H. G. DALES and A. ÜLGER

Pointwise approximate identities in Banach function algebras

H. G. Dales
Department of Mathematics and Statistics
University of Lancaster
Lancaster, LA1 4YF
United Kingdom
E-mail: g.dales@lancaster.ac.uk

A. Ülger
Department of Mathematics
Boğaziçi University
34450 Bebek, Istanbul
Turkey
E-mail: ali.ulger@boun.edu.tr

Contents

Abstract

In this memoir, we shall study Banach function algebras that have bounded pointwise approximate identities, and especially those that have contractive pointwise approximate identities. A Banach function algebra A is (pointwise) contractive if A and every non-zero, maximal modular ideal in A have contractive (pointwise) approximate identities.

Let A be a Banach function algebra with character space Φ_A. We shall show that the existence of a contractive pointwise approximate identity in A depends closely on whether $\|\varphi\| = 1$ for each $\varphi \in \Phi_A$. The linear span of Φ_A in the dual space A' is denoted by $L(A)$, and this is used to define the BSE norm $\|\cdot\|_{\mathrm{BSE}}$ on A; the algebra A has a BSE norm if this norm is equivalent to the given norm. We shall then introduce and study in some detail the quotient Banach function algebra $\mathcal{Q}(A) = A''/L(A)^{\perp}$; we shall give various examples, especially uniform algebras and those involving algebras that are standard in abstract harmonic analysis, including Segal algebras with respect to the group algebra of a locally compact group.

We shall characterize the Banach function algebras for which $\overline{L(A)} = \ell^1(\Phi_A)$, and then classify contractive and pointwise contractive algebras in the class of unital Banach function algebras that have a BSE norm; they are uniform algebras with specific properties. We shall also give examples of such algebras that do not have a BSE norm.

Finally we shall discuss when some classical Banach function algebras of harmonic analysis have non-trivial reflexive closed ideals, and make some remarks on weakly compact homomorphisms between Banach function algebras.

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1. Introduction

Let A be a Banach function algebra, so that A is a commutative, semi-simple Banach algebra, with character space Φ_A. There are many notions of 'approximate identity' associated with A and its maximal modular ideals. Here we concentrate on studying bounded pointwise approximate identities (BPAIs) and contractive pointwise approximate identities (CPAIs) in A. A net (e_α) in a Banach function algebra A is a CPAI if $\|e_\alpha\| \leq 1$ for each α and $\lim_\alpha e_\alpha(\varphi) = 1 \ (\varphi \in \Phi_A)$; as in [18], a Banach function algebra is defined to be pointwise contractive if A and all its non-zero, maximal modular ideals have a CPAI. These nets play an important role in the study of so-called BSE algebras, of the BSE norm, and the determination of the space $\overline{L(A)}$ and the algebra $A''/L(A)^\perp$, where $L(A) = \text{lin}\,\Phi_A$ in A'. Here A'' is the bidual space of a Banach algebra A, taken with the first Arens product.

In this memoir, we shall characterize in various ways Banach function algebras that have a CPAI and that are pointwise contractive, and give various properties of Banach function algebras with a CPAI. In particular, we shall obtain new results about Segal algebras with respect to a Banach function algebra. We shall also define and study the quotient algebra $\mathcal{Q}(A) = A''/L(A)^\perp$, showing that it is also always a Banach function algebra; we shall discuss when this quotient algebra is a uniform algebra.

We shall give various examples of Banach function algebras to illustrate the above concepts. These will mainly be uniform algebras and algebras that arise in harmonic analysis, including group algebras, measure algebras, and Segal algebras.

Summary In Chapter 2, we shall recall some definitions and establish our notation; in particular, we shall define Banach function algebras and recall some of their standard properties, including regularity and strong regularity; we shall define multipliers on and the bidual of a Banach function algebra using the Arens products, noting that sometimes a Banach function algebra is an ideal in its bidual. We shall also define dual Banach function algebras.

In Chapter 3, we shall give some preliminary results on pointwise contractive Banach function algebras and, in particular, pointwise contractive uniform algebras. We shall define an abstract Segal algebra with respect to a given Banach function algebra.

In Chapter 4, we shall define the separating ball property and weak separating ball property for Banach function algebras, and shall obtain some results, in particular involving strong boundary points and Gleason parts for uniform algebras.

In Chapter 5, we shall introduce the key space $L(A)$ for a Banach function algebra A, and use it to define the BSE norm on A. The terminology 'BSE norm' arises from the

Bochner–Schoenberg–Eberlein theorem, which shows, in particular, that the BSE norm on a group algebra $L^1(G)$ for a locally compact abelian group G is equal to the given norm. For example, in this chapter, we shall show that a Banach function algebra that is an ideal in its bidual and has a bounded pointwise approximate identity has a BSE norm.

The space $L(A)$ will lead us in Chapter 6 to the definition of the Banach space $\mathcal{Q}(A) = A''/L(A)^{\perp}$, an apparently new abstract definition; we shall prove that $\mathcal{Q}(A)$ is also a Banach function algebra for every Banach function algebra A. The algebra $\mathcal{Q}(A)$ has an identity if and only if A has a bounded pointwise approximate identity. The character space Φ_A of A is naturally regarded as a subset of the character space $\Phi_{\mathcal{Q}(A)}$ of $\mathcal{Q}(A)$ (although the embedding of Φ_A in $\Phi_{\mathcal{Q}(A)}$ is rarely continuous). We shall consider when Φ_A, regarded as a subset of $\Phi_{\mathcal{Q}(A)}$, is open and discrete and when its closure is compact. We shall see that Φ_A is open and discrete when A has the weak separating ball property; the closure of Φ_A in $\Phi_{\mathcal{Q}(A)}$ is compact if and only if Φ_A is weakly closed in A'.

In Chapter 7, we shall give a number of examples of Banach function algebras A, and we shall determine the corresponding Banach function algebras $\mathcal{Q}(A)$ and the character space $\Phi_{\mathcal{Q}(A)}$; most of these examples are uniform algebras or are related to well-known Banach algebras that arise in abstract harmonic analysis. In particular, we shall identify $\mathcal{Q}(A)$ when A is the disc algebra and when A is the group algebra of a locally compact abelian group.

In Chapter 8, we shall characterize those Banach function algebras that have a contractive pointwise approximate identity, and then use this to prove as a main result the equality of two Banach function algebras $\mathcal{Q}(S_1)$ and $\mathcal{Q}(S_2)$, where S_1 and S_2 are Segal algebras with respect to the same Banach function algebra and both have contractive pointwise approximate identities. This implies that a Segal algebra S with respect to a group algebra on a locally compact abelian group G such that S has a contractive pointwise approximate identity has a BSE norm only in the special case that $S = L^1(G)$.

In Chapter 9, we shall first prove in Theorem 9.3 a classification theorem for contractive and for pointwise contractive unital Banach function algebras A that have a BSE norm: these Banach function algebras are equivalent to uniform algebras that have specific properties. It will be shown in Examples 9.11 and 9.12, respectively, that pointwise contractive and contractive Banach function algebras do not necessarily have a BSE norm and may not be equivalent to a uniform algebra.

We shall also in Chapter 9 compare, for a Banach function algebra A, the two Banach spaces $(\overline{L(A)}, \|\cdot\|)$ and $(\ell^1(\Phi_A), \|\cdot\|_1)$, and consider when these spaces are isomorphic or isometrically isomorphic. For example, in Theorem 9.8, we shall show that the canonical linear map $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is an isometric surjection if and only if A is pointwise contractive and its BSE norm is equal to the uniform norm. We shall address the question when $\mathcal{Q}(A)$ is a uniform algebra. In Theorem 9.10(i), we shall show that $\mathcal{Q}(A)$ is a uniform algebra whenever it is the case that the above canonical linear map is an isometric surjection; the converse holds whenever A is dense in the space $(C_0(\Phi_A), |\cdot|_{\Phi_A})$, but the disc algebra will show that the converse is not true in general. Further, Example 9.11 will show that, for a certain Segal algebra M with respect to $C_0((0,1])$, the algebra $\mathcal{Q}(M)$ is a uniform algebra, but M is not itself a uniform algebra.

In Chapter 10, we shall show that, for each Banach function algebra A that has a contractive pointwise approximate identity and whose norm is equal to its BSE norm, the multiplier algebra of A embeds isometrically into the unital Banach function algebra $\mathcal{Q}(A)$. This result extends a known theorem in which it is supposed that the algebra A has a contractive approximate identity.

In Chapter 11, we shall use the earlier results to consider when certain Banach function algebras contain non-trivial closed ideals that are reflexive Banach spaces, and consider when there are non-trivial weakly compact homomorphisms between two Banach function algebras.

We shall conclude with a list of some questions that we cannot resolve.

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2. Notation and terminology

We shall now recall some definitions and notations that we shall use in this memoir; in general, we shall follow the notation of the monograph [12] and our earlier paper [18].

The natural numbers and integers are $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}$, respectively; the complex plane is denoted by $\mathbb{C}$; the open unit disc in $\mathbb{C}$ is denoted by

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

and the unit circle is $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The closed unit interval $[0, 1]$ in the real line $\mathbb{R}$ is denoted by $\mathbb{I}$. For $n \in \mathbb{N}$, set $\mathbb{N}_n = \{1, 2, \dots, n\}$. The cardinality of a set S is denoted by $|S|$.

All linear spaces are taken to be over $\mathbb{C}$ unless stated otherwise. Let E be a linear space, with a non-empty subset S. Then the linear span of S is $\operatorname{lin} S$ and the convex hull in E of S is denoted by $\operatorname{co} S$.

Let E be a normed space. Then we denote the Banach space which is the dual space of E by E', with the duality specified by

$$(x, \lambda) \mapsto \langle x,\, \lambda\rangle\,, \quad E \times E' \to \mathbb{C}\,;$$

sometimes this duality is written as $\langle\,\cdot\,,\,\cdot\,\rangle_{E,E'}$. The second dual space, or *bidual space*, of E is $E'' = (E')'$, and we regard E as a subspace of E''; the canonical embedding is $\kappa_E : E \to E''$, where

$$\langle \kappa_E(x), \lambda\rangle = \langle x, \lambda\rangle \quad (x \in E, \lambda \in E')\,.$$

The closed ball in E that is centred at 0 and of radius $r \geq 0$ is $E_{[r]}$; the weak topology on E is $\sigma(E, E')$ and the weak-$*$ topology on E' is $\sigma(E', E)$. Thus the closed ball $E'_{[r]}$ is weak-$*$ compact and $E_{[r]}$ is weak-$*$ dense in $E''_{[r]}$ for each $r \geq 0$.

Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a linear space E. Then we say that $\|\cdot\|_1 \preceq \|\cdot\|_2$ if there is a constant $C >$ such that $\|x\|_1 \leq C\,\|x\|_2$ $(x \in E)$, and

$$\|\cdot\|_1 \sim \|\cdot\|_2$$

when the two norms are equivalent, so that $\|\cdot\|_1 \preceq \|\cdot\|_2$ and $\|\cdot\|_2 \preceq \|\cdot\|_1$.

Let E and F be two Banach spaces. The Banach space of all bounded linear operators from E to F is denoted by

$$(\mathcal{B}(E, F), \|\cdot\|_{\mathrm{op}})\,,$$

with $\mathcal{B}(E)$ for $\mathcal{B}(E, E)$. Let $T \in \mathcal{B}(E, F)$. Then $T' \in \mathcal{B}(F', E')$ and $T'' \in \mathcal{B}(E'', F'')$ are the *adjoint* and the *second adjoint* of T, respectively; certainly, we have

$$\|T\|_{\mathrm{op}} = \|T'\|_{\mathrm{op}} = \|T''\|_{\mathrm{op}}\ .$$

The two spaces E and F are *isomorphic* if there is an operator $T \in \mathcal{B}(E, F)$ that is bijective, and we then write

$$E \sim F\,;$$

the spaces are *isometrically isomorphic* if there is an isometry $T \in \mathcal{B}(E, F)$ that is bijective, and we then write

$$E \cong F\,.$$

Let E be a Banach space with a closed linear subspace F. Then the quotient space E/F is a Banach space with respect to the quotient norm. The *annihilator* in E' of a non-empty subset S of E is

$$S^{\perp} = \{\lambda \in E' : \lambda \mid S = 0\},$$

so that $S^{\perp}$ is a weak-$*$ closed linear subspace of E' and $E'/S^{\perp} \cong F'$, where F is the closed linear span of S. For each closed linear subspace F of E, we have

$$\left\|\lambda + F^{\perp}\right\| = \inf\{\|\lambda + \mu\| : \mu \in F^{\perp}\} = \sup\left\{|\langle \zeta,\, \lambda\rangle| : \zeta \in F_{[1]}\right\} \tag{2.1}$$

for each $\lambda \in E'$; further, we always have $(E/F)' \cong F^{\perp}$. Indeed, $\lambda \in F^{\perp}$ acts on E/F by setting

$$\langle x + F,\, \lambda\rangle = \langle x,\, \lambda\rangle \quad (x \in E). \tag{2.2}$$

The space F'' is identified with the space $(F^{\perp})^{\perp}$.

A closed subspace F of a Banach space E is *complemented in* E if there is a closed subspace G of E such that $E = F \oplus G$, or, equivalently, such that there is a projection $P \in \mathcal{B}(E)$ with $P(E) = F$. We shall later consider Banach spaces that are complemented in their biduals. For example, suppose that E is isomorphically a dual Banach space, so that $E \sim F'$ for a Banach space F. Then the dual of the canonical embedding of F into F'' is a bounded projection

$$P : \Lambda \mapsto \Lambda \mid \kappa_F(F), \quad E'' \to E,$$

such that $\|P : F''' \to F'\|_{\mathrm{op}} = 1$ (this is the *Dixmier projection*), and so we can regard F as a closed subspace of E' and write

$$E'' = \kappa_E(E) \oplus F^{\perp} = E \oplus F^{\perp} \tag{2.3}$$

as a Banach space; this shows that E is complemented in its bidual.

All algebras considered here are linear (over $\mathbb{C}$) and associative. The *centre* and *character space* of an algebra A are denoted by $\mathfrak{Z}(A)$ and Φ_A, respectively; the maximal modular ideal that is the kernel of a character φ is denoted by M_{φ}. An *idempotent* in A is an element $p \in A$ such that $p^2 = p$. Let B be a subalgebra and I an ideal of A such that $A = B \oplus I$ as linear spaces. Then A is the *semi-direct product* of B and I, and we write

$$A = B \ltimes I.$$

The algebra formed by adjoining an identity to a non-unital algebra A is denoted by $A^{\sharp}$; in the case where A already has an identity, we set $A^{\sharp} = A$.

Let A be a Banach algebra. Each character φ on A is continuous, with $\|\varphi\| \leq 1$, and Φ_A is a locally compact subspace of the dual space A' of A (when A' has the weak-$*$ topology); in fact, $\Phi_A \cup \{0\}$ is always a weak-$*$-compact subset of $A'_{[1]}$, and Φ_A is compact when A has an identity.

Let I be a closed ideal in a Banach algebra A, with quotient map $q : A \to A/I$. Then the identification of $\varphi \in \Phi_{A/I}$ with $\varphi \circ q \in \Phi_A$ gives a homeomorphism that identifies $\Phi_{A/I}$ as a closed subset of $\Phi_A \cup \{0\}$.

A *multiplier* of a commutative algebra A is a linear map $T : A \to A$ such that

$$T(ab) = aT(b) = T(a)b \quad (a, b \in A).$$

The collection of all the multipliers on A is a unital, commutative subalgebra of the algebra of all linear maps on A; it is called the *multiplier algebra* of A, and it is denoted by $\mathcal{M}(A)$. For a study of multipliers on general (non-commutative) Banach algebras, see the texts [12, 41, 42, 44], for example.

We denote by $(C^{\,b}(K), |\cdot|_K)$ the commutative Banach algebra of all bounded, continuous functions on a non-empty, locally compact space K (always taken to be Hausdorff), where $|\cdot|_K$ is the uniform norm on K. The function that is constantly 1 on K is 1_K, so that 1_K is the identity of $C^{\,b}(K)$. We denote by $C_0(K)$ the algebra of all continuous functions that vanish at infinity on K (with $C(K)$ for $C_0(K)$ when K is compact), so that $C_0(K)$ is a closed ideal in $C^{\,b}(K)$. The ideal in $C_0(K)$ consisting of the functions of compact support is $C_{00}(K)$, so that $C_{00}(K)$ is uniformly dense in $C_0(K)$.

Let S be a non-empty set, and let E be a subset of $\mathbb{C}^S$. The weakest topology τ on S such that each $f \in E$ is continuous with respect to τ is the *E-topology* on S; it is denoted by τ_E.

Definition 2.1. Let K be a non-empty, locally compact space. A *function algebra* on K is a non-zero subalgebra A of $C^{\,b}(K)$ that separates strongly the points of K, in the sense that, for each $x, y \in K$ with $x \neq y$, there exists $f \in A$ with $f(x) = 0$ and $f(y) = 1$, and is such that the given topology on K is τ_A.

In the case where A is a subalgebra of $C_0(K)$ that separates strongly the points of K, the topology τ_A is necessarily equal to the given topology on K [12, Proposition 4.1.2].

Definition 2.2. Let K be a non-empty, locally compact space. A *Banach function algebra* on K is a function algebra A on K that is also a Banach algebra with respect to a norm $\|\cdot\|$. In the case where $1_K \in A$ and $\|1_K\| = 1$, the algebra A is a *unital Banach function algebra on K.*

Let $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ be Banach function algebras on the same locally compact space K. Then we write write $A = B$ to show that A and B consist of the same functions on K; in this case, $\|\cdot\|_A \sim \|\cdot\|_B$. We write $((A, \|\cdot\|_A) = (B, \|\cdot\|_B)$ to show that, further, the norms $\|\cdot\|_A$ and $\|\cdot\|_B$ are equal on the algebra, so that $A \cong B$ as Banach spaces.

Each Banach function algebra A on a locally compact space is a commutative, semisimple Banach algebra and $\|f\| \geq |f|_K \quad (f \in A)$. Further, A is algebraically isomorphic by the Gel'fand transform to a Banach function algebra on Φ_A that is contained in $C_0(\Phi_A)$, and we shall usually identify A with its image by this transform. The space Φ_A is compact if and only if A has an identity. See [12, 18], etc.

Let A be a Banach function algebra. A subset Ω of Φ_A is *determining for A* if $f = 0$ whenever $f \in A$ and $f \mid \Omega = 0$.

Definition 2.3. A Banach function algebra $(A, \|\cdot\|)$ on a non-empty, locally compact space K is a *uniform algebra* if $\|f\| = |f|_K \quad (f \in A)$, and A is *equivalent to a uniform algebra* on K if there is a constant $C > 0$ such that $\|f\| \leq C\,|f|_K \quad (f \in A)$, so that $\|\cdot\| \sim |\cdot|_K$.

Thus $C^{\,b}(K)$ is a unital uniform algebra on K, and $C_0(K)$ is a uniform algebra on K. The Gel'fand transform of a uniform algebra A is a uniform algebra on Φ_A.

The classic texts on uniform algebras include those of Browder [7], Gamelin [31], and Stout [51]. Note that, in these texts, a uniform algebra is, by definition, a closed subalgebra A of $C(K)$ for a (non-empty) compact space K such that A contains the constant functions and separates the points of K, so our definition is somewhat more general.

Let A be a Banach function algebra on a non-empty, locally compact space K. The *evaluation character* at $x \in K$ is the map

$$\varepsilon_x : f \mapsto f(x)\,, \quad A \to \mathbb{C}\,. \tag{2.4}$$

The map $x \mapsto \varepsilon_x$, $K \to \Phi_A$, is a homeomorphic embedding, and we regard K as a subspace of Φ_A. The algebra A is *natural* if $K = \Phi_A$. For $x \in K$, we write M_x for the corresponding maximal modular ideal of A that is the kernel of the character ε_x, and, for convenience, we also set $M_\infty = A$. In the case where $|K| \geq 2$, the ideal M_x is a Banach function algebra on $K \setminus \{x\}$ for each $x \in K$.

Let M be a uniform algebra, and suppose that M does not have an identity, so that Φ_M is not compact. Set $K = \Phi_M \cup \{\infty\}$, the one-point compactification of Φ_M, and regard M as a subalgebra of $C(K)$ by setting $f(\infty) = 0$ $(f \in M)$. We identify $A = M^\sharp$ with $\{z1_K + f : z \in \mathbb{C},\ f \in A\}$, and define

$$\|z1_K + f\| = |z1_K + f|_K \quad (z \in \mathbb{C},\ f \in M)\,,$$

so that A is a natural, unital uniform algebra on K and A contains M as a maximal ideal.

The Banach space of all complex-valued, regular Borel measures on a locally compact space K with the total variation norm is $M(K)$, identified with the dual space, $C_0(K)'$, of $C_0(K)$ by the Riesz representation theorem. A particular closed subspace of $M(K)$ is $\ell^1(K)$, identified with

$$\left\{\mu = \sum_{x\in K} \alpha_x \delta_x \in M(K) : \|\mu\| = \sum_{x\in K} |\alpha_x| < \infty\right\}\,,$$

where δ_x is the point mass at x and $\alpha_x \in \mathbb{C}$ for $x \in K$.

Let K be a non-empty, locally compact space. Then $\overline{f}(x) = \overline{f(x)}$ $(x \in K)$ for $f \in C^b(K)$. A natural Banach function algebra $(A, \|\cdot\|)$ on K is *self-adjoint* if $\overline{f} \in A$ and $\|\overline{f}\| = \|f\|$ for each $f \in A$, so that the map $f \mapsto \overline{f}$ is an isometric involution on A. Of course, every self-adjoint Banach function algebra on K is dense in $(C_0(K), |\cdot|_K)$.

Let $(A, \|\cdot\|)$ be a Banach function algebra, with multiplier algebra $\mathcal{M}(A)$. We regard $\mathcal{M}(A)$ as a subalgebra of $C^b(\Phi_A)$ by setting

$$\mathcal{M}(A) = \{f \in C^b(\Phi_A) : fA \subset A\}\,,$$

so that $(\mathcal{M}(A), \|\cdot\|_{\mathrm{op}})$ is a unital Banach function algebra on Φ_A, and we regard A as an ideal in $\mathcal{M}(A)$ by identifying a function $f \in A$ with the multiplier $L_f : g \mapsto fg$, $A \to A$, in $\mathcal{M}(A)$. Clearly we have

$$|f|_{\Phi_A} \leq \|f\|_{\mathrm{op}} \leq \|f\| \quad (f \in A)\,. \tag{2.5}$$

Let A be a Banach function algebra on a non-empty, locally compact space K. A non-empty, closed subset S of K is a *peak set* for A if there is a function $f \in A$ with

$f(x) = 1\ (x \in S)$ and $|f(y)| < 1\ (y \in K \setminus S)$. A point $x \in K$ is a *peak point* if $\{x\}$ is a peak set and a *strong boundary point* if, for each open neighbourhood U of x, there exists $f \in A$ with $f(x) = |f|_K = 1$ and $|f|_{K \setminus U} < 1$. In the case where X is metrizable, every strong boundary point for A is a peak point.

Let K be a non-empty, compact subspace of $\mathbb{C}^n$, where $n \in \mathbb{N}$. The coordinate projections on K are denoted by $Z_1, \dots, Z_n$. Consider the algebra of restrictions to K of all polynomials; the uniform closure in $C(K)$ of this algebra is denoted by $P(K)$. The uniform closure of the restrictions to K of the rational functions that are analytic on a neighbourhood of K is denoted by $R(K)$, and the space consisting of all functions in $C(K)$ that are analytic on $\operatorname{int} K$, the interior of K, is denoted by $A(K)$. Thus $P(K)$, $R(K)$, and $A(K)$ are unital uniform algebras on K with

$$P(K) \subset R(K) \subset A(K) \subset C(K)\,.$$

In particular, we shall mention the *disc algebra*, defined to be $P(\overline{\mathbb{D}}) = A(\overline{\mathbb{D}})$; $A(\overline{\mathbb{D}})$ is a natural uniform algebra on $\overline{\mathbb{D}}$.

A Banach function algebra A on a locally compact space K is *regular on* K if, for each non-empty, closed subspace S of K and each $x \in K \setminus S$, there exists $f \in A$ with $f(x) = 1$ and $f \mid S = 0$; a Banach function algebra A is *regular* if it is regular on Φ_A. For $x \in K \cup \{\infty\}$, we set

$$J_x = J_x(A) = \{f \in A \cap C_{00}(K) : x \notin \operatorname{supp} f\}\,,$$

so that J_x is also an ideal in A, with $J_x \subset M_x$. Here, $\operatorname{supp} f$, the *support* of $f \in A$, is the closure in K of the set $\{x \in K : f(x) \neq 0\}$. The Banach function algebra A is *strongly regular at* x if J_x is dense in M_x, and A is *strongly regular* on K if this holds for each $x \in K \cup \{\infty\}$. Every strongly regular Banach function algebra is natural and regular. The algebra A is *Tauberian* if $J_\infty(A)$ is dense in A, and so a strongly regular Banach function algebra is Tauberian. For proofs of these remarks and more, related properties of Banach function algebras, see [12, §4.1].

Let A be a Banach function algebra, and suppose that I is a closed ideal in A. The *hull* of I is the closed subset

$$h(I) = \{\varphi \in \Phi_A : \varphi(f) = 0\ (f \in I)\}$$

of Φ_A. We can identify Φ_I with $\Phi_A \setminus h(I)$ and $\Phi_{A/I}$ with $h(I)$. For a closed subset S of Φ_A, set

$$I(S) = \{f \in A : f \mid S = 0\}\,,$$

so that $I(S)$ is a closed ideal in A, and also set

$$J(S) = \{f \in J_\infty : \operatorname{supp} f \cap S = \emptyset\}\,,$$

an ideal in A. When A is a regular Banach function algebra, $h(I(S)) = S$ and

$$\overline{J(S)} \subset I \subset I(S)\,, \quad \text{where} \quad S = h(I)\,, \tag{2.6}$$

for each closed ideal I in A. The set S is a *set of synthesis* if $\overline{J(S)} = I(S)$.

A *Banach sequence algebra* on a non-empty set S is a Banach function algebra A on S (taken with the discrete topology) such that $c_{00}(S) \subset A \subset \ell^\infty(S)$, where $c_{00}(S)$ denotes the algebra of functions of finite support on S and $\ell^\infty(S)$ is the algebra of all bounded functions on S. A natural Banach sequence algebra on S is contained in $c_0(S)$. For Banach sequence algebras, $J_\infty(A) = c_{00}(S)$, and so a Banach sequence algebra is Tauberian if and only if $c_{00}(S)$ is dense in A. A Tauberian Banach sequence algebra is natural, a natural Banach sequence algebra is always regular, and it is strongly regular if and only if it is Tauberian. For example, the space ℓ^p of all p–summable sequences on $\mathbb{N}$ is a Tauberian Banach sequence algebra whenever $1 \leq p < \infty$.

Let A be a Banach algebra. Then the dual and bidual spaces A' and A'' are Banach A-bimodules for operations denoted by $\,\cdot\,$. There are two products, $\Box$ and $\Diamond$, on the Banach space A'', called the *first* and *second Arens products*, that extend the module actions on A''. We recall the definition of the product $\Box$. Let $a \in A$, $\lambda \in A'$, and $\mathrm{M} \in A''$. Then $a \cdot \lambda \in A'$ and $\lambda \cdot \mathrm{M} \in A'$ are defined by

$$\langle b,\, a \cdot \lambda\rangle = \langle ba,\, \lambda\rangle \quad (b \in A)\,, \quad \langle a,\, \lambda \cdot \mathrm{M}\rangle = \langle \mathrm{M},\, a \cdot \lambda\rangle \quad (a \in A)\,,$$

and then, for $\mathrm{M}, \mathrm{N} \in A''$, we define $\mathrm{M} \,\Box\, \mathrm{N} \in A''$ by

$$\langle \mathrm{M} \,\Box\, \mathrm{N},\, \lambda\rangle = \langle \mathrm{M},\, \mathrm{N} \cdot \lambda\rangle \quad (\lambda \in A')\,.$$

The basic theorem of Arens is that $\kappa_A : A \to A''$ is an isometric algebra monomorphism of a Banach algebra A into both $(A'', \Box)$ and $(A'', \Diamond)$. The algebra A is *Arens regular* if the two products $\Box$ and $\Diamond$ agree on A''.

