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Construction of weavings in the plane

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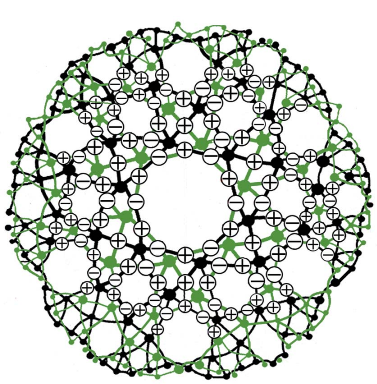


This work develops, in graph-theoretic terms, a methodology for systematically constructing weavings of overlapping nets derived from 2-colorings of the plane. From a 2-coloring, two disjoint simple, connected graphs called nets are constructed. The union of these nets forms an overlapping net, and a weaving map is defined on the intersection points of the overlapping net to form a weaving. Furthermore, a procedure is given for the construction of mixed overlapping nets and for deriving weavings from them.

1. Introduction

In recent years, scientists have gained a strong motivation to study molecular-scale weavings, as coordination networks such as metal–organic frameworks (MOFs) are produced using principles in crystal engineering (Carlucci *et al.*, 2003). These materials often have a complicated architecture of multiple disjoint molecular networks, which puzzled scientists as they typically defy simple enumeration and explanation (Hyde *et al.*, 2016). Batten & Robson (1998) did the earliest investigation of the MOF catenation by reducing the structures of the interpenetrating nets to their topological essence by just looking at the relation on the intersection of the component nets. This network or topological approach to crystal chemistry was also employed by Carlucci *et al.* (2003) to re-examine some curious topological properties of polycatenated networks. Using this approach, they discovered unexpected topological features and non-conventional new linkages that had previously been overlooked. In a more recent work, Hyde *et al.* (2016) described a systematic but simple enumeration scheme for generating some two-dimensional and three-dimensional MOF weavings using two-dimensional surfaces. Their analysis uncovered a remarkable simplicity beneath the seemingly complex structure of the MOFs they studied. Furthermore, they stress that this discovery 'suggests the importance of weavings and the relevance of lower-dimensional descriptions of three-dimensional structures in this technologically important class of advanced materials'. This work is an attempt to contribute to this endeavor by providing a systematic and algorithmic approach to constructing symmetric weavings from colorings of the plane. Moreover, we also wish to construct weavings that may be used to build three-periodic nets, since Ramsden *et al.* (2009) showed that three-dimensional crystalline Euclidean nets may be constructed from weavings on the hyperbolic plane.

According to Hyde *et al.* (2016), there is no definite meaning of molecular-scale weaving yet. In their study, a weaving consists of multiple (unbounded) components that are interlaced *via* regularly patterned under/over-crossings of disjoint edges. In this study, we take a more formal and abstract approach. We construct weavings of two sets of lines such that each set forms a simple, connected infinite graph called a net derived from a 2-coloring of a plane X. In fact, the methodology describes an explicit geometric embedding of the graph in the plane X. Moreover, the construction ensures that the vertices and edges of the nets are invariant under the group of isometries in the plane used to construct the 2-coloring of the plane. Thereupon, we say that the resulting embedding is symmetric.

In Miro *et al.* (2014), the authors discussed a procedure for simultaneously constructing two geometrically identical nets to form an overlapping net from 2-colorings of the plane using an index-2 subgroup of a triangle group. A weaving map is then defined on the overlapping net to form a weaving. In this study, the authors take an alternative approach in the construction of weavings by setting and developing the framework of construction in graph-theoretic terms. The framework is convenient as this allows us to show that, indeed, the methodology yields simple, connected, infinite graphs. Furthermore, the procedure for the construction of overlapping nets is modified – an overlapping net is now defined as a union of nets. This modification allows us to construct interesting weavings from what we refer to as *mixed overlapping nets*. In §5, we define a mixed overlapping net as a union of nets derived from possibly different index-2 subgroups of the triangle group. We show that the modified procedure can yield more general weavings when mixed overlapping nets are considered.

2. Triangle tilings and their colorings

The springboard for the development of the methodology is the relationship between colorings of the triangle tilings of the plane and the subgroups of their corresponding triangle groups studied by Provido *et al.* (2013), Decena *et al.* (2008) and De las Peñas *et al.* (2007).

Consider a triangle Δ with interior angles $\pi/p, \pi/q$ and π/r where p, q and r are integers ≥ 2. Reflecting Δ repeatedly on its sides yields a triangle tiling $\mathcal{T} := \mathcal{T}(p, q, r)$ of the plane X by copies of Δ. The plane X is the Euclidean, elliptic or hyperbolic plane depending on whether $\pi/p + \pi/q + \pi/r$ is equal to, greater than, or less than π. When $p = 6$, $q = 2$ and $r = 4$, triangle Δ shown in Fig. 1(a) exists in the hyperbolic plane, and repeatedly reflecting it on its sides gives us the triangle tiling $\mathcal{T}(6, 2, 4)$ given in Fig. 1(b).

Here we use the *Poincare disc model* of the hyperbolic plane $\mathcal{H}$. In this model, the set of points are the points that lie inside the unit disc; that is, $\mathcal{H} = \{(x, y)|x^2 + y^2 < 1\}$. Note that the points on the boundary of the disc are not in the hyperbolic plane. Moreover, the lines of the hyperbolic plane are arcs of circles orthogonal to the boundary of the unit disc as well as the diameters of the disc. In Fig. 1(b), the arcs and diameters in the interior of the disc are examples of hyperbolic lines. This model of the hyperbolic plane is conformal, which means that the hyperbolic measure of an angle is just its Euclidean angle. However, the distance is distorted. In fact, distances get exponentially larger as you approach the boundary of the disc. For example, in Fig. 1(b), the triangles appear smaller the closer one gets to the boundary of the disc, but they are actually congruent in hyperbolic geometry. For an introduction to hyperbolic geometry, see Anderson (2005).

Let P, Q and R be the reflections on the lines along the sides of Δ opposite the angles $\pi/p, \pi/q$ and π/r, respectively, as shown in Fig. 1(a). The group $G := G(p, q, r)$ of isometries generated by P, Q, R is called the triangle group and has group presentation $\langle P, Q, R | P^2 = Q^2 = R^2 = (QR)^p = (RP)^q = (PQ)^r = I\rangle$, where I is the identity transformation.

Note that the triangle group G acts transitively on the tiling $\mathcal{T}$. Since any tile in $\mathcal{T}$ is the image of Δ under some element of G, the G-orbit of Δ is the whole tiling $\mathcal{T}$; that is, $\mathcal{T} = \{g\Delta | g \in G\}$. Moreover, by construction, the only element of G that maps Δ to itself is the identity I; hence, $\mathrm{stab}_G(\Delta) = \{I\}$, where $\mathrm{stab}_G(\Delta)$ is the stabilizer of Δ in G. Consequently, it can be shown that the mapping $g \mapsto g\Delta$ is a one-to-one correspondence between $G := G(p, q, r)$ and $\mathcal{T} := \mathcal{T}(p, q, r)$. This implies that we can identify each tile in the triangle tiling $\mathcal{T}$ with exactly one element in the triangle group G, and *vice versa*. We will take advantage of this correspondence in the following discussion.

By the construction of the triangle tiling $\mathcal{T} = \{g\Delta | g \in G\}$, we have that $\bigcup_{g \in G} g\Delta = X$ and $\Delta^\circ \cap g\Delta^\circ = \emptyset$ for every non-identity $g \in G$, where Δ° is the interior of Δ. Thus, Δ is a fundamental region of the triangle group G.

We now define a coloring of a triangle tiling $\mathcal{T} := \mathcal{T}(p, q, r)$ by an index-2 subgroup of its corresponding triangle group $G := G(p, q, r)$. Consider an index-2 subgroup H of G and let

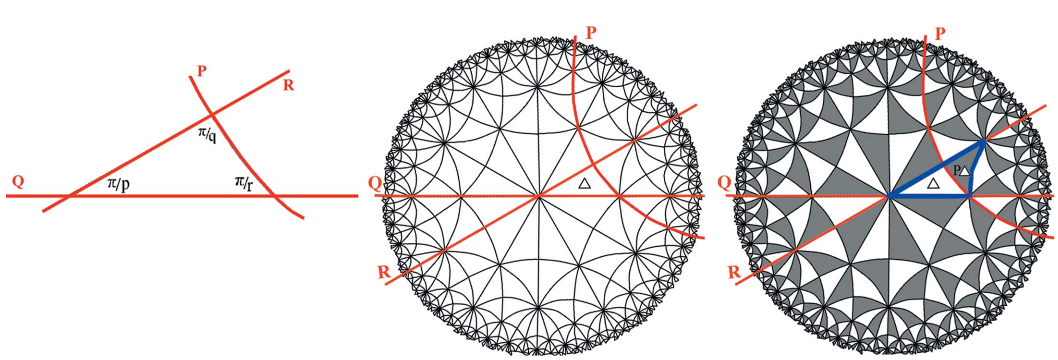


Figure 1
(a) The reflections P, Q and R on the sides of triangle Δ, (b) the triangle tiling $\mathcal{T}(6, 2, 4)$ and (c) the 2-coloring $\mathcal{T}_{H_1}(6, 2, 4)$.

$\{I, \hat{g}\}$ be a complete set of left coset representatives of H in G. Then $G = H \cup \hat{g}H$ and $H \cap \hat{g}H = \emptyset$. For $g \in \{I, \hat{g}\}$, let $gH\Delta = \{(gh)\Delta | h \in H\}$. Then $\mathcal{T} = H\Delta \cup \hat{g}H\Delta$ and $H\Delta \cap \hat{g}H\Delta = \emptyset$, so that $H\Delta$ and $\hat{g}H\Delta$ constitute a partition of the triangle tiling $\mathcal{T}$. We use this partition to define a coloring of $\mathcal{T}$ by H.

