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Jayson D. Tolentino

Reginaldo M. Marcelo

Mark Anthony C. Tolentino

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Twin chromatic indices of some graphs with maximum degree 3

J D Tolentino[1], **R M Marcelo**[2], **and M A C Tolentino**[2]

[1] Eulogio "Amang" Rodriguez Institute of Science and Technology, Manila, Philippines
[2] Ateneo de Manila University, Quezon City, Philippines

E-mail: `jaysondelunatolentino@yahoo.com`, `rmarcelo@ateneo.edu`, `mtolentino@ateneo.edu`

Abstract. Let $k \geq 2$ be an integer and G be a connected graph of order at least 3. A twin k-edge coloring of G is a proper edge coloring of G that uses colors from $\mathbb{Z}_k$ and that induces a proper vertex coloring on G where the color of a vertex v is the sum (in $\mathbb{Z}_k$) of the colors of the edges incident with v. The smallest integer k for which G has a twin k-edge coloring is the twin chromatic index of G and is denoted by $\chi'_t(G)$. In this paper, we determine the twin chromatic indices of circulant graphs $C_n(1, \frac{n}{2})$, and some generalized Petersen graphs such as $GP(3s, k)$, $GP(m, 2)$, and $GP(4s, l)$ where $n \geq 6$ and $n \equiv 0 \pmod 4$, $s \geq 1$, $k \not\equiv 0 \pmod 3$, $m \geq 3$ and $m \notin \{4, 5\}$, and l is odd. Moreover, we provide some sufficient conditions for a connected graph with maximum degree 3 to have twin chromatic index greater than 3.

1. Introduction

Let $G = (V, E)$ be a simple graph. A *proper vertex coloring* of G is a function from V to a given set of colors such that adjacent vertices are colored differently. On the other hand, a *proper edge coloring* of G is a function from E to a given set of colors such that adjacent edges are colored differently. The minimum number of colors needed in a proper vertex coloring and a proper edge coloring of G are the *chromatic number* and *chromatic index* of G and are denoted by $\chi(G)$ and $\chi'(G)$, respectively. Thus $\chi(G) \leq \Delta(G) + 1$ and $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

Aside from the original notions of proper colorings, various graph colorings that use the sum of colors to induce certain types of vertex colorings have also been studied in the literature. Some of these studies are the works of Agustin *et al.* [1] and Slamin [7]. In [1], Agustin *et al.* investigated the local edge antimagic coloring of comb product of some graphs and in [7], Slamin introduced the distance irregular labelling of graphs.

In this paper, we focus on a relatively new kind of graph coloring called the twin edge coloring which was introduced by Chartrand [9] and was initially studied in [2–4].

Definition 1.1. For a connected graph G of order at least 3, a proper k-edge coloring $c : E(G) \to \mathbb{Z}_k$ for some integer $k \geq 2$ is called a *twin k-edge coloring* of G if the induced vertex coloring $c' : V(G) \to \mathbb{Z}_k$ defined by

$$c'(v) = \sum_{e \in E_v} c(e) \text{ in } \mathbb{Z}_k,$$

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where E_v is the set of edges of G incident with v, is proper as well. The minimum k for which G has a twin k-edge coloring is the *twin chromatic index* of G, denoted by $\chi'_t(G)$.

Since a twin edge coloring of G is a proper edge coloring of G, $\chi'_t(G) \geq \Delta(G)$. It has been shown in [2] that every connected graph of order at least 3 has a twin edge coloring.

In [2], Andrews *et al.* obtained the twin chromatic indices of paths, cycles, complete graphs, and complete bipartite graphs. Based on the results in [2], Andrews *et al.* [3] formulated Conjecture 1.2 and verified it for permutation graphs of 5-cycle, grids and prisms, and trees with maximum degree at most 6. Likewise, in [4], Conjecture 1.2 was also verified for several types of trees such as brooms, double stars, and some regular trees. Also, the twin chromatic indices of most of the graphs discussed in [3, 4] are determined.

Conjecture 1.2. [3] *If G is a connected graph of order at least 3 that is not a 5-cycle, then $\chi'_t(G) \leq \Delta(G) + 2$.*

Recently, in 2016, Lakshmi and Kowsalya [5] determined the twin chromatic index of wheel graphs while Rajarajachozhan and Sampathkumar [6] determined the twin chromatic indices of the square graphs P_n^2, where $n \geq 4$, and C_n^2, where $n \geq 6$ and the twin chromatic index of the Cartesian product $C_m \Box P_n$, where $m, n \geq 3$. Moreover, Tolentino *et al.* [8] verified Conjecture 1.2 for all trees of order at least 3.

In this paper, the twin chromatic indices of some graphs with maximum degree 3 will be discussed, beginning with the circulant graphs $C_n(1, \frac{n}{2})$ with $n \geq 6$ is even.

2. Circulant Graphs $C_n(1, \frac{n}{2})$

Definition 2.1. Let n, m, and $a_1, \ldots, a_m$ be positive integers. An undirected graph with vertex set $V = \{v_0, \ldots, v_{n-1}\}$ and edge set $E = \{v_i v_{i+a_j} : 0 \leq i \leq n-1, 1 \leq j \leq m\}$, where subscripts are modulo n, is called a ***Circulant Graph*** and is denoted by $C_n(a_1, \ldots, a_m)$.

The following observation will be helpful.

Observation 2.2. [2] *If a connected graph G contains two adjacent vertices of degree $\Delta(G)$, then $\chi'_t(G) \geq 1 + \Delta(G)$. In particular, if G is a connected r-regular graph for some integer $r \geq 2$, then $\chi'_t(G) \geq 1 + r$.*

Lemma 2.3. *If $n \geq 6$ is an even integer, then $\chi'_t(C_n(1, \frac{n}{2})) \geq 5$.*

Proof. Let $n \geq 6$ be an even integer and $G = C_n(1, \frac{n}{2})$. Let $E(G) = \{e_i = v_i v_{i+1}, f_i = v_i v_{i+\frac{n}{2}} \mid 0 \leq i \leq n-1\}$ (subscripts are computed modulo n). Since G is a cubic graph, by Observation 2.2, $\chi'_t(G) \geq 4$. We will show that $\chi'_t(G) \neq 4$.