For detailed terminology and a full definition of the two Arens products, see [12, §2.6].

Let A be a Banach algebra. We shall usually identify A with $\kappa_A(A)$ and write just A'' for $(A'', \Box)$. For each $\mathrm{N} \in A''$, the map

$$R_{\mathrm{N}} : \mathrm{M} \mapsto \mathrm{M} \,\Box\, \mathrm{N}\,, \quad A'' \to A''\,,$$

is always weak-$*$ continuous on A'', and the map $L_a : \mathrm{M} \mapsto a \cdot \mathrm{M},\ A'' \to A''$, is weak-$*$ continuous for each $a \in A$. The algebra A is Arens regular if and only if the map

$$L_{\mathrm{N}} : \mathrm{M} \mapsto \mathrm{N} \,\Box\, \mathrm{M}\,, \quad A'' \to A''\,,$$

is also weak-$*$ continuous for each $\mathrm{N} \in A''$. (For certain 'strongly Arens irregular' Banach function algebras, including the group algebra of a locally compact group, the map $L_{\mathrm{N}} : \mathrm{M} \mapsto \mathrm{N} \,\Box\, \mathrm{M},\ A'' \to A''$, is continuous only when $\mathrm{N} \in A$; see [16, 17].)

Let A be a commutative Banach algebra. We see that

$$\mathrm{M} \cdot \varphi = \varphi \cdot \mathrm{M} = \langle \mathrm{M},\, \varphi\rangle\, \varphi \quad (\mathrm{M} \in A'',\, \varphi \in \Phi_A)\,, \tag{2.7}$$

and so

$$\langle \mathrm{M} \,\Box\, \mathrm{N},\, \varphi\rangle = \langle \mathrm{M},\, \varphi\rangle\, \langle \mathrm{N},\, \varphi\rangle \quad (\mathrm{M}, \mathrm{N} \in A'',\, \varphi \in \Phi_A)\,. \tag{2.8}$$

For each $\varphi \in \Phi_A$, define $\widetilde{\varphi}$ on A'' by setting $\widetilde{\varphi}(\mathrm{M}) = \langle \mathrm{M},\, \varphi\rangle$ $(\mathrm{M} \in A'')$. Then it follows from (2.8) that $\widetilde{\varphi} \in \Phi_{A''}$, and so we shall regard Φ_A as a subset of $\Phi_{A''}$ via the embedding $\varphi \mapsto \widetilde{\varphi}$. However the embedding of Φ_A into $\Phi_{A''}$ is not necessarily continuous; this occurs if and only if the weak and weak-$*$ topologies of A' coincide when restricted to Φ_A. There is a natural projection

$$\pi_A : \varphi \to \varphi \mid \kappa_A(A)\,, \quad \Phi_{A''} \to \Phi_A \cup \{0\}\,.$$

A commutative Banach algebra A is Arens regular if and only if $(A'', \Box)$ is commutative, i.e., if and only if $\mathfrak{Z}(A'') = A''$. For example, for each locally compact space K, the algebra $C_0(K)$ is Arens regular, and the bidual $C_0(K)''$ is identified with $C(\widetilde{K})$, where $\widetilde{K}$ is the compact space which is the *hyper-Stonean envelope* of the locally compact space K; see [13] for a discussion and several 'constructions' of $\widetilde{K}$. Since closed subalgebras (and quotients) of Arens regular Banach algebras are Arens regular, it follows that every uniform algebra A is Arens regular, and A'' is a closed subalgebra of $C(\widetilde{K})$ (although it may not separate the points of $\widetilde{K}$). Thus A'' is a uniform algebra on $\Phi_{A''}$.

A Banach function algebra A is *an ideal in its bidual* if $\kappa_A(A)$ is an ideal in $(A'', \Box)$. This is the case if and only if the map

$$L_f : g \mapsto fg, \quad A \to A,$$

is weakly compact for each $f \in A$. Let A be a Tauberian Banach sequence algebra. Then it is easy to see that A is an ideal in its bidual [18, Proposition 2.8].

For the next definition, we follow the new book of V. Runde [49, Chapter 5], where references to earlier work are given.

Let A be a Banach algebra, with dual module A', and take a closed subspace F of A'. Then there is a canonical operator $\theta : a \mapsto \kappa_A(a) \mid F,\ A \to F'$, so that

$$\theta(a)(\lambda) = \langle a, \lambda \rangle \quad (a \in A, \lambda \in F).$$

Clearly θ is a contraction, and θ is a module homomorphism when F is a submodule of A'. A *predual* for A is a closed submodule F of A' such that the above map θ is an isomorphism. The algebra A is a *dual Banach algebra* if it has a predual; A is an *isometric dual Banach algebra* if the map θ is an isometry. A predual for A is *unique* if it is the only closed submodule of A' with respect to which A is a dual Banach algebra.

We note that a predual of a Banach algebra is not necessarily unique. For example, let E be any non-zero Banach space, so that E is a Banach algebra for the zero product. Then any Banach space F such that $F' = E$, regarded as a closed subspace of $F'' = E'$ is a predual for E, so that E is a dual Banach algebra. Certainly the Banach space ℓ^1 has many Banach-space preduals no two of which are mutually isomorphic as Banach spaces. Thus, strictly, we should refer to a pair (A, F) when discussing dual Banach algebras. However, when the predual is clear from the context, as will almost always be the case in this memoir, we shall not indicate F in the notation, and just say that 'A is a dual Banach algebra'. A *dual Banach function algebra* is a Banach function algebra that is a dual Banach algebra.

For example, let G be a locally compact group. Then it is standard that the measure algebra $(M(G), \star, \|\cdot\|)$ on G is a dual Banach algebra with predual $C_0(G)$; for details, see [49, Example 5.1.3]. The uniqueness of preduals for various examples, including $M(G)$ and some semigroup algebras, is explored in the papers [20, 21]. A C^*-algebra is a dual Banach algebra if and only if it is a von Neumann algebra.

Let A be a Banach algebra that is a dual Banach algebra with predual F. Then it is clear that the product in A is separately $\sigma(A, F)$-continuous. On the other hand, as in [49], a Banach space F that is an isomorphic predual of A and such that the product in A is separately $\sigma(A, F)$-continuous is a predual of A in a natural way.

Let (A, F) be a dual Banach algebra. Then it follows that

$$A'' = \kappa_A(A) \oplus F^{\perp} = A \oplus F^{\perp} \tag{2.9}$$

as a Banach space, where the projection $P : A'' \to A$ is the restriction map given by $P(\mathrm{M}) = \mathrm{M} \mid F$ $(\mathrm{M} \in A'')$, and hence $F^{\perp}$ is a weak-$*$-closed ideal in A'', and

$$A'' = A \ltimes F^{\perp} \tag{2.10}$$

as an algebra.

Let A be a Banach algebra. Then it is immediate from the definition that A'' is a dual Banach algebra, with Banach-algebra predual A', if and only if A is Arens regular. In particular, in the case where A is a uniform algebra, the bidual A'' is a uniform algebra on $\Phi_{A''}$ that is a dual Banach function algebra.

We shall several times use the following famous *Markov–Kakutani fixed-point theorem*; see [23, V.10.6] or [45, Proposition (0.14)] for classical proofs.

THEOREM 2.4. *Let L be a non-empty, compact, convex set in a locally convex space. Suppose that $\mathcal{F}$ is a commuting family of continuous, affine maps from L to L. Then the operators in $\mathcal{F}$ have a common fixed point in L.* ∎

We shall also use the *Schauder–Tychonoff fixed-point theorem* [23, V.10.5].

THEOREM 2.5. *Let K be a non-empty, compact, convex set in a locally convex space. Then every continuous function from K to K has a fixed point.* ∎

3. Approximate identities

We shall recall the definitions of various types of approximate identities that we shall consider in a commutative Banach algebra, and give some preliminary results.

DEFINITION 3.1. Let A be a commutative Banach algebra. A net (e_α) in A is an *approximate identity* if $\lim_\alpha e_\alpha a = a \ \ (a \in A)$; it is a *bounded approximate identity* (BAI) if, further, $\sup_\alpha \|e_\alpha\| < \infty$, and in this case the *bound* is $\sup_\alpha \|e_\alpha\|$. A BAI is a *contractive approximate identity* (CAI) if the bound is 1.

The following result is immediate.

PROPOSITION 3.2. *Let A be a Banach function algebra with a bounded approximate identity of bound m. Then*

$$\|f\|_{\rm op} \leq \|f\| \leq m\,\|f\|_{\rm op} \quad (f \in A)\,,$$

and so $\|\cdot\|$ and $\|\cdot\|_{\rm op}$ are equivalent on A. ∎

DEFINITION 3.3. Let $(A, \|\cdot\|_A)$ be a Banach function algebra. A Banach function algebra $(S, \|\cdot\|_S)$ on Φ_A is a *Segal algebra* (with respect to A) if S is an ideal in A and if there is a net in S that is an approximate identity for both $(A, \|\cdot\|_A)$ and $(S, \|\cdot\|_S)$.

Thus, in this situation, S is dense in A; the Banach function algebra S is natural on Φ_A, and we may, and shall, suppose that

$$\|f\|_A \leq \|f\|_S \quad (f \in S) \quad \text{and} \quad \|fg\|_S \leq \|f\|_A\,\|g\|_S \quad (f \in A,\, g \in S)\,.$$

This shows that a Segal algebra is a Banach A-module in the sense of [12, p. 239].

Examples of Segal algebras will be given later. For more information on Segal algebras, see [12, pp. 409–410, 491–492].

The following two definitions were essentially given as [18, Definitions 2.3 and 2.11].

DEFINITION 3.4. Let A be a Banach function algebra. Then A is *contractive* if A and each of its non-zero, maximal modular ideals have a contractive approximate identity.

For example, for any non-empty, locally compact space K, the Banach function algebra $C_0(K)$ is contractive. Note that $C_0(K) = \mathbb{C}$ when $|K| = 1$.

DEFINITION 3.5. Let A be a Banach function algebra. A net (e_α) in A is a *pointwise approximate identity* (PAI) if

$$\lim_\alpha e_\alpha(\varphi) = 1 \quad (\varphi \in \Phi_A)\,;$$

the pointwise approximate identity (e_α) is *bounded*, with *bound* $m > 0$, if

$$\sup_\alpha \|e_\alpha\| = m\,,$$

and then (e_α) is a *bounded pointwise approximate identity* (BPAI); a bounded pointwise approximate identity of bound 1 is a *contractive pointwise approximate identity* (CPAI). The algebra A is *pointwise contractive* if A and each of its non-zero, maximal modular ideals have a contractive pointwise approximate identity.

We note that, despite the use of the term ‘approximate identity’ in the above definition, a bounded pointwise approximate identity need not be an approximate identity in the sense of Definition 3.1.

Thus a Banach function algebra $(A, \|\cdot\|)$ with $|\Phi_A| \geq 2$ is pointwise contractive if and only if, for each $\varphi \in \Phi_A \cup \{\infty\}$, each non-empty, finite subset F of Φ_A with $\varphi \notin F$, and each $\varepsilon > 0$, there exists $f \in M_\varphi$ with $\|f\| < 1+\varepsilon$ and $|1 - f(\psi)| < \varepsilon \ (\psi \in F)$, and this holds if and only if, for each $\varphi \in \Phi_A \cup \{\infty\}$, there exists $\mathrm{M} \in M_\varphi''$ such that $\|\mathrm{M}\| = 1$ and $\langle \mathrm{M}, \psi\rangle = 1 \ (\psi \in \Phi_A \setminus \{\varphi\})$.

Examples of Banach function algebras with contractive pointwise approximate identities, but no approximate identities, were first given by Jones and Lahr in [37]; further examples are given in [18] and [35]. We shall note in Example 7.7 that there are natural, unital uniform algebras A on a compact space with maximal ideals M such that M has a contractive pointwise approximate identity, but such that M has no approximate identity.

The following is clear.

Proposition 3.6. *Let A be a Banach function algebra, and take $\varphi \in \Phi_A$ such that M_φ is non-zero. Suppose that A and M_φ have bounded pointwise approximate identities. Then there exists an element $\mathrm{M} \in A''$ such that $\langle \mathrm{M}, \varphi\rangle = 1$ and $\langle \mathrm{M}, \psi\rangle = 0 \ (\psi \in \Phi_A \setminus \{\varphi\})$.* ∎

Proposition 3.7. *Let A be a Banach function algebra that is an ideal in its bidual, and suppose that A has a bounded pointwise approximate identity. Then A has a bounded approximate identity with the same bound.*

Proof. This is [18, Proposition 3.1]. ∎

Proposition 3.8. *Let A be a pointwise contractive Banach function algebra.*

(i) *Take F and G to be disjoint, non-empty, finite subsets of Φ_A, and take $\varepsilon > 0$. Then there exists $f \in A_{[1]}$ such that $|1 - f(\varphi)| < \varepsilon \ (\varphi \in F)$ and such that $f(\psi) = 0 \ (\psi \in G)$.*

(ii) *Let $\varphi_1, \dots, \varphi_n$ be distinct points in Φ_A, take $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{D}}$, and take $\varepsilon > 0$. Then there exists $f \in A_{[4]}$ such that $|f(\varphi_i) - \alpha_i| < \varepsilon \ (i \in \mathbb{N}_n)$.*

Proof. (i) Set $k = |G|$, and choose $\eta \in (0, \varepsilon/k)$. For each $\psi \in G$, there exists $f_\psi \in A_{[1]}$ with $f_\psi(\psi) = 0$ and $|1 - f_\psi(\varphi)| < \eta \ (\varphi \in F)$. Now define

$$f = \prod\{f_\psi : \psi \in G\}.$$

Then clearly $f \in A_{[1]}$ and $f(\psi) = 0 \ (\psi \in G)$. For each $\varphi \in F$, we have

$$|1 - f(\varphi)| \leq \sum\{|1 - f_\psi(\varphi)| : \psi \in G\} < k\eta < \varepsilon,$$

as required.

(ii) First suppose that $\alpha_1, \dots, \alpha_n \in [0,1]$, say $0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq 1$, and set $\alpha_0 = 0$. By (i), for each $j \in \mathbb{N}_n$, there exists $f_j \in A$ with $\|f_j\| \leq \alpha_j - \alpha_{j-1}$, with $f_j(\varphi_i) = 0 \ (i = 1, \dots, j-1)$, and with

$$|f_j(\varphi_i) - (\alpha_j - \alpha_{j-1})| < \frac{\varepsilon}{4n} \quad (i = j, \dots, n).$$

Define $f = f_1 + \cdots + f_n$. Then $f \in A_{[1]}$ and $|f(\varphi_i) - \alpha_i| < \varepsilon/4 \ (i \in \mathbb{N}_n)$.

Now consider the general case, where $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{D}}$. For $j \in \mathbb{N}_n$, there exist $\alpha_{1,j}, \alpha_{2,j}, \alpha_{3,j}, \alpha_{4,j} \in [0,1]$ such that

$$\alpha_j = \alpha_{1,j} - \alpha_{2,j} + \mathrm{i}(\alpha_{3,j} - \alpha_{4,j}) .$$

For each $i = 1,2,3,4$, choose $f_i \in A_{[1]}$ with $|f_i(x_j) - \alpha_{i,j}| < \varepsilon/4 \ (j \in \mathbb{N}_n)$, and then set $f = f_1 - f_2 + \mathrm{i}(f_3 - f_4)$, so that $f \in A_{[4]}$. Clearly we have $|f(\varphi_j) - \alpha_j| < \varepsilon \ (j \in \mathbb{N}_n)$, as required. ∎

LEMMA 3.9. *Fix $\varepsilon > 0$ and $\zeta_0 \in \mathbb{T}$. Then there exist $\delta > 0$ and $h \in A(\overline{\mathbb{D}})_{[1]}$ with $|h(0) - \zeta_0| < \varepsilon$ and $|h(z) - 1| < \varepsilon$ whenever $z \in \overline{\mathbb{D}}$ with $|z-1| < \delta$.*

Proof. Take $r \in (1-\varepsilon, 1)$, and set

$$h(z) = \left(\frac{1 - r\overline{\zeta}}{1 - r\zeta}\right) \cdot \left(\frac{z - r\zeta}{1 - r\overline{\zeta}z}\right) \quad (z \in \overline{\mathbb{D}}) ,$$

where $\zeta \in \mathbb{T}$ is to be specified. Then $h \in A(\overline{\mathbb{D}})_{[1]}$. Since $h(1) = 1$, clearly there exists $\delta > 0$ such that $|h(z) - 1| < \varepsilon$ whenever $z \in \overline{\mathbb{D}}$ is such that $|z-1| < \delta$. We see that $h(0)/r = (r - \zeta)/(1 - r\zeta)$. Since the map

$$\zeta \mapsto (r - \zeta)/(1 - r\zeta) , \quad \mathbb{T} \to \mathbb{T},$$

is a continuous surjection, there is a choice of $\zeta \in \mathbb{T}$ such that $h(0) = r\zeta_0$, and this implies that $|h(0) - \zeta_0| = 1 - r < \varepsilon$, as required. ∎

We shall now show that, in the case where A is a uniform algebra, we can reduce the bound '4' that occurs in Proposition 3.8(ii) to '1'.

PROPOSITION 3.10. *Let A be a pointwise contractive, unital uniform algebra. Take $\varepsilon > 0$, $n \in \mathbb{N}$, $\varphi_1, \dots, \varphi_n$ to be distinct points in Φ_A, and $\zeta_1, \dots, \zeta_n \in \mathbb{T}$. Then there exists $f \in A_{[1]}$ with*

$$|f(\varphi_j) - \zeta_j| < \varepsilon \quad (j \in \mathbb{N}_n) .$$

Proof. We have shown in Lemma 3.9 that there are $\delta_1 > 0$ and $h_1 \in A(\overline{\mathbb{D}})_{[1]}$ with $|h_1(0) - \zeta_1| < \varepsilon/n$ and $|h_1(z) - 1| < \varepsilon/n$ whenever $z \in \overline{\mathbb{D}}$ with $|z-1| < \delta_1$. By Proposition 3.8(i), there is $g_1 \in A_{[1]}$ with $g_1(\varphi_1) = 0$ and $|g(\varphi_j) - 1| < \delta_1 \ (j = 2, \dots, n)$. Set $f_1 = h_1 \circ g_1$, so that $f_1 \in A_{[1]}$. Then $|f_1(\varphi_1) - \zeta_1| < \varepsilon/n$ and $|f_1(\varphi_j) - 1| < \varepsilon/n$ for $j = 2, \dots, n$.

Similarly, there exist functions $f_1, \dots, f_n \in A_{[1]}$ such that $|f_j(\varphi_j) - \zeta_j| < \varepsilon/n$ and $|f_i(\varphi_j) - 1| < \varepsilon/n$ for $i, j \in \mathbb{N}_n$ with $i \neq j$.

Set $f = f_1 \cdots f_n$, so that $f \in A_{[1]}$ and

$$|f(\varphi_j) - \zeta_j| \leq |f_j(\varphi_j) - \zeta_j| + \sum_{i=1, i\neq j}^{n} |f_i(\varphi_j) - 1| < n \cdot (\varepsilon/n) = \varepsilon$$

for each $j \in \mathbb{N}_n$, as required. ∎

4. The separating ball property

In this section, we shall introduce the separating ball property for Banach function algebras. The notion originates in the paper [57]. We shall also introduce a related notion by defining when a Banach function algebra 'has norm-one characters'.

DEFINITION 4.1. Let A be a Banach function algebra, and take $\varphi \in \Phi_A$. Then A has the *separating ball property at* φ if, given $\psi \in \Phi_A \cup \{\infty\}$ with $\psi \neq \varphi$, there is $f \in (M_\psi)_{[1]}$ with $f(\varphi) = 1$. The algebra A has the *separating ball property* if it has the separating ball property at each $\varphi \in \Phi_A$.

Many Banach function algebras that arise in the theory of harmonic analysis have the separating ball property. For example, the Fourier algebra $A(\Gamma)$ has the separating ball property for each locally compact group Γ [57, Proposition 2.5] (see also Example 11.3(ii)), but it has a bounded pointwise approximate identity if and only if Γ is amenable [8]. The Banach sequence algebras ℓ^p for $1 \leq p < \infty$ also have the separating ball property, but no bounded pointwise approximate identity.

Note that, for each non-empty, locally compact space K, the algebra $C_0(K)$ has the separating ball property; in particular, the one-dimensional algebra $(\mathbb{C}, |\cdot|)$ has the separating ball property.

PROPOSITION 4.2. *Let A be a Banach function algebra, and take $\varphi \in \Phi_A$. Suppose that A has the separating ball property at φ. Then φ is a strong boundary point for A.*

Proof. There exists $f_0 \in A_{[1]}$ with $f_0(\varphi) = 1$. Set $L = \{\psi \in \Phi_A : |f_0(\psi)| \geq 1/2\}$, so that L is a compact subset of Φ_A.

Take an open neighbourhood U of φ in Φ_A. For each $\psi \in L \setminus U$, there exists $f_\psi \in (M_\psi)_{[1]}$ with $f_\psi(\varphi) = 1$, and then there is a neighbourhood U_ψ of ψ such that $|f_\psi(x)| < 1/2 \ (x \in U_\psi)$. Since $L \setminus U$ is compact, there are $n \in \mathbb{N}$ and $\psi_1, \dots, \psi_n \in L \setminus U$ such that

$$\bigcup\{U_{\psi_i} : i \in \mathbb{N}_n\} \supset \Phi_A \setminus U\,.$$

Set $f = f_0 f_{\psi_1} \cdots f_{\psi_n}$, so that $f \in A_{[1]}$, $f(\varphi) = 1$, and $|f(\psi)| < 1/2 \ (\psi \in \Phi_A \setminus U)$. This shows that φ is a strong boundary point for A. ■

We shall see in Theorem 4.12 that the converse of the above proposition holds for certain uniform algebras, but it does not hold for arbitrary Banach function algebras.

The first main result of this section is an extension of [57, Lemma 5.1].

THEOREM 4.3. *Let A be a Banach function algebra.*

(i) *Take $\varphi \in \Phi_A$, and suppose that A has the separating ball property at φ. Then there is an idempotent $\mathrm{E}_\varphi \in A''_{[1]}$ with $\langle \mathrm{E}_\varphi, \varphi\rangle = 1$ and $\langle \mathrm{E}_\varphi, \psi\rangle = 0 \ (\psi \in \Phi_A \setminus \{\varphi\})$.*

(ii) *Suppose that A has the separating ball property. Then the space Φ_A is discrete with respect to the relative weak topology, $\sigma(A', A'')$.*

Proof. (i) Define the set

$$S = \{f \in A_{[1]} : f(\varphi) = 1\}\,.$$

Since A has the separating ball property at φ, the set S is not empty, and it is clearly a convex subset of the space A. Consider S as a subset of $A''_{[1]}$, and take L to be its weak-$*$ closure in A''. Then L is a non-empty, compact, convex set in the locally convex space $(A'', \sigma(A'', A'))$. For each $f \in S$, the map

$$T_f : \mathrm{M} \mapsto f \cdot \mathrm{M}, \quad A'' \to A'',$$

is linear, and so $T_f \mid L$ is affine, and T_f is continuous with respect to the weak-$*$ topology $\sigma(A'', A')$. Further, $T_f(S) \subset S$, and hence $T_f(L) \subset L$, for each $f \in S$, and the operators T_f commute. By the Markov–Kakutani theorem, Theorem 2.4, there exists $\mathrm{E}_\varphi \in L \subset A''_{[1]}$ such that $f \cdot \mathrm{E}_\varphi = \mathrm{E}_\varphi \ (f \in S)$. By taking a net in S that converges to E_φ weak-$*$, we see that E_φ is an idempotent in A''. Clearly $\langle \mathrm{E}_\varphi, \varphi \rangle = 1$. For each $\psi \in \Phi_A \setminus \{\varphi\}$, there exists $f \in S$ with $f(\psi) = 0$, and so $\langle \mathrm{E}_\varphi, \psi \rangle = f(\psi)\langle \mathrm{E}_\varphi, \psi \rangle = 0$.

From this, it follows that φ is isolated in Φ_A with respect to the weak topology.

(ii) It follows immediately that Φ_A is discrete with respect to the weak topology when A has the separating ball property. ■

As a first application of the above theorem, we present the following result.

COROLLARY 4.4. *Let A be a Banach function algebra with the separating ball property, and suppose that A has a bounded pointwise approximate identity with bound m. Then:*

(i) *each non-zero, maximal modular ideal of A has a bounded pointwise approximate identity with bound $m + 1$;*

(ii) *the algebra A is reflexive if and only if it is a finite-dimensional space.*

Proof. (i) Take $\varphi \in \Phi_A$, and consider the maximal modular ideal M_φ, assumed to be non-zero. Let E_φ be the idempotent in $A''_{[1]}$ specified in Theorem 4.3(i), and let E be a weak-$*$ accumulation point in $A''_{[m]}$ of the BPAI in A. Set $\mathrm{F}_\varphi = \mathrm{E} - \mathrm{E}_\varphi$, so that $\mathrm{F}_\varphi \in A''_{[m+1]}$. Further, $\langle \mathrm{F}_\varphi, \varphi \rangle = 0$ and $\langle \mathrm{F}_\varphi, \psi \rangle = 1 \ (\psi \in \Phi_A \setminus \{\varphi\})$. A net in $(M_\varphi)_{[m+1]}$ that converges weak-$*$ to F_φ is the required BPAI in M_φ.

(ii) Suppose that A is reflexive, and hence an ideal in A''. By Proposition 3.7 and clause (i), the algebra A and every non-zero, maximal modular ideal of A has a BAI; since A is reflexive, each of these ideals has an identity, and so Φ_A is compact and each point of Φ_A is isolated. Hence Φ_A is finite, and so A is a finite-dimensional space. ■

A bound for a bounded pointwise approximate identity in M_x that arises in clause (i) of the above corollary is $\|\mathrm{E} - \mathrm{E}_\varphi\|$, which is at most $m + 1$. In general, the bound $m + 1$ cannot be improved. For example, let Γ be an infinite, locally compact abelian group. Then the Fourier algebra $A(\Gamma)$ has a contractive approximate identity, and so, by the corollary, every non-zero, maximal modular ideal in $A(\Gamma)$ has a bounded pointwise approximate identity of bound 2. It is shown in [18, Example 3.15] that 2 is the minimum such bound.

The maximal ideal $\{f \in A(\overline{\mathbb{D}}) : f(0) = 0\}$ in the disc algebra does not have a bounded pointwise approximate identity, and so $A(\overline{\mathbb{D}})$ is not pointwise contractive; this shows that we cannot remove the hypothesis that A have the separating ball property in Corollary 4.4 when obtaining clause (i).

We also introduce the following definition.

DEFINITION 4.5. Let A be a Banach function algebra, and take $\varphi \in \Phi_A$. Then A has the *weak separating ball property at* φ if, given $\psi \in \Phi_A \cup \{\infty\}$ with $\psi \neq \varphi$, there is net (f_ν) in $(M_\psi)_{[1]}$ such that $\lim_\nu f_\nu(\varphi) = 1$. The algebra A has the *weak separating ball property* if it has the weak separating ball property at each $\varphi \in \Phi_A$.

Each pointwise contractive Banach function algebra and each Banach function algebra with the separating ball property has the weak separating ball property.

Let A be a Banach function algebra with $|\Phi_A| \geq 2$, and take $\varphi \in \Phi_A$. Suppose that, for each $\psi \in \Phi_A \setminus \{\varphi\}$, there is a net (g_ν) in $A_{[1]}$ such that $\lim_\nu g_\nu(\varphi) = 1$ and $\lim_\nu g_\nu(\psi) = 0$. Set

$$f_\nu = (g_\nu^2 - g_\nu(\psi)g_\nu)/\left\|g_\nu^2 - g_\nu(\psi)g_\nu\right\|$$

for each ν. Then the net (f_ν) is contained in $(M_\psi)_{[1]}$ and $\lim_\nu f_\nu(\varphi) = 1$. It follows that A has the weak separating ball property.

DEFINITION 4.6. Let A be a Banach function algebra. Then A has *norm-one characters* if $\|\varphi\| = 1 \ (\varphi \in \Phi_A)$.