Consider the coloring map $\iota : \mathcal{T} \to \mathcal{T} \times \{c_1, c_2\}$ defined by $\iota(t) = (t, c_1)$ if $t \in H\Delta$ and $\iota(t) = (t, c_2)$ if $t \in \hat{g}H\Delta$, where c_1 represents the color white and c_2 represents the color black. Intuitively, this mapping assigns the label c_1 to the elements of the set $H\Delta$, and the label c_2 to the elements of the set $\hat{g}H\Delta$. Note that the mapping ι is well defined because $H\Delta$ and $\hat{g}H\Delta$ form a partition of $\mathcal{T}$. This results in a 2-coloring of the triangle tiling $\mathcal{T}(p, q, r)$ by H, which we denote by $\mathcal{T}_H(p, q, r) = (H\Delta \times \{c_1\}) \cup (\hat{g}H\Delta \times \{c_2\})$. Furthermore, we now refer to $H\Delta$ as the set of all white tiles and $\hat{g}H\Delta$ as the set of all black tiles.

Note that H acts on the colored tiling $\mathcal{T}_H(p, q, r)$, and if $\Delta_H = \Delta \cup \hat{g}\Delta$, then $\bigcup_{h\in H} h(\Delta_H) = X$ and $\Delta_H^\circ \cap (h\Delta_H^\circ) = \emptyset$ for any non-identity $h \in H$. Thus, $\Delta_H = \Delta \cup \hat{g}\Delta$ is a fundamental region of H on the colored tiling $\mathcal{T}_H(p, q, r)$.

To illustrate the methodology, consider the triangle group $G := G(6, 2, 4)$ and its associated triangle tiling $\mathcal{T} := \mathcal{T}(6, 2, 4)$ as shown in Fig. 1(*b*). Let us construct the 2-coloring of $\mathcal{T}$ by the index-2 subgroup $H_1 = \langle QR, RP, QP\rangle$. Since P is not in H_1, the cosets H_1 and PH_1 partition G. Consequently, $H_1\Delta$ and $PH_1\Delta$ partition the triangle tiling $\mathcal{T}$. In this case, we assign color c_1 (white) to the tiles in $H_1\Delta$ and color c_2 (black) to the tiles in $PH_1\Delta$. The resulting 2-coloring $\mathcal{T}_{H_1} = \mathcal{T}_{H_1}(6, 2, 4)$ is shown in Fig. 1(*c*). A fundamental region of H_1 is $\Delta_{H_1} = \Delta \cup P\Delta$, which is a union of the adjacent white tile Δ and black tile $P\Delta$ bounded by blue lines in Fig. 1(*c*).

2.1. Patches from colorings of triangle tilings

In this section, we define a patch of a colored triangle tiling.

Definition 1. A *black patch* or *B-patch* of a (black-and-white) colored tiling is the union of a collection $B_1, \ldots, B_m$ of black tiles which share a common vertex such that tiles B_1 and B_m are adjacent and, for $i = 1, \ldots, m-1$, the tiles B_i and B_{i+1} are also adjacent. A *white patch* or *W-patch* is similarly defined.

For example, the B-patches and the W-patches of $\mathcal{T}_{H_1}$ are the black and white tiles of the colored tiling, respectively. Meanwhile, for an index-2 subgroup $H_2 = \langle R, Q, PQP, PRP\rangle$ of $G(p, q, r)$, a B-patch of $\mathcal{T}_{H_2}$ is composed of $2p$ adjacent black tiles and a W-patch is similarly composed of $2p$ adjacent white tiles. In Figs. 2(*a*)–2(*g*), we illustrate the seven types of B- and W-patches determined by the colorings of $\mathcal{T}$ by the index-2 subgroups of $G(6, 2, 4)$.

Remark 1. If t is a black tile, then it is part of a unique B-patch; similarly, t is part of a unique W-patch if it is a white tile. Another important observation is that since each subgroup H_i sends the colored tiling $\mathcal{T}_{H_i}$ to itself, any element $h \in H_i$ sends a B-patch to a B-patch and a W-patch to a W-patch.

3. General setting for the weaving construction

Before we present the weaving construction, we first give some definitions from graph theory. A *graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a non-empty set $\mathcal{V} := \mathcal{V}(G)$ of vertices together with a set $\mathcal{E} := \mathcal{E}(G)$ of edges. Each edge $e \in \mathcal{E}(G)$ is associated with an unordered pair $\{u, v\}$ of (not necessarily distinct) elements of $\mathcal{V}$. If there are two edges associated with the same pair of vertices, then such edges are called multiple edges. An edge e associated with the unordered pair $\{u, v\}$ with $u = v$ is called a *loop*. A *simple graph* is a graph containing no loops or multiple edges. In this study, the vertices and edges of a graph $\mathcal{G}$ will be represented by points on a plane and undirected line

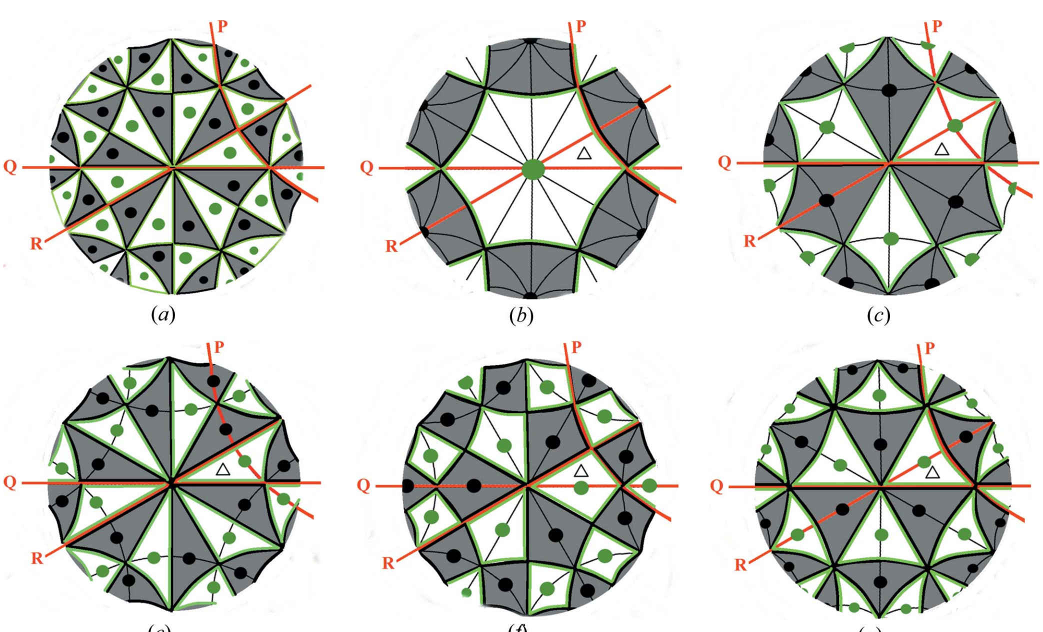


Figure 2
The seven types of B- and W-patches by the coloring of $\mathcal{T}(6, 2, 4)$ using the index-2 subgroups H_i of $G(6, 2, 4)$ for $i = 1, 2, \ldots, 7$. (*a*) $H_1 = \langle QR, RP\rangle$, (*b*) $H_2 = \langle R, Q, PQP\rangle$, (*c*) $H_3 = \langle P, R, QRQ, QPQ\rangle$, (*d*) $H_4 = \langle RQR, P, Q\rangle$, (*e*) $H_5 = \langle QR, P\rangle$, (*f*) $H_6 = \langle RP, Q, PQP\rangle$, (*g*) $H_7 = \langle PQ, R, QRQ\rangle$.

segments joining two points, respectively. Furthermore, in this study, we define the intersection of edges of two graphs as the intersection of their line segment representations.

A *path* $\mathcal{P}$ in a graph $\mathcal{G}$ is an alternate sequence of distinct vertices and distinct edges which begins and ends with vertices; that is, $\mathcal{P}$ is a sequence $v_0 - e_1 - v_1 - e_2 - v_2 - \ldots - v_{k-1} - e_k - v_k$, where each v_i is a vertex and $e_i = \{v_{i-1}, v_i\}$ is an edge for $i = 1, \ldots, k$. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *connected* if, for any two vertices u and v, there is a path from u to v.

In this study, we define a *net* as a simple, connected graph. Since the vertices and edges of a net are represented by points and line segments on a plane, respectively, then a net is a geometric embedding of a graph on a plane. In particular, in this study, a net is derived from a coloring of a plane X and, by construction, it is a symmetric embedding of the graph on the plane X.

We say that two nets $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{N}' = (\mathcal{V}', \mathcal{E}')$ are *disjoint* if $\mathcal{N}$ and $\mathcal{N}'$ have no vertices in common. Furthermore, we are interested in the union of disjoint nets $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{N}' = (\mathcal{V}', \mathcal{E}')$. We call such a union an *overlapping net* and denote it by $\mathcal{O} = \mathcal{N} \cup \mathcal{N}'$.

Given an overlapping net $\mathcal{O}$ of two disjoint nets $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{N}' = (\mathcal{V}', \mathcal{E}')$, we define the set of *weaving points* $\mathfrak{T}$ of $\mathcal{O}$ as $\mathfrak{T} := \{(e, e') \in \mathcal{E} \times \mathcal{E}' | e \text{ and } e' \text{ intersect in } \mathcal{O}\}$. We may interpret each element (e, e') of $\mathfrak{T}$ as the intersection point of the edges e and e'. Consequently, the set $\mathfrak{T}$ may be interpreted as the set of all intersection points of the edge sets $\mathcal{E}$ and $\mathcal{E}'$. We use this interpretation in this study. We now define a weaving on a given overlapping net.

Definition 2. Let $\mathcal{O}$ be an overlapping net derived from nets $\mathcal{N}$ and $\mathcal{N}'$, and let $\mathfrak{T}$ be the set of weaving points of $\mathcal{O}$. A *weaving map* ω on $\mathcal{O}$ is a function from $\mathfrak{T}$ to the two-point set $\{\oplus, \ominus\}$. A *weaving* $\mathcal{W}$ on $\mathcal{O}$ is a tuple $\mathcal{W} = (\mathcal{O}, \omega)$ of the overlapping net $\mathcal{O}$ together with a weaving map ω on $\mathcal{O}$. We say that $\mathcal{W}$ is a *proper weaving* if $\omega(\mathfrak{T}) = \{\oplus, \ominus\}$. We say that e is above e' if $\omega(e, e') = \oplus$; while if $\omega(e, e') = \ominus$, we say that e is below e'.

