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PROBABILITY DENSITY FUNCTION OF M/G/1 QUEUES UNDER (0,K) CONTROL POLICIES: A SPECIAL CASE

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Abstract

In this paper we present probability density function of vacation period of M/G/1 queueing process that operates under (0,k) vacation policy, wherein the server goes on the vacation when the system becomes empty and re-opens for service immediately at the arrival of the k^{th} customer. The number of lattice paths when last arrival is an arrival has also been derived.
The transient analysis is based on approximating the general service time distribution by Coxian two-phase distribution and representing the corresponding queueing process as a lattice path. Finally the lattice path combinatorics is used to present the number of lattice paths.

Keywords-lattice path; Coxian distributiont; control policies

INTRODUCTION

Study of performance analysis of a queueing system i.e. a system consisting of one or several queues, is mainly devoted into two parts i.e. the steady-state performance and the transient behaviour. Many authors have considered the steady-state analysis. We may mention Takagi [1,2], Neuts [3,4], Gross and Harris [5], Kleinrock[6,7] among those who have presented the steady-state analysis for more general queueing systems. Buzacott and Shantikumar [8] concerned on the steady-state analysis in the area of manufacturing systems.

Feature of actual situation in manufacturing models often follows this M/G/1 model. When the failed machines happened and the operators have to repair, then the amount of time needed to finish the job has a general distribution. Buzacott and Shanthikumar [8] presented some probability distributions commonly used in the modeling of manufacturing systems. These are including exponential, Erlang, hyperexponential, general exponential, phase type and two-phase Coxian etc

The queueing system when the server has to work on primary and secondary customers are useful in model building in many real life situations such as digital communication, computer network and production/inventory systems e.g., see Takagi [1,2] and Doshi [9]. For example in a simple case a server serving at a bank teller may continue to serve the customers till no more customers are waiting. At this point server may switch on to other task like counting cash and return back to service as soon as a new customer arrives. In this case we say that the queueing system is operating under server vacation and a number of different rules (control policies) can be set for the server to return to the main systems. Some of the papers that considered server's vacations are Kumar and Madheswari [10], Kao and Narayanan [11], Agarwal and Dshalalow [12], Chae and Kim [13], Chae and Lim [14], Sridharan and Mohanadivu [15], Choudury [16] and Garj [17].

Sen et al. [18] studied this system for M/M/1 queues. The M/G/1 queueing system under this control policy was studied by Agarwal and Dshalalow [12].

2. THE TRANSIENT ANALYSIS

2.1 The (0,K) Control Policy

In this paper we will be concentrating on control policies namely (0, K) control policies. Under this policy the server goes on the vacation when the system becomes empty and re-opens for service immediately at the arrival of the kth costumer.

2.2 Two-phase Coxian distribution

The Coxian distribution describes duration until an event occurs in terms of a process consisting of 2 latent phases, leading to the Markovian structure. Figure 1 presents the Cox-2 phase approximation, where μ_1 and μ_2 are the average rate of serving the customers in phase 1 and phase 2 respectively. The α_1 is the probability that a customer will move from phase 1 to phase 2 and β_1 is the probability that a customer will depart the system after completing the phase 1.

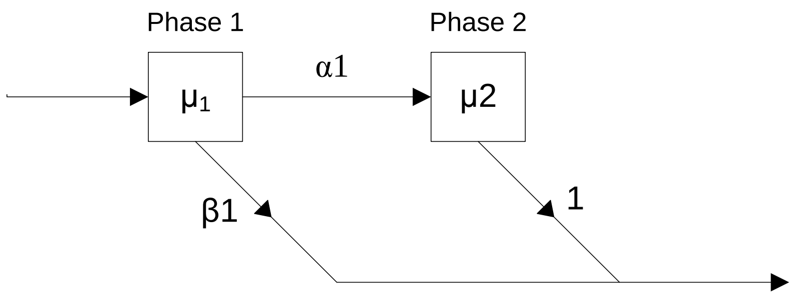

Fig.1. Two-phase Coxian distribution

The Coxian distribution plays an important role. Their importance is in large part due to their universality i.e. any distribution function can be approximated arbitrarily closely by a Coxian distribution, Cox [19], Khosgooftar and Perros [20], Agarwal et al. [21, 22], Sen and Agarwal [23], Sen [24], Harris et al. [25] Due to the memoryless of the exponential distribution, stochastic processes involving Coxian distribution whose branching probabilities are real are Markovian. Therefore, they can be analysed by well known technique, Khosgooftar and Perros [20], Harris et al. [25].

2.3 Lattice path (LP)

Lattice path simply can be defined as the translation of the behaviour of the sample paths of the random process involved. This implies many characters of a queueing system become characteristics of lattice paths, like maximum height, the number of steps until the barrier is reached etc.

