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An Interval Efficiency Analysis with Dual-Role Factors

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Abstract:

Data envelopment analysis (DEA) is a data-driven and benchmarking tool for evaluating the relative efficiency of production units with multiple outputs and inputs. Conventional DEA models are based on a production system by converting inputs to outputs using input-transformation-output processes. However, in some situations, it is inescapable to think of some assessment factors, referred to as dual-role factors, which can play simultaneously input and output roles in DEA. The observed data are often assumed to be precise although it needs to consider uncertainty as an inherent part of most real-world applications. Dealing with imprecise data is a perpetual challenge in DEA that can be treated by presenting the interval data. This paper develops an imprecise DEA approach with dual-role factors based on revised production possibility sets. The resulting models are a pair of mixed binary linear programming problems that yield the possible relative efficiencies in the form of intervals. In addition, a procedure is presented to assign the optimal designation to a dual-role factor and specify whether the dual-role factor is a nondiscretionary input or output. Given the interval efficiencies, the production units are categorized into the efficient and inefficient sets. Beyond the dichotomized classification, a practical ranking approach is also adopted to achieve incremental discrimination through evaluation analysis. Finally, an application to third-party reverse logistics providers is studied to illustrate the efficacy and applicability of the proposed approach.

An Interval Efficiency Analysis with Dual-Role Factors

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Abstract

Data envelopment analysis (DEA) is a data-driven and benchmarking tool for evaluating the relative efficiency of production units with multiple outputs and inputs. Conventional DEA models are based on a production system by converting inputs to outputs using input-transformation-output processes. However, in some situations, it is inescapable to think of some assessment factors, referred to as dual-role factors, which can play simultaneously input and output roles in DEA. The observed data are often assumed to be precise although it needs to consider uncertainty as an inherent part of most real-world applications. Dealing with imprecise data is a perpetual challenge in DEA that can be treated by presenting the interval data. This paper develops an imprecise DEA approach with dual-role factors based on revised production possibility sets. The resulting models are a pair of mixed binary linear programming problems that yield the possible relative efficiencies in the form of intervals. In addition, a procedure is presented to assign the optimal designation to a dual-role factor and specify whether the dual-role factor is a nondiscretionary input or output. Given the interval efficiencies, the production units are categorized into the efficient and inefficient sets. Beyond the dichotomized classification, a practical ranking approach is also adopted to achieve incremental discrimination through evaluation analysis. Finally, an application to third-party reverse logistics providers is studied to illustrate the efficacy and applicability of the proposed approach.

Keywords: Data envelopment analysis; Dual-role factors; Production possibility sets; imprecise data; Third-Party Reverse Logistics Providers.

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1 Introduction

Due to the growth of new technologies, complex business environment and environmental legislation, the firms are involved in recycling and re-manufacturing functions through third-party logistics providers (3PLPs), which may positively influence the performance of the firm. Reverse logistics (Figure 1) is a group of processes for moving a new array of products, goods and parts from one destination to another with the aim of creating value at the end of common direct supply chains (Rogers & Tibben-Lembke, 2001; Dowlatshahi, 2000). For example, this new array of products at the point of consumption include (i) failed products and goods that can be repaired or reused (ii) obsolete products and parts that still have value and (iii) unsold products by retailers (Du & Evans, 2008). Needless to say, these products, goods and parts provide increasingly economic values at the end of the direct supply chain. Reverse logistics is practised in numerous industries, businesses, commercial and consumer organizations as economic opportunities and support of environmental protection (Du & Evans, 2008). The success in reverse logistics brings about notable advantages of the firm, including improved customer satisfaction, reductions in resource consumption levels, and decreased inventory and distribution costs (Autry, Daugherty, & Glenn Richey, 2001; Andel, 1997). Many companies are not able to surmount the intricate reverse logistics processes and all or part of reverse logistics processes are outsourced to 3PLPs (Krumwiede & Sheu, 2002). The basic process of 3PLPs performance is depicted in Figure 2; as shown, in a supply chain a part of productions (used or unused) may return to a reproduction process. This part must be first collected by a 3PLP, then the company buys the collected productions which can undergo various reverse logistics operations before being reused.

Due to a plethora of 3PLPs in the market, it is of extreme importance to evaluate and select the best 3PLPs leading to an effective reverse logistics (Meade & Sarkis, 2002; Govindan, Palaniappan, Zhu, & Kannan, 2012).

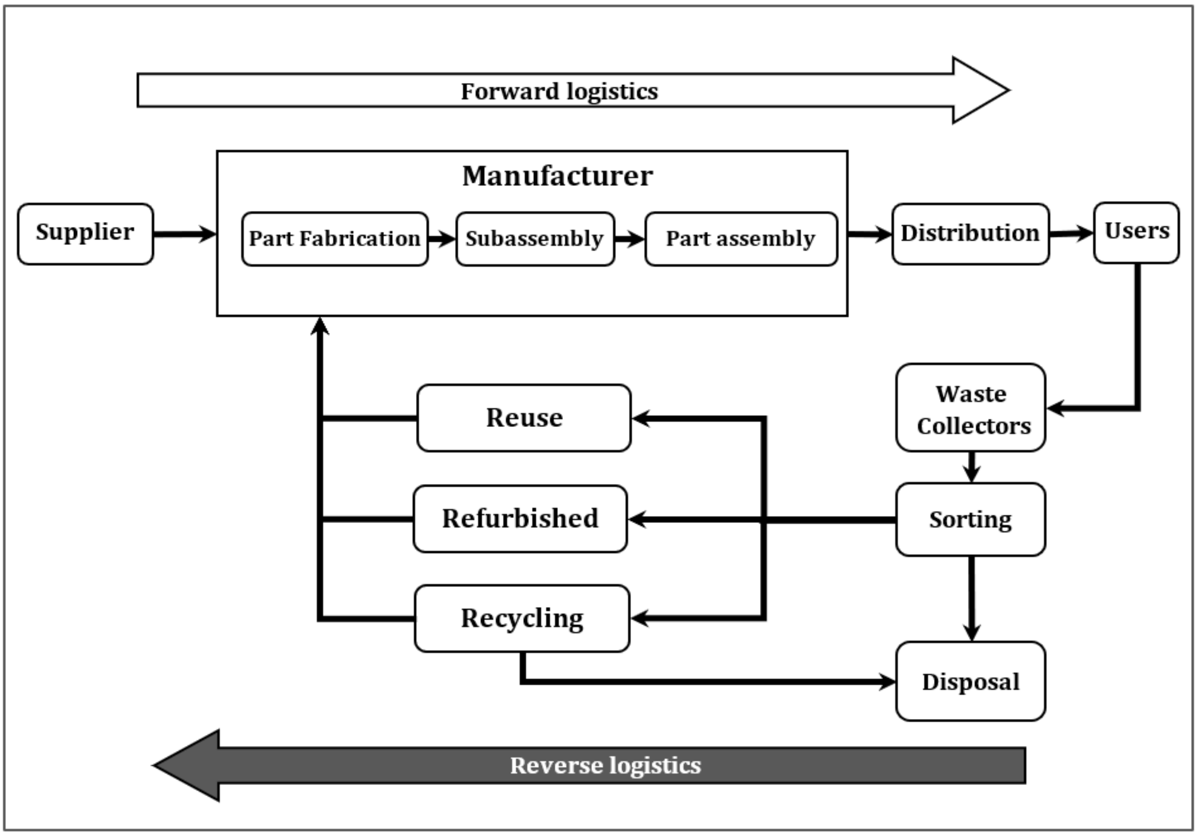

Figure 1. Forward and reverse logistics

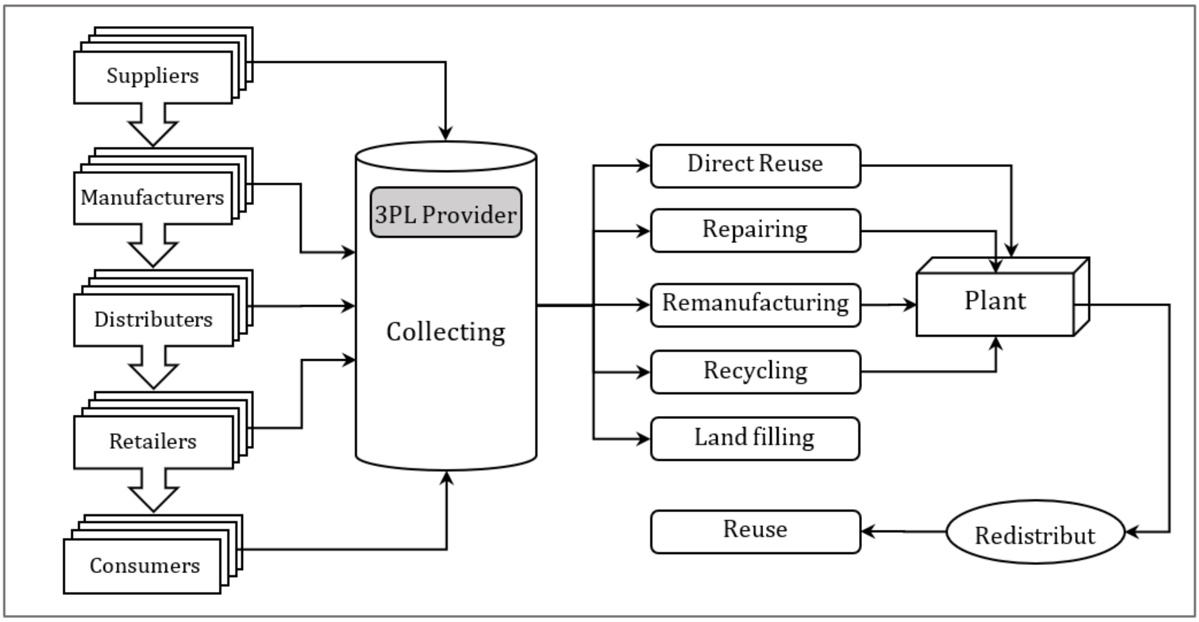

Figure 2. the role of 3PL provider

There has been much effort made to improve the performance of data analytics frameworks to uncover information and insights that can help organizations make informed business decisions and increase revenues. Data envelopment analysis (DEA) is a popular data-oriented and analytics methodology for measuring the relative efficiency of a set of similar production units, called decision making units (DMUs). Charnes, Cooper, & Rhodes (1978) first developed the original DEA model, called the CCR (Charnes, Cooper and Rhodes) model, based on the engineering idea of technical efficiency introduced by Farrell (1957). The authors generalized the single-output, single-input ratio measure of the efficiency of a DMU to a multiple-inputs multiple-outputs setting. In reality, the production function is unknown, particularly in the public sector and DEA strives to estimate the production possibility set (PPS) based upon all observations and some axioms. So, DEA is focusing on evaluating the efficiency of DMUs relative to an estimated production possibility frontier. One direct result of DEA is to divide DMUs into two groups: *efficient* DMUs and *inefficient*. DEA has gained substantial attention as a managerial tool amongst researchers and practitioners for measuring the performance of organizations in public and private sectors such as banks, airports, schools, hospitals, universities, high-tech businesses and so on (Emrouznejad, Parker, & Tavares, 2008).

The standard production process consumes inputs (resources) to produce outputs (products or service). However, some situations include the particular assessment factor(s) that has the simultaneous roles of inputs and outputs (Beasley, 1990; Beasley, 1995; Cook, Green, & Zhu, 2006). For example, when measuring the hospital's performance is deemed, the number of nurse trainees on staff can play the role of both inputs and outputs. This type of factor which is called *dual-role factor* or *flexible measure* has been split the relevant literature into two streams. These streams associated with *dual-role factor* and *flexible measures* started with Cook et al. (2006) and Cook & Zhu (2007), respectively.

Beasley (1990, 1995) was the first to consider *research funding* as both the input and output for measuring the efficiency of university departments. However, Cook et al. (2006) found the model proposed by Beasley (1990, 1995) inappropriate on account of two shortcomings; (i) all DMUs turn out to be efficient in the absence of weight restrictions, and (ii) the dual-role factor is assumed to be a discretionary variable which leads to illogical behaviour for projecting inefficient units on the efficient frontier. To get through the problem in input-oriented models, Cook et al. (2006) considered each dual-role factor as a nondiscretionary variable on the input side. The authors extended the Beasley's approach in a way that all DMUs are classified into three clusters according to whether such a dual-factor is designated as an output, an (non-discretionary) input, or is in equilibrium. In another direction, Cook & Zhu (2007) and Toloo (2012) modified the multiplier and envelopment forms of original DEA model, respectively, in the presence of flexible measures in order to decide the status of these measures. The authors defined the binary variables for each flexible measure for multiplier and envelopment forms to develop a pair of optimistic and pessimistic approaches, respectively, by assigning the most appropriate status (output or input) to these measures (for more details see Toloo, 2014a). Our study in this paper contributes to the existing literature on the dual-role factors-based models.

