2020

Sigma chromatic number of graph coronas involving complete graphs

Agnes Garciano

Maria Czarina T. Lagura

Reginaldo M. Marcelo

Journal of Physics: Conference Series

PAPER • OPEN ACCESS

Sigma chromatic number of graph coronas involving complete graphs

To cite this article: A D Garciano *et al* 2020 *J. Phys.: Conf. Ser.* **1538** 012003

Sigma chromatic number of graph coronas involving complete graphs

A D Garciano[1], M C T Lagura[2] and R M Marcelo[3]

Ateneo de Manila University, Katipunan Avenue, Quezon City, 1108 Metro Manila, Philippines

E-mail: (agarciano[1], rmarcelo[3])@ateneo.edu, mariaczarinalagura@yahoo.com[2]

Abstract. Let $c : V(G) \to \mathbb{N}$ be a coloring of the vertices in a graph G. For a vertex u in G, the color sum of u, denoted by $\sigma(u)$, is the sum of the colors of the neighbors of u. The coloring c is called a sigma coloring of G if $\sigma(u) \neq \sigma(v)$ whenever u and v are adjacent vertices in G. The minimum number of colors that can be used in a sigma coloring of G is called the sigma chromatic number of G and is denoted by $\sigma(G)$. Given two simple, connected graphs G and H, the corona of G and H, denoted by $G \odot H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and where the ith vertex of G is adjacent to every vertex of the ith copy of H. In this study, we will show that for a graph G with $|V(G)| \geq 2$, and a complete graph K_n of order n, $n \leq \sigma(G \odot K_n) \leq \max\{\sigma(G), n\}$. In addition, let P_n and C_n denote a path and a cycle of order n respectively. If $m, n \geq 3$, we will prove that $\sigma(K_m \odot P_n) = 2$ if and only if $m \leq n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$. If n is even, we show that $\sigma(K_m \odot C_n) = 2$ if and only if $m \leq n - 2\left\lceil \frac{n}{4} \right\rceil + 2$. Furthermore, in the case that n is odd, we show that $\sigma(K_m \odot C_n) = 3$ if and only if $m \leq H\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$ where $H(r, s)$ denotes the number of lattice points in the convex hull of points on the plane determined by the integer parameters r and s.

1. Introduction

In this paper, we consider only finite, simple, connected and undirected graphs. Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$ respectively. For a vertex $v \in V(G)$, the *neighborhood* of v in G, denoted by $N_G(v)$ is the set of all vertices in G that are adjacent to v. The *degree* of v, denoted by $\deg_G(v)$ is the cardinality of $N_G(v)$. Given two disjoint graphs G and H, the *corona* of G and H, denoted by $G \odot H$, is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and making the ith vertex of G adjacent to every vertex in the ith copy of H.

The corona of two graphs was introduced in 1970 by Frucht and Harary [3]. Since then, various types of colorings have been studied on this graph [6], [8]. In 2010, Chartrand, Okamoto and Zhang [1] introduced a neighbor-distinguishing type of coloring, called *sigma coloring* of a graph. Let $c : V(G) \to \mathbb{N}$ be a coloring of the vertices of a graph G in which two adjacent vertices may be assigned the same color. The *color sum* of a vertex v in G is the sum of the colors of the vertices in $N_G(v)$ and is denoted by $\sigma(v)$. When it is necessary to highlight the coloring c or the graph G, the color sum of v will also be denoted by $\sigma_c(v)$ or $\sigma_G(v)$. A coloring c of G is called a *sigma coloring* if $\sigma(u) \neq \sigma(v)$ whenever u and v are adjacent in G. The least number of colors required in a sigma coloring of a graph G is called the *sigma chromatic number* of G, and is denoted by $\sigma(G)$.

A number of researches have focused on the study of the sigma chromatic number of a graph. Dehgan, Sadeghi and Ahadi [2] showed that the problem of determining whether $\sigma(G) = 2$ for a 3-regular graph G is NP-complete. In 2016, Luzon, Ruiz and Tolentino [5] determined the sigma chromatic number of some families of circulant graphs. On the other hand, Slamin [7] introduced a coloring similar to sigma coloring where the colors used are those in the set $\{1, .., k\}$, for some positive integer k.

It was shown in [1] that $\sigma(G) \leq \chi(G)$, where $\chi(G)$ is the chromatic number of G. In the same paper, the sigma chromatic numbers of a path P_n, a cycle C_n and a complete graph K_n were determined as follows: $\sigma(P_n) = 2$ if $n \geq 4$, $\sigma(C_n) = 2$ if n is even, $\sigma(C_n) = 3$ if n is odd and $\sigma(K_n) = n$ for any positive integer n. In this paper, sigma colorings of $G \odot K_n$ and $K_n \odot G$ are considered. In particular, when G is either a path P_n or a cycle C_n, the values of $\sigma(G \odot K_n)$ will be determined. Furthermore, necessary and sufficient conditions for $\sigma(K_n \odot P_n)$ and $\sigma(K_n \odot C_n)$ to be 2 or 3 will be given.

2. Sigma Color Distribution

Suppose c_1 and c_2 are sigma colorings on disjoint graphs G and H, respectively. Define $c = (c_1, c_2)$ as the coloring of $G \odot H$ given by $c(v) = c_1(v)$, if $v \in V(G)$, and $c(v) = c_2(v)$, if v is a vertex in any copy of H. By definition, each copy of H is colored uniformly by c_2.

Lemma 1 *Let c_1 and c_2 be sigma colorings of disjoint graphs G and H respectively, and consider the coloring $c = (c_1, c_2)$ on $G \odot H$. If u and v are adjacent vertices that are both in G or both in H, then $\sigma_c(u) \neq \sigma_c(v)$.*

Proof: If $u, v \in V(H)$ and $uv \in E(H)$, then $\sigma_c(u) = \sigma_{c_2}(u) + x \neq \sigma_{c_2}(v) + x = \sigma_c(v)$ where $x = c_1(y)$ for some $y \in V(G)$. If $u, v \in V(G)$ and $uv \in E(G)$, then $\sigma_c(u) = \sigma_{c_1}(u) + k \neq \sigma_{c_1}(v) + k = \sigma_c(v)$ where $k = \sum\limits_{x \in V(H)} c_2(x)$. □

The above lemma implies that in order to show that c is a sigma coloring of $G \odot H$, it suffices to show that $\sigma_c(u) \neq \sigma_c(v)$ for any adjacent vertices $u \in V(G)$ and $v \in V(H)$.