LPC starts with representing the behavior of the queueing process through a sequence of steps represented as lattice path. For example a lattice path can be constructed by representing an arrival into the system by a horizontal step and departure by vertical step. In this case the system size at any point is the difference between the number of horizontal and vertical steps, or the distance between the end point of the path and the barrier $Y = X$. Thus arrival of a customer during any phase of service, departure of a customer that can occurs at any phase of service and entry into phase 2 will be denoted by a horizontal unit step, a vertical unit step and a diagonal of $\sqrt{2}$ unit step, respectively.

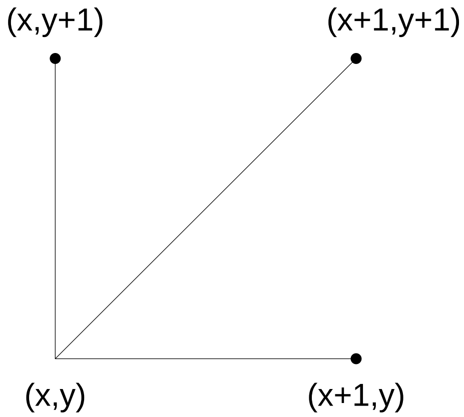

Fig. 2 A sequence of steps represented as lattice path

Therefore, if $(x, y),(x \geq k)$ denotes a vertex of the LP representing the M/C2/1 queueing process at any point, $(x, y+1)$ denotes a departure after any phase, $(x+1, y)$ denotes an arrival during any phase and $(x+1, y+1)$ denotes a customer enters phase 2. The possibility of movements will be as shown in Fig. 2.

2.4 Definition

Run (Agarwal et al., [22]): A sequence of consecutive horizontal (vertical) steps bounded on each side by a vertical (horizontal) step is called a horizontal (vertical) runs of arrivals (departures), respectively. The sequences of arrivals starting from the origin and preceding the first vertical step as well as the sequence of departures at the end following the last arrival are also called the run of arrivals (departures), respectively.

2.5 Notation and terminology.

We follow notation and terminology used by Agarwal et al. [22].

While inserting the diagonals, we have to observe that two or more consecutive diagonal do not occurs, in any horizontal run not more than one diagonal occurs, in vertical run any number of diagonal may occurs, the first vertical step following a diagonal step has to be dotted vertical step, two or more consecutive dotted vertical steps cannot occurs, and a dot vertical step can not immediately preceded by a vertical step (departure after phase 2 cannot be preceded by departure after phase 1).

In a Lattice Path (LP), let

k initial number of customers at the start of busy period

r number of horizontal runs, $r-1$vertical runs, if after r^{th} horizontal run t there is no departure otherwise r vertical runs $(r \geq 1)$.

p total number of diagonal representing entry into phase 2 inserted in horizontal and/or vertical runs $(p \geq 0)$

q total number of diagonal representing entry into phase 2 inserted in horizontal runs

p-q number of diagonal representing entry into phase 2 inserted in vertical runs

li length of the ith horizontal run (i=1,2,…,r1)

Li length of the ith vertical run (i=1,2,…,r2)

$\underset{\sim}{L}$ $\left(l_1, l_2, ..., l_{r_1}; L_1, L_2, ..., L_{r_2}\right)$

$\underset{\sim}{i}$ $\left(i_1, i_2, ..., i_q\right)$

i.e q horizontal runs in each of which a diagonal representing into phase 2 is inserted

$l_{\underset{\sim}{i}}$ $\left(l_1, l_2, ..., l_q\right)$

i.e lengths of horizontal runs

$\underset{\sim}{p}_i$ $\left(p_{i_1}, p_{i_2}, ..., p_{i_q}\right)$

i.e distances from extreme left end points where diagonals representing entry into phase

2 are inserted in horizontal runs $\underset{\sim}{i}$ including vertices at both ends of the runs)

3. RESULTS

Theorem 1. (Vacation period probability density function)

Let $f_0\left(t_1\right)$ denote the probability density function that the system M/C$_2$/1 has vacation period of length t_1 before starting service initially with k customers. Then we have

$$f_0(t_1) = \frac{e^{\lambda t_1}\left(\lambda\right)^{k-1} t_1^{k-2}}{\Gamma\left(k-1\right)} \tag{0.1}$$

Proof: This term corresponds to the case when customers arrive before the system starts service. The total number of arrivals is $k-1$, and the total number of departures is 0. Therefore total number of transition during vacation period is $k-1$.

At time To, Poisson process starts with rate λ. The probability of an arrival is 1 and the probability of a departure is 0. The probability density function of t_1 is $\left(k-1\right)$-Erlang with parameter λ given by

$$f_0(t_1) = \frac{e^{\lambda t_1}\left(\lambda\right)^{k-1} t_1^{k-2}}{\Gamma\left(k-1\right)}$$

(0.2)

Therefore the vacation density function for the case becomes

$$f_0(t_1) = \frac{e^{\lambda t_1}(\lambda)^{k-1} t_1^{k-2}}{\Gamma(k-1)} \quad , \tag{0.3}$$

since the number of lattice paths starting from (0,0) to $(k-1,0)$ is 1.