Suppose on the contrary that $\chi'_t(G) = 4$; that is, G has a twin 4-edge coloring c. If two different elements of $\mathbb{Z}_4$ are used to color the edges e_i, e_{i+1}, $e_{i+\frac{n}{2}}$, and $e_{i+\frac{n}{2}+1}$ for some $i \in \{0, \ldots, n-1\}$, then $c'(v_{i+1}) = c(e_i) + c(e_{i+1}) + c(f_{i+1}) = c(e_{i+\frac{n}{2}}) + c(e_{i+\frac{n}{2}+1}) + c(f_{i+1}) = c'(v_{i+\frac{n}{2}+1})$, which would make c' improper. On the other hand, if all elements of $\mathbb{Z}_4$ are used to color the edges e_i, e_{i+1}, $e_{i+\frac{n}{2}}$, and $e_{i+\frac{n}{2}+1}$ for some $i \in \{0, \ldots, n-1\}$, then any element of $\mathbb{Z}_4$ can no longer be used to color the edge f_{i+1}. Therefore, for each $i \in \{0, \ldots, n-1\}$, the edges e_i, e_{i+1}, $e_{i+\frac{n}{2}}$, and $e_{i+\frac{n}{2}+1}$ are colored using exactly three different elements of $\mathbb{Z}_4$. Let w, x, y, and z be the four distinct elements of $\mathbb{Z}_4$.

First, we arrange the vertices v_0, v_1, $\ldots$, v_{n-1} of G consecutively in a regular n-gon.

Claim 1: $c(e_i) \neq c(e_{i+2})$ for each $i \in \{0, 1, 2, \ldots, n-1\}$.

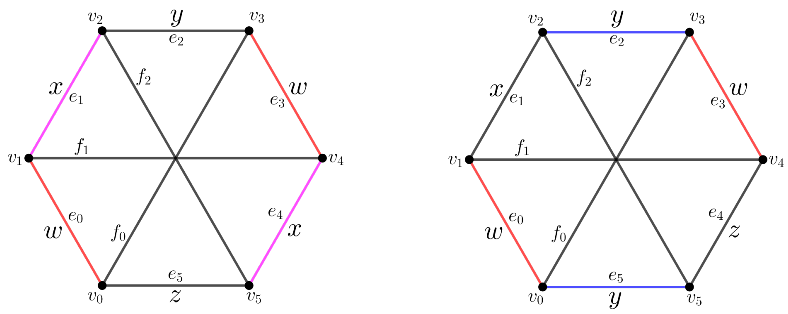
Suppose $c(e_i) = w = c(e_{i+2})$ and $c(e_{i+1}) = x$ for some i. Then we may assume $c(f_{i+1}) = y$ and $c(f_{i+2}) = z$. Since the edge $e_{i+\frac{n}{2}+1}$ is adjacent to the edges f_{i+1} and f_{i+2}, $c(e_{i+\frac{n}{2}+1}) = w$ or

x. Without loss of generality, let $c(e_{i+\frac{n}{2}+1}) = w$. Then $c(e_{i+\frac{n}{2}}) \in \{x, z\}$ and $c(e_{i+\frac{n}{2}+2}) \in \{x, y\}$. Since the edges e_i, e_{i+1}, $e_{i+\frac{n}{2}}$, and $e_{i+\frac{n}{2}+1}$ must be colored using exactly three different elements of $\mathbb{Z}_4$, $c(e_{i+\frac{n}{2}}) = z$. Using the same argument, for the edges e_{i+1}, e_{i+2}, $e_{i+\frac{n}{2}+1}$, and $e_{i+\frac{n}{2}+2}$, we conclude that $c(e_{i+\frac{n}{2}+2}) = y$. Therefore, $c'(v_{i+\frac{n}{2}+1}) = w + y + z = c'(v_{i+\frac{n}{2}+2})$ and we get a contradiction.

Suppose $n = 6$. By Claim 1, $c(e_i) \neq c(e_{i+2})$ for each $i \in \{0, 1, 2, 3, 4, 5\}$. Without loss of generality, let $c(e_0) = w$, $c(e_1) = x$, and $c(e_2) = y$. Since $c(e_2) \neq c(e_4)$ and $c(e_4) \neq c(e_6 = e_0)$, $c(e_4) \in \{x, z\}$.

Case 1: If $c(e_4) = x$, then the coloring is as shown in Figure 1.(a). In this case, $c'(v_1)$ would be equal to $c'(v_4)$.

Case 2: If $c(e_4) = z$, a similar argument (see Figure 1.(b)) gives $c'(v_0) = c'(v_3)$.



(a) (b)

Figure 1. Illustrating the coloring c when $n = 6$.

Hence, $\chi'_t(G) \geq 5$ if $G = C_6(1, 3)$. Assume now that $n \geq 8$.

Claim 2: $c(e_i) \neq c(e_{i+\frac{n}{2}})$ for each $i \in \{0, 1, 2, \ldots, n-1\}$.