Certainly each unital Banach function algebra has norm-one characters; also, A has this property whenever A has the weak separating ball property. On the other hand, consider the maximal ideal $M = \{f \in A(\overline{\mathbb{D}}) : f(0) = 0\}$ in the disc algebra, so that $\Phi_M = \overline{\mathbb{D}} \setminus \{0\}$. Then, for each $z \in \mathbb{C}$ with $0 < |z| \leq 1$, we have $\|\varepsilon_z\| = |z|$, and so M does not have norm-one characters.

The following remark is obvious.

PROPOSITION 4.7. *Let A be a Banach function algebra with $|\Phi_A| \geq 2$. Then A has the weak separating ball property if and only if M_φ has norm-one characters for each $\varphi \in \Phi_A$.* ∎

PROPOSITION 4.8. *Let A be a Banach function algebra that is reflexive as a Banach space and is such that Φ_A is connected. Then the following are equivalent;*

(a) *A is unital;*

(b) *A has norm-one characters;*

(c) *there exists $\varphi \in \Phi_A$ such that $\|\varphi\| = 1$.*

Proof. (a) ⇒ (b) ⇒ (c) These implications are immediate.

(c) ⇒ (a) Take $\varphi \in \Phi_A$ with $\|\varphi\| = 1$. Since A is reflexive, it follows from the Hahn–Banach theorem that there exists $u \in A$ with $\|u\| = 1$ and $u(\varphi) = 1$. Define

$$K = \overline{\mathrm{co}}\,\{u^n : n \in \mathbb{N}\},$$

so that K is a convex and weakly compact set in A with $K \subset A_{[1]}$, and $v(\varphi) = 1 \ (v \in K)$. The map $L_u \mid K : K \to K$ is weakly continuous on K, and so, by the Schauder–Tychonoff fixed point theorem, Theorem 2.5, this map has a fixed point: there exists $v \in K$ such that $uv = v$. Clearly $wv = v \ (w \in K)$, and, in particular, $v^2 = v$, so that v is an idempotent in A. Since $v(\varphi) = 1$ and Φ_A is connected, the element v is the identity of A. Also $\|v\| = 1$, and so A is unital. ∎

In Example 3.3 of [18], there are examples of Banach function algebras that are reflexive as Banach spaces and are defined on connected, compact, infinite spaces. These algebras are unital.

THEOREM 4.9. *Let A be a dual Banach function algebra, with predual $F \subset A'$. Suppose that $\Phi_A \cap F \neq \emptyset$ and that $\varphi \in \Phi_A \cap F$ is such that $\|\varphi\| = 1$. Then there is an idempotent $e \in A_{[1]}$ such that $e(\varphi) = 1$ and $fe = e$ for each $f \in A_{[1]}$ with $f(\varphi) = 1$.*

Proof. Again consider the set

$$S_\varphi = \{f \in A_{[1]} : f(\varphi) = 1\}\,.$$

Since $\|\varphi\| = 1$, there is a net (f_ν) in $A_{[1]}$ with $\lim_\nu f_\nu(\varphi) = 1$. Let $f \in A_{[1]}$ be an accumulation point of (f_ν) with respect to the topology $\sigma(A, F)$. Then $f \in S_\varphi$ because $\varphi \in F$, and so S_φ is not empty. Clearly S_φ is convex, and it is compact with respect to the topology $\sigma(A, F)$. For $f \in S_\varphi$, the maps

$$L_f : g \mapsto fg\,, \quad S_\varphi \to S_\varphi\,,$$

form a commuting family of affine maps that are continuous with respect to $\sigma(A, F)$, and so, again by Theorem 2.4, the family has a fixed point, say $e \in S_\varphi \subset A_{[1]}$. Thus $fe = e$ for each $f \in S_\varphi$, and, in particular, $e^2 = e$. ∎

The following example shows that the condition that $\varphi \in F$ in the above theorem cannot be removed when seeking an idempotent.

EXAMPLE 4.10. Let $A = (\ell^1(\mathbb{Z}^+), \star)$ be the standard semigroup algebra on $\mathbb{Z}^+$, where $\star$ denotes the convolution product, so that A is isomorphic to the natural, unital Banach function algebra on $\overline{\mathbb{D}}$ consisting of the continuous functions on $\mathbb{T}$ with absolutely convergent Taylor series. The maximal ideal M of functions in A that vanish at 0 corresponds to the algebra $\ell^1 = \ell^1(\mathbb{N})$; $c_0 = c_0(\mathbb{N})$ is a closed submodule of $M' = \ell^\infty(\mathbb{N})$, and clearly M is a dual Banach function algebra with predual c_0. Let φ be a character on M that corresponds to evaluation at a point of $\mathbb{T}$. Then $\|\varphi\| = 1$, but $\varphi \notin c_0$. There is no non-zero idempotent in A. ∎

The following example shows that the condition that $\varphi \in F$ in the above theorem cannot be removed when obtaining the conclusion that $fe = e$ for each $f \in S_\varphi$.

EXAMPLE 4.11. Let $A = \ell^\infty = C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone–Čech compactification of $\mathbb{N}$, so that A is a dual Banach function algebra, with predual ℓ^1. Of course, $\Phi_A = \beta\mathbb{N}$. Take $x \in \beta\mathbb{N}$. In the case where $x \in \mathbb{N}$, so that $\varepsilon_x \in \ell^1$, the corresponding idempotent of the above theorem is the characteristic function of x. However, when $x \in \beta\mathbb{N} \setminus \mathbb{N}$, so that $\varepsilon_x \notin \ell^1$, there is no idempotent $e \in A_{[1]}$ such that $fe = e$ for each $f \in A_{[1]}$ with $f(x) = 1$. ∎

4.1. Uniform algebras. We now recall some background concerning uniform algebras.

Let A be a uniform algebra on a non-empty, locally compact space K. The set of strong boundary points for A is now called the *Choquet boundary* of A, and is denoted by $\Gamma_0(A)$. A closed subset S of K is a *closed boundary* for A if $|f|_S = |f|_K \quad (f \in A)$; the intersection of all the closed boundaries for A is a closed boundary, called the *Šilov boundary*, $\Gamma(A)$. Suppose that K is compact. Then, by [12, Corollary 4.3.7(i)], $\Gamma(A) = \overline{\Gamma_0(A)}$ and $\Gamma(A)$ is a closed boundary. For example, let $A = A(\overline{\mathbb{D}})$ be the disc algebra. Then $\Gamma_0(A) = \Gamma(A) = \mathbb{T}$.

The next theorem relates approximate identities in a maximal ideal of a unital uniform algebra to strong boundary points. We recall that M_x'' is commutative because uniform algebras are Arens regular, and that M_x'' is itself a natural uniform algebra on $\Phi_{A''} \setminus \{x\}$. The result is an extension of [18, Theorem 4.7].

Theorem 4.12. *Let A be a unital uniform algebra on a non-empty, compact space K such that $|K| \geq 2$, and take $x \in K$. Then the following conditions on x are equivalent:*

(a) *$x \in \Gamma_0(A)$;*

(b) *M_x has a bounded approximate identity;*

(c) *M_x'' has an identity;*

(d) *M_x has a contractive approximate identity;*

(e) *x is an isolated point of $\Phi_{A''}$;*

(f) *A has the separating ball property at x.*

Proof. The equivalence of (a) and (b) is [12, Theorem 4.3.5, (d) $\Leftrightarrow$ (e)]. Clauses (b) and (c) are clearly equivalent; trivially, (d) implies (b), and (f) implies (a) by Proposition 4.2.

Now suppose that (c) holds; the identity of M_x'' is E. Since E is a non-zero idempotent in the uniform algebra $C(\widetilde{K})$, necessarily $|E|_{\widetilde{K}} = 1$, and so a net in $(M_x)_{[1]}$ that converges weak-$*$ to E is a contractive approximate identity for M_x, giving (d).

Clearly (c) $\Rightarrow$ (e). On the other hand, suppose that (e) holds. Then, by Šilov's idempotent theorem, there exists $E \in A''$ that is the characteristic function of $\Phi_{A''} \setminus \{x\}$, and then E is the identity of M_x'', giving (c).

Suppose that (a) holds, so that x is a strong boundary point for A, and take $y \in K$ with $y \neq x$. Then there exists $f \in A_{[1]}$ with $f(x) = 1$ and $|f(y)| < 1$, say $f(y) = \alpha$. Define

$$B(\zeta) = \frac{(1-\overline{\alpha})(\zeta - \alpha)}{(1-\alpha)(1-\overline{\alpha}\zeta)} \quad (\zeta \in \overline{\mathbb{D}}).$$

Then $B \in A(\overline{\mathbb{D}})_{[1]}$, and so $g := B \circ f \in A_{[1]}$. Clearly $g(x) = 1$ and $g(y) = 0$, and hence A has the separating ball property at x, giving (f). ∎

Definition 4.13. A natural uniform algebra A on a non-empty, compact space K is a *Cole algebra* if $\Gamma_0(A) = K$.

It was a long-standing conjecture, called the *peak-point conjecture* that $C(K)$ is the only Cole algebra on a compact space K. The first counter-example was due to Cole [10], and is described in [51, §19]; an example of Basener [3], also described in [51, §19], gives a compact space K in $\mathbb{C}^2$ such that $R(K)$ is a non-trivial Cole algebra. Further, Feinstein [25] obtained examples of non-trivial, regular Cole algebras on compact, metrizable spaces.

The next result is an immediate consequence of Theorem 4.12.

Corollary 4.14. *Let A be a natural uniform algebra on a non-empty, compact space K. Then the following conditions are equivalent:*

(a) *A is a Cole algebra;*

(b) *A has the separating ball property;*

(c) *A is contractive;*

(d) *each point of K is an isolated point of $\Phi_{A''}$.* ■

Let A be a natural uniform algebra on a non-empty, compact space K, and take $x, y \in K$. Then

$$x \sim y \quad \text{if} \quad d_A(x,y) < 2\,,$$

where d_A is the *Gleason metric* given by

$$d_A(x,y) = \|\varepsilon_x - \varepsilon_y\| \quad (x, y \in K)\,.$$

Then $\sim$ is an equivalence relation on K; the equivalence classes with respect to this relation are the *Gleason parts* for A. These parts form a partition of K, and each part is a completely regular and σ-compact topological space with respect to the Gel'fand topology; by a theorem of Garnett, these are the only topological restrictions on Gleason parts. For a discussion of Gleason parts, including Garnett's theorem, see [31, Chapter VI] and [51, §16]. Clearly, $\{x\}$ is a one-point Gleason part whenever x is a strong boundary point, but the converse fails, as we shall see below and in examples to be given in Chapter 7.

The Gleason parts of $\overline{\mathbb{D}}$ for the disc algebra $A(\overline{\mathbb{D}})$ are the one-point parts that correspond to points of the circle $\mathbb{T}$ together with the open disc $\mathbb{D}$. We shall write H^∞ for $H^\infty(\mathbb{D})$, the uniform algebra of all bounded, analytic functions on $\mathbb{D}$ (taken with respect to the uniform norm on $\mathbb{D}$); take Φ to be the character space of H^∞. Then the Gleason parts of Φ are well known; an early fine exposition is given in [34]; see also [31, 51]. In particular, each point of the Šilov boundary $\Gamma(H^\infty)$ is a strong boundary point, but there are one-point parts that are not in $\Gamma(H^\infty)$. In fact, each part of Φ is either a one-point part or an 'analytic disc'.

A related paper on Gleason parts and biduals of uniform algebras is [40].

The next proposition is standard [51, Lemma 16.1]; our proof is slightly different.

Proposition 4.15. *Let A be a natural uniform algebra on a non-empty, compact space K, and take $x, y \in K$. Then the following are equivalent:*

(a) $\|\varepsilon_x - \varepsilon_y\| = 2\,$;

(b) *for each $\varepsilon > 0$, there exists $f \in (M_y)_{[1]}$ with $|f(x)| > 1 - \varepsilon$.*

Proof. (a) $\Rightarrow$ (b) Take $\varepsilon > 0$. There exists $g \in A_{[1]}$ with $|g(x) - g(y)| > 2 - \varepsilon$. Set $f = (g - g(y)1_K)/2$. Then $f \in (M_y)_{[1]}$ and $|f(x)| > (2 - \varepsilon)/2 > 1 - \varepsilon$.

(b) $\Rightarrow$ (a) Take $\varepsilon > 0$. By Lemma 3.9, there exist $\delta > 0$ and $h \in A(\overline{\mathbb{D}})_{[1]}$ with $|h(0)+1| < \varepsilon$ and $|h(z)-1| < \varepsilon$ whenever $z \in \overline{\mathbb{D}}$ with $|z-1| < \delta$. Take $f \in (M_y)_{[1]}$ with $|f(x)| > 1-\delta$, and set $g = h \circ f \in A_{[1]}$. Then

$$\|\varepsilon_x - \varepsilon_y\| \geq |g(x) - g(y)| > 2 - 2\varepsilon\,.$$

This holds for each $\varepsilon > 0$, and so $\|\varepsilon_x - \varepsilon_y\| = 2$. ∎

COROLLARY 4.16. *Let A be a natural uniform algebra A on a non-empty, compact space K. Then a point $x \in K$ is such that $\{x\}$ is a one-point part if and only if A has the weak separating ball property at x.* ∎

The following theorem is an extension of Theorem 4.3(i) to uniform algebras; in general these algebras do not have the separating ball property.

THEOREM 4.17. *Let A be a natural, uniform algebra on a non-empty, compact space K.*

(i) *Let P be a Gleason part for A, and take $x_0 \in P$. Then there is an idempotent element $E \in A''_{[1]}$ such that $E \mid K = \chi_P$ and such that $F\,E = E$ whenever $F \in A''_{[1]}$ with $F(x_0) = 1$.*

(ii) *For each Gleason part P, there is an idempotent $E_P \in A''_{[1]}$ with $E_P \mid K = \chi_P$ and such that $E_P\,E_Q = 0$ in A'' whenever P and Q are distinct parts.*

Proof. (i) We may suppose that $P \neq K$. Set

$$S = \{F \in A''_{[1]} : F(x_0) = 1\}\,.$$

Since $1_K \in S$, the set S is not empty. Since A'' is a dual uniform algebra, with predual A', and since $\varepsilon_{x_0} \in \Phi_{A''} \cap A'$ with $\|\varepsilon_{x_0}\| = 1$, it follows from Theorem 4.9 that there exists an idempotent $E \in S$ such that $F\,E = E$ whenever $F \in S$

Suppose that $y \in K$ and $E(y) = 0$, so that $E \in (M''_y)_{[1]}$. Then there is a net (f_ν) in $(M_y)_{[1]}$ such that $\lim_\nu f_\nu(x_0) = \langle \mathrm{E}, \varepsilon_{x_0}\rangle = 1$, and so, by Proposition 4.15, $y \notin P$.

Suppose that $y \in K \setminus P$. By Proposition 4.15, there is a sequence (f_n) in $(M_y)_{[1]}$ with $f_n(x_0) \to 1$ as $n \to \infty$. Let F be an accumulation point of (f_n) in A''. Then $F \in S$ and $F(y) = 0$, and so $E(y) = FE(y) = 0$.

It follows that $\{y \in K : E(y) = 0\} = K \setminus P$, and so $E \mid K = \chi_P$, as required.

(ii) Now choose $x_P \in P$ for each part P, so that there exists an idempotent element $E_P \in A''_{[1]}$ such that $E_P \mid K = \chi_P$ and such that $F\,E_P = E_P$ whenever $F \in A''_{[1]}$ with $F(x_P) = 1$. Take distinct parts P and Q, and set $F = 1 - E_P$. Then F is an idempotent in A'', and so $F \in A''_{[1]}$. Also $F(x_Q) = 1$, and so $(1 - E_P)\,E_Q = FE_Q = E_Q$, and hence $E_P\,E_Q = 0$. ∎

COROLLARY 4.18. *Let A be a natural uniform algebra on a non-empty, compact space K. Then each Gleason part of K is clopen in K with respect to the relative weak topology, $\sigma(A', A'')$. Further, each weakly compact subset of K has non-empty intersection with only finitely many parts.*

Proof. Let the Gleason part be P. Then the idempotent E of Theorem 4.17(i) belongs to $C(\Phi_{A''})$ and $E \mid K$ is the characteristic function of P, and so the result holds. ∎

The following theorem is a main result of this section; for a generalization, see Theorem 8.4.

THEOREM 4.19. *Let A be a uniform algebra. Then A has a contractive pointwise approximate identity if and only if A has norm-one characters.*

Proof. The uniform algebra A'' is a dual Banach function algebra, with predual A'.

Suppose that A has norm-one characters, and take a non-empty, finite subset S of Φ_A and $\varepsilon > 0$. Take $\varphi \in S$. Then $\|\varphi\| = 1$ when we regard φ as an element of $\Phi_{A''}$. Since $\varphi \in \Phi_{A''} \cap A'$, it follows from Theorem 4.9 that there is an idempotent $E_\varphi \in A''$ such that $E_\varphi(\varphi) = 1$. For each $\varphi \in S$, set $U_\varphi = \{\psi \in \Phi_{A''} : E_\varphi(\psi) = 1\}$, so that U_φ is a compact and open subset of $\Phi_{A''}$ with $\varphi \in U_\varphi$. Thus $S \subset U := \bigcup\{U_\varphi ; \varphi \in S\}$. The set U is compact and open in $\Phi_{A''}$, and so, by Šilov's idempotent theorem, the characteristic function of U, say E, belongs to A''; E is an idempotent and $E(\varphi) = 1 \ (\varphi \in S)$. Since A'' is a uniform algebra, $|E|_{\Phi_{A''}} = 1$. It follows that there exists $f \in A_{[1]}$ with

$$|f(\varphi) - 1| = |f(\varphi) - E(\varphi)| < \varepsilon \quad (\varphi \in S)\,.$$

Thus A has a CPAI.

The converse is immediate. ∎

COROLLARY 4.20. *Let A be a uniform algebra with $|\Phi_A| \geq 2$. Then the following conditions on A are equivalent:*

(a) *A is pointwise contractive;*

(b) *A has the weak separating ball property;*

(c) *M_φ has norm-one characters for each $\varphi \in \Phi_A \cup \{\infty\}$.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) These are trivial.

(c) $\Rightarrow$ (a) This follows from Theorem 4.19. ∎

The following results extend those given in [18, Theorem 4.6]; they are also immediate from the above results.

PROPOSITION 4.21. *Let A be a natural uniform algebra on a compact space K with $|K| \geq 2$.*

(i) *Take $x \in K$. Then $\{x\}$ is a one-point part if and only if M_x has a contractive pointwise approximate identity if and only if M_x has norm-one characters.*

(ii) *The algebra A is pointwise contractive if and only if each point of K is a one-point part.*

(iii) *Take $x \in K$, and suppose that M_x has a bounded pointwise approximate identity. Then x is isolated in (K, d_A).* ∎

We note that it will be shown in Example 7.6 that, in the above notation, it may be that x is an isolated point of (K, d_A), but $\{x\}$ is not a one-point part. We do not know whether a maximal ideal M_x in a natural, uniform algebra on K has a bounded pointwise approximate identity whenever x is isolated in (K, d_A), but we have the following result that applies in a special case.

A unital uniform algebra A with Šilov boundary $\Gamma(A)$ is *logmodular* if the set of functions $\log|f|$, where f is an invertible function in A, when restricted to $\Gamma(A)$, forms a dense subset of $C_{\mathbb{R}}(\Gamma(A))$, the space of real-valued continuous functions on $\Gamma(A)$; see [51, §17]. For example, H^∞ is a logmodular algebra [51, Example 17.4(d)].

PROPOSITION 4.22. *Let A be a natural, logmodular uniform algebra on a compact space K with $|K| \geq 2$. Take $x \in K$, and suppose that x is isolated in (K, d_A). Then M_x has a contractive pointwise approximate identity.*

Proof. By Proposition 4.21(i), we must show that $\{x\}$ is a one-point part. Assume to the contrary that there is a part P with $x \in P$ and $|P| \geq 2$. Consider the metric d_A on P. Then x is an isolated point of (P, d_A). Since A is logmodular, it follows from [57, Lemma 4.5] (extending [51, Theorem 17.1]) that there is a homeomorphism between $\mathbb{D}$ and (P, d_A). Since $\mathbb{D}$ has no isolated point, the point x is also not isolated in (P, d_A), a contradiction. Thus $\{x\}$ is a one-point part. ∎

Let A be a natural, uniform algebra on a non-empty, compact space K, and take $x \in K$ such that $\{x\}$ is a one-point part. Then, by Theorem 4.17, there is an idempotent element $E_x \in A''$ such that $E_x(x) = 1$ and $E_x(y) = 0$ $(y \in K \setminus \{x\})$. As in Theorem 4.12, in the case where x is a strong boundary point for A, we may suppose that $E_x \in A''$ is the characteristic function of $\{x\}$ as a subset of $\Phi_{A''}$, but this is not the case when x is not a strong boundary point. Indeed, suppose that the characteristic function of $\{x\}$ belongs to A''. Then x is an isolated point of $\Phi_{A''}$, and hence, by Theorem 4.12, (e) $\Rightarrow$ (a), x is a strong boundary point for A.

Let $\{x\}$ be a one-point part of K (with respect to A). Then we *claim* that x is always an isolated point of $\overline{K}$, the closure of K in $\Phi_{A''}$. Indeed, take $\varphi \in \overline{K} \setminus \{x\}$. Then there is a net (y_α) in $K \setminus \{x\}$ such that $\lim_\alpha y_\alpha = \varphi$ in $(\Phi_{A''}, \sigma(A''', A''))$. In particular,

$$0 = \lim_\alpha E_x(y_\alpha) = \langle E_x, \varphi \rangle ,$$

and so $E_x \mid \overline{K}$ is the characteristic function of $\{x\}$ as a subset of $\overline{K}$, giving the claim. In the case where x is not a strong boundary point for A, it follows that

$$\{\varphi \in \Phi_{A''} : \langle E_x, \varphi \rangle = 1\} \not\subset \overline{K}$$

and so $\overline{K}$ is not an open set in $\Phi_{A''}$.

This gives the following result.

PROPOSITION 4.23. *Let A be a unital uniform algebra such that every point of Φ_A is a one-point part. Then Φ_A is open in $\Phi_{A''}$ if and only if each point of Φ_A is a strong boundary point for A.* ∎

We shall exhibit in Example 7.7 several examples that show that one-point parts in the character space of a uniform algebra are not necessarily strong boundary points, or even in the Šilov boundary.

5. The space $L(A)$ and BSE norms

In this section, we shall introduce the key space $L(A)$ for a Banach function algebra A, and use it to define the BSE norm on A.

DEFINITION 5.1. Let A be a Banach function algebra, and take a non-empty subset Ω of Φ_A. Then $L(A,\Omega)$ is the linear span of Ω as a subset of A', with $L(A)$ for $L(A,\Phi_A)$.

Clearly
$$L(A)^{\perp} = \Phi_A^{\perp} = \{\mathrm{M} \in A'' : \langle \mathrm{M},\, \varphi\rangle = 0 \quad (\varphi \in \Phi_A)\}\,.$$
As Banach spaces, we have
$$L(A)' = A''/L(A)^{\perp}\,. \tag{5.1}$$
The space $L(A)$ is always weak-$*$ dense in A'; $L(A)$ is an A-submodule of A' because $f\,\cdot\,\varphi = \varphi(f)\,\varphi\ \ (f \in A,\, \varphi \in \Phi_A)$, and so
$$L(A) \subset AA' := \mathrm{lin}\,\{f\,\cdot\,\lambda : f \in A,\, \lambda \in A'\}\,. \tag{5.2}$$
Note that it follows from equation (2.7) that $\mathrm{M}\,\cdot\,\lambda = \lambda\,\cdot\,\mathrm{M} \in L(A,\Omega)$ whenever $\mathrm{M} \in A''$ and $\lambda \in L(A,\Omega)$. The space $L(A,\Omega)^{\perp}$ is a closed ideal in A'' for each non-empty subset Ω of Φ_A.

PROPOSITION 5.2. *Let A be a Banach function algebra that is an ideal in its bidual. Then $\overline{AA'} = \overline{L(A)}$, $A + L(A)^{\perp}$ is an ideal in A'', and*
$$(f+\mathrm{M})\,\Box\,(g+\mathrm{N}) = fg \quad (f,g \in A,\ \mathrm{M},\mathrm{N} \in L(A)^{\perp})\,. \tag{5.3}$$
Proof. Always $\overline{L(A)} \subset \overline{AA'}$.

Take $f \in A$ and $\lambda \in A'$, and assume towards a contradiction that $f\,\cdot\,\lambda \notin \overline{L(A)}$. Then there exists $\mathrm{M} \in A''$ with $\langle \mathrm{M},\, f\,\cdot\,\lambda\rangle = 1$ and with $\langle \mathrm{M},\, \varphi\rangle = 0\ \ (\varphi \in \Phi_A)$. Now $\mathrm{M}\,\cdot\,f \in A$ because A is an ideal in A'', and so
$$\langle \mathrm{M}\,\cdot\,f,\, \varphi\rangle = \langle \mathrm{M},\, f\,\cdot\,\varphi\rangle = \langle \mathrm{M},\, \varphi(f)\varphi\rangle = \varphi(f)\langle \mathrm{M},\, \varphi\rangle = 0 \quad (\varphi \in \Phi_A)\,.$$
Thus $\mathrm{M}\,\cdot\,f = 0$, a contradiction of the fact that $\langle \mathrm{M}\,\cdot\,f,\, \lambda\rangle = 1$. This shows that $A\,\cdot\,A' \subset \overline{L(A)}$, and hence that $\overline{AA'} \subset \overline{L(A)}$.

Thus $\overline{AA'} = \overline{L(A)}$.

Since both A and $L(A)^{\perp}$ are ideals in A'', so is $A + L(A)^{\perp}$.

Now take elements $f,g \in A$ and $\mathrm{M},\mathrm{N} \in L(A)^{\perp}$. We have $\lambda\,\cdot\,f \in A'A = AA' \subset \overline{L(A)}$ for each $\lambda \in A'$, and so $\langle f\,\cdot\,\mathrm{N},\, \lambda\rangle = \langle \mathrm{N},\, \lambda\,\cdot\,f\rangle = 0$. Hence $f\,\cdot\,\mathrm{N} = 0$. Similarly. $\mathrm{M}\,\cdot\,g = 0$. Since R_{N} is weak-$*$ continuous, it follows that $\mathrm{M}\,\Box\,\mathrm{N} = 0$. This gives equation (5.3). ∎

COROLLARY 5.3. *Let A be a Banach function algebra that is an ideal in its bidual and has a bounded approximate identity. Then $\overline{L(A)} = A\,\cdot\,A'$.*

Proof. Since A has a BAI, it follows from Cohen's factorization theorem [12, §2.9] that $\overline{AA'} = A\,\cdot\,A'$. ∎

PROPOSITION 5.4. *Let A be a dual Banach function algebra, with predual $F \subset A'$, and suppose that $\Omega \subset \Phi_A$ is a determining set for A with $\Omega \subset F$. Then*

$$\overline{L(A,\Omega)} = F \quad \text{and} \quad A'' = A \ltimes L(A,\Omega)^{\perp}\,.$$

Proof. By equation (2.10), $A'' = A \ltimes F^{\perp}$. Since $\Omega \subset F$, we have $L(A,\Omega) \subset F$. Now take $\mu \in F'$ with $\mu \mid L(A,\Omega) = 0$. Then μ is an element, say f, of A with $\varphi(f) = 0 \ (\varphi \in \Omega)$, and so $\mu = f = 0$ because Ω is a determining set for A. It follows from the Hahn–Banach theorem that $\overline{L(A,\Omega)} = F$, and then $A'' = A \ltimes L(A,\Omega)^{\perp}$. ■

For example, let G be a compact, abelian group, and set $A = (M(G), \star)$, the measure algebra on G, regarded as the Banach sequence algebra $B(\Gamma)$ consisting of the Fourier–Stieltjes transforms of measures in $M(G)$ on the discrete dual group $\Gamma = \widehat{G}$ of G. Then Γ is determining for A and $\Gamma \subset C(G)$, and so, by the proposition, $\overline{L(A,\Gamma)} = C(G)$ and

$$M(G)'' = M(G) \ltimes C(G)^{\perp}\,.$$

We now define the BSE norm of a Banach function algebra.

