Lietuvos matematikos rinkinys. LMD darbai
52 tomas, 2011, 353–358

ISSN 0132-2818
www.mii.lt/LMR/

skirstinių asimptotinė analizė

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Santrauka. Diskretų k-matį atsitiktinį vektorių $\vec{\xi_1} = (\xi_1, \xi_2, \dots, \xi_k)$ vadinsime kvazigardeliniu, jeigu jo koordinatės $\xi_1, \dots, \xi_k$ yra kvazigardeliniai atsitiktiniai dydžiai ξ_i, t. y. ξ_i įgyja reikšmes $(\vec{\beta}_i, \vec{\nu}_i) = \beta_{1i}\nu_{1i} + \dots + \beta_{k_i i}\nu_{k_i i}$, $\nu_{ji} = 0, \pm 1, \pm 2, \dots$, $\beta_{ji} > 0$ ir vektoriaus $\vec{\beta}_i = (\beta_{1i}, \dots, \beta_{k_i i})$ koordinatės yra tiesiškai nesurištos racionalių skaičių kūne. Yra įrodyta, kad vietoje kvazigardelinio vektoriaus $\vec{\xi_1}$, galima nagrinėti gardelinį atsitiktinį vektorių $\vec{\theta} = (\theta_1, \dots, \theta_m)$ erdvėje R^m, kur $m = k_1 + k_2 + \dots + k_k$. Toliau mes kalbėsime apie kvazigardelinius atsitiktinius vektorius, kurių klasei priklauso ir gardeliniai atsitiktiniai vektoriai (žiūr. [1]).

Raktiniai žodžiai: kvazigardelinis skirstinys, be galo dalus skirstinys, Appelio daugianariai.

Įvadas

Nagrinėsime k-mačių kvazigardelinių tikimybinių skirstinių $F(\vec{x}) = P\{\vec{\xi_1} < \vec{x}\}$, $\vec{x} \in R^k$, sąsūkas

$$F_n(\vec{x}) = F^{*n}(\vec{x}\sqrt{n}), \quad n = 1, 2, \dots.$$

Tikimybinis skirstinys gali priklausyti nuo n.

Tarkime charakteringoji funkcija

$$\hat{F}_n\left(\frac{\vec{t}}{\sqrt{n}}\right) = \int_{R^k} \mathrm{e}^{i(\frac{\vec{t}}{\sqrt{n}}, \vec{x})}\, dF^{*n}(\vec{x})$$

visiems $|\vec{t}| \leqslant T$, $T > 0$, konverguoja į be galo dalaus skirstinio $G(\vec{x}) = P\{\vec{\eta} < \vec{x}\}$, $\vec{x} \in R^k$, charakteringąją funkciją

$$\hat{G}(\vec{t}) = \exp\left\{ i(\vec{\gamma}, \vec{t}) + \int_{R^k} \left(\mathrm{e}^{i(\vec{t}, \vec{x})} - 1 - i(\vec{t}, \vec{x})\right)\nu(d\vec{x}) \right\}$$

[2, 39 psl.]. Čia matas $\nu(d\vec{x})$ tenkina sąlygą

$$\int_{|\vec{x}|>1} |\vec{x}|\nu(d\vec{x}) < \infty.$$

Čia $|\vec{x}|^2 = x_1^2 + \dots + x_k^2$, $(\vec{t}, \vec{x}) = t_1x_1 + \dots + t_kx_k$ – skaliarinė sandauga.

Levy-Chinčino išdėstyme praleidžiame Gauso komponentę $-\frac{1}{2}\vec{t}^T \varSigma \vec{t}$.

Tarkime, kad tikimybiniai skirstiniai $F_n(\vec{x})$ ir $G(\vec{x})$ turi lygias nuliui matematines viltis ir vienodą neišsigimusią antrųjų momentų matricą $\varSigma$.

1 Charakteringosios funkcijos $\hat{F}_n(\frac{\vec{t}}{\sqrt{n}})$ formalus skleidinys

Konstruodami funkcijos

$$\hat{F}_n\left(\frac{\vec{t}}{\sqrt{n}}\right) = \left(\hat{F}\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)^n$$

formalų asimptotinį skleidinį mes priimame, kad $\vec{t}$ kinta baigtinėje sferoje ($|\vec{t}| < T$) ir kad egzistuoja baigtiniai momentai. Laikysime, kad tokias pat savybes turi be galo dalus tikimybinis skirstinys $G(\vec{x})$.

