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On the set chromatic number of the join and comb product of graphs

Bryan Ceasar L. Felipe

Agnes Garciano

Mark Anthony C. Tolentino

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B C L Felipe, A D Garciano and M A C Tolentino

Department of Mathematics, School of Science and Engineering, Loyola Schools,
Ateneo de Manila University, Philippines

E-mail: bryanceasarfelipe@gmail.com, agarciano@ateneo.edu, mtolentino@ateneo.edu

Abstract. A vertex coloring $c : V(G) \to \mathbb{N}$ of a non-trivial connected graph G is called a *set coloring* if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for any pair of adjacent vertices u and v. Here, $\mathrm{NC}(x)$ denotes the set of colors assigned to vertices adjacent to x. The *set chromatic number* of G, denoted by $\chi_S(G)$, is defined as the fewest number of colors needed to construct a set coloring of G. In this paper, we study the set chromatic number in relation to two graph operations: join and comb prdocut. We determine the set chromatic number of wheels and the join of a bipartite graph and a cycle, the join of two cycles, the join of a complete graph and a bipartite graph, and the join of two bipartite graphs. Moreover, we determine the set chromatic number of the comb product of a complete graph with paths, cycles, and large star graphs.

1. Introduction

A vertex or edge coloring c of a graph G is said to be *neighbor-distinguishing* if c induces a vertex labelling in which every pair of adjacent vertices in G is assigned distinct labels. Aside from the well-studied *proper vertex coloring*, various neighbor-distinguishing colorings have also been introduced and discussed in the literature (for examples, see [3], [4], [6], [7], [8]).

In [5], Chartrand, Okamoto, and Zhang introduced and explored the following neighbor-distingushing vertex coloring that makes use of neighborhood color sets.

Definition 1.1. [5] *For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be assigned the same color.*

(i) *For a set $S \subseteq V(G)$, define the set $c(S)$ of colors assigned to the vertices of S by*

$$c(S) = \{c(v) : v \in S\}.$$

(ii) *The **neighborhood color set** of v, denoted by $NC(v)$, is the set $c(N(v))$; that is, $NC(v)$ is the set of colors of the neighbors of v.*

(iii) *The coloring c is called **set neighbor-distinguishing**, or simply a **set coloring**, if $NC(u) \neq NC(v)$ for every pair u, v of adjacent vertices of G.*

(iv) *The minimum number of colors required in a set coloring of G is called the **set chromatic number** of G and is denoted by $\chi_s(G)$.*

In their paper, Chartrand et al. studied the set chromatic numbers of some families of graphs and established some bounds for the set chromatic number in terms of other graph parameters.

It is clear that $\chi_s(G) \leq \chi(G)$. Moreover, results from [5] on the lower bound for χ_s are presented below.

Proposition 1.2. [5] *For every graph G,*

(i) $\chi_s(G) \geq \lceil \log_2(\chi(G) + 1) \rceil$,

(ii) $\chi_s(G) \geq 1 + \lceil \log_2(\omega(G)) \rceil$,

where $\chi(G)$ and $\omega(G)$ are the chromatic and clique numbers of G, respectively.

In this paper, we will study the set chromatic number of graphs in relation to two well-studied graph operations: join and comb product. Section 2 will focus on the join of graphs. Our results cover wheels and the join of a bipartite graph and a cycle, the join of two cycles, the join of a complete graph and a bipartite graph, and the join of two bipartite graphs. Meanwhile, section 3 focuses on the comb product of graphs. We will determine the set chromatic number of the comb product of a complete graph with paths, cycles, and large star graphs.

2. On the set chromatic number of the join of graphs

We present the definition of the join of two graphs below:

Definition 2.1. *Let G and H be two vertex-disjoint graphs. The **join** $G + H$ of G and H is the graph whose vertex and edge sets are given by*

$$V(G + H) = V(G) \cup V(H),$$

$$E(G + H) = E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}.$$

It is well-known that $\chi(G + H) = \chi(G) + \chi(H)$. In relation to the set colorings, the join of graphs has been studied in [10] by Okamoto et al. In particular, they have established the following sharp bounds:

Theorem 2.2. [10] *For every two graphs G and H,*

$$\max\{\chi_s(G) + \lceil \log_2 \omega(H) \rceil, \chi_s(H) + \lceil \log_2 \omega(G) \rceil\} \leq \chi_s(G + H) \leq \chi_s(G) + \chi_s(H) + 1,$$

where $\omega(\cdot)$ denotes the clique number of a graph.

Moreover, they also considered the join of graphs with complete graphs.