4. The weaving construction

As before, we let H be an index-2 subgroup of the triangle group $G := G(p, q, r)$ associated with the triangle tiling $\mathcal{T} := \mathcal{T}(p, q, r)$. In the preceding sections, we have discussed the construction of the 2-coloring $\mathcal{T}_H := \mathcal{T}_H(p, q, r)$ and the identification of the B-patches and W-patches. In this section, from the B- and W-patches of a colored tiling $\mathcal{T}_H$, we construct two disjoint nets: the B-net $\mathcal{N}_B$ using the B-*motif* Δ_{bH} and the W-net $\mathcal{N}_W$ using the W-*motif* Δ_{wH}. These constructions will then yield an overlapping net $\mathcal{O}_H$. The penultimate step is to define a weaving map ω_H on the set of weaving points $\mathfrak{T}_H$ of $\mathcal{O}_H$. Finally, we get a weaving $\mathcal{W}_H = (\mathcal{O}_H, \omega_H)$.

4.1. Net construction: *B*- and *W*-vertices

In §2.1, we defined the B- and W-patches determined by a 2-coloring $\mathcal{T}_H$ of the tiling $\mathcal{T}$ by a subgroup H of G. Using these patches, we now define the vertex and edge sets of two disjoint nets.

Definition 3. The B-*vertex set* denoted by $\mathcal{V}_B$ is the set of the centroids of the B-patches in $\mathcal{T}_H$, and the W-*vertex set* $\mathcal{V}_W$ is the set of centroids of the W-patches in $\mathcal{T}_H$. The elements of $\mathcal{V}_B$ and $\mathcal{V}_W$ are referred to as the B- and W-*vertices*, respectively.

In Figs. 2(*a*)–2(*g*), we also illustrate the seven B- and W-vertex sets corresponding to the colorings of $\mathcal{T}(6, 2, 4)$ by the index-2 subgroups of $G(6, 2, 4)$. The B-vertices are the black points, while the W-vertices are the green points.

Remark 2. By Definition 3 and Remark 1, each tile t in $\mathcal{T}_{H_i}$ contains a B-vertex if it is a black tile, or a white W-vertex if it is a white tile. Moreover, any element $h \in H_i$ sends a B-vertex to a B-vertex and a W-vertex to a W-vertex.

4.2. Net construction: *B*- and *W*-edges

Now we describe a method for constructing B- and W-edges. Note that the generating set of any subgroup H of the triangle group G need not be unique. In addition, any pair of a black and a white tile in a 2-coloring $\mathcal{T}_H$ is a fundamental region of H. However, a natural choice would be the white tile Δ and one of its adjacent black tiles $g\Delta$ for some $g \notin H$. Before we construct the B- and W-motifs on Δ_H which we will use to construct the B- and W-edges, we impose conditions in choosing a generating set of H and an element $g \notin H$ in constructing a fundamental region (denoted as FR) of H on $\mathcal{T}_H$.

The FR condition. For an index-2 subgroup H, choose a generating set $\{h_1, \ldots, h_l\}$ and an element $g \in G$ but $g \notin H$ satisfying both conditions:

(FR1) Δ and $g\Delta$ are edge-adjacent, and

(FR2) for each $i = 1, \ldots, l$, we have $f_i \cap \Delta \neq \emptyset$ and $f_i \cap g\Delta \neq \emptyset$ where $f_i = \{x \in X | h_i x = x\}$.

We refer to $\Delta_H = \Delta \cup g\Delta$ as a *fundamental region of H with respect to the generating set* $\{h_1, \ldots, h_l\}$, and we say that Δ_H is *constructible*.

The importance of the FR condition will be explained at the end of the section.

In Table 1, we give a constructible fundamental region determined by a given generating set of H_i.

Remark 3. By Remark 2, for a constructible fundamental region $\Delta_H = \Delta \cup g\Delta$, the white tile Δ contains a unique W-vertex, say w_H, and the black tile $g\Delta$ contains a unique B-vertex, say b_H. We call the vertices w_H and b_H the *initial* W-vertex and the *initial* B-vertex of Δ_H, respectively.

We now construct B- and W-motifs on Δ_H.

The motif construction. Let Δ_H be a constructible fundamental region of H with respect to a given generating set, say $\{h_1, \ldots, h_l\}$. Let $b := b_H$ be the *initial* B-vertex. For each generator h_i, if h_i does not fix b, construct edges $\{b, h_i b\}$ and

Table 1
Some possible constructible fundamental regions for H_i.

Subgroup H_i of $G(p, q, r)$	Generating sets	Constructible FR
H_1	$\langle QR, RP\rangle$	$\Delta \cup R\Delta$
	$\langle QR, PQ\rangle$	$\Delta \cup Q\Delta$
	$\langle RP, PQ\rangle$	$\Delta \cup P\Delta$
H_2	$\langle R, Q, PQP, PRP\rangle$	$\Delta \cup P\Delta$
H_3	$\langle P, R, QRQ, QPQ\rangle$	$\Delta \cup Q\Delta$
H_4	$\langle RQR, RPR, P, Q\rangle$	$\Delta \cup R\Delta$
H_5	$\langle QR, P, RPR\rangle$	$\Delta \cup R\Delta$
H_6	$\langle RP, Q, PQP\rangle$	$\Delta \cup P\Delta$
H_7	$\langle PQ, R, QRQ\rangle$	$\Delta \cup Q\Delta$

$\{b, h_i^{-1}b\}$ by connecting the B-vertex b to the B-vertices h_ib and $h_i^{-1}b$ using black line segments. The *B-motif* Δ_{bH} consists of the B-vertices together with the segment of the edges $\{b, h_ib\}$ and $\{b, h_i^{-1}b\}$ in Δ_H for all $h_i \in \{h_1, \ldots, h_l\}$. The *W-motif* Δ_{wH} is similarly obtained.

Consider the 2-coloring of $\mathcal{T}(6, 2, 4)$ by subgroup $H_2 = \langle R, Q, PQP\rangle$ of $G(6, 2, 4)$ shown in Fig. 3(a), with its constructible fundamental region $\Delta_{H_2} = \Delta \cup P\Delta$. Let $b := b_{H_2}$ be the initial B-vertex of Δ_{H_2}. Since among the given generators of H_2 only Q does not fix b, we only construct the B-edge $\{b, Qb\}$ shown in Fig. 3(b). If we let $w := w_{H_2}$ be the initial W-vertex of Δ_{H_2}, we only construct the edge $\{w, (PQP)w\}$ shown in Fig. 3(d) since R and Q fix w. Note that the B-motif Δ_{bH_2} consists of the segment of the B-edge $\{b, Qb\}$ in Δ_{H_2} and the B-vertex b, shown in Fig. 3(c); similarly, the W-motif Δ_{wH_2} consists of the segment of the W-edge $\{w, (PQP)w\}$ in Δ_{H_2} and the W-vertex w shown in Fig. 3(e).

4.3. Overlapping net construction

Since we assume that the set $\{h_1, \ldots, h_l\}$ is a generating set of the subgroup H of G, this means that any $h \in H$ can be written as a finite product of h_i and h_i^{-1} for some $i = 1, 2, \ldots, l$. We use these generators and the B- and W-motifs to construct the edges of two disjoint nets.

The edge construction. Let $K = \{h_1, \ldots, h_l\} \cup \{h_1^{-1}, \ldots, h_l^{-1}\}$ and $b := b_H$ the initial B-vertex of Δ_H. Consider the B-motif Δ_{bH}, and let H act on Δ_{bH} such that when black line segments abut, continuously join them to form B-edges. The action of H on the motif Δ_{bH} produces a graph $\mathcal{N}_B = (\mathcal{V}_B, \mathcal{E}_B)$ called a B-net, where $\mathcal{V}_B = \{hb | h \in H\}$, $\mathcal{E}_B = \{h\{b, kb\} | h \in H$ and $k \in K\}$ and where $h\{b, kb\} := \{hb, hkb\}$. The action of H on the motif Δ_{wH} produces a graph $\mathcal{N}_W = (\mathcal{V}_W, \mathcal{E}_W)$ called a W-net which is similarly constructed.

In Fig. 3(c), we have seen the B-motif Δ_{bH_2}, where $H_2 = \langle R, Q, PQP\rangle$ is a subgroup of $G(6, 2, 4)$. We now construct the B-net $\mathcal{N}_B$ determined by H_2. The action of Q on Δ_{bH_2} yields the B-edge $\{b, Qb\}$ shown in Fig. 4(a). Letting R act on $[\Delta_{bH_2} \cup Q\Delta_{bH_2}]$ yields another B-edge $\{b, (RQ)b\}$ given in Fig. 4(b). Using the generator PQP, we get the edge $\{b, (PQP)(RQ)b\}$ shown in Fig. 4(c). Continuing the process, we get the B-net $\mathcal{N}_B$ determined by H_2 shown in Fig. 4(d).

In the following proposition, we show that the resulting graphs $\mathcal{N}_B$ and $\mathcal{N}_W$ are indeed nets.

Proposition 1. The graphs $\mathcal{N}_B = (\mathcal{V}_B, \mathcal{E}_B)$ and $\mathcal{N}_W = (\mathcal{V}_W, \mathcal{E}_W)$ are nets.