Visiems $n = 1, 2, \ldots$ yra

$$G(\vec{x}) = \left(G^{*\frac{1}{n}}(\vec{x})\right)^{*n}.$$

Čia $G^{*\frac{1}{n}}(\vec{x}) = G_n(\vec{x})$ yra be galo dalus tikimybinis skirstinys su charakteringąja funkcija

$$\hat{G}_n(\vec{t}) = \exp\left\{\frac{1}{n}\int_{R^k}\left(\mathrm{e}^{i(\vec{t},\vec{x})} - 1 - i(\vec{t},\vec{x})\right)\nu(d\vec{x})\right\}.$$

Atlikę kintamųjų pakeitimą $\vec{x} = \frac{\vec{y}}{\sqrt{n}}$, gauname

$$\hat{G}_n(\vec{t}) = \exp\left\{\int_{R^k}\left(\mathrm{e}^{i(\frac{\vec{t}}{\sqrt{n}},\vec{y})} - 1 - i\left(\frac{\vec{t}}{\sqrt{n}},\vec{y}\right)\right)\frac{\nu(d\frac{\vec{y}}{\sqrt{n}})}{n}\right\} = \hat{Q}_n\left(\frac{\vec{t}}{\sqrt{n}}\right).$$

Čia $\hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})$ pažymi be galo dalią charakteringąją funkciją su tripletu $\mathcal{L}\{0, 0, \frac{1}{n}\nu(\frac{d\vec{y}}{\sqrt{n}})\}$.

Pažymėkime $Q_n(\vec{x}\sqrt{n})$ tikimybinis skirstinys, kurio charakteringoji funkcija yra $\hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})$. Tokiu būdu gavome, kad

$$\hat{G}(\vec{t}) = \left(\hat{G}_n(\vec{t})\right)^n = \left(\hat{Q}_n\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)^n$$

arba

$$\hat{G}(\vec{x}) = Q_n^{*n}(\vec{x}\sqrt{n}).$$

Charakteringosios funkcijos $\hat{F}_n(\frac{\vec{t}}{\sqrt{n}})$ skleidimui panaudosime Appel daugianarius:

$$\begin{aligned}\hat{F}\left(\frac{\vec{t}}{\sqrt{n}}\right)^n =& \hat{Q}_n^n\left(\frac{\vec{t}}{\sqrt{n}}\right)\left(1 + \frac{\hat{F}(\frac{\vec{t}}{\sqrt{n}}) - \hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})}{\hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})}\right)^n \\ =& \hat{G}(\vec{t})\exp\left\{\Delta\left(\frac{\vec{t}}{\sqrt{n}}\right)\right\}\left(1 + \sum_{j=1}^{\infty}\left(\frac{1}{n}\right)^j A_j\left(\Delta\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)\right) \qquad (1)\end{aligned}$$

visiems $\vec{t}$, kurie tenkina nelygybę

$$\left|\Delta\left(\frac{\vec{t}}{\sqrt{n}}\right)\right| < 1.$$

Čia

$$\Delta\left(\frac{\vec{t}}{\sqrt{n}}\right) = n\left(\frac{\hat{F}(\frac{\vec{t}}{\sqrt{n}}) - \hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})}{\hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})}\right),$$

$$A_j\left(\Delta_n\left(\frac{\vec{t}}{\sqrt{n}}\right)\right) = (-1)^j\left(\Delta\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)^{j+1}\sum_{l=0}^{j-1} q_{jl}\left(\Delta\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)^l,$$

$$q_{jl} = \frac{(j+l)q_{j-1,l} + q_{j-1,l-1}}{j+l+1}, \quad j = 1,2,\ldots,\ l = 1,2,\ldots,j-2,$$

$q_{j0} = \frac{1}{j+1}$, $q_{jl} = 0$, kai $l < 0$.

Pažymėkime

$$h(\vec{t}) = \int_{R^k} \left(\mathrm{e}^{i(\vec{t},\vec{x})} - 1 - i(\vec{t},\vec{x})\right)\nu(d\vec{x}).$$

Tuomet

$$\hat{Q}_n\left(\frac{\vec{t}}{\sqrt{n}}\right) = \mathrm{e}^{\frac{1}{n}h(\vec{t})}, \quad \vec{t} \in R^k.$$

Funkcijos $(\Delta(\frac{\vec{t}}{\sqrt{n}}))^r$ skleidimas.