Let c be a coloring of a graph G using distinct colors a and b. For $u \in V(G)$, consider the ordered pair (α_u, β_u) where α_u and β_u represent the number of neighbors of u colored a, and b respectively. Then $\deg_G(u) = \alpha_u + \beta_u$ and the color sum of u is $\sigma(u) = \alpha_u a + \beta_u b$. If c is a sigma 2-coloring of G, then for any two adjacent vertices u and v, $\sigma(u) \neq \sigma(v)$, and so, $(\alpha_u, \beta_u) \neq (\alpha_v, \beta_v)$. Now, in general, it is possible that $\sigma(u) = \sigma(v)$ even if $(\alpha_u, \beta_u) \neq (\alpha_v, \beta_v)$. However, it was shown in [4] that by choosing the colors a and b appropriately, it follows that if $(\alpha_u, \beta_u) \neq (\alpha_v, \beta_v)$ then $\sigma(u) \neq \sigma(v)$. Hence, to show that the color sums of two adjacent vertices are not equal, it is enough to show that the ordered pairs (α_u, β_u) and (α_v, β_v) are not equal. In particular, such is the case when $\deg_G(u) \neq \deg_G(v)$. Thus, from hereon, we identify $\sigma(u)$ with the ordered pair (α_u, β_u). Also, we will assume that whenever we refer to two colors a and b, they have the desired property that $(\alpha_u, \beta_u) \neq (\alpha_v, \beta_v)$ implies $\sigma(u) \neq \sigma(v)$ for any two vertices u and v in the graph.

The above notion may be extended analogously to a sigma 3-coloring of a graph using distinct colors a,b and d, for instance. In this case, the color sum $\sigma(u)$ of a vertex u is a triple $(\alpha_u, \beta_u, \gamma_u)$, where α_u, β_u, and γ_u, are the number of neighbors of u which are colored a, b, and d respectively. We note that the identification of color sums with tuples is consistent with the equivalence of sigma colorings and multiset colorings as discussed by Zhang in [9].

Let c be a 2-coloring of a graph G using distinct colors a and b. Then, c induces an ordered pair (n_a, n_b) where $n_a = |\{v \in V(G) : c(v) = a\}|$, $n_b = |\{v \in V(G) : c(v) = b\}|$ and $n_a + n_b = n$. The pair (n_a, n_b) is called the *color distribution* associated to c. If c is a sigma coloring, we call (n_a, n_b) a *sigma color distribution* associated to c. An ordered pair (x, y) is said to be *acceptable* for G if there exists a sigma coloring that induces it.

In any sigma 2-coloring of a path P_n or an even cycle C_n, at least one in every set of four consecutive vertices must be assigned a different color. Hence we have the following observations:

Observation 2 *Let c be a sigma 2-coloring of P_n using colors a and b where $n \geq 4$. Then,* $\left\lfloor \frac{n}{4} \right\rfloor \leq n_a, n_b \leq n - \left\lfloor \frac{n}{4} \right\rfloor$.

Observation 3 *Let c be a sigma 2-coloring of C_n, where n is even, using colors a and b with $n \geq 4$. Then* $\left\lceil \frac{n}{4} \right\rceil \leq n_a, n_b \leq n - \left\lceil \frac{n}{4} \right\rceil$.

Lemma 4 *Let P_n be a path with $n \geq 4$ and suppose k is a positive integer such that $\left\lfloor \frac{n}{4} \right\rfloor \leq k \leq n - \left\lfloor \frac{n}{4} \right\rfloor$. Then, the pair $(n_a, n_b) = (k, n-k)$ is acceptable for P_n.*

Proof: Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ with edges v_iv_{i+1} where $1 \leq i \leq n-1$. Define the coloring c as follows: for $v_i \in V(P_n)$,

$$c(v_i) = \begin{cases} a, & \text{if 4 divides } i, \\ b, & \text{if 4 does not divide } i. \end{cases}$$

Then, c is a sigma 2-coloring of P_n, where $n_a = \left\lfloor \frac{n}{4} \right\rfloor$ and $n_b = n - \left\lfloor \frac{n}{4} \right\rfloor$. Without loss of generality, we may assume $\left\lfloor \frac{n}{4} \right\rfloor \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Consider the sequence $s : c(v_1), c(v_2), ..., c(v_n)$. Define a *block* of s as a maximal subsequence consisting of terms of the same color. In particular, we refer to a block of b's as a *b-block*. It follows that the number of b-blocks in P_n is $\left\lfloor \frac{n}{4} \right\rfloor$ or $\left\lfloor \frac{n}{4} \right\rfloor + 1$. To obtain a sigma coloring in which $n_a = k$, we change the color of $k - \left\lfloor \frac{n}{4} \right\rfloor$ vertices colored b into a. This can be accomplished by changing the color of the second vertex in $k - \left\lfloor \frac{n}{4} \right\rfloor$ of the b-blocks in P_n to a. If $n \equiv 0 \bmod 4$, the number of b-blocks is exactly $\frac{n}{4}$. It is possible to choose $k - \left\lfloor \frac{n}{4} \right\rfloor$ such vertices since $k - \left\lfloor \frac{n}{4} \right\rfloor \leq \frac{n}{4}$. If $n \not\equiv 0 \bmod 4$, the number of b-blocks is $\left\lfloor \frac{n}{4} \right\rfloor + 1$. Since $k \leq \left\lfloor \frac{n}{2} \right\rfloor$, we have $k - \left\lfloor \frac{n}{4} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$. Hence, it is also possible to choose $k - \left\lfloor \frac{n}{4} \right\rfloor$ such vertices from the b-blocks. The resulting new coloring is a sigma 2-coloring in which $n_a = k$ and $n_b = n - k$. $\square$

The proof of the next lemma is similar to that of the Lemma 4 and is omitted here.

Lemma 5 *Let $n \geq 4$ be an even integer and k a positive integer such that $\left\lceil \frac{n}{4} \right\rceil \leq k \leq n - \left\lceil \frac{n}{4} \right\rceil$. Then, the pair $(n_a, n_b) = (k, n-k)$ is acceptable for C_n.*

3. Main Results

This section is divided into two subsections. The first subsection presents the sigma chromatic number of the corona graph $G \odot K_n$, while the second subsection discusses the sigma chromatic number of $K_n \odot G$. We first make the following observation.