Theorem 2.3. [10] *For a graph G and a positive integer p,*

$$\chi_s(G) + p - 1 \leq \chi_s(G + K_p) \leq \chi_s(G) + p.$$

While Theorems 2.2 and 2.3 focused on bounds for the set chromatic number of joins, our results focus on the exact set chromatic number of a join of two graphs from well-known graph families. We begin with the following.

Proposition 2.4. *Let W_n be the wheel graph of order n. Then $\chi_s(W_4) = 4$ and $\chi_s(W_n) = 3$ for $n \geq 5$.*

Proof. The wheel graph W_4 is a complete graph of order 4. Hence, $\chi_s(W_4) = 4$. For $n \geq 5$, without loss of generality, suppose $W_n = C_n + K_1$ such that $C_n = v_1, v_2, ..., v_n, v_1$ and $V(K_1) = \{v_0\}$. Suppose n is even; then $\chi(W_n) = 3$, which implies that $\chi_s(W_n) = 3$ as well. Now, suppose n is odd. Then $\chi(W_n) = 4$ and $\chi_s(W_n) \geq 3$. We define a coloring $c : V(W_n) \to \{1, 2, 3\}$ as follows:

$$c(v_i) = \begin{cases} 1, & i \in \{0, 1, 2, n\}, \\ 2, & i \text{ is odd and } 3 \leq i \leq n - 1, \\ 3, & i \text{ is even and } 3 \leq i \leq n - 1. \end{cases}$$

Then (refer to Fig. 1), it is easy to see that c is a set 3-coloring of W_n. Therefore, $\chi_s(W_n) = 3$ for $n \geq 5$. $\square$
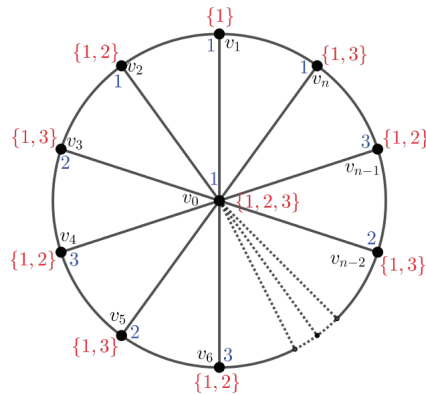
Figure 1. A set 3-coloring of W_n, with odd $n \geq 5$.

We now consider the join of a bipartite graph with another bipartite graph, an odd cycle, or a complete graph.

Theorem 2.5. *Let B be a bipartite graph.*

(i) If H is also a bipartite graph, then $\chi_s(B + H) = 4$.

(ii) If C_n is an odd cycle, then $\chi_s(B + C_3) = 5$ and $\chi_s(B + C_n) = 4$ for $n \geq 5$.

(iii) If p is a positive integer, then $\chi_s(B + K_p) = p + 2$.

Proof. (i) Since $\chi(B + H) = 4$, it follows that $\chi_s(B + H)$ is 3 or 4. Suppose there is a set 3-coloring c of $B + H$; then $c_1 := c|_{V(B)}$ and $c_2 := c|_{V(H)}$ are set colorings of B and H, respectively. Clearly, neither c_1 nor c_2 can use all 3 colors; so we can assume that c_1 uses colors 1 and 2 while c_2 uses colors 1 and 3.
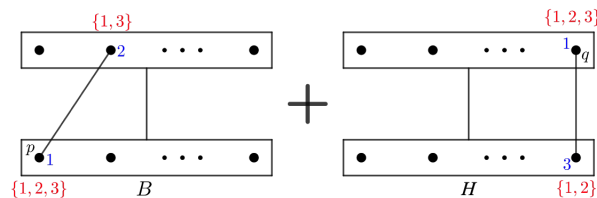

Figure 2. A join of two bipartite graphs B and H.

As shown in Fig. 2, if $v \in V(B)$, then $\mathrm{NC}(v) \in \{\{1,3\},\{1,2,3\}\}$. On the other hand, if $w \in V(H)$, then $\mathrm{NC}(w) \in \{\{1,2\},\{1,2,3\}\}$. Since $\chi(B) = \chi(H) = 2$, there must be a vertex p in B and a vertex q in H whose NC is $\{1,2,3\}$. Since p and q are adjacent in $B + H$, this proves that we cannot use 3 colors. Therefore, $\chi_s(B + H) = 4$.

(ii) First, we have

$$4 = \chi_s(C_3) + \lceil \log_2 \omega(B) \rceil \leq \chi_s(B + C_3) \leq \chi(B + C_3) = 5.$$

Then $\chi_s(B + C_3) = 4$ or 5. By using an argument similar to that in (i), we can show that $B + C_3$ has no set 4-coloring; hence, $\chi_s(B + C_3) = 5$.