DEFINITION 5.5. Let A be a Banach function algebra. Then

$$\|f\|_{\mathrm{BSE}} = \|f\|_{\mathrm{BSE},A} = \sup\{|\langle f,\, \lambda\rangle| : \lambda \in L(A)_{[1]}\} \quad (f \in A)\,.$$

The function $\|\cdot\|_{\mathrm{BSE}}$ is clearly a norm on A such that

$$|f|_{\Phi_A} \le \|f\|_{\mathrm{BSE}} \le \|f\| \quad (f \in A)\,.$$

By equation (2.1), $\|f\|_{\mathrm{BSE}} = \left\|f + L(A)^{\perp}\right\| \ (f \in A)$. Further, it is clear that $\|\cdot\|_{\mathrm{BSE}}$ is an algebra norm, in the sense that

$$\|fg\|_{\mathrm{BSE}} \le \|f\|_{\mathrm{BSE}}\,\|g\|_{\mathrm{BSE}} \quad (f,g \in A)\,.$$

This was first proved in [52]. The norm $\|\cdot\|_{\mathrm{BSE}}$ is called the BSE *norm* on A.

DEFINITION 5.6. A Banach function algebra A *has a* BSE *norm* if $\|\cdot\|_{\mathrm{BSE}}$ is equivalent to the given norm, in the sense that there is a constant $C > 0$ such that

$$\|f\| \le C\,\|f\|_{\mathrm{BSE}} \quad (f \in A)\,.$$

Clearly every uniform algebra has a BSE norm.

The notion of a BSE norm and the related notion of a BSE algebra were introduced by Takahasi and Hatori in [52] and further studied in [39, 53, 54]; see also [19].

The BSE norm has particular significance because key examples in harmonic analysis have a BSE norm. For example, let G be a locally compact group, and write $A = (L^1(G), \star, \|\cdot\|_1)$ for the group algebra on G. Then A has a contractive approximate identity. In the case where G is abelian, A is identified via the Fourier transform with the Banach function algebra $A(\Gamma)$, where $\Gamma = \widehat{G}$, and then $\overline{L(A)}$ is identified with $AP(G)$, the subspace of $L^\infty(G) = L^1(G)'$ consisting of the almost periodic functions, so that $AP(G)$ is a self-adjoint, closed subalgebra of $C^b(G)$. Here $\|\cdot\|_{\mathrm{BSE}} = \|\cdot\|_1$ on A; this is a consequence of the *Bochner–Schoenberg–Eberlein theorem* that is proved in [47, Theorem 1.9.1] and also follows easily from Kaplansky's density theorem, which shows that $AP(G)_{[1]}$ is weak-$*$ dense in $C_0(G)''_{[1]}$. Of course, it is this theorem that leads to the terminology 'BSE norm'.

More generally, let Γ be an arbitrary locally compact group, and let $A(\Gamma)$ and $B(\Gamma)$ denote the Fourier and Fourier–Stieltjes algebras, respectively, on Γ (see the classic thesis of Eymard [24] and the new book [38] of Kaniuth and Lau, where proofs of the following statements can be found). Then $A(\Gamma)$ is the closed ideal $\overline{J_\infty(B(\Gamma))}$ in $B(\Gamma)$, $A(\Gamma)$ is a natural, self-adjoint, strongly regular Banach function algebra on Γ, and $\Phi_{B(\Gamma)} = \Gamma \cup H$, where H is the hull of $A(\Gamma)$ as an ideal in $B(\Gamma)$. Let $C^*(\Gamma)$ be the group C^*-algebra of Γ. Then $C^*(\Gamma)' \cong B(\Gamma)$. Further, by [8], the following are equivalent:

(a) Γ is amenable;

(b) $A(\Gamma)$ has a contractive approximate identity;

(c) $A(\Gamma)$ has a contractive pointwise approximate identity;

(d) $A(\Gamma)$ has a bounded pointwise approximate identity.

In this case, it is also true that $\|f\| = \|f\|_{\mathrm{BSE}}\ \ (f \in B(\Gamma))$, and so $B(\Gamma)$ and $A(\Gamma)$ have BSE norms; this also follows from Kaplansky's density theorem, as in [24, Lemma (2.13)].

For other examples of Banach function algebras that have a BSE norm, see [39, 52, 53, 54]. There will be an account in [19], where more general results will be established.

Proposition 5.7. *Let A be a Banach function algebra with a* BSE *norm, and let B be a Banach function algebra that is isomorphic to a closed subalgebra of A. Then B has a* BSE *norm.*

Proof. There are constants $C_1, C_2 > 0$ such that

$$\|f\|_A \leq C_1 \|f\|_B \ \ (f \in B) \quad \text{and} \quad \|f\|_A \leq C_2 \|f\|_{\mathrm{BSE},A} \ \ (f \in A)\,.$$

Take $\lambda \in L(A)_{[1]}$. Then $\lambda \mid B \in L(B)_{[C_1]}$, and so $\|f\|_{\mathrm{BSE},A} \leq C_1 \|f\|_{\mathrm{BSE},B}$ for each $f \in B$. Thus $\|f\|_B \leq C_1^2 C_2 \|f\|_{\mathrm{BSE},B}$ for $f \in B$, and so B has a BSE norm. ∎

Although $\|f\|_{\mathrm{BSE}} \leq \|f\|\ \ (f \in A)$, we have the following result.

Proposition 5.8. *Let A be a Banach function algebra. Then*

$$\|\lambda\| = \sup\{|\langle f,\, \lambda\rangle| : f \in A,\ \|f\|_{\mathrm{BSE}} \leq 1\} \quad (\lambda \in L(A))\,. \tag{5.4}$$

Proof. Take $\lambda \in L(A)$, and let the supremum on the right be k.

Since $\|f\|_{\mathrm{BSE}} \leq \|f\|\ \ (f \in A)$, certainly $\|\lambda\| \leq k$. On the other hand,

$$\|\lambda\| = \sup\left\{\left|\langle \mathrm{M} + L(A)^\perp,\, \lambda\rangle\right| : \mathrm{M} \in A'',\ \left\|\mathrm{M} + L(A)^\perp\right\| \leq 1\right\}$$

because $L(A)' \cong A''/L(A)^\perp$. But $\|f\|_{\mathrm{BSE}} = \left\|f + L(A)^\perp\right\|\ \ (f \in A)$, and so $\|\lambda\| \geq k$.

Equation (5.4) follows. ∎

Theorem 5.9. *Let A be a Banach function algebra with a bounded pointwise approximate identity of bound m. Then $\|f\|_{\mathrm{BSE}} \leq m\,\|f\|_{\mathrm{op}}\ \ (f \in A)$.*

Proof. Take $f \in A$. For each $\lambda = \sum_{i=1}^n \alpha_i \varphi_i \in L(A)_{[1]}$ and $\varepsilon > 0$, there exists $g \in A_{[m]}$ such that

$$\sum_{i=1}^n |\alpha_i|\,|f(\varphi_i)|\,|1 - g(\varphi_i)| < \varepsilon\,,$$

and so

$$|\langle f,\,\lambda\rangle| \le |\langle fg,\,\lambda\rangle| + |\langle f - fg,\,\lambda\rangle| \le \|fg\| + \sum_{i=1}^{n} |\alpha_i|\,|f(\varphi_i)|\,|1-g(\varphi_i)| \le m\,\|f\|_{\mathrm{op}} + \varepsilon\,.$$

It follows that $\|f\|_{\mathrm{BSE}} \le m\,\|f\|_{\mathrm{op}}$, as required. ∎

COROLLARY 5.10. *Let $(A, \|\cdot\|)$ be a Banach function algebra, and suppose that A has a* BSE *norm and a bounded pointwise approximate identity. Then $\|\cdot\|$ and $\|\cdot\|_{\mathrm{op}}$ are equivalent on A.*

Proof. By Theorem 5.9, there is $m > 0$ such that $\|f\|_{\mathrm{BSE}} \le m\,\|f\|_{\mathrm{op}}\ \ (f \in A)$. By hypothesis, there is a constant $C > 0$ such that $\|f\| \le C\,\|f\|_{\mathrm{BSE}}\ \ (f \in A)$, and so

$$\|f\|_{\mathrm{op}} \le \|f\| \le C\,\|f\|_{\mathrm{BSE}} \le Cm\,\|f\|_{\mathrm{op}} \quad (f \in A)\,.$$

Thus $\|\cdot\|$ and $\|\cdot\|_{\mathrm{op}}$ are equivalent on A. ∎

EXAMPLE 5.11. Let A be a natural uniform algebra on a compact space K with $|K| \ge 2$, and take $x \in K$ such that $\{x\}$ is a one-point part. By Proposition 4.21(i), M_x has a contractive pointwise approximate identity. Then it follows from the above corollary (with control of the constants) that the uniform norm and the operator norm on M_x are equal. However, in the case where x is not a strong boundary point, it follows from Theorem 4.12 that the maximal ideal M_x does not have a bounded approximate identity. ∎

THEOREM 5.12. *Let $(A, \|\cdot\|)$ be a Banach function algebra that is an ideal in its bidual. Then*

$$|\cdot|_{\Phi_A} \le \|\cdot\|_{\mathrm{op}} \le \|\cdot\|_{\mathrm{BSE}} \le \|\cdot\|\ .$$

Proof. Certainly $|\cdot|_{\Phi_A} \le \|\cdot\|_{\mathrm{BSE}} \le \|\cdot\|$ and $|\cdot|_{\Phi_A} \le \|\cdot\|_{\mathrm{op}} \le \|\cdot\|$, and so it suffices to show that $\|\cdot\|_{\mathrm{op}} \le \|\cdot\|_{\mathrm{BSE}}$.

Take $f \in A$, $\mathrm{M} \in L(A)^{\perp}$, and $g \in A_{[1]}$. Then there exists $\lambda \in A'_{[1]}$ with $\|fg\| = \langle fg,\,\lambda\rangle$. By Proposition 5.2, $\overline{AA'} = \overline{L(A)}$, and so $g\cdot\lambda \in \overline{L(A)}$, and this implies that $\langle \mathrm{M},\, g\cdot\lambda\rangle = 0$. Since $g\,\cdot\,\lambda \in A'_{[1]}$, necessarily

$$\|f + \mathrm{M}\| \ge |\langle f + \mathrm{M},\, g\,\cdot\,\lambda\rangle| = |\langle f,\, g\,\cdot\,\lambda\rangle| = |\langle fg,\,\lambda\rangle| = \|fg\|\ .$$

Thus $\sup\{\|fg\| : g \in A_{[1]}\} \le \inf\{\|f + \mathrm{M}\| : \mathrm{M} \in L(A)^{\perp}\}$, i.e., $\|f\|_{\mathrm{op}} \le \|f\|_{\mathrm{BSE}}$. The result follows. ∎

COROLLARY 5.13. *Let A be a Banach function algebra that is an ideal in its bidual.*

(i) *Suppose that A has a bounded pointwise approximate identity. Then A has a* BSE *norm.*

(ii) *Suppose that A has a contractive pointwise approximate identity. Then*

$$\|f\|_{\mathrm{BSE}} = \|f\| \quad (f \in A)\,.$$

Proof. (i) By Proposition 3.7, A has a BAI, say the bound is m. By Proposition 3.2, $\|\cdot\| \le m\,\|\cdot\|_{\mathrm{op}}$. Thus it follows from Theorem 5.12 that $\|\cdot\|$ and $\|\cdot\|_{\mathrm{BSE}}$ are equivalent.

(ii) This follows by taking $m = 1$ in the above calculation. ∎

6. The algebra $A''/L(A)^{\perp}$

Let A be a Banach function algebra. In this section, we shall introduce the Banach space

$$\mathcal{Q}(A) = A''/L(A)^{\perp} = L(A)'$$

(taken with the quotient norm, $\|\cdot\|_{\mathcal{Q}(A)}$), which we shall see is also a Banach function algebra, and establish some general results. We find it to be somewhat surprising that it seems that there has been little earlier study of this Banach algebra in an abstract setting; however, there is an implicit definition of our algebra $\mathcal{Q}(A)$ in [44, Theorem 3.1.14], where some of the properties of $\mathcal{Q}(A)$ that we develop are given.

For $\mathrm{M} \in A''$, the corresponding element in $\mathcal{Q}(A)$ is denoted by $[\mathrm{M}]$. Note that, given $f \in \mathcal{Q}(A)$, there exists $\mathrm{M} \in A''$ with $[\mathrm{M}] = f$ and $\|\mathrm{M}\| = \|f\|_{\mathcal{Q}(A)}$. Indeed, for each $n \in \mathbb{N}$, there exists $\mathrm{M}_n \in A''$ with $[\mathrm{M}_n] = f$ and $\|\mathrm{M}_n\| < \|f\|_{\mathcal{Q}(A)} + 1/n$, and a $\sigma(A'', A')$-accumulation point $\mathrm{M} \in A''$ of the sequence (M_n) has the required properties.

The space $\overline{L(A)}$ is canonically embedded in $\overline{L(A)}'' = \mathcal{Q}(A)'$ by setting

$$\langle [\mathrm{M}],\, \lambda\rangle = \langle \mathrm{M},\, \lambda\rangle \quad (\mathrm{M} \in A'')$$

for $\lambda \in \overline{L(A)}$, so that

$$\|[\mathrm{M}]\|_{\mathcal{Q}(A)} = \sup\{|\langle \mathrm{M},\, \lambda\rangle| : \lambda \in L(A)_{[1]}\}\,. \tag{6.1}$$

First we verify that $\mathcal{Q}(A)$ is always a semi-simple, commutative Banach algebra.

It follows from (2.8) that the space $L(A)^{\perp}$ is a weak-$*$ closed ideal in A''. Thus $\mathcal{Q}(A) = A''/L(A)^{\perp}$ is a Banach algebra.

Take $\mathrm{M}, \mathrm{N} \in A''$. Then it follows from equation (2.8) that $\mathrm{M} \,\Box\, \mathrm{N} - \mathrm{N} \,\Box\, \mathrm{M} \in L(A)^{\perp}$, and so $[\mathrm{M}]\,[\mathrm{N}] = [\mathrm{N}]\,[\mathrm{M}]$. Hence $\mathcal{Q}(A)$ is a commutative algebra. For each $\varphi \in \Phi_A$, the map

$$\widetilde{\varphi} : \mathrm{M} + L(A)^{\perp} \mapsto \langle \mathrm{M},\, \varphi\rangle\,, \quad \mathcal{Q}(A) \to \mathbb{C}\,,$$

is a well-defined character on $\mathcal{Q}(A)$, and $\mathrm{M} + L(A)^{\perp} = 0$ whenever $\widetilde{\varphi}(\mathrm{M} + L(A)^{\perp}) = 0$ for all $\varphi \in \Phi_A$, and so $\mathcal{Q}(A)$ is semi-simple. Hence $\mathcal{Q}(A)$ is (identified with) a Banach function algebra and as a subalgebra of $\ell^{\infty}(\Phi_A)$. Take $\varphi \in \Phi_A$. By identifying φ with $\widetilde{\varphi}$, defined above, we can, and shall, regard Φ_A as a determining subset of $\Phi_{\mathcal{Q}(A)}$.

Clearly $A \cap L(A)^{\perp} = \{0\}$ in A'', and so $A + L(A)^{\perp}$ is an algebraic direct sum. Each element $f \in A$ determines $[f]$ in $\mathcal{Q}(A)$, and $[f] \mid \Phi_A = f$, and so we can regard A as a subalgebra of $\mathcal{Q}(A)$ by identifying $f \in A$ with $[f] \in \mathcal{Q}(A)$. It follows that

$$\|f\|_{\mathcal{Q}(A)} = \|f\|_{\mathrm{BSE}} \quad (f \in A)\,.$$

Since $\mathcal{Q}(A)' = L(A)''$ and $L(A)''$ is identified with the annihilator of $L(A)^{\perp}$ in A''', each $\Lambda \in L(A)''$ acts on $\mathcal{Q}(A)$ through the formula

$$\langle [\mathrm{M}],\, \Lambda\rangle_{\mathcal{Q}(A),\mathcal{Q}(A)'} = \langle \mathrm{M},\, \Lambda\rangle_{A'',A'''} \quad (\mathrm{M} \in A'')\,. \tag{6.2}$$

In particular, equation (6.2) holds when $\Lambda \in \Phi_{\mathcal{Q}(A)}$.

It follows that $\overline{L(A)}$ is a closed submodule of $\mathcal{Q}(A)'$, and it is clear that $\mathcal{Q}(A)$ is a dual Banach function algebra, with isometric predual $\overline{L(A)}$.

Given $[\mathrm{M}] \in \mathcal{Q}(A)$, say with $\|[\mathrm{M}]\|_{\mathcal{Q}(A)} = \|\mathrm{M}\| = m$, there is a net (f_α) in $A_{[m]}$ that converges weak-$*$ to M in A''. Since

$$m \le \liminf_\alpha \|f_\alpha\| \le \limsup_\alpha \|f_\alpha\| \le m\,,$$

it follows that $\lim_\alpha \|f_\alpha\| = m$. Further, $\lim_\alpha f_\alpha = [\mathrm{M}]$ in the space

$$\left(\mathcal{Q}(A), \sigma(\mathcal{Q}(A), \overline{L(A)}\right)\,,$$

and so $A_{[1]}$ is weak-$*$ dense in $\mathcal{Q}(A)_{[1]}$.

The relative topology on Φ_A from $\Phi_{\mathcal{Q}(A)}$ is the weak topology $\sigma(A', A'')$, and so

$$\mathcal{Q}(A) = \{\mathrm{M} \mid \Phi_A : \mathrm{M} \in A''\} \subset \ell^\infty(\Phi_A)\,.$$

As we shall see in Example 7.3, the embedding of Φ_A in $\Phi_{\mathcal{Q}(A)}$ need not be continuous. Since

$$\Phi_A \subset \Phi_{\mathcal{Q}(A)} \subset L(A)'' \subset A'''\,,$$

each $\psi \in \Phi_{\mathcal{Q}(A)}$ is of the form $\psi = \varphi + \xi$, where $\varphi = \psi \mid A \in \Phi_A \cup \{0\}$ and $\xi \in A^\perp$, and so $\langle [f],\, \psi\rangle = \langle f,\, \varphi\rangle$ $(f \in A)$. In particular, it follows that

$$f(\Phi_A) \subset [f](\Phi_{\mathcal{Q}(A)}) \subset f(\Phi_A) \cup \{0\} \quad \text{and} \quad |\,[f]\,|_{\Phi_{\mathcal{Q}(A)}} = |f|_{\Phi_A} \quad (f \in A)\,. \tag{6.3}$$

We see that $\mathcal{Q}(A)'' = L(A)''' = L(A)' \oplus L(A)^\perp = \mathcal{Q}(A) \oplus L(A)^\perp$ as Banach spaces, and that $L(A)^\perp$ is an ideal in $\mathcal{Q}(A)''$, so that

$$\mathcal{Q}(A)'' = \mathcal{Q}(A) \ltimes L(A)^\perp\,.$$

We have established the following theorem.

Theorem 6.1. *Let A be a Banach function algebra. Then $L(A)^\perp$ is a closed ideal in A'' and $A + L(A)^\perp$ is a subalgebra of A''. The quotient Banach algebra*

$$\mathcal{Q}(A) := A''/L(A)^\perp = L(A)'$$

is commutative and semi-simple, so that $\mathcal{Q}(A)$ is a Banach function algebra, $\mathcal{Q}(A)$ is a subalgebra of $\ell^\infty(\Phi_A)$, and $\mathcal{Q}(A)$ contains A as a subalgebra. Further, $\mathcal{Q}(A)$ is a dual Banach function algebra, with isometric predual $\overline{L(A)}$, $A_{[1]}$ is $\sigma(\mathcal{Q}(A), \overline{L(A)})$-dense in $\mathcal{Q}(A)_{[1]}$, and

$$\mathcal{Q}(A)'' = L(A)' \oplus L(A)^\perp \quad \text{and} \quad \mathcal{Q}(A)'' = \mathcal{Q}(A) \ltimes L(A)^\perp\,. \qquad \blacksquare$$

In fact, the quotient algebra $A''/L(A,\Omega)^\perp$ is a Banach function algebra for each non-empty subset Ω of Φ_A. The set Ω is always a determining set for $A''/L(A,\Omega)^\perp$.

Since $\Phi_{\mathcal{Q}(A)} \subset L(A)''$, the Gel'fand topology on $\Phi_{\mathcal{Q}(A)}$ is the relative weak-$*$ topology $\sigma(L(A)'', L(A)')$. We shall write $\overline{\Phi_A}$ for the weak-$*$-closure of Φ_A in $\Phi_{\mathcal{Q}(A)}$. In the case where $\mathcal{Q}(A)$ is regular as a Banach function algebra on $\Phi_{\mathcal{Q}(A)}$, it is clear that $\overline{\Phi_A} = \Phi_{\mathcal{Q}(A)}$. However, in general, $\overline{\Phi_A} \subsetneq \Phi_{\mathcal{Q}(A)}$; for example, see Example 7.8.

In the following result, τ_p is the topology of pointwise convergence on the space Φ_A.

Proposition 6.2. *Let $(A, \|\cdot\|)$ be a Banach function algebra. Then, given $[\mathrm{M}] \in \mathcal{Q}(A)$ with $\|[\mathrm{M}]\| = m$, there is a net (f_α) in $A_{[m]}$ such that $\lim_\alpha f_\alpha = \mathrm{M}$ with respect to τ_p and $\lim_\alpha \|f_\alpha\| = \lim_\alpha \|f_\alpha\|_{\mathcal{Q}(A)} = m$.*

Proof. Take $\mathrm{M} \in A''$ with $\|[\mathrm{M}]\| = m$. Then, as noted above, there is a net (f_α) in $A_{[m]}$ such that $\lim_\alpha f_\alpha = \mathrm{M}$ with respect to τ_p and $\lim_\alpha \|f_\alpha\| = m$. The net (f_α) is contained in $\mathcal{Q}(A)_{[m]}$, and so $\lim_\alpha f_\alpha = [\mathrm{M}]$ in the topology $\sigma(\mathcal{Q}(A), \overline{L(A)})$. Hence

$$m = |[\mathrm{M}]\|_{\mathcal{Q}(A)} \leq \liminf_\alpha \|f_\alpha\|_{\mathcal{Q}(A)} \leq \limsup_\alpha \|f_\alpha\|_{\mathcal{Q}(A)} \leq m\,,$$

and this gives the required result. ■

The above proposition immediately gives the following corollary, which characterizes the elements of $\mathcal{Q}(A)$ in $\ell^\infty(\Phi_A)$.

Corollary 6.3. *Let A be a Banach function algebra. Then $\mathcal{Q}(A)$ is the set of functions $f \in \ell^\infty(\Phi_A)$ for which there is a bounded net (f_α) in A with $\lim_\alpha f_\alpha = f$ in $(\ell^\infty(\Phi_A), \tau_p)$; for $f \in \mathcal{Q}(A)$, the infimum of the bounds of such nets is equal to $\|f\|_{\mathcal{Q}(A)}$.* ■

Proposition 6.4. *Let A be a Banach function algebra. Then:*

(i) $\|[\mathrm{M}]\|_{\mathcal{Q}(A)} = \|[\mathrm{M}]\|_{\mathrm{BSE},\mathcal{Q}(A)}$ $(\mathrm{M} \in A'')$, *and so $\mathcal{Q}(A)$ has a* BSE *norm;*

(ii) *the algebra A has a* BSE *norm if and only if A is isomorphic to a closed subalgebra of $\mathcal{Q}(A)$;*

(iii) *the algebra $\mathcal{Q}(A)$ has an identity if and only if A has a bounded pointwise approximate identity.*

Proof. (i) The space $\overline{L(\mathcal{Q}(A))}$ is the closure in $L(A)''$ of $L(\mathcal{Q}(A))$, and

$$\overline{L(A)} \subset \overline{L(\mathcal{Q}(A))} \subset L(A)''$$

isometrically. Thus, for each $\mathrm{M} \in A''$, we have

$$\left\|\mathrm{M} + L(A)^\perp\right\| = \sup\{|\langle \mathrm{M},\, \lambda\rangle| : \lambda \in L(A)_{[1]}\}\,.$$

Since $\sup\left\{\left|\langle \mathrm{M} + L(A)^\perp, \Lambda\rangle\right| : \Lambda \in L(\mathcal{Q}(A))_{[1]}\right\} \leq \left\|\mathrm{M} + L(A)^\perp\right\|$, we see that

$$\left\|\mathrm{M} + L(A)^\perp\right\| = \sup\{\left|\langle \mathrm{M} + L(A)^\perp,\, \Lambda\rangle\right| : \Lambda \in L(\mathcal{Q}(A))_{[1]}\}\,,$$

and so

$$\|[\mathrm{M}]\|_{\mathcal{Q}(A)} = \|[\mathrm{M}]\|_{\mathrm{BSE},\mathcal{Q}(A)} \quad (\mathrm{M} \in A'')\,.$$

In particular, this shows that $\mathcal{Q}(A)$ has a BSE norm.

(ii) We have $\|[f]\|_{\mathcal{Q}(A)} = \|f\|_{\mathrm{BSE},A} \leq \|f\|_A$ $(f \in A)$, and so A has a BSE norm if and only if A is closed in $\mathcal{Q}(A)$.

(iii) First, suppose that A has a BPAI, say (f_α), and let E be a weak-$*$ accumulation point of the net (f_α) in A''. Clearly $\langle \mathrm{E},\, \varphi\rangle = 1$ $(\varphi \in \Phi_A)$, and so

$$\mathrm{E} \,\Box \mathrm{M} - \mathrm{M} \in L(A)^\perp \quad (\mathrm{M} \in A'')\,.$$

Hence [E] is the identity of the commutative Banach algebra $\mathcal{Q}(A)$.

Conversely, suppose that $\mathrm{E} \in A''$ and that [E] is the identity of $\mathcal{Q}(A)$. Then

$$\langle \mathrm{E},\, \varphi\rangle = \langle \mathrm{E} + L(A)^\perp,\, \varphi\rangle = 1 \quad (\varphi \in \Phi_A)\,.$$

There is a net, say (f_α), in A that is bounded in norm by $\|\mathrm{E}\|$ and that converges weak-$*$ to E in A'', and clearly (f_α) is a BPAI for A. ■

The norm of the identity of $\mathcal{Q}(A)$ that arises in clause (iii), above, is not necessarily equal to 1. Of course, in the case where A has a contractive pointwise approximate identity, the corresponding element $\mathrm{E} \in A''$ is such that $\|[\mathrm{E}]\| = 1$, and so $\mathcal{Q}(A)$ is a unital Banach function algebra.

It follows from clause (iii), above, that $\Phi_{\mathcal{Q}(A)}$ and $\overline{\Phi_A}$ are compact when A has a bounded pointwise approximate identity.

COROLLARY 6.5. *Let A be a Banach function algebra, and suppose that (e_α) and (f_β) are bounded pointwise approximate identities in A. Then:*

(i) $[\mathrm{E}] = [\mathrm{F}]$ *for any weak-$*$ accumulation points* E *and* F *of* (e_α) *and* (f_β), *respectively, in* A'';

(ii) *for each weak-$*$ accumulation point* E *of* (e_α) *in* A'', *there exists* $\mu \in L(A)^\perp$ *with* $\|\mathrm{E} + \mu\| = \|[\mathrm{E}]\|$.

Proof. (i) This follows because an identity of $\mathcal{Q}(A)$ is uniquely defined.

(ii) This follows because $\|[\mathrm{E}]\| = \inf\{\|\mathrm{E} + \mu\| : \mu \in L(A)^\perp\}$ and the space $L(A)^\perp$ is weak-$*$ closed in A'', so that the infimum is attained. ∎

In particular, when A has a bounded pointwise approximate identity, the norm of the identity in $\mathcal{Q}(A)$ does not depend on the choice of the bounded pointwise approximate identity.

In the following corollary, note that neither A nor A'' is necessarily unital.