Suppose $c(e_i) = w = c(e_{i+\frac{n}{2}})$ for some i. Then, we may assume that $c(e_{i-1}) = x$ and $c(e_{i+\frac{n}{2}-1}) = y$. Then $c(f_i) = z$ and $c(e_{i-2}) \in \{y, z\}$. If $c(e_{i-2}) = z$, then $c(f_{i-1}) = w$. Then $c'(v_{i-1}) = w + x + z = c'(v_i)$. Suppose $c(e_{i-2}) = y$. Then $c(e_{i+\frac{n}{2}-2}) \in \{w, z\}$. But by Claim 1, $c(e_{i+\frac{n}{2}-2}) \neq c(e_{i+\frac{n}{2}}) = w$ so $c(e_{i+\frac{n}{2}-2}) = z$. Thus $c(f_{i-1}) = w$. Therefore, $c'(v_{i+\frac{n}{2}-1}) = w + y + z = c'(v_{i+\frac{n}{2}})$. In any case, c' becomes improper, a contradiction.

Claim 3: $c(e_i) \in \{c(e_{i+\frac{n}{2}-1}), c(e_{i+\frac{n}{2}+1})\}$ for each $i \in \{0, 1, 2, \ldots, n-1\}$.

Suppose $w = c(e_i) \notin \{c(e_{i+\frac{n}{2}-1}), c(e_{i+\frac{n}{2}+1})\}$ for some i. By Claim 1 and Claim 2, we may assume that $c(e_{i+\frac{n}{2}-1}) = x$, $c(e_{i+\frac{n}{2}}) = y$, and $c(e_{i+\frac{n}{2}+1}) = z$. Then $c(f_i) = z$ and $c(f_{i+1}) = x$. Therefore, $c'(v_{i+\frac{n}{2}}) = x + y + z = c'(v_{i+\frac{n}{2}+1})$ and we get a contradiction.

Suppose $c(e_0) = w$. By Claims 1 and 3, exactly one of the edges $e_{\frac{n}{2}-1}$ and $e_{\frac{n}{2}+1}$ must be colored using the color w. Without loss of generality, let $c(e_{\frac{n}{2}-1}) = w$. Moreover, we suppose $c(e_{\frac{n}{2}}) = x$. Then $c(e_{n-1}) \neq x$; so by Claim 3, $c(e_1) = x$. By Claim 3, $c(e_{\frac{n}{2}+1}) \in \{c(e_0) = w, c(e_2)\}$. Since $c(e_{\frac{n}{2}-1}) = w$ and $c(e_{\frac{n}{2}-1}) \neq c(e_{\frac{n}{2}+1})$, $c(e_{\frac{n}{2}+1}) = c(e_2)$. Similarly, $c(e_{\frac{n}{2}+2}) \in \{c(e_1) = x, c(e_3)\}$. Since $c(e_{\frac{n}{2}}) = x$ and $c(e_{\frac{n}{2}}) \neq c(e_{\frac{n}{2}+2})$, $c(e_{\frac{n}{2}+2}) = c(e_3)$. Continuing this argument gives us $c(e_j) = c(e_{j+\frac{n}{2}+1})$ for each $j \in \{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n-1\}$. Then $c(e_{n-2}) = c(e_{(n-2)+\frac{n}{2}+1} = e_{\frac{n}{2}-1}) = w$. Therefore $c(e_{n-2}) = w = c(e_0 = e_n)$ (see Figure 2), contradicting Claim 1. Hence, $\chi'_t(G) \geq 5$. $\square$

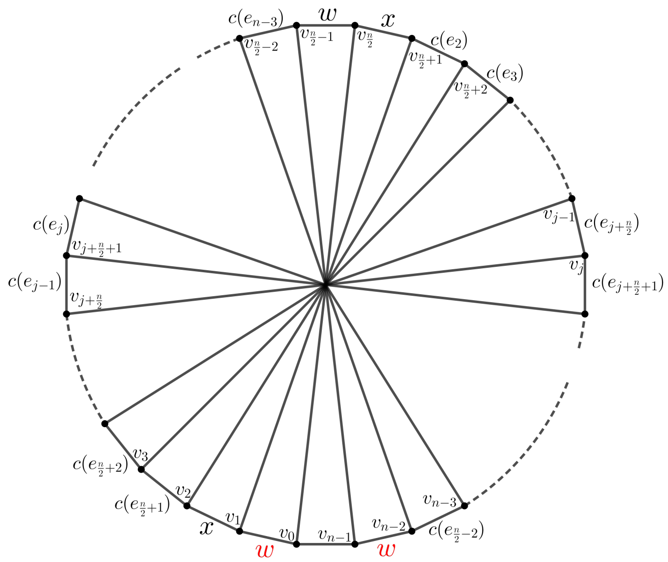


Figure 2. Illustrating the coloring c when $c(e_1) = x$.

We will show in the next theorem that the twin chromatic index of $C_n(1, \frac{n}{2})$, where $n \geq 8$ and $n \equiv 0 \pmod 4$, is 5.

Theorem 2.4. *If $n \geq 8$ and $n \equiv 0$ $(mod\ 4)$, then $\chi'_t(C_n(1, \frac{n}{2})) = 5$.*

Proof. Let $n \geq 8$, $n \equiv 0 \pmod 4$, and $G = C_n(1, \frac{n}{2})$. By Lemma 2.3, $\chi'_t(G) \geq 5$. We will show that $\chi'_t(G) \leq 5$, that is, G has a twin 5-edge coloring. We construct a 5-edge coloring $c : E(G) \to \mathbb{Z}_5$ in G. Let $E(G) = \{e_i = v_i v_{i+1}, f_i = v_i v_{i+\frac{n}{2}} : 0 \leq i \leq n-1\}$. Let

$$c(e_i) = \begin{cases} 0 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd and } 1 \leq i < \frac{n}{2}, \\ 4 & \text{if } i \text{ is odd and } \frac{n}{2} < i \leq n-1, \end{cases}$$

and

$$c(f_i) = \begin{cases} 2 & \text{if } i \text{ is odd,} \\ 3 & \text{if } i \text{ is even.} \end{cases}$$

It is straightforward to see that, by definition, c is a proper edge coloring. Moreover,

$$c'(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd and } \frac{n}{2} < i \leq n-1, \\ 2 & \text{if } i \text{ is even and } \frac{n}{2} < i \leq n, \\ 3 & \text{if } i \text{ is odd and } 1 \leq i < \frac{n}{2}, \\ 4 & \text{if } i \text{ is even and } 2 \leq i \leq \frac{n}{2}. \end{cases}$$

Then c' is also proper. Hence c is a twin 5-edge coloring of G. $\square$

Figure 3 shows twin 5-edge colorings of the circulant graphs $C_6(1,3)$, $C_{10}(1,5)$, and $C_{14}(1,7)$. Therefore, $\chi'_t(C_n(1, \frac{n}{2})) = 5$ for $n \in \{6, 10, 14\}$.