Farzipoor Saen (2011) utilized Cook et al. (2006)'s model with multiple dual-role factors for selecting the appropriate third-party reverse logistics (3PL) provider(s). In supply chain management, Farzipoor Saen & Farzipoor Sean (2010) considered weight restrictions proposed by Wong & Beasley (1990) in addition to multiple dual-role factors for opting for the best suppliers. Kumar, Jain, & Kumar (2014) slightly modified Farzipoor Saen & Farzipoor Sean (2010)'s model to solve the supplier selection problem in which carbon footprints of suppliers is behaving as a dual-role factor. Toloo & Barat (2015) formulated a model to designate the status of dual-role factors for effective supply chain network design.

DEA presumes that the assessment factors are measured by an immense precision tool and every measure would represent its precise value. However, in real-world evaluation problems, this assumption is rarely acceptable and a random bias embedded in inputs and outputs often lead to overestimation or underestimation of true values. One solution to deal with imprecise data is to define the intervals in which the actual values fall in their intervals. Cooper, Park, & Yu (1999, 2001) were the first to discuss the case of interval data in DEA and the methodology has been commonly called imprecise DEA (IDEA). Soon afterwards, the body of literature has documented a great deal of research on IDEA (see e.g., Zhu, 2003; Despotis & Smirlis, 2002; Wang, Greatbanks, & Yang, 2005). Cooper, Park, & Yu (1999, 2001) used a scale transformation technique coupled with the change of variables to convert the nonlinear IDEA model into a linear one. Zhu (2003) first criticized the need for many data transformations and variable alternations in Cooper, Park, & Yu (1999, 2001) due to the computational burden, and then made an attempt to improve the IDEA model in the presence of weight restrictions in a way that the model correctly treats the scale transformation and variable alternation based upon the CCR model. Toloo & Nalchigar (2011) and Toloo (2014b) extended the approach of Zhu (2003) to cope with the supplier selection problem with imprecise data. The efficiencies calculated from the above IDEA models are consistently exact values while it is envisioned that the interval data give rise to the interval efficiency measures in practice. To tackle this problem, Despotis & Smirlis (2002) first proposed a pair of DEA-based models from the optimistic and pessimistic viewpoints to compute the upper and lower bounds for the efficiency of DMUs, and then the interval efficiencies were used to categorize each DMU as a pre-designed group. The idea of Despotis & Smirlis (2002) has been the base for handling the imprecise data in a variety of DEA models (see e.g., Toloo, Aghayi, & Rostamy-malkhalifeh, 2008; Hatami-Marbini, Emrouznejad, & Agrell, 2014; Hatami-Marbini, Emrouznejad, & Tavana, 2011; Shokouhi, Hatami-Marbini, Tavana, & Saati, 2010; Hadi-Vencheh, Hatami-Marbini, Ghelej Beigi, & Gholami, 2015). The main criticism of Despotis & Smirlis (2002) is to use two different technologies for a given DMU to measure the upper and lower bounds of efficiency. Wang et al. (2005) first represented this issue by usinga simple numerical example and then developed a pair of models that includes a unique technology for the sake of equitable assessment. Looking into the literature, it was found that Wang et al. (2005)'s model has received extensive attention among researchers in recent years (see e.g., Hatami-Marbini, Beigi, Fukuyama, & Gholami, 2015; Shokouhi, Shahriari, Agrell, & Hatami-Marbini, 2014).

Cook et al. (2006)'s model was adapted by Toloo, Keshavarz, & Hatami-Marbini (2018) to deal with imprecise dual-role factors. Toloo et al. (2018) lately developed a framework which contributes to imprecise DEA models in the presence of dual-role factors. Their methodology entails the development of a pair of optimistic and pessimistic models based on Despotis & Smirlis (2002)'s approach to gain the interval efficiencies for all DMUs, along with designating the status of dual-role factors. The authors integrated both models to identify the status of each dual-role factor which may be designated as either an input or output from optimistic and pessimistic respects. The authors finally made use of a fuzzy decision-making method to specify an identical role for each dual-role factor.

In summary, the above-mentioned overview of ancillary studies shows that IDEA models with dual-role factors have received less attention from the research community and the related literature is not rich as to how the status of a dual-role factor is identified due to complexity and instability of results calculated from IDEA models. This paper looks into the complexities of dual-role factors in IDEA and presents some alternative models for dealing with data uncertainties.

As mentioned earlier, Toloo et al. (2018)'s models are built on Despotis & Smirlis (2002) which can be severely criticized for the use of incommensurate constraint sets, viz., different technologies, for a certain DMU to obtain the upper and lower efficiency measures. Though it seems that this study is similar to Toloo et al. (2018), with a glance it shows that there is a fundamental difference between them and this study deals with incommensurability in the Toloo et al. (2018) models.

Our method includes a pair of mixed binary linear programming (MBLP) models based upon revising PPSs and the adaptation of developed IDEA proposed by Wang et al. (2005) for the purpose of calculating the lower and upper bounds of the best relative efficiencies from optimistic and pessimistic viewpoints in a commensurable way. Assigning the optimal designation to a dual-role factor in IDEA with the intention of identifying whether it is behaving as an input or output is also in question. So, we make an attempt to fill this gap in the literature by developing a procedure for the designation of the dual-role factors. Given that analysing the efficiency changes in the intervals is a complicated process for decision-makers, particularly determining a complete ranking, a ranking approach is accommodated to the obtained interval efficiencies to yield better insight into the performance analysis. A case study is presented to select and evaluate the best 3PLPs by using the proposed method in this study.

This paper is organized into six sections: Section 2 presents three DEA models with precise and interval data. Section 3 includes the mathematical details of the IDEA proposed in this study as well as presenting a procedure for determining the unique status of each dual-role factor. Section 4 presents an application and managerial implications of the developed method. Lastly, Section 5 is composed of our conclusions and future research directions.

2 Background: Three DEA models

This section provides an overview of three preliminary DEA models with the constant returns to scale (CRS) technology. The first DEA model is assumed to utilize precise input and output data, the second model thinks of impreciseness by interval data, and the last DEA model is presented in the presence of dual-role factors as well as precise inputs and outputs.

2.1 CCR model

Charnes et al. (1978) initially developed the CRS model by assuming full proportionality between the inputs and outputs. Consider the problem of performance evaluation of n DMUs in which each DMU consumes m various inputs to produce s various outputs. Let $\boldsymbol{x}_j = [x_{ij}], i = 1, \dots, m$ and $\boldsymbol{y}_j = [y_{rj}], r = 1, \dots, s$ denote the input and output vectors of $\text{DMU}_j, j = 1, \dots, n$, respectively. It is supposed that the input and output vectors are semi-positive, i.e. $\boldsymbol{x}_j \geq \boldsymbol{0}_m$[2]$, \boldsymbol{x}_j \neq \boldsymbol{0}_m$ and $\boldsymbol{y}_j \geq \boldsymbol{0}_s, \boldsymbol{y}_j \neq \boldsymbol{0}_s$. A pair of such semi-positive input vector $\boldsymbol{x} \in \mathbb{R}^m$ and output vector $\boldsymbol{y} \in \mathbb{R}^s$ is literally called an *observation* and expressed by the notation $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{m+s}$. The PPS or technology is the set of feasible observations as follows:

$$T = \{(\boldsymbol{x}, \boldsymbol{y}) | \boldsymbol{y} \text{ can be produced by } \boldsymbol{x}\} \tag{1}$$

Given that the technology is not known in practice, DEA as an axiomatic approach estimates a technology using a combination of observations and a set of axioms. Assuming the feasibility, CRS, convexity and free disposability axioms with the minimum extrapolation principle (see Cooper, Seiford, & Tone, 2007), the empirical technology is given as:

$$T = \left\{(\boldsymbol{x}, \boldsymbol{y}) \middle| \boldsymbol{x} \geq \textstyle\sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j, \boldsymbol{y} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j, \lambda_j \geq 0 \ \forall j\right\} \tag{2}$$

Using the technology defined in (2), the Farrell measure of technical efficiency (Farrell ,1957) for DMU_o ($o \in \{1, \dots, n\}$) is defined as:

$$e(\boldsymbol{x}_o, \boldsymbol{y}_o) = \min\{\theta | (\theta \boldsymbol{x}_o, \boldsymbol{y}_o) \in T\} \tag{3}$$

Charnes et al. (1978) developed the following linear programming to measure the technical efficiency of DMU_o:

$$\begin{aligned}
&e_o = \min \theta \\
&\text{s.t.} \\
&\textstyle\sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j \leq \theta \boldsymbol{x}_o \\
&\textstyle\sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y}_o \\
&\lambda_j \geq 0 \qquad\qquad \forall j
\end{aligned} \tag{4}$$

The optimal solution θ^* as an efficiency measure of DMU_o, describes the maximal proportional reduction of all inputs that allow us to produce the given outputs. Let $(\theta^*, \boldsymbol{\lambda}^*) = (\theta^*, \lambda_1^*, \dots, \lambda_n^*)$ be the optimal solution of model (4); DMU_o is *CCR-efficient* if

[2] $\boldsymbol{0}_m = (0, \dots, 0) \in \mathbb{R}^m$.

$e_o = 1$ with zero-slack; otherwise, DMU_o is *CCR-inefficient* (Cooper et al. ,2007). The CCR model (4) is called the *envelopment* problem in the literature and its dual is called the *multiplier* problem as formulated below:

$$\begin{aligned} & e_o = \max \textstyle\sum_{r=1}^{s} u_r y_{ro} \\ & \text{s.t.} \\ & \textstyle\sum_{i=1}^{m} v_i x_{io} = 1 \\ & \textstyle\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \quad && \forall j \\ & v_i \geq 0 && \forall i \\ & u_r \geq 0 && \forall r \end{aligned} \tag{5}$$

where u_r and v_i are the weights associated with r^{th} output and i^{th} input, respectively, and e_o represents the efficiency measure of DMU_o. Assume that model (5) is solved and its optimal solution $(\boldsymbol{u}^*, \boldsymbol{v}^*) = (u_1^*, \dots, u_s^*, v_1^*, \dots, v_m^*)$ is at hand; DMU_o is *CCR-efficient* if $e_o = 1$ and there exists at least a strictly positive optimal solution $(\boldsymbol{u}^*, \boldsymbol{v}^*)$. Otherwise, DMU_o is *CCR-inefficient* (Cooper et al. ,2007).

2.2 *Imprecise CCR model*

Consider the situation that all input and output data $x_{ij} (i = 1, \dots, m)$, and $y_{rj} (r = 1, \dots, s, j = 1, \dots, n)$ cannot be precisely gauged in view of uncertainty. However, the true values lie within bounded intervals $x_{ij} \in \left[x_{ij}^l, x_{ij}^u\right]$ and $y_{rj} \in \left[y_{rj}^l, y_{rj}^u\right]$ where $x_{ij}^l \geq 0$ and $y_{rj}^l \geq 0$. The main body of the literature in IDEA is devoted to seek lower and upper bounds of efficiency for all DMUs using two distinct linear programming models from the optimistic and pessimistic points of view. One popular method was developed by Despotis & Smirlis (2002), in this regard where two distinct production frontiers are deployed to measure the upper and lower bounds of efficiency for each DMU. Let T_o^{pes} and T_o^{opt} be the technologies for pessimistic and optimistic production frontiers, respectively.

$$T_o^{pes} = \left\{(\boldsymbol{x}, \boldsymbol{y}) \middle| \boldsymbol{x} \geq \textstyle\sum_{j=1 (j \neq o)}^{n} \lambda_j \boldsymbol{x}_j^l + \lambda_o \boldsymbol{x}_o^u, \boldsymbol{y} \leq \sum_{j=1 (j \neq o)}^{n} \lambda_j \boldsymbol{y}_j^u + \lambda_o \boldsymbol{y}_o^l, \lambda_j \geq 0 \; \forall j \right\} \tag{6}$$

$$T_o^{opt} = \left\{(\boldsymbol{x}, \boldsymbol{y}) \middle| \boldsymbol{x} \geq \textstyle\sum_{j=1 (j \neq o)}^{n} \lambda_j \boldsymbol{x}_j^u + \lambda_o \boldsymbol{x}_o^l, \boldsymbol{y} \leq \sum_{j=1 (j \neq o)}^{n} \lambda_j \boldsymbol{y}_j^l + \lambda_o \boldsymbol{y}_o^u, \lambda_j \geq 0 \; \forall j \right\} \tag{7}$$