Now, consider $B + C_n$, where $n \geq 4$. Then $\chi_s(B + C_n) \geq \chi_s(C_n) + \lceil \log_2 \omega(B) \rceil = 4$. Suppose B has partite sets V_1 and V_2 and $C_n = v_1, v_2, ..., v_n, v_1$. Define a coloring $c : V(G) \to \{1, 2, 3, 4\}$ as follows:

$$c(v) = \begin{cases} 1, & v \in V_1, \\ 2, & v \in V_2, \end{cases} \quad \text{and} \quad c(v_i) = \begin{cases} 2, & i \in \{1, 2, n\}, \\ 3, & i \text{ is odd and } 3 \le i \le n-1, \\ 4, & i \text{ is even and } 3 \le i \le n-1. \end{cases}$$

Then, it is easy to see that c is a set 4-coloring of $B + C_n$; refer to Fig. 3. Hence, $\chi_s(B + C_n) = 4$.
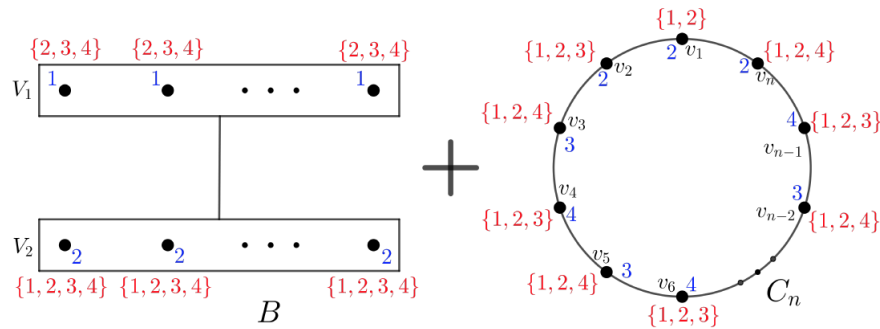

Figure 3. A set 4-coloring of $B + C_n$, $n \ge 4$.

(iii) By Theorem 2.3, $p + 1 \le \chi_s(B + K_p) \le p + 2$. Suppose c is a set coloring of $B + K_p$ that uses only $p+1$ colors. Then $|c(K_p)| = p$ and we can assume that $c(K_p) = \mathbb{N}_p := \{1, 2, ..., p\}$. Let B_1 and B_2 be the partite sets of B. Then there is a non-isolated vertex v in B_1 (without loss of generality) for which $c(v) = p + 1$. If $y \in B_2$ is a neighbor of v, then $c(y) \in \mathbb{N}_p$. Suppose $c(y) = j \in \mathbb{N}_p$. Let z be the vertex in K_p whose color is also j. Then $NC(z) = \mathbb{N}_{p+1} = NC(y)$, which is not possible. Therefore, $\chi_s(B + K_p) = p + 2$. $\qquad\square$

We also determine the set chromatic number of the join of two odd cycles.

Proposition 2.6. *Let $n, k \ge 3$ be odd positive integers. Then*

$$\chi_s(C_n + C_k) = \begin{cases} 6, & \text{if } n = k = 3, \\ 5, & \text{otherwise.} \end{cases}$$

Proof. Suppose $C_n = v_1, v_2, ..., v_n, v_1$ and $C_k = u_1, u_2, ..., u_k, u_1$. Let $G = C_n + C_k$. Then $\chi_s(G) \ge \chi_s(C_n) + \lceil \log_2 \omega(C_k) \rceil = 4$. It is easy to show that G is not set 4-colorable. We now consider four cases:

Case 1: $n = k = 3$. Since $G = C_3 + C_3 \cong K_6$, $\chi_s(G) = 6$.

Case 2: $n = 3$ and $k \ge 5$. Define a coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ as follows:

$$c(v_i) = \begin{cases} 1, & i = 1, \\ 2, & i = 2, \\ 3, & i = 3, \end{cases} \quad \text{and} \quad c(u_i) = \begin{cases} 1, & i \in \{1, 2, k\}, \\ 4, & i \text{ is odd and } 3 \le i \le k-1, \\ 5, & i \text{ is even and } 3 \le i \le k-1. \end{cases}$$

Then (refer to Fig. 4), it is easy to see that c is a set 5-coloring of G. Therefore, $\chi_s(C_3 + C_k) = 5$ for $k \ge 5$.