We now formulate the following conjecture.

Conjecture 2.5. *If $n \geq 6$ and $n \equiv 2$ $(mod\ 4)$, then $\chi'_t(C_n(1, \frac{n}{2})) = 5$.*

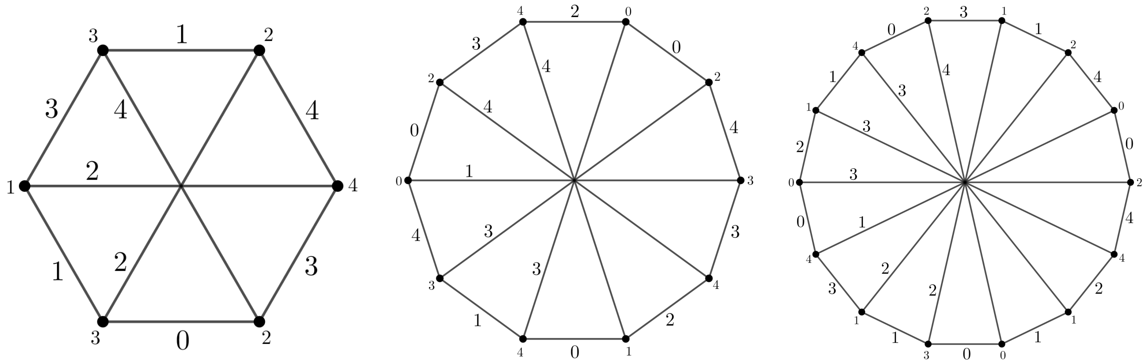


Figure 3. Twin 5-edge colorings of $C_6(1,3)$, $C_{10}(1,5)$, and $C_{14}(1,7)$.

3. Some Generalized Petersen Graphs

In this section, the twin chromatic indices of some generalized Petersen graphs will be discussed. We begin with the generalized Petersen graphs $G = GP(ms,k)$ where $m \geq 3$, $s \geq 1$, and $\gcd(m,k) = 1$.

Definition 3.1. The *Generalized Petersen Graph* $GP(n,k)$, $n \geq 3$ and $1 \leq k \leq n-1$ and $k \neq \frac{n}{2}$, has vertex set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : 0 \leq i \leq n-1\}$ with subscripts reduced modulo n.

Remark 3.2. Since $GP(n,k)$ is a cubic graph, $\chi'_t(GP(n,k)) \geq 4$.

Throughout this section, for each i with $0 \leq i \leq n-1$, we let $e_i = u_iu_{i+1}$, $f_i = u_iv_i$ and $e'_i = v_iv_{i+k}$. Let $E = \{e_i : 0 \leq i \leq n-1\}$, $F = \{f_i : 0 \leq i \leq n-1\}$ and $E' = \{e'_i : 0 \leq i \leq n-1\}$.

Lemma 3.3. *Let $G = GP(ms,k)$ where $m \geq 3$ and $s \geq 1$. If $\gcd(m,k) = 1$, then G has a twin $(m+1)$-edge coloring. Therefore, $\chi'_t(G) \leq m+1$.*

Proof. Since $\gcd(m,k) = 1$, $k \not\equiv 0 \pmod{m}$. We construct a twin $(m+1)$-edge coloring $c : E(G) \to \mathbb{Z}_{m+1}$ in G. For each $i \in \{0,1,2,\ldots,ms-1\}$, let $c(e_i) = i \bmod m$ and let $c(f_i) = m$. Then by definition of c, $c(e_i) \neq c(e_{i-1})$ and $c(f_i) = m \notin \{c(e_i), c(e_{i-1})\}$ for each $i \in \{0,1,2,\ldots,ms-1\}$. Since $m \geq 3$, $i \neq i+2$ in $\mathbb{Z}_m$ so $c(e_i) \neq c(e_{i+2})$ for each $i \in \{0,1,2,\ldots,ms-1\}$. Therefore,

$$c'(u_i) = c(e_{i-1}) + [c(e_i) + m] \neq [c(e_i) + m] + c(e_{i+1}) = c'(u_{i+1})$$

for each $i \in \{0,1,2,\ldots,ms-1\}$. Note that $GP(ms,k) \cong GP(ms,-k)$. If $k \equiv 1 \pmod{m}$, then $-k \not\equiv 1 \pmod{m}$ and we will consider $GP(ms,-k)$ instead of $GP(ms,k)$.