From the pessimistic viewpoint, the production frontier is built by deeming the worst situation for DMU under evaluation and the best situation for the outstanding DMUs while from the optimistic viewpoint the production frontier is made up by considering the best situation for DMU under evaluation and the worst situation for the other DMUs. The following pair of envelopment models are accordingly formulated to measure the lower and upper bounds of the interval efficiency for DMU_o, based on T_o^{pes} and T_o^{opt} (Despotis & Smirlis, 2002):

$$\begin{aligned} E_o^l &= \min\theta \\ \text{s.t.}& \\ \textstyle\sum_{j=1(j\neq o)}^{n}\lambda_j\boldsymbol{x}_j^l+\lambda_o\boldsymbol{x}_o^u &\le \theta\boldsymbol{x}_o^u \\ \textstyle\sum_{j=1(j\neq o)}^{n}\lambda_j\boldsymbol{y}_j^u+\lambda_o\boldsymbol{y}_o^l &\ge \boldsymbol{y}_o^l \\ \lambda_j \ge 0 & \qquad \forall j \end{aligned} \tag{8}$$

$$\begin{aligned} E_o^u &= \min\theta \\ \text{s.t.}& \\ \textstyle\sum_{j=1(j\neq o)}^{n}\lambda_j\boldsymbol{x}_j^u+\lambda_o\boldsymbol{x}_o^l &\le \theta\boldsymbol{x}_o^l \\ \textstyle\sum_{j=1(j\neq o)}^{n}\lambda_j\boldsymbol{y}_j^l+\lambda_o\boldsymbol{y}_o^u &\ge \boldsymbol{y}_o^u \\ \lambda_j \ge 0 & \qquad \forall j \end{aligned} \tag{9}$$

The use of two distinct technologies for evaluating the efficiency of a certain DMU has been criticized in Wang et al. (2005). To overcome the problem, Wang et al. (2005) proposed a unified technology to measure both lower and upper bounds of efficiency for all DMUs as follows:

$$\bar{T}=\left\{(\boldsymbol{x},\boldsymbol{y})\middle|\boldsymbol{x}\ge\textstyle\sum_{j=1}^{n}\lambda_j\boldsymbol{x}_j^l,\boldsymbol{y}\le\sum_{j=1}^{n}\lambda_j\boldsymbol{y}_j^u,\lambda_j\ge 0\ \forall j\right\} \tag{10}$$

Contrary to Despotis & Smirlis (2002)'s approach, the above technology, $\bar{T}$, is independent of DMU under evaluation and constructed by means of the lower bound of inputs $\boldsymbol{x}_j^l$ and upper bound of outputs $\boldsymbol{y}_j^u$ of all DMUs. The following pair of envelopment models are accordingly formulated to obtain the lower and upper bounds of the interval efficiency for DMU_o:

$$\begin{aligned} \bar{E}_o^l &= \min\theta \\ \text{s.t.}& \\ \textstyle\sum_{j=1}^{n}\lambda_j\,\boldsymbol{x}_j^l &\le \theta\boldsymbol{x}_o^u \\ \textstyle\sum_{j=1}^{n}\lambda_j\,\boldsymbol{y}_j^u &\ge \boldsymbol{y}_o^l \\ \lambda_j \ge 0 & \qquad \forall j \end{aligned} \tag{11}$$

$$\begin{aligned} \bar{E}_o^u &= \min\theta \\ \text{s.t.}& \\ \textstyle\sum_{j=1}^{n}\lambda_j\,\boldsymbol{x}_j^l &\le \theta\boldsymbol{x}_o^l \\ \textstyle\sum_{j=1}^{n}\lambda_j\,\boldsymbol{y}_j^u &\ge \boldsymbol{y}_o^u \\ \lambda_j \ge 0 & \qquad \forall j \end{aligned} \tag{12}$$

It is noteworthy that model (12) is based on n precise observations $\left(\boldsymbol{x}_j^l,\boldsymbol{y}_j^u\right), j=1,\dots,n$, which is used to build the corresponding technology, and model (11) is based on $n+1$ observations where a single different observation $(\boldsymbol{x}_o^u,\boldsymbol{y}_o^l)$ related to DMU_o and n other observations $\left(\boldsymbol{x}_j^l,\boldsymbol{y}_j^u\right), j=1,\dots,n$ establish the technology.

2.3 CCR model with dual-role factors

Although the conventional DEA method makes use of a set of factors, which takes inputs and outputs role for the performance assessment, in some situations, it is likely to think of the dual-role factors that have the simultaneous input and output roles. Let w_k $(k=1,\dots,K)$ denote the dual-role factors. According to Cook et al. (2006), a dual-role factor must turn into a nondiscretionary input $(\boldsymbol{w}_q^{in})$, an output $(\boldsymbol{w}_q^{out})$, or an indifferent role $(\boldsymbol{w}_q^{equ})$. Note that an indifferent role for the dual-role factor bespeaks either an output or an input, which is at an equilibrium level. Resultantly, it is a need for defining 3^K technologies for three distinct roles of each dual-role factor as follows:

$$T_{DL}^{q} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}_q^{in}, \boldsymbol{w}_q^{out}) : (\boldsymbol{x}, \boldsymbol{w}_q^{in}) \text{ can produce } (\boldsymbol{y}, \boldsymbol{w}_q^{out})\} \tag{13}$$

where $q = 1, \dots, 3^K$ is the number of possible permutations for the technologies. For example, consider a case with two dual-role factors. Since each dual-role factor can have three choices among input role, output role and equilibrium situation, the product of three by three results in nine possible technologies. In equation (13), $\boldsymbol{w}_q^{in}$ and $\boldsymbol{w}_q^{out}$ are the vectors of dual-role factors, which have the nondiscretionary input and output roles respectively in the q^{th} technology. It should be noted that even though $(\boldsymbol{w}_q^{in}, \boldsymbol{w}_q^{out}, \boldsymbol{w}_q^{equ})$ is the complete vector of dual-role factors, $\boldsymbol{w}_q^{equ}$ with a neutral role can be ignored when counting the possible number of technologies. If the role of each dual-role factor is specified, the general form of the technology in (13) under the CRS assumption can be rewritten as follows:

$$T_{DL} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}^{in}, \boldsymbol{w}^{out}) | \boldsymbol{x} \geq \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_j, \boldsymbol{w}^{in} \geq \sum_{j=1}^{n} \lambda_j \boldsymbol{w}_j^{in}, \boldsymbol{y} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_j, \boldsymbol{w}^{out} \leq \sum_{j=1}^{n} \lambda_j \boldsymbol{w}_j^{out}, \lambda_j \geq 0\ \forall j\} \tag{14}$$

Therefore, the Farrell input efficiency measure of a unit $(\boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{w}_o^{in}, \boldsymbol{w}_o^{out})$ relative to a technology T_{DL} is defined as

$$e_o = \min\{\theta | (\theta \boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{w}_o^{in}, \boldsymbol{w}_o^{out}) \in T_{DL}\} \tag{15}$$

It is essential to be noted that $\boldsymbol{w}_o^{in}$ is viewed as nondiscretionary or exogenously fixed variables such as weather, number of competitors and age of permanent workers that are not under the control of a DMU's management. The most appropriate mathematical way to deal with nondiscretionary inputs in Farrell model is to disregard the minimization of the θ associated with $\boldsymbol{w}_o^{in}$ (Banker & Morey, 1986).

Since the status of dual-role factors is unknown, we develop the following MBLP model to maximize the efficiency score of DMU$_o$ along with accommodating dual-role factors:

$$
\begin{aligned}
& e_o = \min \theta \\
& \text{s.t.} \\
& \textstyle\sum_{j=1}^{n} \lambda_j\, x_{ij} \le \theta x_{io} && \forall i \\
& \textstyle\sum_{j=1}^{n} \lambda_j\, y_{rj} \ge y_{ro} && \forall r \\
& \textstyle\sum_{j=1}^{n} \lambda_j\, w_{kj} \le w_{ko} + M b_k && \forall k \\
& \textstyle\sum_{j=1}^{n} \lambda_j\, w_{kj} \ge w_{ko} - M d_k && \forall k \\
& b_k + d_k \ge 1 && \forall k \\
& b_k, d_k \in \{0,1\} && \forall k \\
& \lambda_j \ge 0 && \forall j
\end{aligned} \tag{16}
$$

where M is a sufficiently large positive number, and b_k and d_k are the auxiliary binary variables associated with the k^{th} dual-role factor. If $b_k = 0$, then the constraint $\sum_{j=1}^{n} \lambda_j\, w_{kj} \le w_{ko} + M b_k$ is changed into $\sum_{j=1}^{n} \lambda_j\, w_{kj} \le w_{ko}$, showing that the dual-role factor w_k is treated as a nondiscretionary input since θ is not included on the right-hand side of this constraint; otherwise (i.e., $b_k = 1$) this constraint is redundant. Similarly, if $d_k = 0$, then the constraint $\sum_{j=1}^{n} \lambda_j\, w_{kj} \ge w_{ko} - M d_k$ is transformed into $\sum_{j=1}^{n} \lambda_j\, w_{kj} \ge w_{ko}$, and w_k is considered as an output or else (i.e., $d_k = 1$) this constraint is redundant. We point out that by defining the constraint $b_k + d_k \ge 1$ the dual-role factor w_k is imposed to take at most one role at a time, and in the case of $b_k = d_k = 1$, the dual role factor w_k is at an equilibrium level. Let $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$ be the optimal solution of model (16), $K_1 = \{k: b_k^* = 0\}$ and $K_2 = \{k: d_k^* = 0\}$. Accordingly, we arrive at the following technology that can be used to evaluate the performance of DMU_o:

$$
T_{DL}^{o} = \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}^{in}, \boldsymbol{w}^{out}) \left| \begin{array}{l} x_i \ge \sum_{j=1}^{n} \lambda_j x_{ij}, \forall i;\ \ w_k^{in} \ge \sum_{j=1}^{n} \lambda_j w_{kj}^{in}, \forall k \in K_1; \\ y_r \le \sum_{j=1}^{n} \lambda_j y_{rj}, \forall r;\ w_k^{out} \le \sum_{j=1}^{n} \lambda_j w_{kj}^{out}, \forall k \in K_2; \lambda_j \ge 0, \forall j \end{array} \right. \right\} \tag{17}
$$

3 Modeling dual-role factors: An Imprecise DEA model

In this section, we first present the mathematical details of the interval DEA models in the existence of dual-role factors to measure the lower and upper bounds of the best relative efficiency of each DMU. Second, we discuss how the unique status of each dual-role factor can determine based on the interval efficiencies.

3.1 Interval CCR model

Assume that there are n homogeneous DMUs to be assessed, and each DMU_j employs m interval inputs $x_{ij} \in [x_{ij}^l, x_{ij}^u]$ to produce s interval outputs $y_{rj} \in [y_{rj}^l, y_{rj}^u]$. In addition to interval inputs and outputs, there are K interval dual-role factors $w_{kj} \in [w_{kj}^l, w_{kj}^u]$ for

DMU$_j$. If the role of all dual-role factors is specified, then the corresponding production technology can be defined as follows:

$$T_{int} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}_q^{in}, \boldsymbol{w}_q^{out}): (\boldsymbol{x}, \boldsymbol{w}_q^{in}) \text{ can produce } (\boldsymbol{y}, \boldsymbol{w}_q^{out})\} \qquad (18)$$

The technology T_{int} is assumed to fulfil the following standard axioms:

(A1) Feasibility: For any $j = 1, \dots, n$, $(\boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{w}_j^{in}, \boldsymbol{w}_j^{out}) \in T_{int}$, in which $\boldsymbol{x}_j \in [\boldsymbol{x}_j^l, \boldsymbol{x}_j^u]$[3], $\boldsymbol{y}_j \in [\boldsymbol{y}_j^l, \boldsymbol{y}_j^u]$, and $\boldsymbol{w}_j^{in}, \boldsymbol{w}_j^{out} \in [\boldsymbol{w}_j^l, \boldsymbol{w}_j^u]$.

(A2) Free disposability: All dominated observations of a feasible observation are feasible, i.e. if $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}^{in}, \boldsymbol{w}^{out}) \in T_{int}$, then $\forall \overline{\boldsymbol{x}} \geq \boldsymbol{x}, \forall \overline{\boldsymbol{y}} \leq \boldsymbol{y}, \forall \overline{\boldsymbol{w}}^{in} \geq \boldsymbol{w}^{in}, \forall \overline{\boldsymbol{w}}^{out} \leq \boldsymbol{w}^{out}$, $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{w}}^{in}, \overline{\boldsymbol{w}}^{out}) \in T_{int}$.

(A3) Constant returns to scale: If an observation $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}^{in}, \boldsymbol{w}^{out})$ is feasible, then $\forall t > 0$, $(t\boldsymbol{x}, t\boldsymbol{y}, t\boldsymbol{w}^{in}, t\boldsymbol{w}^{out})$ is a feasible observation.