Pasinaudosime lema: *jeigu tikimybinių skirstinių $F(\vec{x})$ ir $Q_n(\vec{x})$ sutampa du pirmieji momentai, tai*

$$\int_{R^k} (\vec{t},\vec{x})^l\, d\big(F(d\vec{x}) - Q_n(\vec{x})\big)^{*m} = 0,$$

kai $l = 0,1,\ldots,3m-1$.

Gauname

$$\begin{aligned}
\left(n\left(\hat{F}\left(\frac{\vec{t}}{\sqrt{n}}\right) - \hat{Q}\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)\right)^m &= \int_{R^k} \mathrm{e}^{i(\frac{\vec{t}}{\sqrt{n}},\vec{x})}\, d\big(n(F(\vec{x}) - Q_n(\vec{x}))\big)^{*m} \\
&= \sum_{l=3m}^{\infty} \frac{1}{l!}\left(\frac{1}{\sqrt{n}}\right)^l \int_{R^k} (i\vec{t},\vec{x})^l\, d\big(n(F(\vec{x}) - Q_n(\vec{x}))\big)^{*m} \\
&= \sum_{l=3m}^{\infty} \frac{1}{l!}\left(\frac{1}{\sqrt{n}}\right)^{l-2m} \int_{R^k} (i\vec{t},\vec{x})^l\, d\big(F(\vec{x}) - Q_n(\vec{x})\big)^{*m}.
\end{aligned}$$

Atliekame kintamųjų pakeitimą $r = l - 3m$:

$$\begin{aligned}
&\left(n\left(\hat{F}\left(\frac{\vec{t}}{\sqrt{n}}\right) - \hat{Q}\left(\frac{\vec{t}}{\sqrt{n}}\right)\right)\right)^m \\
&\quad = \sum_{r=0}^{\infty} \frac{1}{(r-3m)!}\left(\frac{1}{\sqrt{n}}\right)^{r+m} \int_{R^k} (i\vec{t},\vec{x})^{r+3m}\, d(F\big(\vec{x}) - Q_n(\vec{x})\big)^{*m}.
\end{aligned}$$

Dabar skleidžiame funkciją

$$\left(\frac{1}{\hat{Q}_n(\frac{\vec{t}}{\sqrt{n}})}\right)^m = \mathrm{e}^{\frac{m}{n}h(\vec{t})} = \sum_{l=0}^{\infty} \frac{(-m)^l}{l!} h^l(\vec{t})\left(\frac{1}{n}\right)^l = \sum_{l=0}^{\infty} \frac{(-mh(\vec{t}))^l}{l!}\left(\frac{1}{\sqrt{n}}\right)^{2l}.$$

Tokiu būdu

$$\Big(\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^m = \sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(r+3m)!l!}\Big(\frac{1}{\sqrt{n}}\Big)^{r+m+2l}\big(-mh(\vec{t})\big)^l \cdot \\ \times \int_{R^k}(i\vec{t},\vec{x})^{r+3m}\,d\big(F(\vec{x})-Q_n(\vec{x})\big)^{*m}. \tag{2}$$

Liko išnagrinėti sandaugą

$$Y_r = \Big(\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^r \mathrm{e}^{\varDelta_n(\frac{\vec{t}}{\sqrt{n}})}, \quad r = 0,1,2,\ldots.$$

Gauname

$$Y_r = \sum_{j=0}^{\infty}\frac{1}{j!}\Big(\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^{r+j} = \sum_{\alpha=r}^{\infty}\frac{1}{(\alpha-r)!}\Big(\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^{\alpha}. \tag{3}$$