Observation 6 *Let c be a sigma coloring on $G \odot H$ where G and H are disjoint graphs. Then the restriction of c on H is a sigma coloring.*

3.1. On the Sigma Chromatic Number of $G \odot K_n$

Theorem 7 *Let G be a simple connected graph with $|V(G)| \geq 2$. Then, $n \leq \sigma(G \odot K_n) \leq \max\{\sigma(G), n\}$ where $n \geq 2$.*

Proof: By Lemma 1, the restriction of any sigma coloring of $\sigma(G \odot K_n)$ to K_n is a sigma coloring of K_n, hence $n \leq \sigma(G \odot K_n)$.

Let $\sigma(G) = m$. First, we assume that $m \geq n$. Let c_1 be a sigma m-coloring of G and c_2 be a sigma n-coloring of K_n such that $c_2(K_n) \subseteq c_1(G)$. Now let $c = (c_1, c_2)$ be the m-coloring of $G \odot K_n$. We claim that c is a sigma m-coloring of $G \odot K_n$. Let u and v be adjacent vertices in $G \odot K_n$. By Lemma 1, it is enough to show that $\sigma(u) \neq \sigma(v)$ where $u \in V(G)$ and $v \in V(K_n)$. However, it is clear that $deg(v) = n < n+1 \leq deg(u)$. Thus, from Section 2, c is a sigma coloring on $G \odot K_n$ using m colors. This shows that $\sigma(G \odot K_n) \leq m$.

If $m < n$, we modify c_1 and c_2 defined above so that $c_1(G) \subset c_2(K_n)$. A similar technique shows that c is a sigma n-coloring of $G \odot K_n$, hence $\sigma(G \odot K_n) \leq n$. □

The following corollary is a direct consequence of Theorem 7 and the fact that $\sigma(P_n) = 2$ for $n \geq 2$ and $\sigma(P_3) = 1$.

Corollary 8 *Let m and n be positive integers with $m, n \geq 2$. Then $\sigma(P_m \odot K_n) = n$.*

Figure 1 presents a sigma coloring of the corona graph $P_m \odot K_4$, where $m \geq 2$ using distinct colors a, b, d and e, where $a, b, d, e \in \mathbb{N}$. By Corollary 8, $\sigma(P_m \odot K_4) = 4$.

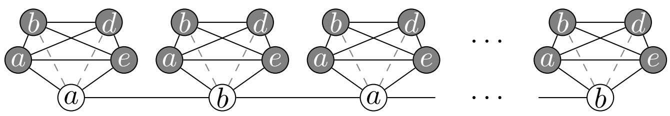

Figure 1. Sigma 4-coloring of $P_m \odot K_4$ where $m \geq 2$

Corollary 9 *Let m be a positive integer with $m \geq 3$. Then*

(i) $\sigma(C_m \odot K_n) = n$ *if* $n \geq 3$

(ii) $\sigma(C_m \odot K_2) = \sigma(C_m)$.

Proof: By Theorem 7, (i) and (ii) hold if m is even, since $\sigma(C_m) = 2$ in this case. Likewise, (i) holds when m is odd since $\sigma(C_m) = 3$.

We now prove (ii) when m is odd. By Theorem 7, $2 \leq \sigma(C_m \odot K_2) \leq 3$. Suppose c is a sigma coloring of $C_m \odot K_2$. By Observation 6, the restriction of c on K_2 must be a sigma coloring of K_2, hence c will assign 2 distinct colors a and b to the vertices of (each copy) of K_2. If u and v are adjacent vertices of C_m, then $\sigma(u) = \sigma_{C_m}(u) + a + b$. Similarly, $\sigma(v) = \sigma_{C_m}(v) + a + b$. Since c is a sigma coloring on $G \odot K_2$, then $\sigma(u) \neq \sigma(v)$. This implies that $\sigma_{C_m}(u) \neq \sigma_{C_m}(v)$, hence the restriction of c to C_m is also a sigma coloring of C_m. But this implies that at least 3 colors are needed since m is odd. Hence, $\sigma(C_m \odot K_2) \geq 3$. Thus, $\sigma(C_m \odot K_2) = 3 = \sigma(C_m)$. □

3.2. On the Sigma Chromatic Number of $K_m \odot G$

Theorem 10 *Given a graph G and $n \geq 2$, then $\sigma(G) \leq \sigma(K_n \odot G) \leq \max\{n, \sigma(G)\}$.*

Proof: By Observation 6, $\sigma(G) \leq \sigma(K_n \odot G)$. Suppose $n \geq \sigma(G)$. Let c_1 be a sigma coloring of K_n using n colors and c_2 be a sigma coloring of G using $\sigma(G)$ colors such that $c_2(G) \subset c_1(K_n)$. We will show that $c = (c_1, c_2)$ is a sigma n-coloring of $K_n \odot G$. By Lemma 1, we only need to show that $\sigma_c(u) \neq \sigma_c(v)$ where u and v are adjacent vertices and $u \in V(K_n)$ and $v \in V(G)$. But then $\deg(u) = (n-1) + |V(G)| > |V(G)| \geq \deg(v)$. From Section 2, $\sigma(u) \neq \sigma(v)$. This shows that $\sigma(K_n \odot G) \leq n = \max\{n, \sigma(G)\}$.

Now suppose $n < \sigma(G)$. Let c_1 and c_2 be as defined previously such that $c_1(K_n) \subset c_2(G)$. A similar argument as in the previous case shows that $c = (c_1, c_2)$ is a sigma coloring on $K_n \odot G$ and thus, $\sigma(K_n \odot G) \leq \sigma(G)$. □

The next observation can be easily seen by considering the vertices in K_n.

Observation 11 *Let c be a 2-coloring of K_m using the colors a and b, and let (m_a, m_b) be the color distribution associated to c. For a vertex $u \in V(K_m)$, recall that $\alpha_u = |\{v \in N(u) : c(v) = a\}|$. Then, $\alpha_u = m_a - 1$ if $c(u) = a$, or $\alpha_u = m_a$ if $c(u) = b$.*

Lemma 12 *Let $n \geq 3$. Using 2 distinct colors, say a and b, the number of distinct acceptable ordered pairs for a path P_n is given by $n - 2\left\lfloor \frac{n}{4} \right\rfloor + 1$. For an even cycle C_n, the number of distinct acceptable ordered pairs is $n - 2\left\lceil \frac{n}{4} \right\rceil + 1$.*

Proof: Suppose c is a sigma 2-coloring of a path $G = P_n$ with $n \geq 3$, using distinct colors a and b. By Lemma 2, $\left\lfloor \frac{n}{4} \right\rfloor \leq n_a, n_b \leq n - \left\lfloor \frac{n}{4} \right\rfloor$ and by Lemma 4, any pair (n_a, n_b) is acceptable if it satisfies this inequality. Since $n_a + n_b = n$, the total number of acceptable ordered pairs for P_n is $n - 2\left\lfloor \frac{n}{4} \right\rfloor + 1$.