Case 3: $n, k \ge 5$. Define a coloring $f : V(G) \to \{1, 2, 3, 4, 5\}$ as follows:

$$f(v_i) = \begin{cases} 1, & i \in \{1, 2, n\}, \\ 2, & i \text{ is odd}, 3 \le i \le n-1, \\ 3, & i \text{ is even}, 3 \le i \le k-1, \end{cases} \quad \text{and} \quad f(u_i) = \begin{cases} 1, & i \in \{1, 2, k\}, \\ 4, & i \text{ is odd}, 3 \le i \le k-1, \\ 5, & i \text{ is even}, 3 \le i \le k-1. \end{cases}$$
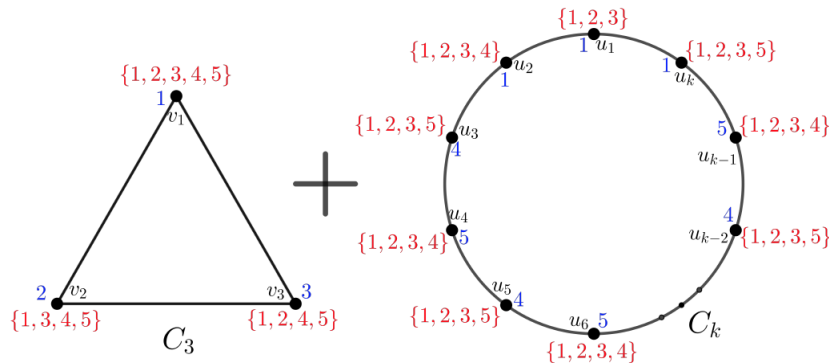
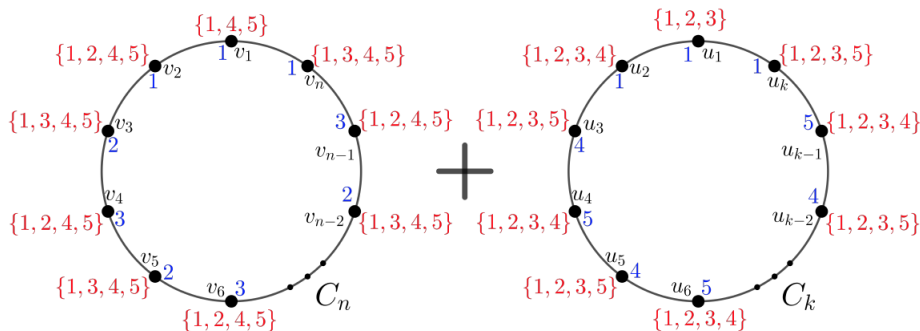**Figure 4.** A set 5-coloring of $C_3 + C_k$, with odd $k \geq 5$.

Figure 5. A set 5-coloring of $C_n + C_k$, with odd $n, k \geq 5$.

Then (refer to Fig. 5), it is easy to see that c is a set 5-coloring of G. Therefore, $\chi_s(C_n + C_k) = 5$ for $n, k \geq 5$.

$\square$

3. On the set chromatic number of comb products of graphs

In [5], Chartrand et al. studied the set chromatic number of a family of graphs denoted by $G_{n,t}$, which is defined as follows: for an integer $n \geq 2$, $G_{n,t}$ is the graph of order $n + t$ obtained from K_n with $V(K_n) = \{v_1, v_2, ..., v_n\}$ by adding t new vertices $u_1, u_2, ..., u_t$ (if $t \geq 1$) and joining each u_i to v_i for $1 \leq i \leq t$. Their result for this family of graphs is as follows:

Proposition 3.1 (Chartrand et al. [5])**.** *For $n \geq 2$ and $0 \leq t \leq n$, $\chi_s(G_{n,t}) = n$.*

When $t = n$, the graph $G_{n,n}$ is, in fact, $\mathrm{cor}(K_n)$—the *corona* of K_n; that is, it is the graph obtained by adding a pendant edge at each vertex of K_n. The notion of the corona a graph can be generalized using the notion of comb product (or rooted product), which we define below.

Definition 3.2. *The **comb product** (or **rooted product**) of a graph G and a rooted graph H is the graph obtained by taking $|V(G)|$ copies of H, and for every vertex v_i of G, identifying v_i with the root node of the ith copy of H.*

More formally, if $V(G) = \{g_1, ..., g_n\}$, $V(H) = \{h_1, ..., h_m\}$, and h_1 is the root of H, then the rooted product of G with H, denoted by $G \rhd_{h_1} H$, has vertex set $V = \{(g_i, h_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and edge set $E = \{(g_i, h_1)(g_k, h_1) : g_i g_k \in E(G)\} \cup \bigcup_{i=1}^{n} \{(g_i, h_j)(g_i, h_k) : h_j h_k \in E(H)\}$.

Hence, $\mathrm{cor}(K_n)$ is, in fact, $K_n \rhd_r P_2$, where r is any vertex of P_2. A direct consequence of Proposition 3.1 is the following.

Corollary 3.3. *Let $n \geq 2$, $m \geq 2$, and r be a pendant vertex of P_m. Then $\chi_s(K_n \rhd_r P_m) = n$.*

The comb product of graphs is a well-studied graph operation and has been considered in relation to several graph parameters and labellings (for examples, see [1], [2], [9], [11], [12]).