Iš (2) ir (3) seka

$$\begin{aligned}
&\mathrm{e}^{\varDelta_n(\frac{\vec{t}}{\sqrt{n}})}A_j\Big(\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)\\
&= (-1)^j\sum_{l=0}^{j-1}q_{jl}\Big(\varDelta\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^{l+j+1}\mathrm{e}^{\varDelta(\frac{\vec{t}}{\sqrt{n}})}\\
&= (-1)^j\sum_{l=0}^{j-1}q_{jl}\sum_{\alpha=l+j+1}^{\infty}\frac{1}{(\alpha-l-j-1)!}\Big(\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^{\alpha}\\
&= (-1)^j\sum_{\beta=0}^{j-1}q_{j\beta}\sum_{\alpha=\beta+j+1}^{\infty}\frac{1}{(\alpha-\beta-j-1)!}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(r+3\alpha)!l!}\Big(\frac{1}{\sqrt{n}}\Big)^{r+\alpha+2l}\\
&\quad\times(-\alpha h(\vec{t}))^l\int_{R^k}(i\vec{t},\vec{x})^{r+3\alpha}\,d\big(F(\vec{x})-Q_n(\vec{x})\big)^{*\alpha}.
\end{aligned}$$

Šią išraišką įstatome į (1) lygybę ir gauname, kad

$$\begin{aligned}
&\Big(\hat{F}\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^n\\
&= \hat{G}(\vec{t})\Big(1+\sum_{m=1}^{\infty}\frac{1}{m!}\Big(\varDelta\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\Big)^m + \sum_{j=1}^{\infty}\Big(\frac{1}{n}\Big)^j A_j\varDelta_n\Big(\frac{\vec{t}}{\sqrt{n}}\Big)\mathrm{e}^{\varDelta_n(\frac{\vec{t}}{\sqrt{n}})}\Big)\\
&= \hat{G}(\vec{t})\Big\{1+\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{l!(r+3m)!}\Big(\frac{1}{\sqrt{n}}\Big)^{r+m+2l}(-mh(\vec{t}))^l\\
&\quad\times\int_{R^k}(i\vec{t},\vec{x})^{r+3m}\,d\big(F(\vec{x})-Q_n(\vec{x})\big)^{*m} + \sum_{j=1}^{\infty}\Big(\frac{1}{n}\Big)^j(-1)^j
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\beta=0}^{j-1} \sum_{\alpha=\beta+j+1}^{\infty} \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} q_{j\beta} \frac{1}{(\alpha-\beta-j-1)!(r+3\alpha)!l!} \left(\frac{1}{\sqrt{n}}\right)^{r+\alpha+2l+2j} \\
&\times \big(-\alpha h(\vec{t})\big)^l \int_{R^k} (i\vec{t},\vec{x})^{r+3\alpha}\, d\big(F(\vec{x})-Q_n(\vec{x})\big)^{*\alpha} \Bigg\} \\
={}& \hat{G}(\vec{t}) + \sum_{\gamma=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^{\gamma} \sum_{r+m+2l=\gamma} \frac{(-m)^l}{m!l!(r+3m)!} \\
&\times \int_{R^k} \left[\frac{\partial^{l+r+3m}}{\partial\varepsilon^l \partial\varrho^{r+3m}} \mathrm{e}^{(1+\varepsilon)h(\vec{t})+\varrho(i\vec{t},\vec{x})}\right]_{\substack{\varepsilon=0\\ \varrho=0}} d\big((F(\vec{x})-Q_n(\vec{x})\big)^{*m} \\
&+ \sum_{\gamma=4}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^{\gamma} \sum_{r+\alpha+2l+2j=\gamma} (-1)^j q_{j\beta} \frac{(-\alpha)^l}{(\alpha-\beta-j-1)!(r+3\alpha)!l!} \\
&\times \int_{R^k} \left[\frac{\partial^{l+r+3\alpha}}{\partial\varepsilon^l \partial\varrho^{r+3\alpha}} \mathrm{e}^{(1+\varepsilon)h(\vec{t})+\varrho(i\vec{t},\vec{x})}\right]_{\substack{\varepsilon=0\\ \varrho=0}} d\big((F(\vec{x})-Q_n(\vec{x})\big)^{*\alpha}. \qquad (4)
\end{aligned}
$$

Dabar būtina aptarti charakteringąją funkciją

$$\exp\big\{i(\vec{t},\varrho\vec{x}) + (1+\varepsilon)h(\vec{t})\big\},$$

kuri yra be galo dali su Levy tripletu

$$\mathcal{L}\big(\varrho, 0, (1+\varepsilon)\nu\big),$$

čia $0 \leqslant \varrho < 1$, $0 \leqslant \varepsilon < 1$.