(A4) Convexity: The convex combination of each two feasible observations is a feasible observation, i.e. if $(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{w}_1^{in}, \boldsymbol{w}_1^{out})$, $(\boldsymbol{x}_2, \boldsymbol{y}_2, \boldsymbol{w}_2^{in}, \boldsymbol{w}_2^{out}) \in T_{int}$ then $\forall \lambda \in [0,1]$, $\lambda(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{w}_1^{in}, \boldsymbol{w}_1^{out}) + (1-\lambda)(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{w}_1^{in}, \boldsymbol{w}_1^{out}) \in T_{int}$.

The technology also satisfies the *minimum extrapolation* principle. The implication of the above axioms in line with the *minimum extrapolation* principle results in the following full formulation of technology T_{int}:

$$T_{int} = \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}^{in}, \boldsymbol{w}^{out}) \middle| \begin{array}{c} \boldsymbol{x} \geq \sum_{j=1}^n \lambda_j \boldsymbol{x}_j^l, \boldsymbol{w}^{in} \geq \sum_{j=1}^n \lambda_j \boldsymbol{w}_j^l, \boldsymbol{y} \leq \sum_{j=1}^n \lambda_j \boldsymbol{y}_j^u, \\ \boldsymbol{w}^{out} \leq \sum_{j=1}^n \lambda_j \boldsymbol{w}_j^u, \lambda_j \geq 0, \forall j \end{array} \right\} \qquad (19)$$

It is plain that if one relaxes the assumption of knowing the status of dual-role factors, then the number of possible technologies is 3^K. At present, the lower and upper bounds of the input-based Farrell efficiency of an observation under assessment, DMU$_o$, relative to the technology T_{int} is defined as follows:

$$e_o^l = \min\{\theta | (\theta \boldsymbol{x}_o^u, \boldsymbol{y}_o^l, \boldsymbol{w}_o^u, \boldsymbol{w}_o^l) \in T_{int}\} \qquad (20)$$

$$e_o^u = \min\{\theta | (\theta \boldsymbol{x}_o^l, \boldsymbol{y}_o^u, \boldsymbol{w}_o^l, \boldsymbol{w}_o^u) \in T_{int}\} \qquad (21)$$

The lower and upper degrees of efficiency of DMU$_o$, denoted by e_o^l and e_o^u, are measured from the pessimistic and optimistic viewpoints, respectively, when the defined technology T_{int} is unique and independent of the observation under evaluation. Although Toloo et al. (2018) latterly developed a method based on Despotis & Smirlis (2002)'s

[3] $\boldsymbol{x}_j \in [\boldsymbol{x}_j^l, \boldsymbol{x}_j^u]$ stands for $x_{ij} \in [x_{ij}^l, x_{ij}^u], \forall i$.

approach to determine the status of dual-role factors in the presence of imprecise data and dual factors, this has not been richly studied in the relevant literature, especially from the envelopment-based respect by the virtue of its complexity. The present study aims to place an emphasis on Wang et al. (2005)'s approach to tackle incommensurability arguably viewed in Despotis & Smirlis (2002). As far as we know, this is the first endeavour to estimate the PPS in the presence of dual-role factors in an uncertain environment. To this end, we propose a pair of MBLP models to measure the extreme bounds of the best relative efficiency of DMU$_o$ in the envelopment form as follows:

$$\begin{aligned} e_o^l = {} & \min \theta \\ \text{s.t.} \\ & \textstyle\sum_{j=1}^n \lambda_j x_{ij}^l \le \theta x_{io}^u && \forall i \\ & \textstyle\sum_{j=1}^n \lambda_j y_{rj}^u \ge y_{ro}^l && \forall r \\ & \textstyle\sum_{j=1}^n \lambda_j w_{kj}^l \le w_{ko}^u + Mb_k && \forall k \\ & \textstyle\sum_{j=1}^n \lambda_j w_{kj}^u \ge w_{ko}^l - Md_k && \forall k \\ & b_k + d_k \ge 1 && \forall k \\ & b_k, d_k \in \{0,1\} && \forall k \\ & \lambda_j \ge 0 && \forall j \end{aligned} \tag{22}$$

$$\begin{aligned} e_o^u = {} & \min \theta \\ \text{s.t.} \\ & \textstyle\sum_{j=1}^n \lambda_j x_{ij}^l \le \theta x_{io}^l && \forall i \\ & \textstyle\sum_{j=1}^n \lambda_j y_{rj}^u \ge y_{ro}^u && \forall r \\ & \textstyle\sum_{j=1}^n \lambda_j w_{kj}^l \le w_{ko}^l + Mb_k && \forall k \\ & \textstyle\sum_{j=1}^n \lambda_j w_{kj}^u \ge w_{ko}^u - Md_k && \forall k \\ & b_k + d_k \ge 1 && \forall k \\ & b_k, d_k \in \{0,1\} && \forall k \\ & \lambda_j \ge 0 && \forall j \end{aligned} \tag{23}$$

where b_k and d_k are the auxiliary binary variables associated with the k^{th} dual-role factor. The third and fourth sets of constraints in models (22) and (23), which are known as *either-or* constraints, are built to determine the status of the dual-role factors in a way that we arrive at the best interval efficiency measure for DMU$_o$. In addition, the third set of constraints in models (22) and (23) demonstrates that the dual-role factors are treated as nondiscretionary on the input side. In model (22), if $b_k = 0$ $(d_k = 0)$, the k^{th} dual-role factor has an input (output) role and if $b_k = 1$ $(d_k = 1)$, it shows that the k^{th} dual-role factor does not play an input (output) role. Put differently, the mathematical analysis of these constraints in model (22) is dependent on b_k and d_k as summed up below: It is clear that $\boldsymbol{w}_o^l$ is treated as nondiscretionary variable on the input side.

$$\sum_{j=1}^n \lambda_j w_{kj}^l \le w_{ko}^u + Mb_k = \begin{cases} \sum_{j=1}^n \lambda_j w_{kj}^l \le w_{ko}^u \text{ (Active)} & \text{if } b_k = 0 \\ \sum_{j=1}^n \lambda_j w_{kj}^l \le w_{ko}^u + M \text{(Redundant)} & \text{if } b_k = 1 \end{cases}$$

$$\sum_{j=1}^n \lambda_j w_{kj}^u \ge w_{ko}^l - Md_k = \begin{cases} \sum_{j=1}^n \lambda_j w_{kj}^u \ge w_{ko}^l \text{ (Active)} & \text{if } d_k = 0 \\ \sum_{j=1}^n \lambda_j w_{kj}^u \ge w_{ko}^l - M \text{(Redundant)} & \text{if } d_k = 1 \end{cases}$$

Analogously, the dual-role constraints of model (23) can be analysed as follows:

$$\sum_{j=1}^{n}\lambda_j\, w_{kj}^{l} \le w_{ko}^{l} + Mb_k = \begin{cases} \sum_{j=1}^{n}\lambda_j\, w_{kj}^{l} \le w_{ko}^{l}\ \text{(Active)} & \text{if } b_k = 0 \\ \sum_{j=1}^{n}\lambda_j\, w_{kj}^{l} \le w_{ko}^{l} + M\text{(Redundant)} & \text{if } b_k = 1 \end{cases}$$

$$\sum_{j=1}^{n}\lambda_j\, w_{kj}^{u} \ge w_{ko}^{u} - Md_k = \begin{cases} \sum_{j=1}^{n}\lambda_j\, w_{kj}^{u} \ge w_{ko}^{u}\ \text{(Active)} & \text{if } d_k = 0 \\ \sum_{j=1}^{n}\lambda_j\, w_{kj}^{u} \ge w_{ko}^{u} - M\text{(Redundant)} & \text{if } d_k = 1 \end{cases}$$

The following theorem reveals the relations between models (22) and (23).

Theorem 1. $e_o^l \le e_o^u$.

Proof. See Appendix A.

Referring to the strong duality theorem of linear programming, the multiplier and envelopment forms in traditional DEA which establish a pair of mutually dual problems have an identical optimal objective value (Cooper et al., 2007). There is a unified way in linear programming to formulate a dual problem based upon a given primal (original) problem. However, dual problems for integer programming cannot uniquely defined and all relationships available in the linear programming duality may not hold for such a primal-dual pair (Walukiewicz, 1981). Generally, in duality theory of optimization problems, a dual (primal) variable is associated with a primal (dual) constraint. We utilize this property to develop the multiplier forms of the proposed envelopment models (22) and (23). Since these models include either-or constraints, it is envisaged that their dual models entail either-or decision variables. For instance, the either-or constraints $\sum_{j=1}^{n}\lambda_j\, w_{kj}^{l} \le w_{ko}^{u} + Mb_k$ and $\sum_{j=1}^{n}\lambda_j\, w_{kj}^{u} \ge w_{ko}^{l} - Md_k$ along with the constraint $b_k + d_k \ge 1$ in the envelopment model (22) make provision for the condition that at most one of the input constraints $\sum_{j=1}^{n}\lambda_j\, w_{kj}^{l} \le w_{ko}^{u}$ or the output constraints $\sum_{j=1}^{n}\lambda_j\, w_{kj}^{u} \ge w_{ko}^{l}$ holds for the k^{th} dual-role factor. Likewise, we define either-or decision variables in the multiplier models with the aim of satisfying the condition that, for each dual-role factor, at most one respective input weight or output weight can take the positive value. To this end, we employ auxiliary binary variables to introduce the either-or decision variables in the multiplier models.

The proposed envelopment models (22) and (23) are MBLP models and their optimal objective values may differ from those of their multiplier DEA models. We hence formulate the multiplier form of models (22) and (23) in order to verify whether the optimal objective values are less than e_o^l or greater than e_o^u. To this end, we formulate the following mixed binary non-linear programming model which is the multiplier form of model (23):

$$
\begin{array}{ll}
\bar{e}_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \bar{d}_k \gamma_k w_{ko}^u - \sum_{k=1}^{K} \bar{b}_k \delta_k w_{ko}^l & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io}^l = 1 & \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{K} \bar{d}_k \gamma_k w_{kj}^u - \sum_{k=1}^{K} \bar{b}_k \delta_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 & \forall j \\
\bar{b}_k + \bar{d}_k \leq 1 & \forall k \\
u_r, v_i \geq 0 & \forall r, i \\
\bar{b}_k, \bar{d}_k \in \{0,1\} & \forall k
\end{array}
\tag{24}
$$

where u_r and v_i are the weights assigned to the r^{th} output and i^{th} input, and γ_k and δ_k are the weights assigned to the k^{th} dual-role factors when the dual-role factor plays the role of an output or an input, respectively. Note that, similar to model (23), model (24) treats dual-role factors as being nondiscretionary on the input side. The objective of model (24) is to find an upper bound for efficiency score of DMU_o in the presence of imprecise data along with to identify the status of dual-role factors. We adapt the same idea of model (23) to the above model in order to provide the best situation for DMU_o relative to a fixed technology. As a matter of fact, defining the auxiliary binary variables, denoted by $\bar{b}_k$ and $\bar{d}_k$, along with the constraint $\bar{b}_k + \bar{d}_k \leq 1$ provide a situation where w_k takes at most one role, that is, if $\{\bar{b}_k = 1, \bar{d}_k = 0\}$ and $\{\bar{b}_k = 0, \bar{d}_k = 1\}$, the dual-role factor w_k is treated as an input and output, respectively. In addition, if $\{\bar{b}_k = 0, \bar{d}_k = 0\}$, then w_k is at the equilibrium status. However model (24) is non-linear due to terms $\bar{b}_k \delta_k$ and $\bar{d}_k \gamma_k$, one can replace $\bar{b}_k \gamma_k$ and $\bar{d}_k \delta_k$ with γ_k and δ_k, respectively, together with imposing the constraints $0 \leq \delta_k \leq M \bar{b}_k$ and $0 \leq \gamma_k \leq M \bar{d}_k$ where M is a large enough number. The variable alterations thereby transform model (24) to the following MBLP:

$$
\begin{array}{ll}
\bar{e}_o^u = \max \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io}^l = 1 & \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{K} \gamma_k w_{kj}^u - \sum_{k=1}^{K} \delta_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 & \forall j \\
0 \leq \delta_k \leq M \bar{b}_k & \forall k \\
0 \leq \gamma_k \leq M \bar{d}_k & \forall k \\
\bar{b}_k + \bar{d}_k \leq 1 & \forall k \\
u_r, v_i \geq 0 & \forall r, i \\
\bar{b}_k, \bar{d}_k \in \{0,1\} & \forall k
\end{array}
\tag{25}
$$

From the duality point of view, the multiplier model (25) can be considered as a dual model of the envelopment model (23). The difference between the optimal values of primal and dual problems is called a *duality gap* (Wolsey & Nemhauser, 2014). The following theorem aims to investigate the duality gap by comparing the optimal objective values of models (23) and (25) for DMU_o:

Theorem 2. $e_o^u \leq \bar{e}_o^u \leq 1$.