Theorem 2. (Counting the number of LPs when last event is an arrival)

For non-negative integers $k,m,n;p,q;r(r\geq 1),l_1,l_2,...,l_r;L_1,L_2,...,L_{r-1};$ let $\left(LP^*_{(k,n;l,m;r,\underset{\sim}{L}^*)}\right)$, where $\underset{\sim}{L}^* = (l_1,l_2,...,l_r;L_1,L_2,...,L_{r-1})$, denote the number of LP$_s$ from *(k,0)* to *(m,n), m>n*, remaining below the line Y=X, each comprising of *m-p* horizontal steps $(\text{including those from } (0,0) \text{ to } (k,0))$, *n-p* vertical steps and p diagonals, such that

(a) $m-p$ horizontal steps form r runs of lengths $l_1,l_2,...,l_r$, respectively, satisfying $l_1 \geq i_0, l_2,...,l_r > 0$ and $\sum_{i=1}^{r} l_i = m-p$

(b) $n-p$ vertical steps form $r-1$ runs of lengths $L_1,L_2,...,L_{r-1}$, respectively, satisfying $L_1,L_2,...,L_{r-1} > 0$ and $\sum_{i=1}^{r-1} L_i = n-p$

(c) $l_1 \geq Max(i_0, L_1+1), \sum_{i=1}^{u} l_i > \sum_{i=1}^{u} L_i, u=1,2,...,r-1,$ $\sum_{i=1}^{r} l_i = m-p,$ $\sum_{i=1}^{r-1} L_i = n-p$

(d) *q* diagonals representing into phase 2 are inserted each in any *q* out of r horizontal runs (including the vertices at both ends of the runs),

(e) The remaining *p-q* diagonals representing into phase 2 are inserted each at any $n-p-r+1$ vertices available along the vertical runs,

Then, for $r \geq 1$ and $m > k$,

$$\left(LP^*_{(k,m,n;p,q;r,\underset{\sim}{L}^*)}\right) = \sum_{R_7}\sum_{R_8}\binom{n-p-r+1}{p-q} \tag{1.4}$$

Proof. If we delete all the diagonal steps, and compress to form a skeleton path, then there must be $(m-p)$ horizontal steps, and $(n-p)$ vertical steps. To get from $(k,0)$ to $(m-p,n-p)$ we suppose this skeleton consists of r horizontal runs and r vertical runs of lengths l_i $(i=1,2,...,r)$, and L_j $(j=1,2,...,r)$ respectively. One unique path will be produced by this scenario. For the purpose of insertion, suppose *q* diagonals are inserted into runs numbered i_1, $i_2,\dots,$ i_q, respectively with lengths of l_i, $l_2,\dots,$ l_q at distances $p_{i_1}, p_{i_2},..., p_{i_q}$ from the extreme left end points. The remaining *p-q* diagonals will be inserted into any *p-q* vertices out of $n-p-r$. The number to do this is $\binom{n-p-r}{p-q}$.

Now summing $\binom{n-p-r}{p-q}$ over all possible q-tuples, ($i_1, i_2, \ldots, i_q$) and $p_{i_1}, p_{i_2}, \ldots, p_{i_q}$, we get (1.4).

Lemma 1. Let $\left(LP^*_{(k,m,n;p)}\right)$ the number of LP_s from *(k,0)* to *(m,n)*, *m>n*, remaining below the line Y=X, each comprising of *m-p* horizontal steps (including those from $(0,0)$ to $(k,0)$, *n-p* vertical steps and *p* diagonals, then summing (1.4) over r, q and L^*, we find

$$\left(LP^*_{(k,m,n;p)}\right) = \sum_{R_4}\sum_{R_5}\sum_{R_6^*}\left(LP^*_{\left(k,m,n;p,q;r,\underset{\sim}{L}^*\right)}\right) \tag{1.5}$$

where

$$\begin{aligned} R_6^* = \{\underset{\sim}{L} : \; & l_1 \geq \max(k, L_1+1), \\ & l_1 + l_2 > L_1 + L_2, \ldots, l_1 + l_2 + \ldots + l_r > L_1 + L_2 + \ldots + L_{r-1}, \\ & \textstyle\sum_{i=1}^{r} l_i = m-p, \sum_{i=1}^{r-1} L_i = n-p\} \end{aligned}$$

For the case $r \geq 1$ and p=0, we get

$$\left(LP_{(k,m,n;0,0)}\right) = \left(LP^*_{(k,m,n;0,0)}\right) = \binom{m+n-k}{n} - \binom{m+n-k}{m} \tag{1.6}$$

ACKNOWLEDGMENT

Isnandar Slamet thanks to Directorate General of Higher Education (Direktorat Jendaral Pendidikan Tinggi) the Republic of Indonesia for providing him with scholarship under the agreement with Sebelas Maret University Surakarta.

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