) & \text{if } (j,i) \notin \mathcal{A} \end{cases}.$$

We assume that $x_i^e = 0 = [0, 0]$ and with interval min-plus algebra notation we have

$$x_i^e = \begin{cases} 0 & \text{if } i = 1 \\ \bigoplus_{1\le j\le n} (A_{ij} \overline{\otimes} x_j^e) & \text{if } i > 1 \end{cases}. \tag{1}$$

Let A is the interval weight matrix of the interval-valued weighted graph of the networks, $\mathbf{x}^e = [x_1^e, x_2^e, \ldots, x_n^e]^T$ dan $\mathbf{b}^e = [0, \varepsilon, \ldots, \varepsilon]^T$, then equation (1) can be written in an iterative system of interval max-plus linear equations

$$\mathbf{x}^e = A \overline{\otimes} \mathbf{x}^e \overline{\otimes} \mathbf{b}^e \tag{2}$$

Since the project networks is acyclic directed graph, then there are no circuit, so according to the result in Rudhito(2011), A is definite. And then according to Theorem 1,

$$\mathbf{x}^e = A^* \overline{\otimes} \mathbf{b}^e \approx [\underline{A}^* \otimes \underline{\mathbf{b}}^e, \overline{A}^* \otimes \overline{\mathbf{b}}^e]$$

$$= [(\underline{E} \oplus \underline{A} \oplus \ldots \underline{A}^{\otimes n-1}) \otimes \underline{\mathbf{b}}^{e}, (\overline{E} \oplus \overline{A} \oplus \ldots \oplus \overline{A}^{\otimes n-1}) \otimes \overline{\mathbf{b}}^{e}]$$

is a unique solution of the system (2), that is the vector of earliest starting times interval for point i can be traversed.

Notice that x_n^e is the *fastest times interval to traverse* the network. We summarize the description above in the Theorem 2.

Teorema 2. *Given a one-way path network network with interval travel times, with n node and* A *is the weight matrix of the interval-valued weighted graph of networks. The interval vector of earliest starting times interval for point i can be traversed is given by*

$$\mathbf{x}^{e} \approx [(\underline{E} \oplus \underline{A} \oplus \ldots \underline{A}^{\otimes n-1}) \otimes \underline{\mathbf{b}}^{e}, (\overline{E} \oplus \overline{A} \oplus \ldots \oplus \overline{A}^{\otimes n-1}) \otimes \overline{\mathbf{b}}^{e}]$$

with $\mathbf{b}^{e} = [0, \varepsilon, \ldots, \varepsilon]^{T}$. *Furthermore,* x_n^e *is the fastest times interval to traverse the network.*

Bukti: (see description above) . ■

Example 1 *Consider the project network in Figure 1.*

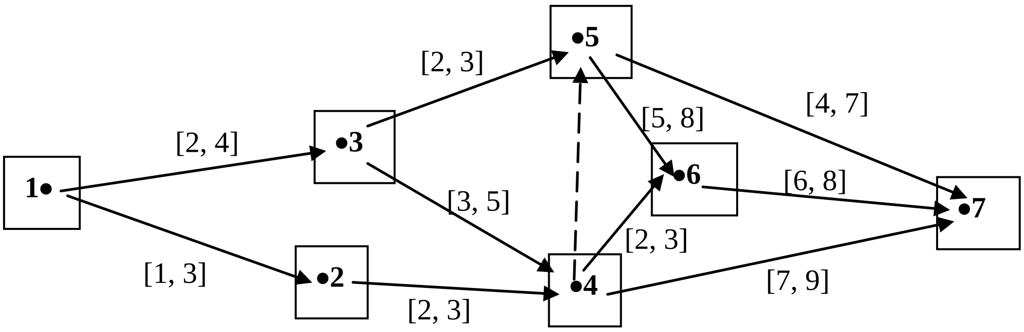

Figure 1. A one-way path network network with interval travel times

We have

$$A = \begin{bmatrix} \varepsilon, & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ [1,3] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ [2,4] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & [2,3] & [3,5] & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & [2,3] & [0,0] & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & [2,3] & [4,7] & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & [7,9] & [5,8] & [6,8] & \varepsilon \end{bmatrix}.$$

Using MATLAB computer program, we have

$$\underline{A}^* = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & 2 & 3 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 3 & 2 & 2 & 0 & 0 & \varepsilon & \varepsilon \\ 5 & 4 & 5 & 2 & 4 & 0 & \varepsilon \\ 8 & 7 & 7 & 5 & 5 & 6 & 0 \end{bmatrix}, \ \overline{A}^* = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 4 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 6 & 3 & 5 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 6 & 3 & 3 & 0 & 0 & \varepsilon & \varepsilon \\ 9 & 6 & 8 & 3 & 7 & 0 & \varepsilon \\ 14 & 11 & 11 & 8 & 8 & 8 & 0 \end{bmatrix},$$

$\underline{\mathbf{x}}^e = [0, 1, 2, 3, 3, 5, 8]^T$ dan $\overline{\mathbf{x}}^e = [0, 3, 4, 6, 6, 9, 14]^T$.