In this work, we focus on the comb product of complete graphs with well-known graph families. We present below our first result, which concerns the comb products of complete graphs with cycles or paths. The proof is based on a technique used for the proof of Proposition 3.1 in [5].

Theorem 3.4. *Let H be a cycle C_s ($s \geq 4$) rooted at any vertex r, or a path P_t ($t \geq 3$) rooted at a vertex r that is not a leaf. Then for $n \geq 3$, $\chi_s(K_n \rhd_r H) = \left\lceil \frac{1}{2}(1 + \sqrt{8n+1}) \right\rceil$.*

Proof. Denote the quantity $\left\lceil \frac{1}{2}(1 + \sqrt{8n+1}) \right\rceil$ by k_n. First, notice that in constructing $K_n \rhd_r H$, each vertex of K_n is connected to two *nonadjacent* vertices of a copy of H. Thus, since $\chi_s(H) \leq 3$ and $k_n \geq 3$, the result follows directly from the case where $H = P_3$, rooted at a non-pendant vertex r.

Now, suppose $H = P_3$, rooted at a non-pendant vertex r. For simplicity, we denote $K_n \rhd_r H$ by G_n and we label its vertices as shown in Figure 6.
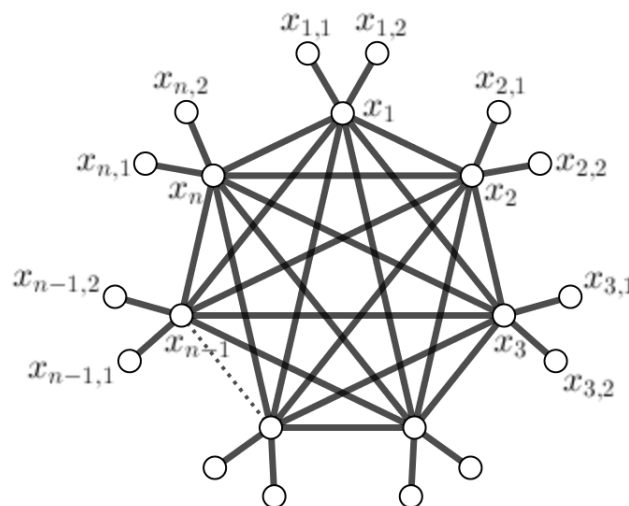

Figure 6. Labels of vertices in $K_n \rhd_r P_3$.

First, we show that $\chi_s(G_n) \leq k_n$ by constructing a set coloring c of G_n that uses at most k_n colors. Let $S_1, S_2, ..., S_m$ be all the 1-subsets and 2-subsets of $\{2, 3, ..., k_n\}$. Then $m = \binom{k_n - 1}{1} + \binom{k_n - 1}{2}$ and it follows that $m \geq n$. Hence, we can define the coloring c so that $c(x) = 1$ for all $x = x_i$, $i = 1, 2, ..., n$, and $\{c(x_{i,j}) : j = 1, 2\} = S_i$ for $i = 1, 2, ..., n$. Thus, $NC(x_i) = \{1\} \cup S_i$ and $NC(x_{i,j}) = \{1\}$ for $i = 1, 2, ..., n$ and $j = 1, 2$. Since the S_i's are nonempty and disjoint, it follows that c is a set coloring.

We now show that $\chi_s(G_n) \geq k_n$ by showing that any set coloring of G_n uses at least k_n colors. To this end, suppose c is a set coloring of G_n that uses p colors. We split the proof into three cases:

Case 1. Suppose c colors the vertices $x_1, x_2, ..., x_n$ using n colors. Then $p \geq n$. Since $n \geq 3$ implies $n \geq k_n$, we must have $p \geq k_n$.

Case 2. Suppose $c(x) = 1$ for all $x = x_i$, $i = 1, 2, ..., n$. Then for each i,

$$NC(x_i) = \begin{cases} \{1\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1\}, \\ \{1, a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1, a\} \text{ or } \{a\}, \\ \{1, a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{a, b\}, \end{cases}$$

where $1 < a, b \le p$. Clearly, the case where $NC(x_i) = \{1\}$ is not possible since this would also make $NC(x_{i,j}) = \{1\}$ for $j = 1, 2$. Hence, since c is a set coloring, we should have at least n different neighborhood color sets of the form $\{1, a\}$ or $\{1, a, b\}$, where $1 < a, b \le p$. Then $\binom{p-1}{1} + \binom{p-1}{2} \ge n$, which is equivalent to $p \ge k_n$.