COROLLARY 6.6. *Let A be a Banach function algebra, and suppose that A has a bounded pointwise approximate identity. Take an element $f \in A$ such that $f(\varphi) \neq 1$ $(\varphi \in \Phi_A)$. Then there exists* $\mathrm{M} \in A''$ *such that*

$$\langle \mathrm{M}, \varphi\rangle (1 - f(\varphi)) = 1 \quad (\varphi \in \Phi_A)\,.$$

Proof. By Proposition 6.4(iii), $\mathcal{Q}(A)$ has an identity, say $[\mathrm{E}]$. It follows from the hypothesis and equation (6.3) that $\langle [f], \psi\rangle \neq 1$ $(\psi \in \Phi_{\mathcal{Q}(A)})$, and hence

$$\langle [\mathrm{E}] - [f], \psi\rangle \neq 0 \quad (\psi \in \Phi_{\mathcal{Q}(A)})\,.$$

Thus $\mathrm{E} - [f]$ is invertible in $\mathcal{Q}(A)$, i.e., there exists $\mathrm{M} \in A''$ with $[\mathrm{M}] \cdot ([\mathrm{E}] - [f]) = [\mathrm{E}]$ in $\mathcal{Q}(A)$. The result follows. ∎

We shall now regard $\mathcal{Q}(A)$ as a Banach function algebra, and usually write f, u, etc., for generic elements of $\mathcal{Q}(A)$.

PROPOSITION 6.7. *Let A be a Banach function algebra, and take $\varphi \in \Phi_A$ with $\|\varphi\| = 1$. Then there is an idempotent $u \in \mathcal{Q}(A)_{[1]}$ such that $u(\varphi) = 1$ and $fu = u$ whenever $f \in \mathcal{Q}(A)_{[1]}$ with $f(\varphi) = 1$.*

Proof. Since $\mathcal{Q}(A)$ is a dual Banach function algebra with predual $\overline{L(A)}$ and since $\varphi \in L(A)$, this is a special case of Theorem 4.9. ∎

The following is immediate from Proposition 5.4.

Proposition 6.8. *Let A be a dual Banach function algebra, with predual $F \subset A'$, and suppose that $\Phi_A \subset F$. Then $\overline{L(A)} = F$, $A'' = A \ltimes L(A)^{\perp}$, and $\mathcal{Q}(A) = A$.* ∎

6.1. The subset Φ_A of $\Phi_{\mathcal{Q}(A)}$. In the next results, we continue to regard Φ_A as a subset of $\Phi_{\mathcal{Q}(A)}$, and will give conditions that imply that it is an open subset of $\Phi_{\mathcal{Q}(A)}$.

Lemma 6.9. *Let A be a Banach function algebra, and let S be a non-empty subset of Φ_A. Then the following are equivalent:*

(a) *each point of S is isolated in $\Phi_{\mathcal{Q}(A)}$;*

(b) *S is open and discrete in $\Phi_{\mathcal{Q}(A)}$.*

Proof. (a) ⇒ (b) This is immediate.

(b) ⇒ (a) Take $\varphi \in S$, and suppose that (φ_α) is a net in $\Phi_{\mathcal{Q}(A)}$ that converges to φ. Since S is open in $\Phi_{\mathcal{Q}(A)}$, the net (φ_α) is eventually in S. Since S is discrete in $\Phi_{\mathcal{Q}(A)}$, eventually $\varphi_\alpha = \varphi$. Thus φ is isolated in $\Phi_{\mathcal{Q}(A)}$. ∎

Theorem 6.10. *Let A be a Banach function algebra.*

(i) *Suppose that $\varphi \in \Phi_A$ and that A has the weak separating ball property at φ. Then φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$.*

(ii) *Suppose that A has the weak separating ball property. Then Φ_A is the set of isolated points of $\Phi_{\mathcal{Q}(A)}$, and so Φ_A is open and discrete as a subspace of $\Phi_{\mathcal{Q}(A)}$.*

Proof. (i) Note that $\|\varphi\| = 1$ because A has the weak separating ball property at φ, and so, by Proposition 6.7, there is an idempotent $u \in \mathcal{Q}(A)_{[1]}$ such that $u(\varphi) = 1$ and $fu = u$ whenever $f \in \mathcal{Q}(A)_{[1]}$ with $f(\varphi) = 1$.

We may suppose that $|\Phi_A| \geq 2$. Take $\psi \in \Phi_A$ with $\psi \neq \varphi$. Then there is a net (g_ν) in $(M_\psi)_{[1]}$ with $\lim_\nu g_\nu(\varphi) = 1$. Since $\|g_\nu\|_{\mathcal{Q}(A)} \leq \|g_\nu\|_A$, this net has a weak-$*$ accumulation point, say g, in $\mathcal{Q}(A)_{[1]}$ with $g(\varphi) = 1$; further, $g(\psi) = 0$. It follows that $u(\psi) = 0$, and so $(fu - f(\varphi)u)(\psi) = 0$ for each $f \in \mathcal{Q}(A)$. Thus $(fu - f(\varphi)u) \mid \Phi_A = 0 \;\; (f \in \mathcal{Q}(A))$. Since Φ_A is a determining set for $\mathcal{Q}(A)$, we have

$$fu = f(\varphi)u \quad (f \in \mathcal{Q}(A))\,.$$

Suppose that $\Phi_A \subsetneq \Phi_{\mathcal{Q}(A)}$, and take $\psi \in \Phi_{\mathcal{Q}(A)}$ with $\psi \neq \varphi$. Then there exists $f \in \mathcal{Q}(A)$ with $f(\varphi) = 1$ and $f(\psi) = 0$, and hence $u(\psi) = 0$. This shows that u is the characteristic function of $\{\varphi\}$ in $\Phi_{\mathcal{Q}(A)}$, and hence φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$. This is immediate when $\Phi_A = \Phi_{\mathcal{Q}(A)}$.

(ii) By (i), each point of Φ_A is an isolated point of $\Phi_{\mathcal{Q}(A)}$. Since Φ_A is determining for $\mathcal{Q}(A)$, each isolated point of $\Phi_{\mathcal{Q}(A)}$ belongs to Φ_A. By Lemma 6.9, Φ_A is open and discrete as a subspace of $\Phi_{\mathcal{Q}(A)}$. ∎

Proposition 6.11. *Let A be a Banach function algebra.*

(i) *Take $\varphi \in \Phi_A$ such that M_φ is non-zero, and suppose that M_φ has a bounded pointwise approximate identity. Then φ is weakly isolated in Φ_A and an isolated point of the space $\Phi_{\mathcal{Q}(A)}$.*

(ii) *Each isolated point of* $\Phi_{\mathcal{Q}(A)}$ *is in* Φ_A*, and is weakly isolated in* Φ_A.

Proof. (i) Let E be a weak-$*$ accumulation point of the BPAI of M_φ in A'', so that E $\mid \Phi_A$ is the characteristic function of $\Phi \setminus \{\varphi\}$. This implies that φ is weakly isolated in Φ_A.

For each F $\in A''$ with F$(\varphi) = 0$, we have (FE $-$ F) $\mid \Phi_A = 0$, and so [F] [E] = [F] in $\mathcal{Q}(A)$. It follows that [E] is the characteristic function of $\Phi_{\mathcal{Q}(A)} \setminus \{\varphi\}$, and so φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$.

(ii) Let ψ be an isolated point of $\Phi_{\mathcal{Q}(A)}$. By Šilov's idempotent theorem, the characteristic function χ_ψ of $\{\psi\}$ is in $\mathcal{Q}(A)$. Since Φ_A is a determining set for $\Phi_{\mathcal{Q}(A)}$, necessarily $\psi \in \Phi_A$, and clearly ψ is weakly isolated in Φ_A. ∎

Theorem 6.12. *Let A be a Banach function algebra such that $|\Phi_A| \geq 2$, and take $\varphi \in \Phi_A$. Then the following are equivalent:*

(a) *the ideal M_φ has a bounded pointwise approximate identity;*

(b) *the algebra A has a bounded pointwise approximate identity and φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$.*

Proof. The characteristic function of $\{\varphi\}$ on $\Phi_{\mathcal{Q}(A)}$ is denoted by χ_φ.

(a) $\Rightarrow$ (b) Take E to be a weak-$*$ accumulation point in A'' of a BPAI in M_φ. By Proposition 6.11(i), φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$, and so $\chi_\varphi \in \mathcal{Q}(A)$. Then [E] $+ \chi_\varphi$ is the identity of $\mathcal{Q}(A)$, and so, again by Proposition 6.4(iii), A has a BPAI.

(b) $\Rightarrow$ (a) Since A has a BPAI, there exists E $\in A''$ such that [E] is the identity of $\mathcal{Q}(A)$. Since φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$, it follows that $\chi_\varphi \in \mathcal{Q}(A)$, and so there exists F $\in A''$ such that [F] $= \chi_\varphi$. The element E $-$ F is in M_φ'', and is such that $\langle \mathrm{E} - \mathrm{F}, \psi \rangle = 1 \ \ (\psi \in \Phi_A \setminus \{\varphi\})$, and so M_φ has a bounded pointwise approximate identity. ∎

Corollary 6.13. *Let A be a Banach function algebra with a bounded pointwise approximate identity and such that $|\Phi_A| \geq 2$, and take $\varphi \in \Phi_A$. Then the ideal M_φ has a bounded pointwise approximate identity if and only if φ is an isolated point of $\Phi_{\mathcal{Q}(A)}$.* ∎

For example, let A be the disc algebra, so that A has an identity. The maximal ideal M_0 does not have a bounded pointwise approximate identity, and so 0 is not an isolated point of $\Phi_{\mathcal{Q}(A)}$. We shall identify $\mathcal{Q}(A)$ in Example 7.5. Again, take Γ to be a locally compact group. Then $A(\Gamma)$ has the separating ball property (see Example 11.3(ii)), and so, by Theorem 6.10(i), each point of Γ is isolated in $\Phi_{\mathcal{Q}(A(\Gamma))}$. However, as noted on page 30, in the case where Γ is not amenable, $A(\Gamma)$ does not have a bounded pointwise approximate identity, and so no maximal modular ideal in $A(\Gamma)$ has a bounded pointwise approximate identity.

Let A be a natural uniform algebra on a non-empty, compact space K. In the following proposition, we again regard K as a subset of $\Phi_{\mathcal{Q}(A)}$. Also, we write $G(A)$ for the set of points $x \in K$ such that $\{x\}$ is a one-point Gleason part in K. Recall that d_A denotes the Gleason metric on K.

Proposition 6.14. *Let A be a natural uniform algebra on a non-empty, compact space K.*

(i) *Each point of $G(A)$ is an isolated point of $\Phi_{\mathcal{Q}(A)}$, and so the set $G(A)$ is open and discrete in $\Phi_{\mathcal{Q}(A)}$.*

(ii) *Each isolated point of $\Phi_{\mathcal{Q}(A)}$ is an isolated point of (K, d_A).*

(iii) *In the case where $\mathcal{Q}(A)$ is a uniform algebra on $\Phi_{\mathcal{Q}(A)}$, the space $G(A)$ is equal to the set of isolated points of $\Phi_{\mathcal{Q}(A)}$.*

Proof. (i) Suppose that $x \in G(A)$. Then, by Corollary 4.16, A has the weak separating ball property at x, and so, by Theorem 6.10(i), x is an isolated point of $\Phi_{\mathcal{Q}(A)}$.

(ii) Suppose that x is an isolated point of $\Phi_{\mathcal{Q}(A)}$, so that $x \in K$ by Proposition 6.11(ii). Then there exists $F \in A''$ such that $F \mid \Phi_{\mathcal{Q}(A)}$ is the characteristic function of $\{x\}$ as a subset of $\Phi_{\mathcal{Q}(A)}$, and again $x \in K$. Set $r = \|F\|$, and take $y \in K$ with $y \neq x$. Then

$$d_A(y, x) \geq |\langle F, \varepsilon_y \rangle - \langle F, \varepsilon_x \rangle| / r = 1/r ,$$

and so x is an isolated point of (K, d_A).

(iii) Now suppose that $\mathcal{Q}(A)$ is a uniform algebra on $\Phi_{\mathcal{Q}(A)}$. By (i), we must show that each isolated point φ of $\Phi_{\mathcal{Q}(A)}$ is in $G(A)$. Such a point φ is an isolated point of (K, d_A) by (ii).

Since x is isolated in $\Phi_{\mathcal{Q}(A)}$, the function χ_x belongs to $\mathcal{Q}(A)$; since $\mathcal{Q}(A)$ is a uniform algebra, $\|\chi_x\| = 1$. Take $F \in A''_{[1]}$ with $F \mid \Phi_{\mathcal{Q}(A)} = \chi_x$, and again take $y \in K \setminus \{x\}$. Then $\langle F, \varepsilon_y \rangle = 0$ and $\langle F, \varepsilon_x \rangle = 1$, and so there is a net (f_ν) in $A_{[1]}$ with $\lim_\nu f_\nu(y) = 0$ and $\lim_\nu f_\nu(x) = 1$. It follows that y is not in the same Gleason part as x, and so $x \in G(A)$. ■

We shall give an example in Example 7.6, below, to show that an isolated point of $\Phi_{\mathcal{Q}(A)}$ need not belong to $G(A)$, and so we shall have an example of a uniform algebra A such that $\mathcal{Q}(A)$ is not a uniform algebra.

6.2. Compactness in $\Phi_{\mathcal{Q}(A)}$. Let A be a Banach function algebra, and consider the closure, $\overline{\Phi_A}$, of Φ_A in the space $\Phi_{\mathcal{Q}(A)}$, where the latter space has its usual weak-$*$ topology, $\sigma(\mathcal{Q}(A)', \mathcal{Q}(A))$, identified with $\sigma(L(A)'', L(A)')$. Here we consider the question when this closure is compact. We shall show that this is the case if and only if Φ_A is weakly closed in A'.

Let A be a Banach function algebra, and suppose that Φ_A is not weakly closed in A'. Then, as noted several times before, the weak closure of Φ_A in A' is $\Phi_A \cup \{0\}$.

Now let F be a Banach space such that its dual $B = F'$ is a Banach function algebra. Then $B' = F''$. Let Ω be a non-empty subset of Φ_B that is contained in F. We note that a net (φ_α) in Ω converges to zero in the topology $\sigma(B', B)$ if and only if (φ_α) converges to zero in the weak topology, $\sigma(F, F')$. On the other hand, no net (φ_α) in Ω converges to zero in the weak-$*$ topology $\sigma(B', B)$ of B' if and only if the closure $\overline{\Omega}$ of Ω in Φ_B is compact.

These comments prove the following result.

LEMMA 6.15. *Let F be a Banach space such that its dual $B = F'$ is a Banach function algebra. Suppose that the set $\Omega := \Phi_B \cap F$ is non-empty, and let $\overline{\Omega}$ be its closure in Φ_B. Then $\overline{\Omega}$ is compact as a subset of Φ_B if and only if Ω is weakly closed in F.* ■

Now let A be a Banach function algebra, and set $F = \overline{L(A)}$ and $\mathcal{Q}(A) = A''/F^{\perp}$, as before.

THEOREM 6.16. *Let A be a Banach function algebra. Then the following conditions are equivalent:*

(a) *Φ_A is weakly closed in A';*

(b) *the closure of Φ_A in $\Phi_{\mathcal{Q}(A)}$ is compact.*

Proof. This follows from Lemma 6.15, taking $B = \mathcal{Q}(A)$ and $\Omega = \Phi_A$. ■

PROPOSITION 6.17. *Let A be a Banach function algebra, and suppose that A has a bounded pointwise approximate identity. Then Φ_A is weakly closed in A'.*

Proof. By Proposition 6.4(iii), $\mathcal{Q}(A)$ has an identity, and so $\Phi_{\mathcal{Q}(A)}$ is compact. Hence, the closure of Φ_A in $\Phi_{\mathcal{Q}(A)}$ is compact, and so Φ_A is weakly closed in A' by Theorem 6.16. ■

We do not know whether the converse of the above proposition holds.

7. Examples

We now present some examples of classical Banach function algebras A, and describe the corresponding Banach function algebra $\mathcal{Q}(A) = A''/L(A)^{\perp}$ and its character space $\Phi_{\mathcal{Q}(A)}$.

7.1. Elementary examples. We first give examples that show that, for a Banach function algebra A, we can have $\mathcal{Q}(A) = A$ and that we can have $\mathcal{Q}(A) = A''$.

EXAMPLE 7.1. Let $A = (\ell^1, \cdot\,)$, the space of summable sequences with pointwise product, so that A is a natural, Tauberian Banach sequence algebra on $\mathbb{N}$, and hence an ideal in its bidual. Further, A is a dual Banach function algebra with predual c_0. Here

$$A' = \ell^{\infty} = C(\beta\mathbb{N})\,,$$

so that $L(A) = c_{00} \subset c_0$ and $A'' = M(\beta\mathbb{N})$. Thus $L(A)^{\perp} = c_0^{\perp} = M(\mathbb{N}^*)$, where $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ is the *growth* of $\mathbb{N}$ in $\beta\mathbb{N}$, and $A'' \cong A \oplus_1 L(A)^{\perp}$. As in equation (5.3), the product in A'' is given by

$$(\alpha, \mu) \,\Box\, (\beta, \nu) = (\alpha\beta, 0) \quad (\alpha, \beta \in \ell^1,\ \mu, \nu \in M(\mathbb{N}^*))\,.$$

Thus the algebra A is Arens regular, $\mathcal{Q}(A) = A$, and $\Phi_A = \Phi_{\mathcal{Q}(A)} = \mathbb{N}$. Also we have $A'' = \mathcal{Q}(A) \ltimes L(A)^{\perp}$ as a Banach algebra.

Since $L(A)_{[1]}$ is weak-$*$ dense in $A'_{[1]}$, the algebra A has a BSE norm; this also follows from Proposition 6.4(i). ■

EXAMPLE 7.2. Take α such that $0 < \alpha < 1$, and consider the Banach function algebras $A = \mathrm{lip}_{\alpha}\mathbb{I}$ and $\mathrm{Lip}_{\alpha}\mathbb{I}$ of Lipschitz functions on the closed interval $\mathbb{I}$, as in [12, §4.4]. Then the Lipschitz algebra A is Arens regular and $A'' = \mathrm{Lip}_{\alpha}\mathbb{I}$ [12, Theorem 4.4.34]. The Banach function algebras A and A'' are regular, natural, and self-adjoint on $\mathbb{I}$. However these algebras do not have the separating ball property, and maximal ideals in them do not have a bounded pointwise approximate identity. Here $L(A)^{\perp} = \{0\}$ and $\mathcal{Q}(A) = A''$, so that $\Phi_{\mathcal{Q}(A)} = \mathbb{I}$. ■

7.2. Uniform algebras. We now consider the case where A is a uniform algebra on a locally compact space.

EXAMPLE 7.3. Set $A = C_0(K)$, where K is a non-empty, locally compact space, so that A has the separating ball property. Then $A' = M(K)$ and $A'' = C(\widetilde{K})$, where $\widetilde{K}$, the hyper-Stonean envelope of K, is a hyper-Stonean space, as we noted earlier. Thus $\Phi_{A''} = \widetilde{K}$.

We recall that $M(K) = M_c(K) \oplus_1 M_d(K)$, where $M_c(K)$ and $M_d(K)$ denote the closed subspaces of $M(K)$ consisting of the continuous and discrete measures, respectively. We have $M_d(K) = \ell^1(K)$, and so $M_d(K)' = \ell^{\infty}(K) = C(\beta K_d)$, where K_d denotes the space K with the discrete topology and βS denotes the Stone–Čech compactification of a discrete space S. We regard βK_d as a clopen subspace of $\widetilde{K}$, and set $\widetilde{K}_c = \widetilde{K} \setminus \beta K_d$, so that $M_c(K)' = C(\widetilde{K}_c)$. For details of these remarks, see [13].

Here it is clear that $\overline{L(A)} \cong \ell^1(K)$, and hence that

$$\mathcal{Q}(A) = \ell^\infty(K) = C(\beta K_d) \quad \text{and} \quad \Phi_{\mathcal{Q}(A)} = \beta K_d\,.$$

Thus the predual of $\mathcal{Q}(A)$ is $\ell^1(K)$, and Φ_A and $\Phi_{\mathcal{Q}(A)}$ can be identified with K and βK_d, respectively, so that $\overline{\Phi_A} = \Phi_{\mathcal{Q}(A)}$ and Φ_A is the set of isolated points in $\Phi_{\mathcal{Q}(A)}$, in accord with Theorem 6.10(ii). It follows that

$$L(A)^\perp = \{F \in C(\widetilde{K}) : F \mid \beta K_d = 0\} = I(\beta K_d)$$

and $A'' = \mathcal{Q}(A) \ltimes L(A)^\perp$ as a Banach algebra. In fact, since βK_d is a clopen subset of $\widetilde{K}$, we can identify $\mathcal{Q}(A)$ with the closed ideal

$$\{F \in C(\widetilde{K}) : F \mid \widetilde{K}_c = 0\} = I(\widetilde{K}_c)$$

in A'', so that $\mathcal{Q}(A)$ is a uniform algebra, and hence is itself Arens regular.

Here the embedding of Φ_A in $\Phi_{\mathcal{Q}(A)}$ is continuous only in the special case that K is discrete. In particular, consider the case where $A = c_0$. Then

$$\mathcal{Q}(A) = A'' = \ell^\infty = C(\beta\mathbb{N})\,.$$

Again set $A = C(K)$ for a compact space K, so that $\mathcal{Q}(A) = C(\beta K_d)$. Then $\mathcal{Q}(\mathcal{Q}(A))$ is equal to $C(\beta((\beta K_d)_d))$, usually a far bigger space than $\mathcal{Q}(A)$.

The natural continuous surjection from $\widetilde{K}$ onto K_∞, the one-point compactification of K, is denoted by π_K. Take $x \in K_\infty$. Then

$$K_{\{x\}} = \pi_K^{-1}(\{x\}) = \{p \in \widetilde{K} : \pi_K(p) = x\}$$

is the *fibre* in $\widetilde{K}$ at x. Each fibre $K_{\{x\}}$ is a closed subspace of $\widetilde{K}$, and clearly we have $\widetilde{K} = \bigcup\{K_{\{x\}} : x \in K_\infty\}$. It is easy to see that, when $C_0(K)$ is regarded as a subspace of $C(\widetilde{K})$ via the canonical embedding, the space $C_0(K)$ consists of the functions $F \in C(\widetilde{K})$ such that $F \mid K_{\{x\}}$ is constant for each $x \in K$ and also such that $F \mid K_{\{\infty\}} = 0$. ∎

Example 7.4. Let A be a natural uniform algebra on a compact space K. Then A'' is a closed subalgebra of $C(\widetilde{K})$, and A is Arens regular; the canonical image of A in A'' consists of the functions in A'' that are constant on each fibre in $\widetilde{K}$. However A'' does not necessarily separate the points of $\widetilde{K}$, and so A'' may not be a uniform algebra on $\widetilde{K}$. The character space of A'' is again denoted by $\Phi_{A''}$, and we again regard K as a subset of $\Phi_{A''}$; its closure in $\Phi_{A''}$ is $\overline{K}$. For a study of the algebra A'' (for a special class of 'tight' uniform algebras), see [11].

Denote by $I = I(\overline{K})$ the closed ideal in A'' (when defined on $\Phi_{A''}$) consisting of all functions in A'' that vanish on $\overline{K}$, so that $\mathcal{Q}(A) = A''/I$. The hull of I in $\Phi_{A''}$ is $h(I)$, so that $\mathcal{Q}(A)$ is a natural Banach function algebra on $h(I)$. ∎

We now determine $\mathcal{Q}(A)$ and $\Phi_{\mathcal{Q}(A)}$ in the case where A is the disc algebra. We are greatly indebted to Professor Ken Davidson for some valuable explanations.

Example 7.5. Let $A = A(\overline{\mathbb{D}})$ be the disc algebra. Our main source for results that we use in this example is the book of Garnett [30].

We shall write H^1 for the Hardy space $H^1(\mathbb{D})$ that consists of the analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^1} := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right| \, d\theta < \infty ,$$

with H_0^1 for $\{f \in H^1 : f(0) = 0\}$. For these spaces, see [30, Chapter II and p. 133].

We regard A as a closed subalgebra of $C(\mathbb{T})$, and we denote the Lebesgue measure on $\mathbb{T}$ by m. The space of measures that are absolutely continuous with respect to m is identified with $L^1(\mathbb{T}, m)$ and $M_{sc}(\mathbb{T})$ is the space of continuous measures on $\mathbb{T}$ that are singular with respect to m, and then

$$C(\mathbb{T})' = M(\mathbb{T}) = L^1(\mathbb{T}, m) \oplus_1 \ell^1(\mathbb{T}) \oplus_1 M_{sc}(\mathbb{T}) .$$

The space $A^\perp$ is the annihilator of A in $M(\mathbb{T})$. The fact that $A^\perp$ can be identified as a closed subspace of $L^1(\mathbb{T}, m)$ is the classical F. and M. Riesz theorem for the disc algebra, and, as explained in [30, p. 133], this implies that $A^\perp$ can be identified with the space H_0^1 (see also [31, Theorem II.7.10]). Thus we conclude that

$$A' = (L^1(\mathbb{T}, m)/H_0^1) \oplus_1 \ell^1(\mathbb{T}) \oplus_1 M_{sc}(\mathbb{T}) . \tag{7.1}$$

By a theorem of Ando [2] that is given in [30, Theorem V.5.4], $L^1(\mathbb{T}, m)/H_0^1$ is the unique isometric predual of H^∞ as a Banach space. Thus it follows from equation (7.1) that

$$A'' = H^\infty \oplus_\infty \ell^\infty(\mathbb{T}) \oplus_\infty M_{sc}(\mathbb{T})'$$

as a Banach space.

Each character on A, given by $z \in \overline{\mathbb{D}}$, has a unique representing measure, say μ_z, on $\mathbb{T}$, so that

$$f(z) = \int_{\mathbb{T}} f \, d\mu_z \quad (f \in A) ;$$

see [30, p. 200].

Take $z \in \mathbb{T}$. Then the unique representing measure for ε_z is the point mass, δ_z, at z, and so $\overline{\mathrm{lin}}\,\{\varepsilon_z : z \in \mathbb{T}\} = \ell^1(\mathbb{T})$, i.e., $\overline{L(A, \mathbb{T})} = \ell^1(\mathbb{T})$.

Each element of $L(A, \overline{\mathbb{D}})'$ extends by Hahn–Banach to an element of A''. The restriction to $L(A, \mathbb{D})$ of each element of $\ell^\infty(\mathbb{T}) \oplus_\infty M_{sc}(\mathbb{T})'$ is the zero functional, and so $L(A, \mathbb{D})'$ is a identified with a subspace of H^∞. Now take $f \in H^\infty$. Then the map

$$\Lambda_f : \lambda = \sum_{i=1}^n \alpha_i \varepsilon_{z_i} \mapsto \sum_{i=1}^n \alpha_i f(z_i) , \quad L(A, \mathbb{D}) \to \mathbb{C} ,$$

is a linear functional on $L(A, \mathbb{D})$. Fix $\varepsilon > 0$. There exists $z \in \mathbb{D}$ with $|f(z)| > |f|_{\mathbb{D}} - \varepsilon$, and so $\|\Lambda_f\| \geq |\langle \Lambda_f, \varepsilon_z \rangle| = |f(z)| > |f|_{\mathbb{D}} - \varepsilon$, whence $\|\Lambda_f\| \geq |f|_{\mathbb{D}}$. Set $f_r(z) = f(rz) \ (z \in \mathbb{D})$ for $0 < r < 1$, so that $f_r \in A$, and take $\lambda \in L(A, \mathbb{D})$. Then $|\lambda(f_r)| \leq \|\lambda\| \, |f_r|_{\mathbb{D}} \ (0 < r < 1)$, and hence

$$|\langle \Lambda_f, \lambda \rangle| = \lim_{r \to 1} |\lambda(f_r)| \leq \|\lambda\| \, |f|_{\mathbb{D}} .$$

Thus $\Lambda_f \in L(A, \mathbb{D})'$ with $\|\Lambda_f\| = |f|_{\mathbb{D}}$, and the element of H^∞ corresponding to Λ_f is f. It follows that $L(A, \mathbb{D})' \cong H^\infty$.

It now follows from equation (7.1) that $\overline{L(A)} = (L^1(\mathbb{T}, m)/H^1_0) \oplus_1 \ell^1(\mathbb{T})$, and hence that

$$\mathcal{Q}(A) = L(A)' = H^\infty(\mathbb{D}) \oplus_\infty \ell^\infty(\mathbb{T})$$

as a Banach space.