So the vector of earliest starting times interval for point i can be traversed is

$\mathbf{x}^e$ = [[0, 0], [1, 3], [2, 4], [3, 6], [3, 6], [5, 9], [8, 14]]^T and the fastest times interval to traverse the network x_n^e = [16, 25].

Next given shortest path interval definition and theorem that gives way determination. Definitions and results is a modification of the definition of critical path-interval and theorem to determine the critical path method-interval, as discussed in Chanas and Zielinski (2001) and Rudhito (2011).We also give some examples for illustration.

Definition 2. *A path $p \in P$ is called an* ***interval-shortest path*** *in* ***S*** *if there exist a set of travel times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that p is shortest path, after replacing the interval travel times* A_{ij} *with the travel time A_{ij} .*

Definisi 3. *A pathway $(k, l) \in \mathcal{A}$ is called an* ***interval-shortest pathway*** *in* ***S*** *if there exist a set of travel times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that (k, l) is shortest pathway, after replacing the interval travel times* A_{ij} *with the travel time A_{ij} .*

The following theorem is given which relates the interval-shortest path and interval-shortest pathway.

Teorema 3. *If path $p \in P$ is an interval-shortest path, then all pathways in the p are interval-shortest pathway.*

Proof : Let path $p \in P$ is an interval shortest path, then according to Definition 2, there exist a set of times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that p is shortest path, after replacing the interval travel times A_{ij} with the travel time A_{ij} . Next, according to the definition of shortest path above, all pathways in p are shortest pathways for a set of travel times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$. Thus according to Definiton 3, all pathways in p are interval-shorstest pathways. ■

The following theorem is given a necessary and sufficient condition a path is an interval-shortest path.

Teorema 4. *A path $p \in P$ is an* ***interval-shortest path*** *in* ***S*** *if and only if p is a shortest path, with interval travel times $\mathrm{A}_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, have been replace with travel times A_{ij} which is determined by the following formula*

$$A_{ij} = \begin{cases} \overline{A}_{ij} \text{ jika } (i, j) \notin p \\ \underline{A}_{ij} \text{ jika } (i, j) \in p \end{cases}. \tag{3}$$

Bukti : $\Rightarrow$: Let p is an interval-shortest path, then according to Definition 2, there exist a set of travel times A_{ij} , $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that p is shortest pathway, after replacing the interval travel times A_{ij} with travel times A_{ij} , $(i, j) \in \mathcal{A}$. If the travel times for all pathway is located at p is reduced from A_{ij} to $\underline{A}_{ij}$ and for all pathway is not located p is increased from A_{ij} to $\overline{A}_{ij}$, then p is a path with minimum weight in $\boldsymbol{S}$ for new travel time formation. Thus path p is a shortest path.

$\Leftarrow$: Since path p a shortest path with a set of travel times $A_{ij} \in [\underline{\mathrm{A}}_{ij}, \overline{\mathrm{A}}_{ij}]$, which is determined by the formula (9), then according to Definition 2, path p is an interval-shortest path. ∎

Example 2. We consider the network in Example 1. We will determine all interval-shortest path in this network. For path 1→3→5→7, by applying formula (9), we have weight

$$\begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 3 & 8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 9 & 4 & 8 & \varepsilon \end{bmatrix}.$$

Using MATLAB computer program, we have a shortest path 1→3→5→7 with minimum weight of path is 8. Thus 1→3→5→7 is an interval-shortest path. The results of the calculations for all possible path in the network are given in Table 1 below.

Tabel 1 Calculation results of all path

No	**Path *p***	**Weight Interval *p***	**Shortest-path p^*** (with formula (9))	**Weight of p^***	**Conclusion**
1	1→3→5→7	[8,14]	1→3→5→7 ,	8	Interval-shortest
2	1→3→5→6→7	[15, 23]	1→3→5→7	11	Not interval-shortest
3	1→3→4→5→7	[9, 16]	1→3→4→5→7 1→3→5→7	9	Interval-shortest
4	1→3→4→5→6→7	[16, 25]	1→3→4→5→7 1→3→5→7	12	Not interval-shortest
5	1→3→4→6→7	[13, 20]	1→3→4→5→7 1→3→5→7	12	Not interval-shortest
6	1→3→4→7	[12, 18]	1→3→4→5→7 1→3→4→7 1→3→5→7	12	Interval-shortest

7	1→2→4→5→7	[7, 13]	1→2→4→5→7	7	Interval-shortest
8	1→2→4→5→6→ 7	[14, 22]	1→2→4→5→7	10	Not interval-shortest
9	1→2→4→6→7	[11, 17]	1→2→4→5→7	10	Not interval-shortest
10	1→2→4→7	[10, 15]	1→2→4→5→7 1→2→4→7	10	Interval-shortest

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