Proof. See Appendix B.

Based on Theorems 1 and 2, we arrive at $e_o^l \leq e_o^u$ and $e_o^u \leq \bar{e}_o^u$, respectively, and the following corollary is therefore deduced from these theorems.

Corollary 1. $e_o^l \leq \bar{e}_o^u$.

At present, let us also formulate the following multiplier form of model (22) to discuss the embedded properties against its envelopment model:

$$
\begin{aligned}
&\bar{e}_o^l = \max \sum_{r=1}^{s} u_r y_{ro}^l + \sum_{k=1}^{K} \gamma_k w_{ko}^l - \sum_{k=1}^{K} \delta_k w_{ko}^u \\
&\text{s.t.} \\
&\sum_{i=1}^{m} v_i x_{io}^u = 1 \\
&\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{K} \gamma_k w_{kj}^u - \sum_{k=1}^{K} \delta_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 && \forall j \\
&0 \leq \delta_k \leq M\bar{b}_k && \forall k \qquad (26) \\
&0 \leq \gamma_k \leq M\bar{d}_k && \forall k \\
&\bar{b}_k + \bar{d}_k \leq 1 && \forall k \\
&u_r, v_i \geq 0 && \forall r, i \\
&\bar{b}_k, \bar{d}_k \in \{0,1\} && \forall k
\end{aligned}
$$

Analogous to model (22), the above formulation evaluates DMU_o from the pessimistic standpoint, in which all DMUs except for DMU_o are viewed from the optimistic standpoint. The following theorem shows the duality gap between the lower bounds of interval efficiency for DMU_o:

Theorem 3. $e_o^l \leq \bar{e}_o^l$.

Proof. The proof is similar to that of Theorem 2 (omitted). □

In brief, models (23) and (25) (models (22) and (26)) are the envelopment and multiplier forms of conventional DEA models in the presence of dual-role factors, which have been presented in this paper to achieve the best practice of the upper (lower) bounds for efficiencies of DMU_o. Though in the original DEA models the optimal objective values associated with the envelopment (primal) and multiplier (dual) models are identical, the existing dual-role factors are made a change in one way that e_o^u (e_o^l) is less than or equal to $\bar{e}_o^u$ ($\bar{e}_o^l$) as shown in Theorems 2 and 3. This occurrence arises from the binary variables, which convert the problems from LP to MBLP models. Also, we explore the relation between the optimal objective values of models (25) and (26) using Theorem 4.

Theorem 4. $\bar{e}_o^l \leq \bar{e}_o^u$.

Proof. See Appendix C. □

According to the above theorems, e_o^l and $\bar{e}_o^u$ calculated by MBLP models (22) and (25), respectively, are the best possible [relative] interval efficiencies for DMU$_o$ under the most unfavourable (pessimistic) and the most favourable (optimistic) situations , respectively, and so $[e_o^l, \bar{e}_o^u]$ is the interval efficiency score of DMU$_o$ in the presence of dual-role factors. We here rely on the definition introduced by Wang et al. (2005) to dichotomize all the DMUs into two sets: efficient and inefficient. Consequently, DMU$_o$ is said to be *technically efficient* if its upper bound of efficiency is equal to one, i.e. $\bar{e}_j^u = 1$; otherwise, it is said to be *inefficient*.

It is worthwhile to note that there is no specific relation between the optimal objective values of models (23) and (26). In other words, though in some cases $\bar{e}_o^l$ can be less than or equal to e_o^u, the reverse, which is irrational, may occur for some cases as well (see Table 2).

It is of interest to explore whether the proposed models (22) and (25) are still valid without input and/or output but there is at least a single dual-role factor. Lovell & Pastor (1999) looked into radial DEA models without inputs (or without outputs). The authors showed that the CRS models in DEA without inputs (or without outputs) are meaningless because all units are classified as infinitely inefficient. Likewise, it can be simply verified that model (22) without inputs (or without outputs) has a feasible solution but its objective function value tends to be negative infinity (infinitely inefficient). Regarding model (25) without inputs is transformed to a multiplier DEA model without the normalization constraint and in view of the remaining constraints, it is clear that its objective function value is equal to zero, which is meaningless. Compellingly, the objective function of model (25) without outputs is $\sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l$ which leads to a meaningful and valid model.

3.2 Determining the status of each dual-role factor

Models (22) and (25) enable us to yield the set of interval efficiencies $\{[e_j^l, \bar{e}_j^u], j = 1, \dots, n\}$ for DMUs. Nevertheless, the specified status of a dual-role factor may not be unique because (i) models (22) and (25) are independently solved for an evaluated DMU and their optimal solutions (i.e., b_k and d_k for model (22) and $\bar{b}_k$ and $\bar{d}_k$ for model (25)) used for identifying the status of the k^{th} dual-role factor may result in inconsistent outcomes. (ii) the optimal solutions for models (22) and (25) would differ from one DMU to another. The most common idea for coping effectively with this problem is to consider a *majority decision rule* that is extensively observed in the literature (Cook & Zhu, 2007). Since the upper bound of efficiency ($\bar{e}_j^u$) is used to define the efficient DMUs, we hence take into account the number of frequency of the status of a given dual-role factor obtained from the optimal solutions of model (25) for all DMUs. That is to say, one assigns a certain status to a dual-role factor based on the maximal occurrence of this status that can be counted after evaluating the efficiencies of all DMUs. Despite the straightforward applicability of the method, it is plausible to face some ties that make the decision about the status of the dual-role factor problematic. Therefore, in the case of ties selecting the role of each dual-role factor can be made in a way that the total deviation from the efficient frontier for all DMUs is minimized. More precisely, let ρ_j be the j^{th} deviation from efficient frontier for DMU$_j$ ($j = 1, \dots, n$) and the following model can minimize the sum of deviations:

$$
\begin{array}{lr}
\min \sum_{j=1}^{n} \rho_j & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i \left(\sum_{j=1}^{n} x_{ij}^l\right) = 1 & \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k=1}^{K} \gamma_k w_{kj}^u - \sum_{k=1}^{K} \delta_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l + \rho_j = 0 & \forall j \\
0 \le \delta_k \le M \bar{b}_k & \forall k \\
0 \le \gamma_k \le M \bar{d}_k & \forall k \\
\bar{b}_k + \bar{d}_k \le 1 & \forall k \\
u_r, v_i \ge 0 & \forall r, i \\
\rho_j \ge 0 & \forall j \\
\bar{b}_k, \bar{d}_k \in \{0,1\} & \forall k
\end{array}
\qquad (27)
$$

Since the upper bound of the efficiency plays a crucial role, particularly in defining the efficient DMUs, the above model is extended on the basis of model (25). It is painless to implement the same configuration for the lower bound (viz. model (22)) in some situations that are essential for the decision-maker. It must be emphasized that the integrated MBLP model (27) is only used in a specific situation (scilicet, ties are observed based on models (22) and (25)) with the aim of yielding the specific role for each dual-role factor on a common base.

In summary, the procedure proposed in this study for identifying the status of dual-role factors can be described in a series of structured and successive steps depicted in Figure 3.

Inputs: *Lower and upper bounds of data*: $\{x_{ij}^l, x_{ij}^u, y_{rj}^l, y_{rj}^u, w_{kj}^l, w_{kj}^u: i = 1, \ldots, m;\ r = 1, \ldots, s;\ k = 1, \ldots, K;\ j = 1, \ldots, n\}$.

begin

for $k = 1, \ldots, K$ **do** $J_{in}^k = \emptyset, J_{out}^k = \emptyset$ and $J_{equ}^k = \emptyset$ **end for;**

for $j = 1, \ldots, n$ **do**

Solve model (25) for DMU$_j$ and find the optimal solution $(\bar{e}_j^u, \boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \overline{\boldsymbol{b}}^*, \overline{\boldsymbol{d}}^*)$.

for $k = 1, \ldots, K$ **do**

if $\bar{b}_k^* = 1$ **then** $J_{in}^k = J_{in}^k \cup \{j\}$;

else if $\bar{d}_k^* = 1$ **then** $J_{out}^k = J_{out}^k \cup \{j\}$;

else $J_{equ}^k = J_{equ}^k \cup \{j\}$.

end for;

end for;

for $k = 1, \ldots, K$ **do**

if $|J_{in}^k| > |J_{out}^k|$ **and** $|J_{in}^k| > |J_{equ}^k|$ **then** consider w_k as a nondiscretionary input and exit.

else if $|J_{out}^k| > |J_{in}^k|$ **and** $|J_{out}^k| > |J_{equ}^k|$ **then** consider w_k as an output and exit.

else if $|J_{equ}^k| > |J_{in}^k|$ **and** $|J_{equ}^k| > |J_{out}^k|$ **then** let w_k in an equilibrium situation and exit.

else solve model (27) and find the optimal solution $(\boldsymbol{\rho}_j^*, \boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \overline{\boldsymbol{b}}^*, \overline{\boldsymbol{d}}^*)$

if $\bar{b}_k^* = 1$ **then** consider w_k as a nondiscretionary input and exit.

else if $\bar{d}_k^* = 1$ **then** consider w_k as an output and exit.

else let w_k in an equilibrium situation and exit.

end for;

end procedure.

Figure 3. The proposed procedure for identifying the status of dual-role factors.

In Figure 3,

$$J_{in}^k = \{j: \bar{b}_j^* = 1\}$$
$$J_{out}^k = \{j: \bar{d}_j^* = 1\}$$
$$J_{equ}^k = (J_{in}^k \cup J_{out}^k)^C$$

In other words, the partition sets J_{in}^k, J_{out}^k and J_{equ}^k represent the index of units that w_k should play as input, output, or equilibrium roles, respectively, which are identified by the optimal values of binary variables b_k and d_k, in model (25). Cardinalities of these index sets, i.e., $|J_{in}^k|$, $|J_{out}^k|$ and $|J_{equ}^k|$, specify the number of elements in each set. Indeed, an index set with maximum cardinality gives the final status for the k^{th} dual-role factor. If this maximum cardinality criterion includes a tie, then we make use of model (27). After determining the unique role of dual-role factors, we apply models (11) and (12) to find the interval efficiencies $\left[E_j^l, E_j^u\right]$ for $j = 1, \dots, n$.[4]

4 Case study

In this section, we study the performance evaluation of third-party reverse logistics (3PL) providers with the intention of developing a long-standing relationship. To this end, we chose the appropriate provider(s) by means of our developed mathematical programming models when both dual-role factors and imprecise data are available. We discuss the detailed calculation steps of our models and then compare the obtained results with those obtained by the method of Toloo et al. (2018). Furthermore, some managerial implications are drawn from our findings.

4.1 3PLP evaluation

3PLP evaluation is an overriding responsibility of purchasing and supply managers in designing an effective supply chain. The supplier data set utilized in this paper is originally reported in Kleinsorge, Schary, & Tanner (1992) and then considered by Talluri & Baker (2002) and Farzipoor Saen (2011) with a slight change. In this paper, our focus is limited to the supplier data set with uncertainty to illustrate the applicability and advantages of our proposed method against the existing study in the literature. Our 3PLP data set consists of eighteen 3PLPs with two cardinal inputs, namely *unit operation cost* (x_1) and *environmental expenditures* (x_2), two cardinal outputs, namely *recycling capacity* (y_1) and *revenue from the sale of recyclables* (y_2), a cardinal dual-role factor, namely *size of the solid waste stream* (w_1) and an ordinal dual-role factor, namely *3PLP reputation* (w_2). A scale transformation proposed by Wang et al. (2005) is applied to convert ordinal data (w_2) to interval data such that the transformation preserves the ordinal preference relationships of ordinal data. Table 1 presents the dataset of eighteen 3PLPs in which the values of x_2, w_2 and y_2 are interval, and x_1, w_1 and y_1 have precise values with the identical lower and upper bounds.

Table 1. Data for eighteen 3PL providers.

DMU	Inputs		Dual-role factors		Outputs	
	x_1	x_2	w_1	w_2	y_1	y_2
01	253	[950, 2000]	24900	[0.01574,0.22917]	2	[50, 65]
02	268	[800, 1800]	64300	[0.02773, 0.40388]	13	[60, 70]

[4] It should be noted that $\left[E_j^l, E_j^u\right] \subseteq \left[e_j^l, \bar{e}_j^u\right]$.