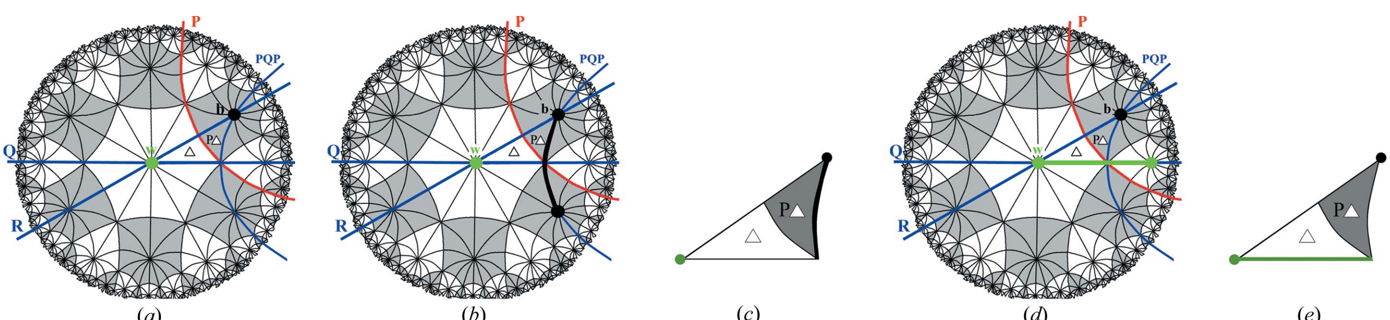


Figure 3
(a) The colored tiling $\mathcal{T}_{H_2}(6, 2, 4)$; ($b$) edge $\{b, Qb\}$ in black and (c) the B-motif Δ_{bH_2}; and (d) edge $\{w, (PQP)w\}$ in green and (e) the W-motif Δ_{wH_2}.

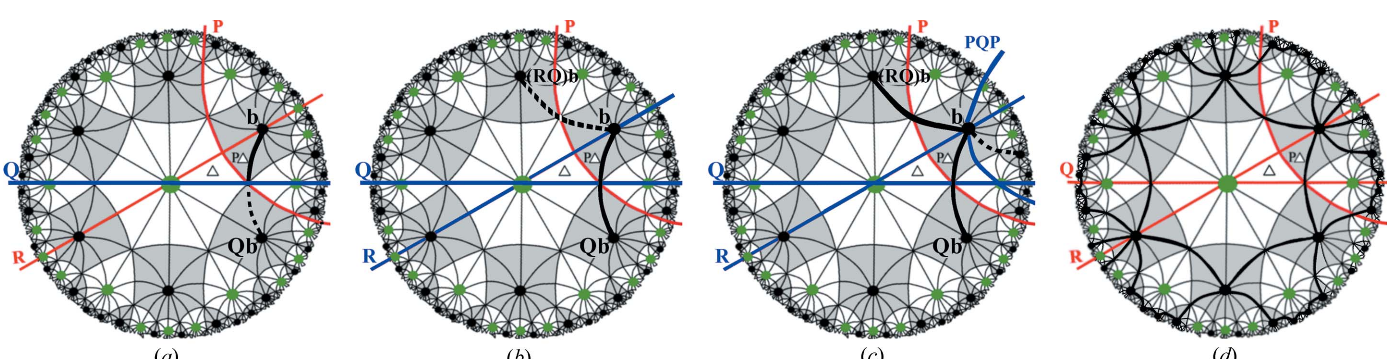


Figure 4
The recursive construction of B-net $\mathcal{N}_B$.

Proof. The simplicity of the graph follows from construction and the fact that the elements of H are isometries. We now show that the graph $\mathcal{N}_B$ is connected. To do this, it is sufficient to show that there exists a path from the initial B-vertex b to any vertex in $\mathcal{V}_B$.

Let b be the initial B-vertex of Δ_H and $K = \{h_1, \ldots, h_l\} \cup \{h_1^{-1}, \ldots, h_l^{-1}\}$ with $H = \langle h_1, h_2, \ldots, h_l \rangle$. Let $v \in \mathcal{V}_B \setminus \{b\}$ such that $v = hb$, for some $h \in H$. Suppose $h = k_1 \ldots k_r$ where $k_i \in K$ for $i = 1, \ldots, r$. For the case when $r = 1$, by assumption, we have that $v = k_1 b$ is not the point b, which implies that the pair $\{b, k_1 b\}$ is a B-edge in $\mathcal{E}_B$ by construction. Consequently, the sequence $b - \{b, k_1 b\} - k_1 b = v$ is a path as desired. Note that for $i = 1, \ldots, r$, if $b \neq k_i b$, the pair $\{b, k_i b\}$ is a B-edge in $\mathcal{E}_B$, and that for any $h' \in H$, the pair $h'\{b, k_i b\} = \{h'b, h'k_i b\}$ is also a B-edge in $\mathcal{E}_B$. Define

$$e_i := \big\{(k_1 \ldots k_{i-1})b, (k_1 \ldots k_{i-1}k_i)b\big\} = (k_1 \ldots k_{i-1})\{b, k_i b\},$$

for $i = 1, 2, \ldots, r$ and with k_0 the identity isometry I. If $b \neq k_i b$, e_i is a B-edge in $\mathcal{E}_B$ whose endpoints are the B-vertices $(k_1 \ldots k_{i-1})b$ and $(k_1 \ldots k_i)b$. On the other hand, suppose $b = k_i b$, the B-vertices $(k_1 \ldots k_{i-1})b$ and $(k_1 \ldots k_i)b$ are the same so that e_i is not a B-edge since we do not consider loops in $\mathcal{E}_B$. Now, construct the sequence

$$\begin{aligned} &b - e_1 - k_1 b - e_2 - k_1 k_2 b - \ldots - (k_1 k_2 \ldots k_{i-1})b \\ &= (k_1 k_2 \ldots k_{i-1})k_i b - \ldots - e_r - k_1 k_2 \ldots k_{r-1} k_r b = hb \end{aligned}$$

where all $e_i \notin \mathcal{E}_B$ are not included.

It is easy to see that this sequence is a path from b to $v = hb$ in $\mathcal{N}_B$. Thus, $\mathcal{N}_B$ is connected. Hence, $\mathcal{N}_B$ is a net. By a similar argument, one can show that $\mathcal{N}_W$ is also a net. □

At this point, we illustrate what happens when the FR condition is not satisfied. Basically, the FR condition is imposed to avoid *wild nets*. We consider a net $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ *wild* if any of these conditions are not satisfied:

(E1) the endpoints of an edge $e \in \mathcal{E}$ are points associated with vertices in $\mathcal{V}$; and

(E2) if two line segments associated with two distinct edges intersect, their intersection must be a point in $\mathcal{V}$.

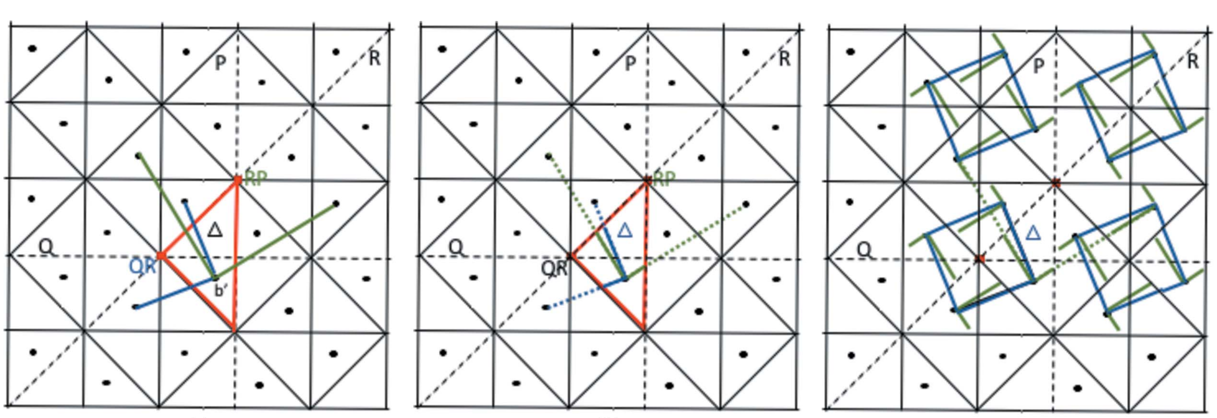
In the literature, we find that these are the types of nets that are relevant to the study of the architecture of MOFs and others.

Note that, by the motif and edge constructions, a B-net $\mathcal{N}_B$ is derived from a B-motif Δ_{bH} that lies inside a constructible fundamental region Δ_H by letting H act on Δ_{bH}. Thus, to ensure that E2 is satisfied by the B-net $\mathcal{N}_B$, we only need to make sure that in the B-motif Δ_{bH} the segments of any two distinct B-edges satisfy E2. However, the motif construction exposes us to the possibility of violating E1. This is where the FR condition becomes important.

To illustrate this point, let us consider the index-2 subgroup $H_1 = \langle QR, RP \rangle$ of $G(4, 4, 2)$ and its non-constructible fundamental region $\Delta'_{H_1} = \Delta \cup Q\Delta$ (bounded by red in Fig. 5*a*). Note that Δ'_{H_1} is not constructible since $Q\Delta \cap f_{RP} = \emptyset$, where f_{RP} is the fixed set of the rotation RP. In Fig. 5(*a*), we have the pair of B-edges $\{b', (QR)b'\}$ and $\{b', (RQ)b'\}$ in blue and the pair of B-edges $\{b', (RP)b'\}$ and $\{b', (PR)b'\}$ in green. To get the motif Δ'_{bH_1}, we basically disregard the segments of these B-edges that lie outside Δ'_{H_1}. The resulting B-motif Δ'_{bH_1} consists of segments of these B-edges in Δ'_{H_1} as shown in Fig. 5(*b*). Proceeding with the edge construction, we let H_1 act on the B-motif Δ'_{bH_1} and a partial result is shown in Fig. 5(*c*). Note that while the blue B-edges are fully formed, the green B-edges are not and fail to satisfy E1 as one of the endpoints of the green B-edges are not B-vertices. We have no problem with the blue B-edges $\{b', (QR)b'\}$ and $\{b', (RQ)b'\}$ precisely because Δ and $Q\Delta$ both intersect with the fixed set of QR (condition FR2 is satisfied on generator QR). But, as noted earlier, we have $Q\Delta \cap f_{RP} = \emptyset$ (fails condition FR2 on generator RP), which resulted in the problematic scenario with the green B-edges. Note that segments of the green edges $\{b', (RP)b'\}$ and $\{b', (PR)b'\}$ are not recovered.