Take $f \in \mathcal{Q}(A)$. It is clear that f is identified with the pair $(f \mid \mathbb{D}, f \mid \mathbb{T})$ in the space $H^\infty(\mathbb{D}) \oplus_\infty \ell^\infty(\mathbb{T})$ and that the product in this latter space is given by

$$(F_1, G_1)(F_2, G_2) = (F_1F_2, G_1G_2) \quad (F_1, F_2 \in H^\infty(\mathbb{D}),\ G_1, G_2 \in \ell^\infty(\mathbb{T}))\,,$$

and so $\mathcal{Q}(A)$ is identified with the uniform algebra $H^\infty(\mathbb{D}) \oplus_\infty \ell^\infty(\mathbb{T})$.

The character space of $\mathcal{Q}(A)$ is the disjoint union of Φ_{H^∞} and $\beta\,\mathbb{T}_d$. We recall that, by Carleson's corona theorem [30, Chapter VIII], the character space of H^∞ is a compact space containing $\mathbb{D}$ as a dense subset, and so the set $\overline{\mathbb{D}}$ is dense in $\Phi_{\mathcal{Q}(A)}$ and $\Phi_{\mathcal{Q}(A)}$ is exactly the hull of $L(A(\overline{\mathbb{D}}))^\perp$. ∎

The above proof can be generalized to apply to uniform algebras defined on suitable subsets of $\mathbb{C}^n$ using the techniques of [9], [11], and [48].

Example 7.6. In [26, Theorem 2.1], Feinstein constructed a separable, strongly regular, natural uniform algebra A on a compact space K such that there is a two-point Gleason part, say $P := \{x_1, x_2\}$, in K and such that all other points of K are peak points, and hence one-point Gleason parts. Here $\Gamma(A) = K$ and $\Gamma_0(A) = K \setminus P$.

Take a finite set F in $K \setminus \{x_1\}$; we may suppose that $x_2 \in F$, so that F has the form $\{x_2, x_3, \dots, x_n\}$, where $x_2, x_3, \dots, x_n$ are distinct points in K. Take $f_2 \in M_{x_2}$ with $f_2(x_1) = 1$, and set $m = |f_2|_K$. Fix $\varepsilon \in (0, 1)$, and take $\delta > 0$ such that $(1 - \delta)^n < \varepsilon$. For $j = 3, \dots, n$, take $f_j \in M_{x_j}$ with $|f_j(x_1) - 1| < \delta$ and $|f_j|_K = 1$, and then define $f = f_2 f_3 \cdots f_n \in A$, so that $|f|_K \le m$ and $|f(x_1) - 1| < (1 - \delta)^n < \varepsilon$. Finally, define $g = f(x_1)1_K - f$, so that $g \in M_{x_1}$ with $|g(x_j) - 1| < \varepsilon$ and $|g|_K \le m + 1 + \varepsilon$. It follows that M_{x_1} has a bounded pointwise approximate identity with bound $m + 1$. Similarly, M_{x_2} has a bounded pointwise approximate identity, and each other maximal ideal of the algebra A has a contractive pointwise approximate identity.

By Theorem 4.17(i), there exists an idempotent $E \in A''$ such that $\langle E,\, \varepsilon_x\rangle = 1\ \ (x \in P)$ and $\langle E,\, \varepsilon_y\rangle = 0\ \ (y \in K \setminus P)$. For $M \in A''$, consider the element

$$F_M = (M\,E - \langle M,\, \varepsilon_{x_1}\rangle\, E)\,(M - \langle M,\, \varepsilon_{x_2}\rangle\, E) \in A''\,.$$

As in Theorem 6.10, $\langle [F_M], \varepsilon_y\rangle = 0\ \ (y \in K)$, and this implies that $[F_M] = 0$ in $\mathcal{Q}(A)$. Take $\varphi \in \Phi_{\mathcal{Q}(A)} \setminus P$, so that $\langle [E], \varphi\rangle = 0$ or $\langle [E], \varphi\rangle = 1$, and assume towards a contradiction that $\langle [E], \varphi\rangle = 1$. Then there exists $M \in A''$ with $\langle [M], \varepsilon_{x_1}\rangle = \langle [M], \varepsilon_{x_2}\rangle = 1$ and $\langle [M], \varphi\rangle = 0$, and hence $\langle F_M,\, \varphi\rangle = 1$, a contradiction. We conclude that $[E]$ is the characteristic function of P in $\Phi_{\mathcal{Q}(A)}$, and so each of the two points x_1 and x_2 is isolated in $\Phi_{\mathcal{Q}(A)}$.

Certainly x_1 and x_2 are isolated in (K, d_A), but $\{x_1\}$ and $\{x_2\}$ are not one-point parts. As we remarked, Proposition 6.14(iii) implies that $\mathcal{Q}(A)$ is not a uniform algebra, but it is equivalent to a uniform algebra. It follows from Proposition 4.4 that the maximal ideal M_{x_1} does not have norm-one characters because it clearly does not have the weak separating ball property. Nevertheless, $\Phi_{M_{x_1}}$ is weakly closed in M'_{x_1}. Indeed, there exists

$F \in M''_{x_1}$ with $\langle F, \varepsilon_y \rangle = 1$ $(y \in K \setminus P)$, and this implies that $\Phi_{M_{x_1}}$ is weakly closed in M'_{x_1}. ∎

An example that is a development of the above example will give a uniform algebra A such that $\mathcal{Q}(A)$ is not even equivalent to a uniform algebra on $\Phi_{\mathcal{Q}(A)}$; see [15].

Example 7.7. In [50], Sidney constructed a natural, separable uniform algebra on a compact space K and a point $x \in K$ such that $\{x\}$ is a one-point Gleason part, but such that M_x^2 is not dense in M_x, and so M_x does not have an approximate identity.

In [32], it is shown that there is a natural, separable, regular uniform algebra A on a compact space K and $x \in K$ such that $\Gamma_0(A) = K \setminus \{x\}$ (so that each point of K is a one-point part, and hence A is pointwise contractive), but again M_x^2 is not dense in M_x.

In [14, Theorem 2.3], it is shown that there is a natural, separable uniform algebra on a compact metric space K such that each point of K is a one-point Gleason part, but $\Gamma(A) \subsetneq K$, so that A is not a Cole algebra, and hence not contractive.

Recall that it was shown in Proposition 4.21(i) that a maximal ideal M_x in a natural uniform algebra on a non-empty, compact space has a contractive pointwise approximate identity if and only if $\{x\}$ is a one-point part. Thus the above examples show that there are maximal ideals in uniform algebras that have a contractive pointwise approximate identity, but such that they do not have any approximate identity. ∎

7.3. Harmonic analysis. Here we give some examples related to Banach function algebras that arise in harmonic analysis. Again the group algebra of a locally compact group G is $(L^1(G), \star)$ and $A(\Gamma)$ and $B(\Gamma)$ are the Fourier and Fourier–Stieltjes algebras of a locally compact group Γ.

Example 7.8. Let G be a compact, abelian group, and set $A = (L^1(G), \star)$, so that $\Phi_A = \Gamma$, the dual group of G. Then Γ is discrete and $A(\Gamma)$ is a Tauberian Banach sequence algebra on Γ, and hence an ideal in its bidual. Also $A' = L^\infty(G)$, a C^*-algebra for the pointwise product, $\overline{L(A)} = C(G)$, and $A'' = M(G) \oplus_1 C(G)^\perp$, so that

$$\mathcal{Q}(A) = \mathcal{Q}(L^1(G)) = M(G)\,. \tag{7.2}$$

Since $(M(G), \star)$ is a closed subalgebra of $(A'', \Box)$, we again see that

$$A'' = \mathcal{Q}(A) \ltimes L(A)^\perp$$

as a Banach algebra. We have $\Phi_{\mathcal{Q}(A)} = \Gamma \cup H$, where H is the hull of A when A is considered as an ideal in $M(G)$. Here, the embedding of Φ_A into $\Phi_{\mathcal{Q}(A)}$ is continuous. Since $M(G)$ is a unital Banach algebra, $\Phi_{\mathcal{Q}(A)}$ is compact and so $\overline{\Phi_A}$ is also compact. In the case where G is infinite, the Wiener–Pitt phenomenon shows that $\overline{\Phi_A}$ is a proper subset of $\Phi_{\mathcal{Q}(A)}$. ∎

Example 7.9. Let G be a locally compact abelian group, with dual group Γ. Then bG, the dual group to Γ_d, is the compact abelian group that is the Bohr compactification of G, so that the Banach function algebra $B(\Gamma_d)$ is identified with $(M(bG), \star)$.

Set $A = (L^1(G), \star)$, identified with the Banach function algebra $A(\Gamma)$. We have noted that $\overline{L(A)}$ is identified with $AP(G)$, and hence with $C(bG)$.

As Banach spaces, we have

$$\mathcal{Q}(A) = L(A)' = C(bG)' = M(bG),$$

and $(\mathcal{Q}(A), \cdot)$ is Banach-algebra isometrically isomorphic to $(M(bG), \star)$ and hence to $(B(\Gamma_d), \cdot)$. ■

Example 7.10. Let Γ be a locally compact group. The dual space $A(\Gamma)'$ is identified with $VN(\Gamma)$, the group von Neumann algebra of Γ; see [38, Theorem 2.3.9]. For $x \in \Gamma$ and $h \in L^2(\Gamma)$, set

$$(\lambda_x h)(y) = h(x^{-1}y) \quad (y \in \Gamma),$$

so that $\lambda_x \in VN(\Gamma) \subset \mathcal{B}(L^2(\Gamma))$ and $VN(\Gamma)$ is the weak-operator closure of the space $\text{lin}\{\lambda_x : x \in \Gamma\}$ [24, Définition (3.9)]. Each operator λ_x acts as a character on $A(\Gamma)$, and indeed we can identify $\Phi_{A(\Gamma)}$ with $\{\lambda_x : x \in \Gamma\}$; for this, see [38, Lemma 2.3.1 and Theorem 2.3.8]. Thus $\overline{L(A(\Gamma))}$ can be identified with the C^*-subalgebra of $VN(\Gamma)$ generated by $\{\lambda_x : x \in \Gamma\}$. This latter C^*-algebra is denoted by $C^*_\delta(\Gamma)$ in the literature; see [43, §4] and [8]. Thus

$$\mathcal{Q}(A(\Gamma)) = C^*_\delta(\Gamma)'.$$

We shall make some remarks on the identification of $C^*_\delta(\Gamma)'$.

The reduced C^*-algebra of Γ is denoted by $C^*_\rho(\Gamma)$; when Γ is amenable, we have $C^*_\rho(\Gamma) = C^*(\Gamma)$. It was shown by Bédos in [4] that there is a natural surjection

$$R : C^*_\delta(\Gamma) \to C^*_\rho(\Gamma_d),$$

where Γ_d is the group Γ with the discrete topology, and so we can say that the algebra $\mathcal{Q}(A(\Gamma))$ contains the reduced Fourier–Stieltjes algebra $B_\rho(\Gamma_d) = C^*_\rho(\Gamma_d)'$, and hence

$$B_\rho(\Gamma_d) \subset \mathcal{Q}(A(\Gamma)).$$

In the two cases where Γ is discrete and where Γ_d is amenable, it is shown in [4, Theorem 3] that $C^*_\delta(\Gamma) = C^*_\rho(\Gamma_d)$. It follows that, in the case where Γ is discrete,

$$\mathcal{Q}(A(\Gamma)) = C^*_\delta(\Gamma)' = C^*_\rho(\Gamma)' = B_\rho(\Gamma),$$

and that, when Γ_d is amenable,

$$\mathcal{Q}(A(\Gamma)) = C^*_\delta(\Gamma)' = C^*_\rho(\Gamma_d)' = C^*(\Gamma_d)' = B(\Gamma_d).$$

These equations recover the previous example, equation (7.2), in the case where Γ is abelian and discrete.

In the case where the locally compact group Γ is amenable, the above map R is an injection if and only if Γ_d is amenable; in this case, we have $\mathcal{Q}(A(\Gamma)) = B_\rho(\Gamma_d)$. An extension of this result was given in [5, Theorem 1]: for an arbitrary locally compact group Γ, the map $R : C^*_\delta(\Gamma) \to C^*_\rho(\Gamma_d)$ is an isomorphism if and only if Γ contains an open subgroup Δ such that Δ_d is amenable. It seems to be an open question whether $\mathcal{Q}(A(\Gamma)) = B(\Gamma_d)$ in the case where $\Gamma = SO(3)$; in this case Γ is amenable, but Γ_d is not amenable.

We now give a separate, explicit identification of the isometric algebra isomorphism from $B(\Gamma_d)$ onto $\mathcal{Q}(A(\Gamma))$ in the case where Γ_d is amenable.

Indeed, let Γ be a locally compact group such that Γ_d is amenable. For $u \in B(\Gamma_d)$, define

$$L_u : \lambda \mapsto \langle u, \lambda \rangle, \quad C^*_\delta(\Gamma) \to \mathbb{C}.$$

Since $C^*_\delta(\Gamma) = C^*(\Gamma_d)$ and $C^*(\Gamma_d)' = B(\Gamma_d)$, we see that

$$\|L_u\| = \|u\| \quad (u \in B(\Gamma_d)).$$

Since $C^*_\delta(\Gamma)$ is a C^*-subalgebra of $VN(\Gamma)$, the functional L_u on $C^*(\Gamma_d)$ has a Hahn–Banach extension $\widetilde{u} \in VN(\Gamma)' = A(\Gamma)''$ with $\|\widetilde{u}\| = \|u\|$. Suppose that $v \in A(\Gamma)''$ is another such extension of u. Then $\widetilde{u} - v \in L(A(\Gamma))^\perp$, and so the map

$$\theta : u \mapsto [\widetilde{u}], \quad B(\Gamma_d) \to \mathcal{Q}(A(\Gamma)),$$

is a well-defined linear isometry, easily seen to be an algebra homomorphism.

To show that θ is a surjection, take $\mathrm{M} \in A(\Gamma)''$. Then there is a bounded net, say (f_α), in $A(\Gamma)$ that converges weak-$*$ to M. Since

$$A(\Gamma) \subset B(\Gamma) \subset B(\Gamma_d)$$

isometrically, the net (f_α) is bounded in $B(\Gamma_d) = C^*(\Gamma_d)'$, and so has a weak-$*$ accumulation point, say u, in $B(\Gamma_d)$, and $\theta(u) = [\mathrm{M}] \in \mathcal{Q}(A(\Gamma))$. Thus $\theta : B(\Gamma_d) \to \mathcal{Q}(A(\Gamma))$ is an isometric algebra isomorphism.

Again suppose that Γ is a locally compact group such that Γ_d is amenable. Then

$$\Phi_{\mathcal{Q}(A(\Gamma))} = \Phi_{B(\Gamma_d)},$$

and so $\Gamma = \Phi_{A(\Gamma)} \subset \Phi_{B(\Gamma_d)}$. Since $A(\Gamma_d)$ is a closed ideal in $B(\Gamma_d)$, it follows that $\Phi_{\mathcal{Q}(A(\Gamma))} = \Gamma_d \cup H$, where H is the hull of $A(\Gamma_d)$ considered as a closed ideal in $B(\Gamma_d)$, and $\Phi_{A(\Gamma)}$ is identified with the set of isolated points in $\Phi_{\mathcal{Q}(A(\Gamma))}$, as in Theorem 6.10. ∎

Note that Example 7.9 is a special case of the above example.

EXAMPLE 7.11. Let $\omega = (\omega_n : n \in \mathbb{Z})$ be a weight on the group $(\mathbb{Z}, +)$, so that the map $\omega : \mathbb{Z} \to [1, \infty)$ is such that $\omega_0 = 1$ and $\omega_{m+n} \le \omega_m \omega_n \ (m, n \in \mathbb{Z})$, and let B_ω be the weighted space $\ell^1(\mathbb{Z}, \omega)$, with convolution product $\star$, so that B_ω is a subalgebra of $(\ell^1(\mathbb{Z}), \star)$. The commutative Banach algebra B_ω is an example of a *Beurling algebra*; for a study of considerably more general versions of these algebras, see [12, §4.6] and [16].

The algebras B_ω can be identified with a natural Banach function algebra on a certain compact subspace of $\mathbb{C}$; the algebra B_ω is a dual Banach function algebra with predual

$$c_0(\mathbb{Z}, 1/\omega) = \{(\beta_n : n \in \mathbb{Z}) : \lim_{|n|\to\infty} |\beta_n| / \omega_n = 0\}.$$

In fact, we shall consider just the case where

$$\inf\{\omega_n^{1/n} : n \in \mathbb{N}\} = \sup\{\omega_{-n}^{-1/n} : n \in \mathbb{N}\} = 1, \tag{7.3}$$

so that B_ω is identified with a natural Banach function algebra on the unit circle $\mathbb{T}$. Suppose further that $\lim_{|n|\to\infty} \omega_n = \infty$ (for example, we can take $\omega = (\omega_n : n \in \mathbb{Z})$, where

$$\omega_n = (1 + |n|)^\alpha \quad (n \in \mathbb{Z})$$

for some $\alpha > 0$). Then each character on B_ω has the form $(\zeta^n : n \in \mathbb{Z})$ for some $\zeta \in \mathbb{T}$, and so $\Phi_{B_\omega} \subset c_0(\mathbb{Z}, 1/\omega)$. It follows from Proposition 5.4 that $\overline{L(B_\omega)} = c_0(\mathbb{Z}, 1/\omega)$ and that $\mathcal{Q}(B_\omega) = B_\omega$.

We are not clear on the identification of $\mathcal{Q}(B_\omega)$ when ω satisfies (7.3) and is unbounded, but it is not the case that $\lim_{|n|\to\infty} \omega_n = \infty$; such weights exist. ■

Example 7.12. Let Γ be a locally compact group, and take p such that $1 < p < \infty$. Then the Figà-Talamanca–Herz algebra $A = A_p(\Gamma)$ is defined in [12, Definition 4.5.29]; it was studied by Herz [33], and is described in the book of Derighetti [22, Chapter 3]. It is known that $(A_p(\Gamma), \,\cdot\,)$ is a natural, self-adjoint, strongly regular Banach function algebra on Γ with the separating ball property; for this, see [22, Chapter 3], [33, Propositions 2 and 3], and [57, Proposition 2.5].

As noted in [8], $\Phi_A = \Gamma$ is weakly closed in A' if and only if Γ is amenable, and so, by Theorem 6.16, the closure of Φ_A in $\Phi_{\mathcal{Q}(A)}$ is compact if and only if Γ is amenable.

Let Γ be a discrete, amenable group, and again set $A = A_p(\Gamma)$. Then A is an ideal in A'' and A has a bounded approximate identity. Let $B_\rho(\Gamma)$ be the multiplier algebra of A. Then, as in Example 7.8, we have $\mathcal{Q}(A) = B_\rho(\Gamma)$ and

$$A'' = B_\rho(\Gamma) \ltimes L(A)^\perp\,,$$

so that $\Phi_{\mathcal{Q}(A)} = \Gamma \cup H$, where H is the hull of A when A is regarded as an ideal in $B_\rho(\Gamma)$. ■

8. Existence of contractive pointwise approximate identities

As we stated in the introduction, we are interested in finding necessary and sufficient conditions on a Banach function algebra to have a bounded pointwise approximate identity and, especially, a contractive pointwise approximate identity. We shall obtain such conditions in this section, and then give some applications.

Recall that a Banach function algebra A with a contractive pointwise approximate identity has norm-one characters.

THEOREM 8.1. *Let A be a Banach function algebra. Then A has a contractive pointwise approximate identity if and only if $\|\lambda\| = 1$ $(\lambda \in \operatorname{co}\Phi_A)$.*

Proof. Certainly $\|\lambda\| \leq 1$ $(\lambda \in \operatorname{co}\Phi_A)$.

Suppose that A has a CPAI. Then there exists an element $\mathrm{E} \in A''$ such that $\|\mathrm{E}\| = 1$ and $\langle \mathrm{E},\, \varphi\rangle = 1$ $(\varphi \in \Phi_A)$. Take $\lambda \in \operatorname{co}\Phi_A$. Then $\langle \mathrm{E},\, \lambda\rangle = 1$, and so $\|\lambda\| \geq 1$. Hence $\|\lambda\| = 1$.

Conversely, suppose that $\|\lambda\| = 1$ $(\lambda \in \operatorname{co}\Phi_A)$. Then, for each $\lambda \in \operatorname{co}\Phi_A$, there exists $\mathrm{M} \in A''$ with $\|\mathrm{M}\| = 1$ and $\langle \mathrm{M},\, \lambda\rangle = 1$. In particular, take $n \in \mathbb{N}$ and a subset $S = \{\varphi_1, \dots, \varphi_n\}$ of Φ_A. Set $\lambda_S = (\sum_{j=1}^{n} \varphi_j)/n \in \operatorname{co}\Phi_A$. Since $\|\lambda_S\| = 1$, there exists $\mathrm{M}_S \in A''$ with $\|\mathrm{M}_S\| = 1$ and $\langle \mathrm{M}_S,\, \lambda_S\rangle = 1$. Since $|\langle \mathrm{M}_S,\, \varphi\rangle| \leq 1$ $(\varphi \in \Phi_A)$, the latter is possible only if $\langle \mathrm{M}_S,\, \varphi_j\rangle = 1$ $(j \in \mathbb{N}_n)$. Thus, for each non-empty, finite subset S of Φ_A, there is $\mathrm{M}_S \in A''_{[1]}$ with $\langle \mathrm{M}_S,\, \varphi\rangle = 1$ $(\varphi \in S)$, and this easily gives a CPAI in A. ∎

As a first application of the above theorem, we present the following result.

Let A and B be Banach function algebras, and suppose that $\theta : A \to B$ is a homomorphism. We recall the standard fact [12, Theorem 2.3.3] that θ is automatically continuous, and so it has a dual map $\theta' : B' \to A'$ that restricts to a continuous map $\theta' : \Phi_B \to \Phi_A \cup \{0\}$.

THEOREM 8.2. *Let A and B be Banach function algebras, and suppose that $\theta : A \to B$ is a monomorphism with dense range such that $\theta'(\Phi_B) = \Phi_A$. Then the map $\theta' : \overline{L(B)} \to \overline{L(A)}$ is a surjective isometry if and only if*

$$\|\theta(f)\|_{\mathrm{BSE},\, B} = \|f\|_{\mathrm{BSE},\, A} \quad (f \in A)\,. \tag{8.1}$$

In this case, $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are isometrically isomorphic as Banach algebras.

Proof. Since $\overline{\theta(A)} = B$, it follows that $\theta(A)$ is also dense in $(B, \|\cdot\|_{\mathrm{BSE},\, B})$, and so $\{\theta(f) : f \in A,\ \|\theta(f)\|_{\mathrm{BSE},\, B} \leq 1\}$ is dense in the set $\{g \in B : \|g\|_{\mathrm{BSE},\, B} \leq 1\}$.

Now suppose that equation (8.1) holds. It follows that $\{\theta(f) : f \in A,\ \|f\|_{\mathrm{BSE},\, A} \leq 1\}$ is dense in the set $\{g \in B : \|g\|_{\mathrm{BSE},\, B} \leq 1\}$.

Next take $\lambda \in L(B)$, so that $\theta'(\lambda) \in L(A)$. By Proposition 5.8,

$$\|\theta'(\lambda)\|_{A'} = \sup\{|\langle f,\, \theta'(\lambda)\rangle| : f \in A,\ \|f\|_{\mathrm{BSE},\, A} \leq 1\}\,,$$

and so

$$\|\theta'(\lambda)\|_{A'} = \sup\{|\langle g,\, \lambda\rangle| : g \in B,\ \|g\|_{\mathrm{BSE},\, B} \leq 1\}\,.$$

Thus, again by Proposition 5.8, $\|\theta'(\lambda)\|_{A'} = \|\lambda\|_{B'}$, and so $\theta' : \overline{L(B)} \to \overline{L(A)}$ is an isometry. Since $\theta'(\Phi_B) = \Phi_A$, this map is a surjection.

The converse is immediate from the definitions.

Set $\mu = \theta' \mid \overline{L(B)} : \overline{L(B)} \to \overline{L(A)}$. In the case where μ is a surjective isometry, the dual map $\mu' : L(A)' \to L(B)'$ is also a surjective isometry, and so $\mu' : \mathcal{Q}(A) \to \mathcal{Q}(B)$ is also a surjective isometry. Moreover, μ' is a homomorphism because θ is a homomorphism. Hence $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ are isometrically isomorphic as Banach algebras. ■

COROLLARY 8.3. *Let A and B be Banach function algebras, and suppose that $\theta : A \to B$ is a monomorphism with dense range such that $\theta'(\Phi_B) = \Phi_A$ and such that*

$$\|\theta(f)\|_{\mathrm{BSE},\,B} = \|f\|_{\mathrm{BSE},\,A} \quad (f \in A)\,.$$

Then A has a contractive pointwise approximate identity if and only if B has a contractive pointwise approximate identity.

Proof. This now follows from Theorems 8.1 and 8.2. ■

THEOREM 8.4. *Let A be a Banach function algebra such that $\|f\|_{\mathrm{BSE}} = |f\,|_{\Phi_A}$ $(f \in A)$. Then A has a contractive pointwise approximate identity if and only if A has norm-one characters.*

Proof. Suppose that A has norm-one characters. Take B to be the uniform closure of A in $C_0(\Phi_A)$, so that $\Phi_B = \Phi_A$ and B has norm-one characters. Since $\|\cdot\|_{\mathrm{BSE},\,B} = |\cdot|_{\Phi_A}$, it follows from the hypothesis that $\|f\|_{\mathrm{BSE},\,B} = \|f\|_{\mathrm{BSE},\,A}$ $(f \in A)$. By Corollary 8.3, A has a CPAI if and only if B has a CPAI. By Theorem 4.19, B has a CPAI, and so A has a CPAI.

The converse is immediate. ■

Let $(A, \|\cdot\|_A)$ be a Banach function algebra. Then $(A, \|\cdot\|_{\mathrm{BSE}})$ is complete if and only if A has a BSE norm, and this is not always the case. Let $(B, \|\cdot\|_B)$ be the Banach function algebra that is the completion of $(A, \|\cdot\|_{\mathrm{BSE}})$, so that $\Phi_B = \Phi_A$ and $L(B) = L(A)$. It follows from equation (5.4) that $\|\lambda\|_{B'} = \|\lambda\|_{A'}$ $(\lambda \in L(A))$, and so

$$\|f\|_{\mathrm{BSE},A} = \|f\|_B = \|f\|_{\mathrm{BSE},B} \quad (f \in A)\,.$$

Let $\theta : A \to B$ be the identity map. Then equation (8.1) is satisfied, and so Theorem 8.2 applies, and so we have the following corollary, which also uses Corollary 8.3.

COROLLARY 8.5. *Let A be a Banach function algebra, and let B be the Banach function algebra that is the completion of $(A, \|\cdot\|_{\mathrm{BSE}})$. Then*

$$(\mathcal{Q}(A), \|\cdot\|_{\mathcal{Q}(A)}) = (\mathcal{Q}(B), \|\cdot\|_{\mathcal{Q}(B)})\,,$$

and B has a contractive pointwise approximate identity if and only if A has a contractive pointwise approximate identity. ■

The following isomorphic form of Theorem 8.2 also holds.

PROPOSITION 8.6. *Let A and B be Banach function algebras, and suppose that $\theta : A \to B$ is a monomorphism with dense range such that $\theta'(\Phi_B) = \Phi_A$. Suppose that A and B both have* BSE *norms. Then $\theta' : \overline{L(B)} \to \overline{L(A)}$ is an isomorphism if and only if θ is a surjection.* ■

For example, let Γ_1 and Γ_2 be two locally compact groups, and suppose that there is a monomorphism $\theta : A(\Gamma_1) \to A(\Gamma_2)$ such that θ has dense range. Since $A(\Gamma_1)$ is a regular Banach function algebra, it follows that $\theta'(\Phi_{A(\Gamma_2)}) = \Phi_{A(\Gamma_1)}$. Thus Proposition 8.6 shows that $\theta' : C'_\delta(\Gamma_2) \to C'_\delta(\Gamma_1)$ is an isomorphism if and only if θ is a surjection.