03	259	[1000, 2100]	71400	[0.01254, 0.1827]	3	[40, 50]
04	180	[820, 2150]	180900	[0.01762, 0.25668]	3	[100, 160]
05	257	[735, 1900]	23800	[0.01405, 0.20462]	24	[45, 55]
06	248	[650, 2500]	24100	[0.0112, 0.16312]	28	[85, 115]
07	272	[450, 2200]	140400	[0.02211, 0.32197]	1	[70, 95]
08	330	[400, 1900]	98400	[0.03106, 0.45235]	24	[100, 180]
09	327	[607, 2040]	64100	[0.02476, 0.36061]	11	[90, 120]
10	330	[455, 1890]	58800	[0.01974, 0.28748]	53	[50, 80]
11	321	[830, 2000]	24100	[0.05474, 0.79719]	10	[250, 300]
12	329	[650, 1950]	56700	[0.04363, 0.63552]	7	[100, 150]
13	281	[960, 2350]	56700	[0.04887, 0.71178]	19	[80, 120]
14	309	[1200, 2300]	96700	[0.03896, 0.56743]	12	[200, 350]
15	291	[880, 2000]	63500	[0.03479, 0.50663]	33	[40, 55]
16	334	[655, 2010]	79500	[0.0613, 0.89286]	2	[75, 85]
17	249	[800, 1990]	68900	[0.01, 0.14564]	34	[90, 180]
18	216	[645, 2153]	91300	[0.06866, 1]	9	[90, 150]

Figure 4 illustrates the problem of evaluating and ranking of 3PLPs in the presence of imprecise inputs, outputs and dual-role factors.

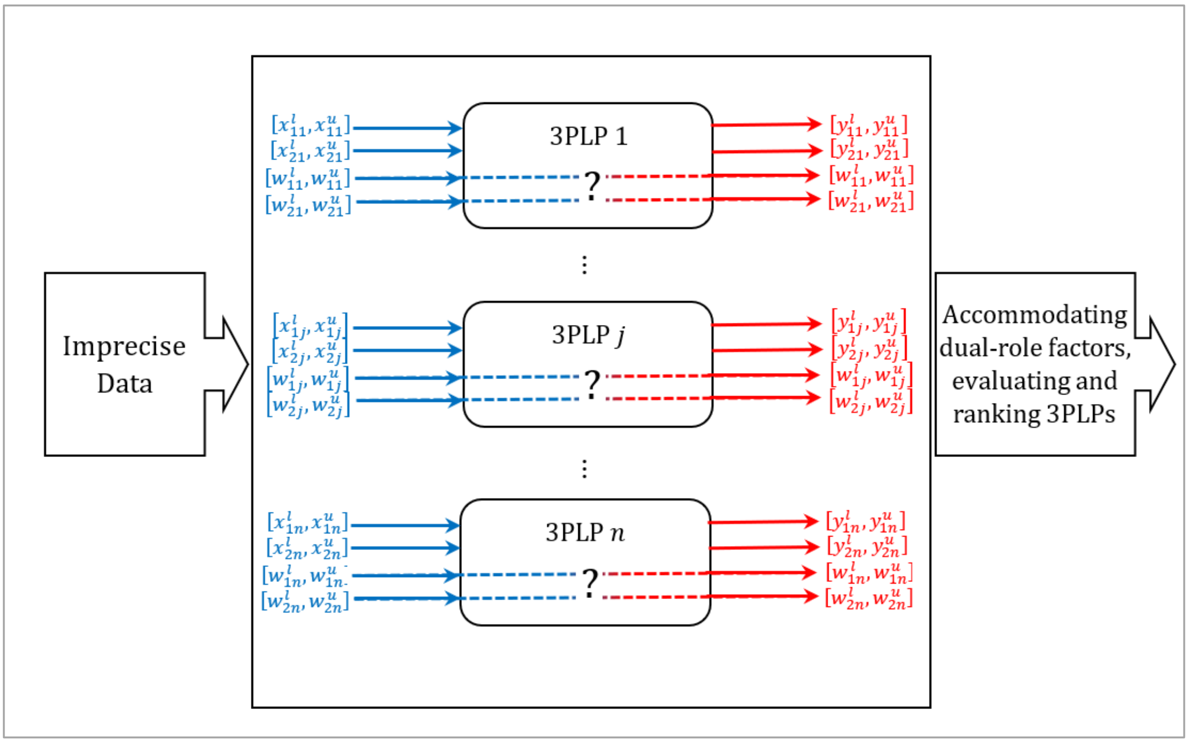

Figure 4. Evaluation and ranking process of 3PLPs

To underline and illustrate the relationship between the four different models proposed in this study especially in terms of proposed theorems and corollary, we report Table 2 which includes the results of all the proposed models. The 2nd-5th columns of

Table 2 represent the optimal objective function values for models (22), (26), (23) and (25), denoted by $e_j^l, \bar{e}_j^l, e_j^u$ and $\bar{e}_j^u$, respectively. Comparing the second and third columns of Table 2 which are obtained from models (22) and (26), it is viewed that e_j^l is invariably less than or equal to $\bar{e}_j^l$ which is in line with Theorem 3. Given the results calculated by models (23) and (25) and represented in the fourth and fifth columns of Table 2, $\bar{e}_j^u$ is always greater than or equal to e_j^u as shown in Theorem 2. Therefore, the best [relative] lower and upper bounds efficiencies, $[e_o^l, \bar{e}_o^u]$, are shown in the second and last columns of Table 2 as stated in Corollary 1. Although $\bar{e}_j^l$ (e_o^l) is less than or equal to $\bar{e}_o^u$ (e_j^u) all the time as shown in Theorem 4 (Theorem 1), it is remarkable that $\bar{e}_j^l$ is not invariably less than or equal to e_j^u; for instance, the third and fourth columns shows that the $\bar{e}_j^l$ values of 3PLPs {2, 3, 4, 5, 6, 15} are greater than their e_j^u values.

Table 2. Results of the proposed models (22), (26), (23) and (25).

DMU	e_j^l Model (22)	$\bar{e}_j^l$ Model (26)	e_j^u Model (23)	$\bar{e}_j^u$ Model (25)
01	0.17813	0.19268	0.23082	0.28217
02	0.34570	0.47110	0.37316	0.57263
03	0.15670	0.31089	0.18651	0.34554
04	0.49048	1	0.78476	1
05	0.59135	0.69778	0.60263	0.83882
06	0.75294	1	0.79968	1
07	0.22721	0.51361	0.52070	1
08	0.50885	0.65067	1	1
09	0.32134	0.36120	0.52794	0.57520
10	1	1	1	1
11	0.70199	0.84144	1	1
12	0.30349	0.33083	0.59851	0.75703
13	0.47395	0.54251	0.54851	0.94008
14	0.62531	0.66677	1	1
15	0.70609	0.79003	0.71126	0.90563
16	0.19825	0.25260	0.33590	0.89922
17	0.89522	0.96800	1	1
18	0.45649	0.60358	0.74784	1

Another aim of our analysis is to seek the status of the dual-role factors, which can be studied from the pessimistic and optimistic standpoints. From the pessimistic standpoint, the dual-role factor w_k $(k = 1, 2)$ serves as an input, output, and at the equilibrium if we arrive at $\{b_k^* = 0, d_k^* = 1\}$, $\{b_k^* = 1, d_k^* = 0\}$ and $\{b_k^* = d_k^* = 1\}$, respectively. In what follows, the optimal values of b_1^*, b_2^*, d_1^* and d_2^* calculated from model (22) are listed in Table 3.

Table 3. Status of dual-role factors of eighteen 3PLPs from the pessimistic and optimistic standpoints

DMU	Pessimistic (model (22))				Optimistic (model (25))			
	w_1		w_2		w_1		w_2	
	b_1^*	d_1^*	b_2^*	d_2^*	$\bar{b}_1^*$	$\bar{d}_1^*$	$\bar{b}_2^*$	$\bar{d}_2^*$
01	1	1	1	1	1	0	0	1
02	1	1	1	1	0	1	0	1
03	1	1	1	1	0	1	0	1
04	1	1	1	1	0	1	1	0
05	1	0	1	0	1	0	0	1
06	1	0	1	0	1	0	0	1
07	1	1	1	1	0	1	0	1
08	1	1	1	1	0	1	0	1
09	1	1	1	1	0	1	0	1
10	1	1	1	1	0	1	1	0
11	1	0	1	0	0	1	0	1
12	1	1	1	1	0	1	0	1
13	1	1	1	1	1	0	0	1
14	1	1	1	1	0	1	0	1
15	1	1	1	1	1	0	0	1
16	1	1	1	1	1	0	0	1
17	1	1	1	1	0	1	1	0
18	1	1	1	1	0	1	0	1

According to the 2nd-5th columns of Table 3, both the dual-role factors for all DMUs with the exception of DMU_5, DMU_6 and DMU_{11}, are designated as both the input and output, namely at the equilibrium, i.e., $\{b_k^* = d_k^* = 1, k = 1, 2\}$, practically meaning that *size of the solid waste stream* (w_1) and *3PLP reputation* (w_2) are at an optimal level (neither wishing to acquire or to lose them). For DMU_5, DMU_6 and DMU_{11}, the dual-role factors w_1 and w_2 play the output role, that is, increased *size of the solid waste stream* and *reputation* would improve these 3PL providers' efficiencies. On the other hand, one is able to specify the status of the dual-role factors from the optimistic standpoint using the optimal values of $\bar{b}_1^*$, $\bar{b}_2^*$, $\bar{d}_1^*$ and $\bar{d}_2^*$ calculated from model (25) in a way that the dual-role factor w_k $(k = 1, 2)$ plays the role of an input, output, and being at the equilibrium if $\{\bar{b}_k^* = 1, \bar{d}_k^* = 0\}$, $\{\bar{b}_k^* = 0, \bar{d}_k^* = 1\}$ and $\{\bar{b}_k^* = \bar{d}_k^* = 0\}$, respectively. As presented in the last four columns of Table 3, the first dual-role factor, w_1, is designated as an output for twelve units and as an input for six units, and the second dual-role factor, w_2, is classified as an output for fifteen units and as an output for three units. It is worth investigating the role of dual-role factors, *size of the solid waste stream* (w_1) and *3PLP reputation* (w_2), from the practical viewpoint. Consider a given 3PLP and identify the decision-maker's preference (optimism or pessimism) in terms of the *internal environment* such as operation cost, stock availability and range of products/services, and the *external environment* such as competitors, regulations and customers. The *size of the solid waste stream* (and/or *3PLP reputation*) as the dual-factor is *acting as an input* and its decline would lead to performance improvement, *acting as an output* and more of the factor is better and would improve the performance, or is at an equilibrium/optimal level and no change of the dual-factor would be preferable.

Since the dual-role factors do not include a unique role from the pessimistic and optimistic standpoint, we apply the proposed procedure that is summed up in Figure 3 to identify a unique role to each dual-role factor. Given that the cardinalities of sets $|J_{in}^k|$, $|J_{out}^k|$ and $|J_{equ}^k|$ are 6, 12 and 0 for $k = 1$, and 3, 15, and 0 for $k = 2$, we conclude that both dual-role factors are designated as the output. The detailed designation procedure is explained as follows:

Begin

Let $J_{in}^1 = \emptyset, J_{out}^1 = \emptyset, J_{equ}^1 = \emptyset, J_{in}^2 = \emptyset, J_{out}^2 = \emptyset$ and $J_{equ}^2 = \emptyset$.

Step 1.

As a result of solving model (25) for DMU_1, we have:

$$\bar{e}_1^u = 0.282171, \quad (u_1^*, u_2^*; v_1^*, v_2^*) = (0, 0.002966;\ 0.003953, 0),$$
$$(\delta_1^*, \delta_2^*; \gamma_1^*, \gamma_2^*) = (0.000001, 0\ ;\ 0, 0.508145),$$
$$\left(\bar{b}_1^*, \bar{b}_2^*; \bar{d}_1^*, \bar{d}_2^*\right) = (1, 0\ ;\ 0, 1).$$

Then, $J_{in}^1 = \{1\}$ and $J_{out}^2 = \{1\}$ since $\bar{b}_1^* = 1, \bar{d}_1^* = 0, \bar{b}_2^* = 0$ and $\bar{d}_2^* = 1$.

Step 2.

As a result of solving model (25) for DMU_2, we have:

$$\bar{e}_2^u = 0.572633, \quad (u_1^*, u_2^{**}; v_1^*, v_2^*) = (0.017974, 0\ ;\ \ 0.003731, 0),$$
$$(\delta_1^*, \delta_2^*; \gamma_1^*, \gamma_2^*) = (0\ , 0\ ;\ \ 0.000003, 0.381916),$$
$$\left(\bar{b}_1^*, \bar{b}_2^*; \bar{d}_1^*, \bar{d}_2^*\right) = (0, 0\ \ ;\ \ 1, 1).$$

Then, $J_{out}^1 = \{2\}$ and $J_{out}^2 = \{1, 2\}$ since $\bar{b}_1^* = 0\ , \bar{d}_1^* = 1, \bar{b}_2^* = 0$ and $\bar{d}_2^* = 1, .$

Step 3.