Definition 4. Let $\mathcal{N}_B = (\mathcal{V}_B, \mathcal{E}_B)$ and $\mathcal{N}_W = (\mathcal{V}_W, \mathcal{E}_W)$ be the nets derived from a 2-coloring by H. The *overlapping net* $\mathcal{O}_H$ determined by H is the union of $\mathcal{N}_B$ and $\mathcal{N}_W$. The *motif* Δ_{mH} of an overlapping net $\mathcal{O}_H := \mathcal{N}_B \cup \mathcal{N}_W$ is the union of the B-motif Δ_{bH} of $\mathcal{N}_B$ and the W-motif Δ_{wH} of $\mathcal{N}_W$.

Note that, in this methodology, the component nets $\mathcal{N}_B$ and $\mathcal{N}_W$ are first constructed independently, and then the over-



(*a*) (*b*) (*c*)

Figure 5
The problematic consequence when the FR condition is not satisfied.

lapping net $\mathcal{O}_H$ is constructed by getting the union of the nets $\mathcal{N}_B$ and $\mathcal{N}_W$. Observe that if, on the other hand, we consider the motif $\Delta_{mH} := \Delta_{bH} \cup \Delta_{wH}$ and let H act on it, then we get the overlapping net $\mathcal{O}_H$. This simultaneous construction of the B- and W-nets to form $\mathcal{O}_H$ is similar to the construction discussed in Miro *et al.* (2014). In §5, we will show that the construction of $\mathcal{O}_H$ as a union of independently constructed nets is fruitful as it allows us to construct more interesting weavings.

4.4. Weaving construction

Let $\mathfrak{T}_H = \{(e_b, e_w) \in \mathcal{E}_B \times \mathcal{E}_W | e_b \text{ and } e_w \text{ intersect in } \mathcal{O}_H\}$ be the set of weaving points of an overlapping net $\mathcal{O}_H$, and let $\mathcal{P}$ be the set of weaving points in the motif $\Delta_{mH} = \Delta_{bH} \cup \Delta_{wH}$ of $\mathcal{O}_H$. We say that two weaving points p and p' are *identifiable under H* if $p = hp'$ for some $h \in H$.

We now define the weaving map $\omega_H : \mathfrak{T}_H \to \{\oplus, \ominus\}$ by first assigning values to $\omega_H(p)$, for each $p \in \mathcal{P}$, with every pair of identifiable weaving points in $\mathcal{P}$ assigned identical values. We then extend ω_H to $\mathfrak{T}_H$ as follows: for each weaving point $q \in \mathfrak{T}_H$, let $\omega_H(q) = \omega_H(p)$ where $p \in \mathcal{P}$ such that $q = hp$ for some $h \in H$. In the following, we argue that this extension is well defined.

Proposition 2. Any weaving point $q \in \mathfrak{T}_H$ of $\mathcal{O}_H$ is of the form hp for some $p \in \mathcal{P}$ and $h \in H$.

Proof. Suppose $q \in \mathcal{O}_H$ is a weaving point; that is, q is an intersection point of a B-edge and a W-edge. Since $\mathcal{O}_H$ can be constructed by letting H act on $\Delta_{mH} = \Delta_{bH} \cup \Delta_{wH}$, there exist $p_q \in \Delta_{mH}$ and $h_q \in H$ such that $q = h_q p_q$. Since h_q is an isometry, its inverse h_q^{-1} is also an isometry and so preserves incidence. Furthermore, as an element of H, h_q^{-1} sends B-edges to B-edges and W-edges to W-edges. Thus, $p_q = h_q^{-1} q$ must also be an intersection of a B-edge and a W-edge in Δ_{mH}. This implies that $p_q \in \mathcal{P}$. □

Note that, since Δ_{mH} is a fundamental region of H on the colored tiling $\mathcal{T}_H(p, q, r)$, Δ_{mH} contains exactly one point from the orbits of H, except on its boundary which may contain duplicates of representatives of some orbits. Hence, in the proof, if p_q is in the interior of Δ_{mH}, then it follows that it is unique so that we define $\omega_H(q) := \omega_H(p_q)$ unambiguously. On the other hand, if p_q is on the boundary of Δ_{mH}, then there may exist duplicate orbit representatives p'_q with $q = h'_q p'_q$ for some $h'_q \in H$. However, since p_q and p'_q belong to the same orbit H, they are identifiable weaving points. Then, as in the previous case, we define $\omega_H(q) := \omega_H(p_q) = \omega_H(p'_q)$. Thus, in either case, the extension is well defined.

Clearly, if the motif Δ_{mH} has only one weaving point, then it will not yield a proper weaving as $\mathfrak{T} = \{\oplus\}$ or $\mathfrak{T} = \{\ominus\}$. To construct a weaving when there are two unidentifiable weaving points in the motif Δ_{mH}, we assign to one weaving point the $\oplus$ value and to the other the $\ominus$ value and then let H act on Δ_{mH} with assigned values.

Consider a 2-coloring $\mathcal{T}_{H_3}(6, 2, 4)$ of the triangle tiling by subgroup H_3 of $G(6, 2, 4)$ shown in Fig. 2(*c*), where $H_3 = \langle P, R, QRQ, QPQ \rangle$. A constructible fundamental region of H_3 is $\Delta_{H_3} = \Delta \cup Q\Delta$. The overlapping net $\mathcal{O}_{H_3}$ is shown in Fig. 6(*a*) and its motif Δ_{mH_3} is given in Fig. 6(*b*). Note that in Δ_{mH_3} there are two weaving points denoted by p_1 and p_2. Since p_1 is on an edge of Δ while p_2 is on the vertex with angle $\pi/4$, no element of H_3 will send one to the other. Hence, p_1 and p_2 are non-identifiable weaving points. We can then define two weaving maps $\omega_{H_3}, \omega'_{H_3}$: $\mathfrak{T}_{H_3} \to \{\oplus, \ominus\}$ as $(\omega_{H_3}(p_1), \omega_{H_3}(p_2)) = (\oplus, \ominus)$ and $(\omega'_{H_3}(p_1), \omega'_{H_3}(p_2)) = (\ominus, \oplus)$. The resulting weavings $\mathcal{W}_{H_3} = (\mathcal{O}_{H_3}, \omega_{H_3})$ and $\mathcal{W}'_{H_3} = (\mathcal{O}_{H_3}, \omega'_{H_3})$ are shown in Figs. 6(*c*)–6(*d*), respectively.

4.5. Improper to proper weavings

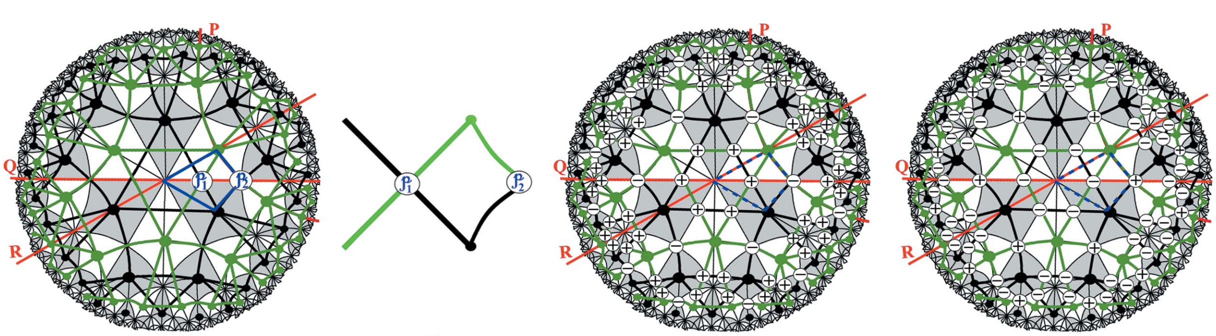
By Definition 2, a weaving $\mathcal{W} = (\mathcal{O}, \omega)$ is said to be proper if $\omega(\mathfrak{T}) = \{\oplus, \ominus\}$. In the methodology just presented, however, there are instances when $\omega(\mathfrak{T}) = \{\oplus\}$ or $\omega(\mathfrak{T}) = \{\ominus\}$, resulting in what we call an *improper weaving*. Certainly, we have an improper weaving when

(i) the set of weaving points $\mathcal{P}$ in Δ_{mH} contains only one element, or

(ii) every pair of weaving points in $\mathcal{P}$ is identifiable.

To obtain a proper weaving when either (i) or (ii) occurs, we use the fact that a fundamental region Δ_H of H in $\mathcal{T}(p, q, r)$ contains n copies of Δ if H is an index-n subgroup of the triangle group G. In the following, we describe the process of extracting a proper weaving from an improper one.

From improper to proper weaving. When either condition (i) or (ii) above occurs for an overlapping net $\mathcal{O}_H$, consider an



(*a*) (*b*) (*c*) (*d*)

Figure 6
(*a*) The overlapping net $\mathcal{O}_{H_3}$, (*b*) the motif Δ_{mH_3}, and the weavings (*c*) $\mathcal{W}_{H_3}(\mathcal{O}_{H_3}, \omega_{H_3})$ and (*d*) $\mathcal{W}'_{H_3}(\mathcal{O}_{H_3}, \omega'_{H_3})$.

index-2 subgroup N of H. Then a fundamental region Δ_N of N consists of four copies of Δ or, equivalently, Δ_N consists of two copies of Δ_H. The resulting motif Δ_{mN} determined by N on $\mathcal{O}_H$ consists of two copies of the original motif Δ_{mH}. This means that the number of weaving points in Δ_{mN} is more than the number in Δ_{mH}. Note, however, that the set of all weavings points is still the same; that is, $\mathfrak{T}_H = \mathfrak{T}_N$. If not every pair of weaving points in Δ_{mN} is identifiable under N, then we are done. But should an improper weaving still occur, the process is repeated either by considering another index-2 subgroup of H or an index-2 subgroup of N. Once at least two unidentifiable weaving points are obtained, assign values $\oplus$ and $\ominus$ to said weaving points and we are done.