A similar argument may be used to show that for a cycle C_n where n is even, the number of acceptable ordered pairs is $n - 2\left\lceil \frac{n}{4} \right\rceil + 1$. □

Since $\sigma(K_m) = m$, its value increases without bound as m increases. However, the sigma chromatic number of its corona with P_n may be kept low. We now present necessary and sufficient conditions for the sigma chromatic number of $K_m \odot P_n$ to be 2.

Theorem 13 *Let $m, n \geq 3$ be positive integers. Then $\sigma(K_m \odot P_n) = 2$ if and only if $m \leq n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$.*

Proof: ($\Rightarrow$) Suppose $\sigma(K_m \odot P_n) = 2$ and let c be a sigma 2-coloring of $K_m \odot P_n$ using a and b. By Observation 6, the restriction of c to P_n is also a sigma coloring. For $1 \leq i \leq m$, let P_n^i, denote the ith copy of P_n in $K_m \odot P_n$ and let (n_a^i, n_b^i) denote the color distribution associated to c on P_n^i. If u_i is the ith vertex in K_m, then by Observation 11, $\alpha_{u_i} = m_a - 1 + n_a^i$ or $m_a + n_a^i$. But by Lemma 2, $\left\lfloor \frac{n}{4} \right\rfloor \leq n_a^i \leq n - \left\lfloor \frac{n}{4} \right\rfloor$. Thus, $\min\limits_{1 \leq i \leq m} \{\alpha_{u_i}\} = m_a - 1 + \left\lfloor \frac{n}{4} \right\rfloor$ and $\max\limits_{1 \leq i \leq m} \{\alpha_{u_i}\} = m_a + n - \left\lfloor \frac{n}{4} \right\rfloor$. Hence, the total number of possible values of α_{u_i} for all i, $1 \leq i \leq m$, is $(m_a + n - \left\lfloor \frac{n}{4} \right\rfloor) - (m_a - 1 + \left\lfloor \frac{n}{4} \right\rfloor) + 1 = n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$. Since c is a sigma 2-coloring, each vertex in K_m must have a distinct color sum. Hence, the number of possible values of α_{u_i} must be greater than or equal to the number of vertices in K_m, that is, $m \leq n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$.
($\Leftarrow$) Suppose $m \leq n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$. We will construct a sigma 2-coloring c on $K_m \odot P_n$. First, denote the vertices of K_m as $u_1, u_2, \ldots, u_m$ and the ith copy of P_n in $K_m \odot P_n$ as P_n^i where $1 \leq i \leq m$. By Lemma 12, $n - 2\left\lfloor \frac{n}{4} \right\rfloor + 1$ gives the number of distinct acceptable ordered pairs for a path P_n.

If $m \leq n - 2\left\lfloor \frac{n}{4} \right\rfloor + 1$, there is a sigma 2-coloring c_i on each P_n^i which induces a unique color distribution (n_a^i, n_b^i). Now, if K_m has order $m = n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$, then for $1 \leq i \leq m - 1$, we can define c_i on P_n^i as in the previous case and let c_m be a sigma coloring on P_n^m with a maximum number of vertices colored a, that is, $n_a^m = n - \left\lfloor \frac{n}{4} \right\rfloor$.

Next, let c_{m+1} be the coloring on K_n such that all vertices are colored a except for the last vertex u_m which is colored b. Let c be the sigma coloring on $K_m \odot P_n$ such that

$$c(x) = \begin{cases} c_i(x), & \text{if } x \in P_n^i, 1 \leq i \leq m \\ c_{m+1}(x), & \text{if } x \in K_m. \end{cases}$$

We claim that c is a sigma 2-coloring of $K_m \odot P_n$. Clearly, no two adjacent vertices in P_n^i have the same color sum by our definition of c. If v and u_i are adjacent vertices where $v \in V(P_n^i)$, and $u_i \in V(K_m)$, then $\deg_G(v)$ is 2 or 3, whereas $\deg_G(u_i) = m - 1 + n \geq 5$ since $m, n \geq 3$. Hence, $\sigma_G(v) \neq \sigma_G(u_i)$.

So, what is left to show that the color sum of any two vertices in K_m are not equal. Suppose u_i and u_j are vertices in K_m where $1 \leq i < j \leq m - 1$. By Observation 11, as a vertex in K_m, $\alpha_{u_i} = m_a - 1 = \alpha_{u_j}$. This implies that as a vertex in $K_m \odot P_n$, $\alpha_{u_i} = m_a - 1 + n_a^i$ and $\alpha_{u_j} = m_a - 1 + n_a^j$. But since $n_a^i \neq n_a^j$, then $\alpha_{u_i} \neq \alpha_{u_j}$. Hence, $\sigma(u_i) \neq \sigma(u_j)$. If $m \leq n - 2\left\lfloor \frac{n}{4} \right\rfloor + 1$, we are done.

Now suppose $m = n - 2\left\lfloor \frac{n}{4} \right\rfloor + 2$ and consider the last vertex u_m in K_m. Since $c(u_m) = b$, then by Observation 11, as a vertex in K_m, $\alpha_{u_m} = m_a$. Since $n_a^m = n - \left\lfloor \frac{n}{4} \right\rfloor$, then as a vertex in $K_m \odot P_n$, we have $\alpha_{u_m} = m_a + n - \left\lfloor \frac{n}{4} \right\rfloor$. Since $\left\lfloor \frac{n}{4} \right\rfloor \leq n_a^i \leq n - \left\lfloor \frac{n}{4} \right\rfloor$, then $\alpha_{u_m} = m_a + n - \left\lfloor \frac{n}{4} \right\rfloor \geq m_a + n_a^i > m_a - 1 + n_a^i = \alpha_{u_i}$ for all $1 \leq i \leq m$. This proves that c is a sigma 2-coloring of $K_m \odot P_n$. □

The next theorem is a counterpart of the previous theorem for the corona of a complete graph with an even cycle. The proof is analogous.