Akivaizdu, kad

$$\big(\hat{G}(\vec{t})\big)^{1+\varepsilon} \mathrm{e}^{i(\vec{t},\varrho\vec{x})} = \exp\big\{i(\vec{t},\varrho\vec{x}) + (1+\varepsilon)h(\vec{t})\big\}.$$

Pasinaudodami formuliu asimptotiniu skleidiniu (4) mes gauname tikimybės

$$P\big\{\vec{\xi}_1 + \cdot + \vec{\xi}_n = \big((\vec{\beta}_1,\vec{\nu}_1),\dots,(\vec{\beta}_k,\vec{\nu}_k)\big)\big\}$$

asimptotinį skleidinį.

Tarkime, kad egzistuoja tankiai

$$g^{*(1+\varepsilon)}(\vec{y}) = \frac{1}{(2\pi)^k} \int_{R^k} \mathrm{e}^{-i(\vec{t},\vec{y})} \mathrm{e}^{(1+\varepsilon)h(\vec{t})}\, d\vec{t}$$

ir

$$p^{*n}(\vec{y}) = \left(\frac{1}{2\pi}\right)^k \int_{R^k} \mathrm{e}^{-i(\vec{t},\vec{y})} \hat{F}_n\left(\frac{\vec{t}}{\sqrt{n}}\right) d\vec{t},$$

$0 \leqslant \varepsilon < 1$.

Tuomet nepriklausomų vienodai pasiskirsčiusių kvazigardelinių atsitiktinių dydžių sekai $\xi_1, \ldots, \xi_n$ yra teisingas formalus skleidinys

$$\begin{aligned}
&\frac{(\sqrt{n})^k}{\beta_1 \ldots \beta_k} P\bigl\{\xi_1 + \cdots + \xi_n = (\vec{\beta}, \vec{\nu})\bigr\} \\
&\quad = g\biggl(\frac{\vec{\beta}\vec{\nu}}{\sqrt{n}}\biggr) + \sum_{\gamma=1}^{\infty} \biggl(\frac{1}{\sqrt{n}}\biggr)^{\gamma} \sum_{r+m+2l=\gamma} \frac{(-m)^l}{m!l!(r+3m)!} \\
&\qquad \times \int_{R^k} \biggl[\frac{\partial^{l+r+3m}}{\partial\varepsilon^l \partial\varrho^{r+3m}} g^{*(1+\varepsilon)}\biggl(\frac{\vec{\beta}\vec{\nu}}{\sqrt{n}} - \varrho\vec{x}\biggr)\biggr]\bigg|_{\substack{\varepsilon=0\\ \varrho=0}} d\bigl(F(\vec{x}) - Q_n(\vec{x})\bigr)^{*m} \\
&\qquad + \sum_{\gamma=4}^{\infty} \biggl(\frac{1}{\sqrt{n}}\biggr)^{\gamma} \sum_{r+\alpha+2l+2j=\gamma} (-1)^j q_{j\beta} \frac{(-\alpha)^l}{(\alpha-\beta-j-1)!(r+3\alpha)!l!} \\
&\qquad \times \int_{R^k} \biggl[\frac{\partial^{l+r+3\alpha}}{\partial\varepsilon^l \partial\varrho^{r+3\alpha}} g^{*(1+\varepsilon)}\biggl(\frac{\vec{\beta}\vec{\nu}}{\sqrt{n}} - \varrho\vec{x}\biggr)\biggr]\bigg|_{\substack{\varepsilon=0\\ \varrho=0}} d\bigl(F(\vec{x}) - Q_n(\vec{x})\bigr)^{*\alpha} + \cdots
\end{aligned}$$

(žiūr. (4)). Čia $g(\vec{y}) = g^{*1}(\vec{y})$, $\vec{\beta}\vec{\nu} = (\beta_1\nu_1, \ldots, \beta_k\nu_k)$,

$$\sum_{r+m+2l=\gamma} = \sum_{m=1}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty},$$

$$\sum_{r+\alpha+2l+2j=\gamma} = \sum_{j=1}^{\infty}\sum_{\beta=0}^{j-1}\sum_{\alpha=\beta+j+1}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{\infty}.$$

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SUMMARY

Asymptotic expansions for distributions of sums of a double sequence of quasi-lattice random variables

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We consider the formal asymptotic expansion of probability distribution of the sums of independent random variables. The approximation was made by using infinitely divisible probability distributions.

Keywords: Quasi-lattice distribution, infinitely divisible distribution, Appel polynomials.