As an example, let us consider a 2-coloring $\mathcal{T}_{H_4}(6, 2, 4)$ of the triangle tiling $\mathcal{T}(6, 2, 4)$ by H_4 shown in Fig. 2(*d*), where $H_4 = \langle RQR, P, Q\rangle$. It has a constructible fundamental region $\Delta_{H_4} = \Delta \cup R\Delta$. In Fig. 7(*a*), we have the overlapping net $\mathcal{O}_{H_4}$ with its motif Δ_{mH_4} bounded by broken blue lines. Note that the motif Δ_{mH_4} only has one weaving point; that is, $\mathcal{P}_{H_4} = \{\varphi_1\}$. Thus, we have an improper weaving.

Now, we consider an index-2 subgroup $N = \langle P, RQRQ\rangle$ of H_4. A fundamental region $\Delta_N = \Delta_{H_4} \cup Q\Delta_{H_4}$ of N is bounded by violet lines in Fig. 7(*a*). The resulting motif Δ_{mN} determined by N on $\mathcal{O}_{H_4}$ is given in Fig. 7(*b*). The bigger motif Δ_{mN} now has two weaving points denoted by φ_1 and φ_2. These two weaving points are non-identifiable in Δ_{mN} since $Q \notin N$. Note that $\mathfrak{T}_N = \mathfrak{T}_{H_4}$, but $\mathcal{P}_{H_4} \subset \mathcal{P}_N = \{\varphi_1, \varphi_2\}$. As in the previous example, we can define two weaving maps ω_N, ω'_N: $\mathfrak{T}_N \to \{\oplus, \ominus\}$ as $(\omega_N(\varphi_1), \omega_N(\varphi_2)) = (\oplus, \ominus)$ and $(\omega'_N(\varphi_1), \omega'_N(\varphi_2)) = (\ominus, \oplus)$, which yields the weavings $\mathcal{W}_N = (\mathcal{O}_{H_4}, \omega_N)$ and $\mathcal{W}'_N = (\mathcal{O}_{H_4}, \omega'_N)$ shown in Figs. 7(*c*)–7(*d*), respectively.

4.6. Weaving patterns

In this section we give some of the weaving patterns we derived from 2-colorings of the triangle tiling $\mathcal{T}(6, 2, 4)$ using the seven index-2 subgroups H_1 to H_7 of the triangle group $G(6, 2, 4)$. These weavings are shown in Fig. 8. The weavings $\mathcal{W}_M$ and $\mathcal{W}_N$ are derived from improper weavings $\mathcal{W}_{H_2}$ and $\mathcal{W}_{H_4}$, respectively, where $M = \langle Q, PQP, RQR, PRQRP\rangle$ is an index-2 subgroup of H_2 and $N = \langle P, RQRQ\rangle$ is an index-2 subgroup of H_4.

5. Weavings from mixed overlapping nets

In this section, we define and construct mixed overlapping nets by combining (*B*- or *W*-) nets of possibly distinct index-2 subgroups of $G(p, q, r)$. The following definition makes sense because of Proposition 1.

Definition 5. Let $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{N}' = (\mathcal{V}', \mathcal{E}')$ be (*B*- or *W*-) nets of index-2 subgroups H and H' of $G(p, q, r)$, respectively. The nets $\mathcal{N}$ and $\mathcal{N}'$ are said to be *compatible* if $\mathcal{E} \cap \mathcal{E}' = \emptyset$. A *mixed overlapping net* $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$ is the union of compatible nets $\mathcal{N}$ and $\mathcal{N}'$.

Consider the triangle group $G(4, 4, 2) = \langle P, Q, R | P^2 = Q^2 = R^2 = (QR)^4 = (RP)^4 = (PQ)^2 = I\rangle$, where I is the identity transformation together with its associated triangle tiling $\mathcal{T}(4, 4, 2)$. In Fig. 9, we give some of the (*B*- or *W*-) nets of the index-2 subgroups of $G(4, 4, 2)$ with the tiling $\mathcal{T}(4, 4, 2)$ in the background. For the purposes of our discussion, we use the following notations: let $\mathcal{N}_1$ be the *B*-net of $H_1 = \langle QR, RP\rangle$, $\mathcal{N}'_1$ the *W*-net of $H_1 = \langle QR, PQ\rangle$, $\mathcal{N}_2$ the *B*-net of $H_2 = \langle R, Q, PRP\rangle$, $\mathcal{N}_3$ the *B*-net of $H_3 = \langle P, R, QRQ\rangle$, $\mathcal{N}_4$ the *B*-net of $H_4 = \langle P, Q, RQR, RPR\rangle$, $\mathcal{N}_5$ the *B*-net of $H_5 = \langle P, QR, RPR\rangle$, $\mathcal{N}_6$ the *B*-net of $H_6 = \langle Q, RP\rangle$, and $\mathcal{N}_7$ the *B*-net of $H_7 = \langle R, PQ, QRQ\rangle$.

For example, the nets $\mathcal{N}_2$ and $\mathcal{N}_3$, in Figs. 9(*c*) and 9(*d*), are not compatible. If we combine these nets, some edges of $\mathcal{N}_2$ will lie on top of some edges of $\mathcal{N}_3$. We do not consider non-compatible nets in the construction of mixed overlapping nets to avoid such a scenario. On the other hand, the net $\mathcal{N}_1$ is compatible with all the other nets in Fig. 9. In Fig. 10, we give the mixed overlapping nets derived by combining the net $\mathcal{N}_1$ with each of the nets in Figs. 9(*b*)–9(*h*).

We now construct weavings on mixed overlapping nets. Let $\mathcal{O}_{\mathcal{N},\mathcal{N}'} = \mathcal{N} \cup \mathcal{N}'$ be a mixed overlapping net, where $\mathcal{N} = (\mathcal{V}, \mathcal{E})$ and $\mathcal{N}' = (\mathcal{V}', \mathcal{E}')$ are (*B*- or *W*-) nets of H and H', respectively. Note that, by construction, the subgroups H and H' act on the nets $\mathcal{N}$ and $\mathcal{N}'$, respectively. Consequently, $H_{\mathcal{N},\mathcal{N}'} = H \cap H'$ acts on the mixed overlapping net $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$. We define a *motif* $\Delta_{\mathcal{N},\mathcal{N}'}$ of a mixed overlapping net $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$ as a fundamental region of $H_{\mathcal{N},\mathcal{N}'} = H \cap H'$ (acting on $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$) contained in a constructible fundamental region Δ_H of H or $\Delta_{H'}$ of H' [both acting on $\mathcal{T}(p, q, r)$]. Thus, we can reconstruct $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$ from its motif $\Delta_{\mathcal{N},\mathcal{N}'}$ by letting $H_{\mathcal{N},\mathcal{N}'}$ act on $\Delta_{\mathcal{N},\mathcal{N}'}$. We

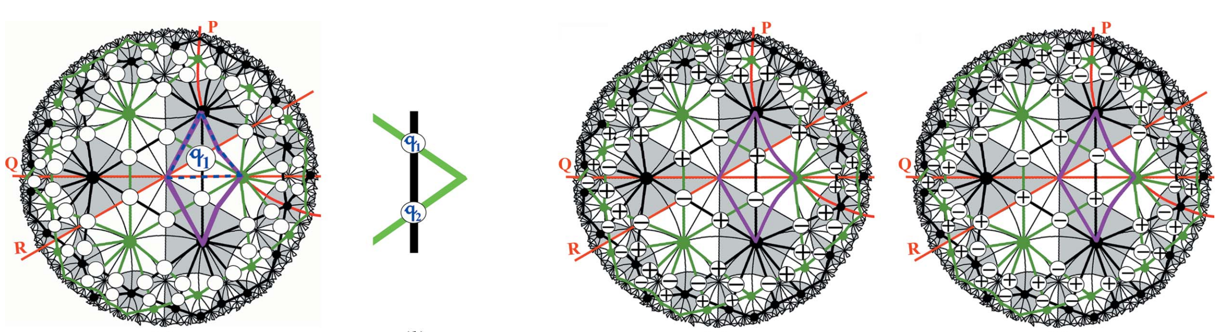


Figure 7
(*a*) The overlapping net $\mathcal{O}_{H_4}$, (*b*) the bigger motif Δ_N, and the weavings (*c*) $\mathcal{W}_N = (\mathcal{O}_{H_4}, \omega_N)$ and (*d*) $\mathcal{W}'_N = (\mathcal{O}_{H_4}, \omega'_N)$.