As a result of solving model (25) for DMU_3, we have:

$$\bar{e}_3^u = 0.345542, \quad (u_1^*, u_2^*; v_1^*, v_2^*) = (0.008047, 0\ \ ;\ \ 0.001303, 0.000662),$$
$$(\delta_1^*, \delta_2^*; \gamma_1^*, \gamma_2^*) = (0\ , 0\ ;\ \ 0.000004, 0.294089),$$
$$\left(\bar{b}_1^*, \bar{b}_2^*; \bar{d}_1^*, \bar{d}_2^*\right) = (0, 0\ \ ;\ \ 1, 1).$$

Then, $J_{out}^1 = \{2,3\}$ and $J_{out}^2 = \{1, 2,3\}$ since $\bar{b}_1^* = 0, \bar{d}_1^* = 1, \bar{b}_2^* = 0$ and $\bar{d}_2^* = 1$.

We need to carry on with the steps as long as we satisfy the exit conditions (See Figure 3). After 18 steps, we arrive at $J_{in}^1 = \{1, 5, 6, 13, 15, 16\}$, $J_{out}^1 = \{2, 3, 4, 7, 8, 9, 10, 11, 12, 14, 17, 18\}$, and $J_{equ}^1 = \emptyset$ where $|J_{in}^1| = 6$, $|J_{out}^1| = 12$ and $|J_{equ}^1| = 0$. Since $|J_{out}^1| > |J_{in}^1|$ and $|J_{out}^1| > |J_{equ}^1|$, w_1 is considered as an output. Analogously, for the second dual-role factor, we obtain $J_{in}^2 = \{4, 10, 17\}$, $J_{out}^2 = \{1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 18\}$ and $J_{equ}^2 = \emptyset$ where $|J_{in}^2| = 3$, $|J_{out}^2| = 15$, $|J_{equ}^2| = 0$. Therefore, w_2 plays the output role due to the fact that $|J_{out}^1| > |J_{in}^1|$ and $|J_{out}^1| > |J_{equ}^1|$.

Given both w_1 and w_2 as the outputs for all DMUs, we can calculate the interval efficiencies $[E_j^l, E_j^u], j = 1, \dots, 18$ by the use of models (11) and (12) as presented in Table 4.

Table 4. Interval efficiencies of eighteen 3PL providers

DMU	$\bar{E}_j^l$	$\bar{E}_j^u$
01	0.19268	0.28085
02	0.47111	0.57263
03	0.31089	0.34554
04	1	1
05	0.59135	0.64045
06	0.75294	0.80717
07	0.51361	1
08	0.65067	1
09	0.36120	0.57520
10	1	1
11	0.70199	1
12	0.33083	0.72381
13	0.54251	0.78998
14	0.66677	1
15	0.79003	0.89543
16	0.25260	0.87923
17	0.96800	1
18	0.60358	1

We therefore take $\bar{E}_j^l$ and $\bar{E}_j^u$ into consideration to measure the interval efficiencies of 3PL providers. It is obvious that $[\bar{E}_j^l, \bar{E}_j^u] \subseteq [e_j^l, \bar{e}_j^u]$ for all DMUs. According to the upper bound of efficiencies, 3PLPs $\{4, 7, 8, 10, 11, 14, 17, 18\}$ are efficient since $E_j^u = 1$. As it is shown, dichotomizing the DMUs based on the upper bound of the interval efficiency is viable; however, a decision-maker frequently seeks a complete ranking beyond the dichotomized classification. To improve the discriminating power, we adopt the *Minimax Regret Approach* (MRA) proposed by Wang et al. (2005) to compare and rank the efficiency interval $[\bar{E}_j^l, \bar{E}_j^u]$ of 3PL providers. Let $A_j = [\bar{E}_j^l, \bar{E}_j^u], j = 1, \dots, 18$, be the efficiency interval of eighteen 3PL providers. The *maximum loss of efficiency* (MLE) of each A_j is defined as $R(A_j) = \max\left\{\max_{q \neq j}\{\bar{E}_q^u\} - \bar{E}_j^l, 0\right\}$ and the minimum among $R(A_j)$ is referred to as *the most desirable interval efficiency*. The MRA ranking procedure is detailed in Figure 5. In brief, the first step opts for a most desirable DMU on the basis of the minimum amount of MLE and then this DMU is eliminated from the set of DMUs in the next step and re-calculated MLE of 17 remaining DMUs to identify the second most desirable DMU with the smallest MLE. We need to iterate the eliminating process until only one efficiency interval (DMU) remains. Obviously, the algorithm terminates after n iterations.

Inputs: *Set of interval efficiencies* $\{[\bar{E}_j^l, \bar{E}_j^u]: j = 1, \dots, n\}$.
begin
Let $J = \{1,2, \dots, n\}, p = 1$.
while $J \neq \emptyset$ **do**
  for $j \in J$ **do** $R(\mathrm{DMU}_j) = \max\left\{\max_{q \in J, q \neq j}\{\bar{E}_q^u\} - \bar{E}_j^l, 0\right\}$ **end for**;
    find $j^* \in J$ which $R(\mathrm{DMU}_{j^*}) = \min\{R(\mathrm{DMU}_j): j \in J\}$, let $\mathrm{Rank}(j^*) = p$, $J = J - \{j^*\}$ and $p = p + 1$.
end while;
end procedure.

Figure 5. The MRA procedure for comparing and ranking interval efficiencies.

Table 5 presents the maximum loss of efficiency for each 3PLP and the corresponding ranking order of eighteen 3PL providers.

Table 5. Results of MRA procedure to rank all DMUs based on maximum loss of efficiency for eighteen 3PLPs and interval efficiencies $[\bar{E}_j^l, \bar{E}_j^u]$ obtained by models (11) and (12).

$p = Rank$ (Iteration)	DMU_{j^*}	$R(\mathrm{DMU}_{j^*}) = \min\{R(\mathrm{DMU}_q): q \in J\}$
1	DMU10	**0.00000**=min{0.80732,0.52890,0.68911,0.00000,0.40865,0.24706,0.48639,0.34933,0.63880,0.00000,0.29801,0.66917,0.45749,0.33323,0.20997,0.74740,0.03200,0.39642}
1	DMU4	**0.00000**=min{0.80732,0.52890,0.68911,0.00000,0.40865,0.24706,0.48639,0.34933,0.63880,0.29801,0.66917,0.45749,0.33323,0.20997,0.74740,0.03200,0.39642}
2	DMU17	**0.03200**=min{0.80732,0.52890,0.68911,0.40865,0.24706,0.48639,0.34933,0.63880,0.29801,0.66917,0.45749,0.33323,0.20997,0.74740,0.03200,0.39642}
3	DMU15	**0.20997**=min{0.80732,0.52890,0.68911,0.40865,0.24706,0.48639,0.34933,0.63880,0.29801,0.66917,0.45749,0.33323,0.20997,0.74740,0.39642}
4	DMU6	**0.24706**=min{0.80732,0.52890,0.68911,0.40865,0.24706,0.48639,0.34933,0.63880,0.29801,0.66917,0.45749,0.33323,0.74740,0.39642}
5	DMU11	**0.29801**=min{0.80732,0.52890,0.68911,0.40865,0.48639,0.34933,0.63880,0.29801,0.66917,0.45749,0.33323,0.74740,0.39642}
6	DMU14	**0.33323**=min{0.80732,0.52890,0.68911,0.40865,0.48639,0.34933,0.63880,0.66917,0.45749,0.33323,0.74740,0.39642}

7	DMU8	**0.34933**=min{0.80732,0.52890,0.68911,0.40865,0.48639,0.34933,0.63880,0.66917,0.45749,0.74740,0.39642}
8	DMU18	**0.39642**=min{0.80732,0.52890,0.68911,0.40865,0.48639,0.63880,0.66917,0.45749,0.74740,0.39642}
9	DMU7	**0.36562**=min{0.80732,0.52890,0.68911,0.40865,0.36562,0.63880,0.66917,0.45749,0.74740}
10	DMU5	**0.28788**=min{0.68655,0.40812,0.56834,0.28788,0.51803,0.54840,0.33672,0.53738}
11	DMU13	**0.33672**=min{0.68655,0.40812,0.56834,0.51803,0.54840,0.33672,0.53738}
12	DMU2	**0.40812**=min{0.68655,0.40812,0.56834,0.51803,0.54840,0.47121}
13	DMU16	**0.47121**=min{0.68655,0.56834,0.51803,0.54840,0.47121}
14	DMU12	**0.24437**=min{0.53113,0.41292,0.36261,0.24437}
15	DMU9	**0.00000**=min{0.38252,0.26431,0.00000}
16	DMU3	**0.00000**=min{0.15286,0.00000}
17	DMU1	**0.00000**=min{0}

Let us now lay emphasis on Toloo et al. (2018)'s method to make a comparison with the interval efficiencies calculated from our developed approach in this study. Toloo et al. (2018) drew mainly on Despotis & Smirlis (2002) to deal with dual-role factors for imprecise data. The lower bound (ξ_j^l) and upper bound (ξ_j^u) of efficiencies obtained from Toloo et al. (2018)'s models are presented in Table 6.

Table 6. Interval efficiencies of Toloo et al. (2018)'s models for eighteen 3PLPs

DMUs	Regardless of the w_1 and w_2 roles		w_1(output) and w_2 (input)	
	ξ_j^l	ξ_j^u	E_j^l	E_j^u
01	0.17813	1	0.17813	1
02	0.34570	1	0.34570	1
03	0.15670	1	0.15670	1
04	0.49048	1	0.49048	1
05	0.59135	1	0.69778	1
06	0.75294	1	1	1
07	0.22721	1	0.22720	1
08	0.50885	1	0.50885	1
09	0.32134	1	0.32134	1
10	1	1	1	1
11	0.70199	1	1	1
12	0.30349	1	0.30349	1
13	0.47395	1	0.47395	1
14	0.72732	1	0.72732	1
15	0.70609	1	0.70609	1
16	0.19825	1	0.19825	1
17	0.95985	1	0.95985	1
18	0.45649	1	0.45649	1

Although the dual-role factors in Toloo et al. (2018)'s method can take different roles from the pessimistic and optimistic viewpoint, the role of dual-role factors w_1 and w_2 are finally found to be the output and input, respectively. Accordingly, we can calculate the interval efficiencies $\left[E_j^l, E_j^u\right], j = 1, \dots, 18$ via models (8) and (9) as reported in the last two columns of Table 6. Contrary to Table 4, our results improve the discriminatory power regarding the upper bound of efficiency intervals. Put differently, all the upper bounds of efficiencies obtained from Toloo et al. (2018)'s method are 1 leading to no valuable information. Interestingly, DMU 10 is invariably efficient from both the pessimistic and optimistic viewpoint regardless of models. Similarly, the MRA ranking method is applied to the interval efficiencies $\left[E_j^l, E_j^u\right]$ presented in Table 6 to make the comparison simpler. The maximum loss of efficiencies of 3PLPs and their ranking order is therefore detailed in Table 7. A Wilcoxon's test shows weak evidence against the null hypothesis, so we fail to reject the null hypothesis. In other words, that there is no significant difference between the result of our study and Toloo et al. (2018)'s method.

Table 7. Results of MRA procedure to rank all DMUs based on maximum loss of efficiency for eighteen 3PLPs and interval efficiencies $\left[E_j^l, E_j^u\right]$ obtained by models (8) and (9).