Theorem 14 *Let $m \geq 3$ and $n \geq 4$ be positive integers where n is even. Then $\sigma(K_m \odot C_n) = 2$ if and only if $m \leq n - 2\left\lceil \frac{n}{4} \right\rceil + 2$.*

We now consider sigma colorings on the corona graph $K_m \odot C_n$, where n is odd. By Observation 6, a sigma coloring on this graph must be a sigma coloring on C_n, and will thus require the use of at least 3 colors. As in the case of sigma 2-colorings, if c is a sigma 3 coloring of a graph G using colors a, b and d, we define a sigma color distribution associated to c as a triple (n_a, n_b, n_d) where n_a, n_b, n_d are the numbers of vertices colored a, b and d respectively. Likewise, a triple (x, y, z) is said to be acceptable for G if there exists sigma 3-coloring which induces it.

Lemma 15 *Let n be an odd integer and $n \geq 3$. Suppose c is a sigma 3-coloring of C_n, using distinct colors a, b and d. Then the following hold: (i) $1 \leq n_a, n_b, n_d \leq n - \left\lceil \frac{n}{4} \right\rceil$; and (ii) $\left\lceil \frac{n}{4} \right\rceil \leq n_a + n_b, n_a + n_d, n_b + n_d \leq n - 1$.*

Proof: Since c is a sigma 3-coloring, then $n_a, n_b, n_d \geq 1$. Since $n_a + n_b = n - n_d$, then $n_a + n_b \leq n - 1$. Since no four consecutive vertices must have the same color, then $n_a, n_b \leq n - 1 - \left\lfloor \frac{n-1}{4} \right\rfloor = n - \left\lceil \frac{n}{4} \right\rceil$. Similarly, $n_d \leq n - \left\lceil \frac{n}{4} \right\rceil$, hence, $n_a + n_b = n - n_d \geq \left\lceil \frac{n}{4} \right\rceil$. The other inequalities are similar. □

Lemma 16 *Let n be a positive odd integer. Suppose $x, y \in \mathbb{N}$ and (i) $1 \leq x, y \leq n - \left\lceil \frac{n}{4} \right\rceil$, (ii) $\left\lceil \frac{n}{4} \right\rceil \leq x + y \leq n - 1$ and (iii) $z = n - x - y$. Then, (x, y, z) is acceptable for C_n.*

Proof: First, we make two claims.
Claim 1: $1 \leq z \leq n - \left\lceil \frac{n}{4} \right\rceil$
Proof of Claim 1: From the assumptions (*i*) to (*iii*) above, $1 = n-(n-1) \leq n-(x+y) \leq n-\left\lceil \frac{n}{4} \right\rceil$. Since $z = n - (x + y)$, then $1 \leq z \leq n - \left\lceil \frac{n}{4} \right\rceil$.
Claim 2: $\left\lceil \frac{n}{4} \right\rceil \leq y + z \leq n - 1$ and $\left\lceil \frac{n}{4} \right\rceil \leq x + z \leq n - 1$
Proof of Claim 2: From assumption (*i*), $\left\lceil \frac{n}{4} \right\rceil = n-(n-\left\lceil \frac{n}{4} \right\rceil) \leq n-y \leq n-1$. Since $x+z = n-y$, then $\left\lceil \frac{n}{4} \right\rceil \leq x + z \leq n - 1$. Similarly, it can be shown that $\left\lceil \frac{n}{4} \right\rceil \leq y + z \leq n - 1$.

Without loss of generality, we assume that $x \geq y \geq z$. In the following, we will show that there is a sigma coloring of C_n using distinct colors a, b and d, and with sigma color distribution (x, y, z). We consider two cases: first, when $z = 1$ and second, when $z \geq 2$.
Case 1: Suppose $z = 1$. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ where $v_i v_{i+1} \in E(C_n)$ for $1 \leq i \leq n$, where addition is done modulo n. Since $z = 1$, we pick a vertex, say v_1 and assign the color d to it. Consider the remaining $n - 1$ vertices $v_2, \ldots, v_n$. These induce a subgraph isomorphic to P_{n-1}. We claim that there is a sigma 2-coloring of this subgraph using colors a and b with color distribution (x, y). By Lemma 4, such a coloring exists if and only if x and y satisfy

$$\left\lfloor \frac{n-1}{4} \right\rfloor \leq x, y \leq n - 1 - \left\lfloor \frac{n-1}{4} \right\rfloor. \tag{1}$$

Note that since n is odd, then $\left\lceil \frac{n}{4} \right\rceil = \left\lfloor \frac{n-1}{4} \right\rfloor + 1$. By assumption (*i*), we have $x \leq n - \left\lceil \frac{n}{4} \right\rceil = n - 1 - \left\lfloor \frac{n-1}{4} \right\rfloor$. Furthermore, by Claim 2, $\left\lceil \frac{n}{4} \right\rceil \leq x + z = x + 1$, hence $\left\lfloor \frac{n-1}{4} \right\rfloor = \left\lceil \frac{n}{4} \right\rceil - 1 \leq x$. This shows that x satisfies the inequality given in inequality (1). A similar proof can be given to show that $\left\lfloor \frac{n}{4} \right\rfloor \leq y \leq n - 1 - \left\lfloor \frac{n-1}{4} \right\rfloor$. Thus, a sigma coloring c' on the subgraph P_{n-1} exists using colors a and b.

Now consider the 3-coloring c on C_n defined by $c(v_1) = d$ and $c(v_i) = c'(v_i)$ for $i \neq 1$. Since c' is a sigma coloring on P_{n-1}, then to show that c is a sigma 3-coloring on C_n, it is enough to show that $\sigma_c(v_i) \neq \sigma_c(v_{i+1})$ for $i \in \{n-1, n, 1, 2\}$. Equivalently, we need to show that $(\alpha_{v_i}, \beta_{v_i}, \gamma_{v_i}) \neq (\alpha_{v_{i+1}}, \beta_{v_{i+1}}, \gamma_{v_{i+1}})$ for this set of values of i. The values of the indicated triples are shown in Figure 2.

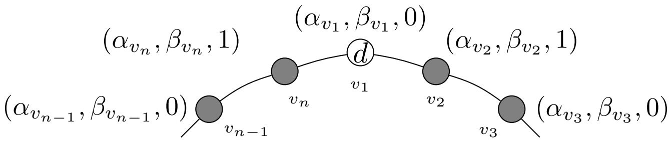

Figure 2. 3-coloring of C_n with $z = 1$.