We may then assume that $k \not\equiv 1 \pmod{m}$. Next, we define $c(e'_i) = c(e_i)$ for each $i \in \{0,1,2,\ldots,ms-1\}$. Then, $c(f_i) \notin \{c(e'_{i-k}), c(e'_i)\}$ for each $i \in \{0,1,2,\ldots,ms-1\}$. Since $k \not\equiv 0 \pmod{m}$, $i-k \not\equiv i \pmod{m}$ so $c(e'_{i-k}) \neq c(e'_i)$ for each $i \in \{0,1,2,\ldots,ms-1\}$. Therefore, c is a proper $(m+1)$-edge coloring of G. Moreover, $i-k \not\equiv i+k \pmod{m}$ so $c(e'_{i-k}) \neq c(e'_{i+k})$ for each $i \in \{0,1,2,\ldots,ms-1\}$. Therefore,

$$c'(v_i) = c(e'_{i-k}) + [c(e'_i) + m] \neq [c(e'_i) + m] + c(e'_{i+k}) = c'(v_{i+k})$$

for each $i \in \{0,1,2,\ldots,ms-1\}$. Since $k \not\equiv 1 \pmod{m}$, $i-1 \not\equiv i-k \pmod{m}$ so $c(e_{i-1}) \neq c(e'_{i-k})$ and

$$\begin{aligned} c'(u_i) &= c(e_{i-1}) + [c(e_i) + m] \\ &\neq c(e'_{i-k}) + [c(e_i) + m] \\ &= c(e'_{i-k}) + [c(e'_i) + m] \\ &= c'(v_i) \end{aligned}$$

for each $i \in \{0, 1, 2, \ldots, ms-1\}$. Therefore, c' is also proper and c is a twin $(m+1)$-edge coloring of G. □

Example 3.4. Figure 4 shows twin 6-edge colorings of $GP(10, 3)$ and $GP(10, 6)$. Since $6 \equiv 1 \pmod 5$, we apply the coloring on $GP(10, 4)$ instead of $GP(10, 6)$.

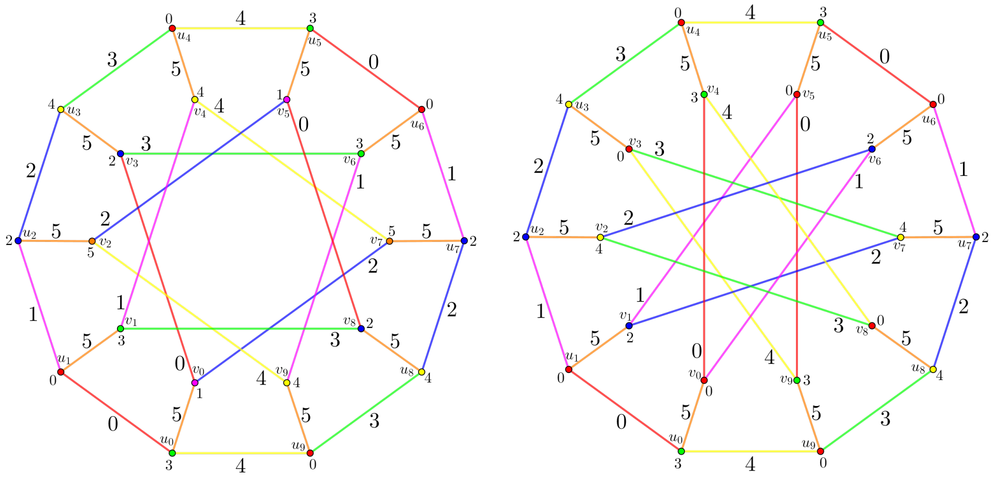


Figure 4. Twin 6-edge colorings of $GP(10, 3)$ and $GP(10, 6)$.

Observe that if $G = GP(3s, k)$ where k is not divisible by 3, then $\chi'_t(G) = 4 = 3 + 1$ by Remark 3.2 and Lemma 3.3. We formally state this observation in the following theorem.

Theorem 3.5. *If $G = GP(3s, k)$ where $s \geq 1$ and $k \not\equiv 0 \pmod 3$, then $\chi'_t(G) = 4$.*

Theorem 3.6. *Let $n \geq 3$ be an integer. If $n \notin \{4, 5\}$, then $\chi'_t(GP(n, 2)) = 4$.*

Proof. If $n \equiv 0 \pmod 3$, then by Theorem 3.5, $\chi'_t(G) = 4$. Therefore, we can just assume that $n \not\equiv 0 \pmod 3$. Since $\chi'_t(G) \geq 4$, we only need to show that $\chi'_t(G) \leq 4$. We do this by constructing a twin 4-edge coloring $c : E(G) \to \mathbb{Z}_4$ in G.

Case 1: Suppose $n \equiv 1 \pmod 3$.

Let $c(e_{n-1}) = 0$. For each $i \in \{0, 1, 2, \ldots, n-2\}$, let

$$c(e_i) = \begin{cases} i \bmod 3 & \text{if } i \not\equiv 0 \pmod 3 \\ 3 & \text{otherwise.} \end{cases}$$

Moreover, for each $i \in \{0, 1, 2, \ldots, n-1\}$, let $c(e'_i) = c(e_{i+1})$. Finally, let $c(f_0) = 2$, $c(f_{n-1}) = 1$, $c(f_{n-2}) = 3$, and $c(f_i) = 0$ for each $i \in \{1, 2, \ldots, n-3\}$. Therefore, by definition of c, it is straightforwar to see that c is a proper edge coloring of G. Observe that $c'(u_{n-2}) = 2$, $c'(u_{n-1}) = 3$ and for each $i \in \{0, 1, 2, \ldots, n-3\}$,

$$c'(u_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 3 \\ 0 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Moreover, $c'(v_{n-1}) = 2$ and for each $i \in \{0, 1, 2, \ldots, n-2\}$,

$$c'(v_i) = \begin{cases} 3 & \text{if } i \equiv 0 \pmod 3 \\ 1 & \text{if } i \equiv 1 \pmod 3 \\ 0 & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Hence, c' is also proper and c is a twin 4-edge coloring of G.

Case 2: Suppose $n \equiv 2 \pmod 3$.