Case 3. Suppose $c(\{x_1, x_2, ..., x_n\}) = \mathbb{N}_\ell := \{1, ..., \ell\}$, where $2 \le \ell \le n-1$. Let S be the subset of $\{x_1, x_2, ..., x_n\}$ satisfying the following condition: for any $x \in S$, there is a vertex $y \ne x$ in S for which $c(x) = c(y)$. For convenience, we assume that $S = \{x_1, x_2, ..., x_{|S|}\}$. Since $2 \le \ell \le n-1$, we must have $|S| \ge n - \ell + 1$. Moreover, for $x_i \in S$,

$$NC(x_i) = \begin{cases} \mathbb{N}_\ell, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} \subseteq \mathbb{N}_\ell, \\ \mathbb{N}_\ell \cup \{a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a\}, \\ \mathbb{N}_\ell \cup \{a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a, b\}, \end{cases}$$

where $\ell < a, b \le p$. Since c is a set coloring, only at most one x_i in S can have $NC(x_i) = \mathbb{N}_\ell$. Hence, we should have at least $|S| - 1$ different neighborhood color sets of the form $\mathbb{N}_\ell \cup \{a\}$ or $\mathbb{N}_\ell \cup \{a, b\}$. Then $\binom{p-\ell}{1} + \binom{p-\ell}{2} \ge |S| - 1 \ge (n - \ell + 1) - 1 = n - \ell$, which is equivalent to $p \ge \left\lceil \frac{2\ell-1}{2} + \sqrt{2n + \frac{1}{4} - 2\ell} \right\rceil$. Then $p \ge \min\limits_{\ell=2,3,...,n-1} \left\lceil \frac{2\ell - 1}{2} + \sqrt{2n + \frac{1}{4} - 2\ell} \right\rceil$. It can easily be verified that this minimum occurs when $\ell = 2$; hence

$$p \ge \left\lceil \frac{3}{2} + \sqrt{2n - \frac{15}{4}} \right\rceil \ge k_n,$$

since $n \ge 3$.

Therefore, in any case, any set coloring of G_n that uses p colors must have $p \ge k_n$. □

In Theorem 3.4, the case where H is the cycle C_3 was not considered. This is because, in this case, in constructing the comb product $K_n \triangleright_r C_3$, each vertex of K_n is connected to two *adjacent* vertices of a copy of C_3. For example, the comb product $K_3 \triangleright_r C_3$ has set chromatic number 3; an optimal set coloring is shown in Figure 7. We consider the rest of this exceptional case in Proposition 3.5.

Proposition 3.5. *Let H be the cycle C_3, rooted at any vertex r. Then for $n \ge 4$, $\chi_s(K_n \triangleright_r H) = \left\lceil \frac{1}{2}(3 + \sqrt{8n - 7}) \right\rceil$.*

Proof. Denote the quantity $\left\lceil \frac{1}{2}(3 + \sqrt{8n - 7}) \right\rceil$ by k_n and the graph $K_n \triangleright_r H$ by G_n. First, note that $k_n \ge 4$ for $n \ge 4$. We show that $\chi_s(G_n) \le k_n$ by constructing a set coloring c of G_n that uses at most k_n colors. Let $S_1, S_2, ..., S_m$ be all the 1-subsets and 2-subsets of $\{3, 4, ..., k_n\}$. Then $m = \binom{k_n-2}{1} + \binom{k_n-2}{2}$ and since $n \ge 4$, it follows that $m \ge n - 1$. Hence, we can define the coloring c so that $c(x_1) = 2$, $c(x) = 1$ for all $x = x_i$ ($i = 2, 3, ..., n$), $\{c(x_{1,j}) : j = 1, 2\} = \{3, 4\}$, and

$$\{c(x_{i,j}) : j = 1, 2\} = \begin{cases} \{1\} \cup S_i, & \text{if } |S_i| = 1, \\ S_i, & \text{if } |S_i| = 2, \end{cases}$$
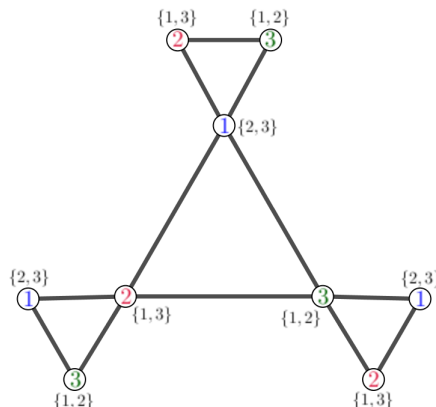
Figure 7. An optimal set coloring of $K_3 \rhd_r C_3$.

for $i = 2, ..., n$. Thus, $NC(x_1) = \{1, 3, 4\}$, $NC(x_i) = \{1, 2\} \cup S_i, i = 2, 3, ..., n$, $\{NC(x_{1,j}) : j = 1, 2\} = \{\{2, 3\}, \{2, 4\}\}$, and

$$\{NC(x_{i,j}) : j = 1, 2\} = \begin{cases} \{\{1\}, \{1, a\}\}, & \text{if } S_i = \{a\}, \\ \{\{1, a\}, \{1, b\}\}, & \text{if } S_i = \{a, b\}, \end{cases}$$

where a and b are distinct integers from $3, 4, ..., k_n$. Since the S_i's are nonempty and disjoint, it follows that c is a set coloring.