$p = Rank$ (Iteration)	DMU_{j^*}	$R\left(\mathrm{DMU}_{j^*}\right) = \min\left\{R\left(\mathrm{DMU}_q\right): q \in J\right\}$
1	DMU6	**0.00000**=min{0.82187,0.65430,0.84330,0.50952,0.30222, 0.00000, 0.77280, 0.49115,0.67866,0.00000,0.00000,0.69651,0.52605,0.27268,0.29391,0.80175,0.04015,0.54351}
1	DMU10	**0.00000**=min{0.82187,0.65430,0.84330,0.50952,0.30222,0.77280,0.49115,0.67866,0.00000, 0.00000,0.69651,0.52605,0.27268,0.29391,0.80175,0.04015,0.54351}
1	DMU11	**0.00000**=min{0.82187,0.65430,0.84330,0.50952,0.30222,0.77280,0.49115,0.67866,0.00000,0.69651,0.52605,0.27268,0.29391,0.80175,0.04015,0.54351}
2	DMU17	**0.04015**=min{0.82187,0.65430,0.84330,0.50952,0.30222,0.77280,0.49115,0.67866,0.69651,0.52605,0.27268,0.29391,0.80175,0.04015,0.54351}
3	DMU14	**0.27268**=min{0.82187,0.65430,0.84330,0.50952,0.30222,0.77280,0.49115,0.67866,0.69651,0.52605,0.27268,0.29391,0.80175,0.54351}
4	DMU15	**0.29391**=min{0.82187,0.65430,0.84330,0.50952,0.30222,0.77280,0.49115,0.67866,0.69651,0.52605,0.29391,0.80175,0.54351}
5	DMU5	**0.30222**=min{0.82187,0.65430,0.84330,0.50952,0.30222,0.77280,0.49115,0.67866,0.69651,0.52605,0.80175,0.54351}
6	DMU8	**0.49115**=min{0.82187,0.65430,0.84330,0.50952,0.77280,0.49115,0.67866,0.69651,0.52605,0.80175,0.54351}
7	DMU4	**0.50952**=min{0.82187,0.65430,0.84330,0.50952,0.77280,0.67866,0.69651,0.52605,0.80175,0.54351 }
8	DMU13	**0.52605**=min{0.82187,0.65430,0.84330,0.77280,0.67866,0.69651,0.52605,0.80175,0.54351 }
9	DMU18	**0.54351**=min{0.82187,0.65430,0.84330,0.77280,0.67866,0.69651,0.80175,0.54351 }
10	DMU2	**0.65430** =min{0.82187,0.65430,0.84330,0.77280,0.67866,0.69651,0.80175 }
11	DMU9	**0.67866** =min{0.82187,0.84330,0.77280,0.67866,0.69651,0.80175 }
12	DMU12	**0.69651** =min{0.82187,0.84330,0.77280,0.69651,0.80175 }
13	DMU7	**0.77280** =min{0.82187,0.84330,0.77280,0.80175 }
14	DMU16	**0.80175**=min{0.82187,0.84330,0.80175 }
15	DMU1	**0.82187**=min{0.82187,0.84330}
16	DMU3	**0.00000**=min{0.00000}

4.2 Managerial implications

Today, the overriding concern of a business is to pinpoint the capabilities and core competencies as a source of sustained competitive advantage over its rivals. Allocating more resources to the core competencies is the key action taken by the strategic leaders in the firm. Outsourcing the partial or overall logistics processes to 3PLPs can be regarded as a distinctive competency of a firm that relates to the organizational function. The for-hire transportation industry includes a great many carriers who are professional in product movement between geographic locations. 3PLPs embrace all logistics activities needed to serve customers, from order entry to product delivery. The increased availability growth of 3PLPs makes the formation of supply chain arrangements easier. Such outsourcing indeed helps ease process-focused integrative management. What seems very essential is to an efficient 3PLP to collaborate with the organization in the reverse logistics process.

Mathematical optimization models can serve as a catalyst to make strategic or operational decisions. The use of the efficiency models allows managers to identify best practice across the 3PLPs with the aim of creating further effective and sustainable collaboration. In this regard, the appropriate selection of assessment factors including inputs, outputs and dual-role factors can severely affect the efficiency and productivity of the evaluated firms. The foregoing example includes two dual-role factors; *size of the solid waste stream* and *3PLP reputation,* and identifying their roles are controversial and vague for managers. Besides, managers need to take account of imprecise data to make correct decisions. To this end, managers can benefit from our developed imprecise DEA approach by determining the most appropriate *3PLP(s)* where dual-role factors and imprecise data are available. In our case study, the nine efficient 3PLPs as the result of Toloo et al. (2018)'s models are narrowed down to two efficient 3PLPs {4,10} as well as ranking the remaining efficient 3PL providers. This shows that the proposed approach enhances the discriminatory power of the model in a relatively uncomplicated way. Furthermore, our models in this paper help managers specify the input or output role of the dual-role factors. It is found that both "size of the solid waste stream" and "3PLP reputation" factors in the example are designated as output factors.

It is worthwhile to point out that the DEA modelling approach provides managers with deeper insight into the performance of potential 3PLPs and the computational tractability for practical efficiency analysis with the objective of minimizing cost and enhancing the supply chain performance.

5 Conclusions

Uncertainty is an essential and inseparable part of most real-life evaluation problems which has been paid less attention in the non-parametric models, particularly DEA models. Today, the remedies for such significant negligence would be an ongoing challenge in DEA. In this paper, we developed an imprecise DEA approach when the observations include the dual-role factors besides the inputs and outputs. Our method in this study was based on revising technology to result in a pair of MBLP models. These models were formulated to calculate the best lower and upper bound of efficiency from the pessimistic and optimistic viewpoints. In addition, we presented a stepwise procedure for specifying the status of each dual-factor, i.e., whether it is an input, output or at the equilibrium. The proposed method was lastly demonstrated and compared with an existing study in the literature through an application on customer-supplier relationships.

Intuitively, the interval approach is more appropriate and affirmative to the firms being evaluated since the collection of interval data is not only simpler, faster and less costly, interval efficiency may also better describe the real situation. Therefore, we plan to implement the proposed framework in the real-world and discuss in a follow-up paper demonstrating the practical implications of our model in real-life problems. Besides, the inputs and outputs in our work are only measured by interval values. In future research, the production technology may be perfected by including categorical, environmental and integer-valued for the inputs, outputs and dual factors. We need to point out that the development in this research is merely related to dual-role factors and an in-depth discussion about flexible measures proposed by Cook & Zhu (2007) could be left as a further research topic.

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Appendix A

Proof of Theorem 1 . Let S_1 and S_2 be the feasible region of models (22) and (23), respectively. It is easy to verify that if $(\boldsymbol{\lambda}, \boldsymbol{b}, \boldsymbol{d}) \in S_2$, then $(\boldsymbol{\lambda}, \boldsymbol{b}, \boldsymbol{d}) \in S_1$. Hence, the feasible region of model (23) is a subset of the feasible region of model (22) and subsequently the optimal objective value of the former model is greater than or equal to the optimal objective value of the later model. This completes the proof. □

Appendix B

Proof of Theorem 2. Let $(\theta^*, \boldsymbol{\lambda}^*, \boldsymbol{b}^*, \boldsymbol{d}^*)$ and $\left(\boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \overline{\boldsymbol{b}}^*, \overline{\boldsymbol{d}}^*\right)$ be the optimal solutions of models (23) and (25), respectively. By defining the index sets $K_b = \left\{k: \bar{b}_k^* = 1\right\}$ and $K_d = \left\{k: \bar{d}_k^* = 1\right\}$, it is clear that $K_b \cap K_d = \emptyset$ thanks to the constraint $\bar{b}_k + \bar{d}_k \leq 1$ of model (25). Consider the following LP model:

$$\begin{array}{lr}
\max\ \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k\in K_d} \gamma_k w_{ko}^u - \sum_{k\in K_b} \delta_k w_{ko}^l & \\
\text{s.t.} & \\
\sum_{i=1}^{m} v_i x_{io}^l = 1 & \qquad (28) \\
\sum_{r=1}^{s} u_r y_{rj}^u + \sum_{k\in K_d} \gamma_k w_{kj}^u - \sum_{k\in K_b} \delta_k w_{kj}^l - \sum_{i=1}^{m} v_i x_{ij}^l \leq 0 \quad \forall j & \\
u_r, v_i, \delta_k, \gamma_k \geq 0 \qquad\qquad \forall r, i, k &
\end{array}$$

Let $\widehat{\boldsymbol{\delta}}^* = (\dots, \delta_k^*, \dots)_{k\in K_b}$ and $\widehat{\boldsymbol{\gamma}}^* = (\dots, \gamma_k^*, \dots)_{k\in K_d}$. It is painless to show that the optimal solution of model (28) is $\left(\boldsymbol{u}^*, \boldsymbol{v}^*, \widehat{\boldsymbol{\gamma}}^*, \widehat{\boldsymbol{\delta}}^*\right)$ and its objective value is $\bar{e}_o^u$. The dual problem of model (28) can be formulated as follows:

$$\begin{array}{llr}
\min \theta & & \\
\text{s.t.} & & \\
\sum_{j=1}^{n} \lambda_j x_{ij}^l \leq \theta x_{io}^l & \forall i & \\
\sum_{j=1}^{n} \lambda_j y_{rj}^u \geq y_{ro}^u & \forall r & \qquad (29) \\
\sum_{j=1}^{n} \lambda_j w_{kj}^l \leq w_{ko}^l & \forall k \in K_b & \\
\sum_{j=1}^{n} \lambda_j w_{kj}^u \geq w_{ko}^u & \forall k \in K_d & \\
\lambda_j \geq 0 & \forall j &
\end{array}$$

Let $(\theta', \boldsymbol{\lambda}')$ be the optimal solution of model (29). By the deployment of the strong duality theorem of linear programming (for more details see Bazaraa, Jarvis, & Sherali (2010)), the optimal objective value of model (29) is $\bar{e}_o^u$. Let

$$b_k' = \begin{cases} 0 & k \in K_b \\ 1 & k \notin K_b \end{cases}, \quad d_k' = \begin{cases} 0 & k \in K_d \\ 1 & k \notin K_d \end{cases}, \quad k = 1, \dots, K.$$

A simple computation clarifies that $(\theta', \boldsymbol{\lambda}', \boldsymbol{b}', \boldsymbol{d}')$ is a feasible solution of model (23) with the objective value $\bar{e}_o^u$ and hence $e_o^u \leq \bar{e}_o^u$. The inequality $\bar{e}_o^u \leq 1$ is straightforwardly stemmed from the first and second constraints of model (25), indeed if $\left(\boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \overline{\boldsymbol{b}}^*, \overline{\boldsymbol{d}}^*\right)$ is an optimal solution of model (25), then the first constraint ensures that $\sum_{i=1}^{m} v_i x_{io}^l = 1$ and from the second constraint we obtain $\bar{e}_o^u = \sum_{r=1}^{s} u_r y_{ro}^u + \sum_{k=1}^{K} \gamma_k w_{ko}^u - \sum_{k=1}^{K} \delta_k w_{ko}^l \le \sum_{i=1}^{m} v_i x_{io}^l = 1$, which completes the proof.□

Appendix C

Proof of Theorem 4. Let $(\boldsymbol{u}^*, \boldsymbol{v}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*, \bar{\boldsymbol{b}}^*, \bar{\boldsymbol{d}}^*)$ be the optimal solution of model (26), and define $t = \sum_{i=1}^{m} v_i^* x_{io}^l$, $(\boldsymbol{u}', \boldsymbol{v}', \boldsymbol{\gamma}', \boldsymbol{\delta}', \bar{\boldsymbol{b}}', \bar{\boldsymbol{d}}') = \left(\frac{\boldsymbol{u}^*}{t}, \frac{\boldsymbol{v}^*}{t}, \frac{\boldsymbol{\gamma}^*}{t}, \frac{\boldsymbol{\delta}^*}{t}, \bar{\boldsymbol{b}}^*, \bar{\boldsymbol{d}}^*\right)$. Therefore, we have $t = \sum_{i=1}^{m} v_i^* x_{io}^l \le \sum_{i=1}^{m} v_i^* x_{io}^u = 1$, $\sum_{i=1}^{m} v_i' x_{io}^l = \sum_{i=1}^{m} \frac{v_i^*}{t} x_{io}^l = \frac{1}{t} \sum_{i=1}^{m} v_i^* x_{io}^l = 1$, and $\sum_{r=1}^{s} u_r' y_{rj}^u + \sum_{k=1}^{K} \gamma_k' w_{kj}^u - \sum_{k=1}^{K} \delta_k' w_{kj}^l - \sum_{i=1}^{m} v_i' x_{ij}^l = \frac{1}{t}(\sum_{r=1}^{s} u_r^* y_{rj}^u + \sum_{k=1}^{K} \gamma_k^* w_{kj}^u - \sum_{k=1}^{K} \delta_k^* w_{kj}^l - \sum_{i=1}^{m} v_i^* x_{ij}^l) \le 0$ which verify that $(\boldsymbol{u}', \boldsymbol{v}', \boldsymbol{\gamma}', \boldsymbol{\delta}', \bar{\boldsymbol{b}}', \bar{\boldsymbol{d}}')$ is a feasible solution of model (25). We arrive at $\sum_{r=1}^{s} u_r' y_{ro}^u + \sum_{k=1}^{K} \gamma_k' w_{ko}^u - \sum_{k=1}^{K} \delta_k' w_{ko}^l \le \bar{e}_o^u$ and also $\bar{e}_o^l = \sum_{r=1}^{s} u_r^* y_{ro}^l + \sum_{k=1}^{K} \gamma_k^* w_{ko}^l - \sum_{k=1}^{K} \delta_k^* w_{ko}^u \le \sum_{r=1}^{s} u_r^* y_{ro}^u + \sum_{k=1}^{K} \gamma_k^* w_{ko}^u - \sum_{k=1}^{K} \delta_k^* w_{ko}^l = t\left(\sum_{r=1}^{s} u_r' y_{ro}^u + \sum_{k=1}^{K} \gamma_k' w_{ko}^u - \sum_{k=1}^{K} \delta_k' w_{ko}^l\right) \le t\bar{e}_o^u \le \bar{e}_o^u$. This completes the proof. □

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