For $i \in \{0, 1, 2, \ldots, n-6\}$, let

$$c(e_i) = \begin{cases} i \bmod 3 & \text{if } i \not\equiv 0 \pmod 3 \\ 3 & \text{otherwise.} \end{cases}$$

and let $(c(e_{n-5}), c(e_{n-4}), c(e_{n-3}), c(e_{n-2}), c(e_{n-1})) = (0, 3, 1, 2, 0)$. Moreover, for $i \in \{0, 1, 2, \ldots, n-1\}$, let $c(e'_i) = c(e_i)$. Finally, let

$$(c(f_1), c(f_0), c(f_{n-1}), c(f_{n-2}), c(f_{n-3}), c(f_{n-4}), c(f_{n-5})) = (2, 1, 3, 0, 2, 1, 3)$$

and let $c(f_i) = 0$ for each $i \in \{2, 3, \ldots, n-6\}$. By definition of c, it is straightforward to see that c is a proper edge coloring of G. Observe that, for $i \in \{2, 3, \ldots, n-6\}$,

$$c'(u_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 3 \\ 0 & \text{if } i \equiv 1 \pmod 3 \\ 3 & \text{if } i \equiv 2 \pmod 3 \end{cases} \quad \textit{and} \quad c'(v_i) = \begin{cases} 0 & \text{if } i \equiv 0 \pmod 3 \\ 3 & \text{if } i \equiv 1 \pmod 3 \\ 1 & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

Moreover,

$$c'(u_1, u_0, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}, u_{n-5}) = (2, 0, 1, 3, 2, 0, 1)$$

and

$$c'(v_1, v_0, v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}, v_{n-5}) = (3, 2, 0, 1, 3, 2, 0).$$

Hence, c' is proper and c is a twin 4-edge coloring of G. □

Theorem 3.7. *If $G = GP(4s, k)$, where $s \geq 1$ and k is odd, then $\chi'_t(G) = 4$.*

Proof. Let $G = GP(4s, k)$ where $s \geq 1$ and k is odd. Then $k \equiv 1 \pmod 4$ or $k \equiv 3 \pmod 4$. Note that $GP(4s, k) \cong GP(4s, 4s-k)$. If $k \equiv 1 \pmod 4$, $4s - k \equiv 3 \pmod 4$. Therefore, we may just assume that $k \equiv 3 \pmod 4$. We construct a twin 4-edge coloring $c : E(G) \to \mathbb{Z}_4$.

For each $i \in \{0, 1, 2, \ldots, 4s-1\}$, let $c(e_i) = i \bmod 4$. Moreover, let $c(f_i) = c(e_{i+1})$ and $c(e'_i) = c(e_{i+2})$ for each $i \in \{0, 1, 2, \ldots, 4s-1\}$. By definition of c, the colors $c(e_i)$, $c(e_{i+1})$, $c(e_{i+2})$, and $c(e_{i+3})$ are the four distinct elements of $\mathbb{Z}_4$ for each $i \in \{0, 1, 2, \ldots, 4s-1\}$. Therefore, $c(f_i) = c(e_{i+1}) \notin \{c(e_{i-1}), c(e_i), c(e_{i+2}) = c(e'_i)\}$ for each $i \in \{0, 1, 2, \ldots, 4s-1\}$. Moreover,

$$\begin{aligned} c'(u_i) &= c(e_{i-1}) + c(e_i) + c(f_i) \\ &\neq c(e_{i+2}) + c(e_i) + c(f_i) \\ &= c(f_{i+1}) + c(e_i) + c(e_{i+1}) \\ &= c'(u_{i+1}) \end{aligned}$$

for each $i \in \{0, 1, 2, \ldots, 4s-1\}$. Since $k \equiv 3 \pmod 4$, $i - k + 2 \equiv i - 1 \pmod 4$, $i + k + 2 \equiv i + 1 \pmod 4$ and $i + k + 1 \equiv i \pmod 4$. Therefore, $c(e'_{i-k}) = c(e_{(i-k)+2}) = c(e_{i-1})$. Thus, $c(f_i) \neq c(e_{i-1}) = c(e'_{i-k})$ and $c(e'_{i-k}) = c(e_{i-1}) \neq c(e_{i+2}) = c(e'_i)$ for each $i \in \{0, 1, 2, \ldots, 4s-1\}$. Hence, c is a proper edge coloring. It remains to show that $c'(u_i) \neq c'(v_i)$ and $c'(v_i) \neq c'(v_{i+k})$ for each $i \in \{0, 1, 2, \ldots, 4s-1\}$.

For each $i \in \{0, 1, 2, \ldots, 4s-1\}$,

$$\begin{array}{rcl} c'(u_i) &=& c(e_{i-1}) + c(e_i) + c(f_i) \\ &=& c(e'_{i-k}) + c(e_i) + c(f_i) \\ &\neq& c(e'_{i-k}) + c(e_{i+2}) + c(f_i) \\ &=& c(e'_{i-k}) + c(e'_i) + c(f_i) \\ &=& c'(v_i) \end{array} \quad \textit{and} \quad \begin{array}{rcl} c'(v_i) &=& c(e'_{i-k}) + c(e'_i) + c(f_i) \\ &=& c(e_{i-1}) + c(e'_i) + c(e_{i+1}) \\ &\neq& c(e_i) + c(e'_i) + c(e_{i+1}) \\ &=& c(f_{i+k}) + c(e'_i) + c(e'_{i+k}) \\ &=& c'(v_{i+k}). \end{array}$$

Hence, c is a twin 4-edge coloring of G. □

4. Sufficient Conditions for Graphs with Maximum Degree 3

In this section, we provide some sufficient conditions for a graph with maximum degree 3 to have twin chromatic index greater than 3. We begin by defining the following terms.