As seen in Figure 2, no triples corresponding to adjacent vertices are equal. Hence, $(x, y, 1)$ is acceptable for C_n.

Case 2: Assume that $z \geq 2$. We make a third claim.

Claim 3: There exist two consecutive values of x' that satisfy the following.

(i) $\left\lceil \frac{y}{3} \right\rceil - 1 \leq x' \leq 3(y+1)$

(ii) $x - 3(z-1) \leq x' \leq \begin{cases} x - \left\lceil \frac{z}{3} \right\rceil - 1, & \text{if } z \not\equiv 2 \bmod 3, \\ x - \left\lceil \frac{z}{3} \right\rceil, & \text{if } z \equiv 2 \bmod 3. \end{cases}$

Proof of Claim 3: Let $r_1 = \left\lceil \frac{y}{3} \right\rceil - 1, r_2 = 3(y+1), s_1 = x - 3(z-1), s_2 = x - \left\lceil \frac{z}{3} \right\rceil - 1$ if $z \not\equiv 2 \bmod 3, x - \left\lceil \frac{z}{3} \right\rceil$ if $z \equiv 2 \bmod 3$. Clearly, $r_1 < r_2$ and $s_1 < s_2$. It is enough to show that $s_1 < r_2$, and $r_1 < s_2$, since it will follow that the two closed intervals $[r_1, r_2]$ and $[s_1, s_2]$ will intersect in at least two consecutive integers.

Since n is odd and $n = x + y + z$, then $4\left(\frac{x+y+z}{4}\right) < 4\left\lceil \frac{x+y+z}{4} \right\rceil$. By Claim 2, $4\left\lceil \frac{x+y+z}{4} \right\rceil \leq 4(y+z)$. From this, it follows that $x - 3(z-1) < 3(y+1)$ or equivalently $s_1 < r_2$. Since $x \geq y \geq z$, then $\left\lfloor \frac{n}{3} \right\rfloor \geq z$, and since $z \geq 2$, then $n \geq 7$ and $x \geq 3$. It follows that $\left\lceil \frac{y}{3} \right\rceil + \left\lceil \frac{z}{3} \right\rceil \leq \left\lceil \frac{y+z}{3} \right\rceil + 1 \leq \left\lceil \frac{x+y+z}{3} \right\rceil \leq x$. Thus, $\left\lceil \frac{y}{3} \right\rceil - 1 \leq x - \left\lceil \frac{z}{3} \right\rceil - 1$ and clearly, $\left\lceil \frac{y}{3} \right\rceil - 1 < x - \left\lceil \frac{z}{3} \right\rceil$.

We will show that $\left\lceil \frac{y}{3} \right\rceil - 1 \neq x - \left\lceil \frac{z}{3} \right\rceil - 1$, if $z \not\equiv 2 \bmod 3$. On the contrary, suppose that $\left\lceil \frac{y}{3} \right\rceil - 1 = x - \left\lceil \frac{z}{3} \right\rceil - 1$ when $z \not\equiv 2 \bmod 3$. Then, $x = \left\lceil \frac{y}{3} \right\rceil + \left\lceil \frac{z}{3} \right\rceil \leq \left\lceil \frac{x+y+z}{3} \right\rceil$. Since $3\left\lceil \frac{y}{3} \right\rceil \leq y+2$ and $3\left\lceil \frac{z}{3} \right\rceil \leq z+2$, we have $(y+2)+(z+2) \geq 3\left\lceil \frac{x+y+z}{3} \right\rceil \geq x+y+z$. Thus, $x \leq 4$. Since $z \geq 2$, then $(x, y, z) \in \{(3,3,3), (4,4,3)\}$. Any one of these ordered triples results to $\left\lceil \frac{y}{3} \right\rceil - 1 \neq x - \left\lceil \frac{z}{3} \right\rceil - 1$, a contradiction. This proves Claim 3.

Let x' satisfy the conditions in Claim 3. We note that we can take x to be odd or even, as necessary. We will show that there exists a sigma coloring on the subgraph $P_{x'+y}$ induced by the vertices $v_1, v_2, v_3, \ldots, v_{x'+y}$, using colors a and b and with color distribution (x', y). By Lemma 4, such a sigma coloring exists if and only if $\left\lfloor \frac{x'+y}{4} \right\rfloor \leq x', y \leq x' + y - \left\lfloor \frac{x'+y}{4} \right\rfloor$. Clearly, it is enough to show that $\left\lfloor \frac{x'+y}{4} \right\rfloor \leq x'$ and $\left\lfloor \frac{x'+y}{4} \right\rfloor \leq y$.

Let $r = y \bmod 3$, where $r \in \{0, 1, 2\}$.

Case 1: Assume that $r = 0$. From Claim 3(i), $x' \geq \left\lceil \frac{y}{3} \right\rceil - 1$. Since $3|y$, $x' \geq \frac{x'+y-3}{4}$, and since x' is an integer, $x' \geq \left\lceil \frac{x'+y-3}{4} \right\rceil = \left\lfloor \frac{x'+y}{4} \right\rfloor$.

Case 2: Suppose $r \neq 0$. By Claim 3(i), $x' \geq \left\lceil \frac{y}{3} \right\rceil - 1 = \left(\frac{y-r}{3} + 1\right) - 1 = \frac{y-r}{3}$. It follows that $x' \geq \frac{x'+y-r}{4}$. Since x' is an integer, then $x' \geq \left\lceil \frac{x'+y-r}{4} \right\rceil \geq \left\lfloor \frac{x'+y}{4} \right\rfloor$.

A similar proof can be used to show that $y \geq \left\lceil \frac{x'+y}{4} \right\rceil$.

Let $x'' = x - x'$. Since x' can be even or odd, x'' can also be even or odd. Our next goal is to give a coloring of the remaining $x'' + z$ vertices of C_n which induce a subgraph isomorphic to

$P_{x''+z}$. We will show that a sigma 2-coloring of $P_{x''+z}$ exists using the colors a and d with color distribution (x'', z). First, we note that from Claim 3, we have

(1) $3(z-1) \geq x'' \geq \begin{cases} \lceil \frac{z}{3} \rceil + 1, & \text{if } z \not\equiv 2 \bmod 3 \\ \lceil \frac{z}{3} \rceil, & \text{if } z \equiv 2 \bmod 3 \end{cases}$

(2) $x'' \equiv 3(z-1) \bmod 2$.