We now show that $\chi_s(G_n) \geq k_n$ by showing that any set coloring of G_n uses at least k_n colors. To this end, suppose c is a set coloring of G_n that uses p colors. We split the proof into four cases:

Case 1. Suppose c colors the vertices $x_1, x_2, ..., x_n$ using n colors. Then $p \geq n$. Since $n \geq 4$ implies $n \geq k_n$, we must have $p \geq k_n$.

Case 2. Suppose $c(x) = 1$ for all $x = x_i$, $i = 1, 2, ..., n$. Then for each i,

$$NC(x_i) = \begin{cases} \{1\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1\}, \\ \{1, a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{1, a\} \text{ or } \{a\}, \\ \{1, a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} = \{a, b\}, \end{cases}$$

where $1 < a, b \leq p$. Clearly, the case where $NC(x_i) = \{1\}$ is not possible since this would also make $NC(x_{i,j}) = \{1\}$ for $j = 1, 2$. Moreover, the case where $NC(x_i) = \{1, a\}$ is also not possible since in this case, we would have $c(x_i) = c(x_{i,j})$ for some j or $c(x_{i,1}) = c(x_{i,2})$. Hence, we should have at least n different neighborhood color sets of the form $\{1, a, b\}$, where $1 < a, b \leq p$. Then $\binom{p-1}{2} \geq n$, which is equivalent to $p \geq \frac{1}{2}(3 + \sqrt{8n + 1}) \geq k_n$.

Case 3. Suppose $c(\{x_1, x_2, ..., x_n\}) = \mathbb{N}_\ell := \{1, ..., \ell\}$, where $2 \leq \ell \leq n - 1$. Let S be the subset of $\{x_1, x_2, ..., x_n\}$ satisfying the following condition: for any $x \in S$, there is a vertex $y \neq x$ in S for which $c(x) = c(y)$. For convenience, we assume that $S = \{x_1, x_2, ..., x_{|S|}\}$. Since $2 \leq \ell \leq n - 1$, we must have $|S| \geq n - \ell + 1$. Moreover, for $x_i \in S$,

$$NC(x_i) = \begin{cases} \mathbb{N}_\ell, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} \subseteq \mathbb{N}_\ell, \\ \mathbb{N}_\ell \cup \{a\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a\}, \\ \mathbb{N}_\ell \cup \{a, b\}, & \text{if } \{c(x_{i,1}), c(x_{i,2})\} - \mathbb{N}_\ell = \{a, b\}, \end{cases}$$

where $\ell < a, b \leq p$.

- *Case 3.1* Suppose $\ell = 2$. Then none of the x_i's can have $NC(x_i) = \mathbb{N}_2$. Then we must have $\binom{p-2}{1} + \binom{p-2}{2} \geq |S| \geq n - 1$, which implies $p \geq k_n$.

- *Case 3.2* Suppose $3 \leq \ell n - 1$. Then only at most one of the x_i's can have $NC(x_i) = \mathbb{N}_\ell$. Then we must have $\binom{p-\ell}{1} + \binom{p-\ell}{2} \geq |S| - 1 \geq n - \ell$, which implies

$$p \geq \left\lceil \frac{2\ell - 1}{2} + \frac{1}{2}\sqrt{8n - 8\ell + 1} \right\rceil \geq \min_{\ell = 3, \ldots, n-1} \left\lceil \frac{2\ell - 1}{2} + \frac{1}{2}\sqrt{8n - 8\ell + 1} \right\rceil.$$

It can be easily verified that the minimum occurs when $\ell = 3$; hence,

$$p \geq \lceil 2.5 + 0.5\sqrt{8n - 23} \rceil \geq k_n$$

since $n \geq 4$.

Therefore, in any case, any set coloring of G_n that uses p colors must have $p \geq k_n$. $\qquad \square$

We now determine the set chromatic number of the comb product of complete graphs with star graphs of large order.

Proposition 3.6. *Let $n \geq 3$ and H be the star graph $K_{1,q}$, $q \geq n$, rooted at the vertex r that is not a leaf. Then $\chi_s(K_n \triangleright_r H) = \lceil 1 + \log_2(n+1) \rceil$.*

Proof. Denote the quantity $\lceil 1 + \log_2(n+1) \rceil$ by k_n and the graph $K_n \triangleright_r H$ by G_n. Label the vertices of K_n as $x_1, x_2, \ldots, x_n$ and for each i, label the leaf vertices adjacent to x_i as $x_{i,1}, x_{i,2}, \ldots, x_{i,q}$.