Recall that, whenever A is a natural Banach function algebra on a locally compact space K and S is Segal algebra with respect to A, the Banach function algebra S is also a natural Banach function algebra on K, and so we can regard $L(A)$ as a subspace of both A' and S'.

Let S be a Segal algebra with respect to a Banach function algebra A, and take $j : L(A) \to L(S)$ to be the identity mapping, so that j is a contraction. The adjoint map is $j' : L(S)' \to L(A)'$, so that

$$j'(\mathrm{M} + L(S)^\perp) = \mathrm{M} + L(A)^\perp \quad (\mathrm{M} \in S'')\,.$$

(In the above equation, the M on the left is an element of $S'' \subset A''$, but the M on the right is regarded as an element of A''.) The map $j' : \mathcal{Q}(S) \to \mathcal{Q}(A)$ is a contractive algebra homomorphism, and it is clearly injective. Thus $j' : \mathcal{Q}(S) \to \mathcal{Q}(A)$ is a Banach-algebra monomorphism, but, in general, the image $j'(\mathcal{Q}(S))$ is not dense in $\mathcal{Q}(A)$. Clearly we have

$$j(\Phi_A) = \Phi_S \quad \text{and} \quad j''(\Phi_{\mathcal{Q}(A)}) \subset \Phi_{\mathcal{Q}(S)} \cup \{0\}\,.$$

In the case where $\Phi_{\mathcal{Q}(A)}$ is compact, $j''(\,\overline{\Phi_A}\,)$ is a compact subset of $\Phi_{\mathcal{Q}(S)} \cup \{0\}$, but it can be that $0 \in j''(\,\overline{\Phi_A}\,)$. For example, take $A = c_0$ and $S = \ell^1$, so that $\Phi_{\mathcal{Q}(A)} = \beta\mathbb{N}$ and $\Phi_{\mathcal{Q}(S)} = \mathbb{N}$.

The following theorem implies that, in the case where S has a contractive pointwise approximate identity, the map $j : L(A) \to L(S)$ is an isometry, and so $j' : \mathcal{Q}(S) \to \mathcal{Q}(A)$ is an isometric Banach-algebra isomorphism.

THEOREM 8.7. *Let $(A, \|\cdot\|_A)$ be a Banach function algebra, and suppose that $(S, \|\cdot\|_S)$ is a Segal algebra with respect to A. Then the following are equivalent:*

(a) *S has a contractive pointwise approximate identity;*

(b) *A has a contractive pointwise approximate identity and*

$$\|\lambda\|_{S'} = \|\lambda\|_{A'} \quad (\lambda \in L(A))\,. \tag{8.2}$$

(c) *A has a contractive pointwise approximate identity and*

$$\|f\|_{\mathrm{BSE},S} = \|f\|_{\mathrm{BSE},A} \quad (f \in S)\,. \tag{8.3}$$

Proof. (a) ⇒ (b) Suppose that S has a CPAI, say (e_α). Then the net (e_α) is also a CPAI for A because $\|f\|_A \le \|f\|_S$ $(f \in S)$. Also we clearly have $\|\lambda\|_{S'} \le \|\lambda\|_{A'}$ $(\lambda \in L(A))$. Now take $\lambda \in L(A)$ and $\varepsilon > 0$. Then there exists $f \in A_{[1]}$ with $|\langle f, \lambda\rangle| > \|\lambda\|_{A'} - \varepsilon$. The net $(e_\alpha f)$ is in S, and $\|e_\alpha f\|_S \le \|e_\alpha\|_S \, \|f\|_A \le 1$, and so

$$|\langle f, \lambda\rangle| = \lim_\alpha |\langle e_\alpha f, \lambda\rangle| \le \limsup_\alpha \|e_\alpha f\|_S \, \|\lambda\|_{S'} \le \|\lambda\|_{S'}\ ,$$

which shows that $\|\lambda\|_{A'} - \varepsilon < \|\lambda\|_{S'}$. It follows that $\|\lambda\|_{A'} \le \|\lambda\|_{S'}$, and then equation (8.2) follows.

(b) ⇒ (a) By Theorem 8.1, $\|\lambda\|_{A'} = 1$ $(\lambda \in \mathrm{co}\,\Phi_A)$, and so, immediately from equation (8.2), $\|\lambda\|_{S'} = 1$ $(\lambda \in \mathrm{co}\,\Phi_A)$. Hence, by Theorem 8.1 again, S has a CPAI.

(b) ⇒ (c) By the definition of the BSE norm, (8.2) implies (8.3).

(c) ⇒ (b) It follows from equation (8.3) that the set $\{f \in S : \|f\|_{\mathrm{BSE},\,A} \le 1\}$ is dense in the set $\{g \in A : \|g\|_{\mathrm{BSE},\,A} \le 1\}$. It now follows from equation (5.4) that equation (8.2) also holds, giving (b). ∎

The above result has some rather unexpected consequences; the next theorem is a main result of this work.

THEOREM 8.8. *Let A be a Banach function algebra, and let S_1 and S_2 be two Segal algebras with respect to A. Suppose that S_1 and S_2 both have contractive pointwise approximate identities. Then*

$$(\overline{L(S_1)}, \|\cdot\|_{S_1'}) = (\overline{L(S_2)}, \|\cdot\|_{S_2'}) = (\overline{L(A)}, \|\cdot\|_{A'})$$

as Banach spaces and

$$(\mathcal{Q}(S_1), \|\cdot\|_{\mathcal{Q}(S_1)}) = (\mathcal{Q}(S_2), \|\cdot\|_{\mathcal{Q}(S_2)}) = (\mathcal{Q}(A), \|\cdot\|_{\mathcal{Q}(A)})\,. \tag{8.4}$$

Proof. By the above theorem, the identifications of $L(S_1)$ and $L(S_2)$ with $L(A)$ are both isometries, and so $\overline{L(S_1)} = \overline{L(S_2)}$, whence $L(S_1)' = L(S_2)' = L(A)'$. Thus the identifications of $\mathcal{Q}(S_1)$ and $\mathcal{Q}(S_2)$ with $\mathcal{Q}(A)$ are both Banach-algebra isometries, and hence equation (8.4) follows. ∎

COROLLARY 8.9. *Let G be a locally compact abelian group, and let S be a Segal algebra with respect to $L^1(G)$. Suppose that S has a contractive pointwise approximate identity. Then $(\overline{L(S)}, \|\cdot\|_{S'}) \cong (AP(G), |\cdot|_G)$ and*

$$(\mathcal{Q}(S), \|\cdot\|_{\mathcal{Q}(S)}) = (M(bG), \|\cdot\|)\,.$$

Further, $\|f\|_{\mathrm{BSE},S} = \|f\|_1$ $(f \in S)$, and hence the Banach function algebra S has a BSE *norm if and only if $S = L^1(G)$.*

Proof. By the Bochner–Schoenberg–Eberlein theorem, $\|f\|_{\mathrm{BSE},L^1(G)} = \|f\|_1$ for each $f \in L^1(G)$), and so this follows from Theorems 8.7 and 8.8 and earlier examples. ∎

Thus, in the case where the Segal algebra S has a contractive pointwise approximate identity, the biduals of S and $L^1(G)$ are equal modulo the ideals $L(S)^\perp$ and $AP(G)^\perp$, respectively.

EXAMPLES 8.10. (i) Set $A = c_0$ and $S = \ell^1$, regarded as natural Banach sequence algebras on $\mathbb{N}$, so that S is a Segal algebra with respect to A. Then A has an obvious contractive approximate identity, but S does not have a bounded pointwise approximate identity; this easy example shows that a Segal algebra with respect to a Banach function algebra that has a contractive pointwise approximate identity need not itself have a contractive pointwise approximate identity.

(ii) Let G be a locally compact abelian group that is neither discrete nor compact; the dual group is Γ. Take p with $1 < p < \infty$.

Set $S_1 = L^1(G) \cap L^p(G)$, taken with the norm given by

$$\|f\|_{S_1} = \max\{\|f\|_1\,, \|f\|_p\} \quad (f \in S_1)\,.$$

Also set $S_2 = \{f \in L^1(G) : \widehat{f} \in L^p(\Gamma)\}$, taken with the norm given by

$$\|f\|_{S_2} = \max\left\{\|f\|_1\,, \left\|\widehat{f}\right\|_p\right\} \quad (f \in S_2)\,.$$

Then S_1 and S_2 are both Segal algebras with respect to $L^1(G)$ [12, Examples 4.5.27].

Neither S_1 nor S_2 has a bounded approximate identity, but it is proved in [35] that both S_1 and S_2 have contractive pointwise approximate identities. Thus we can conclude from the above results that $\overline{L(S_1)} \cong \overline{L(S_2)}$ and

$$(\mathcal{Q}(S_1), \|\cdot\|_{\mathcal{Q}(S_1)}) = (\mathcal{Q}(S_2), \|\cdot\|_{\mathcal{Q}(S_2)}) = (M(bG), \|\cdot\|)\,,$$

that

$$\left\|f + L(S)^{\perp}\right\| = \left\|f + AP(G)^{\perp}\right\| \quad (f \in S)$$

for $S = S_1$ and $S = S_2$, and that neither S_1 nor S_2 has a BSE norm. Indeed, by Theorem 8.7 and the fact that the BSE norm for $L^1(G)$ is equal to the given norm, we have

$$\|f\|_{\mathrm{BSE},S} = \|f\|_1 \quad (f \in S)$$

for $S = S_1$ and $S = S_2$. ■

9. ℓ^1-norms on $L(A)$

In this section, our aim is to compare the spaces $\overline{L(A)}$ and $\ell^1(\Phi_A)$ for a Banach function algebra A, and determine when these two Banach spaces are mutually isometric or isomorphic. Equality of these two spaces is closely related to the equality of the two norms $|\cdot|_{\Phi_A}$ and $\|\cdot\|_{\mathrm{BSE}}$, and is connected to the weak separating ball property.

Let A be a Banach function algebra. Then there is a natural contraction

$$\iota : f \mapsto \sum\{f(\varphi)\,\varphi : (\varphi \in \Phi_A)\}, \quad (\ell^1(\Phi_A), \|\cdot\|_1) \to (A', \|\cdot\|)\,.$$

Clearly $\iota(\ell^1(\Phi_A))$ contains $L(A)$ and is contained in $\overline{L(A)}$, and so $\iota(\ell^1(\Phi_A))$ is a dense subspace of $\overline{L(A)}$. However, the map ι is not always an injection: it may be that there exist a sequence (φ_n) in Φ_A and an element $\alpha = (\alpha_n) \in \ell^1$ such that $\sum_{n=1}^\infty \alpha_n\varphi_n = 0$ (with convergence of the sum in A'), but with $\alpha \neq 0$. For example, this occurs in the case where A is the disc algebra $A(\mathbb{D})$. For a discussion of this point, see [27]. Fortunately, the following lemma shows that this difficulty does not arise in the cases of interest to us.

Recall from Corollary 4.4(i) that a Banach function algebra that has a bounded pointwise approximate identity and has the separating ball property is such that every non-zero, maximal modular ideal has a bounded pointwise approximate identity.

Lemma 9.1. *Let A be a Banach function algebra such that M_φ is non-zero and has a bounded pointwise approximate identity for each $\varphi \in \Phi_A \cup \{\infty\}$. Take a set $\{\alpha_\varphi : \varphi \in \Phi_A\}$ in $\mathbb{C}$ with*

$$\sum\{|\alpha_\varphi| : \varphi \in \Phi_A\} < \infty \quad \text{and} \quad \sum\{\alpha_\varphi\,\varphi : \varphi \in \Phi_A\} = 0\,.$$

Then $\alpha_\varphi = 0 \ \ (\varphi \in \Phi_A)$.

Proof. Assume towards a contradiction that there exists $\varphi_0 \in \Phi_A$ such that $\alpha_{\varphi_0} \neq 0$.

Since both A and M_{φ_0} have bounded pointwise approximate identities, it follows from Proposition 3.6 that there exists an element $\mathrm{M} \in A''$ such that $\langle \mathrm{M},\,\varphi\rangle = \delta_{\varphi,\varphi_0}\ \ (\varphi \in \Phi_A)$. Then

$$0 = \left\langle \mathrm{M},\, \sum_{\varphi\in\Phi_A} \alpha_\varphi\varphi \right\rangle = \sum_{\varphi\in\Phi_A} \alpha_\varphi\langle \mathrm{M},\,\varphi\rangle = \alpha_{\varphi_0}\,,$$

a contradiction. Thus $\alpha_\varphi = 0\ \ (\varphi \in \Phi_A)$. ∎

It follows that, in the case where A satisfies the conditions of the above lemma, the map $\iota : \ell^1(\Phi_A) \to A'$ is an injection, and so we can regard $\ell^1(\Phi_A)$ as a subspace of $\overline{L(A)}$. For $\lambda = \iota(f)$, where $f \in \ell^1(\Phi_A)$, we set $\|\lambda\|_1 = \|f\|_1$; in particular, for an element $\lambda = \sum_{i=1}^n \alpha_i\varphi_i \in L(A)$, we have $\|\lambda\|_1 = \sum_{i=1}^n |\alpha_i|$.

Proposition 9.2. *Let A be a Banach function algebra, and suppose that A is pointwise contractive. Then $\|f\|_{\mathrm{BSE}} \leq 4\,|f|_{\Phi_A}\ \ (f \in A)$.*

Proof. Take $\lambda = \sum_{i=1}^n \alpha_i\varphi_i \in L(A)$, and take $\varepsilon > 0$. Since A is pointwise contractive, Proposition 3.8(ii) shows that there is $f \in A_{[4]}$ such that $|\alpha_i f(\varphi_i) - |\alpha_i|\,| < \varepsilon\ \ (i \in \mathbb{N}_n)$. Hence

$$\|\lambda\|_1 = \sum_{i=1}^n |\alpha_i| \leq |\langle f,\,\lambda\rangle| + n\varepsilon \leq \|f\|\,\|\lambda\| + n\varepsilon \leq 4\,\|\lambda\| + n\varepsilon\,.$$

Thus $\|\lambda\|_1 \leq 4\,\|\lambda\|$.

Now take $f \in A$. For each $\varepsilon > 0$, there exists $\lambda = \sum_{i=1}^{n} \alpha_i \varphi_i$ in $L(A)_{[1]}$ such that $|\langle f,\, \lambda \rangle| > \|f\|_{\mathrm{BSE}} - \varepsilon$. and then

$$\|f\|_{\mathrm{BSE}} \leq \left| \sum_{i=1}^{n} \alpha_i f(\varphi_i) \right| + \varepsilon \leq \sum_{i=1}^{n} |\alpha_i|\,|f|_{\Phi_A} + \varepsilon = \|\lambda\|_1\,|f|_{\Phi_A} + \varepsilon \leq 4\,|f|_{\Phi_A} + \varepsilon\,,$$

and so $\|f\|_{\mathrm{BSE}} \leq 4\,|f|_{\Phi_A}$, giving the result. ∎

We can now give our main classification theorem for unital Banach function algebras that have a BSE norm.

Theorem 9.3. *Let A be a unital Banach function algebra with a* BSE *norm.*

(i) *Suppose that A is contractive. Then A is equivalent to a Cole algebra.*

(ii) *Suppose that A is pointwise contractive. Then A is equivalent to a uniform algebra for which each singleton in Φ_A is a one-point part.*

Proof. The result is trivial when $|\Phi_A| = 1$ (and then A is a Cole algebra), and so we may suppose that $|\Phi_A| \geq 2$.

In both cases, A is pointwise contractive, and so, by Proposition 9.2, A is equivalent to a uniform algebra.

(i) Since A is contractive, $(A, |\cdot|_{\Phi_A})$ is also contractive, and so it is a Cole algebra by Corollary 4.14.

(ii) Since A is pointwise contractive, $(A, |\cdot|_{\Phi_A})$ is also pointwise contractive, and so each singleton in Φ_A is a one-point part by Proposition 4.21(ii). ∎

It is not true that every contractive or pointwise contractive Banach function algebra is necessarily equivalent to a uniform algebra; this will be shown in Examples 9.11 and 9.12. In this case, of course, the algebras cannot have a BSE norm.

Proposition 9.4. *Let A be a self-adjoint Banach function algebra. Then the linear map $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is a surjective isometry if and only if $\|f\|_{\mathrm{BSE}} = |f|_{\Phi_A}$ $(f \in A)$.*

Proof. The embedding of A into $C_0(\Phi_A)$ is a continuous monomorphism with dense range, and so Theorem 8.2 applies. The result follows because $\overline{L(C_0(\Phi_A))} \cong \ell^1(\Phi_A)$. ∎

Proposition 9.5. *Let A be a pointwise contractive, unital uniform algebra. Then the linear map $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is a surjective isometry.*

Proof. Consider $\lambda = \sum_{i=1}^{n} \alpha_i \varphi_i \in L(A)_{[1]}$, and take $\varepsilon > 0$. By Proposition 3.10, there exists $f \in A_{[1]}$ with $|\alpha_i f(x_i) - |\alpha_i|\,| < \varepsilon$ $(i \in \mathbb{N}_n)$ and then, as in Proposition 9.2, $\|\lambda\|_1 = \|\lambda\|$. Thus the specified map is a surjective isometry. ∎

Corollary 9.6. *Let A be a natural uniform algebra on a non-empty, compact space K. Then the following are equivalent:*

(a) *each singleton in K is a one-point Gleason part;*

(b) $(\mathcal{Q}(A), \|\cdot\|_{\mathcal{Q}(A)}) = (\ell^\infty(K), |\cdot|_K)$.

In particular, $\mathcal{Q}(A)$ is a uniform algebra in this case.

Proof. (a) $\Rightarrow$ (b) By Proposition 4.21(ii), A is pointwise contractive, and so the map $\iota : \ell^1(K) \to \overline{L(A)}$ is a surjective isometry by Proposition 9.5. Thus

$$\iota' : \mathcal{Q}(A) = \overline{L(A)}' \to \ell^\infty(K)$$

is a surjective isometry. Since $\mathcal{Q}(A)$ is a subalgebra of $\ell^\infty(K)$, (b) follows.

(b) $\Rightarrow$ (a) Take $x, y \in K$ with $x \neq y$. Then there exists $f \in \ell^\infty(K)_{[1]}$ with $f(x) = 1$ and $f(y) = -1$. By (b), $f \in \mathcal{Q}(A)_{[1]}$, and so there exists $F \in A''_{[1]}$ with $F \mid K = f$. This shows that $\|\varepsilon_x - \varepsilon_y\| = 2$, and so $x \not\sim y$. Thus (a) follows. ∎

PROPOSITION 9.7. *Let A be a Tauberian Banach sequence algebra on a non-empty set S such that A has a bounded pointwise approximate identity. Then the following conditions on A are equivalent:*

(a) *the linear map $\iota : \ell^1(S) \to \overline{L(A)}$ is an isomorphism;*

(b) *$A = c_0(S)$.*

Proof. The norm on A is denoted by $\|\cdot\|$.

(a) $\Rightarrow$ (b) There is a constant $m > 0$ such that $\|\lambda\|_1 \leq m\|\lambda\|$ $(\lambda \in L(A))$. Take $f \in A$. As in the proof of Proposition 9.2, $\|f\|_{\mathrm{BSE}} \leq m\,|f|_{\Phi_A}$, and so $\|\cdot\|_{\mathrm{BSE}} \sim |\cdot|_S$ on A. By Corollary 5.13(i), A has a BSE norm, and so $\|\cdot\| \sim \|\cdot\|_{\mathrm{BSE}}$ on A. Thus $|\cdot|_S$ and $\|\cdot\|$ are equivalent on A. Since A is dense in $c_0(S)$, necessarily $A = c_0(S)$.

(b) $\Rightarrow$ (a) Since $A = c_0(S)$, it follows that $\iota : \ell^1(S) \to A'$ is an isomorphism. As before, $\overline{L(A)} = \ell^1(S)$, and so $A' = \overline{L(A)}$, giving (a). ∎

The following is a further main theorem of this work.

THEOREM 9.8. *Let A be a Banach function algebra. Then the following conditions on A are equivalent:*

(a) *the linear map $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is an isometric surjection;*

(b) *A has the weak separating ball property and $\|f\|_{\mathrm{BSE}} = |f|_{\Phi_A}$ $(f \in A)$;*

(c) *A is pointwise contractive and $\|f\|_{\mathrm{BSE}} = |f|_{\Phi_A}$ $(f \in A)$.*

In the case where $|\Phi_A| \geq 2$, the above conditions are also equivalent to:

(d) *for each $\varphi \in \Phi_A$, the maximal modular ideal M_φ has norm-one characters and $\|f\|_{\mathrm{BSE}} = |f|_{\Phi_A}$ $(f \in A)$.*

Proof. Take B to be the uniform closure of A in $C_0(\Phi_A)$, so that B is a natural uniform algebra on Φ_A. Then the embedding of A into B is a continuous monomorphism with dense range.

(a) $\Rightarrow$ (c) Take $\varphi \in \Phi_A \cup \{\infty\}$, and consider the corresponding ideal M_φ in A.

Define $F_\varphi \in \ell^\infty(\Phi_A) \cong \ell^1(\Phi_A)'$ to be the characteristic function of $\Phi_A \setminus \{\varphi\}$, so that $\|F_\varphi\|_\infty = 1$. Then, as a continuous linear functional on $(L(A), \|\cdot\|)$, we have $\|F_\varphi\| = 1$ because $\overline{L(A)} \cong \ell^1(\Phi_A)$. Extend F_φ to be an element F_φ of A'' with $\|F_\varphi\| = 1$. Then there is a net (f_ν) in $A_{[1]}$ that converges weak-$*$ to F_φ, and we may suppose that (f_ν) is in $(M_\varphi)_{[1]}$. Clearly (f_ν) is a CPAI in M_φ, and hence A is pointwise contractive.

Take $f \in A$. Then

$$\|f\|_{\mathrm{BSE}} = \sup\{|\langle f,\, \lambda\rangle| : \lambda \in L(A)_{[1]}\} = \sup\{|\langle f,\, \lambda\rangle| : \lambda \in \ell^{\,1}(\Phi_A)_{[1]}\} = |f|_{\Phi_A}\ ,$$

as required.

(c) $\Rightarrow$ (b) This is immediate.

(b) $\Rightarrow$ (a) Since A has the weak separating ball property, B also has the weak separating ball property, and so B is pointwise contractive by Corollary 4.20. By Proposition 9.5, $(\overline{L(B)}, \|\cdot\|) \cong (\ell^{\,1}(\Phi_A), \|\cdot\|_1)$.

For each $f \in A$, we have $\|f\|_{\mathrm{BSE},A} = |f|_{\Phi_A}$ by hypothesis, and $\|f\|_{\mathrm{BSE},B} = |f|_{\Phi_A}$, and so $\|f\|_{\mathrm{BSE},A} = \|f\|_{\mathrm{BSE},B}$. It follows from Theorem 8.2 that $\overline{L(B)} \cong \overline{L(A)}$, and so $(\overline{L(A)}, \|\cdot\|) \cong (\ell^{\,1}(\Phi_A), \|\cdot\|_1)$, giving (a).

Now suppose that $|\Phi_A| \geq 2$. Then (b) $\Leftrightarrow$ (d) by Proposition 4.7. ■

Example 9.11, to be given below, will show that a Banach function algebra M such that map $\iota : \ell^{\,1}(\Phi_M) \to \overline{L(M)}$ is an isometric surjection is not necessarily equivalent to a uniform algebra.

The following is the isomorphic analogue of the above theorem.

THEOREM 9.9. *Let A be a Banach function algebra. Then the following conditions on A are equivalent:*

(a) *the linear map $\iota : \ell^{\,1}(\Phi_A) \to \overline{L(A)}$ is an isomorphism;*

(b) *$\|\cdot\|_{\mathrm{BSE}} \sim |\cdot|_{\Phi_A}$ on A and A and each non-zero maximal modular ideal of A has a bounded pointwise approximate identity.*

Proof. (a) $\Rightarrow$ (b) This is essentially the same argument as that contained in the proof of Theorem 9.8, (a) $\Rightarrow$ (c).

(b) $\Rightarrow$ (a) This implication is trivial when $A = (\mathbb{C}, |\cdot|)$, and so we may suppose that $|\Phi_A| \geq 2$. By Lemma 9.1, the map ι is an injection. Since $\|\cdot\|_{\mathrm{BSE}} \sim |\cdot|_{\Phi_A}$, there is a constant $\beta > 0$ such that $\|f\|_{\mathrm{BSE}} \leq 1$ whenever $f \in A$ with $|f|_{\Phi_A} \leq \beta$. It follows from equation (5.4) that $\|\iota(\lambda)\| \geq \beta\,\|\lambda\|_1 \ \ (\lambda \in L(A))$, and so $\|\iota(f)\| \geq \beta\,\|f\|_1 \ \ (f \in \ell^{\,1}(\Phi_A))$ because $L(A)$ is dense in $\ell^{\,1}(\Phi_A)$. This implies clause (a). ■

The next theorem concerns the question when $\mathcal{Q}(A)$ is a uniform algebra on its character space $\Phi_{\mathcal{Q}(A)}$.

THEOREM 9.10. *Let A be a Banach function algebra.*

(i) *Suppose that the linear map $\iota : \ell^{\,1}(\Phi_A) \to \overline{L(A)}$ is an isometric surjection. Then $\mathcal{Q}(A)$ is a uniform algebra on $\Phi_{\mathcal{Q}(A)}$, and $\Phi_{\mathcal{Q}(A)} = \beta K_d$, where $K = \Phi_A$.*

(ii) *Suppose that A is dense in the uniform algebra $(C_0(\Phi_A), |\cdot|_{\Phi_A})$ and that $\mathcal{Q}(A)$ is a uniform algebra on $\Phi_{\mathcal{Q}(A)}$. Then the linear map $\iota : \ell^{\,1}(\Phi_A) \to \overline{L(A)}$ is an isometric surjection.*

Proof. (i) As in Corollary 9.6, $(\mathcal{Q}(A), \|\cdot\|_{\mathcal{Q}(A)}) = (\ell^\infty(K), |\cdot|_K)$, and so $\mathcal{Q}(A)$ is identified with the uniform algebra $C(\beta K_d)$.

(ii) Since $\mathcal{Q}(A)$ is a uniform algebra on $\Phi_{\mathcal{Q}(A)}$, necessarily

$$\|\,[f]\,\|_{\mathcal{Q}(A)} = |f|_{\Phi_{\mathcal{Q}(A)}} \quad (f \in \mathcal{Q}(A))\,.$$

Take $f \in A$. By equation (6.3), $|\,f\,|_{\Phi_{\mathcal{Q}(A)}} = |f|_{\Phi_A}$. Also $\|f\|_{\mathrm{BSE}} = \|\,[f]\,\|_{\mathcal{Q}(A)}$. Hence $\|f\|_{\mathrm{BSE}} = |f|_{\Phi_A}$.

Now take $\lambda \in L(A)$. It again follows from equation (5.4) that

$$\|\lambda\| = \sup\{|\langle f,\, \lambda\rangle| : f \in A,\, |f|_{\Phi_A} \le 1\}\,.$$

Since A is dense in $C_0(\Phi_A)$, the set $\{f \in A : |f|_{\Phi_A} \le 1\}$ is dense in $C_0(\Phi_A)_{[1]}$, and so

$$\|\lambda\| = \sup\{|\langle f,\, \lambda\rangle| : f \in C_0(\Phi_A)_{[1]}\}\,.$$

This implies that $\|\lambda\| = \|\lambda\|_1$, and hence that $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is an isometric surjection. ∎

Let $A = A(\overline{\mathbb{D}})$ be the disc algebra. By Example 7.5, $\mathcal{Q}(A)$ is a uniform algebra on $\Phi_{\mathcal{Q}(A)}$. However, it is not true that each point of $\Phi_A = \overline{\mathbb{D}}$ is a one-point part, and so, by Proposition 4.21(ii), A is not pointwise contractive. By Theorem 9.8, (c) $\Rightarrow$ (a), it is also not true that $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is an isometric surjection. It follows that we cannot delete the hypothesis that A be dense in the space $(C_0(\Phi_A), |\cdot|_{\Phi_A})$ in clause (ii) of Theorem 9.10.