Definition 4.1. *Let $n \geq 2$ and G be a graph with $\Delta(G) \geq 3$. Let $P(u,v) \subset G$ be a $u-v$ path of order n with* $\deg(w) = 2$ *in G for each $w \in V(P(u,v))\backslash\{u,v\}$. Then $P(u,v)$ is called an **internal path** of G if* $\deg(u), \deg(v) \geq 3$ *in G and is called a **pendant path** of G if* $\deg(u) \geq 3$ *and v is a leaf in G.*

If u is a vertex of G and P is an internal or a pendant path of G such that u is an end vertex of P, then P and u are *incident* with each other. Therefore, an internal path or a pendant path $P(u,v)$ is incident with the vertices u and v.

Lemma 4.2. *Let T be a tree with $\Delta(T) = 3$ and suppose that no two vertices of T of degree 3 are adjacent. Then $\chi'_t(T) \geq 4$ if one of the following holds:*

(i) there exists a vertex u of degree 3 such that $d(u,x) \equiv 0 \pmod 3$ for each internal path $P(u,x)$ in T and $d(u,y) \in \{1,k\}$, where $k \equiv 2 \pmod 3$ for each pendant path $P(u,y)$ in T;

(ii) there exists a vertex u of degree 3 such that at least two internal paths of T are incident with u and $d(u,x) \equiv 1 \pmod 3$ for each internal path $P(u,x)$;

(iii) there exists a vertex u of degree 3 such that at least two pendant paths of T are incident with u and $d(u,y) \equiv 0 \pmod 3$ for each pendant path $P(u,y)$;

(iv) there exists a vertex u of degree 3 such that one internal path $P(u,x)$ of T is incident with u with $d(u,x) \equiv 1 \pmod 3$ and one pendant path $P(u,y)$ of T is incident with u with $d(u,y) \equiv 0 \pmod 3$;

(v) there exists an internal path $P(u,v)$ of T with $d(u,v) \equiv 2 \pmod 3$ such that $d(z,x) \equiv 0 \pmod 3$ for each internal path $P(z,x)$ of T ($z \in \{u,v\}$ and $x \notin \{u,v\}\backslash\{z\}$) and $d(z,y) \in \{1,k\}$, where $k \equiv 2 \pmod 3$ for each pendant path $P(z,y)$ of T ($z \in \{u,v\}$); or

(vi) $d(u,v) \not\equiv 1 \pmod 3$ for each internal path $P(u,v)$ in T and $d(x,y) \in \{1,k\}$, where $k \equiv 2 \pmod 3$ for each pendant path $P(x,y)$ in T.

Proof. Let P^1, P^2, P^3, …, P^q be an ordering of all the internal/pendant paths of T ($q \geq 3$). If $P^i = P(u,v)$ ($1 \leq i \leq q$), then we let $P^i = P(u,v) = P^i(u,v)$ with $V(P^i(u,v)) = \{u = w^i_0, w^i_1, \ldots, w^i_{n_i-2}, w^i_{n_i-1} = v\}$ and $E(P^i(u,v)) = \{e^i_j = w^i_j w^i_{j+1} : 0 \leq j \leq n_i - 2\}$ where $\deg(u) = 3$ and n_i is the order of P^i.

If T has a twin 3-edge coloring $c : E(T) \to \mathbb{Z}_3$, then we have the following observations:

- $c(E_u) = \mathbb{Z}_3$ for each $u \in V(T)$ with $\deg(u) = 3$;
- $0 \notin \{c(e^i_{n_i-3}), c(e^j_0)\}$ for any pendant path P^i of order $n_i \geq 3$ and for any pendant path P^j of order 2; and

- $c(e_k^i) \neq -c(e_{k+1}^i)$ for each $k \in \{0, n_i - 3\}$ if P^i is an internal path of order n_i and $c(e_0^j) \neq -c(e_1^j)$ if P^j is a pendant path of order at least 3.

Moreover, for each pendant path $P^j(u,v)$ of order n_j and for each internal path $P^i(x,y)$ of order n_i, the sequences $(c(e_0^j), c(e_1^j), \ldots, c(e_{n_j-2}^j))$ and $(c(e_0^i), c(e_1^i), \ldots, c(e_{n_i-2}^i))$ are periodic of period 3 such that, for $a \in \{1,2\}$

$$(c(e_0^j), c(e_1^j), \ldots, c(e_{n_j-2}^j)) = \begin{cases} (0, a, -a, \ldots, -a) \text{ and } d(u,v) \equiv 0 \pmod 3, \\ (0, a, -a, \ldots, 0) \text{ and } d(u,v) \equiv 1 \pmod 3, \\ (a, 0, -a, \ldots, 0) \text{ and } d(u,v) \equiv 2 \pmod 3, \\ (a, 0, -a, \ldots, a) \text{ and } d(u,v) \equiv 1 \pmod 3 \end{cases} \quad (1)$$

and

$$(c(e_0^i), c(e_1^i), \ldots, c(e_{n_i-2}^i)) = \begin{cases} (0, a, -a, \ldots, a) \text{ and } d(x,y) \equiv 2 \pmod 3, \\ (0, a, -a, \ldots, 0) \text{ and } d(x,y) \equiv 1 \pmod 3, \\ (a, 0, -a, \ldots, 0) \text{ and } d(x,y) \equiv 2 \pmod 3, \\ (a, 0, -a, \ldots, -a) \text{ and } d(x,y) \equiv 0 \pmod 3. \end{cases} \quad (2)$$

For each case, we need to show that $\chi'_t(T) \neq 3$, that is, T has no twin 3-edge coloring. Suppose on the contrary that T has a twin 3-edge coloring $c : E(T) \to \mathbb{Z}_3$ (for each case). Let $a \in \{1,2\}$ and let $c(P^i) = c((e_0^i), c(e_1^i), \ldots, c(e_{n_i-2}^i))$ for each $1 \leq i \leq q$.