Furthermore, note that $3(z-1)$ has the same parity as $\lceil \frac{z}{3} \rceil$ or $\lceil \frac{z}{3} \rceil + 1$, depending on the congruence class of $z \bmod 3$. We want to exhibit a sigma 2-coloring of $P_{x''+z}$ satisfying the following requirements:

(R1) The endpoints of the path are colored d.

(R2) Immediate neighbors of the endpoints are colored a.

(R3) Between any two consecutive vertices colored d, there is/are $0, 1$ or 3 vertices colored a.

(R4) Between any two consecutive vertices colored a, there is/are $0, 1$ or 3 vertices colored d.

The requirements R3 and R4 are imposed to ensure that no color strings of the form *adda* or *daad* occur in the coloring of $P_{x''+z}$, as such strings give rise to adjacent vertices with equal color sums. Moreover, these requirements also ensure that in any given string of four consecutive vertices, not all vertices have the same color.

First, suppose $z \equiv 2 \bmod 3$. We want to define a sigma 2-coloring on $P_{x''+z}$ using colors a and d with color distribution (x'', z), for any possible value of x'' satisfying (1) and (2). From R1,R2, R3 and R4, the minimum number of vertices that can be colored with a is $\frac{z-2}{3} + 1 = \lceil \frac{z}{3} \rceil$. Thus, we can have a sigma coloring with color distribution $\left(\lceil \frac{z}{3} \rceil, z\right)$. If we replace a string of the form *ddd* by *dadad* we obtain a sigma coloring with color distribution $\left(\lceil \frac{z}{3} \rceil + 2, z\right)$. By replacing other strings of the form *ddd* by *dadad* or strings of the form *dad* by *daaad*, we obtain sigma colorings with color distributions $\left(\lceil \frac{z}{3} \rceil + 4, z\right), \left(\lceil \frac{z}{3} \rceil + 6, z\right), \ldots (3(z-1), z)$. The last color distribution corresponds to the color string *daaadaaad*$\ldots$*daaad*. Effectively, each of these possibilities describe a sigma coloring for $P_{x''+z}$ with color distribution (x'', z) where x'' satisfies (1) and (2).

The cases where $z \equiv 1$ or $0 \bmod 3$ may be dealt with following a similar scheme as the one above.

Finally, consider the coloring c on C_n induced by the sigma colorings on $P_{x'+y}$ and $P_{x''+z}$ (Refer to Figure 3) using the colors a, b and d. We will show that c is a sigma 3-coloring on C_n.

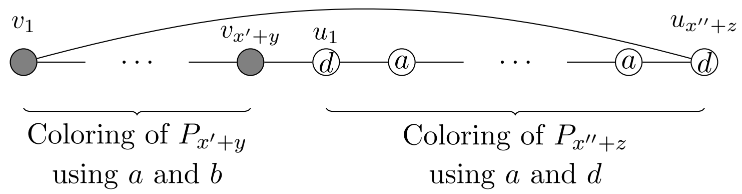

Figure 3. 3-coloring of C_n using colors a, b and d.

Observe that $\sigma_c(v_{x'+y}) = (\alpha_1, \beta_1, 1)$, $\sigma_c(u_1) = (\alpha_2, \beta_2, 0)$, $\sigma_c(v_1) = (\alpha_3, \beta_3, 1)$ and $\sigma_c(u_{x''+z}) = (\alpha_4, \beta_4, 0)$, for some nonnegative integers α_i and β_i, $1 \leq i \leq 4$. Clearly, $\sigma_c(v_{x'+y}) \neq \sigma_c(u_1)$ and $\sigma_c(v_1) \neq \sigma_c(u_{x''+z})$. From the construction, it follows that c is a sigma 3-coloring on C_n. Furthermore, the associated color distribution is (x, y, z). This proves the lemma. $\square$

By Lemma 16, any set of triples (x, y, z) satisfying (i), (ii) and (iii) of the lemma is acceptable for C_n. Note that z is completely determined by $x + y$. Let $\mathcal{N} =$

$\left\{(x,y) : 1 \le x, y \le n - \left\lceil \frac{n}{4} \right\rceil, \left\lceil \frac{n}{4} \right\rceil \le x+y \le n-1\right\}$. Then $N = |\mathcal{N}|$ corresponds to the number of lattice points in the convex hull of the hexagon with vertices $A\left(\left\lceil \frac{n}{4} \right\rceil - 1, 1\right)$, $B\left(1, \left\lceil \frac{n}{4} \right\rceil - 1\right)$, $C\left(1, n - \left\lceil \frac{n}{4} \right\rceil\right)$, $D\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$, $E\left(n - \left\lceil \frac{n}{4} \right\rceil, \left\lceil \frac{n}{4} \right\rceil - 1\right)$ and $F\left(n - \left\lceil \frac{n}{4} \right\rceil, 1\right)$ (Refer to Figure 4).

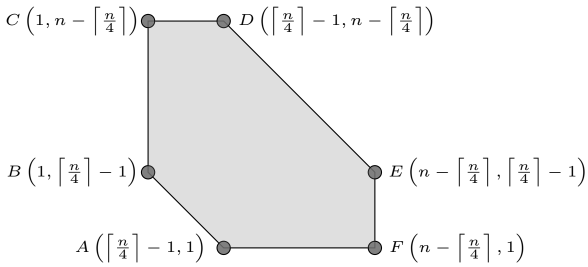

Figure 4. Set $\mathcal{N}$

For two nonnegative integers r, s with $s \ge r$, let $H(r,s)$ be the number of lattice points in the convex hull of A', B', C', D', E' and F' where $A'(r,0)$, $B'(0,r)$, $C'(0,s)$, $D'(r,s)$, $E'(s,r)$ and $F'(s,0)$. Then, $H(r,s) = (s+1)^2 - \binom{r+1}{2} - \binom{s-r+1}{2}$. Thus, $N = H(r,s)$ where $r = \left\lceil \frac{n}{4} \right\rceil - 2$ and $s = n - \left\lceil \frac{n}{4} \right\rceil - 1$.

We are now ready to prove the last theorem.