Suppose that A is a natural uniform algebra on a compact space K such that each singleton in K is a one-point part. Then $\iota : \ell^1(\Phi_A) \to \overline{L(A)}$ is an isometric surjection by Theorem 9.5 and $\mathcal{Q}(A)$ is a uniform algebra by Corollary 9.6. However we noted on page 23 that there are Cole algebras A that are not equal to $C(K)$ (and hence not dense in $C(K)$). Thus the converse to Theorem 9.10(ii) does not hold.

In the following example, we shall exhibit a Banach function algebra M that is not equivalent to a uniform algebra, but is such that $\mathcal{Q}(M)$ is a uniform algebra; the algebra M does not have a BSE norm.

Example 9.11. Let A be the example constructed in [18, Example 5.1] and expounded in [19]. Briefly, A consists of the functions $f \in C(\mathbb{I})$ such that

$$I(f) := \int_0^1 \frac{|f(t) - f(0)|}{t}\,\mathrm{d}t < \infty\,;$$

we define

$$\|f\| = |f|_{\mathbb{I}} + I(f) \quad (f \in A)\,.$$

Then $(A, \|\cdot\|)$ is a natural, self-adjoint, unital Banach function algebra on $\mathbb{I}$. The algebra A is a dense, proper subalgebra of $C(\mathbb{I})$, and so A is not equivalent to a uniform algebra. Set

$$M = \{f \in A : f(0) = 0\}\,.$$

Then the maximal ideal M of A does not have a bounded approximate identity, but it does have a contractive pointwise approximate identity; indeed, it is noted in [18, Example 5.1] that A is pointwise contractive. As in Theorem 8.4, M has norm-one characters.

This example A of a unital Banach function algebra that is pointwise contractive, but not equivalent to a uniform algebra, shows that the requirement in Theorem 9.3 that A have a BSE norm cannot be dropped for the proof of clause (ii).

Set $K = (0,1]$ and $C_0 = C_0(K)$, so that C_0 has a contractive approximate identity. Then M is a Segal algebra with respect to C_0, and so, by Theorem 8.7, (a) $\Rightarrow$ (c),

$$\|f\|_{\mathrm{BSE},M} = |f|_{\mathbb{I}} \quad (f \in M)\,,$$

which shows that M does not have a BSE norm. Further, by Theorem 8.8, we have

$$\overline{L(M)} = \overline{L(C_0)} = \ell^{\,1}(K) \quad \text{and} \quad (\mathcal{Q}(M), \|\cdot\|_{\mathcal{Q}(M)} = (C(\beta K_d), |\cdot|_{\beta K_d})\,.$$

Also $\mathcal{Q}(A) = C(\beta\,\mathbb{I}_d)$, and so $\mathcal{Q}(A)$ and $\mathcal{Q}(M)$ are isometrically isomorphic as Banach algebras.

The Banach function algebra M is such that $\mathcal{Q}(M)$ is a uniform algebra, but M itself is not equivalent to a uniform algebra. ∎

Example 9.12. Let A be the example constructed in [18, Example 5.2]. The algebra A is a natural, unital Banach function algebra on the circle $\mathbb{T}$ such that A is dense in $(C(\mathbb{T}), |\cdot|_{\mathbb{T}})$, but A is not equivalent to the uniform algebra $(C(\mathbb{T}), |\cdot|_{\mathbb{T}})$. It is shown that A is contractive. For this example, $\|f\|_{\mathrm{BSE}} \le 4\,|f|_{\mathbb{T}}$ $(f \in A)$ by Proposition 9.2, and so A does not have a BSE norm. (In fact, $\|f\|_{\mathrm{BSE}} = |f|_{\mathbb{T}}$ $(f \in A)$.)

This example A of a unital Banach function algebra that is contractive, but not equivalent to a uniform algebra, shows that the requirement in Theorem 9.3 that A have a BSE norm cannot be dropped for the proof of clause (i). ∎

10. Embedding multiplier algebras

Our aim in this section is to show that, for each Banach function algebra A that has a contractive pointwise approximate identity and whose norm is equal to its BSE norm, the multiplier algebra of A embeds isometrically into the unital Banach function algebra $\mathcal{Q}(A) = A''/L(A)^{\perp}$.

Let $(A, \|\cdot\|)$ be a natural Banach function algebra on a locally compact space K, with multiplier algebra $\mathcal{M}(A)$, as on page 11, so that

$$\mathcal{M}(A) = \{f \in C^b(K) : fA \subset A\},$$

and

$$|f|_K \leq \|f\|_{\mathrm{op}} \leq \|f\| \quad (f \in A). \tag{10.1}$$

For example, $\mathcal{M}(c_0) = \ell^\infty = C(\beta\mathbb{N})$.

For a general algebra A, there is also a definition of left multipliers on A and of the multiplier algebra, $\mathcal{M}(A)$; see [12, §1.4], for example. In the case where A is a Banach algebra with a contractive approximate identity, it is proved in [12, Theorem 2.9.49] that there is a specific isometric algebra embedding θ of $\mathcal{M}(A)$ into $(A'', \Diamond)$.

Now let A be a Banach function algebra with a contractive approximate identity. Then it follows easily that, in the case where $\|f\|_{\mathrm{BSE}} = \|f\| \ \ (f \in A)$, the above map θ gives an isometric algebra embedding of $\mathcal{M}(A)$ into $\mathcal{Q}(A)$. The theorem below gives the same conclusion when A has just a contractive pointwise approximate identity, rather than a contractive approximate identity.

Let A be a Banach function algebra, and take $T \in \mathcal{M}(A)$. Then

$$T''(\mathrm{M} \,\Box\, \mathrm{N}) = T''(\mathrm{M}) \,\Box\, \mathrm{N} \quad (\mathrm{M}, \mathrm{N} \in A''). \tag{10.2}$$

Suppose that A has a contractive pointwise approximate identity, say (e_α), and let $T = L_f \in \mathcal{M}(A)$. Then

$$\lim_\alpha (Te_\alpha)(\varphi) = f(\varphi) \quad (\varphi \in \Phi_A). \tag{10.3}$$

Take E and F to be weak-$*$ accumulation points of (e_α) in $A''_{[1]}$. Then (10.3) shows that $T''(\mathrm{E}) - T''(\mathrm{F}) \in \Phi_A^{\perp}$, and hence $[T''(\mathrm{E})] = [T''(\mathrm{F})]$ in $\mathcal{Q}(A)$. Thus the map θ in the following theorem does not depend on the choice of E.

THEOREM 10.1. *Let A be a Banach function algebra such that $\|f\|_{\mathrm{BSE}} = \|f\| \ \ (f \in A)$. Suppose that A has a contractive pointwise approximate identity with a weak-$*$ accumulation point* E *in A''. Then the map*

$$\theta : T \mapsto [T''(\mathrm{E})], \quad (\mathcal{M}(A), \|\cdot\|_{\mathrm{op}}) \to (\mathcal{Q}(A), \|\cdot\|_{\mathcal{Q}(A)}),$$

is an isometric algebra embedding.

Proof. It is clear that θ is a linear map. Take $T \in \mathcal{M}(A)$. Since $\mathrm{E} \in A''_{[1]}$, it follows that $\|\theta(T)\|_{\mathcal{Q}(A)} \leq \|T\|_{\mathrm{op}}$, and so θ is a contraction.

Take $S, T \in \mathcal{M}(A)$, and take $\varphi \in \Phi_A$. First note that

$$\langle fg, S'(\varphi)\rangle = \langle f, \varphi\rangle \langle g, S'(\varphi)\rangle \quad (f, g \in A).$$

Thus $\langle f \cdot T''(\mathrm{E}),\, S'(\varphi)\rangle = \langle f,\, \varphi\rangle\, \langle T''(\mathrm{E}),\, S'(\varphi)\rangle$ $(f \in A)$, and hence

$$\langle S''(\mathrm{E} \,\Box\, T''(\mathrm{E})),\, \varphi\rangle = \langle \mathrm{E} \,\Box\, T''(\mathrm{E}),\, S'(\varphi)\rangle = \langle T''(\mathrm{E}),\, S'(\varphi)\rangle = \langle (S'' \circ T'')(\mathrm{E}),\, \varphi\rangle\,.$$

It follows that

$$\begin{aligned}\langle (S \circ T)''(\mathrm{E}),\, \varphi\rangle &= \langle (S'' \circ T'')(\mathrm{E}),\, \varphi\rangle = \langle S''(\mathrm{E} \,\Box\, T''(\mathrm{E})),\, \varphi\rangle \\ &= \langle S''(\mathrm{E}) \,\Box\, T''(\mathrm{E}),\, \varphi\rangle \quad \text{by (10.2)}\,,\end{aligned}$$

and so $(S \circ T)''(\mathrm{E}) - S''(\mathrm{E}) \,\Box\, T''(\mathrm{E}) \in L(A)^{\perp}$. This shows that

$$[(S \circ T)''(\mathrm{E})] = [S''(\mathrm{E})]\, [T''(\mathrm{E})]$$

in $\mathcal{Q}(A)$. Hence $\theta : \mathcal{M}(A) \to \mathcal{Q}(A)$ is an algebra homomorphism.

Now take a multiplier $T \in \mathcal{M}(A)$. For each $\varepsilon > 0$, there exists $f \in A_{[1]}$ such that $\|Tf\| > \|T\|_{\mathrm{op}} - \varepsilon$. Since $\|Tf\|_{\mathrm{BSE}} = \|Tf\|$, there also exists $\lambda \in L(A)_{[1]}$ such that $|\langle Tf,\, \lambda\rangle| > \|T\|_{\mathrm{op}} - \varepsilon$. Now

$$|\langle Tf,\, \lambda\rangle| = |\langle \mathrm{E} \cdot Tf,\, \lambda\rangle| = |\langle T''(\mathrm{E}) \cdot f,\, \lambda\rangle| = |\langle T''(\mathrm{E}),\, f \cdot \lambda\rangle|\,.$$

Since $f \cdot \lambda \in L(A)_{[1]}$, it follows that $|\langle Tf,\, \lambda\rangle| \le \|[T''(\mathrm{E})]\|_{\mathcal{Q}(A)}$, and this implies that $\|T\|_{\mathrm{op}} \le \|[T''(\mathrm{E})]\|_{\mathcal{Q}(A)} + \varepsilon$. Thus $\|T\|_{\mathrm{op}} \le \|[T''(\mathrm{E})]\|_{\mathcal{Q}(A)} = \|\theta(T)\|_{\mathcal{Q}(A)}$, and so the map θ is an isometry. ∎

Example 10.2. In general, the above map θ is not a surjection.

For example, suppose that $A = \mathrm{lip}_{\alpha}\mathbb{I}$, as in Example 7.2, so that $\mathcal{Q}(A) = A'' = \mathrm{Lip}_{\alpha}\mathbb{I}$. Then $\mathcal{M}(A) = A$ because A is unital, and so the range of θ is $A \subsetneq \mathcal{Q}(A)$.

Again, let $A = L^1(G)$ for a locally compact abelian group G. Then $\mathcal{M}(A) = M(G)$ and $\mathcal{Q}(A) = M(bG)$, as in Example 7.9. Clearly, $M(G)$ embeds isometrically into $M(bG)$, but the embedding is rarely a surjection. ∎

Example 10.3. Let M be the Segal algebra mentioned in Example 9.11. Then it is easy to see that $\mathcal{M}(M) = C^b((0,1])$, and so $\mathcal{M}(M)$ embeds isometrically and algebraically in $\ell^{\infty}((0,1]) = \mathcal{Q}(M)$. ∎

Proposition 10.4. *Let A be a Banach function algebra that is an ideal in its bidual and that has a contractive pointwise approximate identity. Then $\mathcal{Q}(A) = \mathcal{M}(A)$.*

Proof. By Corollary 5.13(ii), $\|f\|_{\mathrm{BSE}} = \|f\|$ $(f \in A)$, and so, by Theorem 10.1, the map $\theta : T \mapsto [T''(\mathrm{E})],\ \mathcal{M}(A) \to \mathcal{Q}(A)$, is an isometric algebra embedding.

Take $\mathrm{M} \in A''$. Then the map $R_{\mathrm{M}} : f \mapsto f \cdot \mathrm{M},\ \ A \to A$, is a multiplier on A because A is an ideal in A'', and $\theta(R_{\mathrm{M}}) = [\mathrm{M}]$. Hence the map $\theta : \mathcal{M}(A) \to \mathcal{Q}(A)$ is a surjection. ∎

Example 10.5. Let A be a Banach function algebra with a contractive approximate identity such that A is an ideal in A''. Then $\overline{L(A)} = A \cdot A'$ by Corollary 5.3. In the case where A is also Arens regular, $AA' = A'$ by [56, Corollary 3.2], and so $\overline{L(A)} = A'$ and hence $L(A)^{\perp} = \{0\}$. Thus $\mathcal{Q}(A) = A''$. By Proposition 10.4, $\mathcal{M}(A) = \mathcal{Q}(A)$. Hence $\Phi_{\mathcal{Q}(A)} = \Phi_A \cup H$, where H is the hull of A considered as an ideal in $\mathcal{M}(A)$.

For example, let $A = c_0 \,\widehat{\otimes}\, c_0$. Then the algebra A satisfies the specified conditions, and so $\mathcal{Q}(A) = \mathcal{M}(A) = A''$. Indeed, A is a Tauberian Banach sequence algebra, and so A is an ideal in A''; further, A has a bounded approximate identity, and A is Arens regular by [55, Corollary 4.17(a)]. ∎

EXAMPLE 10.6. Let G be a locally compact abelian group that is neither discrete nor compact, and let S be either of the Segal algebras S_1 or S_2 that were considered in Example 8.10(ii). Then, as we have seen in Corollary 8.9, S does not have a BSE norm, but $\mathcal{Q}(S) = \mathcal{Q}(L^1(G)) = M(bG)$. By [28], $\mathcal{M}(S) = M(G)$. As $M(G)$ embeds in $M(bG)$, we see that $\mathcal{M}(S)$ embeds into $\mathcal{Q}(S)$, although S does not have a bounded approximate identity. ∎

11. Reflexive ideals and weakly compact homomorphisms

In this final section, we shall consider when Banach function algebras contain non-trivial, closed ideals that are reflexive as Banach spaces and also when there are non-zero, weakly compact homomorphisms between two Banach function algebras; our proofs use notions that are given above.

11.1. Reflexive ideals. We first consider the consequences of assumptions that certain closed ideals in and quotients of a Banach function algebra are reflexive.

Definition 11.1. Let A be a Banach function algebra, and let I be a non-zero, closed ideal in A. Then I is *reflexive* if I is reflexive as a Banach space.

Suppose that A is a Banach function algebra and that S is a finite set of isolated points in Φ_A. Then the closed ideal $I(\Phi_A \setminus S)$ is a finite-dimensional Banach space, and hence reflexive. However it is not true that all reflexive ideals in a Banach function algebra are finite dimensional. Indeed, [18, Example 3.3] exhibits an infinite-dimensional, unital, Banach function algebra A that is reflexive as a Banach space and is such that Φ_A is connected. Thus A has many non-trivial reflexive ideals, although Φ_A has no isolated points and no non-zero, finite-dimensional, closed ideals.

Let A be a regular Banach function algebra, and suppose that I is a reflexive ideal. Then it follows from (2.6) that $\overline{J(h(I))}$ is also reflexive, and so we concentrate on closed ideals of this latter form.

Definition 11.2. Let A be a Banach function algebra, and take $\varphi \in \Phi_A$. Then A has the *strong separating ball property at* φ if, for each neighbourhood U of φ in Φ_A, there exists $f \in A_{[1]}$ with $f(\varphi) = 1$ and $\operatorname{supp} f \subset U$. The algebra A has the *strong separating ball property* if it has the strong separating ball property at φ for each $\varphi \in \Phi_A$.

Examples 11.3. (i) Let K be a non-empty, locally compact space. Then $C_0(K)$ has the strong separating ball property.

(ii) Let Γ be a locally compact group. The Figà-Talamanca–Herz algebras $A_p(\Gamma)$ were mentioned in Example 7.12. We *claim* that each Banach function algebra $A_p(\Gamma)$ has the strong separating ball property; in particular, the Fourier algebra $A(\Gamma)$ has this property.

Indeed, take $x \in \Gamma$ and $U \in \mathcal{N}_x$. There is a symmetric, open, relatively compact neighbourhood V of e_Γ such that $xV^2 \subset U$, say $\alpha = 1/m_\Gamma(V)$. Set $f = \alpha^{1/p}\chi_{xV}$ and $g = \alpha^{1/q}\chi_V$, where $q = p'$. Then $\|f\|_p = \|g\|_q = 1$. Set $u = f \star g$, so that $u \in A_p(\Gamma)_{[1]}$. Clearly

$$u(x) = \alpha^{1/p+1/q} \int_V \chi_{xV}(xy)\,\mathrm{d}m_\Gamma(y) = \alpha \cdot (1/\alpha) = 1$$

and $\operatorname{supp} u \subset xV^2 \subset U$. Hence $A_p(\Gamma)$ has the strong separating ball property. ■

Theorem 11.4. *Let A be a Banach function algebra with the strong separating ball property, and take a proper, closed subset S of Φ_A. Suppose that $\overline{J(S)}$ is a reflexive ideal in A. Then the space $\Phi_A \setminus S$ is discrete.*

Proof. Take $\varphi \in \Phi_A \setminus S$, and consider the set

$$K_\varphi = \{f \in \overline{J(S)}_{[1]} : f(\varphi) = 1\}\,.$$

Since A has the strong separating ball property at φ, the set K_φ is non-empty, and clearly it is convex. Since $\overline{J(S)}$ is reflexive, the set K_φ is weakly compact in A. For $g \in K_\varphi$, the maps $L_g : f \mapsto gf,\ \ K_\varphi \to K_\varphi$, form a commuting family of continuous, affine maps, and so, again by Theorem 2.4, the family has a fixed point, say $h \in A$. Thus $h(\varphi) = 1$ and $gh = h$ for each $g \in K_\varphi$

For each $\psi \in \Phi_A \setminus S$ with $\psi \neq \varphi$, there exists $g \in K_\varphi$ with $g(\psi) = 0$. This implies that $h(\psi) = g(\psi)h(\psi) = 0$, and so h is the characteristic function of $\{\varphi\}$. It follows that φ is isolated in Φ_A, and so $\Phi_A \setminus S$ is discrete. ∎

Since there are reflexive Banach function algebras that have connected character space, we cannot delete the hypothesis that A have the strong separating ball property in the above theorem.

Corollary 11.5. *Let Γ be a locally compact group, and take p with $1 < p < \infty$. Then $A_p(\Gamma)$ contains a non-zero, reflexive closed ideal if and only if Γ is discrete.*

Proof. Set $A = A_p(\Gamma)$.

As noted above, A has many reflexive closed ideals when Γ is discrete.

Conversely, suppose that A contains a non-zero, reflexive closed ideal I. Set $S = h(I)$, so that S is a proper, closed subset of $\Gamma = \Phi_A$. Since A is regular, $\overline{J(S)} \subset I$, and so the ideal $\overline{J(S)}$ is reflexive. Since A has the strong separating ball property, it follows from Theorem 11.4 that each point in $\Gamma \setminus S$ is isolated in Γ. Since Γ is a group, each point of Γ is isolated, and so Γ is discrete. ∎

Let E be a Banach space that is weakly sequentially complete. By Rosenthal's ℓ^1-theorem [1, Corollary 10.2.2], either E is reflexive or it contains an isomorphic copy of ℓ^1. The Fourier algebra $A(\Gamma) = A_2(\Gamma)$ is weakly sequentially complete, being the predual of a von Neumann algebra, and so we have the following result.

Corollary 11.6. *Let Γ be a non-discrete, locally compact group. Then every non-zero, closed ideal of $A(\Gamma)$ contains an isomorphic copy of ℓ^1.* ∎

In the case where G is abelian and non-compact, the above corollary (for the algebra $A = L^1(G)$) was obtained by Rosenthal in his seminal memoir [46, Theorem 2.12 and Corollary 2.13] as a consequence of more general results.

Theorem 11.7. *Let A be a Banach function algebra with the separating ball property such that Φ_A is weakly closed in A'. Suppose that I is a proper closed ideal in A such that A/I is reflexive. Then $h(I)$ is finite.*

Proof. Since A/I is reflexive, the space $I^\perp \cong (A/I)'$ is a reflexive subspace of A', and $h(I)$ is a subset of $I^\perp$. Since A has the separating ball property, it follows from Theorem 4.3(ii) that Φ_A is weakly discrete, and so $h(I)$ is relatively weakly compact as a subset of $(I^\perp)_{[1]}$. Since Φ_A is weakly closed, the weak closure of $h(I)$ in A' is contained in the discrete space Φ_A, and hence $h(I)$ is finite. ∎

Corollary 11.8. *Let Γ be a locally compact, amenable group, and take p such that $1 < p < \infty$. Suppose that I is a proper closed ideal in $A_p(\Gamma)$ such that $A_p(\Gamma)/I$ is reflexive. Then $h(I)$ is finite.*

Proof. Set $A = A_p(\Gamma)$. As we remarked in Example 7.12 , $\Phi_A = \Gamma$ is weakly closed in A' whenever Γ is amenable. By Example 11.3(ii), A has the strong separating ball property, and so the result follows from Theorem 11.7. ∎

The above corollary does not hold for an arbitrary locally compact group Γ. Indeed, let Γ be a locally compact group that contains $\mathbb{F}_2$ as a closed subgroup (so that Γ is not amenable). Then $A_p(\mathbb{F}_2)$ is a quotient of $A_p(\Gamma)$ by a result of Herz (see [22, Theorem 5 of §7.8]). The space $\mathbb{F}_2$ contains an infinite so-called Leinert set, say S. It follows from clause (b) of the proof of [6, Proposition 1] that, for $1 < p \leq 2$, the restriction map

$$R : A_p(\Gamma) \to \ell^\infty(S)$$

is such that $\|R(f)\|_q \leq C_p \|f\|_{A_p(\Gamma)} \quad (f \in A_p(\Gamma))$, where $q = p'$. That is, the map $R : A_p(\Gamma) \to \ell^q(S)$ is a bounded linear surjection. Let $I = \ker R$, a closed ideal in $A_p(\Gamma)$. Then I is such that the quotient $A_p(\Gamma)/I$ is a reflexive space, but $h(I)$ is infinite. Since $A_p(\Gamma)$ is isometrically isomorphic to $A_q(\Gamma)$ for each locally compact group Γ, it follows that, whenever Γ contains $\mathbb{F}_2$ as a closed subgroup, the algebra $A_p(\Gamma)$ contains a reflexive ideal I such that $h(I)$ is infinite. Thus, we cannot omit the word 'amenable' in the hypotheses of Corollary 11.8.

The above remark, in the case where $p = 2$, is essentially contained in [57, p. 362].

11.2. Weakly compact homomorphisms. We now consider *weakly compact homomorphisms* between two Banach function algebras; these are algebra homomorphisms that are weakly compact as bounded linear operators. There are many papers in the literature on weakly compact homomorphisms between Banach algebras; for example, see [29, 36].

Theorem 11.9. *Let A and B be Banach function algebras. Suppose that A has norm-one characters and that the only idempotent in B is zero. Then the only weakly compact homomorphism from A into B is the zero homomorphism.*

Proof. Assume towards a contradiction that $\theta : A \to B$ is a non-zero, weakly compact homomorphism. Then $\theta' \mid \Phi_B : \Phi_B \to \Phi_A \cup \{0\}$ is a continuous map, and there exists $\psi_0 \in \Phi_B$ such that $\varphi_0 := \theta'(\psi_0) \in \Phi_A$ because $\theta \neq 0$. By hypothesis, $\|\varphi_0\| = 1$, and so, by Proposition 6.7, there is an idempotent $u \in \mathcal{Q}(A)_{[1]}$ with $u(\varphi_0) = 1$.

Since θ is weakly compact, the range of the map $\theta'' : A'' \to B''$ is contained in B. Take $\mathrm{M} \in L(A)^\perp$. For each $\psi \in \Phi_B$, we have $\langle \theta''(\mathrm{M}), \psi \rangle = \langle \mathrm{M}, \theta'(\psi) \rangle = 0$, and so $\theta''(\mathrm{M}) = 0$ in B. Thus the map

$$\widetilde{\theta} : [\mathrm{M}] \mapsto \theta''(\mathrm{M}), \quad \mathcal{Q}(A) \to B,$$

is well defined; clearly, $\widetilde{\theta}$ is a continuous homomorphism.

Set $g = \widetilde{\theta}(u) \in B$. Then g is an idempotent in B, and so $g = 0$. However we have $g(\psi_0) = u(\phi_0) = 1$, the required contradiction.

Thus the only weakly compact homomorphism from A into B is the zero homomorphism. ∎

Corollary 11.10. *Let A be a Banach function algebra such that A has norm-one characters, and let $B = A(\Gamma)$, where Γ is a locally compact group that is connected and non-compact. Then the only weakly compact homomorphism from A into B is the zero homomorphism.* ∎

12. Open questions

We conclude with a list of some questions that we have not resolved.

1. Let A be a unital uniform algebra. Is every subset P of Φ_A that is a Gleason part with respect to A also a Gleason part with respect to A'' when it is regarded as a subset of $\Phi_{A''}$? In particular, is this true when the part P is a singleton?
2. Let $(A, \|\cdot\|)$ be a Banach function algebra such that $\inf\{\|\varphi\| : \varphi \in \Phi_A\} > 0$. Is there a norm $|||\cdot|||$ on A that is equivalent to $\|\cdot\|$ and such that $(A, |||\cdot|||)$ is a Banach function algebra and $|||\varphi||| = 1$ for each $\varphi \in \Phi_A$?
3. Let A be a Banach function algebra. Since the only possible weak-$*$ accumulation point of Φ_A in A' is 0, the set Φ_A is weakly closed in A' whenever there exists $\mathrm{M} \in A''$ such that $\inf\{|\langle \mathrm{M}, \varphi\rangle| : \varphi \in \Phi_A\} > 0$. Is the converse always true?
4. Let A be a Banach function algebra, and take $\varphi \in \Phi_A$. We know from Proposition 6.11(ii) that φ is weakly isolated in Φ_A whenever φ is an isolated point when regarded as an element of $\Phi_{\mathcal{Q}(A)}$. Is the converse always true?
5. Let A be a Banach function algebra. We know from Proposition 6.17 that Φ_A is weakly closed in A' whenever A has a bounded pointwise approximate identity. Is the converse always true?
6. Let A be a natural uniform algebra on a compact space K. We do not know whether each point x that is isolated in (K, d_A) is also isolated in $\Phi_{\mathcal{Q}(A)}$, and so, by Corollary 6.13, we do not know whether the corresponding maximal ideal M_x always has a bounded pointwise approximate identity.
7. Is there a Banach function algebra A that does not have a bounded pointwise approximate identity, but is such that the norms $\|\cdot\|_{\mathrm{op}}$ and $\|\cdot\|_{\mathrm{BSE}}$ are equivalent on A?
8. Let A be a Banach function algebra that is an ideal in its bidual and is such that $\|f\|_{\mathrm{BSE}} = \|f\|_{\mathrm{op}}$ $(f \in A)$. Does A necessarily have a contractive pointwise approximate identity?
9. In Example 7.5, we identified $\mathcal{Q}(A)$ in the case where A is the disc algebra. It is likely that similar, but more complicated, arguments will identify $\mathcal{Q}(A)$ for more general unital uniform algebras, such as the tight uniform algebras of [11].
10. Let Γ be a locally compact group. In Example 7.10, we identified $\mathcal{Q}(A(\Gamma))$ in certain cases. Can these results be extended to more general groups?
11. Let ω be a weight on the group $(\mathbb{Z}, +)$ such that
$$\inf\{\omega_n^{1/n} : n \in \mathbb{N}\} = \sup\{\omega_{-n}^{-1/n} : n \in \mathbb{N}\} = 1\,,$$
with corresponding Beurling algebra A_ω. In Example 7.11, we showed that we have $\mathcal{Q}(A_\omega) = A_\omega$ for many weights ω. Is this true for all such weights ω?
12. Let Γ be a locally compact group, and take p with $1 < p < \infty$. We do not have any identification of $\mathcal{Q}(A_p(\Gamma))$ for groups that are not both discrete and amenable, save in the case where $p = 2$.

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