First, we show that $\chi_s(G_n) \leq k_n$ by constructing a set coloring c of G_n that uses at most k_n colors. Let $S_1, S_2, \ldots, S_m$ be the set of all nonempty subsets of $\{2, 3, \ldots, k_n\}$. Then $m = 2^{k_n} - 1$ and it follows that $m \geq n$. Hence, we can define the coloring c so that $c(x) = 1$ for all $x = x_i$, $i = 1, 2, \ldots, n$, and $\{c(x_{i,j}) : j = 1, 2, \ldots, q\} = S_i$ for $i = 1, 2, \ldots, n$. Thus, $NC(x_i) = \{1\} \cup S_i$ and $NC(x_{i,j}) = \{1\}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, q$. Since the S_i's are nonempty and disjoint, it follows that c is a set coloring.

We now show that $\chi_s(G_n) \geq k_n$ by showing that any set coloring of G_n uses at least k_n colors. To this end, suppose c is a set coloring of G_n that uses p colors. We split the proof into three cases:

Case 1. Suppose c colors the vertices $x_1, x_2, \ldots, x_n$ using n colors. Then $p \geq n$. Since $n \geq 3$ implies $n \geq k_n$, we must have $p \geq k_n$.

Case 2. Suppose $c(x) = 1$ for all $x = x_i$, $i = 1, 2, \ldots, n$. Then for each i,

$$NC(x_i) = \begin{cases} \{1\}, & \text{if } \{c(x_{i,j}) : j = 1, 2, \ldots, q\} = \{1\}, \\ \{1\} \cup T, & \text{if } \{c(x_{i,j}) : j = 1, 2, \ldots, q\} = \{1\} \cup T \text{ or } T, \end{cases}$$

where T is a nonempty subset of $\{2, \ldots, p\}$. Clearly, the case where $NC(x_i) = \{1\}$ is not possible since this would also make $NC(x_{i,j}) = \{1\}$ for all j. Hence, since c is a set coloring, we should have at least n different sets T. Then $2^{p-1} - 1 \geq n$, which is equivalent to $p \geq k_n$.

Case 3. Suppose $c(\{x_1, x_2, \ldots, x_n\}) = \mathbb{N}_\ell := \{1, \ldots, \ell\}$, where $2 \leq \ell \leq n-1$. Let S be the subset of $\{x_1, x_2, \ldots, x_n\}$ satisfying the following condition: for any $x \in S$, there is a vertex $y \neq x$ in S for which $c(x) = c(y)$. For convenience, we assume that $S = \{x_1, x_2, \ldots, x_{|S|}\}$. Since $2 \leq \ell \leq n - 1$, we must have $|S| \geq n - \ell + 1$. Moreover, for $x_i \in S$,

$$NC(x_i) = \begin{cases} \mathbb{N}_\ell, & \text{if } \{c(x_{i,j}) : j = 1, 2, \ldots, q\} \subseteq \mathbb{N}_\ell, \\ \mathbb{N}_\ell \cup T, & \text{if } \{c(x_{i,j}) : j = 1, 2, \ldots, q\} - \mathbb{N}_\ell = T, \end{cases}$$

where T is a nonempty subset of $\{\ell + 1, ..., p\}$. Since c is a set coloring, only at most one x_i in S can have $NC(x_i) = \mathbb{N}_\ell$. Hence, we should have at least $|S| - 1$ different sets T. Then $2^{p-\ell} - 1 \geq |S| - 1 \geq (n - \ell + 1) - 1 = n - \ell$, which is equivalent to $p \geq \lceil \ell + \log_2(n - \ell + 1) \rceil \geq \min_{\ell=2,3,...,n-1} \lceil \ell + \log_2(n - \ell + 1) \rceil$. It can be easily verified that this minimum occurs when $\ell = 2$; hence

$$p \geq \lceil 2 + \log_2(n - 1) \rceil \geq k_n$$

since $n \geq 3$.

Therefore, in any case, any set coloring of G_n that uses p colors must have $p \geq k_n$. □

4. Conclusion

In this paper, we considered the set chromatic number in relation to two well-studied graph operations: the join and the comb product. While previous research revealed sharp lower and upper bounds for the join of two graphs, our results pertained to the exact set chromatic numbers of the joins of two graphs from well-known families. Moreover, continuing a previous result on the corona of complete graphs, we determined the set chromatic number of the comb product of complete graphs with paths, cycles, and large star graphs.

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