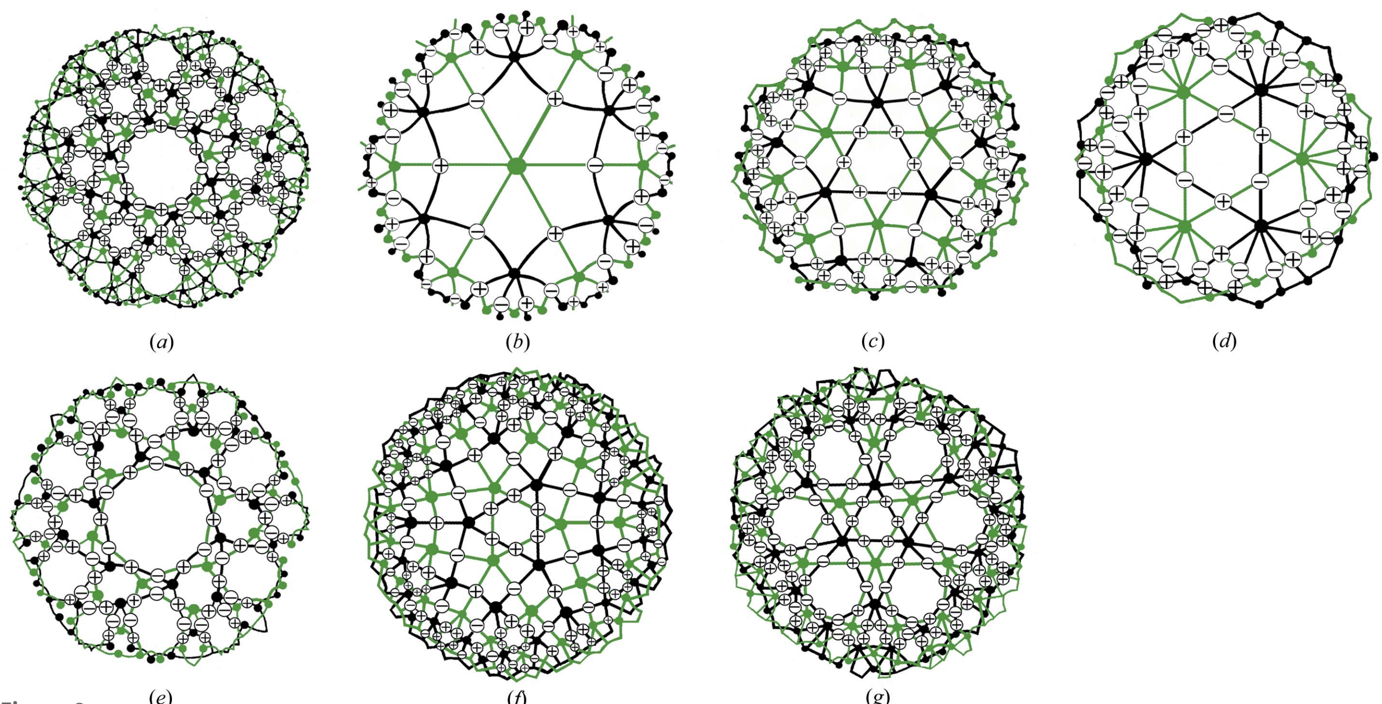


Figure 8
Weavings constructed from the index-2 subgroups of triangle group $G(6, 2, 4)$. (*a*) $\mathcal{W}_{H_1}$, (*b*) $\mathcal{W}_M$ from $\mathcal{W}_{H_2}$, (*c*) $\mathcal{W}_{H_3}$, (*d*) $\mathcal{W}_N$ from $\mathcal{W}_{H_4}$, (*e*) $\mathcal{W}_{H_5}$, (*f*) $\mathcal{W}_{H_6}$, (*g*) $\mathcal{W}_{H_7}$.

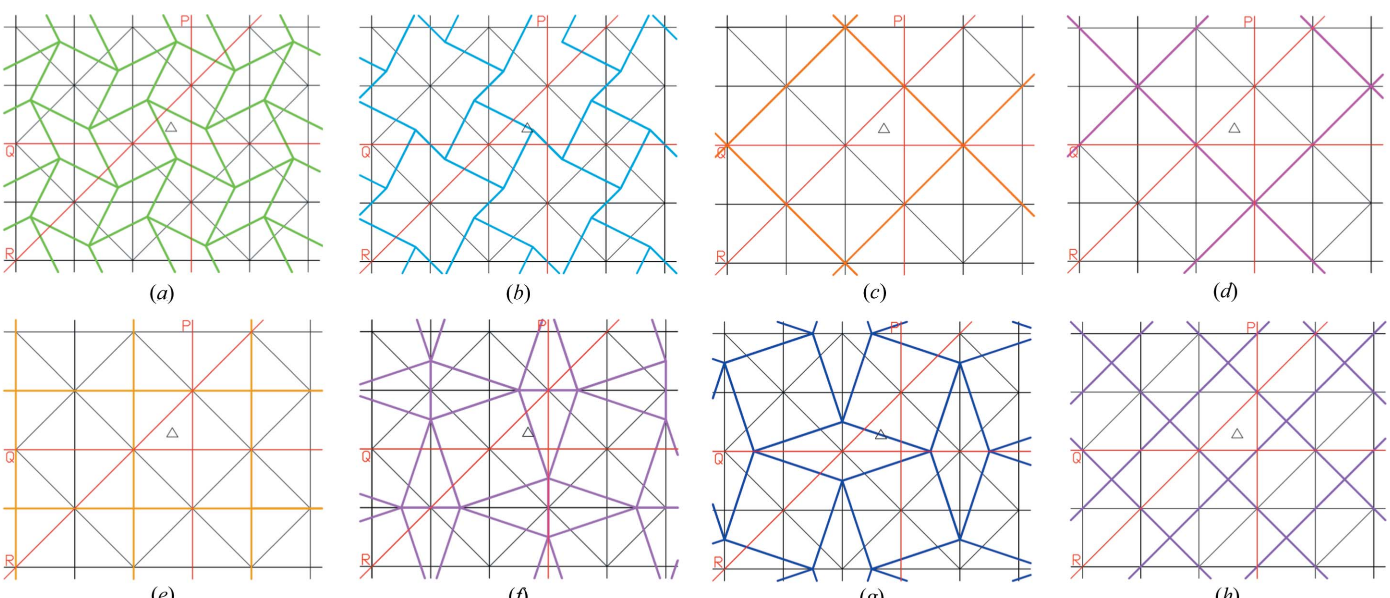


Figure 9
Some of the nets derived from the index-2 subgroups of $G(4, 4, 2)$. (*a*) $\mathcal{N}_1$, (*b*) $\mathcal{N}_1'$, (*c*) $\mathcal{N}_2$, (*d*) $\mathcal{N}_3$, (*e*) $\mathcal{N}_4$, (*f*) $\mathcal{N}_5$, (*g*) $\mathcal{N}_6$, (*h*) $\mathcal{N}_7$.

say that two weaving points p and p' on $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$ are *identifiable under* $H_{\mathcal{N},\mathcal{N}'}$ if $p = hp'$ for some $h \in H_{\mathcal{N},\mathcal{N}'}$.

As in §4.4, we first consider the set $\mathfrak{T}_{\mathcal{N},\mathcal{N}'} = \{(e, e') \in \mathcal{E} \times \mathcal{E}' | e \text{ and } e' \text{ intersect in } \mathcal{O}_{\mathcal{N},\mathcal{N}'}\}$ of weaving points of $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$. Let $\mathcal{P} \subset \mathfrak{T}_{\mathcal{N},\mathcal{N}'}$ be the set of weaving points of $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$ in the motif $\Delta_{\mathcal{N},\mathcal{N}'}$. Now, define the weaving map $\omega_{\mathcal{N},\mathcal{N}'} : \mathfrak{T}_{\mathcal{N},\mathcal{N}'} \to \{\oplus, \ominus\}$. As before, first assign values to $\omega_{\mathcal{N},\mathcal{N}'}(p)$ for each $p \in \mathcal{P}$ with every pair of identifiable weaving points assigned identical value, and then for each weaving point $q \in \mathfrak{T}_H$ define $\omega_{\mathcal{N},\mathcal{N}'}(q) := \omega_{\mathcal{N},\mathcal{N}'}(p)$ where $p \in \mathcal{P}$ such that $q = hp$ for some $h \in H_{\mathcal{N},\mathcal{N}'}$. By the same argument in proof of Proposition 2, this extension is well defined. Then the tuple $\mathcal{W}_{\mathcal{N},\mathcal{N}'} = (\mathcal{O}_{\mathcal{N},\mathcal{N}'}, \omega_{\mathcal{N},\mathcal{N}'})$ is a *weaving on the mixed overlapping net* $\mathcal{O}_{\mathcal{N},\mathcal{N}'}$.

As an illustration, consider the mixed overlapping net $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_7} = \mathcal{N}_1 \cup \mathcal{N}_7$ shown in Fig. 11(*a*), where $\mathcal{N}_1$ is the *B*-net of $H_1 = \langle QR, RP \rangle$ with constructible fundamental region $\Delta_{H_1} = \Delta \cup R\Delta$, and $\mathcal{N}_7$ is the *B*-net of $H_7 = \langle R, PQ, QRQ \rangle$

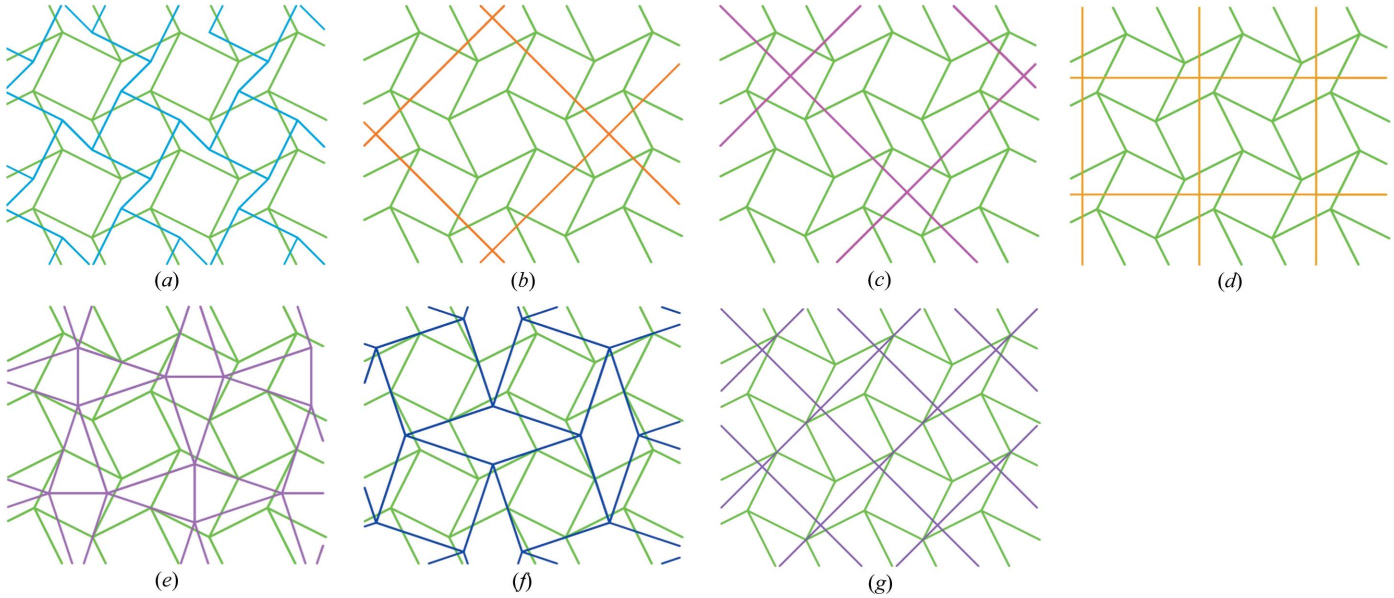


Figure 10
Some mixed overlapping nets. (*a*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_1'}$, (*b*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_2}$, (*c*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_3}$, (*d*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_4}$, (*e*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_5}$, (*f*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_6}$, (*g*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_7}$.