Theorem 17 *Let m and n be positive integers where $m, n \ge 3$ and n is odd. Then, $\sigma(K_m \odot C_n) = 3$ if and only if $m \le H\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$.*

Proof: ($\Rightarrow$) Let $G = K_m \odot C_n$ and suppose $\sigma(G) = 3$. Let c be a sigma 3-coloring of G using the colors a, b and d. Suppose (r,s,t) is the color distribution induced by c on K_m. If $v \in V(K_m)$, then restricted to K_m, $\sigma_{K_m}(v)$ is $(r-1,s,t), (r,s-1,t)$ or $(r,s,t-1)$ according to whether $c(v)$ is a, b or d respectively. Consider the restriction of c to the copy of C_n that is joined to v in G. Note that this restriction is a sigma 3-coloring of C_n. If (x,y,z) is the sigma 3-color distribution on this copy of C_n, then the possible values of $\sigma(v)$ are as follows:

$$(x,y,z) + (r-1,s,t) \text{ or } (x,y,z) + (r,s-1,t) \text{ or } (x,y,z) + (r,s,t-1) \tag{2}$$

Note that we can ignore the third component in each triple in (2) since it is dependent on the first two components. Furthermore, to simplify computations, we can look at the resulting color sums as translations of $(r-1, s-1)$, that is, the possible values of $\sigma(v)$ given in (2), can be simplified to $(x,y) + (0,1), (x,y) + (1,0)$ and $(x,y) + (1,1)$ respectively. The total number of such values corresponds to $H\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$. Since the vertices in K_m should have distinct color sums, then $m \le H\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$.

($\Leftarrow$) Suppose $m = H\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$. Let $\mathcal{A} = \mathcal{N}$,

$$\mathcal{B} = \left\{(x,y) \in \mathcal{N} : \left\lceil \frac{n}{4} \right\rceil - 1 \le x \le n - \left\lceil \frac{n}{4} \right\rceil \text{ and } y = 1, \text{ or } x = n - \left\lceil \frac{n}{4} \right\rceil \text{ and } 1 \le y \le \left\lceil \frac{n}{4} \right\rceil - 1\right\}$$

and $\mathcal{D} = \{(x,y) \in \mathcal{N} : x + y = n - 1\}$. Then, $m = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{D}|$.

We will now exhibit a sigma 3-coloring on G. First, we color the vertices in K_m such that $|\mathcal{A}|$ are colored a, $|\mathcal{B}|$ are colored b and $|\mathcal{D}|$ are colored d. Next, for each vertex colored a in K_m, assign to the corresponding C_n a sigma coloring with distinct color distribution from $\mathcal{A}$. Similarly, assign to the corresponding C_n of each vertex colored b (respectively, d) in K_m a sigma coloring with distinct color distribution from $\mathcal{B}$ (respectively, $\mathcal{D}$).

In the succeeding part of the proof, refer to Figure 5 below. From the construction, the color sums of the vertices in K_m colored a correspond to distinct elements of $\mathcal{A} + (0, 1)$ (region shaded by horizontal lines). Similarly, the color sums of the vertices in K_m colored b correspond to distinct elements of $\mathcal{B} + (1, 0)$ (two perpendicular line segments with a common endpoint). Finally those colored d have color sums that correspond to the elements of $\mathcal{D} + (1, 1)$ (line segment). Clearly, no differently colored vertices in K_m have equal color sums. Furthermore, the total number of distinct color sums generated by these sets is m.

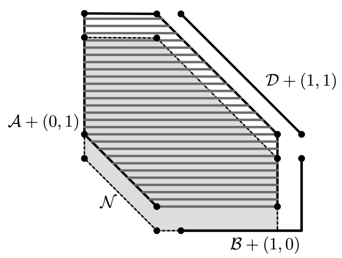

Figure 5. Color sums in $K_m \odot C_n$, where n is odd

If $m < H\left(\left\lceil \frac{n}{4} \right\rceil - 1, n - \left\lceil \frac{n}{4} \right\rceil\right)$, then we can choose subsets of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ such that the union has cardinality m, and assign the colors accordingly. Hence, we have $\sigma(K_m \odot C_n) \leq 3$. By Theorem 10, $\sigma(K_m \odot C_n) \geq 3$. □

4. Conclusion

In this paper, bounds for $\sigma(G \odot K_n)$ and $\sigma(K_n \odot G)$, for which G is arbitrary, were given. Consequently the values of $\sigma(P_m \odot K_n)$ for $m, n \geq 2$ and $\sigma(C_m \odot K_n)$ for $m \geq 3$ and $n \geq 2$ were determined. Moreover, necessary and sufficient conditions for $\sigma(K_m \odot C_n)$ to be 2 or 3 were given. Determining $\sigma(G \odot H)$ and $\sigma(H \odot G)$ for other families of graphs G and H are recommended for further research.

Acknowledgments

The authors would like to thank the Loyola Schools of the Ateneo de Manila University for the LS Scholarly Work Grant to be able to conduct this research. They are also grateful to the reviewers of this paper for their helpful comments and suggestions.

References

[1] Chartrand G, Okamoto F and Zhang P 2010 The sigma chromatic number of a graph *Graphs Combin.* **26** 755-773
[2] Dehgan A, Sadeghi M R and Ahadi A 2014 The complexity of the sigma chromatic number of cubic graphs *arXiv* **1** 1403.6288 (math.CO)
[3] Frucht R and Harary F 1970 On the corona of two graphs *Aequationes Math.* **4** 322-325
[4] Lagura M C 2016 On the sigma chromatic number of the join of paths, cycles and complete multipartite graphs Master's Thesis (Ateneo de Manila University)
[5] Luzon P A, Ruiz M and Tolentino M A 2016 The sigma chromatic number of the circulant graphs $C_n(1, 2)$, $C_n(1, 3)$, and $C_{2n}(1, n)$, *Discrete Comput. Geom.* eds. Akiyama J, Ito H, Sakai T and Uno Y Springer Lect. Notes in Computer Science **9943**
[6] Molam S, Geetha J and Somasundaram K 2017 Total toloring of the corona product of graphs *Australas. J. Combin.* **68**(1) 15-22
[7] Slamin 2017 On distance irregular labelling of graphs *Far East J. Math. Sci.* **102**(5) 919-932
[8] Venkatesam K, Joseph V and Mathiyazhagan V 2015 On star coloring of corona graphs *Appl. Math. E-Notes* **15**(4) 97-104
[9] Zhang P 2016 *A Kaleidoscopic View of Graph Colorings* (Kalamazoo: Springer Nature)