Proof of 1: By equation (2), $c(e_0^i) \in \{1,2\}$ if $P^i(u,x)$ is an internal path of T with $d(u,x) \equiv 0 \pmod 3$ ($w_0^i = u$). Similarly, by our observation and equation (1), $c(e_0^j) \in \{1,2\}$ if $P^j(u,y)$ is a pendant path of T with $d(u,y) \in \{1,k\}$, where $k \equiv 2 \pmod 3$. Therefore $0 \notin c(E_u)$. This contradicts the fact that $c(E_u) = \mathbb{Z}_3$.

Proof of 2: In this case, T has at least three vertices of degree 3. By equation (2), $c(P^i) = (0, a, -a, \ldots, 0)$ if $P^i(u,x)$ is an internal path of T with $d(u,x) \equiv 1 \pmod 3$. Therefore, by our assumption, at least two of the edges incident with u are colored 0. This contradicts the assumption that c is a proper edge coloring of T.

Proof of 3: By equation (1), $c(P^i) = (0, a, -a, \ldots, -a)$ if $P^i(u,y)$ is a pendant path of T with $d(u,y) \equiv 0 \pmod 3$. Since $d(u,y) \equiv 0 \pmod 3$ for each pendant path $P(u,y)$ in T, at least two of the edges incident with u are colored 0 which is impossible since c is proper.

Proof of 4: Let $P^i(u,x)$ be an internal path of T ($w_0^i = u$) with $d(u,x) \equiv 1 \pmod 3$ and let $P^j(u,y)$ be a pendant path of T ($w_0^j = u$) with $d(u,y) \equiv 0 \pmod 3$. Then by equations (1) and (2) we have $c(e_0^i) = 0 = c(e_0^j)$, a contradiction.

Proof of 5: Without loss of generality, let $P^i = P(u,v)$ where $w_0^i = u$. By our observation and equation (1), we have $c(e_0^j) \neq 0$ for any pendant path $P^j(u,y)$ of T. On the other hand, by equation (2), we have $0 \notin \{c(e_0^l), c(e_{n_l-2}^l)\}$ for any internal path $P^l(u,x)$ of T ($x \neq v$). Then we must have $c(e_0^i) = 0$. Using similar arguments, we can say that $c(e_{n_i-2}^i) = 0$. But by equation (2), since $d(u,v) \equiv 2 \pmod 3$, $c(P^i) = (0, a, -a, \ldots, a)$, a contradiction.

Proof of 6: By condition 1, we can just assume that T has at least two vertices of degree 3. Suppose T has at least two vertices of degree 3. We choose a vertex of T of degree 3 that is incident with two pendant paths of T and label it by u_1. By assumption, there exists a unique internal path $P^i(u,v)$ of T. We let $P^1 = P^i(u,v)$. By our observation and by equation (1), we must have $c(e_0^1) = 0$; so by equation (2) we have $d(u_1, v) \equiv 2 \pmod 3$. We now let $v = u_2$. Since $d(u_1, u_2) \equiv 2 \pmod 3$, $c(P^1) = (0, a, -a, \ldots, a)$ ($w_0^1 = u_1$). By condition 5, we can just assume that there exists an internal path $P^j(u_2, w)$ of T with $d(u_2, w) \equiv 2 \pmod 3$ and $w \neq u_1$. We let $P^2 = P^j(u_2, w)$ and let $w = u_3$. Since $d(u_2, u_3) \equiv 2 \pmod 3$, $c(P^2) = (0, a, -a, \ldots, a)$ or $(0, -a, a, \ldots, -a)$ ($w_0^2 = u_2$). Since T is finite, we will have a finite sequence of distinct internal paths P^1, P^2, ..., P^m such that for each $\ell \in \{1, 2, \ldots, m\}$, $P^\ell = P^\ell(u_\ell, u_{\ell+1})$, $d(u_\ell, u_{\ell+1}) \equiv 2 \pmod 3$, $c(P^\ell) = (0, a, -a, \ldots, a)$ or $(0, -a, a, \ldots, -a)$ $(w_0^\ell = u_\ell)$, and $d(u_{m+1}, x) \equiv 0 \pmod 3$ for each internal path $P(u_{m+1}, x)$ $(x \neq u_m)$ (if any). Thus, we have $0 \notin c(E_{u_{m+1}})$ which is impossible. □

It is straighforward to check that Lemma 4.2 is also true for any cyclic connected graph G with maximum degree 3 if one of the conditions $1-5$ of Lemma 4.2 holds for G. On the other hand, condition 6 in Lemma 4.2 is not a sufficient condition for a cyclic connected graph G with maximum degree 3 to have twin chromatic index greater than 3 as the example in Figure 5 shows.

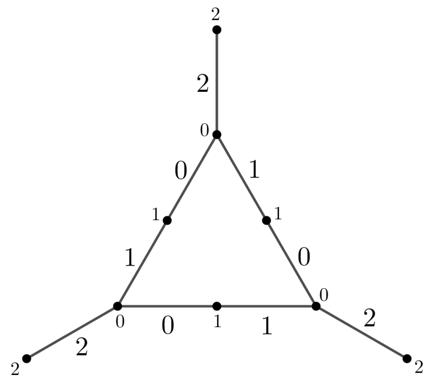


Figure 5. A twin 3-edge coloring of a cyclic connected graph.

5. Conclusion

In this paper, we have shown that the twin chromatic index of a circulant graph $C_n(1, \frac{n}{2})$ is 5 for any even integer $n \geq 6$ with $n \equiv 0 \pmod 4$. On the other hand, we showed that the Pertersen graphs $GP(3s, k)$, $GP(m, 2)$, and $GP(4s, l)$ where $s \geq 1$, $k \not\equiv 0 \pmod 3$, $m \geq 3$ and $m \notin \{4, 5\}$, and l is odd have twin chromatic index 4. Moreover, we provided some sufficient conditions for a graph with maximum degree 3 to have twin chromatic index greater than 3.

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