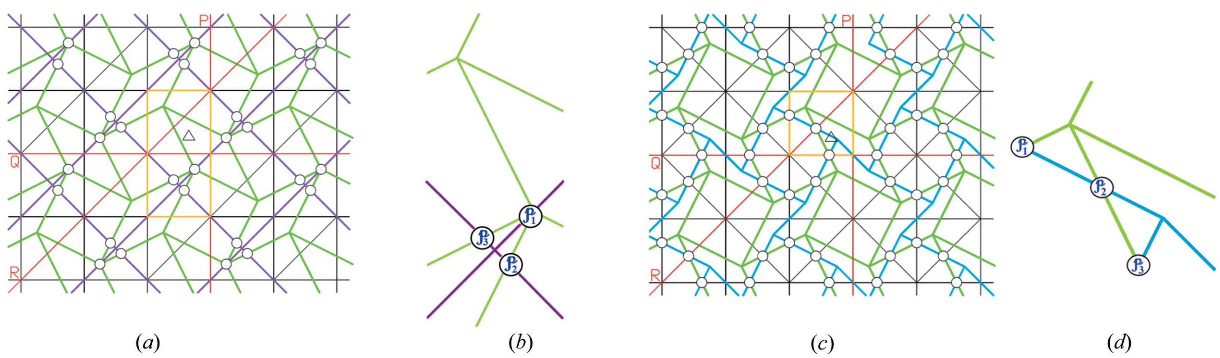


Figure 11
The mixed overlapping nets (*a*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_7}$ and (*c*) $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_1'}$ on $\mathcal{T}(4,4,2)$ with their weaving points and their motifs (*b*) $\Delta_{\mathcal{N}_1,\mathcal{N}_7}$ and (*d*) $\Delta_{\mathcal{N}_1,\mathcal{N}_1'}$.

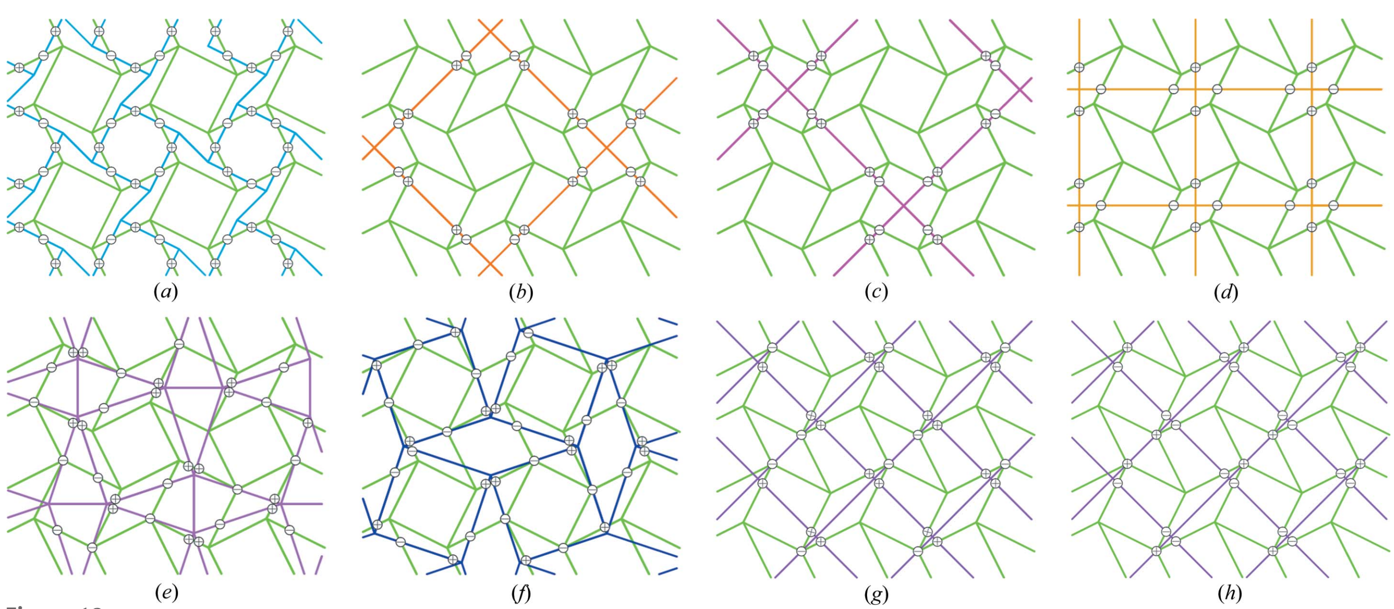


Figure 12
Weavings on mixed overlapping nets. (*a*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_1'}$, (*b*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_2}$, (*c*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_3}$, (*d*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_4}$, (*e*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_5}$, (*f*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_6}$, (*g*) $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_7}$, (*h*) $\mathcal{W}'_{\mathcal{N}_1,\mathcal{N}_7}$.

with constructible fundamental region $\Delta_{H_7} = \Delta \cup Q\Delta$. The area $\Delta_{H_1} \cup Q\Delta_{H_1}$, outlined in yellow in Fig. 11(*a*), contains the motif $\Delta_{\mathcal{N}_1,\mathcal{N}_7}$ of $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_7}$ shown in Fig. 11(*b*). The motif $\Delta_{\mathcal{N}_1,\mathcal{N}_7}$ has three non-identifiable weaving points under $H_{\mathcal{N}_1,\mathcal{N}_7} = \langle QP, RPRP, RQRP \rangle$, so there are eight possible weaving values. We define two weaving maps $\omega_{\mathcal{N}_1,\mathcal{N}_7}, \omega'_{\mathcal{N}_1,\mathcal{N}_7} : \mathfrak{T}_{\mathcal{N}_1,\mathcal{N}_7} \to \{\oplus, \ominus\}$ as $(\omega_{\mathcal{N}_1,\mathcal{N}_7}(p_1), \omega_{\mathcal{N}_1,\mathcal{N}_7}(p_2), \omega_{\mathcal{N}_1,\mathcal{N}_7}(p_3)) = (\ominus, \oplus, \oplus)$ and $(\omega'_{\mathcal{N}_1,\mathcal{N}_7}(p_1), \omega'_{\mathcal{N}_1,\mathcal{N}_7}(p_2), \omega'_{\mathcal{N}_1,\mathcal{N}_7}(p_3)) = (\oplus, \ominus, \ominus)$. The resulting weavings $\mathcal{W}_{\mathcal{N}_1,\mathcal{N}_7} = (\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_7}, \omega_{\mathcal{N}_1,\mathcal{N}_7})$ and $\mathcal{W}'_{\mathcal{N}_1,\mathcal{N}_7} = (\mathcal{O}_{\mathcal{N}_1,\mathcal{N}_7}, \omega'_{\mathcal{N}_1,\mathcal{N}_7})$ are shown in Figs. 12(*g*)–12(*h*).

For our final example, we construct a weaving on the mixed overlapping net $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}'_1} = \mathcal{N}_1 \cup \mathcal{N}'_1$. We note that $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}'_1}$ is a combination of two distinct nets derived from the same subgroup H_1, albeit *via* different generating sets: $\mathcal{N}_1$ is the *B*-net of $H_1 = \langle QR, RP \rangle$ with constructible fundamental region $\Delta_{H_1} = \Delta \cup R\Delta$, while $\mathcal{N}'_1$ is the *W*-net of $H_1 = \langle QR, PQ \rangle$ with constructible fundamental region $\Delta'_{H_1} = \Delta \cup Q\Delta$. In this case, $H_{\mathcal{N}_1,\mathcal{N}'_1}$ is H_1 itself and so we choose the motif $\Delta_{\mathcal{N}_1,\mathcal{N}'_1}$ to be the patch of $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}'_1}$ contained in Δ_{H_1} shown in Fig. 11(*d*). Note that there are three weaving points, say p_1, p_2 and p_3, on $\Delta_{\mathcal{N}_1,\mathcal{N}'_1}$, but p_1 and p_3 are identifiable under H_1 as $p_1 = RQ(p_3)$ and $RQ \in H_1$. A resulting weaving on $\mathcal{O}_{\mathcal{N}_1,\mathcal{N}'_1}$ is given in Fig. 12(*a*). In Fig. 12, we also give some weavings on the other mixed overlapping nets given in Fig. 10.

6. Outlook

In this study, the authors developed in graph-theoretic terms a methodical construction of weavings from combinatorial objects called nets defined on a colored tiling of the plane. It was shown that the methodology yields nets that are simple, connected infinite graphs. The union of these nets called an overlapping net is considered to form a weaving. Furthermore, the authors introduced the construction of mixed overlapping nets, which yield more general and fascinating weavings.

The procedure presented in this study is relevant because it provides an algorithmic approach to constructing and cataloging weavings, which may be useful in understanding the complex architecture of the molecular-scale weavings in MOFs and in covalent organic frameworks. Furthermore, the resulting weavings are of interest because they consist of a symmetric net, and symmetric nets, among other things, are important in the formation of MOFs (Hyde *et al.*, 2016).

In the next part of the study, we are looking at extending the definition of *B*- and *W*-vertices to include more points other than the centroids of the *B*- and *W*-patches. Another goal is to consider constructing nets, (mixed) overlapping nets and weavings from *n*-colorings of triangle tilings for $n > 2$. Moreover, we also seek to investigate the properties of the weavings constructed from the methodology. A notion of equivalent weavings has been developed by the authors in a separate paper, and an investigation on the equivalence of weavings on (mixed) overlapping nets is a consequent goal. Finally, the ultimate goals of this research project are to explore which of the resulting weavings are equivalent to the molecular-scale weavings of coordination networks, and to use these weavings to construct three-periodic patterns using the methodology given in Ramsden *et al.* (2009).

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